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Period Relations for Picard Integrals Defined on a Special Class of Kaehler Manifolds.

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PERIOD RELATIONS FOR PICARD INTEGRALS DEFINED ON A SPECIAL CLASS OF KÄHLER MANIFOLDS

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy in The Department of Mathematics

by

Joseph Clement Wilson
M.S., Louisiana State University, 1950
June, 1954
MANUSCRIPT THESSES

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ABSTRACT

In this dissertation we investigate the relations existing among the periods of the Picard integrals defined on a Kähler manifold; relations analogous to those existing among the periods of the Abelian integrals on a Riemann surface. The chief difficulties encountered in the extension of our work to general compact Kähler manifolds arise from the presence of singular divisors and a general lack of a theory connecting the topology of the manifold with its analytic structure. In order to achieve our results, a special class of Kähler manifolds has been employed. Briefly, this class consists of compact Kähler manifolds each of which possesses a family of analytic curves which intersect the non-singular hyper-surfaces in a proper manner and each curve of which contains a 1-dimensional Betti basis of the manifold. This class of Kähler manifolds certainly includes the algebraic manifolds, but its extent beyond those is not known.

The handling of the existence problems arising in this work is based on the theory of currents, developed by G. deRham, and applied so successfully by K. Kodaira. Also, material from W. V. D. Hodge's classic work, "The Theory and Applications of Harmonic Integrals," plays an essential role in our development.
INTRODUCTION

The first chapter of this dissertation is purely expository, being a brief recount of the results obtained by K. Kodaira and G. de Rham concerning harmonic forms and currents defined on Riemannian and Kählerian manifolds. A considerable part of the material in the second chapter is due to Kodaira, especially that related to Picard integrals of the second kind, for both the non-singular and singular cases, and Picard integrals of the third kind for the non-singular case. By using these results, it has been possible to show the existence of integrals of the third kind for the singular case as well. The chief contributions of this work are contained in chapter three. On the basis of certain assumptions set forth in the second chapter, relations have been obtained for the periods of both integrals of the second and third kind, analogous to those given by Severi for algebraic surfaces.¹

The author wishes to acknowledge here the use he has made of the Kodaira, de Rham, and Severi material, both directly and indirectly, in order to achieve the desired goal.

A. The Complex Analytic Manifold.

Definition: A manifold is a connected, topological space, each neighborhood of which is homeomorphic to an open subset of Euclidean space.

Definition: The dimension of the manifold is defined to be the dimension of the Euclidean space.

Definition: A complex analytic manifold, of complex dimension \( n \), is a manifold carrying a complex analytic structure.

The concept of a complex analytic structure can be defined by the concept of a regular analytic function in a neighborhood of a point and by the axioms:

Axiom 1: If \( f(\mu) \) is an arbitrary function defined in a neighborhood \( U_\mu \) of \( \mu \), \( f(\mu) \) is regular analytic in \( U_\mu \) or it is not.

Axiom 2: For every point \( \mu \) of the manifold, there exists a neighborhood \( U_\mu \) and \( n \) complex valued functions \( z_1(\mu), z_2(\mu), \ldots, z_n(\mu) \) defined in \( U_\mu \) such that:

a) The mapping \( \mu \rightarrow (z_1(\mu), z_2(\mu), \ldots, z_n(\mu)) \) is a topological mapping of \( U_\mu \) on an open subset of the space of \( n \) complex variables;

b) An arbitrary function \( f(\mu) \) defined in \( V \subseteq U_\mu \).
is regular analytic in $V$ if and only if $f(\varphi) = f(z_1(\varphi), z_2(\varphi), \ldots, z_n(\varphi))$ is a holomorphic function of the $n$ complex variables $z_1, z_2, \ldots, z_n$.

$V$ is an arbitrary open subset of $\mathcal{U}_p$.

The functions $z_1, z_2, \ldots, z_n$ will be called "local co-ordinates" in $\mathcal{U}_p$. Axiom 2 asserts the existence of local co-ordinates in $\mathcal{U}_p$ for each $p$ and further implies that in the intersection of two neighborhoods $\mathcal{U}_i \cap \mathcal{U}_j$, the co-ordinates of one system are regular analytic functions of the co-ordinates of the other system and the Jacobian of the transformation is non-vanishing.

In all that follows, the manifolds will be assumed compact.

By putting $z_1 = x_1 + ix_2, z_2 = x_3 + ix_4, \ldots, z_n = x_{2n-1} + ix_{2n}$, real local co-ordinates are introduced on the manifold and on account of the axioms 1 and 2, these co-ordinates determine a $C^\infty$ structure on the manifold $\mathcal{M}_{2n}$.

**Definition:** A scalar of odd kind is determined, at each point $p$ of $\mathcal{M}_{2n}$, by a function $f(x_1, x_2, \ldots, x_{2n})$ which depends on the co-ordinate system in the following way: If $f'$ is its value in another co-ordinate system, then $f' = f$ if the Jacobian $J$ is positive, and $f' = -f$ if the Jacobian is negative. Thus, $f' = \left[\frac{J}{|J|}\right]f$.

---

Definition: $\mathbb{M}_{2n}$ is said to be orientable if there exists a continuous scalar, $\xi$, of odd kind defined on $\mathbb{M}_{2n}$ such that $\xi^2 = 1$.

This definition is equivalent to the usual one, for denote as positive those co-ordinate systems for which $\xi = 1$ and negative those for which $\xi = -1$. The Jacobian relative to the two systems is then positive or negative according to whether the systems are of the same sign or not. Conversely, under a partitioning of the systems into two classes, $\xi$ can be defined by the condition that $\xi = 1$ with respect to any system of the positive class. In order to show that $\mathbb{M}_{2n}$ is orientable, consider the analytic transformation

$$z'_k = z_k'(z_1, \ldots, z_n),$$
$$z'_k = z_k'(z_1, \ldots, z_n),$$
$$\vdots$$
$$z'_n = z_n'(z_1, \ldots, z_n).$$

To this transformation is associated the real transformation

$$x'_1 = x'_1(x_1, \ldots, x_{2n}),$$
$$x'_2 = x'_2(x_1, \ldots, x_{2n}),$$
$$\vdots$$
$$x'_{2n} = x'_{2n}(x_1, \ldots, x_{2n}),$$

where $x'_1 + ix'_2 = z'_1, \ldots, x'_{2n} + ix'_{2n} = z'_n$. Now,

$$\frac{\partial (x'_1, \ldots, x'_{2n})}{\partial (x_1, \ldots, x_{2n})} = \left\{ \frac{\partial (z'_1, \ldots, z'_n)}{\partial (z_1, \ldots, z_n)} \right\}^2,$$

and since the square of the absolute value is greater than zero, the manifold is orientable. Choose the orientation $\xi$.

---

of $\mathcal{M}_{2n}$ so that $\xi = 1$ with respect to the system of coordinates $x_1, \ldots, x_{2n}$. This orientation will be called the natural orientation of the manifold and is uniquely determined by the analytic structure.

It is to be noted that if $f(x)$ is an arbitrary $C^\infty$ function of the real variables $x_1, \ldots, x_{2n}$, one may introduce, symbolically, the new "variables" $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$, and assuming $\bar{z}_1, \ldots, \bar{z}_n$ as formally independent of $z_1, \ldots, z_n$, $f(x)$ can be written as $f(z, \bar{z})$ where the partials are defined as

$$\frac{\partial f}{\partial z_\alpha} = \frac{1}{2} \left( \frac{\partial f}{\partial x_{2\alpha-1}} - i \frac{\partial f}{\partial x_{2\alpha}} \right), \quad \frac{\partial f}{\partial \bar{z}_\alpha} = \frac{1}{2} \left( \frac{\partial f}{\partial x_{2\alpha-1}} + i \frac{\partial f}{\partial x_{2\alpha}} \right),$$

where $\alpha = 1, \ldots, n$. The $C^\infty$ function $f(z, \bar{z})$ is regular analytic with respect to $z_1, \ldots, z_n$ if and only if $\partial f/\partial \bar{z}_\alpha = 0$, $\alpha = 1, \ldots, n$.

**Definition:** A positive definite metric $ds^2 = \sum_{\alpha, \beta = 1}^n g_{\alpha \beta} \, dz^\alpha d\bar{z}^\beta$, where the coefficients $g_{\alpha \beta}$ are $C^\infty$ functions of the $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$, and satisfy the Hermitian condition, $g_{\alpha \beta} = \bar{g}_{\beta \alpha}$, will be called a Kähler metric if

$$\frac{\partial g_{\alpha \beta}}{\partial z^\gamma} = \frac{\partial g_{\gamma \beta}}{\partial z^\alpha} \quad \text{and} \quad \frac{\partial g_{\alpha \beta}}{\partial \bar{z}^\gamma} = \frac{\partial g_{\gamma \beta}}{\partial \bar{z}^\alpha}.$$

It will be assumed that a Kähler metric is defined on $\mathcal{M}_{2n}$ so that the manifold becomes a "Kähler manifold" and will hereafter be designated by the letter $\mathcal{M}$. By means of the real local co-ordinates, the metric can be written in the form $ds^2 = \sum_{i, \bar{j} = 1}^n g_{i \bar{j}} \, dx^i d\bar{x}^\bar{j}$, where $g_{i \bar{j}} = \bar{g}_{\bar{j} i}$. Thus $\mathcal{M}$ is a $2n$-

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Ibid., p. 38.
B. Exterior Differential Forms.

Definition: An exterior differential form of degree p, a p-form, is an expression \[ \frac{1}{p!} \sum_{i_1, \ldots, i_p} a_{i_1, \ldots, i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \] where:

1) The coefficients \( a_{i_1, \ldots, i_p} \) are complex valued \( C^\infty \) functions of the real coordinates \( x_1, \ldots, x_n \) and are skew-symmetric in the indices \( i_1, \ldots, i_p \);

2) The coefficients transform like the components of a covariant tensor;

3) The symbol \( \wedge \) (roof) denotes the exterior product, an operation characterized by the following rules:
   a) The operation is associative and distributive.
   b) \[ dx^i \wedge dx^j = -dx^j \wedge dx^i, \quad 0 \wedge dx^i = dx^i \wedge 0 = cdx^i, \]
   \[ dx^i \wedge adx^j = edx^i \wedge dx^j, \text{where} \ c \ \text{is any scalar}. \]

By using these rules, every form of degree \( p \) can be put in the canonical form \[ \frac{1}{p!} \sum_{i_1, \ldots, i_p} a_{i_1, \ldots, i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \]

Definition: The carrier of a \( p \)-form is the closure of the set of all points of \( \mathbb{M} \) on which this form is different from zero.

Definition: Let \( \varphi = \frac{1}{p!} \sum a_{i_1, \ldots, i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \) and \( \psi = \frac{1}{q!} \sum b_{i_1, \ldots, i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q} \) be two forms of degree \( p \) and \( q \) respectively. Then the exterior product of \( \varphi \) and \( \psi \) is a \((p+q)\)-form and is given by \[ \varphi \wedge \psi = \frac{1}{(p+q)!} \sum c_{i_1, \ldots, i_p, j_1, \ldots, j_q} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q} \]
where \[ c_{i_1, \ldots, i_p, j_1, \ldots, j_q} = \frac{1}{p!} \frac{1}{q!} \sum_{\kappa_1, \ldots, \kappa_p} a_{i_1, \ldots, i_p} b_{j_1, \ldots, j_q} \delta_{\kappa_1, \ldots, \kappa_p} \wedge \delta_{\kappa_1, \ldots, \kappa_p}, \] and where \( \delta_{\kappa_1, \ldots, \kappa_p} \) is the generalized Kronecker delta.
0. The Operators $\partial$, $\mathcal{L}$, $\kappa$, and $\Delta$.

Definition: The differential of a p-form $\omega = \sum_{i_2, \ldots, i_p} a_{i_2, \ldots, i_p} dx^{i_2} \wedge \ldots \wedge dx^{i_p}$ is the $(p + 1)$-form $d\omega = \sum_{i_2, \ldots, i_p} da_{i_2, \ldots, i_p} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}$.

Using this definition, the following rules hold:

1) If $\omega_1$ and $\omega_2$ are both p-forms, then $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.

2) $d(\omega \wedge \beta) = d\omega \wedge \beta + (-1)^p \omega \wedge d\beta$ where $\omega$ is a p-form and $\beta$ is a q-form. To see this let $\omega = \sum_{i_2, \ldots, i_p} a_{i_2, \ldots, i_p} dx^{i_2} \wedge \ldots \wedge dx^{i_p}$ and $\beta = \sum_{j_1, \ldots, j_q} b_{j_1, \ldots, j_q} dx^{j_1} \wedge \ldots \wedge dx^{j_q}$. Then $d(\omega \wedge \beta) = \sum_{i_2, \ldots, i_p, j_1, \ldots, j_q} d(a_{i_2, \ldots, i_p} b_{j_1, \ldots, j_q}) dx^{i_2} \wedge \ldots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q}$ and $d(\omega \wedge \beta) = \sum_{i_2, \ldots, i_p, j_1, \ldots, j_q} d(a_{i_2, \ldots, i_p} b_{j_1, \ldots, j_q}) dx^{i_2} \wedge \ldots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q} + \sum_{i_2, \ldots, i_p, j_1, \ldots, j_q} a_{i_2, \ldots, i_p} db_{j_1, \ldots, j_q} dx^{i_2} \wedge \ldots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q}$.

But, $d(\omega \wedge \beta) = d(\omega \wedge \beta)$, so this last is $\sum_{i_2, \ldots, i_p, j_1, \ldots, j_q} d(a_{i_2, \ldots, i_p} b_{j_1, \ldots, j_q}) dx^{i_2} \wedge \ldots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q} + \sum_{i_2, \ldots, i_p, j_1, \ldots, j_q} a_{i_2, \ldots, i_p} db_{j_1, \ldots, j_q} dx^{i_2} \wedge \ldots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q}$. Now the first sum in the last expression is $d\omega \wedge \beta$. As for the second sum, it is multiplied by $(-1)^p$ if the factor $db_{j_1, \ldots, j_q}$ is moved past the factor $dx$ which occurs in the $p$th place. Then the second sum is $(-1)^p \omega \wedge d\beta$. This result can be generalized to the product of any number of factors.

3) $d^2 \omega = 0$, for again, let $\omega = \sum_{i_2, \ldots, i_p} a_{i_2, \ldots, i_p} dx^{i_2} \wedge \ldots \wedge dx^{i_p}$. It is sufficient to verify this for a monomial term, say $\omega = \sum_{i_2, \ldots, i_p} a_{i_2, \ldots, i_p} dx^{i_2} \wedge \ldots \wedge dx^{i_p}$. Then $d\omega = \sum_{i_2, \ldots, i_p} da_{i_2, \ldots, i_p} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}$ and $d^2 \omega = \sum_{i_2, \ldots, i_p} d(da_{i_2, \ldots, i_p}) \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p} = 0$. Therefore, $d^2 \omega = 0$. 
Then \( d\omega = \omega \lambda_1 \wedge \cdots \wedge \lambda_p \). However, since each factor on the right is an exact differential, the formula for the differential of a product gives the desired result. Also, the definition of \( d\alpha \) is invariant under a change of co-ordinates, for let \( \alpha \) be as above and let \( \omega = \omega_1 \cdots \omega_p \lambda_1 \wedge \cdots \wedge \lambda_p \) be one of the terms of \( \alpha \). After expressing the \( x_i \) in terms of new variables \( y_i \), \( \omega \) will then be the product of \( (p+1) \) factors, the first of which is of degree zero and the rest forms of degree 1. Let \( \omega' \) designate the term \( \omega \) expressed in terms of the \( y_i \). Then, \( d\omega' = \omega_1 \cdots \omega_p \lambda_1 \wedge \cdots \wedge \lambda_p + \omega_1 \cdots \omega_p d(\lambda_1) \wedge \cdots \wedge \lambda_p - \omega_1 \cdots \omega_p \lambda_1 \wedge \cdots \wedge \lambda_p - \omega_1 \cdots \omega_p \lambda_1 \wedge \cdots \wedge \lambda_{p-1} d(\lambda_p) \wedge \lambda_p - \cdots - \omega_1 \cdots \omega_p \lambda_1 \wedge \cdots \wedge \lambda_{p-1} \lambda_p = \omega_1 \cdots \omega_p \lambda_1 \wedge \cdots \wedge \lambda_{p-1} \wedge d(\lambda_p) \). But, each term but the first is zero and \( d\omega' \) is obtained by replacing, in \( d\omega \), the \( x_i \) by the \( y_i \). Since \( \omega \) was any term of \( \alpha \), the assertion is true.

Let \( g_{ij} \) be the fundamental tensor associated with the Riemannian metric. Then, taking the product of \( 2n \) tensors equal to \( g_{ij} \) and making it skew-symmetric, there results the new tensor \( g_{i_1 \ldots i_n j_1 \ldots j_n} \) expressed in terms of the \( g_{ij} \).

Now setting \( e_1, \ldots, e_n = \sqrt{g_{1 \ldots 1n 1 \ldots n}} \), one arrives at the following definition.

**Definition:** The adjoint form to the \( p \)-form \( \alpha = \sum_{i_1, \ldots, i_p} \alpha_{i_1 \ldots i_p} \lambda_1 \wedge \cdots \wedge \lambda_p \) is the \((2n-p)\)-form \( \star \alpha = \sum_{i_1, \ldots, i_{2n-p}} a_{i_1 \ldots i_{2n-p}} \lambda_{i_1} \wedge \cdots \wedge \lambda_{i_{2n-p}} \), where \( a_{i_1 \ldots i_{2n-p}} = \sum_{j_1, \ldots, j_p} a_{i_1 \ldots i_{2n-p}} \ast_{j_1 \ldots j_p} g_{j_1 \ldots j_p} \) and being the contravariant tensor determined from \( a_{i_1 \ldots i_p} \) by the
formula $\sum_{i=1}^{n} \partial_{i} \cdot a_{i} = \sum_{i=1}^{n} a_{i}$, the $g_{i}^{k}$ being determined
by $g_{i}^{k} g_{k}^{j} = \delta^{i}_{j}$.

**Definition:** The adjoint form to one is the volume element,
\[ dx^{1} \wedge dx^{2} \wedge \cdots \wedge dx^{n} \]

If $\alpha$ and $\beta$ are two $p$-forms with coefficients $a_{i_{1}} \cdots i_{p}$
and $b_{i_{1}} \cdots i_{p}$, then $\alpha \wedge \beta = \beta \wedge \alpha = \sum_{i_{1}}^{n} a_{i_{1}} \cdots i_{p} b_{i_{1}} \cdots i_{p} \wedge dx^{1} \wedge dx^{2} \wedge \cdots \wedge dx^{n}$. To show this, we have $\lambda \wedge \alpha = \sum_{i_{1}}^{n} a_{i_{1}} \cdots i_{p} \partial_{i_{1}} \wedge dx^{1} \wedge dx^{2} \wedge \cdots \wedge dx^{n}$ and $\lambda \wedge \beta = \sum_{i_{1}}^{n} b_{i_{1}} \cdots i_{p} \partial_{i_{1}} \wedge dx^{1} \wedge dx^{2} \wedge \cdots \wedge dx^{n}$. Then $\alpha \wedge \beta = \sum_{i_{1}}^{n} a_{i_{1}} \cdots i_{p} b_{i_{1}} \cdots i_{p} \partial_{i_{1}} \wedge dx^{1} \wedge dx^{2} \wedge \cdots \wedge dx^{n}$. On the other hand, $\beta \wedge \alpha = \sum_{i_{1}}^{n} b_{i_{1}} \cdots i_{p} a_{i_{1}} \cdots i_{p} \partial_{i_{1}} \wedge dx^{1} \wedge dx^{2} \wedge \cdots \wedge dx^{n}$. Thus, $\alpha \wedge \beta = \beta \wedge \alpha = \sum_{i_{1}}^{n} a_{i_{1}} \cdots i_{p} b_{i_{1}} \cdots i_{p} \wedge dx^{1} \wedge dx^{2} \wedge \cdots \wedge dx^{n}$. Further, $\lambda \wedge \lambda = (-1)^{p} \lambda$, for $\lambda = \sum_{i_{1}}^{n} a_{i_{1}} \cdots i_{p} \partial_{i_{1}} \wedge dx^{1} \wedge dx^{2} \wedge \cdots \wedge dx^{n}$, so $\lambda(\lambda \wedge \lambda) = \sum_{i_{1}}^{n} a_{i_{1}} \cdots i_{p} \partial_{i_{1}} \wedge dx^{1} \wedge dx^{2} \wedge \cdots \wedge dx^{n}$. By again employing orthogonal co-ordinates, this last can be written $\sum_{i_{1}}^{n} (\sum_{i_{2}}^{n} a_{i_{1}} \cdots i_{p} \partial_{i_{2}} \wedge dx^{1} \wedge dx^{2} \wedge \cdots \wedge dx^{n}) = \sum_{i_{1}}^{n} (\sum_{i_{2}}^{n} a_{i_{2}} \cdots i_{p} \partial_{i_{1}} \wedge dx^{1} \wedge dx^{2} \wedge \cdots \wedge dx^{n}) \wedge dx^{1} \wedge dx^{2} \wedge \cdots \wedge dx^{n}$. But the product in parenthesis is positive or negative according as $(-1)^{p}$ is positive or negative so that the last expression is equal to $(-1)^{p} \lambda \wedge \lambda$. Finally, the operation $\wedge$ is independent of the co-ordinate system employed, due to the tensorial character of the forms.
**Definition:** The co-differential of a p-form is the \((p-1)\)-form \(\delta \omega = (-1)^{p+1} (\star d \star \omega)\).

The following properties hold for the operator \(\delta\):

1) \(\frac{d^2 \omega}{\delta \omega} = \delta \frac{d \omega}{\delta \omega} = 0\), since \(\delta \omega = (-1)^{2p+1} (\star d \star \omega)\), then:
   \[\delta^2 \omega = (-1)^{2p+1} \star d \star ((-1)^{2p+1} \star d \star \omega) = \star d \star (d \star \omega) = (-1)^{2p+1} \star d \star (\star d \star \omega) = 0.\]

2) \(\star d \omega = (-1)^{p+1} (\star d \star \omega)\), since \(\star d \star \omega = \star ((-1)^{2p+1} \star d \star \omega)\) =
   \((-1)^{2p+1} \star (d \star \omega) = (-1)^{p+1} d \star \omega\).

3) \(\star d \omega = (-1)^{p+1} \star \omega \), since \((-1)^{2p+1} \star \omega = (-1)^{p+1} ((-1)^{2p+1} \star d \star \omega)\) = \((-1)^{2p+1} \star d \star \omega = \star d \omega\).

**Definition:** The operator \(\Delta\) is defined as \(\Delta = d \star \delta + \delta \star d\).

If \(\omega\) is a p-form, then \(\Delta \omega\) is a p-form, and further:

1) \(\delta \Delta = \delta (d \star \delta + \delta \star d) = d \star \delta \delta = (d \star \delta + \delta \star d) \delta = \Delta \delta\).

2) \(\delta \Delta = \delta (d \star \delta + \delta \star d) = d \star \delta \delta = (d \star \delta + \delta \star d) \delta = \Delta \delta\).

3) \(\star d \omega = (-1)^{p+1} \star \omega \), since \((-1)^{2p+1} \star \omega = (-1)^{p+1} ((-1)^{2p+1} \star d \star \omega)\) = \((-1)^{2p+1} \star d \star \omega = \star d \omega\).

4) \(\star d \omega = (-1)^{p+1} d \star \omega = (-1)^{p+1} (-1)^{p+1} \star \omega = d \delta \omega\).

5) \(\star \Delta = \star (d \star \delta + \delta \star d) = \star d \delta \star \omega + \delta \star d \delta \omega = \delta \delta \omega + d \delta \omega = (\delta \delta + d \delta) \star \omega = \Delta \omega\).

**D. Integrals of Forms.**

Let \(\omega = a_1 \ldots a_n \, dx^1 \wedge \ldots \wedge dx^n\) be a 2n-form. Under a change of co-ordinates \(\alpha\) becomes \(\alpha = a_1' \ldots a_n' \, dy^1 \wedge \ldots \wedge dy^n\), where

\[a_1' \ldots a_n' = \sum_{b_1, \ldots, b_n} a_{b_1} \ldots a_{b_n} \frac{dx^{b_1}}{dy^1} \cdots \frac{dx^{b_n}}{dy^n}\]

and

\[\sum_{b_1, \ldots, b_n} \Delta_{b_1, \ldots, b_n} \Delta \frac{dx^{b_1}}{dy^1} \cdots \frac{dx^{b_n}}{dy^n} = J > 0,\]

then
Thus, the coefficient changes according to the same rule as a scalar density.

Now, suppose the carrier of $\omega$ is contained in the domain $D$ of a co-ordinate system $x_1', \ldots, x_n'$. Then $a_1, \ldots, a_n$ is a function (locally) of $x_1', \ldots, x_n'$ which can be extended to the entire Euclidean space $E$ by setting $a_1, \ldots, a_n = 0$ outside the domain of $E$ corresponding to $D$.

**Definition:** The integral of $\omega$ is defined to be
$$
\int_E \omega = \int_0^\infty \cdots \int_0^\infty a_1, \ldots, a_n \, dx_1' \cdots dx_n'.
$$

This definition is clearly independent of the co-ordinate system since $\omega$ transforms as above.

**Definition:** A $p$-simplex, $s^p$, in $\mathcal{M}$ is defined by a mapping $\Pi$ of a rectilinear $p$-dimensional simplex $s^p$, contained in Euclidean $p$-space, and an orientation of $s^p$, i.e., $s^p = (s^p, \Pi)$, orientation of $s^p$.

**Definition:** A $p$-simplex is said to be $C^r$, if the mapping $\Pi$ is $C^r$, i.e., if the mapping $\Pi$ can be extended to a mapping into $\mathcal{M}$ of a domain $D$ of $E^p$, containing $s^p$, so that $\Pi$ is $C^r$ in $D$.

It will be hereafter assumed that $r \geq 1$.

**Definition:** A chain on $\mathcal{M}$ is a linear combination of simplices $s^p = \sum k_i s^p_i$, where the $k_i$'s are real numbers and all but a finite number of them are zero.

**Definition:** Let $s^p$ be a rectilinear simplex and let $s^{p-1}_i$, $(i = 0, 1, \ldots, p)$ be the $(p-1)$-dimensional sides of $s^p$. If $s^p$ is a simplex on $\mathcal{M}$ determined by the mapping $\Pi$ of $s^p$, then $\Pi$ applied to $s^{p-1}_i$, along with the relation between the orientation of $s^p$ and $s^{p-1}_i$, determines an oriented simplex $s^{p-1}_i$ on $\mathcal{M}$.
The chain $\sum_k s_{k}^{P-1}$ is the boundary of $s^P$ and will be denoted by $B_s^P$. The boundary of any chain $s^P = \sum_k k_i s_i^P$ is then defined to be $B_s^P = \sum_k k_i B_s_i^P$.

**Definition:** A p-chain is called closed (or called a cycle), if its boundary is the (p-1)-chain 0.

**Definition:** A p-chain is called homologous to zero (written $s^P \sim 0$), if it is the boundary of a (p+1)-chain.

**Definition:** Two p-chains $s^P$ and $c^P$ are said to be homologous provided their difference is homologous to zero.

**Definition:** A finite number of p-chains $s_1^P, s_2^P, \ldots, s_n^P$ are called homologically independent if no homology of the form $a_1 s_1^P + a_2 s_2^P + \ldots + a_n s_n^P \sim 0$ exists unless all the $a_i$s are zero, the $a_i$s being arbitrary real numbers.

**Definition:** The $p^{th}$ Betti number $B_p$ of $\mathcal{M}$ is the maximal number of homologically independent closed p-chains on $\mathcal{M}$.

Suppose the p-simplex $s^P = (s^P, \Pi, \xi)$, where $\xi$ is an orientation of $s^P$, is contained in the domain of a co-ordinate system $x_1, \ldots, x^{2n}$. This can always be achieved by a normal subdivision if it is not already the case. Further, let $\alpha$ be a p-form defined in the co-ordinate domain. Then, $\int_{s^P} \alpha = \int_{s^P} \xi \Pi \alpha = \int_{s^P} f \Pi^x \alpha$, where $f$ is the characteristic function of $s^P$ in $\mathbb{P}$ (i.e., $f = 1$ on $s^P$ and zero elsewhere) and $\Pi^x \alpha$ is the p-form corresponding to $\alpha$ under the mapping $\Pi$.

Clearly, $f \xi \Pi^x \alpha$ is a p-form with compact carrier in $\mathbb{P}$ and the integral is determined by the definition on page 11.

**Definition:** For every chain $s^P = \sum_k k_i s_i^P$ and every p-form
\[ \int_{\gamma} \alpha = \sum k_i \int_{\gamma_i} \alpha. \]

**Theorem 1.1:** If \( \beta \) is a \((2n-1)\)-form, of class \( C^1 \), with a compact carrier, \( \int d\beta = 0. \)

**Stokes' Theorem:** If \( \gamma \) is a \((p-1)\)-form of class \( C^1 \) and \( \sigma^p \) is a finite \( p \)-chain of class \( C^1 \), then \( \int_{\sigma^p} d\gamma = \int_{\partial \sigma^p} \gamma. \)

**E. The Concept of a Current.**

Let \( D \) be an arbitrary domain such that \( D \subseteq \mathbb{R}^n \) and \( \{ \gamma \} \) be the linear space consisting of all \((2n-p)\)-forms \( \gamma \) of class \( C^\infty \) whose carriers are contained in \( D \).

**Definition:** A current, \( T \), of degree \( p \) in \( D \) is a linear functional \( T[\gamma] \) defined on \( \{ \gamma \} \) which is continuous in the following way: For an arbitrary sequence \( \{ \gamma, \gamma', \ldots \} \) of forms belonging to \( \{ \gamma \} \), such that all their carriers are contained in a compact subset \( K \subseteq D \) covered by one coordinate system \( x', \ldots, x^n \), \( T[\gamma^i] \rightarrow 0 \) \( (i \rightarrow \infty) \), if each partial derivative \( \frac{\partial^j \gamma^i}{\partial x_{k_1} \cdots \partial x_{k_j}} \) of each coefficient of \( \gamma^i \) converges uniformly to zero for \( i \rightarrow \infty \) and for \( 0 \leq k_j \leq 2n, j = 0, 1, \ldots, n \).

The number \((2n-p)\) will be called the dimension of the current so that the sum of the degree and the dimension is \[ 5 \]

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5 G. de Rham and K. Kodaira, op. cit., p. 12.
always the dimension of the manifold.

**Definition:** The current $T$ will be said to vanish at a point $\gamma \in D$ if there exists a neighborhood $\mathcal{U}_\gamma$ of $\gamma$ such that $T[\varphi] = 0$ for all $\varphi$ whose carriers are contained in $\mathcal{U}_\gamma$.

**Definition:** The carrier of $T$, denoted by $|T|$, will be the set consisting of all points $\gamma \in D$ such that $T$ does not vanish at $\gamma$.

Every continuous $p$-form $\psi$, defined in $D$, can be considered as a current by indentifying $\psi$ with the linear functional $\psi[\varphi] = \int_D \psi \wedge \varphi$, for every $\varphi$ belonging to the space $\{\varphi\}$ mentioned in the current definition. It is evident that $\psi[\varphi]$ is a current since $\int_D \psi \wedge \varphi$ certainly exists for each $\varphi$ and if $\{\psi^n\}$ is an arbitrary sequence of forms satisfying the derivative conditions stated in the definition, then $\left|\int_K \psi \wedge \psi^n - \int_K \psi \wedge \psi^m\right|$ may be made arbitrarily small for all $n, m$ greater than some given $n$. Further, if $\psi$, considered as a form, does not vanish at some point of $D$, it does not vanish in some neighborhood of this point. Thus, the set of points at which $\psi$ does not vanish is open. Call this set $P$. Now, if $\gamma'$ is any point on the boundary of $P$, then $\psi$, considered as a current, could not vanish there, for otherwise, there would be a neighborhood $\mathcal{U}_{\gamma'}$ of $\gamma'$ such that $\psi[\varphi] = 0$ for all $\varphi$ whose carriers are contained in $\mathcal{U}_{\gamma'}$. But any such neighborhood will contain points of $P$ so that there arises a contradiction. Therefore, as a current, $\psi$ is carried on the closure of $P$. 

Definition: The current \( T [\psi] \) will be said to be equal to
the form \( \psi \) if \( T [\psi] - \psi [\psi] \) is zero at every point \( p \in M \).

Again, we note that every differentiable \((2n-p)\)-chain, \( c \),
can be considered as a \( p \)-current defined by \( c [\psi] = \int_c \psi \).
As a matter of fact, from the definition of the integral of
a form over a chain, it follows that the integral exists and
that the condition for continuity for currents is satisfied.
Also, the chain \( c \), considered as a current, is carried on the
set of points comprising the chain itself. For the integral,
by definition, is zero off of \( c \), regardless of what the carrier of \( \psi \) is.

From the previous results, if \( \alpha \) and \( \beta \) are forms of degree
\( p \) and \( q \), respectively, then \( \alpha \wedge \beta [\psi] = (-1)^{pq}(\beta \wedge \alpha) [\psi] = \int \alpha \wedge \beta \wedge \psi = \alpha [\beta \wedge \psi] \).
This leads to the definition:

\[ T \wedge (\beta [\psi]) = (-1)^{pq}(\beta \wedge T) [\psi] = T[\beta \wedge \psi]. \]

Suppose \( \alpha \) is a \( p \)-form and \( \beta \) is a \((2n-p-1)\)-form with
compact carrier, both defined in a domain \( D \). Then \( \alpha \wedge \beta \) is
a \((2n-1)\)-form and by Theorem 1.1, the integral \( \int_D d(\alpha \wedge \beta) = 0. \)
However, \( d(\alpha \wedge \beta) = d \alpha \wedge \beta + (-1)^p \alpha \wedge d \beta \) and so \( d[\beta] = (-1)^{p+1} \alpha [d \beta] \).

Definition: For any current \( T \) of degree \( p \), \( dT \) is defined as
\( dT [\beta] = (-1)^{p+1} T [d \beta] \).

If \( T \) is equal to a \((2n-p)\)-chain \( c^{2n-p} \), then \( d c^{2n-p} [\beta] = (-1)^{p+1} c^{2n-p} [d \beta] = (-1)^{p+1} \int c^{2n-p} d \beta = (-1)^{p+1} \int_{Bc^{2n-p}} \beta \).
(-1)^p \sum \beta_{2n-p}(\omega) \text{, by Stokes' Theorem. Thus, except for sign, the differential of a chain is its boundary. Also, using the definition of } dT, it follows immediately that } d^2 \nu = 0 \text{ and that } d(T \wedge \omega) = dT \wedge \omega + (-1)^p T \wedge d\omega.

If } T_{\omega_1, \ldots, \omega_p}(\omega_1, \ldots, \omega_p) \text{ are } (2^p) \text{ currents of degree zero, defined in the domain } D \text{ of a co-ordinate system } x', \ldots, x^{2n}, \text{ then for any current } T \text{ of degree } p, T = \sum_{i_{\omega_1}, \ldots, i_{\omega_p}} T_{i_{\omega_1}, \ldots, i_{\omega_p}} dx^{i_{\omega_1}} \wedge \ldots \wedge dx^{i_{\omega_p}}.

This can be shown as follows: Let } T_{i_1, \ldots, i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \text{ and } T_{j_1, \ldots, j_p} dx^{j_1} \wedge \ldots \wedge dx^{j_p} \text{ be any two terms of the sum and set } dx = dx^{i_1} \wedge \ldots \wedge dx^{i_p} \text{ and } dy = dx^{j_1} \wedge \ldots \wedge dx^{j_p}. \text{ Both } dx \text{ and } dy \text{ are } p \text{-forms of class } C^\infty. \text{ Let } \varphi \text{ be a } (2n-p) \text{-form, } C^\infty \text{ with compact carrier contained in } D. \text{ Then } T_{i_1, \ldots, i_p}[dx \wedge \varphi] \text{ is a well-defined current in } D, \text{ and by the product rule, } T_{i_1, \ldots, i_p}[dx \wedge \varphi] = T_{i_1, \ldots, i_p} dx [\varphi], \text{ so that } (T_{i_1, \ldots, i_p} dx) \text{ is a } p \text{-current, defined in } D. \text{ Thus, each term of the sum is a } p \text{-current in } D. \text{ Since } \sum_{i_{\omega_1}, \ldots, i_{\omega_p}} (T_{i_1, \ldots, i_p} dx)[\varphi] = \sum_{i_{\omega_1}, \ldots, i_{\omega_p}} (T_{i_1, \ldots, i_p} dx) \wedge \varphi \text{. Using this last notation, } \sum_{i_{\omega_1}, \ldots, i_{\omega_p}} (T_{i_1, \ldots, i_p} dx)[\varphi] + \sum_{j_{\omega_1}, \ldots, j_{\omega_p}} (T_{j_1, \ldots, j_p} dy)[\varphi] = \sum_{i_{\omega_1}, \ldots, i_{\omega_p}} (T_{i_1, \ldots, i_p} dx) \wedge \varphi + \sum_{j_{\omega_1}, \ldots, j_{\omega_p}} (T_{j_1, \ldots, j_p} dy) \wedge \varphi = \sum_{i_{\omega_1}, \ldots, i_{\omega_p}} (T_{i_1, \ldots, i_p} dx + T_{j_1, \ldots, j_p} dy) \wedge \varphi = (T_{i_1, \ldots, i_p} dx + T_{j_1, \ldots, j_p} dy) \wedge \varphi, \text{ which shows that the sum of the two } p \text{-currents } T_{i_1, \ldots, i_p} dx \text{ and } T_{j_1, \ldots, j_p} dy \text{ is again a } p \text{-current defined in } D. \text{ Therefore, } \sum_{i_{\omega_1}, \ldots, i_{\omega_p}} T_{i_1, \ldots, i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \text{ is a } p \text{-current defined in } D. \text{ Now, any } p \text{-current defined in } D \text{ can be represented by such an expression, for let } T, \text{ of degree } p, \text{ be given in } D.

Define } T_{i_1, \ldots, i_p} \text{ by } T_{i_1, \ldots, i_p}[dx^{i_1} \wedge \ldots \wedge dx^{i_p}] = \sum_{i_{\omega_1}, \ldots, i_{\omega_p}} T_{i_1, \ldots, i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \text{ Then it is immediately seen that } T \text{ has the representation } T = \sum_{i_{\omega_1}, \ldots, i_{\omega_p}} T_{i_1, \ldots, i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \text{ A current of degree zero is a }
distribution in the sense of L. Schwartz. Every current can be represented by a symbolic differential form whose coefficients are distributions.

Definition: The scalar product of two p-forms \( \alpha \) and \( \beta \) is the number \( \langle \alpha, \beta \rangle = \int \alpha \wedge \star \beta \).

Since \( \alpha \wedge \star \beta = \beta \wedge \star \alpha \), then \( \langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle \) and \( \langle \star \alpha, \star \beta \rangle = \langle \alpha, \beta \rangle \) for \( \star \alpha \wedge \star \beta = \beta \wedge \star \alpha = \alpha \wedge \star \beta \).

Definition: If \( \langle \alpha, \beta \rangle = 0 \), then \( \alpha \) is said to be orthogonal to \( \beta \).

The defining formula for \( \star \alpha \), where \( \alpha \) is a p-form, when applied to a symbolic form, gives the definition of the adjoint to a current. Similarly, the operators \( \mathcal{J} \) and \( \Delta \), as defined for forms, carry directly over to currents and all the properties shown to hold for \( \star \), \( \mathcal{J} \), and \( \Delta \) operating on forms, hold for currents.

Definition: The scalar product of a current \( T \) and a form \( \psi \) of class \( C^\infty \) is defined by \( \langle T, \psi \rangle = \langle \psi, T \rangle = T[\star \psi] = \int T \wedge \star \psi \).

If \( T \) is of degree \( p \) and \( \psi \) of degree \( p-1 \) with compact carrier, then, from Theorem 1.1, \( \int \text{d}(\psi \wedge \star T) = 0 \) and so \( \int \text{d} \psi \wedge \star T = (-1)^p \int \text{d} \psi \wedge \star \text{d} \star T \), which can be written \( \langle T, \text{d} \psi \rangle = -\langle \star \text{d} \star T, \psi \rangle \) because the degree of \( \text{d} \star T \) is \( (2n-p-1) \) and \( \star \text{d} \star T = (-1)^{2np+p} \text{d} \star T \). But, \( (-1)^{2np+1} \text{d} \star = \mathcal{J} \), so \( \langle T, \text{d} \psi \rangle = \langle \mathcal{J} T, \psi \rangle \). Also, if \( \psi \) is a \( C^\infty \) form of degree \( p+1 \), with com-

\[6\]
pect carrier, \((T, \varphi \psi) = (dT, \psi)\).

**Definition:** A current \(T\), which is not "regular" at a point \(p\), i.e., which is not equal to a form of class \(C^\infty\) in a neighborhood of \(p\), is called "singular" at \(p\). The singular set, \(T_s\), of the current is the set of all singular points of \(T\).

\(T_s\) is a subset of the carrier of \(T\). If \(T\) is defined in all \(\mathbb{M}\), then evidently \(T_s\) is closed, for otherwise, if \(p\) is a limit point of \(T_s\) and not contained in it, then there would exist a neighborhood of \(p\) in which \(T\) coincides with a form, \(C^\infty\). But in this neighborhood, there are an infinite number of points belonging to \(T_s\) so that it would be impossible to find any neighborhood of \(p\) in which \(T\) is regular. This is a contradiction of the assumption. Also, \(T_s\) must be compact, since \(\mathbb{M}\) is. Let \(U\) be an arbitrary open set containing \(T_s\). Then there exists a form \(\psi\) of class \(C^\infty\) such that the carrier of \(T - \psi\) is contained in \(U\). For example, off of \(T_s\), \(\psi\) can be taken to be \(T\), so that \(T - \psi = 0\) outside \(T_s\).

If \(S\) and \(T\) are two p-currents defined everywhere in \(\mathbb{M}\) and it is assumed that the intersection of the singular sets of \(S\) and \(T\) is empty, there follows the

**Definition:** Let \(\psi\) and \(\varphi\) be p-forms of class \(C^\infty\) chosen so that the intersection of the carriers of \(T - \psi\) and \(S - \varphi\) is empty. Then, the scalar product of \(S\) and \(T\) is defined as

\[
(S, T) = (\psi, \varphi) + (\varphi, T - \psi) + (S - \varphi, \psi).
\]

This definition gives the scalar product even though neither of the currents is regular. Also, the value of \((S, T)\)
does not depend on the choice of \( \psi \) and \( \psi' \). This assertion follows from a direct calculation which employs the definition of \((S,T)\).

F. Harmonic Forms and the Operators \( H \) and \( G \).

**Definition:** By a harmonic \( p \)-form of the first kind we will mean a \( p \)-form of class \( C^\infty \) satisfying, everywhere on \( \mathbb{M} \), the conditions \( d\psi = \psi' = 0 \).

The existence of harmonic forms, defined on \( \mathbb{M} \), was first shown by Hodge. The fact that there are a finite number of linearly independent such forms on \( \mathbb{M} \) is of the utmost importance, since it allows one to set up a 1-1 correspondence between these forms and the co-homology classes of the manifold. This is of particular significance in what is to follow concerning the periods of harmonic integrals.

Let \( \varphi_1, \varphi_2, \ldots, \varphi_\kappa \) be a normed, orthogonal basis of real harmonic \( p \)-forms of the first kind. Then, to each current \( T \) there is associated a well determined harmonic form \( h_1 \), such that \((T,h) = (h_1,h)\) for each harmonic form \( h \). Indeed, set

\[
  h_1 = \sum_{k=1}^\kappa (T, \varphi_k) \varphi_k.
\]

Since \( h \) is harmonic, it can be represented in terms of the basis as \( h = \sum_{i=1}^\kappa a_i \varphi_i \) where the \( a_i \)'s are real constants, not all zero. Now, consider \((h_1,h) = \)

\[
  (\sum_{k} (T, \varphi_k) \varphi_k, \sum a_i \varphi_i).\]

It will be sufficient to deal with a general term of each sum, say \((T, \varphi_i) \varphi_j\) and \( a_j \varphi_j \).

Then, \((T, \varphi_i) \varphi_j, a_j \varphi_j = \int [(T, \varphi_i) \wedge \varphi_i] \wedge a_j \varphi_j = \)
\[ \int (T, \varphi_i) \wedge [\varphi_i \wedge \alpha_j \varphi_j] = (T, \varphi_i) \int \varphi_i \wedge \alpha_j \varphi_j = (T, \varphi_i) \]
a \int \varphi_i \wedge \varphi_j = \alpha_j (T, \varphi_i)(\varphi_i, \varphi_j),\text{but } (\varphi_i, \varphi_j) = \delta_j, \text{so that the last expression is equal to } (T, \alpha_j \varphi_j), which is the jth term of \((T, h)\). Therefore, \((\sum (T, \varphi_i) \varphi_i, \sum \alpha_i \varphi_i) = (T, \sum \alpha_i \varphi_i)\) and so \((h_1, h) = (T, h)\). With this observation, it follows directly that the number of linearly independent harmonic p-forms of the first kind is equal to the pth Betti number of M. 7

**Definition:** The harmonic form \(h_1\) will be called the harmonic part of \(T\) and will be designated as \(HT\).

The operator \(H\) possesses the following properties:

1) \((HT, S) = (HT, HS) = (T, HS)\), which follows directly from the definition of \(H\).

2) \(H\) is orthogonal to \(d\) and \(\mathcal{F}\); as a matter of fact, \(\mathcal{F}HT = HT = 0\) from the definition of harmonic forms. \(HdT = 0\) since \((HdT, h) = (dT, h) = (T, \mathcal{F}h) = 0\) for each harmonic \(h\) and \(\mathcal{F}\mathcal{F}T = 0\) because \((H\mathcal{F}T, h) = (\mathcal{F}T, h) = (T, dh) = 0\).

3) \(\mathcal{F}H = H \mathcal{F}\), since \((H \mathcal{F}T, \mathcal{F}h) = (\mathcal{T}, \mathcal{F}h) = (T, h) = (HT, h) = (\mathcal{F}HT, h)\), where \(h\) is harmonic.

We now introduce the operator \(G\) by the

**Theorem 1.2:** There exists one and only one linear operator \(G\) mapping any p-current \(T\), defined in \(\Omega\), into a p-current \(GT\) having the properties: 1) \(\Delta GT = G \Delta T = \nabla - HT\); 2) \(GHT = GNT = 0\); 3) \(G\) is permutable with \(d, \mathcal{F}\), and \(\mathcal{K}\); 4) \((GT, S) = (S, GT)\). 8

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Theorem 1.2 is extremely important for all further work with harmonic forms. The construction of currents with given singularities, as well as the decomposition formula, to be given below, depend directly on this theorem.

We state next the

**Theorem 1.3:** Let $T$ be a $p$-current defined in a domain $D \subseteq \mathbb{M}$. If $\Delta T$ is regular at a point $\gamma$ of $D$, then $T$ is regular at $\gamma$. If both $dT$ and $\int T$ are regular at $\gamma$, then $T$ is regular at $\gamma$. (H. Weyl's Lemma).

**Definition:** A current $T$ is said to be closed if $dT = 0$.

From Theorem 1.3, it follows that if $T$ is regular at $\gamma$, then $\mathcal{G}T$ is regular at $\gamma$, since $\Delta \mathcal{G}T = T - \mathcal{H}T$. Now, using the identity $T = \Delta \mathcal{G}T + \mathcal{H}T = d \mathcal{G} \mathcal{G}T + \int d \mathcal{G}T + \mathcal{H}T$, set $T_1 = d \mathcal{G} \mathcal{G}T, T_2 = \int d \mathcal{G}T, T_3 = \mathcal{H}T$. Thus any current can be uniquely decomposed into the sum of the currents $T_1 + T_2 + T_3$. If $dT = 0, T_2 = 0$. Conversely, if $T_2 = 0, dT = 0$, since $dT_1 = dT_3 = 0$. Since $T_1$ bounds $\int \mathcal{G}T$, the following theorem holds.

**Theorem 1.4:** Each closed current $T$ is homologous to the harmonic part of $T$.

If $T$ is homologous to zero, i.e., $T = dU$, then $T_3 = 0$ and $T$ bounds the current $\int \mathcal{G}T$, the latter being co-homologous to zero, i.e., $\int^2 \mathcal{G}T = 0$. $\int^2 \mathcal{G}T$ is the only current co-homologous to zero and bounded by $T$, since if $V$ were another such current, $V - \int \mathcal{G}T$ would satisfy $d(V - \int \mathcal{G}T) = 0$ and $V - \int \mathcal{G}T$ would be co-homologous to zero and consequently harmonic. But since $H \int = 0$,  

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9 Ibid., p. 61.
It is also to be noted that if \( T \) is regular in a domain \( D \), then \( d\gamma T, \psi \gamma T, T_1, \) and \( T_2 \) are also regular there. From these last observations come the theorems.

**Theorem 1.5**: A current \( T \) which is homologous to zero bounds one and only one current which is co-homologous to zero, and this current is regular everywhere that \( T \) is regular.

**Theorem 1.6**: A closed current is homologous to zero if and only if its harmonic part vanishes.

In concluding this section, it is worthwhile to mention that if one substitutes \( T = \Delta \gamma T + \gamma T \) into the previously obtained formula for the scalar product of two currents \( \gamma \) and \( T \), there results the new formula: \( (\gamma, T) = (d\gamma, \gamma T) + (\gamma \delta T, \gamma T) + (\gamma \gamma, \gamma T) \). It is assumed here that the singular sets of \( dT \) and \( d\gamma \), as well as those of \( \gamma T \) and \( \gamma \gamma \), do not meet.

3. The Operators \( \gamma \) and \( \Lambda \).

An arbitrary \( p \)-form \( \psi^p = \frac{1}{i_1 \cdots \cdot i_p} \psi_{i_1 \cdots \cdot i_p} \) \( dx^{i_1 \cdots \cdot i_p} \), can be written, by substituting \( x^{a^{r-1}} = \frac{1}{2} (z^a + \bar{z}^{\bar{a}}) \), \( x^{a^n} = \frac{1}{2} (z^a - \bar{z}^{\bar{a}}) \), \( a = 1, \cdots, n \), as

\[
(1.7) \quad \psi^p = \sum_{\alpha_1+\cdots+\alpha_p = p} \psi_{\alpha_1 \cdots \cdot \alpha_p} \frac{dz^{\alpha_1} \cdots \cdot dz^{\alpha_p} \Lambda \cdot d\bar{z}^{\bar{\alpha}_1} \cdots \cdot d\bar{z}^{\bar{\alpha}_p}}{i_1 \cdots \cdot i_p},
\]

where the \( z \) and \( \bar{z} \) are considered as independent parameters, as was mentioned earlier in this chapter (see section A).

Since the operators \( d, \gamma, \delta, \) and \( \Delta \) are independent of the co-ordinate system, the forms \( d \gamma, \delta \gamma, \delta \gamma \), and \( \Delta \gamma \), where \( \gamma \) is written in terms of the above parameters, can be obtained by introducing the above substitution into their
expressions in real co-ordinates. All properties demonstrated to hold for these operators in the real system necessarily carry over to the complex system. We now define the new operators \( C \) and \( \Lambda \) as follows:

\[
C \varphi^p = \sum_{r+s=p} \sum_{\alpha_1 < \ldots < \alpha_r, \beta_1 < \ldots < \beta_s} \frac{1}{r!s!} (-1)^{r+s} \varphi_{\alpha_1 \beta_1} \ldots \varphi_{\alpha_r \beta_s} dz^{\alpha_1} \ldots dz^{\alpha_r} \wedge d\bar{z}^{\beta_1} \ldots d\bar{z}^{\beta_s};
\]

\[
\Lambda \varphi^p = \sum_{r+s=p-1} \sum_{\alpha_1 < \ldots < \alpha_r} (-1)^r \varphi_{\alpha_1 \beta_1} \ldots \varphi_{\alpha_r \beta_s} dz^{\alpha_1} \ldots dz^{\alpha_r} \wedge d\bar{z}^{\beta_1} \ldots d\bar{z}^{\beta_s}.
\]

The \( g_{\alpha \bar{\beta}} \) are defined by the relation \( \sum \delta^{\alpha \beta} g_{\alpha \bar{\beta}} = \delta^\alpha_\beta \). It is clear that \( \Lambda \varphi^p \) is a \((p-2)\)-form and if \( p = 0 \) or \( p = 1 \), \( \Lambda \varphi^p \) is defined to be zero. From the above definitions of \( C \) and \( \Lambda \), it can be shown that the following formulae hold:

1) \( C \Delta = \Delta C \);

2) \( \Lambda \Delta = \Delta \Lambda \);

3) \( CC \varphi^p = (-1)^p \varphi^p \);

4) \( \Lambda d-d \Lambda = C^{-1} \varphi^p \).

Also, by means of the 2-form \( \omega = \frac{1}{2} \sum_{\alpha \beta} g^{\alpha \bar{\beta}} dz^{\alpha} \wedge d\bar{z}^{\beta} \) associated with the Kahler metric, \( \Lambda \varphi^p \) can be written as

\[
\Lambda \varphi^p = (-1)^p \wedge (\omega \wedge \varphi^p).
\]

Hence, \( (-1)^p \Lambda \varphi^p \wedge \varphi^p = \int \wedge (\omega \wedge \varphi^p) \wedge \psi \). Now, employing the relation \( \alpha \wedge \beta = \beta \wedge \alpha \) where, in this case, \( \alpha = \wedge (\omega \wedge \varphi^p) \), a \((2n-p+2)\)-form and \( \wedge \beta = \psi \), another \((2n-p+2)\)-form, there results:

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11. Ibid., p. 489.
\[
\int {\omega \wedge x^p} \wedge \psi = \int (-1)^{p-2} \wedge x^p \wedge (-1)^{2n-p+2} (\omega \wedge x^p) = \\
\int x^p \wedge (\omega \wedge x^p) = \int (x^p \wedge \omega) \wedge x^p = \int x^p \wedge (x^p \wedge \omega).
\]

Since \( \omega \wedge \beta = (-1)^{pq} \psi \wedge \beta \), then \( \omega \wedge x^p = (-1)^{2(p-2)} \wedge x^p \wedge \omega \) = \( x^p \wedge \omega \), and \( \int x^p \wedge (x^p \wedge \omega) = \int x^p \wedge (\omega \wedge x^p) = (-1)^{p-1} x^p [\Lambda \psi] \). Thus, \( \Lambda x^p = x^p \Lambda \psi \). Again, \( C x^p [\Lambda \psi] = \int C x^p \wedge \psi = \int C(x^p \wedge \psi) = \int C \wedge x^p = (-1)^{p} \int x^p \wedge C \psi = \\
(-1)^{p} x^p [C \psi] \), so that \( C x^p = (-1)^{p} x^p C \). The derivation of this last equation follows from the facts, first, that \( C x^p \Lambda \psi \) is a 2n-form, thus \( r = s \), and second, that \( C(\alpha \wedge \beta) = C\alpha \wedge C\beta \) whatever the degrees of \( \alpha \) and \( \beta \). The second fact can be argued as follows: Let \( \alpha^p \) be any \( p \)-form whose coefficients are \( a_1, \ldots, a_r, \ldots, a_n \), where \( r + z = p \), and let \( \beta^q \) be any \( q \)-form whose coefficients are \( b_1, \ldots, b_z, \ldots, b_n \), where \( m + n = q \). Then the coefficients of \( \alpha \wedge \beta \) are given by \( \gamma \), where \( \gamma = \sum \frac{a_1 \ldots a_r \ldots a_n}{(z, \ldots, z)} = b_1 \ldots b_z \ldots b_n \), where \( k + 1 = p + q \). Evidently, \( C \gamma = i^{k-1} \gamma \). On the other hand, \( C\alpha \wedge C\beta \) gives \( f^{x+m-n} \sum \frac{a_1 \ldots a_r \ldots a_n}{(z, \ldots, z)} = b_1 \ldots b_z \ldots b_n \), for coefficients. In both cases, the \( \alpha \)'s and \( \beta \)'s must be some permutation of the \( \alpha \)'s and the same can be said for the \( \beta \)'s, \( \beta \)'s, and \( \beta \)'s. Therefore, \( r + m = k \) and \( n + s = l \), so that the assertion \( C(\alpha \wedge \beta) = C\alpha \wedge C\beta \) is true.

We can now define \( \Lambda \) and \( C \) for arbitrary \( p \)-currents.

**Definition:** \( \Lambda [\psi] = T [\Lambda \psi] \) and \( C[p] = (-1)^{p} \Lambda [C \psi] \).
By using the symbolic form for $T$, the expressions $CT$ and $\Lambda T$ can be written as

$$CT = \sum_{\gamma_1=\gamma_2=\cdots=\gamma_p=1}^{\text{all} \gamma_j} \sum_{\delta_1=\delta_2=\cdots=\delta_p=1}^{\text{all} \delta_j} i^{\gamma - \delta} T_{\gamma_1 \gamma_2 \cdots \gamma_p} \, dz_{\gamma_1} \cdots dz_{\gamma_p} \, d\bar{z}_{\delta_1} \cdots d\bar{z}_{\delta_p}$$

and

$$\Lambda T = \sum_{\gamma_1=\gamma_2=\cdots=\gamma_p=1}^{\text{all} \gamma_j} \sum_{\delta_1=\delta_2=\cdots=\delta_p=1}^{\text{all} \delta_j} (-1)^{\gamma + \delta} T_{\gamma_1 \gamma_2 \cdots \gamma_p} \, dz_{\gamma_1} \cdots dz_{\gamma_p} \, d\bar{z}_{\delta_1} \cdots d\bar{z}_{\delta_p} + \text{terms}.$$

Finally, the operators $C$ and $\Lambda$ are commutative with $H$ and $G$, for $\Delta HCT = C \Delta HT = 0$, since the harmonic part of any current operated on by $\Delta$ is zero. But $C \Delta HT = C \Delta HT = \Delta HCT$. Similarly, $\Delta H \Lambda T = \Lambda \Delta HT = 0$ and since $\Lambda \Delta = \Delta \Lambda$, the operators $H$ and $\Lambda$ are commutative. Further, $\Delta GT = T - HT$, so $\Delta GCT = CT - HCT$. But, $\Delta CGT = C \Delta GT = C(T - HT) = CT - HCT = CT - HCT$. Therefore, $\Delta GCT = \Delta CGT$ and $G$ and $C$ are commutative. On replacing $C$ by $\Lambda$ throughout this last set of equations, it is immediate that $G$ and $\Lambda$ are commutative.

H. Pure Forms and Currents.

**Definition:** An arbitrary $p$-form (or $p$-current) is called pure of type $t$ if expression 1.7 (or symbolic form) is homogeneous of degree $t$ in the $d\bar{z}$'s.

Obviously, any $p$-form or $p$-current is expressible as a sum of pure $p$-forms or $p$-currents of type $t, t = 0, 1, \cdots, p$. $P_t \Phi$ or $P_t T$ will designate the pure component of type $t$ of either the form or the current. For any $p$-form or $p$-current, we have $\Delta P_t = P_t \Delta$. It is important to note here that a complex, normed, orthogonal basis for harmonic $p$-forms can be

$^{12}$ Ibid., pp. 489-91.
chosen from pure p-forms. Any pure component of a harmonic form, \( \varphi^p \), is harmonic and since \( \varphi^p \) has a unique representation in terms of pure forms, the group \( H^p \) of all harmonic p-forms can be uniquely displayed as the direct sum of pure subgroups of type 0 to p, whose total rank is that of \( H^p \). By choosing a basis for each subgroup of type \( t \), we see that any \( \varphi^p \) will be expressible as a linear combination of the basis elements with complex coefficients, not all zero. The pure basis may then be normalized and orthogonalized. As a consequence of this observation, the following results are obtained: Since \( T = \sum_{t=0}^{p} P_t T \), then \( HT = \sum_{t=0}^{p} P_t T = H P_t T + \ldots + H P_t T + \ldots + H P_t T \), because \( HT \) is harmonic, and with the pure basis, it can be expressed as a sum of a finite number of pure components, each being harmonic. But \( P_t HT \) is the projection of \( HT \) onto the \( t \)th pure component, or, in other words, \( P_t HT = H P_t T \). This is true for every \( t = 0, 1, \ldots, p \) in the sum so that \( H \) is commutative with \( P_t \). Likewise, \( G \) and \( P_t \) are commutative since \( \Delta G P_t T = P_t T - H P_t T = P_t \Delta G T = \Delta P_t G T \).

**Definition:** A pure p-form of type zero is said to be holomorphic (or meromorphic) at a point \( p \), if in some neighborhood of \( p \), its coefficients are holomorphic (or meromorphic) functions of \( z^1, \ldots, z^n \).

**Definition:** A p-ple differential, \( \psi \), is a meromorphic p-form defined in a domain \( D \subseteq \mathbb{C} \) satisfying \( d \psi = 0 \) except for the singular points of \( \psi \).

Let \( \psi \) be an arbitrary holomorphic p-ple differential defined in a domain \( D \). Then, as follows directly from the
definition of $C$ and $\bigwedge, \omega \varphi = i^p \varphi$ and $\bigwedge, \varphi = 0$. Moreover,

$$\int \varphi = i^p \int \omega \varphi = i^p \left( \bigwedge, d - d \bigwedge, \right) \varphi = 0 \text{ in } D.$$ 

**Definition:** A holomorphic p-ple differential, $\varphi$, defined everywhere on $\mathcal{M}$, is called a p-ple differential of the first kind, and since $d \varphi = \int \varphi = 0$, $\varphi$ is a harmonic p-form of the first kind.

We now conclude the first chapter with two basic theorems.

**Theorem 1.8:** Let $T$ be a pure $p$-current of type zero, defined in a domain $D \subseteq \mathcal{M}$. If $T$ satisfies $dT = 0$ in $D$, then $T$ is a holomorphic p-ple differential in $D$. $^{13}$

**Theorem 1.9:** Let $T$ be a pure $(p+1)$-current of type one, defined in all $\mathcal{M}$ and satisfying $dT = 0$, $RT = 0$, $p$ being a positive integer $\leq n$. Then, the p-current $\Theta = (d \bigwedge, G + i \omega G)T$ is a pure p-current of type zero and satisfies $d \Theta = iT$; moreover, $\Theta$ is a holomorphic p-ple differential in $\mathcal{M} - \{|T|\}$. Let $Q$ be an arbitrary $p$-current such that $T = dQ$. Then the integral of $\Theta$ over an arbitrary $p$-cycle $Z \subset \mathcal{M} - \{|T|\}$ is given by

$$\int_Z \Theta = i(\star Z, Q) + (-1)^{p-1}lQ[JZ].$$ $^{14}$


$^{14}$ Ibid., p. 820.
CHAPTER II

A. Poincaré Integrals of the First Kind.

Definition: A simple differential of the first kind defined on \( \Omega \) is a holomorphic 1-form, pure of type zero.

From this definition, a simple differential of the first kind is a harmonic 1-form and from Theorem 1.8, any harmonic 1-form, pure of type zero, is a simple differential of the first kind. Thus, the group \( H_{(0)}^{1} \) of all harmonic 1-forms, pure of type zero, defined on \( \Omega \) is identical with the group of simple differentials of the first kind. Now, the group \( H_{(1)}^{1} \) of all harmonic 1-forms of the first kind can be displayed as the direct sum of the subgroups \( H_{(0)}^{1} \) and \( H_{(1)}^{1} \), where \( H_{(1)}^{1} \) is the group of all harmonic 1-forms of the first kind, pure of type one. With these observations, it will follow that if \( A_{1}, \ldots, A_{q} \) is a basis for \( H_{(0)}^{1} \), then \( A_{1}, \ldots, A_{q}, \bar{A}_{1}, \ldots, \bar{A}_{q} \), where the bar denotes conjugate forms, are harmonic, linearly independent, and constitute a basis for all harmonic 1-forms of the first kind. To see this, let \( A \) be any element of \( H_{(0)}^{1} \).

If \( A = a_{1}(z)dz' + a_{2}(z)dz^{2} + \cdots + a_{n}(z)dz^{n} \), then using \( z' = x' + i x' \), \( s^{2} = x^{2} + i x^{2} \), \( \ldots \), \( s^{n} = x^{n} + i x^{n} \), 

\[ A = (a_{1} + i a_{1}')(dx' + idx^{2}) + (a_{2} + i a_{2}')(dx^{3} + idx^{4}) + \cdots + (a_{q} + i a_{q}')(dx^{2m} + idx^{2m+1}) = [(a_{1} - a_{1}''dx' + a_{2}''dx^{2} - a_{2}^{'}dx^{3} + \cdots + a_{q}''dx^{2m}) + i(a_{1} dx' + a_{2} dx^{2} + a_{3} dx^{3} + \cdots + a_{q} dx^{2m})] \]

\[ = \alpha + i \beta. \] Then \( \bar{A} = \alpha - i \beta \) and since \( dA = d(\alpha + i \beta) \)
\[ d\alpha + id\beta = 0 \text{ and } J\alpha = J\alpha + iJ\beta = 0, d\bar{\alpha} = J\bar{\alpha} = 0. \text{ Thus,} \]
\[ \bar{\alpha} \text{ is harmonic and obviously pure of type one, so that } \bar{\alpha} \in H'(\omega). \]
Again, \( A_1, \ldots, A_r, \bar{A}_1, \ldots, \bar{A}_r \) are linearly independent, for by hypothesis, \( A_1, \ldots, A_r \) are and since \( H(\omega) \) and \( H'(\omega) \) have no elements in common, none of the \( A_i's \) can be expressed in terms of the \( \bar{A}_i's \) and vice versa. Also, the set \( \bar{A}_1, \ldots, \bar{A}_r \) are independent, for suppose there was a relation \( c_1\bar{A}_1 + \cdots + c_r\bar{A}_r = 0, \) where not all the \( c_i's \) are zero. Then, using the equalities
\[ \bar{A}_1 = \alpha_1 - i\beta_1, \bar{A}_2 = \alpha_2 - i\beta_2, \ldots, \bar{A}_r = \alpha_r - i\beta_r, \bar{A}_1 = (k_2\alpha_2 + \cdots + k_r\alpha_r) \]
\[ -i(k_2\beta_2 + \cdots + k_r\beta_r), \text{ so that } \alpha_i = (k_2\alpha_2 + \cdots + k_r\alpha_r) \text{ and } \beta_i = \]
\[ (k_2\beta_2 + \cdots + k_r\beta_r). \text{ But this last implies that there exists a relation between the } A_i's, \text{ for consider the equation } c_1'\bar{A}_1 + \cdots + c_r'\bar{A}_r = 0. \text{ This can be written } c_1'(\alpha_1 + i\beta_1) + \cdots + c_r'(\alpha_r + i\beta_r) = 0 \text{ and if } \alpha_i \text{ and } \beta_i \text{ are as above, then by substitution,}
\[ c_1'(k_2\alpha_2 + \cdots + k_r\alpha_r) + i(k_2\beta_2 + \cdots + k_r\beta_r) + \cdots + c_r'(\alpha_r - i\beta_r) = 0 \text{ or}
\[ [(c_1'k_2\alpha_2 + \cdots + c_r'k_r\alpha_r) + i(c_1'k_2\beta_2 + \cdots + c_r'k_r\beta_r)] + \]
\[ [( \alpha_2 - i\beta_2) + \cdots + c_r'(\alpha_r - i\beta_r) + i(c_2'k_2\alpha_2 + \cdots + c_r'k_r\beta_r)] = 0 \text{ and this equation can be satisfied by taking }
\[ c_2' = -c_1'k_2, \ldots, c_r' = -c_1'k_r. \]
Finally, the \( A_i's \) and \( \bar{A}_i's \) form a basis for \( H' \), for suppose \( \alpha \) was a real harmonic 1-form of the first kind, independent of the \( A_i's \) and \( \bar{A}_i's \). Then using complex parameters \( z', \ldots, z^n, \)
\[ \bar{z}', \ldots, \bar{z}^n, \alpha \text{ can be written } \alpha = a_1/2 \ dz' + a_2/2 \ dz^+ + \cdots + a_n/2 \ dz^n + a_1/2 \ d\bar{z}' + a_2/2 \ d\bar{z}^+ + \cdots + a_n/2 \ d\bar{z}^n \text{ and } C\alpha = i(a_1/2 \ dz' + \cdots + a_n/2 \ dz^n - a_1/2 \ d\bar{z}' - \cdots - a_n/2 \ d\bar{z}^n). \text{ Consider the form } (\alpha - iC\alpha). \text{ Since } d(\alpha - iC\alpha) = 0 \text{ and } C(\alpha - iC\alpha) =
\[ i(\alpha - iC\alpha), (\alpha - iC\alpha) \text{ is a simple differential of the first kind, independent of the } A_i's. \text{ But this is a contra-} \]
diction. Therefore, $A_1, \ldots, A_q, \overline{A}_1, \ldots, \overline{A}_q$ form a basis for $H^1$ and $2q = R$, the first Betti number of $\mathcal{M}$.

We consider now the integral of an arbitrary holomorphic simple differential defined on $\mathcal{M}$. This integral will be referred to as a Picard integral of the first kind and will be written as $\int \varphi$, the path of integration being some 1-chain on $\mathcal{M}$. By writing $\varphi = \alpha + i \beta$, where $\alpha$ and $\beta$ are real 1-forms whose co-efficients are $C^\infty$ functions of $x_1, \ldots, x_n$; $\int \varphi$ is defined according to Chapter I, section D. Moreover, the fact that $\varphi$ is closed implies that the conditions of integrability are satisfied so that $\int \varphi = P(z)$ and $dP = \varphi$. $P$ is a many-valued holomorphic function defined on $\mathcal{M}$, since if $P$ were single-valued, then it would be a constant.

Further, there exists a 1-cycle, $\gamma$, of $\mathcal{M}$, such that $\int \varphi = \omega$, $\omega \neq 0$.  

Definition: The number $\omega$ will be called the period of $P(z)$ on $\gamma$.

Let $\gamma_1, \ldots, \gamma_{2q}$ be a one dimensional Betti basis of $\mathcal{M}$. Any integral 1-cycle $\gamma$, not homologous to zero, can be expressed as $\gamma \sim \sum n_i \gamma_i$, where the $n_i$s are integers, not all zero. Then, $\int \varphi = \sum n_i \int \gamma_i$ and the period $\omega$ of $\int \varphi$ is expressible as $\omega = \sum n_i \omega_i$, where $\omega_i$ is the period $\int \gamma_i \varphi$. Further, the periods $\omega_i$ are independent, since if there was a relation

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15 G. de Rham and K. Kodaira, op. cit., p. 41.
a_0 \gamma_0 + \ldots + a_\gamma \gamma_\gamma = 0 \text{ where not all the } a_i's \text{ are zero, then }
\int \gamma = 0, \text{which implies } \sum a_i \gamma_i = 0 \text{ and this is a contradiction, since } \gamma_1, \ldots, \gamma_\gamma \text{ being a basis must be homologously independent. Thus, any Picard integral of the first kind has } 2q \text{ linearly independent periods on } \mathcal{M}.

**Definition:** A finite number of Picard integrals of the first kind are said to be independent if no non-trivial linear combination of them reduces to a constant.

**Theorem:** There are exactly q independent Picard integrals of the first kind on } \mathcal{M}.

**Proof:** First, if } A_1, \ldots, A_\gamma \text{ are a basis for simple differentials of the first kind, then the equation } a_1 \int A_1 + \ldots + a_\gamma \int A_\gamma = k, \text{ must have only the trivial solution for otherwise } a_1 A_1 + \ldots + a_\gamma A_\gamma = 0 \text{ would be satisfied non-trivially. Thus, there are at least } q \text{ independent integrals.}

That there is at most } q \text{ is seen as follows: If } P_1, \ldots, P_{q+1} \text{ are independent such integrals, then } a_1 P_1 + \ldots + a_{q+1} P_{q+1} = k \text{ is satisfied if and only if } a_1 = a_2 = \ldots = a_{q+1} = k = 0. \text{ But this assumption implies } a_1 \partial P_1 + \ldots + a_{q+1} \partial P_{q+1} = 0 \text{ has only the trivial solution, for if it did not, then } \partial P_1 = a_2 / a_1 \partial P_2 + \ldots + a_{q+1} / a_1 \partial P_{q+1} \text{ and } P_1 = a_2 / a_1 P_2 + \ldots + a_{q+1} / a_1 P_{q+1} + c \text{ or } P_1 = a_2 P_2 + \ldots + a_{q+1} P_{q+1} = c \text{ and so the system } P_1, \ldots, P_{q+1} \text{ would be dependent.}

**B. Sub-varieties and Divisors.**
Definition: A sub-variety of a complex manifold $\mathcal{M}$ is a subset $S$ of $\mathcal{M}$ satisfying the following conditions: To each point $p_0$ of $S$ there is a neighborhood $U_{p_0} \subset \mathcal{M}$ and a system $f_1, \ldots, f_n$ of independent functions, each holomorphic in $U_{p_0}$, such that the set of points $p$ in $U_{p_0}$ where $f_1(p) = f_2(p) = \cdots = f_n(p) = 0$ coincides exactly with $S \cap U_{p_0}$.

Definition: A sub-variety will be termed closed if it is a compact point set.

Definition: A point $p_0$ of a sub-variety is called non-singular if there exists a local co-ordinate system $z_1, \ldots, z_n$ in some neighborhood $U_{p_0}$ of $p_0$ such that the defining set of functions $f_1, \ldots, f_n$ in $U_{p_0}$, referred to above, is given by $z_i = f_i, \ldots, z_n = f_n$.

If such a representation is not possible, then $p_0$ will be called a singular point of the sub-variety.

Definition: The number $n-r$ is called the dimension (or complex dimension) of $S$ at $p_0$, written $c\text{-dim}(S)_{p_0}$. The dimension (or complex dimension) of a subvariety is the least upper bound of the dimensions of $S$ at each of its non-singular points, i.e.,

$$c\text{-dim}(S) = \sup_{p_0 \in S} \{c\text{-dim}(S)_{p_0}\}.$$ 

Definition: A sub-variety is called proper if it is closed and not of $c\text{-dim}$. $0$ or $n$.

A proper sub-variety always has non-singular points since the set of all singular points constitutes a sub-variety having $c\text{-dim} \leq (n-r-1)$.

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**Definition:** A closed sub-variety $S$ is said to be reducible if it can be represented as the union $S = S' \cup S''$ of two proper sub-varieties. Otherwise $S$ is irreducible.

Every reducible sub-variety can be uniquely decomposed into the sum of a finite number of irreducible sub-varieties. 17

**Definition:** A proper sub-variety is called homogeneous if it has the same dimension at each of its non-singular points.

**Definition:** A homogeneous proper sub-variety of $c$-dim. $n-1$ is called a hypersurface in $\mathbb{M}$. A homogeneous proper sub-variety of $c$-dim. 1 is called a curve in $\mathbb{M}$.

A hypersurface can be represented in some neighborhood of each of its points by a single equation $f_\varphi(z) = 0$; such an equation is called a local equation of the hypersurface at that point.

**Definition:** A local equation $R_\varphi(z) = 0$ of a hypersurface is said to be minimal if, for every local equation $f_\varphi(z) = 0$, the ratio $f_\varphi/R_\varphi$ is holomorphic in some neighborhood of $\varphi$.

Let $H$ be an irreducible hypersurface of $\mathbb{M}$ and let $R_\varphi(z) = 0$ be the minimal local equation of $H$ at $\varphi$. If $\varphi_0$ is a non-singular point of $H$, then at least one of the partials $\partial R_\varphi/\partial z^\alpha$, $\alpha = 1, \ldots, n$, does not vanish at $\varphi_0$. The vanishing or non-vanishing of all partials does not depend on the co-ordinate system employed for let $\tilde{z}', \ldots, \tilde{z}^n$ be any other co-ordinates at $\varphi_0$. Then,

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This system can be solved uniquely for \( \frac{\partial R_{\rho_0}}{\partial z^1} \) and all partials \( \frac{\partial R_{\rho_0}}{\partial z^\alpha} \) vanish at \( \rho_0 \) if and only if all \( \frac{\partial R_{\rho_0}}{\partial z^\alpha} \) do.

**Definition:** A divisor is an integral \((2n-2)\) cycle whose carrier consists of a finite number of irreducible hypersurfaces.

In notation, a divisor, \( D \), will be written as \( D = \sum m_y D_y \), where the \( m_y \) are integers and the \( D_y \) are irreducible hypersurfaces.

**Definition:** \( F(z) \), defined on \( \mathbb{M} \), is called a meromorphic function if, for every point \( \rho \) of \( \mathbb{M} \), \( F(z) \) can be represented in a neighborhood \( \mathcal{U}_\rho \) of \( \rho \) as a ratio \( g_\rho / f_\rho \) of two holomorphic functions defined in \( \mathcal{U}_\rho \).

Every meromorphic function on \( \mathbb{M} \) determines its divisor as follows: Assuming that the functions \( g_\rho \) and \( f_\rho \) are relatively prime, denote by \( D_\rho \) the hypersurface defined by the system of local equations \( g_\rho f_\rho = 0 \). Decompose \( D_\rho \) into a
sum \( D_f = \sum D_k \) of irreducible hypersurfaces and denote the minimal local equation of each \( D_k \) at \( p \) by \( f_{p,k} = 0 \). Then, for every point \( p \in \mathcal{M} \), \( F(z) \) can be represented in \( \mathcal{U}_p \) as \( F(z) = u_p \prod f_{p,k}^{m_k} \), where \( u_p \) is a unit and the \( m \)'s are integers, independent of \( p \). The divisor of \( F(z) \) will then be, by definition, the \((2n-2)\) cycle \( (F) = \sum m_k D_k \).

**Definition:** The divisor \( D' \) will be called equivalent to \( D \) if there exists on \( \mathcal{M} \) a meromorphic function \( F \) such that \( D' - D = (F) \).

Using this definition, it is clear that all the divisors on \( \mathcal{M} \) can be decomposed into mutually disjoint equivalence classes, called divisor classes, on \( \mathcal{M} \). The class containing \( D \) will be denoted by \( \{D\} \).

**Definition:** A divisor \( D = \sum m_k D_k \) will be called positive if all the \( m \)'s are \( \geq 0 \) and \( D \neq 0 \).

**Definition:** A meromorphic function \( F \) on \( \mathcal{M} \) will be called a multiple of \( D \) if \( (F) - D \geq 0 \).

The set of all meromorphic functions which are multiples of \( -D \) will be denoted by \( \mathcal{F}(D) \), i.e., \( \mathcal{F}(D) = \{F/(F) + D \geq 0\} \).

Since the manifolds to be employed in the next three sections will be of arbitrary c-dim. \( n > 2 \), the divisors to be encountered will be assumed free of singularities. Due to the complicated nature of the singular sets of such manifolds, there is no local representation known which affords a means of handling the convergence problems which arise.
C. Introduction of a Special Class of Manifolds.

We consider the class of manifolds \( \{ \mathcal{M}_s \} \), described as follows: Any member, \( \mathcal{M}_s \), of this class is a compact Kähler manifold containing an analytic curve \( C = \sum C_k \) (where each \( C_k \) is irreducible), having the following properties:

1) If \( \Gamma \) and \( \Gamma' = \Gamma_1 - \Gamma_2 \) are divisors on \( \mathcal{M}_s \), (both non-singular for \( n > 2 \)), and \( \Gamma' \sim 0 \), then \( C \) intersects \( \Gamma \) only at the points \( \Gamma_1, \ldots, \Gamma_m \), and intersects \( \Gamma' \) only at the points \( \Gamma_1', \ldots, \Gamma_m' \), where \( \Gamma_1, \ldots, \Gamma_m \in \Gamma \) and \( \Gamma_1', \ldots, \Gamma_m' \in \Gamma_2 \). Not all the points \( \Gamma \) nor \( \Gamma' \) need be distinct, but the total multiplicity of the intersections must, in each case, be \( m \).

2) There exists a 1-dimensional Betti basis of \( \mathcal{M}_s \) on \( C \).

3) Each \( C_k \) is the holomorphic image of a Riemann surface and is a 1-1 image in a neighborhood of any point of intersection of \( C \) with \( \Gamma \) or \( \Gamma' \).

Only manifolds of class \( \{ \mathcal{M}_s \} \) will hereafter be considered. It can be shown that the class \( \{ \mathcal{M}_s \} \) is also realized by assuming the existence of a certain set of currents defined on \( \mathcal{M} \). Indeed, let \( C = \bigcup C_k \) be a finite sum of 2(real)-dimensional nowhere dense closed subsets of \( \mathcal{M} \) chosen so that \( C \) contains a 1-dimensional Betti basis of \( \mathcal{M} \) and has the intersection points, mentioned above, in common with \( \Gamma \) and \( \Gamma' \). Before proceeding, the following definition is needed.
Definition: A pure 1-current of type zero defined in an open subset $\Omega$ of $\mathcal{M}$ is called an analytic current if it is closed in $\Omega$ except for a nowhere dense closed subset, $E$, of $\Omega$. The current is said to be regular in $\Omega - E$.

By Theorem 1.8, an analytic current coincides with a holomorphic simple differential in $\Omega - E$. The following two assumptions will now be made:

1) To each neighborhood $U_p$ of a point $p \in C$, there is an analytic 1-current $T_p$ defined in $U_p$ such that $T_p$ is regular in $U_p - C$ and $T_p - T_q$ is regular in $U_p \cap U_q$. Under this assumption, $d(T_p - T_q) = 0$ in $U_p \cap U_q$ so that the current $Z = dT_p$ is a well determined 2-current on $\mathcal{M}$, carried on $C$ and satisfying $dZ = 0$.

2) The current $Z$ satisfies $HZ = 0$.

Then, there exists an analytic current $T$, defined on $\mathcal{M}$, such that $T - T_p$ is regular in $U_p$ for all $p$. Since any meromorphic simple differential can be considered as a current, it is possible to assert the existence of a meromorphic 1-form having $C$ as polar curve. Thus $C$ is an analytic curve on $\mathcal{M}$ containing $2q$ homologously independent 1-cycles on $\mathcal{M}$ and $C$ intersects the divisors $\mathcal{M}^i$ and $\mathcal{M}^j$ in the required points.

As a final observation on the class $\{\mathcal{M}_S\}$, we remark that if $\mathcal{M}$ is an algebraic variety, then $\mathcal{M} \in \{\mathcal{M}_S\}$, for let $P(x^0, x^1, \ldots, x^{n+1}) = 0$ be the equation of the variety, where

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$x^0, x', \ldots, x^{n+1}$ are homogeneous co-ordinates. By taking successive hyper-plane sections of $\mathcal{M}$, we arrive at a curve $C$:

$P(x^0, x', x^2, a, b, c, \ldots, u) = 0$. This curve contains a $1$-dimensional Betti basis of the variety. Furthermore, for any $m$ arbitrary non-singular points $p_1, \ldots, p_m$ of $C$, it is possible to determine a hypersurface of the variety which has, in common with $C$, only these points. For example, consider the $m$-dimensional linear system of varieties $\lambda_1 F_1 (x^0, x', \ldots, x^{n+1}) + \lambda_2 F_2 (x^0, x', \ldots, x^{n+1}) + \ldots + \lambda_{m+1} F_{m+1} (x^0, x', \ldots, x^{n+1}) = 0$, where $F_\mu (x^0, x', x^2, a, b, \ldots, u)$, $\mu = 1, 2, \ldots, m+1$, is not identically zero. It is possible to determine a unique set of values of the $\lambda$'s, say $\lambda'_1, \ldots, \lambda'_{m+1}$ such that $\lambda'_1 F_1 + \ldots + \lambda'_{m+1} F_{m+1}$ vanishes only for $p_1, \ldots, p_m$ and does not vanish for any other points of $C$. Thus,

$$\Gamma : \begin{cases} P(x^0, x', \ldots, x^{n+1}) = 0 \\ \lambda'_1 F_1 + \ldots + \lambda'_{m+1} F_{m+1} = 0 \end{cases}$$

is a hypersurface of the variety meeting $C$ at the required points. A $\Gamma'$ may be determined analogously so that the assumptions made in the beginning of the section are fulfilled in the algebraic case.

D. Picard Integrals of the Second Kind

Let $\Gamma'$ be the divisor on $\mathcal{M}_S$ introduced in section C.

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First, there are the following definitions.

**Definition**: A many valued meromorphic function $F(z)$ on $\mathcal{M}_S$ will be called additive if by analytic continuation along an arbitrary closed continuous curve $\gamma$ on $\mathcal{M}_S$, $F(z)$ is changed to $F(z) + c_\gamma$, where $c_\gamma$ is a constant depending on $\gamma$.

An arbitrary additive meromorphic function is represented as the integral $\int_\gamma \frac{df}{f}$, $df$ being a simple differential.

**Definition**: The integral $\int_\gamma \phi$ is called a Picard integral of the second kind if it is locally single valued on $\mathcal{M}_S$ and $\phi$ is a simple meromorphic differential defined on $\mathcal{M}_S$.

Unless it is everywhere holomorphic, an additive meromorphic function is then a Picard integral of the second kind.

The principle objective of this section can now be stated: To introduce a linear space $\mathcal{F}(\Gamma, d)$ consisting of all meromorphic functions $f$ on $\Gamma$ satisfying the conditions $(f) + d \geq 0$, where $d$ is the divisor of a system $\{h_\gamma\}$ of locally meromorphic functions defined on $\Gamma$, and to show first, that for any $f \in \mathcal{F}(\Gamma, d)$, there exists on $\mathcal{M}_S$ at least one additive meromorphic function $A_f$, which is a multiple of $-\Gamma$ in the sense that, for every point $\gamma$ of $\mathcal{M}_S$, $R_{\gamma}A_f$ is holomorphic in some neighborhood of $\gamma$; second, that $A_f$ satisfies the relation $R_{\gamma}A_f/h_{\gamma} = f$ on $\Gamma$. Here, $R_{\gamma}$ is a minimal local equation of $\Gamma$ at $\gamma$. The space of functions $A_f$ satisfying these conditions will be denoted by $\Omega(\Gamma)$, and the mapping $R_{\gamma}A_f/h_{\gamma}$ will be called the Severi residue function of $A_f$ on $\Gamma$.

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In order to achieve these results, it will be necessary to assume that the divisor \( \Gamma \) satisfies one of the following two conditions:

1) Each component of \( \Gamma \) is the divisor of a meromorphic function on \( M_s \), or

2) \( \Gamma \) is an algebraic variety.

When \( \Gamma \) satisfies either of these conditions, it is possible to state the important existence theorem:

**Theorem 2.1:** There exists a system \( \{ h_p / \eta \in \Gamma \} \) of meromorphic functions, \( h_p \), defined respectively in a neighborhood \( U_p \) of \( \cdot \Gamma \), such that \( h_p / \eta = R_p / R \eta = t_p \cdot \eta \) in \( U_p \cap U \eta \), \( t_p \cdot \eta \) being a non-vanishing and holomorphic function in \( U_p \cap U \eta \).

If \( F \) is a meromorphic function on \( M_s \), having \( \Gamma \) as divisor, the system \( \{ h_p \} \) is simply obtained by putting \( h_p = R \cdot F \). If \( \Gamma \) is an algebraic variety, the existence of the system has been shown. 21 The system \( \{ h_p \} \) is not unique. First, it depends on the choice of the system of minimal local equations. Let \( \{ R_p = 0 \} \) be another system of minimal local equations. Then \( M_p(z) = R_p / R \) is a non-vanishing holomorphic function in a neighborhood of \( \eta \) on \( \Gamma \) and the meromorphic functions \( h^{'}_p \) defined by \( h^{'}_p = M_p \cdot h_p \) constitute a system \( \{ h^{'}_p \} \) satisfying Theorem 2.1 with respect to \( \{ R_p \} \).

Second, by multiplying each \( h_p \) by an arbitrary fixed meromorphic function \( f \) on \( \Gamma \), the system \( \{ h^{'}_p / h^{'}_p = f h_p \} \) satisfies...
fies the theorem with respect to \( \{ R_\alpha \} \). Conversely, for
two arbitrary systems \( \{ h_\alpha \} \) and \( \{ h'_\alpha \} \) satisfying Theorem
2.1 with respect to \( \{ R_\alpha \} \), \( h'_\alpha \cdot h_\alpha = h'_\alpha \cdot h_\alpha \) in \( U_\alpha \cap U_\beta \)
so that \( f = h'/h \) is a well-determined meromorphic function on
\( \Gamma \). Thus, for fixed \( \{ R_\alpha \} \), \( \{ h_\alpha \} \) is unique up to multipli-
cation by an arbitrary meromorphic function on \( \Gamma \).

Now, for any meromorphic function defined on \( \Gamma \), it was
shown, in section B, precisely how its divisor is determined.
Corresponding to this, the divisor, \( d \), of the system \( \{ h_\alpha \} \) is
defined to be the \((2n-1)\)-cycle \( d = \sum_{\alpha} ( \sum m_{\alpha} d_{\alpha} ) \), where each
\( d_{\alpha} \) has minimal local equation \( h_{\alpha,\alpha} \). With this definition, we
see that if \( f \) is any function belonging to \( \mathcal{F}( \Gamma, d ) \), then,
since \( (f) + d \geq 0 \), \( fh_\alpha \) is a holomorphic function defined on
\( \Gamma \cap U_\alpha \), \( U_\alpha \) being a neighborhood of \( \alpha \) in \( M_S \). More-
over, \( fh_\alpha \) can be extended to a holomorphic function \( N(z) \),
defined in all \( U_\alpha \) by simply putting \( N(z) = f(z^2, \ldots, z^n) \cdot
h_\alpha(z^2, \ldots, z^n) \).

Turning now to current theory, let \( R_\alpha \) be as above. Then
\( 1/R_\alpha \) can be considered as a zero current defined in some
neighborhood \( U_\alpha \) on \( M_S \). In fact, if \( \psi \) is an arbitrary \( 2n \)-
form of class \( C^\infty \), whose carrier is contained in \( U_\alpha \), then
the integral \( \int 1/R_\alpha \wedge \psi \) converges absolutely and we set
\( 1/R_\alpha \wedge \psi = \int 1/R_\alpha \wedge \psi \). Putting \( Q_\alpha = 1/4i\pi (1+iC)d(1/R_\alpha) \),
we introduce the \( 1 \)-current \( Q_\alpha \). Since \( d(1/R_\alpha) \) is holo-
morphic in \( U_\alpha \), \( Q_\alpha \) is carried on \( \Gamma \). Again, \( C Q_\alpha = -i \otimes \psi \),
since \( C Q_\alpha = 1/4i\pi [ (C-1)\text{d}(1/R_\alpha) ] = -1/4i\pi [ (1+iC)\text{d}(1/R_\alpha) ] \).
= -i Q_p. Thus, Q_p is pure 1 of type 1. If H_p is any holomorphic function in U_p, then the meromorphic function H_p/R_p can be considered as a current in U_p since it is the product of a 0-current and a 0-form. Now, \( d(H_p/R_p) = 1/R_p \Lambda dH_p + H_p \Lambda d(1/R_p) \) and \( dH_p \) satisfies \((1+i\epsilon)dH_p = 0\). Hence,

\[
(2.2) \quad (1+i\epsilon)d(H_p/R_p) = 4i \pi Q_p \Lambda H_p.
\]

In order to obtain an explicit expression for \( Q_p \), there is the following theorem:

**Theorem:** For an arbitrary \((2n-1)\)form \( \psi \) of class \( C^\infty \) whose carrier is contained in \( U_p \),

\[
(2.3) \quad Q_p(\psi) = \int \sum dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n.
\]

Proof: \( 4i \pi Q_p(\psi) = (1+i\epsilon)d(1/R_p) \Lambda \psi = -1/R_p \Lambda (1+i\epsilon) \psi \)

\[
= - \int 1/R_p \cdot d(1-i\epsilon) \psi. \quad \text{But, } d(1/R_p) \Lambda (1-i\epsilon) \psi \text{ vanishes identically in } U_p - \Gamma \text{, so } 1/R_p \Lambda d(1-i\epsilon) \psi = d\{1/R_p \cdot (1-i\epsilon) \psi\} \text{ in } U_p - \Gamma . \text{ Consequently, making use of the fact that } \Gamma \text{ is nonsingular and writing } z' = 0 \text{ for minimal local equation of } \Gamma \text{ at } \varpi, \text{ we have } \]

\[
4i \pi Q_p(\psi) = \lim_{\epsilon \to 0} \int \sum dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n.
\]

On \( \mid z' \mid = \epsilon \), \( z' \bar{z}' = \epsilon^2 \) and so \( \bar{z}' dz' + z' d\bar{z}' = 0 \). The exterior multiplication of this last equation by \( dz' \) yields the result \( d\bar{z}' \wedge dz' = 0 \) on \( \mid z' \mid = \epsilon \). The last integral above then becomes

\[
\lim_{\epsilon \to 0} \int \sum \psi_{\alpha_1 \ldots \alpha_{2n-1}} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n .
\]
Consider now the 1-current \( Q_f \) defined as
\[
(2.4) \quad Q_f[\psi] = - \int_{\Gamma} \text{fh}_f \psi \, dz^1 \ldots dz^n
\]
where \( \psi \) is a variable \((2n-1)\) form of class \( C^\infty \). It is obvious from its definition that \( Q_f \) is pure of type 1 and carried on \( \Gamma \). Furthermore, \( d Q_f \) is a pure 2-current of type 1. Therefore, by virtue of Theorem 1.9, \( \Theta_f = (d \wedge g + i d' g) \).

\( d Q_f \) is a holomorphic simple differential in \( \mathcal{M}_\Sigma - \Gamma \) and the integral of \( \Theta_f \) over an arbitrary 1-cycle \( Y \subset \mathcal{M}_S - \Gamma \) is given by \( \int_Y \Theta_f = \int \mathcal{M}_S \). Putting \( A_f = 2 \pi \int \mathcal{M}_S \), it is seen that \( A_f \) is an additive holomorphic function on \( \mathcal{M}_S - \Gamma \).

It will now be shown that for every \( \gamma \in \Gamma, \text{R}_\gamma A_f \) is holomorphic in a neighborhood \( \mathcal{U}_\gamma \) of \( \gamma \). Let the holomorphic function \( \text{fh}_f \), defined in \( \mathcal{U}_\gamma \cap \Gamma \), be extended to the function \( N(z) \), holomorphic in all \( \mathcal{U}_\gamma \). Then, from (2.2), (2.3), and (2.4), it is immediate that \((1 + iC) d(N/R_\gamma) = -\frac{1}{4\pi} \int \mathcal{M}_S Q_f \). Consider the difference
\[
(2.5) \quad T = 2 \pi \int \mathcal{M}_S - 1/2(1-iC) d(N/R_\gamma).
\]

\( T \) is a pure 1-current of type 0 in \( \mathcal{U}_\gamma \) since \( CT = 2 \pi \int \mathcal{M}_S \)

\[
-1/2(1+iC) d(N/R_\gamma) = 1/2 \pi \int \mathcal{M}_S \}
\]

From Theorem 1.9, \( d \Theta_f = t d Q_f \), so that \( dT = 2 \pi d \Theta_f - 1/2 \cdot d[(1+iC) d(N/R_\gamma)] = 2 \pi d \Theta_f + 1/2 \cdot d[(1+iC) d(N/R_\gamma)]. \)

---

Theorem 1.8. T is a holomorphic simple differential in \( U \) and \( J^2 T = L(z) \) is a holomorphic function in \( U \). On the other hand, in \( U - \Gamma, 1/2(1-10)d(N/R_p) = 1/2d(N/R_p) + 1/2d(N/R_p) = d(N/R_p) \). Using (2.5), we have

\[
(2.6) \quad A_f = N/R_p + L + \text{constant, } (\text{in } U - \Gamma),
\]
showing that \( R_p A_f \) is holomorphic in all \( U \). Moreover, from (2.6),
\[
R_p(A_f - L - \text{constant}) = R_p F = N \quad \text{and so } \quad R_p P(0, z^2, \ldots, z^n) = N(0, \ldots, z^n) = f(z^2, \ldots, z^n)h_p(z^2, \ldots, z^n)
\]
which means that

\[
(2.7) \quad R_p F(0, z^2, \ldots, z^n)/h_p = f \quad (\text{in } U).
\]
Finally, for points of \( \Gamma \), since \( R_p L = R_p \cdot \text{constant} = 0 \), we see that \( A_f \) satisfies \( R_p A_f/h_p = f \). Thus, \( A_f \) belongs to \( \mathcal{Q}(\Gamma) \) and has Severi residue \( f \). Evidently any other function \( A_f' \) which belongs to \( \mathcal{Q}(\Gamma) \) and satisfies \( R_p A_f'/h_p = f \) can be represented as \( A_f' = A_f + \int z \sum \frac{1}{y-1} n_y A_y + \text{constant} \), where \( A_1, \ldots, A_q \) are a base of simple differentials of the first kind. In conclusion, it has been established that there exists a Picard integral of the second kind which has \( \Gamma \) as its only singular locus and any Picard integral belonging to \( \mathcal{Q}(\Gamma) \) and having Severi residue \( f \) can be written as

\[
(2.8) \quad A_f' = 2\pi i \int z \left[ \Theta f + \sum n_y A_y \right] + n_y,
\]
where the \( n_y \) are constants. Conversely, for arbitrary \( n_y \)'s, the function \( A_f' \), defined by (2.8), belongs to \( \mathcal{U}(\Gamma) \) and satisfies \( R_p A_f'/h_p = f \).

**Definition:** A finite number of Picard integrals of the second kind are said to be independent if no non-trivial linear combination of them reduces to a single valued meromorphic function.
From (2.8), it is clear that there are at least \( q \) independent Picard integrals of the second kind which are multiples of \(-\Gamma\).

E. Picard Integrals of the Third Kind.

Definition: The integral \( \int z \varphi \), where \( \varphi \) is a simple meromorphic differential, is said to be a Picard integral of the third kind if it is not locally single valued.

Let \( \Gamma' \) be as in section C. Then we propose to show that there exists a Picard integral of the third kind having only log singularities at the points of the carrier of \( \Gamma' \). First, designate by \( R_{\Gamma_1} \) and \( R_{\Gamma_2} \) the minimal local equations of \( \Gamma_1 \) at \( \varphi \) and \( \Gamma_2 \) at \( \varphi \), respectively, where \( \varphi_1 \) and \( \varphi_2 \) are any points on these divisors. Then, \( \mathcal{T}_{\Gamma} = \frac{1}{2\pi i} (d\log R_{\varphi_1} - d\log R_{\varphi_2}) \) can be considered as a 1-current in some neighborhood \( U_{\Gamma} \) of \( \Gamma' \). This follows from the fact that both \( 1/R_{\varphi_1} \) and \( 1/R_{\varphi_2} \) can be considered as 0-currents defined in neighborhoods of \( \Gamma_1 \) and \( \Gamma_2 \), so \( dR_{\varphi_1}/R_{\varphi_1} \) and \( dR_{\varphi_2}/R_{\varphi_2} \) are 1-currents defined in these neighborhoods, each being the product of a 0-current and a holomorphic 1-form. Also, the current \( \mathcal{T}_{\Gamma} \) satisfies 

\[ C \mathcal{T}_{\Gamma}[\psi] = i \mathcal{T}_{\Gamma}[\bar{\psi}] \]

a fact which may be verified by direct calculation. Furthermore, considered as a current, \( d\mathcal{T}_{\Gamma}[\bar{\psi}] = i \mathcal{T}_{\Gamma'}[\psi] \), where \( \psi \) is an arbitrary \((2n-2)\) form of class \( C^\alpha \) whose carrier is contained in \( U_{\Gamma} \). Consider now the

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G. deRham and K. Kodaira, op. cit., p. 103.
2-current \( \Gamma' [\psi] = \int_\Gamma' \psi \), where \( \psi \) is a variable \((2n-2)\) form of class \( C^\infty \). It is easy to show that \( \mathcal{C} [\Gamma' = \Gamma' \), since \( \Gamma' [\mathcal{C} [\psi] = \sum_{n=0}^{\infty} \int_{\Gamma'} \psi \cdot \omega^0 \cdots \omega^s d\omega^0 \cdots d\omega^s \), and inserting \( z^\omega = z^\omega (t', \cdots, t^{n-1}) \) in this expression, it is immediate that \( \Gamma' [\mathcal{C} [\psi] = \Gamma' [\psi] \). \( \Gamma' \) is then pure 2 of type 1. Since \( \Gamma' \sim 0 \), then, considered as currents, \( d \Gamma' = 0 = 0 \).

By Theorem 1.9, \( \Theta = (d \Lambda \cdot \mathcal{C} + i \Theta \Theta) \Gamma' \) is a holomorphic simple differential on \( \mathcal{M}_s - \Gamma' \), and the integral of \( \Theta \) over an arbitrary 1-cycle \( \gamma \subset \mathcal{M}_s - \Gamma' \) is given by \( \int_\gamma \Theta = i \mathcal{Q} [\mathcal{H} \gamma] + i (\gamma, \mathcal{Q}) \), where \( d \mathcal{Q} = |\Gamma' \). Thus, \( \Theta = \int_\gamma z \Theta \) is an additive holomorphic function on \( \mathcal{M}_s - \Gamma' \). Furthermore, for every point \( \rho \in \Gamma' \), \( \Theta - (\log \mathcal{R}_{\rho_1}, -\log \mathcal{R}_{\rho_2}) \) is holomorphic in a neighborhood of \( \rho \). This may be seen as follows: The difference \( \Theta = \Theta - \Theta \) is a pure 1-current of type 0 since \( \Theta = \Theta - \Theta \Theta = i \Theta - i \Theta = i \Theta \). Also, \( d \Theta = d \Theta - d \Theta = d \Theta - i \Gamma' \). But, by Theorem 1.9, \( d \Theta = i \Gamma' \), so that \( d \Theta = 0 \). By Theorem 1.8, \( \Theta \) is a holomorphic simple differential in \( \mathcal{U}_\rho \) and the integral \( L = \int_\mathcal{U}_\rho \Theta \) is a holomorphic function in \( \mathcal{U}_\rho \). Now, in \( \mathcal{U}_\rho - \Gamma' \), \( \Theta \) satisfies \( d \Theta = 0 \), so that from \( \Theta = \Theta - \Theta \), it is immediate that \( \Theta = \log \mathcal{R}_{\rho_1} - \log \mathcal{R}_{\rho_2} + L + \text{constant} \) in \( \mathcal{U}_\rho - \Gamma' \). This proves that \( \Theta = \Theta = \log \mathcal{R}_{\rho_1} - \log \mathcal{R}_{\rho_2} + L + \text{constant} \) is holomorphic in all \( \mathcal{U}_\rho \).

In summation, there exists at least one Picard integral of the third kind having \( \log \mathcal{R} \) singularities only on \( \Gamma' \), provided \( \Gamma' \sim 0 \). On \( \mathcal{M}_s - \Gamma' \), the integral is an additive holomorphic function of \( z \).

**Definition:** A finite number of Picard integrals of the
third kind are said to be independent if no non-trivial linear combination of them reduces to a log function.

There are at least \( q \) independent Picard integrals of the third kind having \( \Gamma' \) as logarithmic divisor.

\[ \mathcal{V} \text{ of Dimension 2. } \Gamma \text{ and } \Gamma' \text{ Having Arbitrary Singularities.} \]

It is possible to achieve the results of sections D and E while allowing the divisors \( \Gamma \) and \( \Gamma' \) to have arbitrary singularities provided the dimension of the manifold is two. It will also be required, for future needs, that the points of intersection of \( \Gamma \) and \( \Gamma' \) with the curve \( C \), introduced in section C, are not singular points of \( \Gamma \) or \( \Gamma' \). As regards these singularities, it is readily seen that neither \( \Gamma \) nor \( \Gamma' \) can have more than a finite number of them, since the discriminant of \( R_{\Gamma} \), being a polynomial in one variable, has only a finite number of zeroes and both \( \Gamma \) and \( \Gamma' \) can be covered by a finite number of co-ordinate systems.

To begin with, we remark that the results of section D have been obtained even for the singular case, when \( n = 2 \), by Kodaira and what follows will be a brief description of his work.

Let \( \Gamma = \sum_{\gamma=1}^{N} \Gamma_{\gamma} \), where each \( \Gamma_{\gamma} \) is irreducible. Each \( \Gamma_{\gamma} \) is the holomorphic image of a compact Riemann surface \( \overline{\Gamma_{\gamma}} \), i.e., \( \Gamma_{\gamma} = \wp(\overline{\Gamma_{\gamma}}) \), and \( \wp^{-1} \) is single-valued except at the singular points. \( \overline{\Gamma} = \sum_{\gamma} \overline{\Gamma_{\gamma}} \) will be called the non-
singular model of $\Gamma$. For the remainder of this section, points on $\Gamma$ will be denoted by $p, q, \ldots$, and points on $\Gamma$ by $\hat{p}, \hat{q}, \ldots$. Now, for each $p \in \Gamma$, choose a sufficiently small neighborhood $U_p$ of $p$ so that the closure $[\varphi(U_p)]$ of $\varphi(U_p)$ satisfies the conditions:

a) $[\varphi(U_p) - \varphi(p)]$ contains no singular point of $\Gamma$;

b) The local equation $\varphi^*(\rho)$ of $\Gamma$ is useful at every point $\hat{q} \in [\varphi(U_p)]$.

Let $p$ and $q$ be points on $\Gamma$ such that $U_p \cap U_q$ is not empty. Then, the ratio $T_{pq} = \frac{\varphi^*(\rho)}{\varphi^*(q)}$ is a non-vanishing, holomorphic function of $\varphi$, defined in some neighborhood of $[\varphi(U_p \cap U_q)]$. Put $t_{pq} = T_{pq}(\varphi(t), \varphi^*(t))$, where $\varphi(t)$ is a local uniformizer on $\Gamma$. Thus, $t_{pq}$ is a non-vanishing, holomorphic function of $\varphi$, defined in $U_p \cap U_q$, provided $U_p \cap U_q$ is not empty. The following theorem, stated in section B as Theorem 2.1, can now be proved.

**Theorem 2.1a:** There exists a system $\{h_p/p \in \Gamma\}$ of meromorphic functions, defined respectively in $U_p$, such that $h_p/h_q = t_{pq}$ in $U_p \cap U_q$.

The remarks made immediately following Theorem 2.1 of section B concerning the system $\{h_p\}$ also hold in this case.

**Definition:** For an arbitrary meromorphic function $f(\varphi) = \sum a_n(\varphi + a_n + \cdots) (a_n \neq 0)$, defined in $U_p$, the exponent $m$ will be denoted by $V_p(f)$.

---

If $f$ is a meromorphic function defined everywhere on $\mathbb{C}$, the divisor $(f)$ of $f$ can be written $(f) = \sum_{p} V_{p}(f) \cdot p$.

**Definition:** The divisor $d_{s}$ of an arbitrary system $\{h_{p}\}$ satisfying Theorem 2.1 is defined as $d_{s} = \sum_{p} V_{p}(h_{p}) \cdot p$.

The divisor $d_{s}$ is independent of the system $\{R_{p}\}$, since the divisor of $h' = \sum_{p} V_{p} [M \varphi(p) \{\varphi'(\tau), \varphi(\tau) \cdot h_{p}\}] \cdot p = \sum_{p} V_{p}(h_{p}) \cdot p = d_{s}$, and if $d_{s}'$ is the divisor of $\{h_{p}\} = \{fh_{p}\}$, then $d_{s}' - d_{s} = (f)$, so that the divisor class $\{d_{s}\}$ on $\mathbb{C}$ is determined uniquely by $\Gamma$ and does not depend on the choice of the systems $\{h_{p}\}$ and $\{R_{p}\}$. It is clear that the divisor $d_{s}$ introduced here is the analogue of the divisor $d$ of section D.

**Definition:** For an arbitrary meromorphic differential $\xi$, defined in $U_{p}$, we denote by $\text{Res}_{p} [\xi]$ the residue $1/2\pi i \oint \xi$ of $\xi$ at $p$, where $\oint$ denotes the integral taken over a sufficiently small circle with center $p$.

Corresponding to the space $\mathbb{F}(\Gamma, d)$ of section D, we define the space $\mathbb{F}(\Gamma, d_{s})$ as that space consisting of all functions $f$ on $\mathbb{C}$ which satisfy the conditions:

a) $(f) + d_{s} \geq 0$;

b) $\sum_{\varphi(p) = p} \text{Res}_{p} \left[ (\varphi')^{k}(\varphi)^{l} \cdot f_{p} \cdot \sigma_{p} \right] = 0$, for $k, l = 0, 1, 2, \ldots$, where $p$ is any singular point of $\Gamma$ and $\sigma_{p}$ is the differential defined as $\sigma_{p} = d \varphi_{p}/[\partial R_{p}(\varphi', \varphi) / \partial z]$.

The fact that any function which satisfies condition a) must also satisfy condition b) follows directly from the
Theorem: For an arbitrary holomorphic double differential
\[ \Delta_\mu = B_{12}(z)dz_1\overline{dz}_2, \quad \sum \text{Res}_{\mu} B_{12}(\varphi_\mu) \sigma_{\mu} = 0. \]

As was done in section D, it can now be shown, by using current theory, that there exists at least one additive meromorphic function \( \Delta_\mu \) belonging to \( \mathcal{A} (\Gamma) \) and having Severi residue \( f \). First, \( 1/R_\mu \) can be considered as a 0-current in \( \mathcal{U}_\mu \), regardless of whether \( \mu \) is singular or not, for if \( \varphi \) is an arbitrary \( \varphi \)-form of class \( C^\infty \) with carrier contained in \( \mathcal{U}_\mu \), and \( \mu \) is singular, then the Cauchy principal value of the integral \( \int 1/R_\mu \wedge \varphi \) is defined by \( \lim_{\epsilon \to 0} \int_{|z| > \epsilon} 1/R_\mu \wedge \varphi \).

The fact that this limit exists and the current so defined is independent of the choice of the co-ordinates, has been shown. Then, putting \( \varphi_\mu = 1/4\pi (1 + iG)d(1/R_\mu) \), it follows as before that \( \varphi_\mu \) is a pure 1-current of type 1, and if \( H_\mu \) is an arbitrary holomorphic function in \( \mathcal{U}_\mu \), then
\[ (2.2a) \quad (1 + iG)d(H_\mu/R_\mu) = 4\pi \wedge \varphi_\mu \wedge H_\mu. \]

An explicit expression for \( \varphi_\mu \) is given by the

Theorem: Let \( \varphi \) be an arbitrary \( 3 \)-form of class \( C^\infty \) with \( \mathcal{U}_\mu \). Then the current \( \varphi_\mu \) is equal to
\[ (2.3a) \quad \varphi_\mu = \sum \lim_{\epsilon \to 0} \int_{|z| > \epsilon} \Psi_{z,\mu} \wedge \varphi_{\mu}, \]
where \( \varphi_{\mu} \).

\[ (2.3a) \quad \varphi_\mu = \sum \lim_{\epsilon \to 0} \int_{|z| > \epsilon} \Psi_{z,\mu} \wedge \varphi_{\mu} \]

\[ \varphi_{\mu}(p) = \sum_{\nu}(p) \]
Consider now the one current defined by

\[(2.4_8) \quad \mathcal{Q}_f = \int \frac{\varphi}{|\varphi|^3} \, dh_p \sigma_p d \varphi^3_p,\]

where \(\varphi\) is a variable 3-form of class \(C^\infty\). Since \(\mathcal{O}_q = \left[3(z_q^1, z_q^2, z_q^3)/z(z_p^1, z_p^2, z_p^3)\right] \cdot \sigma_p\) in

\[U_p \cap U_q,\]

the integrand in \((2.4_8)\) is a well defined 2-form on \([\Gamma]\). From its expression, \(\mathcal{Q}_f\) is clearly a pure 1-current of type 1, carried on \(\Gamma\). Further \(d \mathcal{Q}_f\) is a pure 2-current of type 1. By virtue of Theorem 1.9, \(\Theta_f = (d - \omega^G + i \varphi^G)d \mathcal{Q}_f\) is a holomorphic simple differential on \(\mathcal{M}_S - \Gamma\) and the integral of \(\Theta_f\) over an arbitrary 1-cycle \(Y \subset \mathcal{M}_S - \Gamma\) is given by \(\int Y \Theta_f = i \mathcal{Q}_f[\lambda Y]\). Therefore, \(A_f = 2\pi i \int \Theta_f\) is an additive holomorphic function on \(\mathcal{M}_S - \Gamma\). For \(\nu\) a simple point, it can be shown, in exactly the same manner as was done in section D, that \(R_{\nu} A_f / h_{\nu} = f\) on \(\Gamma\). If \(\nu\) is singular, then \(R_{\nu} A_f\) is holomorphic in \(U_{\nu} - \nu\) and from a theorem of Hartogs, \(R_{\nu} A_f\) must be holomorphic in all \(U_{\nu}\). Thus, \(A_f \in \mathcal{Q}(\Gamma)\) and has Severi residue \(f\). We can state again the conclusions arrived at in section D, i.e., there exists at least \(q\) independent Picard integrals of the second kind which are multiples of \(-\Gamma\) and have Severi residue \(f\). Kodaira has shown that there are at most \(2q\) of them.

\[\text{Ibid., p. 839.}\]
\[\text{Ibid., p. 842.}\]
\[\text{K. Kodaira, "The Theorem of Riemann-Roch on Compact Analytic Surfaces," p. 846.}\]
Finally, to establish the existence of a Picard integral of the third kind having \( \Gamma' \) as its only log curve, we consider again the expression \( T_\mu = (\log R_\mu - \log R_{\mu_2}) \). Since it has been remarked above that \( 1/R_\mu \) can be considered as a current, regardless of whether \( \mu \) is singular or not, then both \( 1/R_\mu \) and \( 1/R_{\mu_2} \) can be considered as \( \Omega \)-currents, defined respectively in neighborhoods \( \mathcal{U}_\mu \) and \( \mathcal{U}_{\mu_2} \) on \( \Gamma' \) and \( \Gamma_1 \).

Thus, \( dR_\mu/R_\mu \) and \( dR_{\mu_2}/R_{\mu_2} \) are \( 1 \)-currents, defined in \( \mathcal{U}_\mu \), and \( \mathcal{U}_{\mu_2} \), each being the product of a \( \Omega \)-current and a holomorphic \( 1 \)-form. Therefore, \( T_\mu \) is a \( 1 \)-current defined in a neighborhood of \( \Gamma' \) and, as can be directly verified, \( CT_\mu[\psi] = i T_\mu[\psi] \). It remains to show that, as a current, \( d T_\mu[\psi] = i T'_\mu[\psi] \), where \( \psi \) is an arbitrary \( 2 \)-form of class \( C^\infty \), carried in \( \mathcal{U}_\mu \), \( \mu \) being singular. It will be sufficient to show this for one component, say \( \Gamma_1 \), of \( \Gamma' \). Designate by \( R \) a minimal local equation of \( \Gamma_1 \) at \( \mu_1 \), where \( \mu_1 \) is any singular point of \( \Gamma_1 \).

Choose the system of local coordinates \( (z',z^2) \) with origin at \( \mu_1 \) and such that \( R(z',0) \neq 0 \). \( R_1(\mu_1) \) consists of a finite number of points \( \mu_1, \ldots, \mu_4 \), corresponding to the irreducible branches \( \Gamma_1', \ldots, \Gamma_2 \) of \( \Gamma_1 \) passing through \( \mu_1 \). Accordingly, \( R \) is decomposed into the product \( R = \prod R_i \) of a unit \( \mathcal{M} \) and the irreducible distinguished polynomials \( R_1, \ldots, R_4 \) in \( z' \), each representing a branch \( \Gamma_i^2 \). For the present purposes, \( \mathcal{M} \) may be taken as 1. Thus, each \( R_\mu \) can be written

\[ R_\mu = \prod R_i \]

\[ S. \, Bohr and \ W.\, T. \, Martin, \textit{op. cit.}, \textit{p. 57}. \]
as \( R_s = (z')^{m_s} + d_1s(z')(z')^{m_{s-1}} + \ldots + d_{m_s}s(z^2) \), where the \( B_i \)'s are holomorphic functions of \( z \) and \( B_y, s(0) = 0 \). Under these assumptions, \( \Gamma_i \) can be thought of as being decomposed into the union of \( m_s \) "sheets" each having parametric representation \( z' = \Theta_{j_s}(\tau_s), z^2 = \tau_s \), where \( \Theta_{j_s}(0) = 0 \), so that \( R \) may be written as \( R = \frac{1}{m_s} \sum_{j=1}^{m_s} \left[ z' - \Theta_{j_s}(\tau_s) \right] \).

Now, \( 2\pi i \oint_{\Gamma_i} [\psi] = \int d\log R \wedge d\psi \) and we examine the limit
\[
\lim_{\varepsilon \to 0} \int_{|z'| > \varepsilon} d\left\{ \left( \frac{dR}{R} \right) \wedge \psi \right\} = -\left[ \left( \frac{dR}{R} \right) \wedge \psi \right] \text{ since } d\left( \frac{dR}{R} \right) \text{ vanishes in } \mathcal{U}_{\Gamma_i}.
\]
Thus,
\[
\lim_{\varepsilon \to 0} \int_{|z'| > \varepsilon} d\left\{ \left( \frac{dR}{R} \right) \wedge \psi \right\} = -\lim_{\varepsilon \to 0} \int_{|z'| > \varepsilon} d\left\{ \left( \frac{dR}{R} \right) \wedge \psi \right\} = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \left\{ \int_{|z'| > \varepsilon} (\frac{dR}{R}) \wedge \psi \right\} = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \left\{ A + B \right\}.
\]

For integral \( A \), we have
\[
\int_{|z'| = \varepsilon} \left( \frac{dR}{R} \right) \wedge \psi = \int_{|z'| = \varepsilon} \left[ \frac{\partial \psi}{\partial z} \cdot \psi_{\overline{\alpha}} \right] dz' \wedge d\overline{z} + \int_{|z'| = \varepsilon} \left( \frac{dR}{R} \right) \wedge \psi_{\overline{\alpha}} dz' \wedge d\overline{z}
\]

since on \( |z'| = \varepsilon \), \( dz' \wedge d\overline{z} = 0 \). Due to a result obtained by Kodaira, \(^{33}\) it is possible to assert that each of the last two integrals have limit zero as \( \varepsilon \to 0 \), so that integral \( A \) vanishes. On the other hand, for integral \( B \) we have
\[
\lim_{\delta \to 0} \int_{|z'| > \varepsilon} \left( \frac{dR}{R} \right) \wedge \psi = 2\pi i \sum_{s} \sum_{j} \int_{|\tau_s| > \varepsilon_s} \psi'(\Theta_{j_s}, \tau_s) d \tau_s d \overline{\tau_s}, \text{ since for } |z'| > \varepsilon \text{ and for each } s \text{ it has been shown that } \lim_{\delta \to 0} \int_{|z'| > \varepsilon} \left( \frac{dR}{R} \right) \wedge \psi = \]

\[ 2\pi \sum_j \int \psi'(\xi_j, \xi_k) d\xi d\xi_k, \text{ where } \psi' \text{ indicates the form resulting from } \psi \text{ after the parametric substitution, so that } \]
\[ \lim_{s \to 0} \frac{1}{|s|} \int d\xi d\psi \text{ is a summation } 2\pi \sum_j \int \psi' d\xi d\xi_k. \]
\[ \text{Now, the limit } \lim_{\epsilon \to 0} \sum_j \psi' d\xi d\xi_k \text{ certainly exists since each coefficient of } \psi' \text{ is } C^\infty. \]
\[ \text{Thus, the limit } \]
\[ (2.5) \lim_{\epsilon \to 0} 2\pi \sum_j \int \psi' d\xi d\xi_k \]
\[ \text{exists and, in fact, represents the Cauchy principal value of the integral } 2\pi \sum_j \int \psi' d\xi d\xi_k. \]
\[ \text{We can therefore infer the existence of the limit } \]
\[ \lim_{\epsilon \to 0} \int d\log \xi \text{ and it is given by } (2.5). \]
\[ \text{Hence, in any case, we can set } dT_{\psi}(\psi) = 1 \Gamma'(\psi) = i \int \psi, \text{ where the integral is evaluated according to the limit } (2.5) \text{ at } \rho_1. \]

Consider now the current \( \Gamma'(\psi) = \int \psi, \psi \text{ being a variable } 2\text{-form of class } C^\infty \text{ and where the value of the integral is considered as Cauchy principal value in case it does not converge absolutely. Clearly, as defined, } \Gamma' \text{ is a } 2\text{-current and satisfies } C \Gamma' = \Gamma', \text{ this last following directly from a parametric substitution for } z' \text{ and } z^\alpha. \]
\[ \text{Then, proceeding exactly as in section } \| \text{ and employing Theorem 1.9, } \Theta = (d \wedge G + i\phi) \Gamma' \text{ is a holomorphic simple differential on } M_1 \text{ if } \Gamma'. \]
\[ \text{Therefore, } \Theta = \int \Theta' \text{ is an additive holomorphic function on } M_1 \text{ if } \Gamma'. \text{ For every simple point of } \Gamma', \Theta - (\psi' \Gamma' - \log \Gamma') \text{ is holomorphic in some neighborhood of the point and by } \]

\[ \text{G. deRham and K. Kodaira, op. cit., p. 106.} \]
Hartog's theorem, this conclusion holds for all points of \( \Gamma' \).

We can state: There exists at least \( q \) independent Picard integrals of the third kind having \( \Gamma' \) as a pure log curve providing \( \Gamma' \) is homologous to zero.
CHAPTER III

A. Normal Cycles.

We remark first that due to the assumptions made in Chapter II concerning the curve C and the divisors $\Gamma$ and $\Gamma'$, the results to be obtained in this Chapter will hold for either of the previously considered cases, i.e., for $\mathcal{M}_s$ of dimension $n$ or of dimension 2 with $\Gamma$ and $\Gamma'$ having singular points.

Let $C = \sum_k C_k$ be the analytic curve introduced in section C of Chapter II. Designate by $p_k$ the genus of each $C_k$ and set $p = \sum_k p_k$. Consider now the retiassections on each $C_k$. On $C_1$, label each pair which intersect at a point by $\ell_h$, $\ell_{p+h}$, where $h = 1, \ldots, p$; on $C_2$, similarly, where $h$ will go from $p_1 + 1, \ldots, p_1 + p_2$; and so on. On $C_h, h = p_1 + p_2 + \ldots + p_{k-1} + 1, \ldots, p_1 + p_2 + \ldots + p_k$. Let I and J be two arbitrary Picard integrals of the second kind on $\mathcal{M}_s$ having periods $\omega_1, \ldots, \omega_2^p$ and $\epsilon_1, \ldots, \epsilon_2^p$, respectively, on the retiassections. For example, on the pair $\ell_1, \ell_{p+1}$, I has periods $\omega_1, \omega_2^{p+1}$, and so forth. By employing Riemann's method of boundary integration, where the boundary of $C$ will be designated as $BC = \sum_k BC_k$, we have

\[(3.1) \quad \int_{BC} \text{Id}J = \sum_k \int_{BC_k} \text{Id}J = \sum_{j=1}^{p_k} (\omega_j \epsilon_{j+p} - \epsilon_j \omega_{j+p}) + \sum_{j=p_1}^{2^p} (\omega_j \epsilon_{j+p} - \epsilon_j \omega_{j+p}) + \ldots + \sum_{j=p_1}^{p_1+p_2} (\omega_j \epsilon_{j+p} - \epsilon_j \omega_{j+p}) = \]

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\[ \sum_{j=1}^{p} (\omega_j - \omega_{j+p}). \]

However, \( C \) contains 2q 1-cycles, \( \sigma_1, \ldots, \sigma_{2q} \), which are homologically independent on \( M_2 \), and if we designate the periods of \( I \) and \( J \) by \( \tau_1, \ldots, \tau_{2q-1} \) and \( \theta_1, \ldots, \theta_{2q} \), respectively, on \( \sigma_1, \ldots, \sigma_{2q} \), then we have

\[ (3.2) \]

\[
\begin{align*}
\omega_1 &= m_{j+1} \tau_1 + \ldots + m_{j+q} \tau_{j+q} \\
\omega_2 &= m_{j+1} \theta_1 + \ldots + m_{j+q} \theta_{j+q}
\end{align*}
\]

where \( j = 1, \ldots, 2p \) and where the \( m's \) are integers, not all zero. By substitution of (3.2) in (3.1), there results

\[ (3.3) \]

\[
\sum_{j=1}^{p} (\omega_j - \omega_{j+p}) = \left[ \sum_{j=1}^{p} (m_{j+1} m_{j+2p} - m_{j+1} m_{j+2p+1}) \right] \tau_1 \theta_1 + \ldots + \left[ \sum_{j=1}^{p} (m_{j+1} m_{j+2p} - m_{j+1} m_{j+2p+1}) \right] \tau_{j+q} \theta_{j+q}
\]

The general term on the left side of (3.3) is

\[
\sum_{j=1}^{p} (m_{j+1} m_{j+2p}) \tau_1 \theta_1, \]

which shows that the matrix \( (k_{q,p}) \) is skew symmetric. By a method due to Frobenius, it is possible, by using a congruent, unimodular transformation

\[ (3.4) \]

\[
\begin{align*}
\tau_1 &= a_{11} \tau_1 + \ldots + a_{1q} \tau_q \\
\theta_1 &= a_{11} \theta_1 + \ldots + a_{1q} \theta_q
\end{align*}
\]

\[ \]
(where the $a$'s are integers and $h = 1, \ldots, 2q$), to put (3.3) in the form

$$
(3.5) \quad \sum_{a=1}^{\phi} \phi_a \left( \tau_a \theta_{q+a} - \tau_{q+a} \theta_a \right),
$$

where the $\phi$'s are the invariant factors of $(k_a)_a$ and $\phi_a > \phi_{a-1}$. The inverse of (3.4) is

$$
(3.6) \quad \tau_{h} = b_{h,1} \tau_1 + \ldots + b_{h,a_1} \tau_{a_1},
\quad \theta_{h} = b_{h,1} \theta_1 + \ldots + b_{h,a_1} \theta_{a_1},
$$

$h = 1, \ldots, 2q$. Now, consider the cycles defined as

$$
(3.7) \quad \nu_1 = b_{1,1} \sigma_1 + \ldots + b_{1,a_1} \sigma_{a_1},
\quad \nu_2 = b_{2,1} \sigma_1 + \ldots + b_{2,a_1} \sigma_{a_1},
\quad \ldots
\quad \nu_{a_1} = b_{a_1,1} \sigma_1 + \ldots + b_{a_1,a_1} \sigma_{a_1},
$$

the coefficients being the same as those in (3.5). The $\nu$'s are independent on $\mathcal{M}_s$ provided the determinant of the $b$'s does not vanish, but this is assured since there exists no relation between the periods $\tau_i$ nor the periods $\theta_i$.

**Definition:** The cycles $\nu$ will be called, after Severi, normal cycles on $\mathcal{M}_s$. 36

Two normal cycles, the subscripts of whose symbols differ by $q$, will be called associated. The cycles $\nu_1, \ldots, \nu_q$ will be termed normal cycles of the first group and the cycles $\nu_{q+1}, \ldots, \nu_{2q}$ will be called normal cycles of the second group. The normal period will be the period of a Picard integral along a normal cycle.

---

B. Normal Integrals

Let $F_1, \ldots, F_q$ be $q$ independent Picard integrals of the first kind on $\mathcal{M}_S$, and $\tau_1, \ldots, \tau_{h,q}$ be the periods of $F_h$ on the normal cycles. Consider the period matrix

$$
(\tau_{h,k}) = 
\begin{pmatrix}
\tau_{1,1} & \cdots & \tau_{1,q} \\
\vdots & \ddots & \vdots \\
\tau_{q,1} & \cdots & \tau_{q,q}
\end{pmatrix}
$$

The determinant

$$
T = \begin{vmatrix}
\tau_{1,1} & \cdots & \tau_{1,q} \\
\vdots & \ddots & \vdots \\
\tau_{q,1} & \cdots & \tau_{q,q}
\end{vmatrix} \neq 0,
$$

since otherwise the system

$$
\begin{align*}
c_{11}\tau_{1,1} + \cdots + c_{1q}\tau_{1,q} &= 0 \\
\vdots & \quad \vdots \\
c_{q,1}\tau_{q,1} + \cdots + c_{q,q}\tau_{q,q} &= 0,
\end{align*}
$$

would have a non-trivial solution but this is impossible due to the independence of the $\tau_{h,k}$ and the $F_1, \ldots, F_q$.

Let $\lambda_1 = \frac{T_{1h}}{T}, \lambda_2 = \frac{T_{2h}}{T}, \ldots, \lambda_q = \frac{T_{qh}}{T}$, where $T_{1h}, \ldots, T_{qh}$ are minors of the $h$th column, $h = 1, \ldots, q$, of $T$. Now set

$$
I_h = \left[ T_{1h} F_1 + \cdots + T_{qh} F_q \right] / T =
$$

$$
\begin{pmatrix}
\begin{vmatrix}
\tau_{1,1} & \cdots & \tau_{1,h-1} & \tau_{1,h+1} & \cdots & \tau_{1,q} \\
\tau_{2,1} & \cdots & \tau_{2,h-1} & \tau_{2,h+1} & \cdots & \tau_{2,q} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\tau_{q,1} & \cdots & \tau_{q,h-1} & \tau_{q,h+1} & \cdots & \tau_{q,q}
\end{vmatrix} & \begin{vmatrix}
\tau_{1,1} & \cdots & \tau_{1,h-1} & \tau_{1,h+1} & \cdots & \tau_{1,q} \\
\tau_{2,1} & \cdots & \tau_{2,h-1} & \tau_{2,h+1} & \cdots & \tau_{2,q} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\tau_{q,1} & \cdots & \tau_{q,h-1} & \tau_{q,h+1} & \cdots & \tau_{q,q}
\end{vmatrix} \\
\begin{vmatrix}
\tau_{1,1} & \cdots & \tau_{1,h-1} & \tau_{1,h+1} & \cdots & \tau_{1,q} \\
\tau_{2,1} & \cdots & \tau_{2,h-1} & \tau_{2,h+1} & \cdots & \tau_{2,q} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\tau_{q,1} & \cdots & \tau_{q,h-1} & \tau_{q,h+1} & \cdots & \tau_{q,q}
\end{vmatrix} & \begin{vmatrix}
\tau_{1,1} & \cdots & \tau_{1,h-1} & \tau_{1,h+1} & \cdots & \tau_{1,q} \\
\tau_{2,1} & \cdots & \tau_{2,h-1} & \tau_{2,h+1} & \cdots & \tau_{2,q} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\tau_{q,1} & \cdots & \tau_{q,h-1} & \tau_{q,h+1} & \cdots & \tau_{q,q}
\end{vmatrix}
\end{pmatrix}
$$

$$
\begin{pmatrix}
\tau_{1,1} & \cdots & \tau_{1,h-1} & \tau_{1,h+1} & \cdots & \tau_{1,q} \\
\tau_{2,1} & \cdots & \tau_{2,h-1} & \tau_{2,h+1} & \cdots & \tau_{2,q} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\tau_{q,1} & \cdots & \tau_{q,h-1} & \tau_{q,h+1} & \cdots & \tau_{q,q}
\end{pmatrix} \quad \begin{pmatrix}
\tau_{1,1} & \cdots & \tau_{1,h-1} & \tau_{1,h+1} & \cdots & \tau_{1,q} \\
\tau_{2,1} & \cdots & \tau_{2,h-1} & \tau_{2,h+1} & \cdots & \tau_{2,q} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
\tau_{q,1} & \cdots & \tau_{q,h-1} & \tau_{q,h+1} & \cdots & \tau_{q,q}
\end{pmatrix}
$$
On $\nu_h$, the period of $P_1$ is $\tau_{1h}$, the period of $P_2$ is $\tau_{2h}$, and so forth. Thus the period of $I_h$ on $\nu_h$ is

$$\mathcal{O}(I_h) = \frac{T_{1h} \tau_{1h} + T_{2h} \tau_{2h} + \cdots + (-1)^{h+q} T_{qh} \tau_{qh}}{P} = 1.$$ 

On $\nu_k (k=1, \ldots, h-1, h+1, \ldots, q)$, the period of $P_1$ is $\tau_{1k}$; the period of $P_2$ is $\tau_{2k}$, and so forth. Thus, the period of $I_h$ on $\nu_k$ is

$$\mathcal{O}(I_h) = \frac{T_{1h} \tau_{1h} + T_{2h} \tau_{2h} + \cdots + (-1)^{h+q} T_{qh} \tau_{qh}}{P} = 0,$$

since the numerator can be written as

$$\begin{vmatrix}
\tau_{1k} \tau_{1i} \cdots \tau_{i,h-1} \tau_{i,h+1} \cdots \tau_{i,q} \\
\tau_{2k} \tau_{2i} \cdots \tau_{i,h-1} \tau_{i,h+1} \cdots \tau_{i,q} \\
\vdots & & \ddots & & \vdots \\
\tau_{qk} \tau_{qi} \cdots \tau_{i,h-1} \tau_{i,h+1} \cdots \tau_{i,q}
\end{vmatrix}$$

and if $k \neq h$, two columns of this determinant are equal, for every $k$. The period table of the integrals $I_h$ is as follows:

$$
\begin{array}{cccccccc}
& I_1 & I_2 & \cdots & I_q & \cdots & I_{q-1} & I_q \\
\tau_1 & 0 & \cdots & 0 & \omega_1 & \cdots & \omega_{q-1} & \omega_q \\
\tau_2 & 0 & \cdots & 0 & \omega_2 & \cdots & \omega_{q-1} & \omega_q \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\tau_q & 0 & \cdots & 0 & \omega_q & \cdots & \omega_{q-1} & \omega_q \\
\end{array}
$$

**Definition:** The integrals $I_h$ will be called normal Picard integrals of the first kind.
Turning to Picard integrals of the second kind, it has been shown that there exists at least \( q \) independent such integrals which are multiples of \(-f'\). Let \( L \) be any one of these integrals and call \( \theta_1, \ldots, \theta_2^q \) its periods on the normal cycles. Then, the integral

\[
J = L - \theta_1 I_1 - \theta_2 I_2 - \cdots - \theta_q I_q,
\]

where \( I_1, \ldots, I_q \) are normal integrals of the first kind, will have all its periods zero along \( \nu_1, \ldots, \nu_q \). Not all the periods will be zero for otherwise \( L = \theta I + \cdots + \theta_q I_q + \text{meromorphic function} \).

**Definition:** The integral \( J \) will be called a normal integral of the second kind relative to the divisor \( f' \).

Finally, as has been shown, it is possible to construct a Picard integral of the third kind having log singularities at the points of the carrier of \( f' = f_1 - f_2 \). Designate the integral by \( S \) and suppose \( S \) has normal periods \( \alpha_1, \ldots, \alpha_q \). The integral

\[
K = S - \alpha_1 I_1 - \alpha_2 I_2 - \cdots - \alpha_q I_q,
\]

where the \( I \)'s are as above, has periods zero on \( \nu_1, \ldots, \nu_q \). Again, not all the periods will be zero, since then \( S \) would reduce to a log function on \( \nu \setminus \nu_1 \).

**Definition:** The integral \( K \) will be called a normal integral of the third kind relative to \( f' \).

We remark here that in case any of the cycles contains intersection points of \( C \) with \( f' \), or \( f' \), they may be slightly deformed without affecting the value of the integral.
C. **Period Relations for a Picard Integral of the 2nd Kind.**

Let \( T_\alpha \) be the period of the normal integral of the second kind, \( J \), on the cycle \( \psi_{q+\alpha} \) and consider the integral \( \int_{BC} JdI_\alpha \), where \( I_\alpha \) is the normal integral of the first kind which has period 1 on the cycle \( \psi_\alpha \) associated with \( \psi_{q+\alpha} \). From (3.1), (3.3), and (3.5), we have

\[
(3.8) \quad \int_{BC} JdI_\alpha = -\phi_\alpha T_\alpha,
\]

where \( \phi_\alpha \) is the invariant factor defined on page 58 in expression (3.5). On the other hand,

\[
\int_{BC} JdI_\alpha = 2\pi \sum \text{residues of } JdI_\alpha \text{ at the points of } C.
\]

However, the only points of \( C \) at which the integrand has a non-vanishing residue is at the points of intersection \( p_1, \ldots, p_m \) of \( \Gamma \) with \( C \). Let \( h_{p_j} \) be a local uniformizer on \( C \) with center at \( p_j \). In a sufficiently small neighborhood, \( \mathcal{U}_{p_j} \), of \( p_j \), \( J = \frac{A(h_{p_j})}{R_{p_j}\left(h_{p_j}\right)} + \text{regular part} \), where \( R_{p_j} \) is the minimal local equation of \( \Gamma \) at \( p_j \). Since \( p_j \) is simple, we can set \( R_{p_j} = h_{p_j} \) in \( \mathcal{U}_{p_j} \) and \( J \) may be written as \( J = A/h_{p_j} + \text{regular part in } \mathcal{U}_{p_j} \). Then,

\[
\lim_{h_{p_j} \to 0} h_{p_j}JdI_\alpha = \lim_{h_{p_j} \to 0} \left( \frac{A}{h_{p_j}R_{p_j}} \frac{dI_\alpha}{dR_{p_j}} + \frac{dI_\alpha}{dh_{p_j}}(\text{reg. part}) \right).
\]

The second sum on the right is zero in the limit, so that, recalling the Severi residue function, introduced on page 39, it is clear that \( J' = \frac{f h_{p_j}}{R_{p_j}} \) at \( p_j \) and thus the

\[
\lim_{h_{p_j} \to 0} h_{p_j}JdI_\alpha = \left[ f(h_{p_j})h_{p_j}(h_{p_j}) \frac{dI_\alpha}{dR_{p_j}} \right]_{h_{p_j} \to 0}.
\]
Therefore, for $j = 1, \ldots, n$,

\[
(3.9) \quad \int_{BC} j dI_\alpha = 2\pi \sum_j \left[ f(I_{\beta_j}) h_{\gamma_j}(I_{\beta_j}) dI_\alpha / dI_{\beta_j} \right] \bigg|_{I_{\beta_j}=0}.
\]

By combining (3.8) and (3.9), we arrive at the relation:

\[
(3.10) \quad \Pi_\alpha = \frac{-24\pi}{\Phi_\omega} \sum_j \left[ f h_{\beta_j} dI_\alpha \right] \bigg|_{I_{\beta_j}=0},
\]

and since the left side of this expression is a constant, evidently the right side is independent of the uniformizer employed. (3.10) shows that the periods of a normal Picard integral of the second kind can be expressed in terms of simple differentials of the first kind and a residue function. From the expression on line 6, page 61, we see that the periods of any Picard integral of the second kind are expressible in terms of the periods of normal integrals of the first and second kinds.

D. Period Relations for a Picard Integral of the $3^{rd}$ Kind.

Let $K$ be a normal integral of the third kind having log singularities on the carrier of $\Gamma'$. Now, enclose each point of intersection $q_1, \ldots, q_m, q'_1, \ldots, q'_m$, of $C$ with $\Gamma'$ in an arbitrarily small $\gamma$-circle, where $\gamma$ is a local uniformizer at the intersection points. Here, we do not imply that the same $\gamma$ satisfies for all points. Join each circle to the boundary of $C$ by a path which is at least rectifiable, and denote the new contours formed by $s_1$ and $s'_1$ and their sum by $s$. Under a canonical decomposition, the entire boundary of $C$ is now $BC + s$. Let $\gamma_\alpha$ be the period of $K$ on $\gamma_q \gamma_\alpha$ and let $I_\alpha$ be the normal integral of the first kind having period 1 on $\gamma_\alpha$. Then,
Consider first \[ \int_{BC} I_{\alpha} dK. \] Since \( K \) is of the third kind, we must take into account both the cyclic and logarithmic periods of the integral. Thus, the expression (3.5), which was used to obtain (3.8), will not suffice in this case. However, the following may readily be shown: Let \( I \) be a Picard integral of the first kind, having normal periods \( \gamma_1, \ldots, \gamma_{q_1} \), and let \( S \) be a Picard integral of the third kind, having normal periods \( \theta_1, \ldots, \theta_{q_1} \). Then,

\[
(3.12) \quad \int_{BC} IdS = \sum_{\alpha=1}^{q} \left[ \phi_{\alpha} (\gamma_{q_1+\alpha} = \gamma_{q_1+\alpha}\theta_{\alpha}) + (r_{\alpha} \gamma_1 + s_{\alpha} \gamma_{q_1+\alpha}) \right],
\]

where the \( r_1 \) and \( s_1 \) are integers. \(^{37}\) Now, (3.12), applied to \( I_{\alpha} \) and \( K \), gives

\[
(3.13) \quad \int_{BC} I_{\alpha} dK = \phi_{\alpha} \gamma_{q_1+r_{\alpha}} + s_{\alpha} \omega_{\alpha} + \ldots + s_{q_1} \omega_{q_1},
\]

where \( \omega_{\alpha}, \ldots, \omega_{q_1} \) are the periods of \( I_{\alpha} \) on the cycle \( \gamma_{q_1+\alpha} \).

In order to evaluate the \( \int_{S} I_{\alpha} dK \), we proceed as follows: Suppose \( S_1 \) is the contour about \( \gamma_1 \) and let \( S_{q_1} \) be the uniformizer with center at \( \gamma_1 \). Then, in a sufficiently small neighborhood of \( \gamma_1 \), \( K = \log R_{\gamma_1} + \) regular part, where \( R_{\gamma_1} \) is the minimal local equation of \( \gamma_1 \) at \( \gamma_1 \), and since \( \gamma_1 \) is

\(^{37}\) Ibid., p. 278.
simple, we can write for $K$ the equality $K = \log J_{q_i} + $ regular part throughout the neighborhood. Therefore, by Cauchy’s integral formula, $\int_{s_1} I_\alpha \, dK = 21\pi \left[ I_\alpha (J_{q_i}) \right]_{q_i = 0}$. Likewise, if $s'_1$ is the contour about $q'_i$, then $\int_{s'_1} I_\alpha \, dK = -21\pi \left[ I_\alpha (J_{q'_i}) \right]$, evaluated at $J_{q'_i} = 0$. Thus,

$$\int_s I_\alpha \, dK = 21\pi \sum_{i=1}^{m} \left[ I_\alpha (J_{q_i}) - I_\alpha (J_{q'_i}) \right]$$

evaluated at $J_{q_i} = 0$, $J_{q'_i} = 0$. Now, using (3.11), (3.13), and (3.14),

$$J_\alpha = \frac{21\pi}{\phi_\alpha} \left\{ \sum_{i} \left[ I_\alpha (J_{q_i}) - I_\alpha (J_{q'_i}) \right] \right\} - \frac{1}{\phi_\alpha} \left[ r_\alpha + \sum_{k=1}^{2} s_k \omega_\alpha k \right].$$

The relation (3.15) expresses the periods of a normal integral of the third kind in terms of normal integrals of the first kind and their periods. From the expression for $K$ on line 18, page 61, it is clear that the periods of any Picard integral of the third kind can be expressed as a linear combination of those of normal integrals of the first and third kinds.
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