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Density Properties of Sets

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DENSITY PROPERTIES OF SETS

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ABSTRACT

Dr. H. L. Smith in a paper which has not been published as yet shows that by starting with a general function satisfying Caratheodory's first two postulates on an outer measure function, it is possible to construct a function which satisfies all four postulates. In this dissertation we have studied some of the characteristics of this function, principally those which are of use in deriving our theorems on density.

We have set up three general density functions, and have succeeded in showing that for the most general of these, it is true that the Smith measure of the set at which the upper density is less than 1, and the one at which the lower density is greater than 1, is zero. We have also established that under certain circumstances there is a definite relation between this function and the density function defined by Besicovitch.

Considerable attention has been devoted to certain fundamental geometric theorems, which have led us to a generalized form of the Vitali theorem. We have also derived a set of sufficient conditions for the validity of this theorem, including the one for the Smith function.

DENSITY PROPERTIES OF SETS

INTRODUCTION

The past seventy five years have seen a great deal of research into the fundamental concepts of mathematics. In the course of these investigations the theory back of our concept of measure has come in for careful scrutiny. Various theories have been proposed during this time, culminating with Caratheodory's general theory, based on the four postulates included in the body of this thesis.

Dr. H. L. Smith in a paper which is as yet unpublished generalizes this concept further, by showing how it is possible to start with a function satisfying only the first two postulates and then building up one which satisfies all four. In this paper we derive some of the essential characteristics of this function.

A. S. Besicovitch using Cartheodory's measure has shown that the geometric nature of a measurable plane set depends on the value of his density function. We have constructed three new density functions using the Smith measure function in place of Caratheodory's, and have derived some of their essential relations.

In addition to this we have included in this dissertation a section on elementary covering theorems including a generalized form of the Vitali theorem and certain sufficient conditions for its validity.

COVERING THEOREMS

1. PRELIMINARY THEOREMS ON THE SUBSETS OF AN ARBITRARY CLASS.

There are a number of theorems closely related to the covering theorems for which no special hypotheses are required on the space, that is they are true for a perfectly general class $\mathcal{P}$ of elements p . This section shall be devoted to these theorems. In the interest of simplicity we lay down the following definition:

A family $\mathcal{F}$ of sets S is said to be <u>separable</u> if $\Sigma\mathcal{F}$, (the set of all points or elements that belong to the various classes that make up $\mathcal{F}$), contains a countable subset P such that every set S of $\mathcal{F}$ contains at least one point of P . In this definition and throughout this paper we use the word countable to mean either finite or countably infinite.

We now state:

<u>Theorem 1.</u> If $\mathcal{F}$ is a separable family of sets S , then $\mathcal{F}$ contains a countable subset $s_1, s_2, \cdots$ such that

(1.1) $\quad s_m \cdot s_n = 0 \qquad (m < n)$

(1.2) $\quad S\,(s_1 + s_2 + \cdots) > 0$ for every S of $\mathcal{F}$.

<u>Proof:</u> The proof will be divided into two cases.

Case I. There exists a finite set $s_1, \ldots, s_n$ of sets of $\mathcal{F}$ such that

a) $S_m \cdot S_n = 0 \quad (m < n)$

b) $S(s_1 + \cdots + s_n) > 0$ for every S of $\mathcal{F}$.

The theorem is clearly true in this case.

Case II. There is no finite set $S_1, \ldots S_n$ satisfying the conditions of Case I.

Let $p_1, p_2, \ldots$ be a denumeration of the points of P, where P is the countable subset of the definition given above. Let n_1 be the smallest value of n such that p_n belongs to some set say S_1 of $\mathcal{F}$. Let n_2 be the smallest value of n such that p_n belongs to some set say S_2 of $\mathcal{F}$ and such that $S_1 \cdot S_2 = 0$. By induction we secure sequences $\{n_k\}$, $\{S_k\}$ such that

(1.3) $n_1 < n_2 < n_3 < \cdots$

(1.4) $S_k \in \mathcal{F} \quad (k)$

(1.5) $S_k \cdot S_j = 0 \quad (k > j)$

(1.6) $(S_1 + \cdots + S_{k-1})\, S = 0,\ S \in \mathcal{F},\ p_n \in S \ \cdot)\cdot\ n \geqq n_k.$

Now let S be any set of $\mathcal{F}$ and take $p_n \in S$. Let k_0 be the smallest value of k such that $n_k > n$.

Then $(S_1 + \cdots + S_{k_0 - 1})\, S > 0$ for otherwise we would have $n_{k_0} \leqq n$ which is contrary to fact. Hence $S \sum_n S_n > 0$ for every S in $\mathcal{F}$.

There is now a question of notation which we wish to make clear. If $\mathcal{F}$ is any family of sets S and $f(s)$ is a function of the sets S we represent by $\overline{B} f(\mathcal{F})$ the least upper bound of all $f(s)$ such that $S \epsilon \mathcal{F}$. In general we will use $\overline{B}$ and $\underline{B}$ to indicate the least upper bound and greatest lower bound, respectively.

Using this notation we are now able to prove

Theorem 2. If $\mathcal{F}$ is a separable family of sets S and $f(s)$ is a function on S such that

(2.1) $f(s) > 0 \qquad (s \epsilon \mathcal{F})$

(2.2) $\overline{B} f(s) < +\infty$

and e is any positive number, then there wxists in $\mathcal{F}$ a countable subset $s_1, s_2, \ldots$ which satisfies the following conditions:

$$A_1 : S_m \cdot S_n = 0 \qquad (m < n)$$

$$A_2(e, \mathcal{F}) : S \epsilon \mathcal{F} \;\cdot)\cdot\; f(s) \leq (1+e)\, f(s_n) \qquad (n)$$

$$A_3(e, \mathcal{F}) : S \epsilon \mathcal{F},\ S \sum_n S_n = 0 \;\cdot)\cdot\; f(s) < \frac{1}{1+e} \overline{B} f(\mathcal{F})$$

Proof: Set $\mathcal{F}_0 \equiv \left[all\ S \ni S \epsilon \mathcal{F}, f(S) \geq \frac{1}{1+e} \overline{B} f(\mathcal{F}) \right]$

This bracket notation will be used at all times to indicate the set or class of all points or elements whose description is contained within the bracket, thus $\mathcal{F}_0$ is the class of all sets S such that $S \epsilon \mathcal{F}$ and $f(s) \geq \frac{1}{1+e} \overline{B} f(\mathcal{F})$

Now by theorem 1, $\mathcal{F}_0$ contains a countable subset $s_1, s_2, \ldots$ such that A is true and

(3.3) $S \in \mathcal{F}_0 \cdot) \cdot S \sum_n S_n > 0$

In order to prove $A_2(e, \mathcal{F})$, we take $S \in \mathcal{F}$ then $f(s) \leq \bar{B} f(\mathcal{F})$.
and since $S_n \in \mathcal{F}_0$ we have

$$f(S_n) \geq \frac{1}{1+e} \bar{B} f(\mathcal{F})$$

from which it follows that

$$f(s) \leq (1+e) f(S_n) \qquad (n)$$

$A_3(e, \mathcal{F})$ follows from the fact that if $S \sum_n S_n = 0$ then S does not belong to $\mathcal{F}_0$ and therefore $f(s) < \frac{1}{1+e} \bar{B} f(\mathcal{F})$.

<u>Theorem 3.</u> Under the same hypotheses as in theorem 2 there exists a countable subset $s_1, s_2, \ldots$ of $\mathcal{F}$ such that

(3.1) $S_m \cdot S_n = 0 \qquad (m < n)$

(3.2) If S is any set of $\mathcal{F}$ there exists an n such that $S \cdot S_n > 0$ and $f(s) \leq (1+e) f(S_n)$

<u>Proof:</u> Set $\mathcal{F}_0 \equiv \mathcal{F}$, then by theorem 2, there is a sequence of sets $\{S_{0n}\}$ satisfying $A_1, A_2(e, \mathcal{F}), A_3(e, \mathcal{F})$
Now set

$$\mathcal{F}_1 \equiv [\text{all } S \ni S \in \mathcal{F}_0,\ S \sum_n S_{0n} = 0]$$

On applying theorem 2 to $\mathcal{F}_1$ we obtain a sequence $\{S_{1n}\}$ which satisfies $A_1, A_2(e, \mathcal{F}_1), A_3(e, \mathcal{F}_1)$

On continuing this process we obtain a sequence of sets $\{S_{mn} \mid n\}$ $(m = 0, 1, 2 \ldots)$ satisfying $A_1, A_2(e, \mathcal{F}_m), A_3(e, \mathcal{F}_m)$ where $\mathcal{F}_m$ $(m = 0, 1, 2, \ldots)$ are subsets of $\mathcal{F}_0$ such that

(3.3) $\mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \ldots$

Now since $A_3\ (e, \mathcal{F}_m)$ holds we see that
$\overline{B}f(\mathcal{F}_m) \leq \frac{1}{1+e}\,\overline{B}f(\mathcal{F}_{m-1})$
and therefore
$\overline{B}f(\mathcal{F}_m) \leq \frac{1}{(1+e)^m}\,\overline{B}f(\mathcal{F}_0)$
so that

(3.4) $\lim_m \overline{B}f(\mathcal{F}_m) = 0$

Now

(3.5) $S_{m'n'} \cdot S_{m''n''} = 0 \qquad (m', n') \neq (m'', n'')$

For if $m' = m''$ this is clearly true since $\{S_{m'n} | n\}$ satisfies A_1. If $m' \neq m''$ we may suppose $m' < m''$. Then $S_{m''n''} \in \mathcal{F}_{m''}$ and hence $S_{m''n''} \in \mathcal{F}_{m'+1}$ since $m'' \geq m'+1$. But every S in $\mathcal{F}_{m'+1}$ has no point in common with any $S_{m'n}$ by the definition of $\mathcal{F}_{m'+1}$ and so (3.5) is proven.

Now suppose $S \in \mathcal{F}\ (= \mathcal{F}_0)$ and let m be the smallest integer such that $S \in \mathcal{F} - \mathcal{F}_m$. It is evident that such an integer exists by (3.4). Then S does not belong to $\mathcal{F} - \mathcal{F}_{m-1}$ and hence $S \in \mathcal{F}_{m-1}$; further, since S belongs to $\mathcal{F}_{m-1}$ but not to $\mathcal{F}_m$, $S \sum_n S_{m-1,n} > 0$. It is also true that $\{S_{m-1,n}\}$ satisfies $A_2(e, \mathcal{F})$ and therefore $f(S) \leq (1+e) f(S_{m-1,n})$. Hence the double sequence $\{S_{mn} | m, n\}$ satisfies conditions (3.1) and (3.2) which completes the proof of the theorem.

2. COVERING THEOREMS FOR A METRIC SPACE

Up to the present the only thing we have used is that our space was a completely arbitrary class $\mathcal{P}$ but now we are forced to have some additional postulates on it. We shall assume that we have a metric space $\Sigma = (\mathcal{P}, \Delta)$ that is a class $\mathcal{P}$ of elements p and a real-valued function Δ on $\mathcal{P}, \mathcal{P}$ such that

I $\quad \Delta(p_1, p_2) > 0 \ (p_1 \neq p_2); \ \Delta(p, p) = 0$

II $\quad \Delta(p_1, p_2) = \Delta(p_2, p_1)$

III $\quad \Delta(p_1, p_2) + \Delta(p_2, p_3) \geqq \Delta(p_1, p_3)$

If S is a subset of $\mathcal{P}$ and p is any element of $\mathcal{P}$ we define

$$\underline{\Delta}(p, S) \equiv \underline{B}\ [all\ \Delta(p, p') \ni p' \in S]$$

The e -<u>neighborhood</u> of a subset S of $\mathcal{P}$ is defined by the equation

$$N(S, e) \equiv [all\ p \ni \underline{\Delta}(p, S) \leqq e]$$

and the <u>diameter</u> of a set S by

$$d(S) \equiv \overline{B}\ [all\ \Delta(p_1, p_2) \ni p_1 \in S, p_2 \in S]$$

We have the obvious inclusion $N(S, e) \geqq S$.

<u>Theorem 4.</u> If $S_1 \cdot S_2 > 0$ then

$$S_1 \leqq N(S_2, d(S_1))$$

<u>Proof:</u> Let $p_1 \in S_1$, $p_{12} \in S_1 \cdot S_2$ then

$$\underline{\Delta}(p_1, S_2) \leqq \Delta(p_1, p_{12}) \leqq d(S_1)$$

and so $p_1 \in N(S_2, d(S_1))$ which proves the theorem.

<u>Theorem 5</u>. If $\mathcal{F}$ is a separable family of sets S, such that $\overline{B}d(\mathcal{F}) < +\infty$ and e is a positive number, then there exists a countable subset $S_1, S_2 \ldots$ of $\mathcal{F}$ such that

(5.1) $S_m \cdot S_n = 0 \quad (m < n)$

(5.2) $\sum_n N(S_n, (1+e)\, d(S_n)) \supseteq \Sigma\mathcal{F}$

<u>Proof:</u> Let the sets $S_1, S_2 \ldots$ be determined by theorem 3, with $f(S) = d(S)$, then if $S \epsilon \mathcal{F}$, there is an S_n such that

(5.3) $S \cdot S_n > 0$ and $d(S) \leqq (1+e)\, d(S_n)$

But by theorem 4 we have

$$S \subseteq N(S_n, d(S)) \subseteq N(S_n, (1+e)\, d(S_n)) \subseteq \sum_n N(S_n, (1+e)\, d(S_n))$$

which proves the theorem.

The closed sphere with center at a, and radius r is defined as follows:

$$S(a, r) \equiv [\text{all } p \ni \Delta(a, p) \leqq r]$$

As an immediate consequence of this definition we have

<u>Theorem 6.</u> For every sphere $S(a, r)$, it is true that

$$d(S(a, r)) \leqq 2r$$

We may further state

<u>Theorem 7.</u> For every $a \epsilon \mathcal{P}$ and $r > 0$, $e > 0$ it is true that

$$N(S(a, r), e) \subseteq S(a, r+e)$$

<u>Proof:</u> Suppose $p \epsilon N(S(a, r), e)$. Then $\Delta(p, S(a, r)) \leqq e$

Hence there is a sequence $\{p_n\}$ such that

(7.1) $p_n \in S(a,r) \quad (n)$ and $\Delta(p,p_n) \leq e + \frac{1}{n} \quad (n)$

Then

(7.2) $\Delta(p,a) \leq \Delta(p,p_n) + \Delta(p_n,a) \leq e + \frac{1}{n} + r \quad (n)$

so that

(7.3) $\Delta(p,a) \leq e + r$ or $p \in S(a, r+e)$

which completes the proof.

<u>Theorem 8.</u> If $\mathcal{F}$ is a separable family of spheres such that $\bar{B}r(\mathcal{F}) < +\infty$ and e is a positive number then there is a countable subset $S_1, S_2 \ldots\ldots$ of $\mathcal{F}$ such that

(8.1) $S_m \cdot S_n = 0 \qquad (m < n)$

(8.2) If S_n^* is the sphere concentric with S_n and with radius $(3+e)$ times as great then

$$\sum_n S_n^* \geq \sum \mathcal{F}$$

<u>Proof:</u> Let $S_1, S_2, \ldots\ldots$ be the set of spheres obtained by applying theorem 5 with e replaced by $\frac{e}{2}$. Then

$$\sum \mathcal{F} \leq \sum_n N(S_n, (1+\tfrac{e}{2})\, d(S_n)) \leq \sum_n N(S_n, (2+e)\, r(S_n))$$
$$\leq \sum_n S(z(S_n), r(S_n) + (2+e)\, r(S_n)) = \sum_n S_n^*$$

where $z(S_n)$ is the center of S_n.

3. COVERING THEOREMS FOR A METRIC SEPARABLE SPACE

If we strengthen the hypotheses on our space and assume it to be a metric separable space, that is, one which contains a countable subset whose closure is the space itself, we may weaken the hypotheses in our theorems. Thus we may state:

Theorem 9. If $\mathcal{F}$ is a family of spheres, then there is a countable subset $S_1, S_2 \ldots\ldots$ of $\mathcal{F}$ such that

(9.1) $S_m \cdot S_n = 0 \quad (m < n)$

(9.2) $S(S_1 + S_2 + \ldots\ldots) > 0$ for every S of $\mathcal{F}$

This theorem is an immediate consequence of theorem 1.

Theorem 10. If $\mathcal{F}$ is a family of spheres with $\overline{B}d(\mathcal{F}) < +\infty$ and e is a positive number, then there is a countable subset $S_1, S_2 \ldots\ldots$ of $\mathcal{F}$ which satisfies the following three conditions

$$A_1 : S_m \cdot S_n = 0 \quad (m < n)$$

$$A_2(e,\mathcal{F}) : S \in \mathcal{F} \cdot) \cdot d(S) \leqq (1+e)\, d(S_n) \quad (n)$$

$$A_3(e,\mathcal{F}) : S \in \mathcal{F},\ S \sum_n S_n = 0 \cdot) \cdot d(S) < \frac{1}{1+e} \overline{B}d(\mathcal{F})$$

This is nothing but a restatement of theorem 2.

Theorem 11. If $\mathcal{F}$ is a family of spheres with $\overline{B}d(\mathcal{F}) < +\infty$ and e is a positive number, then there is a countable subset $S_1, S_2 \ldots\ldots$ of $\mathcal{F}$ such that

(11.1) $S_m \cdot S_n = 0 \qquad (m < n)$

(11.2) $\sum_n S_n^* \geqq \Sigma \mathcal{F}$

where S_n^* is the sphere concentric with S_n and radius $(3+e)$ times as great.

This theorem is merely a corollary of theorem 8.

4. THE VITALI THEOREM

We shall now devote our attention to the most useful of all covering theorems, the Vitali theorem. We shall assume our space to be the general metric space of section 2. We now make the following definition:

A set A is said to have the Vitali property $(\mu^*, \mathcal{F})$, that is, relative to an outer measure function μ^* and a family $\mathcal{F}$ of closed sets F if there exists a ϑ between 0 and 1 such that to every open set G there corresponds a finite subset of $\mathcal{F}$: $F_1, \ldots\ldots F_n$ such that

I $\quad F_i \cdot F_j = 0 \quad (1 \leq i < j \leq n)$

II $\quad F_i \subseteq G \quad (i = 1, 2, \ldots\ldots n)$

III $\quad \mu^*(AG - \sum_{i=1}^{n} F_i) \leq \vartheta \mu^*(AG)$

It would be well to note at this point that if a set A has the Vitali property $(\mu^*, \mathcal{F})$ and $\mathcal{F}_1 \supseteq \mathcal{F}$ then A has the Vitali property $(\mu^*, \mathcal{F}_1)$

In proving the Vitali theorem we will need the following:

<u>Lemma:</u> If a set A has the Vitali property $(\mu^*, \mathcal{F})$ and also $(\nu^*, \mathcal{F})$ then there exists a number ϑ between 0 and 1 such that to every open set G there corresponds a subset $F_1, \ldots\ldots F_n$ of $\mathcal{F}$ such that

(1) $\quad F_i \cdot F_j = 0 \quad (1 \leq i < j \leq n)$

(2) $\quad F_i \subseteq G \quad (i = 1, 2, \ldots\ldots n)$

(3) $\mu^*(AG - \sum_{i=1}^{n} F_i) \leqq \vartheta \mu^*(AG)$

(4) $\nu^*(AG - \sum_{i=1}^{n} F_i) \leqq \vartheta \nu^*(AG)$

<u>Proof:</u> Let ϑ', ϑ'' be the numbers between 0 and 1 which are given by the hypothesis. Take ϑ to be the larger of the two. Then to any G there corresponds a sequence $F_1, \ldots\ldots F_m$ in $\mathcal{F}$ such that

(5) $F_i \cdot F_j = 0 \qquad (1 \leqq i < j \leqq m)$

(6) $F_i \leqq G \qquad (i = 1, \ldots\ldots m)$

(7) $\mu^*(AG - \sum_{i=1}^{m} F_i) \leqq \vartheta' \mu^*(AG) \leqq \vartheta \mu^*(AG)$

Now take

(8) $G_1 = G - \sum_{i=1}^{m} F_i$

Then to G_1 there corresponds a sequence $F_{m+1}, \ldots\ldots F_n$ in $\mathcal{F}$, such that

(9) $F_i \cdot F_j = 0 \qquad (m+1 \leqq i < j \leqq n)$

(10) $F_i \leqq G_1 \qquad (i = m+1, \ldots\ldots n)$

(11) $\nu^*(AG_1 - \sum_{i=m+1}^{n} F_i) \leqq \vartheta'' \nu^*(AG_1) \leqq \vartheta \nu^*(AG)$

It now follows from (8) and (10) that

(12) $F_i \cdot F_j = 0 \qquad (i < m, \; j > m)$

which with the help of (5) and (9) proves (1).

We also have from (8) and (10) that

(13) $F_i \leqq G_1 \leqq G \qquad (i = m+1, \ldots\ldots n)$

which in connection with (6) gives us (2).

In addition to this it follows that

$$\mu^*(AG - \sum_{i=1}^{n} F_i) \leqq \mu^*(AG - \sum_{i=1}^{m} F_i) \leqq \vartheta \mu^*(AG)$$

and

$$\nu^*(AG - \sum_{i=1}^{n} F_i) \leq \nu^*(AG_1 - \sum_{i=m+1}^{n} F_i) \leq \vartheta\,\nu^*(AG_1) \leq \vartheta\,\nu^*(AG)$$

which completes the proof of the lemma.

Theorem 12. (Vitali) If A has the Vitali property $(\mathcal{M}^*, \mathcal{F})$ and also $(\nu^*, \mathcal{F})$ then there is in $\mathcal{F}$ a countable set $\{F_n \mid n\}$ of sets such that

(12.1) $F_m \cdot F_n = 0 \qquad (m < n)$

(12.2) $\mathcal{M}^*(A - \sum_n F_n) = 0$

(12.3) $\nu^*(A - \sum_n F_n) = 0$

Proof: Let G_0 be the entire space. Then by the lemma there corresponds to G_0 a sequence $F_1, \ldots\ldots F_{n_0}$ from $\mathcal{F}$ such that

(12.4) $F_i \cdot F_j = 0 \qquad (1 \leq i < j \leq n_0)$

(12.5) $\mathcal{M}^*(A - \sum_{i=1}^{n_0} F_i) = \mathcal{M}^*(AG_0 - \sum_{i=1}^{n_0} F_i) \leq \vartheta\mathcal{M}^*(A)$

(12.6) $\nu^*(A - \sum_{i=1}^{n_0} F_i) = \nu^*(AG_0 - \sum_{i=1}^{n_0} F_i) \leq \vartheta\,\nu^*(A)$

Now take

(12.7) $G_1 = G_0 - \sum_{i=1}^{n_0} F_i$

Then to G_1 there corresponds a sequence $F_{n_0+1}, \ldots\ldots F_{n_1}$ from $\mathcal{F}$ such that

(12.8) $F_i \cdot F_j = 0 \qquad (n_0+1 \leq i < j \leq n_1)$

(12.9) $F_i \leq G_1 \qquad (i = n_0+1, \ldots\ldots n_1)$

(12.10) $\mathcal{M}^*(AG_1 - \sum_{i=n_0+1}^{n_1} F_i) \leq \vartheta\mathcal{M}^*(AG_1)$

(12.11) $\nu^*(AG_1 - \sum_{i=n_0+1}^{n_1} F_i) \leq \vartheta\,\nu^*(AG_1)$

From (12.4), (12.8), (12.9) we have

(12.12) $F_i \cdot F_j = 0 \quad (1 \leqq i < j \leqq n_1)$

It is further clear that

(12.13) $\mu^*(A - \sum_{i=1}^{n_1} F_i) \leqq \mu^*(AG_1 - \sum_{i=n_0+1}^{n_1} F_i) \leqq \vartheta \mu^*(AG_1) \leqq \vartheta^2 \mu^*(A)$

and similarly

(12.14) $\nu^*(A - \sum_{i=1}^{n_1} F_i) \leqq \vartheta^2 \nu^*(A)$

On continuing this process we secure sequences $\{F_r | r\}$ $\{n_r\}$ such that

$$\mu^*(A - \sum_i F_i) \leqq \mu^*(A - \sum_{i=1}^{n_r} F_i) \leqq \vartheta^r \mu^*(A)$$

$$\nu^*(A - \sum_i F_i) \leqq \vartheta^r \nu^*(A)$$

from which the theorem follows.

5. SUFFICIENT CONDITIONS THAT A SET HAVE THE VITALI PROPERTY ($\mu^*, \mathcal{F}$).

The question now naturally arises as to when a set A, has the Vitali property. We shall answer this by the theorems in this section. Our space will be taken to be the same metric space that we had in the development of the Vitali theorem itself. We start with

__Theorem 13.__ If A has the Vitali property $(\mu^*, \mathcal{F})$ and if $\mathcal{F}_1$ is a family of closed sets and k a positive number such that to each F in $\mathcal{F}$ there corresponds an F_1 in $\mathcal{F}_1$ such that $F_1 \leqq F$ and $\mu^*(AF_1) \geqq k\,\mu^*(AF)$, then A has the Vitali property $(\mu^*, \mathcal{F}_1)$.

__Proof:__ Let G be any open set. Then there are sets $F_1, \ldots F_n$ of $\mathcal{F}$ such that

(13.1) $F_i \cdot F_j = 0 \quad (1 \leq i < j \leq n)$

(13.2) $F_i \subseteq G \quad (i = 1, \ldots\ldots n)$

(13.3) $\mathcal{M}^* (AG - \sum_{i=1}^{n} F_i) \leq \vartheta \mathcal{M}^* (AG)$

Now take $F_{11}, \ldots\ldots F_{n1}$ in $\mathcal{F}_1$, so that

(13.4) $F_{i1} \subseteq F_i \quad (i = 1, 2, \ldots\ldots n)$

(13.5) $\mathcal{M}^* (A F_{i1}) \geq k \mathcal{M}^* (A F_i) \quad (i = 1, \ldots\ldots n)$

We notice that necessarily $0 < k < 1$.

Now

(13.6) $\mathcal{M}^* (AG) = \mathcal{M}^* (AG - \sum_{i=1}^{n} F_{i1}) + \mathcal{M}^* (\sum_{i=1}^{n} A F_{i1})$

so that by (13.5) and (13.6) we have

(13.7) $\mathcal{M}^* (AG - \sum_{i=1}^{n} F\ ) = \mathcal{M}^* (AG) - \mathcal{M}^* (\sum_{i=1}^{n} A F_{i1})$

$$\leq \mathcal{M}^* (AG) - k \mathcal{M}^* (\sum_{i=1}^{n} A F_i)$$

Similarly by (13.2), (13.3) we have

(13.8) $\mathcal{M}^* (\sum_{i=1}^{n} A F_i) = \mathcal{M}^* (AG) - \mathcal{M}^* (AG - \sum_{i=1}^{n} F_i)$

$$\geq (1 - \vartheta) \mathcal{M}^* (AG)$$

So that we have from (13.7) and (13.8)

$$\mathcal{M}^* (AG - \sum_{i=1}^{n} F_{i1}) \leq \{1 - k (1 - \vartheta)\} \mathcal{M}^* (AG)$$

which completes the proof of the theorem.

<u>Theorem 14.</u> Let $A, \mathcal{M}^*$ and $\mathcal{F} = [F]$ be such that

(14.1) for every G it is true that the set $\mathcal{F}^G$ of all F of $\mathcal{F}$ such that $F \subseteq G$, covers A;

(14.2) there exists a positive number k and to each F of $\mathcal{F}$ there corresponds an F^* not necessarily in $\mathcal{F}$ such that

a) $F \subseteq F^*$

b) $\mu^*(AF) \geqq k\mu^*(AF^*)$

c) any subset $\mathcal{F}_0$ of $\mathcal{F}$ contains a countable sequence $F_1, F_2, F_3, \ldots\ldots$ such that

1') $F_i \cdot F_j = 0 \quad (i < j)$

2') $\sum_i F_i^* \geqq \sum \mathcal{F}_0$

Then A has the Vitali property $(\mu^*, \mathcal{F})$.

Proof: We observe that k as given in (b) is necessarily less than or equal to 1. Now let G be any open set and consider $\mathcal{F}^G$; by (14.1) we have

(14.3) $\sum \mathcal{F}^G \geqq AG$

and by (c) there is a countable subset of $\mathcal{F}^G$; $F_1, F_2, \ldots\ldots$ such that

(14.4) $F_i \cdot F_j = 0 \quad (i < j)$

(14.5) $\sum_i F_i^* \geqq \sum \mathcal{F}^G$

from which it follows that

(14.6) $\sum_i F_i^* \geqq AG$

so that

(14.7) $\sum_i \mu^*(AF_i^*) \geq \mu^*(AG)$

which in connection with (b) gives

(14.8) $\sum_i \mu^*(AF_i) \geq k\mu^*(AG)$

Therefore there is an n such that

(14.9) $\sum_{i=1}^{n} \mu^*(AF_i) \geq \frac{1}{2} k\mu^*(AG)$

But

(14.10) $\mu^*(AG) = \mu^*(AG - \sum_{i=1}^{n} F_i) + \mu^*(\sum_{i=1}^{n} AF_i) = \mu^*(AG \sum_{i=1}^{n} F_i) + \sum_{i=1}^{n} \mu^*(AF_i)$

so that

(14.11) $\mu^*(AG - \sum_{i=1}^{n} F_i) = \mu^*(AG) - \sum_{i=1}^{n} \mu^*(AF_i) \leqq (1 - \frac{1}{2}k)\mu^*(AG)$

which is the theorem.

Before continuing we make the following definitions:

A family of spheres $\mathcal{F}$ is said <u>to cover a set</u> A <u>in the strict Vitali sense</u> if at every point $a \in A$ there is a sequence of spheres $\{S(a, r_n(a))\}$ such that $\lim_n r_n(a) = 0$.

A family $\mathcal{F}$ of spheres is said <u>to cover a set</u> A <u>in the strict Vitali sense relative to</u> μ^* if there is a positive number k such that to every open set G and positive number e there is a U_e such that

I $AG \subseteq U_e$ (e)

II $\mu^*(U_e) \leqq \mu^*(AG) + e$ (e)

III the class of all S of $\mathcal{F}$ such that $\mu^*(U_e S) \geqq k\mu^*(U_e S^*)$ covers AG in the strict Vitali sense. (Here S^* as usual denotes the sphere concentric with S and having radius 4 times as great.)

A family $\mathcal{F}$ of spheres is said <u>to cover a set</u> A <u>in the strict Vitali sense strongly relative to</u> μ^* if there is a positive number k such that the set of all spheres S of $\mathcal{F}$ such that $\mu^*(AS) \geqq k\mu^*(AS^*)$ covers A in the strict Vitali sense.

<u>Theorem 15</u>. If a family $\mathcal{F}$ of spheres covers a set A in the Vitali sense strongly relative to μ^* then it covers A in the strict Vitali sense relative to μ^*.

Proof: Let $\mathcal{F}_1$ denote the class of all S of $\mathcal{F}$ such that $\mu^*(AS) \geq k\, \mu^*(AS^*)$. Then $\mathcal{F}_1$ covers A in the strict Vitali sense.

Now set

(15.1) $\mathcal{F}_1{}^G \equiv [\text{all } S \ni S \in \mathcal{F}_1,\ S^* \subseteq G]$

Then $\mathcal{F}_1{}^G$ covers AG in the strict Vitali sense. But if $S \in \mathcal{F}_1{}^G$

(15.2) $\mu^*(AS) \geq k\, \mu^*(AS^*)$

since $S \in \mathcal{F}_1$ and also

(15.3) $\begin{cases} \mu^*(AS) = \mu^*(AGS) \\ \mu^*(AS^*) = \mu^*(AGS^*) \end{cases}$

since $S \subseteq S^* \subseteq G$, so that

(15.4) $\mu^*(AGS) \geq k\, \mu^*(AGS^*)$

Hence if we set $U_e = AG$ conditions I, II, III above are satisfied and the theorem is proved.

Theorem 16. If $\mu^*(A) < +\infty$ and a family $\mathcal{F}$ of spheres covers A in the strict Vitali sense relative to μ^* then A has the Vitali property $(\mu^*, \mathcal{F})$.

Proof: Let G be any open set such that $\mu^*(AG) > 0$. Take U_e so that

(16.1) $AG \subseteq U_e,\ \mu^*(U_e) \leq \mu^*(AG) + e$

Set

(16.2) $\mathcal{F}_e \equiv [\text{all } S \ni S \in \mathcal{F},\ \mu^*(U_e S) \geq k\, \mu^*(U_e S^*)]$

Then $\mathcal{F}_e$ covers AG in the strict Vitali sense; in particular

(16.3) $\sum \mathcal{F}_e \geqq AG$.

Now by theorem 11 we may take $\{S_{en}\}$ in $\mathcal{F}_e$ so that

(16.4) $S_{em} \cdot S_{en} = 0 \quad (m < n)$, $\sum_n S^*_{en} \geqq \sum \mathcal{F}_e$

then

$$\begin{aligned} \mathcal{M}^*(\sum U_e S_{en}) &= \sum_n \mathcal{M}^*(U_e S_{en}) \geqq k \sum_n \mathcal{M}^*(U_e S^*_{en}) \\ &\geqq k\, \mathcal{M}^*(\sum_n U_e S^*_{en}) \geqq k\, \mathcal{M}^*(\sum_n AG S^*_{en}) = k\, \mathcal{M}^*(AG) \end{aligned} \tag{16.5}$$

Hence

$$\begin{aligned} \mathcal{M}^*(AG - \sum_n S_{en}) &\leqq \mathcal{M}^*(AG - \sum_n U_e S_{en}) \leqq \mathcal{M}^*(U_e - \sum_n U_e S_{en}) \\ &= \mathcal{M}^*(U_e) - \mathcal{M}^*(\sum_n U_e S_{en}) \\ &\leqq \mathcal{M}^*(AG) + e - k\, \mathcal{M}^*(AG) \\ &= (1-k)\, \mathcal{M}^*(AG) + e \\ &\leqq (1 - \tfrac{1}{2} k)\, \mathcal{M}^*(AG) \end{aligned} \tag{16.6}$$

if we take $e \leqq \frac{1}{2} k\, \mathcal{M}^*(AG)$. But since

(16.7) $\mathcal{M}^*(AG - \sum_n S_{en}) = \lim_n \mathcal{M}^*(AG - \sum_{m=1}^{n} S_{em})$

it follows from (16.6) and (16.7) that there exists an n_1 such that

$$\mathcal{M}^*(AG - \sum_{n=1}^{n_1} S_{en}) \leqq (1 - \tfrac{1}{4} k)\, \mathcal{M}^*(AG)$$

which proves the theorem, since if we set $\vartheta = 1 - \frac{1}{4} k$ we have $\frac{3}{4} < \vartheta < 1$.

THE SMITH MEASURE FUNCTION $\varphi^{\sigma\lambda}$

Caratheodory defines an outer measure function μ^* as any set function satisfying the following four postulates.

I $\quad 0 \leqq \mu^*(A) \leqq +\infty, \quad \mu^*(0) = 0$

II $\quad B \subseteq A \ \supset \ \mu^*(B) \leqq \mu^*(A)$

III $\quad \sum_i \mu^*(A_i) \geqq \mu^*(\sum_i A_i)$

IV $\quad \Delta(A,B) > 0 \ \supset \ \mu^*(A+B) = \mu^*(A) + \mu^*(B)$

Dr. Smith in his general theory starts off with a function φ satisfying Caratheodory's first two postulates and then lays down the following definitions:

$$\varphi^{\sigma}(A) \equiv \underline{B}\left[\text{all} \sum_n \varphi(A_n),\ A \subseteq \sum_n A_n\right]$$

$$\varphi^{\lambda}(A) \equiv \overline{B}_{\sigma}\left[\text{all} \sum_{\Delta\sigma} \varphi(A \cdot \Delta\sigma)\right]$$

where σ is any partition of the space into a countable number of cells measurable Caratheodory and $\Delta\sigma$ are the cells of the partition. He then shows that by performing the σ and λ operations successively we obtain a function $\varphi^{\sigma\lambda}(A)$ which satisfies Caratheodory's four postulates.

He further defines a function,

$$\varphi(A,r) \equiv \underline{B}\left[\text{all} \sum_n \varphi(A_n),\ A \subseteq \sum_n A_n,\ d(A_n) \leqq r \quad (n)\right]$$

and shows that

$$\lim_{r \to 0} \varphi(A,r) = \varphi^{\sigma\lambda}(A).$$

Above we mentioned a set being Caratheodory measurable, it would be well to say what is meant by the expression. We say a set A is measurable Caratheodory if it is measurable for every measure function μ^* which satisfies Caratheodory's four postulates.

We are now in a position to continue with our work. We first lay down the following definition

$$\varphi^{\beta}(A) \equiv \underline{B}\left[\text{all } \varphi(B),\ B \supseteq A,\ B \in \beta\right]$$

where β is some class of sets.

<u>Theorem 17.</u> If β is an additive class that includes every closed set, then to every A, there corresponds a B_A such that

$$A \subseteq B_A,\ B_A \in \beta,\ d(B_A) = d(A),\ \varphi^{\beta}(A) = \varphi(B_A).$$

<u>Proof:</u> Take B_n so that

$$A \subseteq B_n,\ B_n \in \beta,\ \varphi^{\beta}(A) \leqq \varphi(B_n) \leqq \varphi^{\beta}(A) + \frac{1}{2^n}$$

Then take

$$B_A = \bar{A} \prod_n B_n$$

where $\bar{A}$ is the closure of A. Then

$$A \subseteq B_A,\ \varphi^{\beta}(A) \leqq \varphi(B_A) \leqq \varphi(B_n) \leqq \varphi^{\beta}(A) + \frac{1}{2^n}$$

$$\therefore \varphi^{\beta}(A) = \varphi(B_A).$$

Also

$$d(A) \leqq d(B_A) \leqq d(\bar{A}) = d(A)$$

$$\therefore d(A) = d(B_A)$$

<u>Theorem 18.</u> If β is an additive class that includes every closed set, then

(18.1) $\varphi^{\beta\sigma}(A) = \underline{B}$ [all $\sum_n \varphi(B_n)$, $A \subseteq \sum_n B_n$, $B_n \in \beta$, (n)]

(18.2) $\varphi^{\beta}(A, r) = \underline{B}$ [all $\sum_n \varphi(B_n)$, $A \subseteq \sum_n B_n$, $B_n \in \beta$ (n), $d(B_n) < r$ (n)]

Proof: We prove (18.2); the proof of (18.1) is similar. Let k denote the right member of (18.2). We show first that,

(18.3) $\varphi^{\beta}(A, r) \leqq k$

To this end take $\{B_{en} | n\}$ so that

(18.4) $\begin{cases} A \subseteq \sum_n B_{en}, \; B_{en} \in \beta, \; d(B_{en}) \leqq r \quad (n) \\ k \leqq \sum_n \varphi(B_{en}) \leqq k + e \end{cases}$

Set

(18.5) $A_{en} = A \cdot B_{en}$

Then

(18.6) $A = \sum_n A_{en}$, $d(A_{en}) \leqq d(B_{en}) \leqq r$

Hence

(18.7) $\varphi^{\beta}(A_{en}) \leqq \varphi^{\beta}(B_{en}) = \varphi(B_{en})$

We have by (18.4) and (18.6)

(18.8) $\varphi^{\beta}(A, r) \leqq \sum_n \varphi^{\beta}(B_{en}) \leqq \sum_n \varphi(B_{en}) \leqq k + e$

and (18.3) follows from this.

We now prove

(18.9) $\varphi^{\beta}(A, r) \geqq k$

In order to do this take $\{A_{en} | n\}$ so that

(18.10) $\begin{cases} A = \sum_n A_{en}, \; d(A_{en}) \leqq r \\ \varphi^{\beta}(A, r) \leqq \sum_n \varphi^{\beta}(A_{en}) \leqq \varphi(A, r) + e. \end{cases}$

Take $\{B_{en} | n\}$ so that

(18.11) $A_{en} \subseteqq B_{en}$, $d(B_{en}) = d(A_{en})$, $B_{en} \in \beta$, $\varphi^{\beta}(A_{en}) = \varphi^{\beta}(B_{en})$

Then

(18.12) $A \subseteqq \sum_n B_{en}$, $d(B_{en}) \leqq r$, $B_{en} \in \beta$, (n), $B_{em} \cdot B_{en} = 0$ $(m \neq n)$

so that

(18.13) $k \leqq \sum_n \varphi(B_{en}) = \sum_n \varphi^{\beta}(A_{en}) \leqq \varphi^{\beta}(A, r) + e$ (e)

which proves (18.9). The theorem follows immediately from (18.3) and (18.9).

<u>Theorem 19:</u> If β is an additive class that includes every closed set, then

(19.1) $\varphi^{\beta\sigma}(A) = \underline{B}\,[\text{all} \sum_n \varphi(B_n),\ A \subseteqq \sum_n B_n,\ B_m \cdot B_n = 0\ (m \neq n),\ B_n \in \beta\ (n)]$

(19.2) $\varphi^{\beta}(A, r) = \underline{B}\,[\text{all} \sum_n \varphi(B_n),\ A \subseteqq \sum_n B_n,\ B_n \in \beta, (n)\, d(B_n) \leqq r, (n)\ B_m \cdot B_n = 0\ (m \neq n)]$

<u>Proof:</u> We prove (19.2); the proof of (19.1) is similar. Let k_1 denote the right member of (19.2), and k_2 the right member of (18.2) theorem 18; then clearly

(19.3) $k_1 \geqq k_2$

In order to show

(19.4) $k_1 \leqq k_2$

take $\{B'_{en} \mid n\}$ so that

(19.5) $\begin{cases} A \subseteqq \sum_n B'_{en},\ d(B'_{en}) \leqq r,\ B'_{en} \in \beta\ (n) \\ k_2 \leqq \sum_n \varphi(B'_{en}) \leqq k_2 + e \end{cases}$

Set

(19.6) $B_{e1} = B'_{e1}$, $B_{e2} = B'_{e2} - B'_{e1}$, ... $B_{en} = B'_{en} - (B'_{e1} + \cdots + B'_{en-1})$

Then

(19.7) $\begin{cases} A \subseteqq \sum_n B'_{en} = \sum_n B_{en},\ d(B_{en}) \leqq d(B'_{en}) \leqq r \\ B_{en} \in \beta,\ (n),\quad B_{em} \cdot B_{en} = 0 \quad (m \neq n) \end{cases}$

We now have from (19.7)

(19.8) $k_1 \leqq \sum_n \varphi(B_{en}) \leqq \sum_n \varphi(B'_{en}) \leqq k_2 + e.$

from which (19.4) follows ; and (19.4) and (19.3) give the theorem.

Theorem 20. If β is an additive class of sets which contains every closed set, then

(20.1) $\varphi^{\beta\sigma}(A) = \underline{B}\,[\text{all } \sum_n \varphi^{\beta}(A\cdot B_n),\ A \leqq \sum_n B_n,\ B_n \in \beta,\ (n)]$

(20.2) $\varphi^{\beta}(A,\lambda) = \underline{B}\,[\text{all } \sum_n \varphi^{\beta}(A\cdot B_n),\ A \leqq \sum_n B_n,\ B_n \in \beta,\ d(B_n) \leqq \lambda\ \ (n)]$

Proof: Let the right member of (20.2) be denoted by k_1 and the right member of (18.3), theorem 18 by k_2 .

Clearly

(20.3) $\varphi^{\beta}(A,\lambda) \leqq k_1$

But since $\varphi^{\beta}(A\cdot B_n) \leqq \varphi(B_n)$ we have that

(20.4) $\sum_n \varphi^{\beta}(A\cdot B_n) \leqq \sum_n \varphi(B_n)$

It follows from this that

(20.5) $k_1 \leqq k_2$

But $k_2 = \varphi^{\beta}(A,\lambda)$ by theorem 17. Hence (20.2) is proved.

We now lay down the following definition

$$\rho(\varphi, A, e) \equiv \tfrac{1}{2}\underline{B}\,[\text{all } \lambda,\ \varphi^{\sigma\lambda}(A) - \varphi(A,\lambda) > e]$$

In view of the definition we have the following obvious relationships.

I $\rho(\varphi, A, e) > 0$

II $\lambda \leqq \rho(\varphi, A, e) \ \cdot\supset\cdot\ \varphi(A,\lambda) \geqq \varphi^{\sigma\lambda}(A) - e$

III $A = \sum_n A_n,\ d(A_n) \leqq \rho(\varphi, A, e) \ \cdot\supset\cdot\ \sum_n \varphi(A_n) \geqq \varphi^{\sigma\lambda}(A) - e.$

<u>Theorem 21.</u> If B is Caratheodory measurable then

$$\rho(\varphi, AB, e) \geqq \rho(\varphi, A, e)$$

<u>Proof:</u> We have

$$(21.1) \quad \varphi^{\sigma\lambda}(A) = \varphi^{\sigma\lambda}(AB) + \varphi^{\sigma\lambda}(A-B)$$

$$(21.2) \quad \varphi(A, \pi) \leqq \varphi(AB, \pi) + \varphi(A-B, \pi)$$

The second of these equations follows from a theorem in Dr. Smith's paper in which he shows that $\varphi(\sum_n A_n, \pi) \leqq \sum_n \varphi(A_n, \pi)$.

From (21.1) and (21.2) we have

$$(21.3) \quad \varphi^{\sigma\lambda}(A) - \varphi(A, \pi) \geqq \{\varphi^{\sigma\lambda}(AB) - \varphi(AB, \pi)\} + \{\varphi^{\sigma\lambda}(A-B) - \varphi(A-B, \pi)\}$$
$$\geqq \varphi^{\sigma\lambda}(AB) - \varphi(AB, \pi)$$

It follows from (21.3) that

$$(21.4) \quad [\text{all } \pi, \varphi^{\sigma\lambda}(A) - \varphi(A, \pi) > e] \supset [\text{all } \pi, \varphi^{\sigma\lambda}(AB) - \varphi(AB, \pi) > e]$$

and from (21.4) we have

$$\rho(\varphi, A, e) \leqq \rho(\varphi, AB, e)$$

<u>Theorem 22.</u> If $\{B_n\}$ is such that

(22.1) $A - \sum_n B_n = AB$ where B is Caratheodory measurable,

(22.2) $d(B_n) \leqq \rho(\varphi, A, e) \quad (n)$

Then

$$\sum_n \varphi(AB_n) \geqq \varphi^{\sigma\lambda}(\sum AB_n) - e$$

<u>Proof:</u> By (22.1) we have $A - B = \sum_n AB_n$. This in connection with (22.2) and theorem 21 gives

$$d(AB_n) \leqq d(B_n) \leqq \rho(\varphi, A, e) \leqq \rho(\varphi, A-B, e)$$

Hence by III in the definition we obtain

$$\sum_n \varphi(AB_n) \geqq \varphi^{\sigma\lambda}(A-B) - e = \varphi^{\sigma\lambda}(\sum_n AB_n) - e$$

<u>Theorem 23</u>. To every A, r, e there corresponds a sequence $\{B_{ren} \mid n\}$ such that

(23.1) $A \subseteq \sum_n B_{ren}$, $d(B_{ren}) \leqq r$, $B_{ren} \in \beta_1$ (n), $B_{rem} \cdot B_{ren} = 0$ $(m \neq n)$

where β_1 is the class of all Caratheodory measurable sets.

(23.2) If $\{B'_{ren} \mid n\}$ is any subsequence of $\{B_{ren} \mid n\}$ then

$$\sum_n \varphi(B'_{ren}) \leqq \varphi^{\beta_1 \sigma r}(A \cdot \sum_n B'_{ren}) + e$$

<u>Proof:</u> Take r_1 to be the smaller of $r, \rho(\varphi^{\beta_1}, A, \frac{e}{2})$. Then take $\{B_{ren} \mid n\}$ so that

(23.3) $A \subseteq \sum_n B_{ren}$, $d(B_{ren}) \leqq r_1$, $B_{ren} \in \beta_1$, $B_{rem} \cdot B_{ren} = 0$ $(m < n)$

(23.4) $\sum_n \varphi(B_{ren}) \leqq \varphi^{\beta_1}(A, r_1) + \frac{e}{2}$

This is possible by theorem 19. From (23.4) we have

(23.5) $\sum_n \varphi(B_{ren}) \leqq \varphi^{\beta_1 \sigma r}(A) + \frac{e}{2}$

Now let $\{B'_{ren} \mid n\}$ be any subsequence from $\{B_{ren} \mid n\}$ and let $\{B''_{ren} \mid n\}$ be the complementary subsequence. Now

(23.6) $\begin{cases} \sum_n \varphi(B_{ren}) = \sum_n \varphi(B'_{ren}) + \sum_n \varphi(B''_{ren}) \\ \varphi^{\beta_1 \sigma r}(A \cdot \sum_n B_{ren}) = \varphi^{\beta_1 \sigma r}(A \sum_n B'_{ren}) + \varphi^{\beta_1 \sigma r}(A \sum_n B''_{ren}) \end{cases}$

since $\sum_n B'_{ren} \cdot \sum_n B''_{ren} = 0$, $\sum_n B'_{ren} \in \beta_1$, $\sum_n B''_{ren} \in \beta_1$. Now by (23.6)

(23.7) $\sum_n \varphi(B'_{ren}) - \varphi^{\beta_1 \sigma r}(A \sum_n B'_{ren})$

$$= \{\sum_n \varphi(B_{ren}) - \varphi^{\beta_1 \sigma r}(A \sum_n B_{ren})\} + \{\varphi^{\beta_1 \sigma r}(A \cdot \sum_n B''_{ren}) - \sum_n \varphi(B''_{ren})\}$$

so that the theorem follows from (23.7), (23.5), and theorem 22.

<u>Theorem 24</u>. If $\{B_n\}$ is such that

(24.1) $B_n \in \beta_1 \quad (n), \quad B_m \cdot B_n = 0 \quad (m < n)$

(24.2) $\varphi^{\beta_1}(A \cdot B_n) < (1 - \frac{1}{k})\, \varphi^{\beta_1 \sigma \lambda}(A B_n) \quad (n)$

(24.3) $d(B_n) \leqq \rho(\varphi^{\beta_1}, A, \frac{d}{k})$

then

$$\varphi^{\beta_1 \sigma \lambda}(\sum_n A B_n) \leqq d$$

<u>Proof:</u> We have by (24.2) and (24.1) that

$$\sum_n \varphi^{\beta_1}(A B_n) < (1 - \tfrac{1}{k}) \sum_n \varphi^{\beta_1 \sigma \lambda}(A B_n) = (1 - \tfrac{1}{k})\, \varphi^{\beta_1 \sigma \lambda}(\sum_n A B_n)$$

Hence by theorem 22 we obtain

$$\varphi^{\beta_1 \sigma \lambda}(\sum_n A B_n) < k \left\{ \varphi^{\beta_1 \sigma \lambda}(\sum_n A B_n) - \sum_n \varphi^{\beta_1}(A B_n) \right\} \leqq d$$

<u>Theorem 25.</u> To every A, λ, d, k there corresponds a sequence $\{B_{\lambda d k n} | n\}$ such that

(25.1) $A \leqq \sum_n B_{\lambda d k n}, \; B_{\lambda d k n} \in \beta_1, \; d(B_{\lambda d k n}) \leqq \lambda, \; B_{\lambda d k m} \cdot B_{\lambda d k n} = 0 \quad (m < n)$

(25.2) If $\{B^k_{\lambda d k n} | n\}$ is the subsequence of $\{B_{\lambda d k n} | n\}$ made up of its elements which satisfy the inequality

(25.3) $\varphi^{\beta_1}(A \cdot B^k_{\lambda d k n}) > (1 + \frac{1}{k})\, \varphi^{\beta_1 \sigma \lambda}(A \cdot B^k_{\lambda d k n})$

then

$$\varphi^{\beta_1 \sigma \lambda}(\sum_n A \cdot B^k_{\lambda d k n}) \leqq d$$

<u>Proof:</u> By theorem 23 we can choose $\{B_{\lambda d k n} | n\}$ so as to satisfy (25.1) and the condition

(25.4) $\sum_n \varphi^{\beta_1}(A \cdot B^k_{\lambda d k n}) \leqq \varphi^{\beta_1 \sigma \lambda}(\sum_n A \cdot B^k_{\lambda d k n}) + \frac{d}{k}$

But by (25.3) and (25.4) we have

$$\varphi^{\beta_1 \sigma \lambda}(A \sum_n B^k_{\lambda d k n}) < k \left\{ \sum_n \varphi^{\beta_1}(A B^k_{\lambda d k n}) - \varphi^{\beta_1 \sigma \lambda}(\sum_n A B^k_{\lambda d k n}) \right\} \leqq d.$$

Before going any further we need the following definitions.

If $f(A)$ is a function of a set A, we define

$$\overline{\lim_{A\to a}} f(A) \equiv \underset{r}{B}\ \overline{\underset{A}{B}}\ [\text{all } f(A),\ a \in A,\ d(A) \leqq r]$$

$$\overline{f}(A,a) \equiv \overline{\lim_{B\to a}} f(A\cdot B)$$

With similar definitions for the lower limits.

<u>Theorem 26.</u> If A, f are such that there exist sequences $\{e_k\}$, $\{B_{rdkn}|n\}$ such that

(26.1) $\lim\limits_{k} e_k = 0$

(26.2) $A \subseteq \sum\limits_{n} B_{rdkn},\ B_{rdkn} \in \beta_1,\ d(B_{rdkn}) \leqq r,\ B_{rdkm}\cdot B_{rdkn} = 0 \quad (m<n)$

(26.3) $\varphi^{\beta_1\sigma\lambda}(A\sum\limits_{n}\cdot B^{k}_{rdkn}) \leqq d$ where $\{B^{k}_{rdkn}|n\}$

is the subsequence of $\{B_{rdkn}|n\}$ made up of the elements which satisfy the inequality $f(A\cdot B^{k}_{rdkn}) < -e_k$

then

$$\varphi^{\beta_1\sigma\lambda}([\text{all } a,\ a \in A,\ \overline{f}(A,a) < 0]) = 0$$

<u>Proof</u>: If in the sequence $\{B_{rdkn}|n\}$ of the hypothesis we replace r by $\frac{1}{2^m}$ and d by $\frac{d}{2^m}$ we secure sequences which we represent by $\{B_{dkmn}|n\}$ and which are such that

(26.4) $A \subseteq \sum\limits_{n} B_{dkmn},\ B_{dkmn} \in \beta_1,\ B_{dkmn'}\cdot B_{dkmn''} = 0 \quad (n'<n'')$

(26.5) $d(B_{dkmn}) \leqq \frac{1}{2^m}$

(26.6) $\varphi^{\beta_1\sigma\lambda}(A\sum\limits_{n} B^{k}_{dkmn}) \leqq \frac{d}{2^m} \quad (d,k,m,n)$

where $\{B^{k}_{dkmn}|n\}$ is the subsequence of $\{B_{dkmn}|n\}$ made up of those elements of it which satisfy the inequality

(26.7) $f(A\cdot B^{k}_{dkmn}) < -e_k$

Set

(26.8) $A_{dk} = \sum_m \sum_n A B^k_{dkmn}$

Then by (26.6) we get

(26.9) $\varphi^{A,\sigma\lambda}(A_{dk}) \leq \sum_m \varphi^{A,\sigma\lambda}(A \sum_n B^k_{dkmn}) \leq \sum_m \frac{d}{2^m} = d.$

Now suppose $a \in A - A_{dk}$. Then $a \in A$ and $a \notin A_{dk} = \sum_m \sum_n A B^k_{dkmn}$ and for some value of n, $a \in A \cdot B_{dkmn}$; let us denote the corresponding B_{dkmn} by the symbol B_{dkma} so that $a \in B_{dkma}$ but B_{dkma} is not in the sequence $\{B^k_{dkmn} | n\}$. On account of the latter we have

(26.10) $f(A \cdot B_{dkma}) \geqq -e_k$

By (26.5) and (26.10) we see that

(26.11) $\bar{f}(A, a) \geqq -e_k \quad (d, k, a \in A - A_{dk})$

Now set

(26.12) $\begin{cases} A_k \equiv [\text{all } a ; a \in A, \bar{f}(A, a) < -e_k] \\ A_0 \equiv [\text{all } a ; a \in A, \bar{f}(A, a) < 0] \end{cases}$

so that

(26.13) $A_0 = \sum_k A_k$

From (26.11) and (26.12) we have

(26.14) $A - A_{dk} \subseteq A - A_k$

which gives

(26.15) $A_k \subseteq A_{dk} \quad (d, k)$

Now by (26.15) and (26.9) we obtain

(26.16) $\varphi^{A,\sigma\lambda}(A_k) \leqq d \quad (k, d)$

and so we have

(36.17) $\varphi^{\rho,\sigma\lambda}(A_k) = 0$

The theorem therefore follows from (36.13) and (36.17).

DENSITY FUNCTIONS

If μ and ν are set functions, we define the following density functions[1]

I $D_0^*(\mu,\nu,A,a) \equiv \overline{\lim_{B\to a}} \dfrac{\mu(A\cdot B)}{\nu(A\cdot B)}$

II $D_{00}^*(\mu,\nu,A,a) \equiv \overline{\lim_{r\to 0}} \dfrac{\mu(A\cdot C(a,r))}{\nu(A\cdot C(a,r))}$

III $D^*(\mu,\nu,A,a) \equiv \overline{\lim_{r\to 0}} \dfrac{\mu(A\cdot C(a,r))}{\nu(C(a,r))}$

with corresponding definitions for the lower densities

$D_{*0}(\mu,\nu,A,a)$, $D_{*00}(\mu,\nu,A,a)$, $D_*(\mu,\nu,A,a)$

We have the following obvious inequalities

(1) $D_0^*(\mu,\nu,A,a) \geqq D_{00}^*(\mu,\nu,A,a) \geqq D^*(\mu,\nu,A,a)$

and

(2) $D_{*0}(\mu,\nu,A,a) \leqq D_{*00}(\mu,\nu,A,a)$, $D_*(\mu,\nu,A,a) \leqq D_{*00}(\mu,\nu,A,a)$

Theorem 27.

(27.1) $\varphi^{\beta,\sigma\lambda}([\text{all } a \ni a \in A,\ D_0^*(\varphi^{\beta,\sigma\lambda};\ \varphi^{\beta};\ A, a) < 1]) = 0$

(27.2) $\varphi^{\beta,\sigma\lambda}([\text{all } a \ni a \in A,\ D_{*0}(\varphi^{\beta,\sigma\lambda};\ \varphi^{\beta};\ A, a) > 1]) = 0$

Proof of (27.1): Set

(27.3) $f(A) = \dfrac{\varphi^{\beta,\sigma\lambda}(A)}{\varphi^{\beta}(A)} - 1$

so that

(27.4) $[\text{all } a \ni a \in A,\ \bar{f}(A,a) < 0] = [\text{all } a \ni a \in A,\ D_0^*(\varphi^{\beta,\sigma\lambda};\ \varphi^{\beta};\ A, a) < 1]$

If we further set

(27.5) $e_k = \dfrac{1}{k+1}$

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[1] I and II are entirely new, III reduces to that of Besicovitch with replaced by the linear measure function of Carathéodory and replaced by the diameter.

then $f(A\cdot B) < -e_k$ is equivalent to

(27.6) $\varphi^{\beta_1}(A\cdot B) > (1+\frac{1}{k})\,\varphi^{\beta_1\sigma\lambda}(A\cdot B)$

Therefore (27.1) follows from (27.4), (27.5), and (27.6) and theorems 25 and 26.

Proof of (27.2): set

(27.7) $f(A) = 1 - \dfrac{\varphi^{\beta_1\sigma\lambda}(A)}{\varphi^{\beta_1}(A)}$;

then $\bar{f}(A,a) = 1 - D_{*0}(\varphi^{\beta_1\sigma\lambda}, \varphi^{\beta_1}, A, a)$, so that

(27.8) $[\text{all } a \,;\, a \in A,\ \bar{f}(A,a) < 0] = [\text{all } a \,;\, a \in A,\ D_{*0}(\varphi^{\beta_1\sigma\lambda}, \varphi^{\beta_1}, A, a) > 1]$

Also if we set

(27.9) $e_k = \frac{1}{k-1}$

we have

(27.10) $f(A\cdot B) < -e_k \;:\equiv:\; \varphi^{\beta_1}(AB) < (1-\frac{1}{k})\,\varphi^{\beta_1\sigma\lambda}(AB)$.

and hence (27.2) follows from (27.8), (27.9), (27.10) and theorems 24 and 26.

Theorem 28. If $\varphi^{\sigma\lambda}(A) < +\infty$ and $\varphi = \varphi^{\beta_1}$ then

(28.1) $\varphi^{\sigma\lambda}([\text{all } a \,;\, a \in A,\ D_{*0}(\varphi^{\sigma\lambda}, \varphi, A, a) > 1]) = 0$

(28.2) $\varphi^{\sigma\lambda}([\text{all } a \,;\, a \in A,\ D^{*}_{0}(\varphi^{\sigma\lambda}, \varphi, A, a) < 1]) = 0$

(28.3) $\varphi^{\sigma\lambda}([\text{all } a \,;\, a \in A,\ D_{*0}(\varphi^{\sigma\lambda}, \varphi, A, a) \leqq 1 \leqq D^{*}_{0}(\varphi^{\sigma\lambda}, \varphi, A, a)]) = \varphi^{\sigma\lambda}(A)$

Proof: It is clear that (28.1) and (28.2) are immediate consequences of theorem 27. To prove (28.3), let A'_0, A''_0, A_0 denote the classes occuring in (28.1), (28.2), and (28.3) respectively.

Then $A \leqq A'_0 + A''_0 + A_0$ and hence we have

(28.4) $\varphi^{\sigma\lambda}(A) \leqq \varphi^{\sigma\lambda}(A'_0) + \varphi^{\sigma\lambda}(A''_0) + \varphi^{\sigma\lambda}(A_0) = \varphi^{\sigma\lambda}(A_0)$

but $A_0 \subseteq A$ so that

(38.5) $\varphi^{\sigma\lambda}(A) \geq \varphi^{\sigma\lambda}(A_0)$

and (38.3) follows from (38.4) and (38.5).

Corollary: If $\varphi^{\sigma\lambda}(A) < +\infty$, $\varphi = \varphi^{\beta_1}$ then

$$\varphi^{\sigma\lambda}([\text{all } a \ni a \in A, D_{*0}(\varphi^{\sigma\lambda}, \varphi, A, a) = D_0^*(\varphi^{\sigma\lambda}, \varphi, A, a)])$$
$$= \varphi^{\sigma\lambda}([\text{all } a \ni D_0^*(\varphi^{\sigma\lambda}, \varphi, A, a) = D_{*0}(\varphi^{\sigma\lambda}, \varphi, A, a) = 1])$$

We will now consider a particular case of the above, and derive a relationship between $D^*(\varphi^{\sigma\lambda}, \varphi, A, a)$ and $D_0^*(\varphi^{\sigma\lambda}, \varphi, A, a)$. In order to do this we will first set

$$(38.6) \quad \begin{cases} r(a,A) \equiv \overline{B}\,[\text{all } \Delta(a,a') \ni a' \in A] \\ S(a,A) \equiv S(a, r(a,A)) \end{cases}$$

Theorem 39. If φ is such that there exists $k > 0$ such that

(39.1) $\varphi(C(a,A)) \leq k\varphi(A) \qquad (A,\ a \in A)$

then

$$D^*(\varphi^{\sigma\lambda}, \varphi, A, a) \geq \frac{1}{k} D_0^*(\varphi^{\sigma\lambda}, \varphi, A, a) \qquad (a \in A)$$

Proof: Take $\{B_n\}$ such that $a \in B_n$ (n), $d(B_n) \leq \frac{1}{2^n}$ and

$$\lim_n \frac{\varphi^{\sigma\lambda}(AB_n)}{\varphi(AB_n)} = D_0^*(\varphi^{\sigma\lambda}, \varphi, A, a)$$

Then
$$\frac{\varphi^{\sigma\lambda}(A\cdot C(a, AB_n))}{\varphi(C(a, AB_n))} \geq \frac{\varphi^{\sigma\lambda}(AB_n)}{\varphi(C(a, AB_n))} = \frac{\varphi^{\sigma\lambda}(A\cdot B_n)}{\varphi(AB_n)} \cdot \frac{\varphi(A\cdot B_n)}{\varphi(C(a, AB_n))}$$
$$\geq \frac{1}{k} \frac{\varphi^{\sigma\lambda}(AB_n)}{\varphi(AB_n)}$$

Hence

$$D^*(\varphi^{r\lambda}, \varphi, A, a) \geq \overline{\lim_n} \frac{\varphi^{r\lambda}(A \cdot C(a, AB_n))}{\varphi(C(a, AB_n))}$$

$$\geq \frac{1}{k} D_0^*(\varphi^{r\lambda}, \varphi, A, a)$$

Theorem 30. If

(30.1) Σ is a metric separable space

(30.2) φ is such that there exists a $k > 0$ so that $\varphi(S^*) \leq k\varphi(S)$ for every sphere S, where S^* is the sphere concentric with S and with radius 4 times as great

(30.3) $A \supseteq B$; (30.4) $\varphi^{r\lambda}(A) < +\infty$; (30.5) $\varphi^{r\lambda}(B) > 0$

(30.6) $D^*(\varphi^{r\lambda}, \varphi, A, a) > d$ $a \in B$

Then

$$d \leq 1$$

Proof: Let $\mathcal{F}_e$ be the family of all spheres $S(a, r)$ such that

(30.7) $a \in B$; (30.8) $2r \leq \frac{1}{4}\rho(\varphi, A, e)$

(30.9) $\varphi^{r\lambda}(A \cdot C(a, r)) \geq d\,\varphi(C(a, r))$

Then

(30.10) $B \subseteq \Sigma\mathcal{F}_e$

Now by theorem 11 there is a set $S_{e1}, S_{e2}, \ldots$ of spheres from $\mathcal{F}_e$ such that

(30.11) $S_{em} \cdot S_{en} = 0$ $(m < n)$; (30.12) $\sum_n S_{en}^* \supseteq \Sigma\mathcal{F}_e$

We have by (30.10), (30.11), and (30.12) that

(30.13) $B = B\Sigma\mathcal{F}_e \subseteq A\Sigma\mathcal{F}_e$

Hence by theorem 22 and (30.2) we get

$$(30.14)\quad \varphi^{\sigma\lambda}(B) \leq \varphi^{\sigma\lambda}(A\sum_n S^*_{en}) \leq \sum_n \varphi(A\cdot S^*_{en}) + e$$
$$\leq \sum_n \varphi(S^*_{en}) + e \leq k\sum_n \varphi(S_{en}) + e$$

Also by theorem 22 and (30.9) we have

$$(30.15)\quad \sum_n \varphi(S_{en}) + e \geq \varphi^{\sigma\lambda}(A\sum_n S_{en}) = \sum_n \varphi^{\sigma\lambda}(AS_{en}) \geq d\sum_n \varphi(S_{en})$$

or

$$(30.16)\quad (d-1)\sum_n \varphi(S_{en}) \leq e$$

and from (30.14) we obtain

$$(30.17)\quad \sum_n \varphi(S_{en}) \geq \frac{1}{k}\{\varphi^{\sigma\lambda}(B) - e\}$$

so that from (30.16) and (30.17) it follows that

$$(30.18)\quad (d-1)\{\varphi^{\sigma\lambda}(B) - e\} \leq ke \qquad (e < \varphi^{\sigma\lambda}(B))$$

It is to be noted that (30.18) is obviously true if $d \leq 1$ since then the left member is negative or zero, while the right member is positive. If $d > 1$ then (30.18) follows from (30.17) and (30.16).

We have from (30.18)

$$(30.19)\quad (d-1)\varphi^{\sigma\lambda}(B) \leq 0$$

which together with (30.14) gives $d-1 \leq 0$ as was to be proved.

<u>Theorem 31</u>. Under the same hypotheses as theorem 30, it is true that $\varphi^{\sigma\lambda}([\text{all } a \ni a \in A, D^*(\varphi^{\sigma\lambda}, \varphi, A, a) > 1]) = 0$

<u>Proof:</u> Set $E_n = [\text{all } a \ni a \in A, D^*(\varphi^{\sigma\lambda}, \varphi, A, a) > 1 + \frac{1}{2^n}]$ $n = 1, 2, \ldots\ldots$

Then

$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots\ldots$$

and

$$A_0 = [\text{all } a \, ; \, a \in A, \; D^* (\varphi^{\sigma\lambda}, \varphi, A, a) > 1] = \lim_n E_n$$

But by the preceeding theorem we have

$$\varphi^{\sigma\lambda}(E_n) = 0 \qquad (n)$$

and hence

$$0 \leqq \varphi^{\sigma\lambda}(A_0) \leqq \sum_n \varphi^{\sigma\lambda}(E_n) = 0$$

which completes the proof of the theorem.

FURTHER SUFFICIENT CONDITIONS THAT A SET HAVE THE VITALI PROPERTY

In a previous chapter we have discussed the sufficient conditions for a set to have the Vitali property $(\mu^*, \mathcal{F})$. Now that we have completed our discussion on density functions we are able to determine some further conditions, in particular the sufficient conditions for a set to have the Vitali property $(\varphi^{\sigma\lambda}, \mathcal{F})$.

Theorem 32. If φ is such that there exists a positive k such that

$$\varphi(S(a,r)) \geq k\,\varphi(S(a,4r)) \qquad (a,r)$$

if A is any set with $\varphi^{\sigma\lambda}(A) < +\infty$ and there exist positive numbers $\underline{M}, \overline{M}$ such that

$$0 < \underline{M} \leq D_*(\varphi^{\sigma\lambda}, \varphi, A, a) \leq D^*(\varphi^{\sigma\lambda}, \varphi, A, a) \leq \overline{M} \qquad (a \in A)$$

and if $\mathcal{F}$ is any family of spheres which covers A in the strict Vitali sense, then A has the Vitali property $(\varphi^{\sigma\lambda}, \mathcal{F})$.

Proof: It is sufficient, in view of theorems 15, 16, to show that $\mathcal{F}$ covers A in the strict Vitali sense strongly relative to $\varphi^{\sigma\lambda}$. We have the identity

$$(32.1) \quad \varphi^{\sigma\lambda}(A \cdot S(a,r)) = R(a,r)\,\varphi^{\sigma\lambda}(A \cdot S(a,4r))$$

where

$$(32.2) \quad R(a,r) \equiv \frac{\varphi^{\sigma\lambda}(A \cdot S(a,r))}{\varphi(S(a,r))} \cdot \frac{\varphi(S(a,4r))}{\varphi^{\sigma\lambda}(A \cdot S(a,4r))} \cdot \frac{\varphi(S(a,r))}{\varphi(S(a,4r))}$$

But

(33.3) $\underline{\lim}_{r \to 0} R(a, r) \geq (\underline{M})\left(\frac{1}{\overline{M}}\right) \cdot k.$

Hence there exists an $(r_e | e)$ such that $r_e \leqq e$ and

$$R(a, r_e) \geqq \frac{1}{2}\left(\underline{M}/\overline{M}\right) k.$$

and hence the set of all S of $\mathcal{F}$ such that

(33.4) $\varphi^{\sigma\lambda}(AS) \geqq \frac{1}{2}\left(\underline{M}/\overline{M}\right) k\, \varphi^{\sigma\lambda}(AS^*)$

covers A in the strict Vitali sense, which proves the theorem.

<u>Theorem 33.</u> Let μ^* be such that $\mu^*(S) < +\infty$ for every sphere and there exists a positive number k such that for every sphere

(33.1) $\mu^*(S) \geqq k\,\mu^*(S^*)$

and further let μ^* be such that for every set A

(33.2) $\mu^*(A) = \underline{B}\,[\text{all } \mu^*(G),\ G \geqq A,\ G^{\text{open}}]$

Then if A is such that $\mu^*(A) < +\infty$ and A is covered in the strict Vitali sense by a family $\mathcal{F}$ of spheres, it is true that A has the Vitali property $(\mu^*, \mathcal{F})$.

<u>Proof:</u> It is sufficient to show that $\mathcal{F}$ covers A in the strict Vitali sense relative to μ^*. Let G be our open set. Then by (33.2) there exists for every e an open set U_e such that

(33.3) $AG \subseteqq U_e,\ \mu^*(U_e) \leqq \mu^*(AG) + e \quad (e)$

Set

$$\mathcal{F}_e \equiv [\text{all } S,\ S \in \mathcal{F},\ S^* \subseteqq U_e]$$

Then $\mathcal{F}_e$ covers AG in the strict Vitali sense. Also if $S \in \mathcal{F}_e$,

(33.4) $U_e S = S$, $U_e S^* = S^*$

so that by (33.1),

(33.5) $\mu^*(U_e S) \geq h \mu^*(U_e S^*)$.

Hence the class of all S of $\mathcal{F}$ that satisfy (33.5) includes $\mathcal{F}_e$ and thus covers AG in the strict sense of Vitali, which completes the proof of the theorem.

Corollary: If a set A of the n-dimensional Euclidean space is covered by a family $\mathcal{F}$ of n-dimensional Euclidean spheres in the strict Vitali sense, then A has the Vitali property relative to the n-dimensional Lebesque outer measure function and the family $\mathcal{F}$.

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BIOGRAPHY

Harry T. Fleddermann was born in New Orleans, Louisiana, February 20, 1910. His primary education was obtained from private tutors and at St. Augustines School in Havana, Cuba. He attended high school at St. Paul's College, Covington, Louisiana, where he graduated in 1925. For the next four years he attended Spring Hill College, Mobile, Alabama, graduating with a Bachelor of Science degree in 1929. He then attended the University of Detroit where he received the degree of Bachelor of Science in Mechanical Engineering in 1931. Since that time he has been teaching in the department of Physics and Mathematics at Loyola University in New Orleans. For the past four and a half years he has been enrolled in the graduate school at Louisiana State University, receiving his Master of Science degree in the department of Mathematics in 1937.

On April 22, 1938 he married Ethel A. Mazerat. They have two children, Ethel born on April 24, 1939, and Harry T. Jr., born on April 25, 1940.

Comment on paper by Harry T. Fleddermann's <u>Density Properties of Sets.</u>

The paper of Fledderman is an interesting and very illuminating discussion of the components of the Vitali's theorem in the theory of measure. The enunciations are made to fit very general assumptions on the underlying space and Vitali's own assumption that the space is finite dimensional is reduced to certain inequalities whose formulation does not depend on this assumption explicitly. The results of the paper were not intended to be novel in its implications. Nevertheless the present study is a valuable contribution to the present knowledge and future study in the field.

S. Bochner

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Princeton, N.J.
May 10, 1940.

EXAMINATION AND THESIS REPORT

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