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A Necessary and Sufficient Condition That a Set Be Homeomorphic to a Plane Region Bounded by a Finite Number of Nonintersecting Circles.

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A NECESSARY AND SUFFICIENT CONDITION THAT A SET BE HOMEOMORPHIC TO A PLANE REGION BOUNDED BY A FINITE NUMBER OF NONINTERSECTING CIRCLES

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ABSTRACT

A number of papers have been written concerning the characterization of sets homeomorphic with a subset (proper or improper) of the plane or sphere. Among these are R. L. Moore, who gives several characterizations of the plane, Leo Zippin, R. L. Wilder, D. W. Hall, G. S. Young and R. H. Bing.

The main result of this paper is the following theorem: A necessary and sufficient condition that a space S be homeomorphic with a closed, compact, connected subset of the plane bounded by a finite number, n, of nonintersecting circles, where $n \geq 1$, is that S be a nondegenerate, compact, continuous curve containing a collection π of n nonintersecting simple closed curves, such that S is not separated by any pair of points or by any element of π, but S is separated by any simple closed curve which is not an element of π. The necessity of this condition being obvious, it is only necessary to prove the sufficiency of the condition.

The proof of the sufficiency is divided into four parts. In the first part is proved the theorem:

Let M be a locally compact, connected, metric space which can be covered by a finite number of connected open sets of diameter less than e when e is positive. Suppose further that M cannot be separated by the omission of any pair of points.

Let C be a compact subset of M and let x and y be common limit points of two components D_A and D_B of M-C such that:

(1) C is locally connected at x and at y;

(2) There is a positive δ_1 such that $C-(C_{x,\delta} + C_{y,\delta})$ has a finite number of components when $\delta \leq \delta_1$ ($C_{x,\delta}$ is the component of $C \cdot U(x,\delta)$ which contains x).

Then given a positive number e_o, there are:

(1) a δ_o less than or equal to δ_1;

(2) connected open supersets $U_{e_o}(C_{x,\delta_o})$ and $U_{e_o}(C_{y,\delta_o})$ of C_{x,δ_o} and C_{y,δ_o} which do not intersect C_1' where $C_1' = C-(C_{x,\delta_o} + C_{y,\delta_o})$; and

(3) a simple closed curve J in $D_A + D_B + U_{e_o} C_{x,\delta_O} + U_{e_o} C_{y,\delta_o}$ such that J intersects C_{x,δ_o},

C_{y,δ_0}, D_A and D_B and J does not separate any point of $M-J$ from C_1'.

The proof of this theorem follows closely Bing's proof of the Kline sphere characterization problem.

In the second section it is shown that if C_1, C_2, $\cdots$, C_n are the elements of π, then $S-\sum_1^n C_i$ is homeomorphic to a subset of the plane. This is accomplished by showing that $S-\sum_1^n C_i$ is a Peano space which contains at least one simple closed curve and that every simple closed curve, but no arc separates $S-\sum_1^n C_i$.

In the third section it is shown that S does not contain any primitive skew curve of the first or second types and hence, by a theorem of Claytor, S is homeomorphic to a subset of the plane. In the last section it is shown that S is homeomorphic to a region of the plane bounded by n nonintersecting circles.

INTRODUCTION

A number of papers have been written concerning the characterization of sets homeomorphic with a subset (proper or improper) of the plane or sphere.

In [11][1] R. L. Moore gives three systems of axioms for plane topology and in [12] he proves that the spaces determined by his systems are really homeomorphic with the plane. In [6] Miss Gawehn proves that certain conditions will define a 2-dimensional manifold without boundary among arbitrary Housdorff spaces. In [16] Leo Zippin shows that in locally compact, locally connected, connected spaces satisfying the Janiszewski theorm, the nondegenerate cyclic elements are homeomorphic with a 2-sphere (or a region of a 2-sphere). In [10], E. R. van Kampen shows that a P-space which contains at least one simple closed curve and which is separated by every simple closed curve but by no closed

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[1] See appendix for footnotes.

are is homeomorphic with a region on a sphere. In [5] S. Claytor shows that a necessary and sufficient condition that a P-continuum K be homeomorphic with a subset of a spherical surface is that:

(1) K does not contain any primitive skew curve of type one or type two;

(2) each cut point P of K is a boundary point of the closure of every component of K-P.

In connection with this theorem, Hall in [9] shows that if M is a locally connected continuum which is separated by no pair of its points and contains no primitive skew curve of type one, then M contains no primitive skew curve of type two. Further characterizations of the sphere and regions of the sphere are given by Wilder [13, 14] and Zippin [17, 18].

In [8, 9] Hall gives a partial solution to the problem of J.R. Klein: Is a nondegenerate, locally connected, compact continuum which is separated by each of its simple closed curves but by no pair of its points homeomorphic with the surface of a sphere? In [4] Bing gives a complete solution to this problem. In [15] Young gives a simple characterization of 2-manifolds, with or without boundaries, using Bing's solution of the

Kline problem as a tool. For further bibliographical refrences to other work of this nature see [4, 10].

The purpose of this paper is to prove the following theorem: A necessary and sufficient condition that a space S be homeomorphic with the closed, connected, compact subset of the plane bounded by a finite number, n, of nonintersecting circles, when $n \geq 1$, is that S be a nondegenerate, compact, continuous curve containing a collection π of n nonintersecting simple closed curves, such that S is not separated by any pair of points or by any element of π, but S is separated by any simple curve which is not an element of π.

It is easily seen that this condition is necessary. The problem has been handled nicely by Bing in [4] when π is empty. In the case that π is not empty, it is easily seen that S cannot be homeomorphic to the entire 2-sphere, and hence if it is homeomorphic to a subset of the sphere, it is homeomorphic to a bounded subset of the plane.

The proof of the sufficiency of this condition is given in four sections. In the first section is given a generalization of an argument used by Bing [4]. This generalization provides a valuable tool for dealing with certain sets which are separated by all but a finite number of simple closed curves, but which are not separated by any pair of points. In the second section it is shown that if $C_1, C_2, \ldots, C_n$ are the elements of π, then $S-\sum_1^n C_i$ is homeomorphic to a subset of the plane. In the third section, it is shown that S is homeomorphic to a subset of the plane and in the fourth section, it is shown that S is homeomorphic to a region, i. e. a closed, connected, compact subset, of the plane bounded by n nonintersecting circles.

In the arguments of the first section, the word disrupt is used several times. It shall be said that the point set M disrupts x from y in D if there is an arc from x to y in D but each such arc contains a point of M. Also, extensive reference is made to Bing's lemma which is given below.

Bing's lemma: Suppose that space is locally connected and cannot be separated by the omission of any pair of its points, that the boundary of the connected domain D is equal to the sum of the mutually exclusive sets M, N and E, each of which is accessible form D, and that D' is a connected subdomain of D such that no point of D either disrupts D' from E+M in D+E+M or disrupts D' from E+N in D+E+N. Then there is an open arc from M to N in D that does not disrupt D' from E in D+E.

The following notations are used throughout the paper:

S is a set satisfying the hypothesis of the problem;

π is a collection of n nonintersecting simple closed curves such that no element of π separates S;

C_i $(1 \leq i \leq n)$ is an element of π;

x, y and z are always points of S;

$U(x)$ is any open subset of S which contains x;

$U(x,\delta)$ is the set of all points of S whose distance from x is less than δ;

$U_{x,\delta}$ is the component of $U(x,\delta)$ which contains x;

$C_{x,\delta}$ (where C is a subset of S) is the component of $C \cdot U(x,\delta)$ which contains x;

C_i' is the set $C-(C_{x,\delta_0}+C_{y,\delta_0})$ where x and y are two points of C;

$U_e C_{x,\delta_0}$ is an open set containing C_{x,δ_0} which is of diameter less than e but which does not intersect C_i';

xy is an arc from x to y including the endpoints x and y;

$\langle xy \rangle$ is the open arc $xy-(x+y)$.

All other notations and theorems used are conventional.

CHAPTER I

§1. **Lemma:** Let M be a locally connected metric space, and let L be a closed subset of M. Let D be a connected open subset of M containing a single component L_1 of L, such that $\overline{D}$ is compact. If M_1 is a component of M-L then $M_1 \cdot D$ has a finite number of components.

Proof: If $M_1 \subseteq D$ then the conclusion is obvious. Suppose then, that M_1 is not a subset of D.

The proof will consist of four steps.

(1) To show that every component of $M_1 \cdot D = \overline{M}_1 \cdot D - L_1$ is open.

(2) To show that every component of $M_1 \cdot D$ has a limit point on L_1.

(3) To show that every component of $M_1 \cdot D$ has a limit point on $M_1 - D$.

(4) To show that, in the light of (2) and (3), the assumption that $M_1 \cdot D$ has an infinite number of components contridicts (1).

Let $D^*=D\cdot\overline{M}_1$.

(1) Let x be any point of a component D_{b_1} of D^*-L. Now D^*-L is an open subset of a locally connected metric space and thus it is locally connected. Therefore there is a neighborhood U of x which is a subset of D_{b_1}, and D_{b_1} is open.

(2) Suppose $\overline{D}_{b_1}\cdot L_1 = \phi$. Set $D=D_{b_1}+\left[(D-D^*)+(D^*-D_{b_1}-L_1)+L_1\right]$. Consider $K=\overline{D}_{b_1}\cdot\left[(D-D^*)+(D^*-D_{b_1}-L_1)+L_1\right.$ $\left.+D_{b_1}\cdot\left[\overline{(D-D^*)}+\overline{(D^*-D_{b_1}-L_1)}+\overline{L}_1\right]\right.$. Now $\overline{D}_{b_1}\cdot L_1+D_{b_1}\cdot\overline{L}_1 = \phi$. Also, $D_{b_1}\subseteq\overline{M}_1$, $D-D^*\subseteq M-\overline{M}_1$, $\overline{D-D^*}\subseteq D-D^*+L_1+\overline{D}\cdot\overline{M-D}$. Therefore, $\overline{D}_{b_1}\subseteq\overline{M}_1$, and $\overline{D}_{b_1}\cdot\left[D-D^*\right]+D_{b_1}\cdot\left[\overline{D-D^*}\right]= \phi$. Now D_{b_1} is a component of D^*-L_1 and hence $\overline{D}_{b_1}\cdot\left[D^*-L_1-D_{b_1}\right]=\phi$. Also, from part 1, no point of D_{b_1} is a limit point of $M-D_{b_1}$. Therefore, $D_{b_1}\cdot\left[\overline{D^*-L_1-D_{b_1}}\right]= \phi$. Thus $K=\phi$ and D has a partition. This is a contradiction.

(3) Suppose $\overline{D}_{b_1}\cdot(M_1-D) = \phi$. Then $M_1=(M_1-D^*)+(D^*-L-D_{b_1})+D_{b_1}$. Consider $K=\overline{D}_{b_1}\cdot\left[(M_1-D^*)+(D^*-L-D_{b_1})\right]+D_{b_1}\cdot\left[\overline{(M_1-D^*)}+\overline{(D^*-L-D_{b_1})}\right]$. As above $\overline{D}_{b_1}\cdot(D^*-L-D_{b_1})+D_{b_1}\cdot(\overline{D^*-D_{b_1}-L}) = \phi$. By supposition $\overline{D}_{b_1}\cdot(M_1-D^*)= \overline{D}_{b_1}\cdot(M_1-D\cdot\overline{M}_1)=\overline{D}_{b_1}\cdot(M_1-D)= \phi$. Now $D_{b_1}\subseteq D^*$, and hence, $M_1-D^*\subseteq M-D_{b_1}$. But no point of D_{b_1} is a limit point of $M-D_{b_1}$. Therefore $D_{b_1}\cdot(\overline{M_1-D^*})\subseteq D_{b_1}\cdot(\overline{M-D_{b_1}}) = \phi$. Therefore,

$K = \phi$ and M_1 has a partition. But M_1 is a component. This is a contradiction.

Now L_1 and $M_1 - D$ are closed disjoint sets and hence, $\rho(L_1, M_1 - D) = k > 0$. Since each component of $D^* - L_1$ has a limit point on L_1 and a limit point on $M_1 - D$, then each component has a diameter greater than or equal to k. Then each component has a point at a distance k/2 from L_1. Now suppose $D^* - L_1$ has an infinite number of components. Then there is an infinite set of points $\{x_n\}$ such that, (1) each x_n belongs to a different component of $D^* - L_1$ and (2) $\rho(L_1, x_n) = k/2$. Now each $x_n \in D$ and hence thereexists an x such that x is a limit point of $\{x_n\}$. Now each x_n belongs to $\overline{M}_1$ so that x belongs to $\overline{M}_1$. Also, $\rho(L_1, x_n) = k/2$ for every x_n and hence x does not belong to L_1 and x does not belong to $M_1 - D$. Therefore, x belongs to D^* and hence to some component D_b of D^*. But by (1) there is a neighborhood U of x which is a subset of D_b. But U contains an infinite number of points of $\{x_n\}$ and hence D_b contains points of an infinite number of components. Therefore, $M_1 \cdot D = \overline{M}_1 \cdot (D - L_1)$ has only a finite number of components.

§2. Lemma: Let M be a locally connected, locally compact metric space which can be covered by a finite number of connected domains all of diameter less than e for every $e > 0$. If z is a point of M and M-z is connected, then M-z can be covered by a finite number of connected domains of diameter less than e, none of which contain z.

Proof: By theorem 3.9, page 106 of Wilder [3], M is locally connected. Since M is locally compact, for each x belonging to M, there is a positive δ_x such that, for δ less than δ_x, $\overline{U(x,\delta)\cdot M}$ is compact.

Let z be any point of M for which M-z is connected, and let e be any positive number. Let e_z be a positive number less than $\min(e/2, \delta_z/2)$. Then M can be covered by a finite number of connected domains of diameter less than e_z. Let $D_1, D_2, \ldots, D_j$ be those which do not contain z and let $D_{j+1}, D_{j+2}, \ldots, D_n$ be those which do. Set $D = D_{j+1} + D_{j+2} + \ldots + D_n$. Then D is a connected open set containing z such that (1) the diameter of D is less than e and (2) $\overline{D}$ is compact. Then by §1, $D\cdot(M-z)$ has a finite number of components. These with $D_1, D_2, \ldots, D_j$ give the required covering.

Definition: If $x \in C$ then $C_{x,\delta}$ shall be the component of $U(x,\delta)\cdot C$ which contains x.

§3. Theorem: Let M be a locally compact, connected metric space which can be covered by a finite number of connected domains of diameter less than e when $e > 0$. Suppose further that M cannot be separated by the omission of any pair of points.

Let C be a compact subset of M and let x and y be common limit points of two components D_A and D_B of M-C such that:

(1) C is locally connected at x and at y;

(2) There is a positive δ_1 such that $C-(C_{x,\delta}+C_{y,\delta})$ has a finite number of components when $\delta \leq \delta_1$.

Then given a positive number e_o, there are:

(1) a δ_o less than or equal to δ_1;

(2) connected open supersets $U_{e_o}(C_{x,\delta_o})$ and $U_{e_o}(C_{y,\delta_o})$ of C_{x,δ_o} and C_{y,δ_o} which do not intersect C_1' where $C_1'=C-(C_{x,\delta_o}+C_{y,\delta_o})$; and

(3) a simple closed curve J in $D_A+D_B+U_{e_o}C_{x,\delta_o}+U_{e_o}C_{y,\delta_o}$ such that (a) J intersects C_{x,δ_o}, C_{y,δ_o}, D_A and D_B; and (b) J does not separate any point of M-J from C_1'.

Proof;

Before giving the details, a brief outline of the proof will be given.

A finite collection H_1 of connected domains will be obtained such that their sum, H_1^*, does not separate any point of $M-H_1^*$ from C_1' in $M-H_1^*$, and such that the sum of any pair of nonintersecting elements separates H_1^*. Collections H_2, H_3,... will be defined which satisfy corresponding conditions and which are such that the closure of an element of H_{n+1} is a subset of H_n^*. The collections $H_1, H_2, \ldots$ will be described in such a way that the common part of their sums is a simple closed curve J which is a subset of $D_A+D_B+U_{e_o}C_{x,\delta_0}+U_{e_o}C_{y,\delta_0}$ that does not separate any point of $M-J$ from C_1' in M.

Now consider the details of the proof:

Description of collection H_1. Let $e_1 \leq e_o$ be a positive number less than one one-hundredth of the distance from x to y. A collection H_1 of connected domains will be described. The sum of the elements of H_1 will be denoted by H_1^*. The collection H_1 of connected domains $h_{1,1}, h_{1,2}, \ldots, h_{1,t}$ $(t > 100)$ will satisfy the following conditions:

(1) $h_{1,i}$ intersects $h_{1,j}$ $(1 \leq i \leq t,\ 1 \leq j \leq t)$ if and only if $i=j-1$ or $i=j$ or $i=j+1$ $(h_{1,t+1} = h_{1,1},\ h_{1,1-1} = h_{1,t})$.

(2) If $z \in M-H_1^*$ then H_1^* does not separate z from C_1' in $M-H_1^*$;

(3) some point of $M-H_1^*$ is accessible from $h_{1,i}$;

(4) the diameter of $h_{1,i}$ is less than e_1 ;

(5) no connected subset of H_1^* that intersects $h_{1,i}$ and $h_{1,i+2}$ $(1 \leqq i \leqq t)$ is of diameter less than $e_1/4$.

Denote by $D_1, D_2, \ldots, D_n$ the elements of a finite collection of connected domains covering $M-(x+y)$ such that the diameter of each is less than $e_1/300$.[2] Suppose that each of the domains $D_1, D_2, \ldots, D_j$ intersects the complement of D_A+D_B, each of the domains $D_{j+1}, D_{j+2}, \ldots, D_k$ is a subset of D_A and each of the domains $D_{k+1}, D_{k+2}, \ldots, D_n$ is a subset of D_B.

Let $\alpha_1, \alpha_2, \ldots, \alpha_j$ be a collection of arcs in the complement of D_A+D_B+x+y such that α_i $(i=1,\ldots,j)$ intersects D_i and $C-(x+y)$.[3] Also there are collections of arcs $\beta_{i,\eta}$ $(i=j+1,\ldots,n\ ;\ \eta=1,2,3)$ such that:

(1) $\beta_{i,\eta}$ intersects D_i and C;

(2) $\beta_{i,1} \subseteq \overline{D}_A-(x+y)(i=j+1,\ldots,k)$;

$\beta_{i,1} \subseteq \overline{D}_B-(x+y) \quad (i=k+1,\ldots,n)$;

(3) $\beta_{i,2} \subseteq \overline{D}_A-x \quad (i=j+1,\ldots,k)$;

$\beta_{i,2} \subseteq \overline{D}_B-x \quad (i=k+1,\ldots,n)$;

(4) $\beta_{i,3} \subseteq \overline{D}_A - y \quad (i=j+1,\ldots,k)$;

$\beta_{i,3} \subseteq \overline{D}_B - y \quad (i=k+1,\ldots,n)$;

(5) $\beta_{i,1}\cdot\beta_{i,2} = \beta_{i,1}\cdot\beta_{i,3} = \beta_{i,2}\cdot\beta_{i,3} = x_i + y_i$ [4]

(x_i and y_i are endpoints of $\beta_{i,\eta}$).

Since $\sum_1^j \alpha_i$, $\sum_{j+1}^n \beta_{i,1}$, $\sum_{j+1}^n \beta_{i,2}$, and $\sum_{j+1}^n \beta_{i,3}$ are all closed compact sets there are three positive numbers δ_2, δ_3 and δ_4 such that:

(1) $\left(U(x,\delta_2) + U(y,\delta_2)\right)\cdot\left(\sum_1^j \alpha_i + \sum_{j+1}^n \beta_{i,1}\right) = \phi$;

(2) $U(x,\delta_3)\cdot \sum_{j+1}^n \beta_{i,2} = \phi$;

(3) $U(y,\delta_4)\cdot \sum_{j+1}^n \beta_{i,3} = \phi$.

Now suppose $0 < \delta_0 < \min(\delta_1, \delta_2, \delta_3, \delta_4, e_1/600)$. Then

(1) $(C_{x,\delta_0} + C_{y,\delta_0})\cdot(\sum_1^j \alpha_i + \sum_{j+1}^n \beta_{i,1}) = \phi$;

(2) $C_{x,\delta_0}\cdot \sum_{j+1}^n \beta_{i,2} = \phi$;

(3) $C_{y,\delta_0}\cdot \sum_{j+1}^n \beta_{i,3} = \phi$;

(4) diameter of C_{x,δ_0} and of C_{y,δ_0} is less than $e_1/300$;

(5) $C_1' = C - (C_{x,\delta_0} + C_{y,\delta_0})$ has a finite number of components.

It will be noted that since C is locally connected at x and at y, there is a neighborhood $U(x,\lambda)$ of x and a neighborhood $U(y,\lambda)$ of y such that $U(x,\lambda)\cdot C$ is a subset of C_{x,δ_0}, and $U(y,\lambda)\cdot C$ is a subset of C_{y,δ_0}. But since, by theorem 2 page 89 in Moore [2], the accessible limit points of D_A and D_B are dense in $C\cdot\overline{D}_A$ and $C\cdot\overline{D}_B$ respectively, then $U(x,\lambda)\cdot C$ and $U(y,\lambda)\cdot C$ both contain accessible limit points of D_A and of D_B. Hence, C_{x,δ_0} contains accessible limit points of D_A and of D_B and C_{y,δ_0} contains accessible limit points of D_A and of D_B.

Owing to the existance of the arcs $\beta_{j+1,1}$, $\beta_{j+1,2}$ and $\beta_{j+1,3}$, no point of D_A disrupts D_{j+1} either from $\overline{D}_A(C_{x,\delta_0}+C_1')$ or from $\overline{D}_A(C_{y,\delta_0}+C_1')$. Considering D_A, D_{j+1}, $\overline{D}_A\cdot C_{x,\delta_0}$, $\overline{D}_A\cdot C_{y,\delta_0}$ and $\overline{D}_A\cdot C_1'$ as D, D', M, N, and E of Bing's lemma, it is seen that there is an arc α_{j+1} from C_1' to D_{j+1} in D_A+C_1' that does not disrupt C_{x,δ_0} from C_{y,δ_0} in $D_A+C_{x,\delta_0}+C_{y,\delta_0}$.[6] Let D' be a component of $D_A-\alpha_{j+1}$ that contains an open arc from C_{x,δ_0} to C_{y,δ_0}. If D_{j+2} is not a subset of D', let α_{j+2} be an arc in $D_A-D'+C_1'$ from a point of D_{j+2} to a point of C_1'.[5] If D_{j+2} is a subset of D', Bing's lemma can be applied to get an arc α_{j+2} from D_{j+2} to C_1' in D_A+C_1' such that $\alpha_{j+1}+\alpha_{j+2}$ does not disrupt C_{x,δ_0} from C_{y,δ_0} in $D_A+C_{x,\delta_0}+C_{y,\delta_0}$. The procedure is described in the following paragraph.

Let R be a point of D'. Since no point of D_A disrupts D_{j+2} from $C_1'+C_{x,\delta_0}$ in $D_A+C_1'+C_{x,\delta_0}$ there is an arc β from D_{j+2} to $C_1'+C_{x,\delta_0}$ in $D_A+C_1'+C_{x,\delta_0}-R$. A subarc of

β in $D'+C_1'+\alpha_{j+1}+C_{x,\delta_0}-R$ intersects D_{j+2} and $C_1'+C_{x,\delta_0}+\alpha_{j+1}$. Hence, R does not disrupt D_{j+2} from $C_1'+\alpha_{j+1}+C_{x,\delta_0}$ in $D'+\overline{D}'(C_1'+\alpha_{j+1}+C_{x,\delta_0})$. Similarly, R does not disrupt D_{j+2} from $C_1'+\alpha_{j+1}+C_{x,\delta_0}$ in $D'+\overline{D}'(C_1'+\alpha_{j+1}+C_{y,\delta_0})$. Applying the lemma, it is found that there is an arc from C_{x,δ_0} to C_{y,δ_0} in $D'+\overline{D}'(C_{x,\delta_0}+C_{y,\delta_0})$ which does not disrupt D_{j+2} from $C_1'+\alpha_{j+1}$ in $D'+\overline{D}'(C_1'+\alpha_{j+1})$. It follows that there is an arc α'_{j+2} from D_{j+2} to $C_1'+\alpha_{j+1}$ in $D'+\overline{D}'(C_1'+\alpha_{j+1})$ which does not disrupt C_{x,δ_0} from C_{y,δ_0} in $D'+\overline{D}'(C_{x,\delta_0}+C_{y,\delta_0})$. If α'_{j+2} intersects C_1', set $\alpha_{j+2}=\alpha'_{j+2}$. If $\alpha'_{j+2}\cdot C_1'=\Phi$, let α_{j+2} be the arc in $\alpha'_{j+2}+\alpha_{j+1}$ from D' to C_1'. In either case $\alpha_{j+1}+\alpha_{j+2}$ does not disrupt C_{x,δ_0} from C_{y,δ_0} in $D_A+C_{x,\delta_0}+C_{y,\delta_0}$.

Likewise there is an arc α_{j+3} from D_{j+3} to C_1' in D_A+C_1' such that $\alpha_{j+1}+\alpha_{j+2}+\alpha_{j+3}$ does not disrupt C_{x,δ_0} from C_{y,δ_0} in $D_A+C_{x,\delta_0}+C_{y,\delta_0}$. A continuation of this process provides arcs $\alpha_{j+1}, \alpha_{j+2},\dots,\alpha_n$ in $D_A+D_B+C_1'$ whose sum does not disrupt C_{x,δ_0} from C_{y,δ_0} in $D_A+C_{x,\delta_0}+C_{y,\delta_0}$ and does not disrupt C_{x,δ_0} from C_{y,δ_0} in $D_B+C_{x,\delta_0}+C_{y,\delta_0}$, and such that α_i $(i + j+1,\dots,n)$ intersects D_i and C_1'.

Let G' be the collection of all domains g such that g is a component of the common part of some domain of $D_1, D_2,\dots, D_n$ and the complement of $C+\alpha_1+\alpha_2+\dots+\alpha_n$.

If P is a point of D_1, there is an arc in D_1 from P to α_1. Hence, if g is an element of G', some point of $C+\alpha_1+\alpha_2+\dots+\alpha_n$ is accessible from g.

Let g_x' and g_y' be connected domains which cover C_{x,δ_0} and C_{y,δ_0}, respectively, such that $\rho(x'; C_{x,\delta_0}) < e_1/300$ and $\rho(y'; C_{y,\delta_0}) < e_1/300$ for every $x' \in g_x'$ and for every $y' \in g_y'$, but which contain no points of $C_1'+\alpha_1,+\dots+\alpha_n$.[7] Now let g_x be the union of g_x' and all members of G' which intersect g_x' and which do not have accessible limit points on $C_1'+\alpha_1+\dots+\alpha_n$ and let g_y be the union of g_y' and members of G' which intersect g_y' and which do not have accessible limit points of $C_1'+\alpha_1+\dots+\alpha_n$. Since every g' has an accessible limit point on $C+\alpha_1+\alpha_2+\dots+\alpha_n$, every g' added to g_x' and every g' added to g_y' must have accessible limit points on C_{x,δ_0} and on C_{y,δ_0}, respectively. Since every element of G' is of diameter less than $e_1/300$ and since C_{x,δ_0} and C_{y,δ_0} are of diameter less than $e_1/300$ and since $\rho(x'; C_{x,\delta_0})$ and $\rho(y'; C_{y,\delta_0})$ are less than $e_1/300$ for every $x' \in g_x'$ and for every $y' \in g_y'$, then the diameters of g_x and g_y are both less than $e_1/100$. Let G be the collection of all elements of G' which have accessible limit points on $C_1'+\alpha_1+\dots+\alpha_n$. If g is an element of G then some point of $C_1'+\alpha_1+\dots+\alpha_n$ is accessible from g and either $g \leq D_A$, $g \leq D_B$ or $g \leq M-(D_A+D_B)$.[8] There exists a finite collection G_A of domains of G such that this collection but no collection of fewer elements of G satisfies the condition that the sum of the elements of G_A is a connected subset of D_A and intersects both g_x and g_y.[9] Denote the elements of G_A by $g_2, g_3, \dots, g_{q-1}$ where g_x intersects g_2; g_i $(i=2,\dots,q-2)$ intersects g_{i+1} and g_{q-1} intersects g_y but g_i does not intersect g_j for $j \geq i+2$.[10] Similarly, there is a collection G_B of elements of G such that this collection, but no collection of fewer elements of G, satisfies the condition that the sum of the elements of G_B is a connected subset of D_B and intersects both g_x and g_y. Denote the elements of G_B by $g_{q+1}, g_{q+2}, \dots, g_r$ where g_y intersects g_{q+1}; g_i $(i=q+1,\dots,r-1)$ intersects g_{i+1}, and g_r intersects g_x but g_i does not intersect g_j for $j \geq i+2$. Denote g_x by g_1 and g_y by g_q.

Let E denote the set $g_1+g_2+\dots+g_r$ plus all points of $M-(g_1+g_2+\dots+g_r)$ that it separates from C_1' in $M-(\overline{C_{x,\delta_0}} + \overline{C_{y,\delta_0}})$. Each component of the common part of E and an element of G intersects an element of $g_1, g_2, \dots, g_r$.[11] However, it is to be noted that no such components intersects two g_i's that do not belong to a consecutive set of three domains of $g_1, g_2, \dots, g_r$.[12] Denote by g_i' the sum of

g_i and all such components that intersect g_i. It will be noted that g_i' is of diameter less than $e_1/33$.

If three is a factor of r, denote the sum of the first three elements of $g_1', g_2', \ldots, g_r'$ by h_1, the sum of the next three elements by h_2, ... , and the sum of the last three elements by h_s. If three is a factor of r-1, then $h_1, h_2, \ldots, h_s$ are defined as before except that h_s is the sum of the last four elements of $g_1', g_2', \ldots, g_r'$ instead of the last three. If three is a factor of r-2, each of h_{s-1} and h_s is the sum of four elements of $g_1', g_2', \ldots, g_r'$. Since each element of h_i contains an element g of G, then a point of $C_1' + \alpha_1 + \alpha_2 + \ldots + \alpha_n$ is accessible from h_i. Now h_i is of diameter less than $e_1/8$ and the collection $h_1, h_2, \ldots, h_s$ satisfies conditions analogus to conditions (1), (2) and (3) to be satisfied by $h_{1,1}, h_{1,2}, \ldots, h_{1,t}$.[13]

Let $h_{1,1}$ be the sum of $h_1, h_2, \ldots, h_n$ where some connected subset of $h_1 + h_2 + \ldots + h_s$ of diameter less than $e_1/4$ intersects h_1 and h_n but no such subset intersects both h_1 and h_{n+1}; let $h_{1,2}$ be the sum of $h_{n+1}, h_{n+2}, \ldots, h_m$ where some connected subset of diameter less than $e_1/4$ intersects h_{n+1} and h_m, but no such subset intersects both h_{n+1} and h_{m+1}, ... , and let $h_{1,t}$ be the sum of $h_{p+1}, h_{p+2}, \ldots, h_s$ where some connected subset of diameter less than $e_1/4$ intersects $h_{p+1}, h_{p+2}, \dots, h_s$. Then the collection $h_{1,1}, h_{1,2}, \dots, h_{1,t}$ satisfies conditions (1), (2), (3), (4) and (5).

<u>Description of collection H_2</u>. Choose a positive number e_2 less than one one-hundredth of the diameter of any connected set in H_1^* that intersects $h_{1,i}$ and $h_{1,i+2}$ ($i=1,\dots,t$; $h_{1,t+1}=h_{1,1}$; $h_{1,t+2}=h_{1,2}$). A collection H_2 of connected domains $h_{2,1}, h_{2,2}, h_{2,3}, \dots, h_{2,s}$ will be obtained such that:

(1) $h_{2,i}$ intersects $h_{2,j}$ if, and only if, $i=j-1$ or $i=j$ or $i=j+1$ ($h_{2,1-1}=h_{2,s}$; $h_{2,s+1}=h_{2,1}$);

(2) if $z \in M-H_2^*$ then H_2^* does not separate z from C_1' in M;

(3) some point of $M-H_2^*$ is accessible from $h_{2,i}$;

(4) the diameter of $h_{2,i}$ is less than e_2;

(5) no connected subset of H_2^* that intersects $h_{2,i}$ and $h_{2,i+2}$ is of diameter less than $e_2/4$;

(6) if $H(n;i,j)$ denotes $h_{n,i-100}+\dots+h_{n,i}+\dots+h_{n,j}+\dots,h_{n,j+100}$ where ($h_{n,t+r}=h_{n,r}$) and if h_{1,i_0} intersects h_{2,m_0} and if h_{1,j_0} intersects h_{2,n_0} then either $\overline{H(2;m_0,n_0)} \subseteq H(1;i_0,j_0)$ and $\overline{H(2;n_0,m_0)} \subseteq H(1;j_0,i_0)$ or else $\overline{H(2;m_0,n_0)} \subseteq H(1;j_0,i_0)$ and $\overline{H(2;n_0,m_0)}$

$\subseteq H(1;i_o,j_o)$.

Denote by L the component of the common part of $h_{1,2}+h_{1,3}+\dots+h_{1,9}$ and the complement of the closure of $h_{1,1}+h_{1,10}$ which contains $h_{1,5}+h_{1,6}$. It will be shown that if P is a point of $h_{1,5}+h_{1,6}$ and if R is a point of L-P then R does not disrupt P from $M-H_1^*$ in $M-H_1^*+L$. If R is not a point of $h_{1,5}+h_{1,6}$ the result is evident. Let PQ be an arc in M-R from P to a point Q of $M-H_1^*$. Let Q' be the first point of PQ in order from P to Q on M-L. If PQ' intersects $h_{1,3}$ then there is an arc from PQ'-Q' to $M-H_1^*$ in $M-H_1^*+h_{1,3}$ because a point of $M-H_1^*$ is accessible from $h_{1,3}$. Also, if PQ' intersects $h_{1,8}$, R does not disrupt P from $M-H_1^*$ in $M-H_1^*+L$. If PQ' intersects neither $h_{1,3}$ nor $h_{1,8}$ then Q' is a point of $M-H_1^*$. This demonstrates that R does not disrupt P from $M-H_1^*$ in $M-H_1^*+L$.

Let G_1 be a finite collection of connected domains of diameter less than $e_2/1200$ which cover M. Let G be the collection of all elements of G_1 which intersects $h_{1,5}+h_{1,6}$. No point of L disrupts an element of G from $M-H_1^*$ in $M-H_1^*+L$. Repeated appleoations of Bing's lemma give that there is a collection of arcs K in $M-H_1^*+L$ such that for each element $g \in G$ there is an element $\alpha \in K$ such that α intersects g and $M-H_1^*$ but such that $\sum_{\alpha \in K} \alpha + M - H_1^*$ does not disrupt $\overline{h_{1,1}}$ from $\overline{h_{1,10}}$ in $h_{1,1}+\dots+h_{1,10}$.[14] Let G' be the set of all domains g' such that g' is a component of (1) the common part of $h_{1,2}+\dots+h_{1,9}$, the complement of K^*, and an element of G, or (2) the common part of the complement of K^*+C and $h_{1,i}$ $(i=2,3,4,7,8,9)$.

There exists a finite collection G" of elements of G' such that the sum of the elements of G" is a connected domain intersecting $h_{1,1}$ and $h_{1,10}$ but the sum of no subcollection of G' having fewer elements than G" is a connected domain intersecting $h_{1,i}$ $(i=1,\dots,10)$.[15] Assume that g_1 of G" intersects $h_{1,1}$, g_i $(=1,\dots,r-1)$ intersects g_{i+1} and g_r intersects $h_{1,10}$.[16]

There exists a collection $g_1', g_2', \dots, g_r'$ of connected domains such that g_1' intersects $h_{1,1}$, g_i' intersects g_{i+1}' $(i=1,\dots,r-1)$, g_r' intersects $h_{1,10}$, and the closure of g_i' is a subset of g_i.[17]

Let E denote $h_{1,1}+g_1'+\dots+g_r'+h_{1,10}+\dots+h_{1,t}$ plus all points of H_1^* which it separates from $M-H_1^*$ in M. Each component of the common part of E and an element of G' intersects one of the domains $h_{1,1}, g_1', \dots, g_r', h_{1,10}$, but no such component intersects two of these domains that do not belong to a consecutive set of three of these domains.[18] Add such components to the ones of $h_{1,1}$, g_1', $h_{1,10}$ that they intersect to form the sets h_1, g_1'', h_{10}. It is noted that the diameters of each g_i'' not intersecting $h_{1,2}+h_{1,3}+h_{1,4}+h_{1,7}+h_{1,8}+h_{1,9}$ is less than $e_2/400$.

Consecutive elements of $g_1'', g_2'', \dots, g_r''$ may be combined by threes and fours in a manner previously described so as to get a collection $g_{1,1}, g_{1,2}, \dots, g_{1,u}$ such that the collection $h_1, g_{1,1}, g_{1,2}, \dots, g_{1,u}, h_{10}, h_{1,11}, \dots, h_{1,t}$ satisfies conditions analogus to conditions (1), (2) and (3) to be satisfied by $h_{2,1}, h_{2,2}, \dots, h_{2,s}$.[19] It is noted that the closure of each $g_{1,i}$ is a subset of $h_{1,1}, \dots, h_{1,10}$ when $i_0 < i < i_1$ (where g_{1,i_0} is the last element of $g_{1,1}, \dots, g_{1,u}$ which intersects $h_{1,2}$ and g_{1,i_1} is the first element of $g_{1,1}, \dots, g_{1,u}$ which follows g_{1,i_0} and intersects $h_{1,9}$).[20]

In a manner similar to that in which $h_{1,1}+\dots+h_{1,10}$ was replaced by $h_1+g_{1,1}+\dots+g_{1,u}+h_{10}$, replace $h_{1,11}+\dots+h_{1,20}$ by $h_{11}+g_{11,1}+\dots+g_{11,v}+h_{20}, \dots,$ and $h_{1,t-m}+\dots+h_{1,t}$ $(9 \leqq m \leqq 18)$ by $h_{t-m}+g_{t-m,1}+\dots+g_{t-m,w}+h_t$.

Let $g_{i,o}$ be the fourth element of $g_{i,1}, g_{i,2}, \dots g_{i,n}$ which follows all of those elements that intersect $h_{1,i+3}$. Note that $g_{i,0}$, the three domains immediately preceding $g_{i,0}$, and the three domains immediately following $g_{i,0}$, are each a subset of $h_{1,i+4}$ of diameter less than $e_2/100$.

In the manner described above, replace $g_{1,0}+\cdots+g_{1,u}+h_{10}+h_{11}+g_{11,1}+\cdots+g_{11,0}$ by $g'_{1,0}+h'_{2,2}+\cdots+h'_{2,r}+g'_{11,0}$; replace $g'_{11,0}+\cdots+g'_{21,0}$ by $h'_{2,r+1}+\cdots+h'_{2,v}+g'_{21,0}$; ... ; and replace $g'_{t-m,0}+\cdots+g'_{1,0}$ by $h'_{2,n}+\cdots+h'_{2,u}+h'_{2,1}$.[21] The closure of $h'_{2,1}+\cdots+h'_{2,r}$ is a subset of $h_{1,1}+\cdots+h_{1,20}$, ... , and the closure of $h'_{2,n}+\cdots+h'_{2,u}+h'_{2,1}$ is a subset of $h_{1,t-m}+\cdots+h_{1,10}$. Consecutive elements of $h'_{2,1}, h'_{2,2}, \ldots, h'_{2,u}$ may be combined in a manner previously described so as to form a collection H_2 of connected domains $h_{2,1}$, $h_{2,2}$, $\ldots, h_{2,s}$ satisfying conditions (1), (2), (3), (4), (5) and (6).[22]

<u>Description of simple closed curve J</u>. For each positive integer i greater than one, a collection H_i of connected domains $h_{i,1}$, $h_{i,2}, \ldots, h_{i,n_i}$ can be described satisfying conditions analogus to those satisfied by H_2, where e_i is a positive number less than one one-hundredth of the diameter of any connected set in H^*_{i-1} intersecting $h_{i-1,j}$ and $h_{i-1,j+2}$. It will be shown that the common part J of H^*_1, $H^*_2, \ldots$ is a simple closed curve in $D_A+D_B+U_{e_0}C_{x,\delta_0}+U_{e_0}C_{y,\delta_0}$ that does not separate any point of M-J from C'_1.

As the closure of H^*_{i+1} is a connected subset of

H_i^* (condition 6) and as each $h_{i,j}$ contains an element of $H_{i,1}$,[23] then J is a nondegenerate continuum. This continuum does not separate any point of $M-J$ from C_1' because no H_i^* separates any point of $M-H_i^*$ from C_1'. Also, since $J \leqq H_1^*$, and since $H_1^* \leqq D_A+D_B+g_x+g_y$ (where g_x and g_y are open sets of diameter less than $e_1/33$, containing C_{x,δ_o} and C_{y,δ_o}, respectively, but not intersecting C_1'), then $J \leqq D_A+D_B+U_{e_o}C_{x,\delta_o}+U_{e_o}C_{y,\delta_o}$. Let P and Q be any pair of points of J. Suppose that h_{i,p_i} and h_{i,Q_i} are elements of $h_{i,1},\ldots,h_{i,n_i}$ that contain P and Q, respectively. For convenience in notation, it will be assumed that it is $H(i;P_i,Q_i)$ that covers the closure of $H(i+1;\ P_{i+1},Q_{i+1})$ and that it is $H(i;\ Q_i,P_i)$ that covers the closure of $H(i+1;\ Q_{i+1},P_{i+1})$. If J_{PQ} is the common part of $H(1;P_1Q_1)$, $H(2;\ P_2Q_2),\ldots$ and J_{QP} is the common part of $H(1;\ Q_1,P_1)$, $H(2;\ Q_2,P_2)\ \ldots$, it is found that $J=J_{PQ}+J_{QP}$ where J_{PQ} and J_{QP} have only the points P and Q in common.[24]

Hence, J is a simple closed curve in $D_A+D_B+U_{e_o}C_{x,\delta_o}+U_{e_o}C_{y,\delta_o}$ which does not separate any point of $M-J$ from C_1'. Also, since J contains points of D_A and points of D_B it must intersect C_{x,δ_o} and C_{y,δ_o}.

CHAPTER II

§4. **Lemma:** Let α be an arc in S which separates the points A and B. Let xy be a minimal subarc of α, its endpoints being x and y, which separates A and B. Then there is a member C_{1_0} of π such that $xy \cdot C_{1_0} = x+y$.

Proof: S-xy has two components S_1 and S_2 which contain A and B, respectively. Now there exists a δ_0 such that $U(x,\delta_0)$ intersects at most one element of π and $U(x,\delta_0) \cdot U(y,\delta_0) = \phi$.

Since no subarc of xy separates A and B, then x and y are common limit points of S_1 and S_2. Then by §3, there is a simple closed curve J in S which intersects $C_{x,\delta}$ and $C_{y,\delta}$ ($C_{x,\delta}$ and $C_{y,\delta}$ are components of $U(x,\delta) \cdot xy$ and $U(y,\delta) \cdot xy$, and which contain x and y, respectively) when $\delta < \delta_0$ which does not separate any point of S-J from $xy-(C_{x,\delta}+C_{y,\delta})$. But the latter is a connected set, and thus S-J is connected. Therefore, J must be an element of π. Let it be C_{1_0}. Since $U(x,\delta_0)$ intersects only one element of π, then there is no other element of π which intersects $C_{x,\delta}$ and $C_{y,\delta}$. But since C_{1_0} intersects $C_{x,\delta}$

and $C_{y,\delta}$ for every $\delta<\delta_o$, then C_{1_o} must contain x and y.

Now let z be any point of xy-(x+y). Then there is a δ_1 less than δ_o such that $z \in xy-(C_{x,\delta_1}, C_{y,\delta_1})$. But by § 3, C_{1_o} does not intersect $xy-(C_{x,\delta_1}+C_{y,\delta_1})$. Hence, $z \notin C_{1_o}$. Therefore, x and y are the only points of xy which belong to C_{1_o} and xy has endpoints only on C_{1_o}.

§ 5. Lemma: Let ab be any arc in S with endpoints only on an element C_1 of π such that no proper subarc of ab has endpoints only on any element C_{1_o} of π. Then ab separates S.

Proof: Suppose ab does not separate S. Let ac_1b and ac_2b be the two different arcs of C_1 with ends a and b. Now consider $S-(ab+ac_1b)$, a set which has a partition. Let x be any point of $ab+ac_1b$ and suppose that x is not a limit point of some component S_1 of $S-(ab+ac_1b)$. Then there is a neighborhood U of x which contains no point of S_1. But then a minimal subarc of $(ab+ac_1b)$ separates S. This subarc must have endpoints only on some element of π (§ 4). But ab is the only such subarc. This is a contradiction. Therefore every point of $ab+ac_1b$ is a limit point of every component of $S-(ab+ac_1b)$.

Lemma 5.1: Let S_1 be the component of $S-(ab+ac_1b)$ which contains $\langle ac_2b\rangle$. Then one component of S_1-ac_2b has limit points on $\langle ac_1b\rangle$ and on $\langle ab\rangle$.

Proof of lemma 5.1: Let Z be a point of $\langle ac_1b\rangle$. Now there exists a δ_1 such that $U(Z,\delta_1)\cdot(ac_2b+ab)=\phi$. In $U(Z,\delta_1)$ there is a point which is an accessible limit point of S_1. Let such a point be Z_1'. Let Z_2 be a point of S_1. Then there is in S_1+Z_1' an arc $Z_1'Z_2$ from Z_1' to Z_2. Since Z_1' does not belong to $ab+ac_2b$, there is a subarc $Z_1'Z_2'$ of $Z_1'Z_2$ which does not intersect $ab+ac_2b$. But then $Z_1'Z_2'-Z_1' \subseteq S_1-(ac_2b)$ which has Z_1' as a limit point. But $Z_1'Z_2'-Z_1'$ is connected and belongs to some component of $S_1-\langle ac_2b\rangle$. Therefore, some component of $S_1-\langle ac_2b\rangle$ has a limit point on $\langle ac_1b\rangle$. Call this component S^*. Suppose S^* has no limit points on $\langle ab\rangle$. Let $S-(ab+ac_1b) = S_1+S_a$. ($S_1\cdot S_a=\phi$). Consider $S-C_1=S^*+[S_1-(S^*+C_1)] + S_a+\langle ab\rangle$. Let $K=\overline{S^*}\cdot\{[S_1-(S^*+C_1)]+S_a+\langle ab\rangle\} + S^*\cdot\{\overline{[S_1-(S^*+C_1)]+S_a+\langle ab\rangle}\}$. Since S^* has no limit points on $\langle ab\rangle$ then $\overline{S^*} \subseteq S^*+C_1$. But then $\overline{S^*}\cdot\{[S_1-(S^*+C_1)] +S_a+\langle ab\rangle\} = \phi$. Now $S^*\cdot\{\overline{[S_1-(S^*+C_1)]}+\overline{S_a+\langle ab\rangle}\}$

$-(S^*+C_1)] +S_a+\langle ab\rangle\} = \phi$. Now $S^*\cdot\{\overline{[S_1-(S^*+C_1)]}+\overline{S_a+\langle ab\rangle}\}$

$\subseteq S^*\cdot[(S_1-C_1)-S^*]+S_1\cdot(S_a) + S_1\cdot(ab)$. Now S_1-C_1 is locally connected and thus $S^*\cdot[\overline{(S_1-C_1)-S^*}] = \phi$.

Also, $S-(ab+ac_1b)$ is locally connected and thus $S_1 \cdot (\overline{S_a}) = \phi$, and since $S_1 \subseteqq S-(ab)$, then $S_1 \cdot ab = \phi$. Therefore, $K = \phi$.

But then $S-C_1$ has a partition. This is a contradiction and thus S^* has limit points on $\langle ab \rangle$. This ends the proof of lemma 5.1.

Now let S_2 be a second component of $S-(ab+ab_1c)$. Since $S_2 \cdot \langle ac_2b \rangle = \phi$, then S_2 is also a component of $S-(C_1+ab)$. Therefore $S-(C_1+ab)$ has as two components S^* and S_2. Now let $x \in ab-(a+b)$ and let $y \in ac_1b-(a+b)$ be such that x and y are limit points of S^*. Since every point of $ab+ac_1b$ is a limit point of S_2, x and y are common limit points of S^* and S_2. Now let δ_0 be such that $[U(x,\delta_0)+U(y,\delta_0)] \cdot (a+b) = \phi$. If $C = C_1+ab$, then it is easily seen that $C-(C_{x,\delta_0}+C_{y,\delta_0})$ is a connected set. Now by § 3, there is a simple closed curve J in S which intersects C_{x,δ_0}, and C_{y,δ_0} such that J does not separate any point of S-J from $C-(C_{x,\delta_0}+C_{y,\delta_0})$. Thus S-J is connected. But J intersects C_{x,δ_0} and hence cannot be C_1; and J intersects C_{y,δ_0} and hence cannot be any element of π different from C_1. This is a contradiction. Therefore ab separates S.

§ 6. Lemma: If ab is an arc in S, then ab separates

S if and only if there is a subarc of ab with endpoints only on some element C_{1_0} of π.

Proof: Suppose ab separates S. Let x and y be two points of S-ab which belong to different components of S-ab. Let a'b' be a minimal subarc of ab which separates x and y. Then by § 4, the arc a'b' has endpoints only on some element C_{1_0} of π.

Suppose ab has a subarc a'b' with endpoints only on an element C_{1_0} of π. It can be assumed that a'b' does not contain a proper subarc with endpoints only on some other element C_1 of π.[25] Now a'b' separates S by § 5, (i. e. $S-a'b'=S_1|S_2$). Now $S_1-(ab)$ cannot be empty, else it would be possible to show that a pair of points would separate S. Also, $S_2-(ab)$ cannot be empty. Then $(S_1-ab)|(S_2-ab)$ is a partition of S-ab.

§ 7. Lemma: Suppose S^* is a domain of S not separated by any pair of points and such that $S-S^*$ has a finite number of components.

Suppose C_1 is a member of π which is a subset of S^*. Suppose every component of S^*-C_1 has limit points on every component of $S-S^*$.

Then S^*-C_1 is connected.

Proof: Let S_1 be a component of S^*-C_1 and suppose that S^*-C_1 has another component S_2. There are two possibilities. In case I will be considered the possibility that every point of C_1 is a common limit point of S_1 and S_2. In case II will be considered the possibility that some point of C_1 is not a common limit point of S_1 and S_2.

Case I. Let x_1 and x_2 be two points of C_1. Let a and b be two points of C_1 which are separated (on C_1) by x_1+x_2. Let $\delta = 1/2\rho(x_1+x_2, a+b)$. If $C=C_1+S-S^*$, then $C_{x_1,\delta}$ and $C_{x_2,\delta}$ are both subsets of C_1 and $C-(C_{x_1,\delta}+C_{x_2,\delta}) = (S-S^*)+C_1-(C_{x_1,\delta}+C_{x_2,\delta})$. Since $C_1-(C_{x_1,\delta}+C_{x_2,\delta})$ consists of two arcs, and since $S-S^*$ has a finite number of components, then by § 3, there is a simple closed curve J in $S_1+S_2+U_eC_{x_1,\delta}+U_eC_{x_2,\delta}$ such that no point of $S-J$ is separated from $C-(C_{x_1,\delta}+C_{x_2,\delta})$ by J. Now in S_2+C_1 there is a subarc J_1 of J which has endpoints only on C_1. Since $J-J_1$ contains limit points of $S-J$, then no point of $S-J_1$ is separated from $C-(C_{x_1,\delta}+C_{x_2,\delta})$ by J_1. But $S_1 \cdot J_1 = \phi$, and S_1 contains limit points on every component of $C-(C_{x_1,\delta}+C_{x_2,\delta})$. Therefore $S-J_1$ is connected. But J_1 is an arc with endpoints only on an element of π. Hence, by § 6, $S-J_1$ has a partition. This is a contradiction, and hence some point of C_1 is not a common limit point of S_1 and S_2.

Case II. Some point of C_1 is not a common limit point of S_1 and S_2.

In this case some subarc of C_1 separates S^*. Let A be a point of S_1 and let B be a point of S_2. Let x_1ax_2 be a minimal subarc of C_1 which separates A and B in S^*. Let S_1' and S_2' be the components of $S^*-x_1ax_2$ which contain A and B, respectively. Now S_1' and S_2' must have x_1 and x_2 as common limit points. Also, S_1' and S_2' have additional limit points on x_1ax_2 for otherwise S^* could be separated by the omission of a pair of points. Let x_3 be such a limit point of S_1' and x_4 be such a limit point of S_2'.

Let $\delta = 1/2\cdot\rho(x_1+x_2,\ x_3+x_4)$. If $C=x_1ax_2+S-S^*$ then $C_{x_1,\delta}+C_{x_2,\delta} \subseteq x_1ax_2$, and $x_1ax_2-(C_{x_1,\delta}+C_{x_2,\delta})$ is a single arc which contains a limit point of S_1' and a limit point of S_2'. Therefore $C-(C_{x_1,\delta}+C_{x_2,\delta})$ has a finite number of components, each of which contains a limit point of S_1' and each of which contains a limit point of S_2'. By §3, there is a simple closed curve J in $S_1'+S_2'+U_eC_{x_1,\delta}+U_eC_{x_2,\delta}$ such that no point of $S-J$ is separated from $C-(C_{x_1,\delta}+C_{x_2,\delta})$. Now some point of J does not belong to C_1. Suppose that such a point lies in the complement of S_1'. Then as in case I there is a subarc J_1 of J in the complement of S_1' such that: (1) J_1 has endpoints only on C_1; and (2) no point of $S-J_1$ is separated from $C-(C_{x_1,\delta}$

$+C_{x_2,\delta})$ by J_1. (Note that J_1 may not have endpoints on $C_{x_1,\delta}$ or on $C_{x_2,\delta}$). But $C-(C_{x_1,\delta}+C_{x_2,\delta})+S_1'$ is a connected set in $S-J_1$. Therefore, $S-J_1$ is connected. But by §6, $S-J_1$ has a partition. This is a contradiction and hence S^*-C_1 is connected.

§8. Lemma: If S cannot be separated by any collection consisting of k elements of π and any finite number of points, then S cannot be separated by any collection consisting of k+1 elements of π.

Proof: Consider a collection of k+1 elements of π. Let these elements be $C_1, C_2, \ldots, C_{k+1}$. Now suppose $S^*=S-\sum_{2}^{k+1} C_i$. Then by the hypotheses S^* is a connected open subset of S which cannot be separated by the omission of any pair of points. Also, $S-\sum_{1}^{k+1} C_i + C_{i_0}$ is connected for $1 < i_0 \leqq k+1$.

Now if M is any connected open subset of S and K is any closed subset of M, then every component of M-K will have limit points on K. Therefore, since $S-\sum_{1}^{k+1} C_i + C_{i_0}$ is connected for $1 < i_0 \leqq k+1$, then every component of $S^*-C_1=S-\sum_{1}^{k+1} C_i$ has limit points on C_{i_0} $(1 < i_0 \leqq k+1)$. (This property will be used several times in subsequent theorems.) Since $S-S^*=\sum_{2}^{k+1} C_i$, then $S-S^*$ has a finite number of components, and every component of S^*-C_1 has limit points on every component of $S-S^*$. also, C_1 is a subset of S^*. therefore, by §7, S^*-C_1 is connected, and thus no collection of k+1 elements of π separates S.

§9. Lemma: If S cannot be separated by any collection consisting of k elements of π and any finite number of points, then S cannot be separated by any collection consisting of k+1 members of π and any finite number of points.

Proof: By induction.

Let $C_1, C_2, C_3, \ldots, C_{k+1}$ be any k+1 elements of π and let x_1 be any point. Suppose $S^*=S-(\sum_2^{k+1} C_i+x_1)$. Then S^* is a connected, open subset of S which cannot be separated by any pair of points. Also, $S-S^* = \sum_2^{k+1} C_i+x_1$, and has a finite number of components. Since $S-(\sum_1^{k+1} C_i+x_1)+x_1$ is connected (§8) and since $S-(\sum_1^{k+1} C_i+x_1)+C_{i_o}$ is connected $(1 < i_o \leq k+1)$, then every component of $S^*-C_1=S-(\sum_1^{k+1} C_i+x_1)$ has x_1 as a limit point and has a limit point on C_{i_o} $(1 < i_o \leq k+1)$. Also, it can be assumed that C_1 is a subset of S^*. But then by §7, S^*-C_1 is connected. Therefore S is not separated by any collection consisting of k+1 elements of π and a single point.

Now suppose that S cannot be separated by any collection consisting of k+1 elements of π' and by any g points. Let $C_1, C_2, \ldots, C_{k+1}, x_1, x_2, \ldots, x_g, x_{g+1}$ be any collection consisting of k+1 elements of π' and any g+1 points. Suppose $S^* = S - \left(\sum_2^{k+1} C_i + \sum_1^{g+1} x_i \right)$. Then S^* is a connected, open subset of S which cannot be separated by any pair of points. Also, $S - S^*$ has a finite number of components.

Now $S - \left(\sum_1^{k+1} C_i + \sum_1^{g+1} x_i \right) + C_{i_0}$ is connected when $1 < i_0 \leq k+1$ and $S - \left(\sum_1^{k+1} C_i + \sum_1^{g+1} x_i \right) + x_{i_0}$ is connected when $1 \leq i_0 \leq g+1$. Therefore, every component of $S^* - C_1$ has x_{i_0} as a limit point when $1 \leq i_0 \leq g+1$, and has a limit point on C_{i_0} when $1 < i_0 \leq k+1$. Also, it can be assumed that C_1 is a subset of S^*. Then by §7, $S^* - C_1$ is connected, and thus S cannot be separated by any collection consisting of k+1 elements of π' and any g+1 points.

Therefore, by induction, S cannot beseparated by any collection consisting of k+1 elements of π' and any finite number of points.

§10. Lemma: S cannot be separated by the omission of any finite number of points.

Proof: Suppose that S cannot be separated by the omission of any set of k points when $k \geqq 2$. Let $x_1, x_2, \ldots x_{k+1}$ be any set of k+1 distinct points. Suppose that $x_1, x_2, \ldots, x_{k+1}$ all belong to the same element C_{i_0} of π for some i_0, when $1 \leqq i_0 \leqq n$. Then $S-C_{i_0} \subseteqq S-(x_1+x_2+\cdots+x_{k+1}) \subseteqq \overline{S-C_{i_0}}$. Since $S-C_{i_0}$ is connected (definition of C_{i_0}) then so is $S-(x_1+x_2+\cdots+x_{k+1})$.

Suppose that two points, say x_1 and x_2, do not belong to the same element C_{i_0} of π. Then any simple closed curve containing x_1 and x_2 is not a member of π.

Suppose $S-\sum_1^{k+1} x_i$ has a partition. Let S_1 and S_2 be two components. Since no set of k points separates S then x_1 and x_2 are common limit points of S_1 and S_2.

Now consider $S-(x_4+x_5+\cdots+x_{k+1})$ (note that $x_4+x_5+\cdots+x_{k+1}$ may be empty). By a repeated application of §2 to S, it can be shown that $S-(x_4+x_5+\cdots+x_{k+1})$ is a locally compact, connected metric space which can be covered by a finite number of connected domains of diameter less that e for every positive e. Also, $S-(x_4+x_5+\cdots+x_{k+1})$ cannot be separated by the omission of any pair of points. Therefore $S-(x_4+x_5+\ldots+x_{k+1})$ satisfies the conditions of M in §3. Also, $x_1+x_2+x_3$ satisfies the conditions for C in §3. Now $C_{x_1,\delta}$ and $C_{x_2,\delta}$ are just the points x_1 and x_2 for and δ. Also, $C-(C_{x_1,\delta}+C_{x_2,\delta})$ is the point x_3. Now by §3, there is in $S_1+S_2+U_eC_{x_1,\delta}+U_eC_{x_2,\delta}$

a simple closed curve J which intersects x_1 and x_2 but which does not separate any point of $S-(x_4+x_5+\dots+x_{k+1})$ from x_3. Hence, $S-(x_4+x_5+\dots+x_{k+1})-J$ is connected. But $S-J \subseteq \overline{S-(x_4+\dots+x_{k+1})-J}$. Therefore, S-J is connected. This is a contradiction. Therefore no set of k+1 points separates S. Since by hypothesis, the lemma is true for k when k=2, it is true for any finite number of points.

§11. Theorem: S cannot be separated by any collection consisting of k elements of π and a finite number of points.

Proof: S cannot be separated by any element of π by hypothesis. By §10, S cannot be separated by any finite collection of points. Then by §9, S cannot be separated by any element of π and a finite collection of points. By §8, S cannot be separated by any two elements of π.

Therefore by mathematical induction using §8 and §9, S cannot be separated by any collection consisting of k elements of π $(k \leqq n)$ and a finite number of points.

§12. Lemma: Let α' be an arc in S with endpoints

a_1' and b_1'. If S^* is a component of $S-\alpha'$, then S^* cannot be separated by the omission of any finite number of points.

Proof: Let S^* be a component of $S-a_1'b_1'$. Let x be a point of S^* and suppose that S^*-x has a partition. Then there are points A and B of S^*-x such that A and B are in separate components of $S-(a_1'b_1'+x)$. Suppose that A belongs to C_{i_0} $(1 \leqq i_0 \leqq n)$. There is an e such that $U_{A,e}\cdot(a_1'b_1'+x)=\phi$. But then $U_{A,e}$ is a subset of the compontnt S_1 of $S-(a_1'b_1'+x)$ which contains A. However, $U_{A,e}$ contains points of $S-\sum_1^n C_i$. Hence, an A' can be picked in S_1 such that $A' \notin C_i$ $(i=1,\ldots,n)$. Therefore, it can be assumed that $A+B \subseteq S-\sum_1^n C_i$.

Since $S-x$ is connected, then $a_1'b_1'$ separates A and B in $S-x$. Let $\alpha=a_1b_1$ be a minimal subarc of $a_1'b_1'$ which separates A and B in $S-x$. Let S_A and S_B be the components of $S-(a_1b_1+x)$ which contain A and B, respectively. It will be noticed that, since no pair of points separates S, α is not a degenerate arc. Since α is a minimal arc separating A and B then a_1 and b_1 are both common limit points of S_A and S_B.

Now $S-x$ satisfies the conditions of §3, and there is a δ_0 such that when $\alpha_1=\alpha-(\alpha_{a_1,\delta}+\alpha_{b_1,\delta})$, then α_1 is connected when $\delta<\delta_0$. Therefore by §3, there is a simple closed curve J in $S_A+S_B+U_e\alpha_{a_1,\delta}+U_e\alpha_{b_1,\delta}$ that does not separate any point of S-x-J from α_1. Since α_1 is connected then S-x-J is connected and S-J is connected ($S-J \subseteq \overline{S-x-J}$). Therefore, J must be one of the simple closed curves that does not separate S. Since there is such a J intersecting $\alpha_{a_1,\delta}$ and $\alpha_{b_1,\delta}$ for every δ less than δ_0, and since there is a δ_1 such that $\alpha_{a_1,\delta}$ intersects at most one element of π when $\delta<\delta_1$, then one element C_1 of π must contain a_1 and b_1.

Let z be any point of $a_1b_1-(a_1+b_1)$. Then there is a δ_2 such that z does not belong to $\alpha_{a_1,\delta_2}+\alpha_{b_1,\delta_2}$. But C_1 belongs to $S_A+S_B+U_e\alpha_{a_1,\delta_2}+U_e\alpha_{b_1,\delta_2}$, and hence z does not belong to C_1. Therefore a_1b_1 has endpoints only on C_1.

Also, e and δ can be made sufficiently small so that $U_e\alpha_{a_1,\delta}+U_e\alpha_{b_1,\delta}$ does not contain x. Hence $C_1 \subseteq S-x$ and x does not belong to C_1.

Now a_1 and x are both common limit points of S_A and S_B and the component of $U(x,e)\cdot(\alpha+x)$ which contains x is just x itself. Also when $\delta<\delta_0$, $\alpha-\alpha_{a_1,\delta}$ is an arc. Then by §3, for any e there is in $S_A+S_B+U_e\alpha_{a_1,\delta}+U(x,e)$ a simple closed curve J which does not separate any point of S-J from $\alpha-\alpha_{a_1,\delta}$. But $\alpha-\alpha_{a_1,\delta}$ is connected and hence $S-J$ is connected. Now δ and e can be chosen sufficiently small that: (1) $U_e \alpha_{a_1,\delta} \cdot C_i = \phi$ $(i=2,\ldots,n)$; and (2) $U(x,e) \cdot C_1 = \phi$. Hence J is not an element of π. This is a contradiction. Hence, no point of S^* separates S^*.

Now suppose that the lemma is true for any set of k points. Let $x_1, x_2, \ldots, x_{k+1}$ be $k+1$ points of S^*, such that $S^* - \sum_1^{k+1} x_i$ has a partition. Then there are points A and B in $S^* - \sum_1^{k+1} x_i$ such that A and B are in separate components of $S-(a_1'b_1' + \sum_1^{k+1} x_i)$. As before, it can be assumed that A and B do not belong to any element of π.

Since $S-\sum_1^{k+1} x_i$ is connected, by §10, $a_1'b_1'$ separates A and B in $S-\sum_1^{k+1} x_i$. Let $\alpha = a_1 b_1$ be a minimal subarc of $a_1'b_1'$ which separates A and B in $S-\sum_1^{k+1} x_i$. Let S_A and S_B be the components of $S-(a_1 b_1 + \sum_1^{k+1} x_i)$ which contain A and B, respectively. It will be noted that since no finite number of points separate S, by §10, α is not a degenerate arc. Since α is the minimal arc separating A and B, and since S^* is not separated by any k points, then a_1, b_1, x_i $(i=1,\ldots,k+1)$ are all common limit points of S_A and S_B.

It can be shown, by repeated application of §2, that $S-\sum_1^{k+1} x_i$ satisfies the conditions of §3, and since there is a δ_0 such that when $\alpha_1 = \alpha - (\alpha_{a_1,\delta} + \alpha_{b_1,\delta})$ then α_1 is connected for every $\delta < \delta_0$, then by §3, there is a simple closed curve J in $S_A+S_B+U_e\alpha_{a_1,\delta}+U_e\alpha_{b_1,\delta}$ that does not separate any point of $S-\sum_1^{k+1}x_i-J$ from α_1. Since α_1 is connected then $S-\sum_1^{k+1}x_i-J$ is connected and $S-J$ is connected $(S-J \subseteq \overline{S-\sum_1^{k+1}x_i-J})$. Therefore, J must be an element of π. Since there is such a J intersecting $\alpha_{a_1,\delta}$ and $\alpha_{b_1,\delta}$ for every δ less than δ_0, and since there is a δ_1 such that $\alpha_{a_1,\delta}$ intersects at most one element of π when $\delta<\delta_1$, then an element C_1 of π must contain a_1 and b_1.

Let z be any point of $a_1b_1-(a_1+b_1)$. Then there are numbers δ_2 and e_2 such that $x_1,x_2,\ldots,x_{k+1}$ and z do not belong to $U_{e_2}\alpha_{a_1,\delta_2}+U_{e_2}\alpha_{b_1,\delta_2}$. But C_1 belongs to $S_A+S_B+U_{e_2}\alpha_{a_1,\delta_2}+U_{e_2}\alpha_{b_1,\delta_2}$ and hence $x_1,x_2,\ldots,x_{k+1}$ and z do not belong to C_1. Therefore, a_1b_1 has endpoints only on C_1 and the points $x_1,x_2,\ldots,x_{k+1}$ do not belong to C_1.

Now a_1 and x_1 are both common limit points of S_A and S_B and the component of $U(x,e)\cdot(\alpha+x_1)$ which contains x_1 is just x_1 itself. Also, when $\delta<\delta_0$, $\alpha-\alpha_{a_1,\delta}$ is an arc. Since $S-\sum_2^{k+1}x_i$ satisfies the conditions of §3, then for any e there is in $S_A+S_B+U_e\alpha_{a_1,\delta}+U(x_1,e)$ a simple closed curve J which does not separate any point of $S-\sum_2^{k+1}x_i-J$ from $\alpha-\alpha_{a_1,\delta}$. But $\alpha-\alpha_{a_1,\delta}$ is connected and hence $S-\sum_2^{k+1}x_i-J$ is connected. Since $S-J \subseteq \overline{S-\sum_2^{k+1}x_i-J}$, then $S-J$ is connected. Now δ and e can be chosen sufficiently small that: (1) $U_e\alpha(a_1, \delta) \cdot C_i = \phi$ $(i=2,\ldots,n)$; and (2) $U(x_1, e) \cdot C_1 = \phi$. Hence, J is not an element of π. This is a contradiction and hence, no finite set of points of S^* separates S^*.

§13. Lemma: Let α be an arc in $S-\sum_1^n C_i$ and let S^* be a component of $S-\alpha$. Then if S^* cannot be separated by the omission of any collection consisting of $k-1$ elements of π and any finite number of points, then S^* is not separated by any collection consisting of k elements of π.

Proof: Notice first of all that $S-S^*$ is connected. Now let $C_1, C_2, \ldots, C_k$ be any set of k elements of π. Set $S' = S^* - \sum_2^k C_i$. Now consider $S'-C_1$.

Since $S-\sum_2^k C_i$ is connected, by §11, then every component of $S'-C_1$ has a limit point on $S-S^*$. Since $S^*-\sum_1^k C_i + C_{i_0}$ is connected when $1 < i_0 \leq k$, then every component of $S'-C_1$ has a limit point on C_{i_0}. It may be assumed that C_1 is a subset of S'. Also, S' is a connected open subset of S which cannot be separated by the omission of any pair of points. Then by §7, $S-C_1$ is connected and hence $S^*-\sum_1^k C_i$ is connected.

§14. Lemma: Let α be an arc of $S-\sum_1^n C_i$ and let S^* be a component of $S-\alpha$. If S^* cannot be separated by the omission of any collection consisting of k-1 elements of π and any finite number of points, then S^* cannot be separated by the omission of any collection consisting of k elements of π and any finite number of points.

Proof: (by induction). Note that $S-S^*$ is just α itself and since $S-\sum_1^n C_i-x$ is connected, then every component of $S^*-\sum_1^n C_i-x$ has limit points on every component of $S-S^*$.

Let $C_1,C_2,\ldots,C_k$ be any collection of k elements of π and let x be any point. Then $S^*-\sum_1^k C_i$ is connected, by §13, and $S^*-\left(\sum_1^k C_i+x\right)+C_{i_o}$ is connected when $1\leq i_o\leq k$. Set $S'=S^*-\left(\sum_2^k C_i+x\right)$. This is a connected open subset of S which cannot be separated by the omission of any pair of points. Since $S^*-\sum_1^k C_i$ is connected, then every component of $S'-C_1$ has x as a limit point, and since $S^*-\left(\sum_1^k C_i+x\right)+C_{i_o}$ is connected, then every component of $S'-C_1$ has a limit point on C_{i_o} when $1<i_o\leq k$. It may be assumed that C_1 is a subset of S' and hence by §7, $S'-C_1$ is connected. Therefore, $S^*-\left(\sum_1^k C_i+x\right)$ is connected.

Suppose that $S^*-\left(\sum_1^k C_i\right)$ is not separated by any collection of m points. Let $x_1,x_2,\ldots,x_{m+1}$ be any set of m+1 points. Since $S-\sum_1^n C_i-\sum_1^{m+1} x_i$ is connected, then every component of $S^*-\sum_1^k C_i-\sum_1^{m+1} x_i$ has a limit point of α, and hence on every component of $S-S^*$. Set S' equal to $S^*-\left(\sum_2^k C_i+\sum_1^{m+1} x_i\right)$. Since $S^*-\left(\sum_1^k C_i+\sum_1^{m+1} x_i\right)+C_{i_0}$ is connected when $1 \leqq i_0 \leqq k$, then every component of $S'-C_1$ has a limit point of C_{i_0}. Since $S^*-\left(\sum_1^k C_i+\sum_1^{m+1} x_i\right)+x_{i_0}$ is connected when $1 \leqq i_0 \leqq m+1$ then every component of $S'-C_1$ has x_{i_0} as a limit point. Also, S' is a connected open subset of S which cannot be separated by any pair of points, and it can be assumed that C_1 is a subset of S'. Then by § 7, $S'-C_1$ is connected and hence $S^*-\left(\sum_1^k C_i+\sum_1^{m+1} x_i\right)$ is connected.

Then, by induction, $S^*-\sum_1^k C_i$ cannot be separated by the omission of any finite set of points.

§ 15. Theorem: Let α be any arc in $S-\sum_1^n C_i$ and let S^* be a component of $S-\alpha$. Then S^* cannot be separated by any collection consisting of k elements of π and any finite set of points.

Proof: By § 12, S^* cannot be separated by the omission of any finite number of points. Hence, the lemma is true for k=0. Now suppose that the lemma is true for k=m. Then by §13 and §14, the lemma is true for k=m+1. Hence, the lemma is true for any finite k.

§16. Lemma: No subarc of $S-\sum_1^n C_i$ separates $S-\sum_1^n C_i$.

Proof: By §6, if α is an arc of $S-\sum_1^n C_i$ then $S-\alpha$ is connected. But by §15, as $S^*=S-\alpha$, $S-\alpha-\sum_1^n C_i = S-\sum_1^n C_i-\alpha$ is connected. Hence, α does not separate $S-\sum_1^n C_i$.

§17. Theorem: The set $S-\sum_1^n C_i$ is homeomorphic to a region on a sphere.

Proof: (1) The set $S-\sum_1^n C_i$ is connected, by §11, locally compact and locally connected. Then $S-\sum_1^n C_i$ is a P-space in the sense used by E. R. van Kampen. (2) The set $S-\sum_1^n C_i$ is a locally compact, locally connected set which cannot be separated by the omission of any point, by §11. If x is any point of C_i, then every $U(x,e)$ contains points of $S-C_i$. Thus $S-\sum_1^n C_i$ is nondegenerate. Since S is a continuous curve, S is separable. Then by 1.25 chapter III of Wilder, [3], S is perfectly separable, and hence any subset of S is perfectly separable. Also, S is normal. Then $S-\sum_1^n C_i$ is a nondegenerate, perfectly separable and normal, locally compact, locally connected and connected set. Hence, by 3.32 chapter III of

Wilder [3], $S-\sum_1^n C_i$ is cyclicly connected. Therefore $S-\sum_1^n C_i$ contains at least one simple closed curve.

(3) Let β be a simple closed curve in $S-\sum_1^n C_i$. Then $\beta \subsetneqq S$ and $S-\beta = S_1 | S_2$. Now β contains a limit point of S_1. Since $S-\sum_1^n C_i$ is open, there is an e_1 such that $U(x_1, e_1) \subsetneqq S-\sum_1^n C_i$. But $U(x_1, e_1)\cdot S_1 \neq \phi$. Therefore $(S-\sum_1^n C_i)\cdot S_1 \neq \phi$. Similarly $(S-\sum_1^n C_i)\cdot S_2 \neq \phi$. But then $(S-\sum_1^n C_i)-\beta = (S-\sum_1^n C_i)\cdot S_1 \big| (S-\sum_1^n C_i)\cdot S_2$. Therefore β separates $S-\sum_1^n C_i$.

(4) Let ab be a closed arc of a simple closed curve β of $S-\sum_1^n C_i$. Then by §16, ab does not separate $S-\sum_1^n C_i$.

Then by a theorem of E. R. van Kampen [10], $S-\sum_1^n C_i$ is homeomorphic with a region on a sphere, and hence, to a region of the Euclidean plane.

CHAPTER III

§18. Lemma: Let C be a finite collection of arcs α_i ($i=1,\ldots,n$) such that: (1) $\alpha_i \cdot \alpha_j \subseteq a_i + b_i$ where a_i and b_i are endpoints of α_i (i.e., the only points of intersection are endpoints); and (2) $\sum_1^n \alpha_i$ is connected. Then the arcs can be rearranged into the order $\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_n}$ so that $\sum_1^k \alpha_{p_i}$ is connected when $1 \leqq k \leqq n$.

Proof: Pick any arc and label it α_{p_1}. Suppose that, for some k_o less than n, k_o arcs have been labeled ($\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_{k_o}}$) such that $\sum_1^k \alpha_{p_i}$ is connected when $1 \leqq k \leqq k_o$. Suppose that no other arc intersects $\sum_1^{k_o} \alpha_{p_i}$. Then $\sum_1^n \alpha_i$ is not connected, contrary to hypothesis. Therefore some arc must intersect $\sum_1^{k_o} \alpha_{p_i}$. Label this arc $\alpha_{p_{k_o+1}}$. Then $\sum_1^k \alpha_{p_i}$ is connected when $1 \leqq k \leqq k_o+1$. Since the conditions of the lemma are obviously satisfied when $k_o=1$, they are satisfied, by mathematical induction, when $k_o=n$.

§19. Lemma: Let C be a collection of a finite number of arcs α_i ($i=1,\ldots,m$) such that: (1) $\sum_1^m \alpha_i$ is connected; and (2) $\alpha_i \cdot \alpha_j \subseteq a_i + b_i$ where a_i and b_i are the endpoints of α_i. If S^* is a component of $S - C^*$ then no pair of points separates S^*.

Proof: By §12, the lemma is true for m=1. Suppose that the lemma is true when m=k-1. Let C be any collection of k arcs satisfying the hypothesis. Let x be any point of S^* and suppose that $S^* - x$ has a partition. There are three cases.

Case I. The collection C contains two arcs α_1 and α_2 where $\alpha_1 = a_1'b_1$ and $\alpha_2 = a_2'b_2$ such that $a_1' \notin \alpha_i$ $(i=2,\ldots,k)$ and $a_2' \notin \alpha_i$ $(i=1,3,\ldots,k)$. Since α_1 and α_2 each have only one point in common with $\sum_3^k \alpha_i$, it is obvious that $\sum_3^k \alpha_i + \alpha_1$ and $\sum_2^k \alpha_i$ are both connected.

If $S^* - x$ has a partition, then there are points A and B in $S^* - x$ such that A and B lie in different components of $S - (C^* + x)$. It may be assumed that A and B belong to $S - \sum_1^n C_i$. Since $\{\alpha_i\}$ $(i=2,\ldots,k)$ is a collection of k-1 arcs which satisfies the conditions for C, and since S^* is a subset of component of $S - \sum_2^k \alpha_i$, then A and B are in the same component of $S - \sum_2^k \alpha_i - x$. Therefore, α_1 separates A and B in $S - \sum_2^k \alpha_i - x$.

Now let $a_1 b_1$ be the minimal subarc of α_1 with endpoint b_1 which separates A and B in $S-\sum_{2}^{k}\alpha_i - x$. Since $\{a_1b_1, \alpha_i\}$ $(i=3,\ldots,k)$ is a collection of $k-1$ arcs which satisfies the conditions for C, and since S^* is a subset of a component of $S-\sum_{3}^{k}\alpha_i - a_1b_1$, then A and B are in the same component of $S-\sum_{3}^{k}\alpha_i - a_1b_1 - x$. Therefore, α_2 separates A and B in $S-\sum_{3}^{k}\alpha_i - a_1b_1 - x$.

Let a_2b_2 be the minimal subarc of α_2, with endpoint b_2 which separates A and B in $S-\sum_{3}^{k}\alpha_i - a_1b_1 - x$. Then A and B belong to separate components of $S-\sum_{3}^{k}\alpha_i - a_1b_1 - a_2b_2 - x$. Call these components S_A and S_B, respectively. Now a_1, a_2 and x are all common limit points of S_A and S_B. Suppose that $C' \equiv \sum_{3}^{k}\alpha_i + a_1b_1 + a_2b_2 + x$ and set δ_0 equal to $1/2 \min[\rho(a_1,b_1), \rho(a_2,b_2)]$. Then $C'-(C'_{a_1,\delta}+C'_{x,\delta})$ is connected and $C'-(C'_{a_2,\delta}+C'_{x,\delta})$ is connected when $\delta<\delta_0$. Also, there is an e such that $(U_eC'_{a_1,\delta})\cdot(U_eC'_{a_2,\delta}) = \phi$, when $\delta<\delta_0$. Then by §3, there are two simple closed curves J_1 and J_2 such that: (1) $J_1 \subseteq S_A+S_B+U_eC'_{a_1,\delta}+U_eC'_{x,\delta}$ and $J_2 \subseteq S_A+S_B+U_eC'_{a_2,\delta}+U_eC'_{x,\delta}$; and (2) $S-J_1$ is connected and $S-J_2$ is connected. Now J_1 contains points of $U_eC'_{a_1,\delta}$ and J_2 contains points of $U_eC'_{a_2,\delta}$. Hence, $J_1 \neq J_2$. But $J_1\cdot J_2$ contains x. Therefore, one of J_1 and J_2, say J_1, is not a member of π. This is a contradiction.

Case II. The collection C contains one arc α_1 where $\alpha_1 = a_1'b_1$ such that $a_1' \notin \alpha_i$ $(i=2,\ldots,k)$; and for every other arc α_{i_0} of C, where $\alpha_{i_0}=a_{i_0}b_{i_0}$ $(i_0 \neq 1)$, $a_{i_0} \in \alpha_{j_1}$ and $b_{i_0} \in \alpha_{j_2}$ for some j_1 different from i_0 and j_2 different from i_0. Again, let A and B be points of S^*-x such that A and B belong to different components of $S-C^*-x$. Now $\{\alpha_i\}$ $(i=2,\ldots,k)$ is a collection of k-1 arcs satisfying the conditions for C. Since S^* is a subset of a component of $S-\sum_2^k \alpha_i$, A and B belong to the same component of $S-\sum_2^k \alpha_i - x$. Therefore, α_1 separates A and B in $S-\sum_2^k \alpha_i - x$. Now let a_1b_1 be the minimal subarc of α_1, with b_1 as one endpoint, which separates A and B in $S-\sum_2^k \alpha_i - x$. If S_A and S_B are the components of $S-\sum_2^k \alpha_i - a_1b_1 - x$ which contains A and B, respectively, then a_1 and x are both common limit points of S_A and S_B.

Suppose that $\delta_0 < \rho(a_1,b_1)$. If $C' = \sum_2^k \alpha_i + a_1b_1 + x$, then $C' - (C'_{a_1,\delta} + C'_{x,\delta})$ is connected when $\delta < \delta_0$. But then, by §3, there is a simple closed curve J in $S_A + S_B + U_e C'_{a_1,\delta} + U_e C'_{x,\delta}$ such that S-J is connected. Then as in the proof of §12, it can be shown that J is one of the elements of π, call it C_1, and that C_1 contains a_1 and x but does not contain any other point of C'.

Due to §13, it can be assumed that $\sum_1^{k-1} \alpha_i$ is connected. Since $\{\alpha_i\}$ $(i=1,\ldots,k-1)$ is a collection of k-1 arcs satisfying the conditions for C, and since S^* is a subset of a component of $S-\sum_1^{k-1} \alpha_i$, then A and B belong to the same component of $S-\sum_{1}^{k-1}\alpha_i-x$. Therefore, α_k separates A and B in $S-\sum_{1}^{k-1}\alpha_i-x$. Let the components of $S-C^*-x$ which contain A and B be S_A' and S_B', respectively. Now let z be a point of $\alpha_k-(a_k+b_k)$ where $\alpha_k=a_kb_k$.

Suppose that z is a common limit point of S_A' and S_B'. Set δ_o equal to 1/2 min $[\rho(a_k,z),\rho(b_k,z)]$. If $C'=C^*+x$, then $C'-(C'_{z,\delta_o}+C'_{x,\delta_o})$ is connected, Then by §3, there is a simple closed curve J in $S_A'+S_B'+U_eC'_{z,\delta_o}+U_eC'_{x,\delta_o}$ such that S-J is connected. But J contains z and hence, is not identical to C_1, and also, J contains x and, hence, intersects C_1. This is a contradiction, hence, z is not a common limit point of S_A' and S_B'.

Now consider the subarc a_kz of α_k. If A and B belong to separate components of $S-\sum_{1}^{k-1}\alpha_i-zb_k-x$, then $\{\alpha_i, zb_k\}$ (i=1,...,k-1) is a collection of k arcs which fall under case I. Hence, A and B must belong to the same component of $S-\sum_{1}^{k-1}\alpha_i-zb_k-x$ and a_kz separates A and B in $S-\sum_{1}^{k-1}\alpha_i-zb_k-x$. Let a_kz' be the minimal subarc of a_kz with a_k as an endpoint which does this. Let S_A and S_B be the components of $S-\sum_{1}^{k-1}\alpha_i-zb_k-a_kz'-x$ which contain A and B, respectively. Then z' and x are common limit points of S_A and S_B. As before, it can be shown that there is a δ_o, and a simple closed curve J in $S_A+S_B+U_eC'_{z',\delta}+U_eC'_{x,\delta}$ such that S-J is connected, where $C'\equiv\sum_{1}^{k-1}\alpha_i+a_kz'+zb_k+x$.

But J contains z' and x, and hence, J is neither C_1 nor C_i $(i=2,\ldots,n)$. This is a contradiction.

Case III. If α_{i_0} is any arc of C with endpoints a_{i_0} and b_{i_0}, then there is an α_{j_1} and an α_{j_2} belonging to C such that: (1) $j_1 \neq i_0$ and $j_2 \neq i_0$; and (2) $a_{i_0} \in \alpha_{j_1}$ and $b_{i_0} \in \alpha_{j_2}$. Because of §18, it can be assumed that $\sum_1^{k-1}\alpha_i$ is connected. Let the components of $S-\sum_1^{k}\alpha_i-x$ which contain A and B be S'_A and S'_B, respectively. As before, it can be shown that α_k separates A and B in $S-\sum_1^{k-1}\alpha_i-x$, and hence, contains limit points of both S'_A and S'_B.

Let α_k be an arc such that $\alpha_k=a_kb_k$, and suppose that there are two distinct points z_1 and z_2 of $a_kb_k-(a_k+b_k)$ which are common limit points of S'_A and S'_B. Set δ_0 equal to $1/2 \min\left[\rho(a_k+b_k, z_1+z_2), \rho(z_1,z_2)\right]$. If $C'=C^*+x$ then $C'-(C'_{z_1,\delta_0}+C'_{x,\delta_0})$ is connected and $C'-(C'_{z_2,\delta_0}+C'_{x,\delta_0})$ is connected. Also, $(C'_{z_1,\delta_0})\cdot(C'_{z_2,\delta_0})=\phi$. Then by §3, there are two simple closed curves J_1 and J_2 such that: (1) $J_1 \subseteq S'_A+S'_B+U_eC'_{z_1,\delta_0}+U_eC'_{x,\delta_0}$ and $J_2 \subseteq S'_A+S'_B+U_eC'_{z_2,\delta_0}+U_eC'_{x,\delta_0}$; and (2) $S-J_1$ is connected and $S-J_2$ is connected. Now J_1 intersects C'_{z_1,δ_0} and J_2 intersects C'_{z_2,δ_0} and, hence, $J_1 \neq J_2$. But J_1 and J_2 both contain x; hence, $J_1\cdot J_2 \neq \phi$. This is a contradiction. Therefore, there are not two distinct points of $a_kb_k-(a_k+b_k)$ which are common limit points of S'_A and S'_B.

Let z be a point of $a_k b_k - (a_k + b_k)$ which is not a common limit point of S'_A and S'_B. Consider the two subarcs $a_k z$ and $z b_k$ of $a_k b_k$. If $a_k z$ or $z b_k$ separates A and B in $S - \sum_1^{k-1} \alpha_i - x$, then this case reduces to case II. Therefore, A and B belong to the same component of $S - \sum_1^{k-1} \alpha_i - x - a_k z$ Let $z_1 b_k$ be the minimal subarc of $z b_k$ which has b_k as an endpoint and which separates A and B in $S - \sum_1^{k-1} \alpha_i - x - a_k z$. Similarly let $a_k z_2$ be the minimal subarc of $a_k z$ which has a_k as an endpoint and which separates A and B in $S - \sum_1^{k-1} \alpha_i - x - z_1 b_k$.

Let S_A and S_B be the usual components of $S - C'$ where $C' \equiv \sum_1^{k-1} \alpha_i + a_k z_2 + z_1 b_k + x$. Then z_1, z_2 and x are common limit points of S_A and S_B. Let δ_0 be less than $\rho(a_k + b_k, z_1 + z_2)$. Then, as before, there are two simple closed curves J_1 and J_2 such that: (1) $J_1 \neq J_2$; and (2) $J_1 \cdot J_2$ contains x; and (3) $S - J_1$ and $S - J_2$ are both connected. This is a contradiction. Hence, in all cases, $S^* - x$ is connected.

Now suppose that x and y are two points of S^* such that $S^* - (x+y)$ has a partition. Let S'_A and S'_B be two components of $S^* - (x+y)$. Then, since no point separates S^*, x and y are common limit points of S'_A and S'_B. Now if $C' \equiv C^* + x + y$, then for any δ_0, $C' - (C'_{x,\delta_0} + C'_{y,\delta_0})$ is connected. Hence, by §3, there is a simple closed curve J_1 in $S_A'+S_B'+U_eC'_{x,\delta_0}+U_eC'_{y,\delta_0}$ such that $S-J_1$ is connected. Since this is true for any δ_0 and e, it can be assumed that $J_1 \cdot C^* = \phi$. There are now two cases.

Case I. There is in C an arc α_1, where $\alpha_1 = a_1'b_1$, such that $a_1' \notin \alpha_i$ $(i=2,\ldots,k)$. Now $\sum_2^k \alpha_i$ is connected and $\{\alpha_i\}$ $(i=2,\ldots,k)$ is a collection of k-1 arcs satisfying the conditions for C. Since S^* is a subset of a component of $S-\sum_2^k \alpha_i$ then A and B are not separated in $S-\sum_2^k \alpha_i - x - y$, where A and B are points of $S^*-(x+y)$ which belong to different components of $S-(C^*+x+y)$. Therefore, α_1 separates A and B in $S-\sum_2^k \alpha_i - x - y$.

Let a_1b_1 be the minimal subarc, with b_1 as an end-point, of $a_1'b_1$ which separates A and B in $S-\sum_2^k \alpha_i - x - y$. Let S_A and S_B be the components of $S-y-C'$ which contain A and B, respectively, where $C' \equiv \sum_2^k \alpha_i + a_1b_1 + x$. Now set $\delta_0 = 1/2\ \rho(a_1b_1)$. Then $C'-(C'_{a_1,\delta_0}+C'_{x,\delta_0})$ is connected. Since S-y satisfies the conditions for §3, then by §3, there is a simple closed curve J_2 in $S_A+S_B+U_eC'_{a_1,\delta_0}+U_eC'_{x,\delta_0}$ such that $S-y-J_2$ is connected. But $S-J_2 \subseteq \overline{S-y-J_2}$ and hence, $S-J_2$ is connected, and J_2 intersects C^* and contains x, hence $J_2 \neq J_1$ and $J_2 \cdot J_1 \neq \phi$. This is a contradiction.

CaseII. If α_{i_0} is an arc of C such that α_{i_0}

$= a_{i_o}b_{i_o}$ then there is an α_{j_1} and an α_{j_2} such that: (1) $j_1 \neq i_o$ and $j_2 \neq i_o$; and (2) $a_{i_o} \in \alpha_{j_1}$ and $b_{i_o} \in \alpha_{j_2}$.

Because of §18, it can be assumed that $\sum_1^{k-1}\alpha_i$ is connected. Also, if A and B are points of $S^*-(x+y)$ which belong to different components of $S-(C^*+x+y)$ then it can be shown as before that α_k separates A and B in $S-\sum_1^{k-1}\alpha_i-x-y$. Let the components of $S-(C^*+x+y)$ which contain A and B be S_A' and S_B'. Suppose that there is a point z on $\alpha_k-(a_k+b_k)$ which is a common limit point of S_A' and S_B'.

Suppose that δ_o is given such that $\delta_o < \rho(a_k+b_k, z)$. If $C' \equiv C^*+x$ then $C'-(C'_{z,\delta_o}+C'_{x,\delta_o})$ is connected. Then by §3, there is a simple closed curve J_2 in $S_A+S_B+U_eC'_{z,\delta_o}+U_eC'_{x,\delta_o}$ such that $S-y-J_2$ is connected. But then $S-J_2$ is connected. However, $J_2 \cdot C'_{z,\delta_o} \neq \phi$, and J_2 contains x. Hence $J_2 \neq J_1$ and $J_2 \cdot J_1 \neq \phi$, a contradiction. Therefore, no point of $\alpha_k-(a_k+b_k)$ is a common limit point of S_A' and S_B'.

Let z be any point of a_kb_k and let a_kz and zb_k be the two subarcs of a_kb_k determined by z. If A and B are separated in $S-\sum_1^{k-1}\alpha_i-a_kz-x-y$ then the case reduces to case I. Suppose, therefore, that A and B are not separated in $S-\sum_1^{k-1}\alpha_i-a_kz-x-y$. Let z_1b_k be the minimal subarc of zb_k which has b_k as an endpoint and which separates

A and B in $S-\sum_{1}^{k-1}\alpha_i-a_kz-x-y$. Set $C'=\sum_{1}^{k-1}\alpha_i+a_kz+z_1b_k+x$ and let S_A and S_B be the components of $S-y-C'$ which contain A and B, respectively. If δ_o is less than $\rho(z_1,b_k)$, then $C'-(C'_{z_1,\delta_o}+C'_{x,\delta_o})$ is connected.

Then by §3, there is a simple closed curve J_2 in $S_A+S_B+C'_{z_1,\delta_o}+C'_{x,\delta_o}$ such that $S-y-J_2$ is connected. However, as before, this can be shown to lead to a contradiction.

Therefore, if the lemma is true for n=k-1, it is true when n=k. Since the lemma is true when n=1, then by induction, it is true for any finite n.

§20. Lemma: If C is a finite collection of arcs α_i (i=1,...,n) where $\alpha_i=a_ia_i^*$ such that $\alpha_i\cdot\alpha_j$ has a finite number of components, then there is a finite collection C' of arcs α'_i (i=1,...,n') such that: (1) $\alpha'_i\cdot\alpha'_j\subseteq a'_i+b'_i$ where a'_i and b'_i are endpoints of α'_i; and (2) $C^*=C'^*$.

Proof: Let K be a collection such that every member of K is a component of some $\alpha_i\cdot\alpha_j$. Then K has a finite number of members. Now every component of $\alpha_i\cdot\alpha_j$ is a point or an arc. Let $b_{p_1},b_{p_2},\ldots,b_{p_k}$ be those members of K which are points and let $b_{p_{k+1}},b_{p_{k+2}},\ldots,b_{p_m}$ be the first and last points of those members of K which are arcs.

Now let $b_1^i, b_2^i, \ldots, b_{q_i}^i$ be the points $\{b_{p_j} | b_{p_j} \in \alpha_i\}$ arranged in the order of occurance on α_i from a_i to a_i^*. Consider the arcs $a_i b_1^i$, $b_{q_i}^i a_i^*$, $b_j^i b_{j+1}^i$, for all $1 \leq i \leq n$ and $1 \leq j < q_i$. This is a set of at most $n(m+1)$ distinct arcs (some arcs may be represented by two different representations). Let the arcs be labeled $\alpha_1', \alpha_2', \ldots, \alpha_{n'}'$, where each distinct arc is counted only once. If $C' = \{\alpha_i'\}$ $(i=1,\ldots,n')$ then C' satisfies the necessary requirements.

§21. Lemma: Let M' be a connected subset of S consisting of the sum of a finite collection C of arcs α_i $(i=1,\ldots,m)$ such that: (1) $M' \cdot C_i$ has a finite number of components when $i=1,\ldots,n$; and (2) if α_i and α_j are two members of C, then $\alpha_i \cdot \alpha_j \subseteq a_i + b_i$ where a_i and b_i are the endpoints of α_i. Let A_1B_1, $A_1'A_2'$ and A_2B_2 be arcs such that: (1) $A_iB_i \cdot M' = B_i$ $(i=1,2)$; (2) $A_1B_1 \cdot A_2B_2 = \phi$; (3) $A_1'A_2' \subseteq C_1$; (4) $A_1'A_2' \cdot (A_1B_1 + A_2B_2 + M') = A_1 + A_2$; (5) $(A_1 + A_2) \cdot (A_1' + A_2') = \phi$. Then in $S - (M' + \sum_1^n C_i)$ there is an arc P_1P_2 joining a point of A_1B_1 and a point of A_2B_2.

Proof: If $M' \subseteq \sum_1^n C_i$, then $S - (M' + \sum_1^n C_i) = S - \sum_1^n C_i$

which is connected by §11. Since $A_1'A_2' \subseteq C_1$; $A_1'A_2' \cdot (A_1B_1) = A_1$ and $(A_1+A_2)\cdot(A_1'+A_2') = \phi$, then A_1B_1 and A_2B_2 each contains points of $S-\sum_1^n C_i$. Let P_1 be such a point on A_1B_1 and let P_2 be such a point of A_2B_2. Then since $S-\sum_1^n C_i$ is connected, there is an arc P_1P_2 in $S-\sum_1^n C_i$.

Suppose, therefore, that $M' \nsubseteq \sum_1^n C_i$. Set M_o equal to $M'+(A_1B_1-A_1)+(A_2B_2-A_2)$. If $M_o\cdot(C_1-A_1'A_2') \neq \phi$, let α_{m+1} be empty, and if $M_o\cdot(C_1-A_1'A_2') = \phi$ let α_{m+1} be an arc in $S-\left(\sum_2^n C_i + A_1'A_2'\right)$ from $C_1-(A_1'A_2')$ to M_o such that $\alpha_{m+1}\cdot C_1$ and $\alpha_{m+1}\cdot M_o$ each consists of a single point. Set $M_1 = M_o + \alpha_{m+1}$.

Now suppose that M_i has been defined when $1 \leq i < k$. If $M_{k-1}\cdot C_k \neq \phi$ let α_{m+k} be the null set, and if $M_{k-1}\cdot C_k = \phi$ let α_{m+k} be an arc in $S-\sum_1^n C_i + C_k$ from C_k to M_{k-1} such that $\alpha_{m+k}\cdot C_k$ and $\alpha_{m+k}\cdot M_{k-1}$ each consists of a single point. Define M_k to be $M_{k-1}+\alpha_{m+k}$. Then, by induction, M_n can be defined such that: (1) M_n contains M'; (2) $M_n\cdot(C_1-A_1'A_2') \neq \phi$; and (3) $M_n\cdot C_i \neq \phi$ $(i=2,\ldots,n)$.

Let B_i' be the first point of intersection from A_i to B_i of A_iB_i with $M'+\sum_1^n \alpha_{m+1}+(C_1-A_1'A_2')+\sum_2^n C_i$. Then the set M consisting of $M_n+(C_1-\langle A_1'A_2'\rangle)+\sum_2^n C_i-(A_1B_1'-B_1')-(A_2B_2'-B_2')$ is a closed connected set consisting of the sum of a collection of a finite number of arcs α_i' $(i=1,\ldots,p')$ such that $\alpha_i'\cdot\alpha_j'$ has a finite number of components. Then by §20, there is a finite collection C of arcs $\{\alpha_i\}$ $(i=1,\ldots,p)$ such that: (1) $\alpha_i \cdot \alpha_j \subseteq a_i+b_i$ where a_i and b_i are endpoints of α_i; and (2) $C^*=M$.

Also, $S-M-A_1'A_2' \subseteq S-M'-\sum_1^n C_i$. Let P_1 be a point of $A_1B_1'-(A_1+B_1')$ and let P_2 be a point of $A_2B_2'-(A_2+B_2')$ and suppose that P_1 and P_2 belong to different components of $S-M-A_1'A_2'$. Now $A_1B_1'+A_1'A_2'+A_2B_2'-(A_1'+A_2'+B_1'+B_2')$ is a connected subset of $S-M$ and hence P_1 and P_2 belong to the same component S^* of $S-M$. Let $A_1''A_2''$ be a minimal subarc of $A_1'A_2'$ which separates P_1 and P_2 in S^*. Let S_1 and S_2 be the components of $S-M-A_1''A_2''$ which contain P_1 and P_2, respectively.

Since by §19, no pair of points separates S^* then: (1) $A_1'' \neq A_2''$ and (2) one of the components, say S_1, has an accessible limit point on $A_1''A_2''-(A_1''+A_2'')$. Let x be such a point. Since $S-(A_1''A_2'')$ is connected, S_1 also has an accessible limit point on M. Let y be such a point. Then there is an arc xy in S_1+x+y which joins x and y.

Since $(\sum_1^n \alpha_{m+i}+M')\cdot A_1'A_2' = \phi$, there is a δ_0 such that $[U(A_1'',\delta_0)U(A_2'',\delta_0)]\cdot(x+\sum_1^n \alpha_{m+i}+M') = \phi$. If $C'=M+A_1''A_2''$ then $C'-(C'_{A_1'',\delta_0}+C'_{A_2'',\delta_0})=M-(C'_{A_1'',\delta_0}+C'_{A_2'',\delta_0})+A_1''A_2''-(C'_{A_1'',\delta_0}+C'_{A_2'',\delta_0})$ which consists of exactly two components. Then by §3, there is a simple closed curve J_1 in $S_1+S_2+U_eC'_{A_1'',\delta_0}+U_eC'_{A_2'',\delta_0}$ such that J_1 does not separate any point of $S-J_1$ from $C'-(C'_{A_1'',\delta_0}+C'_{A_2'',\delta_0})$. Since this is true for any e, it can be assumed that $(U_e C'_{A_1'',\delta_0})\cdot xy=\phi$ $(i=1,2)$. Now there is an arc x_1y_1 of J_1 in $S_2+x_1+y_1$ such that x_1 and y_1 belong to C_1 and $x_1y_1\cdot xy=\phi$. But then xy is an arc joining $M-(C'_{A_1'',\delta_0}+C'_{A_2'',\delta_0})$ and $A_1''A_2''-(C'_{A_1'',\delta_0}+C'_{A_2'',\delta_0})$ in $S-x_1y_1$. Since x_1y_1 does not separate any point of $S-x_1y_1$ from $C'-(C'_{A_1'',\delta_0}+C'_{A_2'',\delta_0})+xy$ then $S-x_1y_1$ is connected. But by §6, this is a contradiction.

Therefore, P_1 and P_2 belong to the same component of $S-M-A_1'B_1'$ and there is an arc P_1P_2 joining P_1 and P_2 in $S-M-A_1'B_1'$ which is a subset of $S - M' - \sum_1^n C_i$.

§22. Lemma: Let a_1b_1 be a minimal separating arc with endpoints only on C_1 and let $\langle a_1r_1b_1\rangle$ and $\langle a_1r_2b_1\rangle$ be the two components of $C_1-(a_1+b_1)$. Then $\langle a_1r_1b_1\rangle$ and $\langle a_1r_2b_1\rangle$ belong to different components of $S-a_1b_1$.

Proof: Since a_1b_1 is a minimal separating arc, a_1 and b_1 are common limit points of all components. Suppose $\langle a_1r_1b_1\rangle$ and $\langle a_1r_2b_1\rangle$ belong to the same component S_1 of $S-a_1b_1$. Then $C_1 \subseteqq S_1+a_1+b_1$. Let δ_0 be such that: (1) $\delta_0 < 1/2\rho(a_1,b_1)$; and (2) $\delta_0 < \rho(a_1+b_1,C_i)$ $(i \neq 1)$. If $\alpha = a_1b_1$, then $(\alpha_{a_1,\delta_0}+\alpha_{b_1,\delta_0})\cdot C_i=\phi$ $(i \neq 1)$ and $\alpha-(\alpha_{a_1,\delta_0}+\alpha_{b_1,\delta_0})$ is connected. But then by §3, there is a simple closed curve J which intersects α_{a_1,δ_0} and α_{b_1,δ_0} and S_2 such that S-J is connected. (S_2 is a second component of $S-a_1b_1$). But this is a contradiction. Therefore $\langle a_1r_1b_1\rangle$ and $\langle a_1r_2b_1\rangle$ belong to different components of $S-a_1b_1$.

§23. Lemma: Let ab be an arc with endpoints only on some element C_1 of π, and let $\langle ar_1b\rangle$ and $\langle ar_2b\rangle$ be the two components of $C_1-(a+b)$. Then $\langle ar_1b\rangle$ and $\langle ar_2b\rangle$ belong to different components of S-ab.

Proof: Let α be an arc with endpoints only on C_1. Let these endpoints be a and b. Let r_1 and r_2 be points of C_1 which are separated in C_1 by a+b. Suppose that r_1 and r_2 belong to the same component of S-ab. Suppose that C_2 is the first element of π that is intersected by ab from a to b. Let a_1 be the first point and b_1 be the last point of $ab\cdot C_2$ from a to b. Then r_1 and r_2 belong to the same component S^* of $S-(aa_1+a_1b)$.

Let a_1yb_1 be a subarc of C_2. Suppose that a_1yb_1 separates r_1 and r_2 in S^*. Obviously $a_1yb_1 \nsubseteq \alpha$. Let S_1 and S_2 be the two components which contain r_1 and r_2, respectively. Suppose that every point of a_1yb_1 is a common limit point of S_1 and S_2.

Let y_1 and y_2 be two points of $a_1yb_1-(a_1+b_1)$ and set δ_1 equal to 1/2 min $[\rho(y_1,y_2), \rho(y_1+y_2, a_1+b_1)]$. If $C=aa_1+a_1yb_1+b_1b$, then $C_{y_1,\delta}$ and $C_{y_2,\delta}$ is a subset of a_1yb_1 when $\delta<\delta_1$. Since a and b are both common limit points of S_1 and S_2, then $C-(C_{y_1,\delta}+C_{y_2,\delta})$ consists of three components, each of which contains a limit point of S_1 and a limit point of S_2.

Now by §3, there is a simple closed curve J in $S_1+S_2+U_eC_{y_1,\delta}+U_eC_{y_2,\delta}$ such that no point of S-J is separated from $C-(C_{y_1,\delta}+C_{y_2,\delta})$. Now $C-(C_{y_1,\delta}+C_{y_2,\delta})$ contains points of C_2 and hence J is not equal to C_2. Therefore, J contains a point which does not belong to C_2. Let z be such a point. Then z lies in the complement of S_1 or in the complement of S_2, say the first. Since $C_{y_1,\delta}+C_{y_2,\delta}$ is a subset of C_2, then there is a subarc J_1 of J in the complement of S_1 which has endpoints only on C_2. Then J_1 does not separate any point of $S-J_1$ from $S_2+C-(C_{y_1,\delta}+C_{y_2,\delta})$. But this set is connected and hence $S-J_1$ is connected. But by §6, $S-J_1$ is not connected. This is a contradiction, and thus some point of a_1yb_1 is not a common limit point of S_1 and S_2.

Let y be such a point. Let ay_1 be the minimal arc of aa_1+a_1y with endpoint a which separates r_1 and r_2 in $S-(yb_1+b_1b)$, and let y_2b be the minimal arc of yb_1+b_1b

with endpoint b which separates r_1 and r_2 in $S-ay_1$. Let S_1' and S_2' be the two components of $S-(ay_1+y_2b)$ which contain r_1 and r_2, respectively. Then y_1 and y_2 are common limit points of S_1' and S_2'. Suppose that y_1 and y_2 both belong to $\langle a_1yb_1\rangle$.

Set δ_1' equal to $1/2\ \rho(y_1+y_2,\ a_1+b_1)$. If $C=ay_1+y_2b$, then $C-(C_{y_1,\delta}+C_{y_2,\delta})$ consists of two components when $\delta<\delta_1'$. One of these components contains a and the other contains b. Therefore, both S_1' and S_2' have limit points on every component of $C-(C_{y_1,\delta}+C_{y_2,\delta})$. Now by §3, there is a simple closed curve J in $S_1'+S_2'+U_eC_{y_1,\delta}+U_eC_{y_2,\delta}$ such that no point of $S-J$ is separated from $C-(C_{y_1,\delta}+C_{y_2,\delta})$. Now $C-(C_{y_1,\delta}+C_{y_2,\delta})$ contains points of C_2 and hence J contains at least one point which does not belong to C_2. Let z be such a point. It can be assumed that z lies in the complement of S_1'. Since $C_{y_1,\delta}+C_{y_2,\delta}$ is a subset of C_2, there is a subarc J_1 of J in the complement of S_1' which has endpoints only on C_2. Then J_1 does not separate any point of $S-J_1$ from $S_2'+C-(C_{y_1,\delta}+C_{y_2,\delta})$. But this set is connected and hence $S-J_1$ is connected. Since this is a contradiction, by §6, then at least one of y_1 and y_2 must lie in the complement of $\langle a_1yb_1\rangle$. However, since aa_1+b_1b does not separate r_1 and r_2 in S, then one of y_1 and y_2, say y_1, still belongs to $\langle a_1yb_1\rangle$. Now suppose that every point of $a_1 y_1 - a_1$ is a common limit point of S_1' and S_2'.

Let y_3 be a point of $\langle a_1 y_1 \rangle$ and set δ_2 equal to 1/2 min $[\rho(y_1, y_3), \rho(y_3, a_1)]$. If $C = (ay_1 + y_2 b)$, then $C - (C_{y_1,\delta} + C_{y_3,\delta})$ consists of three components when $\delta < \delta_2$. Two of these components contain a and b, respectively, and the third contains points of $y_1 y_3$. Now a, b and all points of $y_1 y_3$ are limit points of S_1' and S_2'.

By §3, there is a simple closed curve J in $S_1' + S_2' + U_e C_{y_1,\delta} + U_e C_{y_3,\delta}$ such that no point of $S - J$ is separated from $C - (C_{y_1,\delta} + C_{y_3,\delta})$. Now $C - (C_{y_1,\delta} + C_{y_3,\delta})$ contains points of C_2, and hence some point z of J does not belong to C_2. Assume that z belongs to the complement of S_1'. Then as before, there is a subarc J_1 of J which belongs to the complement of S_1' and has endpoints only on C_2. Also, no point of $S - J_1$ is separated from $S_2' + C - (C_{y_1,\delta} + C_{y_3,\delta})$. But this set is connected and so $S - J_1$ is connected. Since, by §6, this is a contradiction, then soem point of $a_1 y_1 - a_1$ is not a common limit point of S_1' and S_2'.

Let y' be such a point. Let $y_1 y_1'$ be the minimal subarc of $y_1 y'$ which separates r_1 and r_2 in $S - (ay' + y_2 b)$, and let ay_2' be the minimal subarc of ay' which separates r_1 and r_2 in $S - (y_1 y_1' + y_2 b)$. Suppose that y_2' belongs to $\langle a_1 y b_1 \rangle$. Let S_1'' and S_2'' be the components of $S - (ay_2' + y_1' y_1$

$+y_2b)$ which contain r_1 and r_2, respectively.

Set δ_3 equal to $\rho(y_2', a_1)$. If $C=ay_2'+y_1'y_1+y_2b$ then, when $\delta<\delta_3$, $C-(C_{y_1',\delta}+C_{y_2',\delta})$ will consist of at most three components, two of which will contain a and b, respectively, and the third (if it exists) of which will contain y_1. Therefore, every component of $C-(C_{y_1',\delta}+C_{y_2',\delta})$ contains a limit point of S_1'' and a limit point of S_2''.

By §3, there is a simple closed curve J in $S_1''+S_2''+U_eC_{y_1',\delta}+U_eC_{y_2',\delta}$ such that no point of S-J is separated from $C-(C_{y_1',\delta}+C_{y_2',\delta})$. Now $C-(C_{y_1',\delta}+C_{y_2',\delta})$ contains points of C_2, and hence some point z of J does not belong to C_2. Assume that z belongs to the complement of S_1''. Then as before, there is a subarc J_1 of J which belongs to the complement of S_1'' and has endpoints only on C_2. Also, no point of $S-J_1$ is separated from $S''+C-(C_{y_1',\delta}+C_{y_2',\delta})$. But this set is connected and so $S-J_1$ is connected. Since by §6, this is a contradiction, then y_2' does not belong to $\langle a_1yb_1\rangle$. Then ay_2' and y_2b are both subsets of α.

$\langle a_1yb_1\rangle$; suppose y_1+y_1' separates r_1 and r_2 in $S-(ay_2'+y_2b)$. Then y_1+y_1' would separate r_1 and r_2 in $S-\alpha$. But by §19, no pair of points separates any component of $S-\alpha$. Therefore: (1) $y_1 \neq y_1'$ and (2) some point of $y_1y_1'-(y_1+y_1')$ is a limit point of S_1'', and some point of $y_1y_1'-(y_1+y_1')$ is a limit point of S_2''. Let y_4 and y_4' be such limit points of S_1'' and of S_2'', respectively.

Set δ_4 equal to $1/2\ \rho(y_4+y_4',\ y_1+y_1')$. If $C=ay_2'+y_1y_1'+y_2b$, then, when $\delta<\delta_4$, the set $C-(C_{y_1,\delta}+C_{y_1',\delta})$ consists of three components, two of which contain a and b, respectively, and the third of which contains y_4 and y_4'. Therefore, every component of $C-(C_{y_1,\delta}+C_{y_1',\delta})$ contains a limit point of S_1'' and a limit point of S_2''.

By §3, there is a simple closed curve J in $S_1''+S_2''+U_eC_{y_1,\delta}+U_eC_{y_1',\delta}$ such that no point of $S-J$ is separated from $C-(C_{y_1,\delta}+C_{y_1',\delta})$. Now $C-(C_{y_1,\delta}+C_{y_1',\delta})$ contains points of C_2, and hence some point z of J does not belong to C_2. Assume that z belongs to the complement of S_1''. Then as before, there is a subarc J_1 of J which belongs to the complement of S_1'' and has endpoints only on C_2. Also, no point of $S-J_1$ is separated from $S_2''+C-(C_{y_1,\delta}+C_{y_1',\delta})$. But this set is connected and so $S-J_1$ is connected. But by, §6, this is a contradiction, and hence a_1yb_1 does not separate r_1 and r_2 in $S-(aa_1+b_1b)$.

If α_2 is defined to be $aa_1+a_1yb_1+b_1b$, then α_2; (1) has endpoints only on C_1; (2) does not have any subarc with endpoints only on C_2, and (3) does not separate r_1 and r_2.

Since there are only k elements in π, this process need be repeated at most k-1 times to yield an arc α_k

such that (1) α_k has endpoints only on C_1; (2) α_k does not have any subarc with endpoints only on C_i (i=2, ...,k); and (3) α_k does not separate r_1 and r_2. But as a result of §6, α_k is a minimal separating arc. Therefore by §22, α_k must separate r_1 and r_2. This is a contradiction and hence must separate r_1 and r_2. Therefore, $\langle ar_1b \rangle$ and $\langle ar_2b \rangle$ must lie in separate components of $S-\alpha$.

§24. Lemma: If a_1b_1 is a minimal separating arc then $S-a_1b_1$ has only two components.

Proof: Suppose $S-a_1b_1$ has at least three components S_1, S_2 and S_3. Then a_1 and b_1 are common limit points of S_1, S_2 and S_3. If $\alpha = a_1b_1$ then there is a δ_0 sufficiently small that: (1) $\alpha - (\alpha_{a_1,\delta_0} + \alpha_{b_1,\delta_0})$ is connected; and (2) α_{a_1,δ_0} intersects only one element of π. But by §3, there are two simple closed curves J_1 and J_2 such that: (1) $J_1 \subseteq S_1+S_2+U_e\alpha_{a_1,\delta_0}+U_e\alpha_{b_1,\delta_0}$; (2) $J_2 \subseteq S_1+S_2+U_e\alpha_{a_1,\delta_0}+U_e\alpha_{b_1,\delta_0}$; (3) $S-J_1$ is connected; and (4) $S-J_2$ is connected.

Now J_1 and J_2 intersect different components of $S-a_1b_1$ and hence $J_1 \neq J_2$. But J_1 and J_2 both intersect α_{a_1,δ_0} and hence they cannot both be an element of π.

This is a contradiction, and hence $S-a_1b_1$ has only two components.

§25. Lemma: Suppose that the arc xay separates S. Suppose that M is a connected set containing xay such that $M-\langle xay\rangle$ is connected. Let S_o be any component of S-xay. Then there is an arc xby in $S-S_o$ with endpoints x and y such that xby does not separate S.

Proof: Let a_2b_2 be a minimal separating subarc of xay where a_2 is the first point of a_2b_2 from x to y. Then by §4, a_2b_2 has endpoints only on an element of π (call it C_1). Also, by §5, it is seen that no proper subarc of a_2b_2 has endpoints only on any element of π.

Let a_1b_1 be the maximum subarc of xay with the properties that: (1) a_2b_2 is a subarc of a_1b_1; (2) a_1 and b_1 belong to C_1. Let $a_2r_1b_2$ and $a_2r_2b_2$ be the arcs into which a_2+b_2 divides C_1. Now if $a_1=a_2$ and $b_1=b_2$, then by §23, and §24, $S-a_1b_1$ contains exactly two components, S_1 and S_2, one of which, say S_1, contains $\langle a_2r_1b_2\rangle$ and the other of which, S_2, Contains $\langle a_2r_2b_2\rangle$.

Any component of S-xay is a subset of either S_1 or S_2. Let S_o be a componentsof S-xay and suppose that $S_o \subseteq S_1$. Now consider the arc $xa_2r_2b_2y$. This lies in

$S-S_0$, has x and y as endpoints, and intersects C_1 only in a single arc.

Now suppose that $a_1 \neq a_2$ or $b_1 \neq b_2$ or both.

Lemma 25.1. Both of the points a_1 and b_1 belong to the same one of $a_2r_1b_2$ or $a_2r_2b_2$, say to the first.

Proof of lemma 25.1: Other cases being obvious, suppose that $a_1 \in \langle a_2r_1b_2 \rangle$ and $b_1 \in \langle a_2r_2b_2 \rangle$. Now $S-a_2b_2$ has a partition and by § 23, there are components S_1 and S_2 of $S-a_2b_2$ such that $r_1 \in S_1$ and $r_2 \in S_2$. But $M- \langle xay \rangle +xa_1+b_1y$ is a connected set which does not intersect a_2b_2 and this set joins a point of S_1 and a point of S_2. This is a contradiction and hence a_1 and b_1 belong to the same arc $a_2r_1b_2$ of C . The lemma is proved.

Let $C_1=a_1s_1b_1+a_1s_2b_1$ where $a_1s_1b_1$ and $a_1s_2b_1$ are the two arcs into which a_1+b_1 divides C_1. Since a_1 and b_1 both belong to $a_2r_1b_2$ then a_2 and b_2 both belong to one arc of C_1, say $a_1s_1b_1$. Since a_2b_2 is a subset of a_1b_1 then every component of $S-a_1b_1$ is a subset of a component of $S-a_2b_2$.

Suppose that $S'_2 = [x' \mid x' \in S - a_1b_1$ and there is an arc $x'y'$ in $S - a_1b_1 + y'$ which joins x' and a point y' of $a_1s_2b_1 - (a_1 + b_1)]$ and $S'_1 = [x' \mid x' \in S - a_1b_1$ and $x' \notin S'_2]$. It is to be shown that $S'_1 | S'_2$ is a partition of $S - a_1b_1$.

First suppose $a_1s_2b_1 \subseteq a_1b_1$. Then $a_1s_2b_1 = a_1b_1$. But a_2b_2 is a subarc of a_1b_1 which has endpoints only on C_1. This is a contradiction and thus there is an x_2 in $a_1s_2b_1$ such that $x_2 \in S - a_1b_1$. Therefore S'_2 is not empty.

Now since a_2 and b_2 are both points of $a_1s_1b_1$ then either $a_2r_1b_2 \subseteq a_1s_1b_1$ or $a_2r_2b_2 \subseteq a_1s_1b_1$. But since a_1 and b_1 are points of $a_2r_1b_2$ and since either $a_1 \neq a_2$ or $b_1 \neq b_2$, then $a_2r_2b_2 \subseteq a_1s_1b_1$. For similar reasons to those given above, $a_2r_2b_2$ contains at least one point x_1 of $S - a_1b_1$. Let x_1y' be any arc J joining x_1 and a point y' of $\langle a_1s_2b_1 \rangle$. Since $a_2r_2b_2 \subseteq a_1s_1b_1$, then $\langle a_1s_2b_1 \rangle \subseteq \langle a_2r_1b_2 \rangle$, and J joins a point of $\langle a_2r_2b_2 \rangle$ and a point of $\langle a_2r_1b_2 \rangle$. But by §23, $\langle a_2r_1b_2 \rangle$ and $\langle a_2r_2b_2 \rangle$ are subsets of different components of $S - a_2b_2$. Hence J intersects a_2b_2 and hence, a_1b_1. Therefore $J \nsubseteq S - a_1b_1 + y'$. Therefore $x_1 \in S'_1$ and hence S'_1 and S'_2 are not empty. Now $S - a_1b_1$ is locally connected, and hence, no point of one component can be a limit point of any combination of other components. Obviously, if any point of a component of $S - a_1b_1$ belongs to S'_2, the whole component does. Therefore, if any point of a component of $S-a_1b_1$ belongs to S_1' then the whole component does. But then $S_1'\cdot\overline{S_2'}+\overline{S_1'}\cdot S_2' = \phi$. Therefore, $S-a_1b_1 = S_1' \mid S_2'$.

Lemma 25.2: If P is any point of $a_1b_1\cdot C_1$ then P belongs to $a_1s_1b_1$.

Proof of lemma 25.2: Suppose that P belongs to $\langle a_1s_2b_1\rangle$. Since $a_1b_1\cdot\langle a_1s_1b_1\rangle \neq \phi$, there is a subarc PQ of $\langle a_1b_1\rangle$ such that $P \in \langle a_1s_2b_1\rangle$ and $Q \in \langle a_1s_1b_1\rangle$. Let Q' be the first point of PQ from P to Q on $a_1s_1b_1$. Let P' be the first point of PQ' from Q' to P on $a_1s_2b_1$. Since $PQ \subseteq \langle a_1b_1\rangle$, then $a_1 \notin P'Q'$ and $b_1 \notin P'Q'$. Then $P' \neq Q'$ and P'Q' is an arc with endpoints only on C_1. Then by §6, P'Q' separates S. Also as a result of §23, a_1 and b_1 are in separate components of $S-P'Q'$. But $M = \langle xay\rangle + xa_1 + b_1y$ is a connected set joining a_1 and b_1 which does not intersect P'Q'. This is a contradiction, and hence P belongs to $a_1r_1b_1$. Lemma 25.2 is proved.

Lemma 25.3: The set $a_1s_2b_1\cdot S_1' = \phi$ and the set $a_1s_1b_1\cdot S_2' = \phi$.

Proof of lemma 25.3: Obviously $a_1s_2b_1 \cdot S_1' = \phi$. Now suppose that $P \in a_1s_1b_1$. If $P \in a_1b_1$ then obviously $P \notin S_2'$. Suppose then that $P \notin a_1b_1$. Let $\langle P_1PP_2 \rangle$ be the component of $a_1s_1b_1 - a_1b_1$ which contains P. Then P_1 and P_2 belong to a_1b_1. Suppose that P_1 belongs to the subarc a_1P and the subarc P_1b_1 of a_1b_1 intersects Pb_1, where a_1P and Pb_1 are the subarcs of $a_1s_1b_1$. Let P_2' be the first point of intersection of P_1b_1 with Pb_1 from P_1 to b_1. Let P_1' be the last point of intersection of P_1P_2' with a_1P from P_1 to P_2'. Since by lemma 25.2, $P_1'P_2'$ does not intersect $a_1s_2b_1$, then $P_1'P_2'$ has endpoints only on C_1 and by § 6, $S - P_1'P_2'$ is not connected. Also, it is easily seen that by § 23, P and $a_1s_2b_1$ belong to separate components of $S - P_1'P_2'$. Hence any arc joining P and a point y of $\langle a_1s_2b_1 \rangle$ must intersect $P_1'P_2'$ and hence a_1b_1. Therefore P S_2'. Lemma 25.3 is proved.

Now let S_0 be a component of $S - xay$. Then S_0 is a subset of a component of $S - a_1b_1$ which is a subset of one of S_1' or S_2', say S_1'. Then by lemma 25.3, $a_1s_2b_1$ does not intersect S_0. Now consider the arc $xa_1s_2b_1y$. It contains no subarc with endpoints only on C_1. Also, it does not intersect S_0.

Hence, in all cases, it is possible to obtain an arc xb'y in $S-S_o$ which does not have a subarc with endpoints only on C_1. If xb'y separates S then the argument can be repeated with C_2, etc. In a finite number of steps an arc xby will be obtained in $S-S_o$ such that xby does not have a subarc with endpoints only on any C_i $(i=1,\ldots,n)$. Therefore by §6, xby does not separate S.

§26. Lemma: If Θ is a primitive skew curve $(ax)+(xb)+(ay)+(yb)+(az)+(zb)+(xu)+(yu)+(zu)$ of type one in S, then there is a primitive skew curve Θ_1 of type one such that $\Theta_1 = (ax)'+(xb)'+(ay)'+(yb)'+(az)'+(zb)'+(xu)'+(yu)'+(zu)'$ and Θ_1 has the property that neither $(ax)'$, $(xb)'$, $(ay)'$, $(yb)'$, $(az)'$, $(zb)'$, $(xu)'$, $(yu)'$ nor $(zu)'$ separates S.

Proof: Consider any one of the arcs, say ax, and suppose that S-ax is not connected. Now Θ-ax+a+x is connected and by §25, there is an arc $(ax)'$ in $S-S_o$ which does not separate S, where S_o is any component of S-ax.

Since Θ-ax is connected, then it belongs to one component of S-ax. If this component is chosen as S_o, then $(ax)'$ does not intersect Θ-ax. Hence, $\Theta-(ax)+(ax)'$ is a primitive skew curve of type one such that $(ax)'$

does not separate S.

This process can be repeated for each of the other eight arcs.

§27. Lemma: If Θ is a primitive skew curve of type one in S, then there is a primitive skew curve Θ'_1, of type one in $S-\sum_1^n C_i$.

Proof: Suppose that $\Theta =$ ax+xb+ay+yb+az+zb+ux+uy +uz and let Θ be a primitive skew curve of type one in S. Then by §26, it can be assumed that no one of the nine arcs of Θ separates S. Note that each of the points a, b, x, y, z and u is the endpoint of three arcs. Therefore in small neighborhoods of these points, there are associated points a', b', x', y', z' and u' belonging to Θ such that: (1) each primed point belongs to $S-\sum_1^n C_i$; and (2) (using a and a' as examples) either a'=a or else $a'a-a \subseteq S-\sum_1^n C_i$, where a'a is a subarc of either ax or ay or az.

The proof of the lemma will be divided into two parts. In part A it will be shown that Θ can be replaced by a Θ' whose vertices a_1, b_1, x_1, y_1, z_1 and u_1 all lie in $S-\sum_1^n C_i$. In part B it will be shown that Θ' can be replaced by a Θ'_1 which lies in $S-\sum_1^n C_i$.

Part A: Now considerthe point a. If $a \in S-\sum_1^n C_i$ set $a=a_1$. Suppose that a does not belong to $S-\sum_1^n C_i$. Then a' belongs to one of ax, ay, or az, say az. There are now two possibilities.

Case I: The point x' belongs to ax. Now ax does not separate S and hence the intersection of ax with any element of π is a point or an arc or empty. Since x'a is a subset of xa and since aa'-a is a subset of $S-\sum_1^n C_i$, then the intersection of x'a' with any element of π is a point or an arc or empty.

Let AB be such an intersection, say with C_1, where A precedes B on x'aa' from x' to a'. Consider the set M=a'z+x'x+xb+yb+zb+ux+uy+uz. This is a closed connected set composed of a finite number of arcs which intersect only at the endpoints. Also, there is an arc A"B" in C_1 such that: A" $\neq$ A, B" $\neq$ B; (2) AB $\subseteq$ A"B"; and (3) A"B"$\cdot$M= ϕ. Now the arcs x'A and Ba' have endpoints only on A"B" and each intersects M. Hence, by § 21, there is an arc A_1B_1 in $S-(M+\sum_1^n C_i)$ which joins x'A and Ba'. Hence, the arc x'a' can be replaced by an arc $x'A_1B_1a'$ which does not intersect C_1. Since x'a' intersects only a finite number of elements of π, then by repeating this process, an arc (x'a')' will be obtained which is a subset of $S-\sum_1^n C_i$.

Now consider the arc yaa'. Let a_1 be the first point of intersection of yaa' and $(x'a')'$ from y to a'. Note that a_1y may still contain a, but a_1y contains no point of any element of π not previously contained by ay. Then the set $xx'a_1+a_1z+a_1y+xb+yb+zb+xu+yu+zu$ is a primitive skew curve of type one, no arc of which separates S, such that $a_1 \in S-\sum_1^n C_i$.

Case II: The point x' does not belong to ax. In this case $x' \in xb$ or $x' \in xu$, say $x' \in xb$. Since the intersection of xa and any C_i is a point or an arc (if it is not empty) and since $x'x-x$ and $aa'-a$ are subsets of $S-\sum_1^n C_i$, then the intersection of $x'a'$, where $x'a'=x'xaa'$, with any element of π is either a point, an arc, or empty. If $M=x'b+a'z+yb+zb+uy+yz$, then as in case I, the arc $x'a'$ can be replaced by an arc $(x'a')'$ in $S-(M+\sum_1^n C_i)$. Now let x_1 be the first point of intersection from u to x' of the arc uxx' with $(x'a')'$ and let a_1 be the first point of intersection from y to a' of the arc yaa' with $(x'a')'$. Note that while a_1y and ux_1 may still contain a and x, respectively, a_1y and ux_1 do not contain any point of $\sum_1^n C_i$ not previously contained in ay and ux. Then the set $x_1a_1+a_1y+a_1z+x_1b+yb+zb+x_1u+uy+uz$ is a primitive skew curve of type one, no arc of which separates S, such that a_1 $S-\sum_1^n C_i$.

Since this can be repeated for each of the six points a, b, x, y, z and u it can be assumed that there is a primitive skew curve Θ' of type one in S such that: (1) no one of the nine arcs of Θ' separates S; and (2) each of the points a, b, x, y, z and u belongs to $S-\sum_1^n C_i$.

Part B. Consider any arc of Θ', say ay. Now ay does not separate S and hence its intersection with any element of π is either an arc, a point or empty. If $M = \Theta - ay + a + y$, then as in case I of part A, ay can be replaced by an arc $(ay)'$ with endpoints a and y such that $(ay)' \subseteq S-\sum_1^n C_i$. Since this can be done for each of the nine arcs of Θ', then there is a primitive skew curve of type one in $S-\sum_1^n C_i$.

§ 28. Theorem: The set S is homeomorphic to a subset of the plane.

Proof: By §17, $S-\sum_1^n C_i$ is homeomorphic to a subset of the plane. Then according to a theorem by S. Claytor [5] ; $S-\sum_1^n C_i$ does not contain a primitive skew curve of type one.

Now suppose S contains a primitive skew curve of type one. Then by §27, $S-\sum_1^n C_i$ contains a primitive skew curve of type one. This is a contradiction and hence S cannot contain a primitive skew curve of type one.

Hence, by a theorem of Hall [9] , S does not contain a primitive skew curve of type two. Therefore S is a Peanian continuum which does not have any cut points and which does not contain any primitive skew curves of type one or two. Then by Claytor's theorem [5] , S is homeomorphic to a subset of a spherical surface.

If the collection π is empty, then Bing [4] has shown that the set S is homeomorphic to the entire sphere. Assume that π is not empty. Let C_1 be an element of π and let C_1' be the homeomorph of C_1 in S_2, the 2-sphere. Then, by Jordan's curve theorem, S_2-C_1' contains two components D_1 and D_2. Then the homeomorph of $S-C_1$ in S_2 must be a subset of D_1 (or D_2) else it is easy to show that $S-C_1$ is not connected. Therefore S is homeomorphic to a subset of $\bar{D}_1$ and hence to a subset of the plane. Note that S is homeomorphic to a bounded subset of the plane.

CHAPTER IV

§ 29. Two theorems and a lemma by A. Gehman [7].

Definition: Let M and M' be point sets in the planes R and R', respectively. Let f be a homeomorphism which carries M into M'. Then it is said that f can be extended in the sense of Antoine (A-extended) to a correspondence between R and R' if there is a homeomorphism F of R into R' such that F(M)=M'. Note that F(x) is not necessarily equal to f(x) when x belongs to M.

Definition: Two plane continuous curves M and M' are in the same interior class with respect to the planes S and S' in which they lie if there exists: (a) a continuous 1-1 correspondence T such that T(M)=M'; and (b) a 1-1 correspondence between the set of all simple closed curves in M and the set of all simple closed curves in M', which is such that if J is a simple closed curve in M and J' is the corresponding simple closed curve in M', and if N is the set of all points of M which are interior to J and if N' is the set of all points of M' which are interior to J', then there exists a continuous 1-1 correspondence W such that W(N)=N'.

Of course if M and M' are two simple closed curves, as they are in all applications in this paper, they are in the same interior class.

In the following theorem and lemma the author lists several sets of conditions involving the number of simple closed curves, endpoints, and cutpoints, any set of which will satisfy the theorem (or lemma). Only the set of conditions applicable to the situation in this paper have been copied.

Theorem G1: If M is a continuous curve lying in a plane S, and T is a continuous 1-1 correspondence such that T(M)=M', where M' lies in a plane S', then T can be A-extended to a correspondence between the planes S and S' provided that: (1) M and M' are in the same interior class with respect to S and S'; and (2) M has one simple closed curve, less than four endpoints, and the same number of branch points as endpoints.

Lemma G: The plane continuous curve M is <u>reversible</u> if M has one simple closed curve, less than three endpoints, and the same number of branch points as endpoints.

Note: The general notion of reversibility of continuous curves is complicated. All plane simple closed curves are reversible and are the only type considered below.

Theorem G2: Given (1) a point set M lying in a plane S, and a continuous (1-1) correspondence T such that T(M)=M' where M' lies in a plane S'; (2) each component of M is bounded and except for at most one component, each is reversible; (3) if C denotes a simple closed curve in either S or S', C encloses points of at most a finite number of components of either M or M'; (4) for each component M_i of M, the correspondence T between M_i and the component $T(M_i)=M_i'$ of M' can be extended to a correspondence between the planes S and S'; and (5) if M_i and M_j denote any two components of M, then M_i lies in the domain[26] of S-M_j bounded by the subset B of M_j, if and only if $T(M_i)=M_i'$ lies in the domain[26] of S-M_j' bounded by T(B). Under these conditions, the correspondence T between M and M' can be A-extended to a correspondence between the planes S and S'.

§30. Lemma: Let S' be the homeomorph of S in the plane R. Let D be a component of R-S'. Suppose that the boundary of D contains a simple closed curve J. Then $J \subseteq S'$ and S'-J is connected.

Proof: Since D is a component of $R-S'$, and since S' is closed, then J is a subset of S'. Let π' be the collection of elements $\{C_i'\}$ such that C_i' is the homeomorph of C_i $(i=1,\ldots,n)$. Suppose that $S'-J$ has a partition. Then J is not identical to any member of π'. Hence, there are two points P_1 and P_2 of J such that $P_1+P_2 \subseteq S'-\sum_1^n C_i'$. Now let $e_1 > o$ be sufficiently small that (1) $U(P_1,e_1)\cdot U(P_2,e_2)=\phi$; and (2) $U(P_1+P_2, e_1)\cdot \sum_1^n C_i' = \phi$. By theorem 2, chapter II of Moore [2], there is a point P_i' in $U(P_i,e_1)\cdot B$ such that P_i' is an accessible limit point of D, where B is the boundary of D and $i=1,2$.

By §11, $S'-\sum_1^n C_i'$ cannot be separated by any finite number of points, and hence there are three arcs $P_1'x_1P_2'$, $P_1'x_2P_2'$ and $P_1'x_3P_2'$ in $S'-\sum_1^n C_i'$ such that $(P_1'x_iP_2')\cdot(P_1'x_jP_2') = P_1'+P_2'$ whenever $i \neq j$. Also, since P_1' and P_2' are accessible limit points of D, there is an arc $P_1'yP_2'$ in $D+P_1'+P_2'$.

Now one of the three arcs $\{P_1'x_iP_2'\}$ $(i=1,2,3)$, say $P_1'x_1P_2'$, is such that $\langle P_1'x_2P_2'\rangle$ and $\langle P_1'x_3P_2'\rangle$ lie in different components of $R-(P_1'yP_2'+P_1'x_1P_2')$. Since $\langle P_1'x_2P_2'\rangle$ and $\langle P_1'x_3P_2'\rangle$ are subsets of S', then $S'-(P_1'yP_2'+P_1'x_1P_2')$ has a partition. Since $\langle P_1'yP_2'\rangle$ is a subset of D, then $S'-P_1'x_1P_2'$ has a partition.

But $P_1'x_1P_2'$ is a subset of $S'-\sum_1^n C_i'$ and no subarc of $P_1'x_1P_2'$ has endpoints on any element of π'. Therefore, by §6, $S'-P_1'x_1P_2'$ is connected. This is a contradiction and hence $S'-J$ is connected.

§31. Lemma: Let S' be the homeomorph of S in the plane R. If π contains more tha one element then: (1) the boundary of S' is the set $\sum_1^n C_i'$ where C_i' is the homeomorph of C_i; (2) there is an element, say C_1, of π such that $S'-C_1'$ is in the bounded component of $R-C_1'$; (3) C_i' is interior to C_1' when $i=2,\ldots,n$; and (4) C_j' is exterior to C_i' when $i \neq 1$, $j \neq 1$.

Proof: (1) Let π' be the collection of simple closed curves which are homeomorphs of elements of π. Let C_{i_o}' be any element of π'. Since C_{i_o}' is a simple closed curve, then $R-C_{i_o}'$ has exactly two components, D_1 and D_2, such that $C_{i_o}' \subseteq \bar{D}_1$ and $C_{i_o}' \subseteq \bar{D}_2$. Now $S'-C_{i_o}'$ must be a subset of one of D_1 or D_2, say D_1, else $S-C_{i_o}$ would have a partition. Hence, $D_2 \subseteq R-S'$. Thus $C_{i_o}' \subseteq S'$ and $C_{i_o}' \subseteq \overline{R-S'}$. Therefore, $C_{i_o}' \subseteq B(S')$. Since this is true for every i_o $(1 \leq i_o \leq n)$, then $\sum_1^n C_{i_o}' \subseteq B(S')$.

Let D be any component of $R-S'$ and let y be any point of $B(D)$. Now by theorem 41, page 261, of Moore [2], $B(D)$ contains a simple closed curve J. By §30, $J \subseteq S'$

and $S'-J$ is connected. Since J is a simple closed curve then $R-J = D_1 | D_2$. Because $J \subseteq S'$ then D is a subset of one of D_1 or D_2, say D_1.

It is possible to draw arcs x_1x_2 and a_1a_2 such that: (1) $\langle x_1x_2 \rangle \subseteq D$ and $\langle a_1a_2 \rangle \subseteq S'-J$; (2) $a_1+a_2+x_1+x_2 \subseteq J$; and (3) a_1+a_2 separates x_1+x_2 on J. If $S'-J$ were a subset of D_1 then $\langle a_1a_2 \rangle$ and $\langle x_1x_2 \rangle$ would both be subsets of the same component D_1 of $R-J$. Since $\langle a_1a_2 \rangle \cdot \langle x_1x_2 \rangle = \phi$, this is impossible. Thus $S'-J$ is not a subset of D_1. Since $S'-J$ is connected then it must belong to D_2. Therefore $S' \subseteq \overline{D}_2$ and $D_1 \subseteq R-S'$. Thus D_1 is a component of $R-S'$ which contains D. Since D_1 and D are components of $R-S'$ and contain points on common, then $D_1=D$. Therefore $y \in J$. Since $S'-J$ is connected then J is an element of π', and therefore $y \in \sum_1^n C'_i$.

Now let $y \in B(S')$ and suppose $y \notin \sum_1^n C'_i$. Then there are an infinite number of components $D_1, D_2, \ldots$ of $R-S'$ and a sequence $\{y_i\}$ such that $y_i \in D_i$ and $\{y_i\} \to y$. Since for every $i \geqq 1$, y_i belongs to D_i and y belongs to S' and hence to $R-D_i$, there is a sequence $\{y'_{p_i}\}$ such that y'_{p_i} belongs to $B(D_{p_i})$ and $\{y'_{p_i}\} \to y$. But this is a sequence of points of $\sum_1^n C'_i$ which converges to a point of $R-\sum_1^n C'_i$. Since $\sum_1^n C'_i$ is closed, this is a contradiction. Hence, $B(S') \subseteq \sum_1^n C_i'$. But then $B(S') = \sum_1^n C_i'$.

(2) Since S' is bounded, then $R-S'$ contains one unbounded component D'. Then, as in (1), the boundary of D' is an element C_1' of π'. Then $S'-C_1'$ is interior to C_1'.

(3) This follows from (2).

(4) Let C_i' be any element of π' different from C_1'. Let C_j' be any element of π' different from C_i'. Since $S-C_i'$ is connected and since C_1' is exterior to C_i' then $S-C_i'$ is exterior to C_i'. But $C_j' \subseteq S'-C_i'$ and thus C_j' is exterior to C_i'.

§ 32. Lemma: Let M be the sum of a collection of n nonintersecting simple closed curves $C_1, C_2, \ldots, C_n$ in a plane R.

Let M' be the sum of a collection of n nonintersecting simple closed curves $C_{p_1}', C_{p_2}', \ldots, C_{p_n}'$ in a plane R' such that if C_{p_i}' and C_{p_j}' are any two curves of M', then C_{p_i}' is in the interior of C_{p_j}' if and only if C_i is in the interior of C_j.

Then there is a homeomorphism T carrying R into R' such that $T(C_i)=C_i'$ $(i=1,\ldots,n)$, where C_i' is one of $C_{p_1}', C_{p_2}', \ldots, C_{p_n}'$.

Proof: (1) Let C_i be one of the simple closed curves of M. Then there is a homeomorphism f_i' carrying C_i into C'_{p_i}. Now C_i and C'_{p_i} are in the same interior class with respect to R and R'. Therefore by theorem G1 (Gehman), there is a homeomorphism f_i carrying R into R' such that $f_i(C_i)=C'_{p_i}$. Now define $T':M \to M'$ as follows: $T'(x)=f_i(x)$ $(x \in C_i)$. Now T' is a homeomorphism carrying M into M'.

(2) By lemma G a simple closed curve is reversible. Also, every simple closed curve in the plane is bounded. Therefore every component of M is bounded and reversible.

(3) Since M and M' have only a finite number of components, obviously any simple closed curve C can contain points of only a finite number of components of M or of M'.

(4) For each component C_i of M, $T'(C_i)=f_i(C_i)$, and thus $T'(C_i)$ (the notation of Lefschetz, page 2, is $T'\ C_i$) can be extended to the plane R.

(5) Let C_i and C_j be two components of M. Now since C_j is a simple closed curve, there is only one bounded component of R-C_j and that is bounded by C_j. Any point which belongs to this bounded component of R-C_j is interior to C_j. The same is true for C'_{p_j}. Since C_i is

interior to C_j if and only if C'_{p_i} is interior to C'_{p_j}, then C_i belongs to the bounded component of $R-C_j$ if and only if C'_{p_i} belongs to the bounded component of $R-C'_{p_j}$. Hence by theorem G2, there is a homeomorphism $T:R \rightarrow R'$ which carries M into M'. Now each of the simple closed curves of M must go into a simple closed curve of M'. Let the curve of M which C_i goes into be labeled C'_i. Then T carries C_i into C'_i.

§33. Theorem: The set S is homeomorphic to the plane region bounded by n nonintersecting circles.

Proof: The proof will consist of two parts. Part I will consider the case when π consists of a single element. Part II will consider the case when π consists of more than one element.

Part I. The collection π consists of a single element C_1.

In §28 it was shown that S was homeomorphic to a proper subset of the sphere and that $S-C_1$ was connected. It is easily shown, by projection, that S is homeomorphic to a subset of the plane R such that $S'-C'_1$ is in the bounded domain D' of $R-C'_1$. Then $S' \subseteq \overline{D'}$. Now suppose that there is a point x of D' which belongs to $R-S'$. Let

D_x be the component of $R-S'$ which contains x. Since $C_1' \subseteq S'$ then $D_x \subseteq D'$. Also since $S'-C_1'$ is not empty then D_x is a proper subset of D'. Now by theorem 41, page 261 of Moore [2] the boundary of D_x contains a simple closed curve J. Since D_x is a proper subset of D' then the boundary of D' cannot be identical with J. But by §30, $S'-J'$ is connected. Since π contains a single element, this is impossible. Therefore $D' \subseteq S'$. But then $\overline{D'} \subseteq S'$ and $S'=\overline{D'}$. Let C_1'' be a circle in R. Then there is a homeomorphism T of R with itself which carries C_1' into C_1''. If D'' is the bounded component of $R-C_1''$, then T carries D' into D''. But then T carries $\overline{D'}$ into $\overline{D''}$ and S is homeomorphic to $\overline{D''}$. Therefore S is homeomorphic to the plane region bounded by a circle.

Part II. The collection π consists of more than one element.

Let S' be the homeomorph of S in the plane R and let C_i' be the homeomorph of C_i for every element C_i belonging to π. Then by §31: (1) $S'-C_1'$ is a subset of the bounded component of $R-C_1'$ and (2) C_j' is exterior to C_i' when $\{i \neq 1, j \neq 1\}$. Now let $C''_{p_1}, \ldots, C''_{p_n}$ be nonintersecting circles in R such that: (1) C''_{p_i} is interior to C''_{p_1} when i is different from 1 and (2) C''_{p_j} is exterior to C''_{p_i} when $\{i \neq 1, j \neq i\}$. Note that C''_{p_i} is interior to C''_{p_j} if and only if C'_i is interior to C'_j. Then by §32, there is a homeomorphism $T:R \to R$ such that $T(C'_i)=C''_i$, C''_i being one of $C''_{p_1},\dots,C''_{p_n}$.

Suppose S'' is the set into which T carries S'. Then S'' is homeomorphic to S. By §31, $B(S'')=\sum_1^n C''_i$. Since $S''-\sum_1^n C''_i$ is connected, then $S'' \subseteq \overline{D'}$ where D' is the component of $R-\sum_1^n C''_i$ whose boundary is $\sum_1^n C''_i$. Note that there is only one such component and that it is bounded. Now suppose that x is a point of D' which does not belong to S''. Then as in part I, it can be shown that there is in S'' a simple closed curve J which is not a subset of $\sum_1^n C''_i$ such that $S''-J$ is connected. Since this is a contradiction then every point of D' belongs to S''. Therefore $\overline{D'} \subseteq S''$ and $S''=\overline{D'}$. Therefore S is homeomorphic to the plane region bounded by n nonintersecting circles.

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APPENDIX

[1]Numbers in brackets refer to the bibliography.

[2]Since M can be covered by a finite number of connected domains of diameter less than e for every $e > 0$, M is locally connected (Wilder [3], page 106, theorem 3.9). Then by a repeated application of §2, $M-(x+y)$ can be covered by a finite number of connected domains of diameter less than e for $e > 0$.

[3]Let x_1 be a point of D_1. Let y_1 be a point of $C-(x+y)$. Since no pair of points separates M, there are three arcs $\alpha_{1,1}$, $\alpha_{1,2}$, $\alpha_{1,3}$ from x_1 to y_1 in M such that $\alpha_{1,j} \cdot \alpha_{1,k} = x_1+y_1$ $(1 \leq j, k \leq 3, j \neq k)$. Then at least one on $\alpha_{1,j}$ does not intersect $x+y$. Then let $\alpha_{1,1}$ be this arc. Let y_1' be the first point of $\alpha_{1,1}$ from x_1 to y_1 of C. If $x_1 \in M-(D_A+D_B)$ then $x_1y_1'-y_1'$ is a subset of $M-(D_A+D_B)$. If $x_1 \in D_A$, then $x_1y_1'-y_1' \subseteq D_A$; if $x_1 \in D_B$, then $x_1y_1'-y_1' \subseteq D_B$.

[4]Let $\alpha_{i,1}$, $\alpha_{i,2}$ and $\alpha_{i,3}$ $(i=j+1,\ldots,n)$ be the same as in 3. Let $y'_{i,k}$ be the first point from x_i to y_i of $\alpha_{i,k}$ on C'. Let $\beta_{i,k}$ be the subarc of $\alpha_{i,k}$ from x_i to $y'_{i,k}$. Then at most one of $\beta_{i,k}$ contains x and at most one of $\beta_{i,k}$ contains y. Let $\beta_{i,1}$ be the arc which contains neither x nor y, $\beta_{i,2}$ be the arc which does not contain x but may contain y and let $\beta_{i,3}$ be the arc which does not contain y but may contain x. Then these are the desired arcs.

[5]Suppose D_{j+2} contains a point x_1 of D' and a point x_2 of $M-D'$. Then there is an arc x_1x_2 from x_1 to x_2 in D' which does not intersect C. Since x_1x_2 does not intersect C it must intersect α_{j+1}, as $B(D') \subseteq C+\alpha_{j+1}$. Then in $\alpha_{j+1}+x_1x_2$ there is an arc α_{j+2} from x_2 to C'_1 which lies in $D_A-D'+C'_1$.

Suppose $D_{j+2} \subseteq M-D'$. Let x_1 be the first point of $\beta_{j+2,1}$ from D_{j+2} to C'_1 on $C'_1+\alpha_{j+1}$, and let β'_{j+2} be the subarc from D_{j+2} to x_1. Then in $\beta'_{j+2}+\alpha_{j+1}$ there is an arc α_{j+2} from D_{j+2} to C'_1. But $\beta'_{j+2}+\alpha_{j+1} \subseteq D_A-D'+C'_1$. In either case, α_{j+2} is the desired arc.

[6]In the following discussion $i=j+1,\ldots,k$. Let R be any point of D_A. If $R \notin \beta_{i,1}$ then $\beta_{i,1}$ joins D_i and

C_1'. Suppose $R \in \beta_{1,1}$. If $R \neq x_1$ then $R \notin \beta_{1,2} + \beta_{1,3}$. Therefore, R does not disrupt D_1 from $C_1' + C_{x,\delta_0}$ and R does not disrupt D_1 from $C_1' + C_{y,\delta_0}$. If $R = x_1$ then there is a subarc of $\beta_{1,1}$ which joins D_1 and C_1'. In any case R does not disrupt D_1 from $C_1' + C_{x,\delta_0}$ and R does not disrupt D_1 from $C_1' + C_{y,\delta_0}$. By Bing's lemma there is an arc J from C_{x,δ_0} to C_{y,δ_0} in $D_A + C_{x,\delta_0} + C_{y,\delta_0}$ which does not disrupt D_1 from C_1' in $D_A + C_1'$. Let α_1 be an arc from D_1 to C_1' in $D_A + C_1' - J$. Then α_1 does not intersect J and hence α_1 does not disrupt C_{x,δ_0} from C_{y,δ_0} in $D_A + C_{x,\delta_0} + C_{y,\delta_0}$.

[7]Consider C_{x,δ_0}. Since $C_1' + \alpha_1 + \dots + \alpha_n$ is a closed compact set, then for every $x' \in C_{x,\delta_0}$ there is a $\delta_{x'} < e_1/300$ such that $U_{x',\delta_{x'}} \cdot (C_1' + \alpha_1 + \dots + \alpha_n) = \phi$, where $U_{x',\delta_{x'}}$ is the component of $U(x', \delta_{x'})$ which contains x'. Let g_x' be the union of $U_{x',\delta_{x'}}$ for all $x' \in C_{x,\delta_0}$. Then g_x' is a connected domain of diameter less than $e_1/100$ which contains C_{x,δ_0}, but does not intersect $C_1' + \alpha_1 + \dots + \alpha_n$.

A set g_y' containing C_{y,δ_0} can be obtained with similar properties.

[8]Suppose g is an element of G. Suppose also that $g \cdot D_A \neq \phi$ and $g \cdot (M - D_A) \neq \phi$. Then $g \cdot B(D_A) \neq \phi$. But $B(D_A) \subseteqq C$ and $g \cdot C = \phi$. Therefore, either $g \subseteqq D_A$ or $g \subseteqq M - D_A$.

If $g \subseteq M-D_A$, the same argument gives that $g \subseteq D_B$ or $g \subseteq M-D_B$. Hence, $g \subseteq D_A$, or $g \subseteq D_B$, or $g \subseteq M-(D_A+D_B)$.

[9]Since $\alpha_1+\alpha_2+\cdots+\alpha_n$ does not disrupt C_{x,δ_0} from C_{y,δ_0} in $D_A+C_{x,\delta_0}+C_{y,\delta_0}$, there is an arc J from C_{x,δ_0} to C_{y,δ_0} in $D_A+C_{x,\delta_0}+C_{y,\delta_0}-(\alpha_1+\alpha_2+\cdots+\alpha_n)$. Every point of $J\cdot D_A$ belongs to some element of G. Let G' be the set of all elements of G which contain a point of $J\cdot D_A$. Then $G'+g_x+g_y$ is an open covering of J. Since J is compact, there is a finite number of elements of G' which, together with g_x and g_y, cover J. These elements must all be subsets of D_A. Therefore, there is a finite collection of elements of G whose sum is a connected subset of D_A joining g_x and g_y. Since there is a finite number of domains with this property, there is a smallest number which does this. Let G_A be such a collection.

[10]Notation: Let $G_{A-\{\alpha\}}$ be the collection of all elements of G_A except those which have preassigned subscripts belonging to the set $\{\alpha\}$. Now suppose g_x intersects more than one element of G_A. Let these elements be $g_{k_1}, g_{k_2}, \ldots, g_{k_p}$. Let $G^{*}_{k_1}$ be the union of all components of $G^{*}_{A-\{k_1,k_2,\ldots,k_p\}}$ which intersects g_{k_1}. Then one of the $G^{*}_{k_1}$, say $G^{*}_{k_1}$ must intersect g_y. But then $G^{*}_{A-\{k_2,\ldots,k_p\}}$

connects g_x and g_y (i. e. the union of all elements of $G_{A-\{k_2,\ldots,k_p\}}$ contains a connected subset which intersects g_x and g_y). But this is a contradiction, for G_A is the smallest possible collection of domains which does this. Therefore, g_x intersects only one element of G_A. Call this element g_2. Now suppose g_k intersects g_{k+1} but no other element of $G_{A-\{1,\ldots,k\}}$ for some $k < q-2$ ($g_1 \equiv g_x$). Set $K_x = \sum_1^{k+1}{}' g_i$. Suppose K_x intersects more than one element of $G_{A-\{2,\ldots,k+1\}}$. Denote these elements by g_{k_1}, $g_{k_2},\ldots,g_{k_p}$. Suppose one of g_{k_i}, say g_{k_1}, intersects g_y. Then $\{g_2,\ldots,g_{k+1},g_{k_1}\}$ is a subcollection of G_A which joins g_x and g_y. This is a contradiction. Therefore, no g_{k_i} intersects g_y. But then if C'_{k_i} is the union of all components of $G^*_{A-\{2,\ldots,k+1,k_1,\ldots,k_p\}}$ which intersects g_{k_i}, one of the G'_{k_i} intersects g_y. Say G'_{k_1} is the one. Then $G_{A-\{k_2,\ldots,k_p\}}$ is a subcollection of G_A which joins g_x and g_y, a contradiction. Therefore, by induction, the collection G_A may be numbered in the desired manner.

[11]Suppose that g'_k is a component of $E\cdot g_k$ (g_k an element of G) which does not intersect any element of g_1, $g_2,\ldots,g_r$. Let x be a point of g'_k and let J be an arc in g_k+y from x to a point y of $C'_1+\alpha_1+\ldots+\alpha_n$. Since $x \in E-\sum_1^r{}' g_i$ then J must intersect $\sum_1^r{}' g_i$. Let y' be the first

point of J from x to y on $B(\sum_1^r g_i)$. Then xy' is a connected subset of g_k. Also, no point of xy' can be joined to C_1 in $M-\sum_1^r g_i$. Therefore, $xy' \subseteq g_k'$. Since $y' \subseteq g_k'$ there is an η_o such that $U_{y',\eta} \subseteq g_k'$ for $\eta < \eta_o$. If every point of U_{y',η_1} belongs to E for some $\eta_1 < \eta_o$ then $U_{y',\eta_1} \subseteq g_k'$. But U_{y',η_1} contains points of $\sum_1^r g_i$. Therefore, $U_{y,\eta}$ contains points of M-E for every $\eta < \eta_o$. Since C_1' has only a finite number of components, and since every point of M-E can be joined to C_1', then $M-E+C_1'$ has a finite number of components. Let $M_1, M_2, \ldots, M_n$ be the components of $M-E+C_1'$. Now there is an infinite sequence of points $\{y_i\}$ in M-E which converges to y'. It can be assumed that $\{y_i\}$ belongs to M_1. But then $y' \in \overline{M}_1$ and $xy'+M_1$ is a connected set which joins x and C_1' but does not intersect $\sum_1^r g_i$, a contradiction. Therefore, g_k' intersects some g_i $(i=1,\ldots,r)$.

[12]Suppose g_k is an element of G and g_k' is a component of $E \cdot g_k$. Suppose that g_k' intersects g_i and g_j where $j \geqq i+3$. Since one of g_i or g_j is different from g_1 or g_r then g_k' (and hence g_k) is a subset of D_A or of D_B, say D_A. But then $g_1, g_2, \ldots, g_i, g_k, g_j, g_{j+1}, \ldots, g_q$ contains a subcollection of at most r-3 elements of G which lie in D_A and join g_x and g_y. But G_A contains the smallest

number of such elements and it contains $r-2$ elements. This is a contrasiction, and thus g_k' does not intersect two elements of $g_1, g_2, \ldots, g_r$ that do not lie in a consecutive set of three.

[13]The only condition that is not obvious is (2). Suppose $x \in M$ and $x \notin (h_1+h_2+\ldots+h_s)$. Then there is a connected set K in M which joins x and some component of C_1' but does not intersect $g_1+g_2+\ldots+g_r$. Suppose $y \in K(E-(g_1+g_2+\ldots+g_r))$. Then $g_1+g_2+\ldots+g_r$ does not separate y from C_1', a contradiction. Therefore, $K \cdot E = \phi$ and $h_1+h_2+\ldots+h_s$ does not separate x from C_1'.

[14]Let $g_1, g_2, \ldots, g_j$ be the elements of G which intersect $M-H_1^*$ and let $g_{j+1}, g_{j+2}, \ldots, g_n$ be the elements of G which are subsets of H_1^*. For each i $(i=1, \ldots, j)$ let α_j be a degenerate arc consisting of a point of $g_i \cdot (M-H_1^*)$. These arcs do not intersect L and hence cannot disrupt $\overline{h_{1,1}}$ from $\overline{h_{1,10}}$ in $H_1^* \cdot \overline{L}$.

Now let L, g_{j+1}, $\overline{L} \cdot (\overline{h_{1,1}}) \cdot H_1^*$, $\overline{L} \cdot (\overline{h_{1,10}}) \cdot H_1^*$ and $\overline{L} \cdot (M-H_1^*)$ be the sets D, D', M, N, and E, respectively, of Bing's lemma. Since no point of L disrupts g_{j+1} from $\overline{L} \cdot (M-H_1^*)$, there is an arc from $\overline{L} \cdot (\overline{h_{1,1}}) \cdot H_1^*$ to $\overline{L} \cdot (\overline{h_{1,10}}) \cdot H_1^*$ which does not disrupt g_{j+1} from $\overline{L} \cdot (M-H_1^*)$. Therefore, there is an arc α_{j+1} from g_{j+1} to $\overline{L}\cdot(M-H_1^*)$ that does not disrupt $\overline{L}\cdot(\overline{h_{1,1}})\cdot H_1^*$ from $\overline{L}\cdot(\overline{h_{1,10}})\cdot H_1^*$ in $L+h_{1,1}+h_{1,10}$. Let L' be the component of $L-\alpha_{j+1}$ which contains an open arc from $\overline{h_{1,1}}\cdot H_1^*$ to $(\overline{h_{1,10}})\cdot H_1^*$.

Suppose $x \in (g_{j+2})\cdot(M-L')$. Let y be a point of $M-H_1^*$. Let xy be an arc joining x and y. Let y' be the first point of xy from x to y on $M-H_1^*+\alpha_{j+1}$. Then in $xy'+\alpha_{j+1}$ there is an arc from g_{j+2} to $M-H_1^*$ which does not intersect L'. Suppose $g_{j+2} \subseteq L'$. Let R be any point of L'. Then there is an arc in $M-H_1^*+L-R$ from g_{j+2} to $M-H_1^*$, and hence there is in $L'-R+(M-H_1^*)+\alpha_{j+1}$ an arc from g_{j+2} to $(M-H_1^*+\alpha_{j+1})$. Therefore no point of L' disrupts g_{j+2} from $(M-H_1^*+\alpha_{j+1})$. By an application of Bing's lemma, there is in L' an open arc joining $\overline{h_{1,1}}\cdot H_1^*$ and $\overline{h_{1,10}}\cdot H_1^*$ which does not disrupt g_{j+2} from $M-H_1^*+\alpha_{j+1}$. Thus there is in $M-H_1^*+L$ an arc α_{j+2} from g_{j+2} to $M-H_1^*$ such that $\alpha_{j+1}+\alpha_{j+2}$ does not disrupt $\overline{h_{1,1}}\cdot H_1^*$ from $\overline{h_{1,10}}\cdot H_1^*$ in $L+\overline{h_{1,1}}+\overline{h_{1,10}}$.

Continuing this process provides arcs $\alpha_{j+1},\ldots,\alpha_n$ such that $\alpha_1+\alpha_2+\ldots+\alpha_j+\alpha_{j+1}+\ldots+\alpha_n$ does not disrupt $\overline{h_{1,1}}\cdot H_1^*$ from $\overline{h_{1,10}}\cdot H_1^*$ in $L+\overline{h_{1,1}}+\overline{h_{1,t}}$. It is obvious that $\sum_1^n \alpha_i+(M-H_1^*)$ does not disrupt $h_{1,1}\cdot H_1^*$ from from $\overline{h_{1,10}}\cdot H_1^*$ in $h_{1,1}+\ldots+h_{1,t}$.

[15]Let J be an arc from $\overline{h_{1,1}}\cdot H_1^*$ to $\overline{h_{1,10}}\cdot H_1^*$ in $L+h_{1,1}+h_{1,10}$ which lies in L except for endpoints and which does not intersect K^*. Since J is connected, J intersects $h_{1,i}$ $(i=2,\ldots,9)$. Now every point of J lies in $h_{1,2}+\ldots+h_{1,9}-K^*$ and hence belongs to at least one element of G'. Since J is compact there is a finite covering. Since J intersects $\overline{h_{1,1}}$ and $\overline{h_{1,10}}$ at least one element of any open covering of J intersects $h_{1,1}$ and at least one element intersects $h_{1,10}$. Also, any covering of J by open components has for its sum a connected domain. Hence, there is at least one finite collection of elements of G' whose sum is a connected domain intersecting $h_{1,i}$ $(i=1,\ldots,10)$. Since there is one finite collection which will do this, there is a collection G'' such that the sum of the elements of G'', but the sum of no subcollection of G' with fewer elements than G'', is a connected domain intersecting $h_{1,i}$ $(i=1,\ldots,10)$.

[16]The proof that such an assumption can be made is identical, except for obvious changes in notation, to that of 10.

[17]Let x_1 be a point of $h_{1,1}\cdot g_1$ and let y_1 be a point of $g_1\cdot g_2$. Since g_1 is a connected domain, there is

an arc x_1y_1 in g_1 joining x_1 and y_1. Since $M-g_1$ is closed and x_1y_1 is closed and compact, then there is a δ_1 such that $\rho(x_1y_1) = 2\delta_1$. Now let U_{x,δ_1} be the component of $U(x,\delta_1)$ which contains x. Set g_1' equal to $\sum_{x\in x_1y_1} U_{x,\delta_1}$. Then g_1' is a connected open set whose closure is a subset of g_1. Also, g_1' intersects $h_{1,1}$ and g_2.

Now suppose that g_i' has been defined for i less than or equal to m-1 when $m \leqq r$. Let x_m belong to $g_{m-1}'\cdot g_m$ and let y_m belong to $g_m \cdot g_{m+1}$ where $g_{m+1} \equiv h_{1,10}$ if m=r. Then as before, a connected open set g_m' can be constructed such that $\overline{g_m'}$ is a subset of g_m and g_m' intersects g_{m-1}' and g_{m+1}.

[18]The proof of this statement is identical, except for notation, to 11 and 12.

[19] (1) Evident.

(2) Suppose $Z \in M-E$. Then there is a connected set N joining Z and C_1' in $D_A+C_1'-(h_1+g_{1,1}+\dots+g_{1,u}+h_{10}+h_{1,11}+\dots+h_{1,t})$. Since no point of N is separated from C_1' in D_A+C_1' by $h_1+g_{1,1}+\dots+g_{1,u}+h_{10}+h_{1,11}+\dots+h_{1,t}$, then $N\cdot E = \phi$. Therefore, Z is not separated from C_1' in M by E.

(3) Let g' be an element of G'. Then either a point of K^* is accessible from g' or else a point of $M-H_1^*$ is accessible from g'. Since $(M-H_1^*)+K^* \subseteq M-E$ then a point of $M-E$ is accessible from g'.

Let g_{1,i_0} be any element of $g_{1,1},\ldots,g_{1,u}$. Then g_{1,i_0} is made up of at least three consecutive elements g''_{j_0-1}, g''_{j_0}, g''_{j_0+1} of g''_i $(i=1,\ldots,r)$. Now by the method of construction, g''_{j_0-1} contains a subset of g_{j_0-1}, g''_{j_0} contains a subset of g_{j_0} and g''_{j_0+1} contains a subset of g_{j_0+1}. Suppose that x is a point of g''_{j_0} which belongs to g_{j_0}. Since g_{j_0} is an element of G' then there is an arc xy' in $g_{j_0}+y'$ from x to a point y' of $M-H_1^*+K^*$. Let y be the first point from x to y' of xy' which belongs to $M-E$.

Suppose that there is a point z of $xy-y$ which belongs to $E-g_{1,i_0}$. Since g_{j_0} intersects only g_{j_0-1} and g_{j_0+1}, then $xy-y$ does not contain any points of $h_{1,1}$, g'_1, $g'_2,\ldots,g'_{j_0-2}$, $g'_{j_0+2},\ldots,g'_r$, $h_{1,10},\ldots,h_{1,t}$. Thus z must be a point of a component g'' of the intersection of an element g' of G' with E such that g'' does not intersect g'_{j_0-1}, g'_{j_0} or g'_{j_0+1}. Let Z be the set of all points z of $xy-y$ which belong to $E-g_{1,i_0}$. Let z' be the first point from x to y of xy which belongs to $\overline{Z}$. Suppose z' belongs to g_{1,i_0}. Since g_{1,i_0} is open then it contains points of Z. This is a contradiction, and thus $z' \notin g_{1,i_0}$.

Then $z' \in Z$. But from above, z' belongs to some component g'' of the intersection of an element of g' of G' with E such that $g'' \cdot g_{1,i_0} = \phi$. Since g'' is open then it contains points of $xz' - z'$. But then z' is not the first element of $\bar{Z} \cdot xy$ from x to y.

Thus the assumption that some point of $xy - y$ belongs to $E - g_{1,i_0}$ leads to a contradiction. Therefore, y is a point of $M - E$ which is accessible from g_{1,i_0}. Therefore (3) is true for $g_{1,1}, \ldots, g_{1,u}$. Obviously, the condition is true for h_1, h_{10}, $h_{1,i}$ $(i = 11, \ldots, t)$.

[20]Since $g_{1,i}$ $(i_0 < i < i_1)$ and $h_{1,j}$ $(j = 11, \ldots, t)$ are open sets which do not intersect, then $\bar{g}_{1,i} \cdot h_{1,j} = \phi$. Suppose that $\bar{g}_{1,i} \cdot (M - H_1^*) \neq \phi$. Let $y \in \bar{g}_{1,i} \cdot (M - H_1^*)$. Now there exists a connected open set $U(y)$ of diameter less than $e_1/4$ containing y which does not contain any point of $\sum_1^p g_i'$ since $\overline{g_i'} \subseteq g_i \subseteq H_1^*$. Since $U(y)$ is open there is a point x belonging to $U(y) \cdot g_{1,i}$. Since $U(y)$ is connected there is an arc xy in $U(y)$ which joins x and y. Let y' be the first point of xy from x to y which belongs to $M - H_1^*$. Now consider $xy' - y'$. Since $g_{1,i}$ $(i_0 < i < i_1)$ is a subset of $h_{1,3} + \cdots + h_{1,8}$, then no connected subset of H_1^* of diameter less than $e_1/4$ joins $g_{1,i}$ and $h_{1,j}$ $(i = 1, 10, 11, \ldots, t)$. Hence, since $xy' - y'$ is a subset of H_1^*

which is of diameter less than $e_1/4$, and since $xy'-y'$ intersects $g_{1,i}$, it cannot intersect $h_{1,j}$ ($j = 1, 10, 11, \ldots, t$). Since $xy'-y'$ is a subset of $U(y)$, then it does not intersect $\sum_1^p g_i'$. But then $xy'-y'$ is a connected set which joins x and y' and does not intersect $h_{1,1}+g_1'+\ldots+g_p'+h_{1,10}+\ldots+h_{1,t}$. Therefore, x does not belong to E. But this is a contradiction. Therefore, $\overline{g_{1,i_o}}\cdot(M-H_1^*)=\phi$. Thus $\overline{g_{1,i}} \subseteq h_{1,1}+\ldots+h_{1,10}$.

[21]Note here that all of the g's have been replaced by elements of $h_{2,i}'$. Also, note that $h_{2,1}'$ is obtained in the last step of the process.

[22]The proofs of (1), (2) and (3) are the same as in footnote 19. Properties (4) and (5) follow from the method of combining the sets to form $h_{2,i}$.

Now consider property (6). Let h_{1,i_o} intersect h_{2,m_o} and let h_{1,j_o} intersect h_{2,n_o} and suppose $n_o > m_o$. Now either $(h_{1,i_o}+\ldots+h_{1,j_o})\cdot(h_{2,m_o}+\ldots+h_{2,n_o})$ contains a connected set joining h_{1,i_o} and h_{1,j_o} or else $(h_{1,j_o}+\ldots+h_{1,i_o})\cdot(h_{2,m_o}+\ldots+h_{2,n_o})$ contains a connected set joining h_{1,i_o} and h_{1,j_o}. Suppose the first is true.

Consider $H(1;i_o,j_o)$ and $H(2;m_o,n_o)$. There are elements h_{2,k_o}, $h_{2,k_1},\ldots,h_{2,k_p}$ of H_2 where h_{2,k_o} contains $h'_{2,1}$; h_{2,k_1} contains $h'_{2,r+1}$; ...; and h_{2,k_p} contains $h'_{2,n}$. Notice that $k_j+100 < k_{j+1}$. Then $\overline{h_{2,k_j}+h_{2,k_j+1}+\dots+h_{2,k_{j+1}}} \subseteq h_{1,10j+1}+\dots+h_{1,10j+31}$.

Let j_1 and j_2 be such that $h_{2,k_{j_1}}$ and $h_{2,k_{j_2}}$ are not elements of $H(2;m_o,n_o)$, but h_{2,k_j} is an element of $H(2;m_o,n_o)$ when $j_1 < j < j_2$. Then h_{2,m_o} is an element of $\{h_{2,i}\}$ when $k_{j_1} < i < k_{j_1+2}$ and h_{2,n_o} is an element of $\{h_{2,i}\}$ when $k_{j_2-2} < i < k_{j_2}$. Therefore h_{1,i_o} is an element of $h_{1,10j_1}$, $h_{1,10j_1+1}, \dots, h_{1,10j_1+42}$. Also, h_{1,j_o} is an element of $h_{1,10j_2-20}$, $h_{1,10j_2-19}$, ..., $h_{1,10j_2+22}$.

Now $\overline{H(2;m_o,n_o)} \subseteq h_{1,10j_1+1}+h_{1,10j_1+2}+\dots+h_{1,10j_2+20}$. But $h_{1,10j_1+1}+\dots+h_{1,10j_2+20} \subseteq h_{1,i_o-100}+\dots+h_{1,j_o+100}$ and thus $\overline{H(2;m_o,n_o)} \subseteq H(1;i_o,j_o)$. Similarly $\overline{H(2;n_o,m_o)} \subseteq H(1;j_o,i_o)$.

23 Let $h_{1,j}$ be any element of H_1. Then there are elements h_{1,j_o} and h_{1,j_1} such that: (1) $h_{1,j}$ is an element of h_{1,j_o+1}, $h_{1,j_o+2}, \dots, h_{1,j_1-1}$; and (2) there is a subchain H'_2 of H_2 in $h_{1,j_o+1}+\dots+h_{1,j_1-1}$ joining h_{1,j_o} and h_{1,j_1}. Since the sum of the elements of H'_2 is a connected subset of $h_{1,j_o+1}+\dots+h_{1,j_1-1}$ joining h_{1,j_o} and h_{1,j_1} then some element of H'_2 intersects $h_{1,i}$ for every i between j_o and j_1. Let h_{2,n_o} be the first element of H'_2

from h_{1,j_0} to h_{1,j_1} which intersects $h_{1,j+1}$. Let h_{2,m_0} be the first element of H_2^1 from h_{2,n_0} to h_{1,j_0} which intersects $h_{1,j-1}$. Then $h_{2,m_0}+h_{2,m_0+1}+\cdots+h_{2,n_0}$ is a connected set joining $h_{1,j-1}$ and $h_{1,j+1}$. Since the diameter of any such set is greater than or equal to $100e_2$, then the collection $h_{2,m_0}, h_{2,m_0+1},\ldots, h_{2,n_0}$ must contain at least 100 elements. Now consider h_{2,m_0+1}. This element cannot intersect $h_{1,i}$ $(i < j)$ and it cannot intersect $h_{1,i}$ $(i > j)$. Therefore it is a subset of $h_{1,j}$.

[24]Let R be any point of J_{PQ} different from P or Q. Then there exists an i_0 such that $\rho(R,P+Q) < e_{i_0}/200$. Therefore R does not belong to $h_{i_0,P_{i_0}-100}+\cdots+h_{i_0,P_{i_0}+100}$ $+h_{i_0,Q_{i_0}-100}+\cdots+h_{i_0,Q_{i_0}+100}$. But $H(i_0;P_{i_0},Q_{i_0})\cdot$ $H(i_0;Q_{i_0},P_{i_0}) = h_{i_0,P_{i_0}-100}+\cdots+h_{i_0,P_{i_0}+100}+h_{i_0,Q_{i_0}-100}$ $+\cdots+h_{i_0,Q_{i_0}+100}$. Therefore, since R belongs to $H(i_0;P_{i_0},$ $Q_{i_0})$, R does not belong to $H(i_0;Q_{i_0},P_{i_0})$. Therefore, R does not belong to J_{QP}.

[25]Let a_1b_1 be an arc with endpoints only on C_1. If a_1b_1 does not contain any subarc with endpoints only

on C_2 set a_2b_2 equal to a_1b_1. If a_1b_1 does contain a subarc with endpoints only on C_2 set a_2b_2 equal to this subarc. Then $a_2b_2 \cdot C_1 = \phi$. In either case a_2b_2 has endpoints on one element of $\{C_1, C_2\}$ and no subarc of a_2b_2 has endpoints on the other element of $\{C_1, C_2\}$.

Since π contains only a finite number of elements, there is obtained in a finite number of steps an arc a_nb_n which is a subarc of a_1b_1 with endpoints only on some element of π such that no proper subarc of a_nb_n has endpoints only on any element of π.

[26]The author evidently means the bounded domain.

AUTOBIOGRAPHY

The author, Robert L. Broussard, was born on March 9, 1924 to Mr. and Mrs. Lloyd Broussard of Edgerly, Louisiana. He attended High School at Vinton, Louisiana, where he graduated in 1940. From 1940 to 1942 he attended John McNeese Junior College at Lake Charles. In the fall of 1942 he enrolled at L.S.U. where he received his B.S. in 1944. After serving two years in the Navy, he enrolled in the L.S.U. graduate school where he is now a candidate for the degree of Ph. D.

EXAMINATION AND THESIS REPORT

Candidate: Robert Lloyd Broussard

Major Field: Mathematics

Title of Thesis: A Necessary and Sufficient Condition That a Set be Homeomorphic to the Plane Region Bounded by a Finite Number of Non-intersecting Circles.

Approved:

N. C. Rutt
Major Professor and Chairman

Richard J. Russell
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EXAMINING COMMITTEE:

Newton S. Hawley

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Date of Examination:

July 30, 1951.