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ON SOME OPTIMAL CONTROL PROBLEMS
FOR THE CENTROAFFINE GEOMETRY ON THE PLANE

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

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Abstract

A certain parametrization of substantial planar curves yields a centroaffine arclength s and a centroaffine curvature $\kappa(s)$ that remain invariant under $\mathbf{GL}(2, \mathbb{R})$ motions. In Chapter 4 we search for those substantial curves with predetermined position and velocity at the initial and terminal points, which minimize the total square curvature $\int_0^T \frac{\kappa^2(s)}{2} ds$ as κ varies over all square summable functions on each interval $[0, T]$. These curves are called centroaffine elastic curves. Thinking of the curvature κ as a control, we pose our problem as an optimal control problem over the Lie group $\mathbf{GL}(2, \mathbb{R})$ with fixed initial and terminal values but with free terminal time T . To find information about the elastic curves we apply a geometric version of the Pontryagin maximum principle. We find that the optimal κ for these curves must satisfy the third order nonlinear differential equation

$$\frac{d^3}{ds^3}\kappa = \left(\frac{3}{2}\kappa^2 - 4\epsilon\right)\frac{d}{ds}\kappa.$$

To study the nonconstant solutions of this equation we consider it as a second order conservative differential equation depending upon parameters. Using this necessary condition, numerical experiments are carried out to graph representative extremals for the case $\epsilon = 1$. We also pose the minimal centroaffine arclength problem by using the same framework. We apply the maximum principle and the generalized Legendre–Clebsch condition for optimality of singular extremals to show that the minimal centroaffine arclength problem has no solution. This improves a result by Mayer and Myller.

Motivated by the discussion of the curves with minimal centroaffine arclength, we look in Chapter 5 at an extended optimal control problem in which the centroaffine arclength is regarded as an additional control function. This problem

serves as a model for an extension of the minimal arclength problem for which unbounded nonnegative controls are allowed. In this dissertation we show that, in the absence of chattering controls, extremal trajectories for this problem are concatenations of trajectories determined by impulsive controls and null controls. We also describe the trajectories and costs associated with the null control and the impulsive controls for our dynamics.

Chapter 1. Introduction

The geometry of the centroaffine plane is obtained by the action of the group of invertible matrices $\mathbf{GL}(2, \mathbb{R})$ on \mathbb{R}^2 . A pair of centroaffine differential invariants can be singled out for a certain class of curves: the substantial curves. These invariants play the same role as the arclength and curvature functions do for the Euclidean geometry. Substantial curves can be described by the second order equation $\ddot{\mathbf{x}}(s) = -\epsilon\mathbf{x}(s) + \kappa(s)\dot{\mathbf{x}}(s)$, where \mathbf{x} is parametrized by centroaffine arclength and is linearly independent to $\dot{\mathbf{x}}$, the fixed parameter ϵ takes only the values ± 1 and determines the local concavity or convexity of the curve with respect to the origin, and κ denotes the centroaffine curvature. This equation yields a Serret–Frenet type system characterizing the curve up to a $\mathbf{GL}(2, \mathbb{R})$ or centroaffine motion.

The first problem we consider in this dissertation is that of identifying the centroaffine elastic curves, i.e., the curves with predetermined position and velocities at the initial and terminal points, minimizing the total square curvature functional $\int_0^T \frac{\kappa^2(s)}{2} ds$ as κ varies over all square summable functions on each interval $[0, T]$. Under the same initial and terminal data, we also look at the minimal centroaffine arclength problem for substantial curves.

Our key observation is that the centroaffine Serret–Frenet type equations lead to an affine control system over the four dimensional Lie group $\mathbf{GL}(2, \mathbb{R})$,

$$\frac{dg}{ds}(s) = \bar{L}_1(g(s)) + \kappa(s)\bar{L}_3(g(s)). \quad (1.1)$$

Here,

$$\bar{L}_1(g) = g \begin{pmatrix} 0 & -\epsilon \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \bar{L}_3(g) = g \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{respectively.}$$

Thus, these classical problems of the calculus of variations for this geometry can be posed as optimal control problems. In this framework, a coordinate free version of the Pontryagin maximum principle provides first order necessary conditions for optimality analogous to the Euler–Lagrange equations of the Calculus of Variations. Any curve satisfying these conditions is called an extremal. In general the Pontryagin conditions are not sufficient to establish optimality. However, they provide a much narrower class of candidates for optimality.

Mayer and Myller [17] found that the extremals for the variational problem of minimal centroaffine arclength are curves with constant centroaffine curvature and that such an extremal can be optimal for the minimal centroaffine arclength only in the case $\epsilon = 1$ and $|\kappa| < 2$. In this dissertation, our optimal control standpoint enables us to improve this result. We show in Chapter 4 that the problem of minimal centroaffine arclength has no solution.

The minimizers for the corresponding elastic problem for the Euclidean geometry on the plane were known to Euler, via the Calculus of Variations. Bryant and Griffiths [8] obtained interesting results with respect to the existence of free elastica on the hyperbolic plane H^2 by using the theory of exterior differential systems. Langer and Singer [16] used variational methods to obtain the Euler–Lagrange equation for the curvature κ of closed elastica on the sphere S^2 . They obtained existence results for this case by analyzing the solutions of this equation in terms of elliptic functions.

Geometric optimal control theory provides an unifying link between these problems. Recently, Jurdjevic [9] obtained a complete characterization of the extremals for the elastic problem over any complete, simply connected, two dimensional surface S with constant sectional curvature. He posed an optimal control problem over the three dimensional Lie group of isometries of S with the geodesic curvature as

the control function. He exploited the symmetries of the Hamiltonian system associated to the Pontryagin conditions to identify a set of independent constants of the motion that render the system completely integrable.

However, for the centroaffine elastic problem we cannot produce all the necessary constants of the motion to render the system completely integrable. In Section 4.3, using the maximum principle we easily obtain a nonlinear third order differential equation satisfied by any optimizing control κ . It is easily seen from this equation that the substantial curves of constant centroaffine curvature are also extremals for the centroaffine elastic problem. More interestingly, this equation also has nonconstant solutions. The long term behaviour of these nonconstant solutions is analyzed via phase space analysis in Section 4.4. This insight is used to produce numerically the graphs of various types of extremals for our problem in the case $\epsilon = 1$. These numerical experiments provide a glimpse at the nature of the extremals.

Our third problem, studied in Chapter 5, can be described in broad terms as follows. Motivated by the discussion of the curves with minimal centroaffine arclength, we look at an extended optimal control problem in which the centroaffine arclength is regarded as an additional control function. Namely, we consider the dynamics

$$\begin{aligned}
 \dot{x} &= v\tau, & x(0) &= x_0, & x(S) &= x_f, & (1.2) \\
 \dot{v} &= -\epsilon x\tau + \kappa v, & v(0) &= v_0, & v(S) &= v_f, \\
 \dot{y} &= w\tau, & y(0) &= y_0, & y(S) &= y_f, \\
 \dot{w} &= -\epsilon y\tau + \kappa w, & w(0) &= w_0, & w(S) &= w_f, \\
 \dot{t} &= \tau, & t(0) &= 0, & t(S) &= T,
 \end{aligned}$$

where $x_0w_0 - v_0y_0$ and $x_fw_f - v_fy_f$ are both nonzero and have the same sign. Our problem is to minimize

$$\int_0^S \alpha\tau(s) + \beta\kappa(s)ds, \quad \alpha > 0 \text{ and } \beta \geq 0, \quad (1.3)$$

over the set \mathcal{U} of ordered pairs (τ, κ) of nonnegative locally bounded functions. As will be seen, writing equation (1.1) in coordinates and taking $\tau = 1$ in the extended control system (1.2), every trajectory of (1.1) for which $\kappa \geq 0$ is also a trajectory of the extended control system. Moreover, the trajectories of this extended system associated with the choice $\tau = 0$ model the responses of the system (1.1) under an impulsive control. The reparametrization technique leading from (1.1) to this extended optimal control problem has been applied previously by Dorroh and Ferreyra to model impulsive controls in some singular problems with unbounded controls in one [4] and two dimensional [5] Euclidean spaces.

Since the corresponding Hamiltonian function for the optimal control problem (1.3)–(1.2) is linear in the controls, the necessary conditions established by the maximum principle do not completely characterize some of its extremals. These types of extremals which are not completely characterized by the Pontryagin conditions are called singular extremals. In this dissertation we show that in the absence of chattering controls, extremal trajectories for this problem are concatenations of trajectories determined by impulsive controls ($\tau = 0, \kappa = 1$) and null controls ($\tau = 1, \kappa = 0$). The main tool is the generalized Legendre–Clebsch condition for singular vector valued controls, a second order condition for optimality of singular extremals developed by Krener [12]. We also describe the trajectories associated to the null control and the impulsive controls for our dynamics.

This dissertation is structured as follows: Chapter 2 provides a self contained introduction to the subject of optimal control on manifolds. It contains a general

discussion on how to pose optimal control problems in this setting as well as a review of the Hamiltonian formalism needed to state an intrinsic, coordinate free version of the Pontryagin maximum principle. We also review the generalized Legendre–Clebsch condition as stated by Krener in [12].

Chapter 3 reviews the basic facts about the centroaffine geometry and substantial curves. We identify the privileged parametrization which leads to the centroaffine Serret–Frenet system and show its invariance under $\mathbf{GL}(2, \mathbb{R})$ motions.

Chapter 4 deals with the minimal arclength problem and the elastica problem. Here, the result by Mayer and Myller is improved by applying the coordinate free version of the maximum principle and the generalized Legendre–Clebsch condition for scalar controls developed in the second chapter. The maximum principle yields the equation describing the evolution of the extremal curvature κ for the elastica problem. The study of the differential equation associated with the optimal control function κ for the elastica problem is carried out there, as well as numerical experiments showing the graphs of representative extremals. Our results on the extended control problem are in Chapter 5.

Chapter 2. Necessary Conditions for Optimality

This chapter has three main goals. The first one is to establish the general framework on which to pose an optimal control problem on a smooth manifold. The second is to state a coordinate free version of the Pontryagin maximum principle for optimal control problems on manifolds. The Pontryagin maximum principle gives a first order necessary condition for optimality. It will be our main tool for the variational problems we will consider in this dissertation. After looking at the version of the principle stated in canonical coordinates, we shift our attention to our third goal: to state the generalized Legendre–Clebsch condition for scalar and vector controls, a second order optimality condition for singular extremals.

The definitions and general presentation used to achieve the first two goals follow closely the work by Jurdjevic in [10] and Schättler in [21]. The main source for the section dealing with the generalized Legendre–Clebsch conditions is the work of Krener [12].

2.1 Carathéodory Conditions

Definition 2.1. Let h be a function defined on some open set $\mathbf{O} \times J \subset \mathbb{R}^n \times \mathbb{R}$. Let $(x_0, t_0) \in \mathbf{O} \times J$. A solution to the initial value problem

$$\dot{x} = h(x, t), \quad x(t_0) = x_0,$$

is any absolutely continuous function x defined on an interval $I \subset J$, such that (i) $t_0 \in I$, (ii) $\dot{x}(t) = h(x(t), t)$ holds almost everywhere on I , (iii) $x(t) \in \mathbf{O}$ for all $t \in I$ and (iv) $x(t_0) = x_0$.

For a map with domain in a product space, D_1 denotes the partial derivative with respect to the first component.

Definition 2.2. Let $\mathbf{O} \times J \subset \mathbb{R}^n \times \mathbb{R}$ be open. A map $h : \mathbf{O} \times J \rightarrow \mathbb{R}^n$ satisfies the C^1 -Carathéodory conditions if

- (i) for every $t \in J$, the map $y \rightarrow h(y, t)$ is C^1 ;
- (ii) for every $y \in \mathbb{R}^n$ the maps $t \rightarrow h(y, t)$ and $t \rightarrow D_1 h(y, t)$ are measurable;
- (iii) for every compact subset $K \subset \mathbf{O}$ and every compact subinterval $I \subset J$ there exists an integrable function λ in $L^1(I)$ such that for all $t \in I$ and all $y \in K$ we have $\|h(y, t)\| + \|D_1 h(y, t)\| < \lambda(t)$.

A proof of the following theorem may be found in [18].

Theorem 2.3. If h satisfies the C^1 -Carathéodory conditions, then for any (x_0, t_0) belonging to $\mathbf{O} \times J$, the initial value problem (2.1) has a unique local solution $x(t; x_0, t_0)$. This solution is defined on a maximal open interval

$$(\tau_-(x_0, t_0), \tau_+(x_0, t_0)) \subset J.$$

The function $x(t; x_0, t_0)$ is a continuous function of all its variables on its domain

$$D = \{(t; x_0, t_0) : (x_0, t_0) \in \mathbf{O} \times J, \quad t \in (\tau_-(x_0, t_0), \tau_+(x_0, t_0))\}.$$

Furthermore, $x(t; x_0, t_0)$ is continuously differentiable in the variables (x_0, t_0) .

2.2 Manifolds

An n dimensional *topological manifold* \mathcal{M} is a second countable Hausdorff space such that for each $x \in \mathcal{M}$ there is a neighborhood U of x and a homeomorphism ϕ from U onto a subset of \mathbb{R}^n . The pair (U, ϕ) is called a *coordinate chart*. Two coordinate charts (U, ϕ) and (V, ψ) are said to be C^∞ compatible if $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are C^∞ diffeomorphisms between the open subsets $\psi(U \cap V)$ and $\phi(U \cap V)$ of \mathbb{R}^n . An *atlas* is a set of C^∞ compatible charts that covers \mathcal{M} . Two atlases are

said to be equivalent if their union is an atlas. A C^∞ *differentiable structure* on \mathcal{M} is a class of equivalent atlases. A *smooth manifold* is a topological manifold together with a C^∞ differentiable structure.

Example 2.4. \mathbb{R}^n under its usual topology has a natural differentiable structure with $(\mathbb{R}^n, \text{id})$ as its only chart. (Here id is the identity map on \mathbb{R}^n .)

Example 2.5. An open subset U of a smooth manifold M is itself a smooth manifold. The coordinate charts (V', ψ') obtained by restricting ψ to the open set $V' = V \cap U$ for each coordinate neighborhood (V, ψ) of M that intersects U provide an atlas for U , and hence define a C^∞ differentiable structure on U .

Example 2.6. Consider the group of invertible $n \times n$ matrices with real coefficients $\mathbf{GL}(n, \mathbb{R})$. The elements of $\mathbf{GL}(n, \mathbb{R})$ are all the $n \times n$ matrices with nonzero determinant. Since the set of all $n \times n$ matrices $\mathbf{M}_n(\mathbb{R})$ may be identified with \mathbb{R}^{n^2} and the determinant of a matrix is a polynomial function of its entries, $\mathbf{GL}(n, \mathbb{R})$ is the complement of the closed set $\{A \in \mathbf{M}_n(\mathbb{R}) : \det A = 0\}$. In particular, it is a smooth manifold. In fact, it is disconnected and has only two connected components $\mathbf{GL}(n, \mathbb{R})^+ = \{A \in \mathbf{M}_n(\mathbb{R}) : \det A > 0\}$ and $\mathbf{GL}(n, \mathbb{R})^- = \{A \in \mathbf{M}_n(\mathbb{R}) : \det A < 0\}$. A proof of these statements may be found in [25].

In what follows let M and N be smooth manifolds of dimensions n and m respectively.

Definition 2.7. A continuous map $F : M \rightarrow N$ is said to be *smooth* if for any coordinate chart (U, ϕ) at x on M and every chart (V, ψ) at $F(x)$ on N the map $\psi \circ F \circ \phi^{-1}$ is a C^∞ map from the open subset $\phi(U)$ of \mathbb{R}^n to the open subset $\psi(V)$ of \mathbb{R}^m .

For a point $x \in M$, let $\mathcal{C}(x)$ be the class of smooth curves σ defined on a neighborhood of zero which have $\sigma(0) = x$. Define an equivalence relation on $\mathcal{C}(x)$ by saying that two curves $\alpha(t)$ and $\beta(t)$ on $\mathcal{C}(x)$ are equivalent if

$$\frac{d}{dt}\phi \circ \alpha(t)|_{t=0} = \frac{d}{dt}\phi \circ \beta(t)|_{t=0}$$

for any chart (U, ϕ) of M with x in U . An equivalence class under this relation is called a *tangent vector* to M at the point x . A vector space structure is defined on the set of tangent vectors to M at x . This vector space, denoted by $T_x M$, is called the *tangent space to M at x* . It has the same dimension as M . The elements of $T_x M$ may be regarded as derivations. A *derivation* is a linear operator which obeys the product rule and acts on the set of smooth functions defined on a neighborhood of x . In this framework, if x_1, \dots, x_n are local coordinates on M , then $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ are regarded as a local basis for $T_x M$.

Let $F: M \rightarrow N$ be smooth. The *tangent map of F at x* is a linear map $F_{*x}: T_x M \rightarrow T_{F(x)} N$ defined by

$$F_{*x}v = \frac{d}{dt}\psi \circ F \circ \alpha(t)|_{t=0}, \text{ for any } v \in T_x M$$

where (V, ψ) is a chart of N containing $F(x)$ and the curve $\alpha(t) \in \mathcal{C}(x)$ belongs to the equivalence class determined by v . The *rank* of F at x is the *rank* of its tangent map at x . A subset S of M is an *immersed submanifold* if S is a smooth manifold such that the rank of the inclusion map at each point of S is the dimension of S .

The *tangent bundle TM* of M is the union of the tangent spaces to M , i.e., $TM = \cup_{x \in M} T_x M$. It is a smooth manifold of dimension $2n$. A point of TM is a vector v , tangent to M at some point x . If (x_1, \dots, x_n) are local coordinates on M and v_1, \dots, v_n are the components of v with respect to a basis of $T_x M$, then the $2n$ numbers $(x_1, \dots, x_n, v_1, \dots, v_n)$ give a local coordinate system on TM . The

mapping $\pi : TM \rightarrow M$ which takes a tangent vector $v \in T_x M$ to x , is called the natural projection.

Definition 2.8. A smooth map $W : M \rightarrow TM$ such that $\pi \circ W$ is the identity is called a vector field on M .

Suppose a smooth curve $x(t)$ on M satisfies

$$\frac{dx}{dt} = W \circ x, \quad (2.1)$$

where W is a vector field on M . Let f be a smooth function on a neighborhood of $x(t)$ and regard each side of the equation (2.1) as a derivation, that is

$$\frac{dx}{dt} f = W(x(t))f.$$

To express this equality in coordinates, let (U, ϕ) be a coordinate chart around the point $x(t)$ on M , let $(x_1, \dots, x_n) = \phi(U)$ be local coordinates, and let $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ be the basis of $T_{x(t)}M$. Let

$$\frac{dx_1}{dt}, \dots, \frac{dx_n}{dt}$$

be the coordinates of the vector $\frac{dx}{dt}$ and let

$$W_1(x_1, \dots, x_n), \dots, W_n(x_1, \dots, x_n)$$

be the coordinates of the vector field W with respect to the basis of $T_{x(t)}M$. Then

$$\sum_{i=1}^n \frac{dx_i}{dt} \frac{\partial}{\partial x_i} f = \sum_{i=1}^n W_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i} f$$

If we choose the coordinate function x_i as f , we get

$$\frac{dx_i}{dt} = W_i(x_1(t), \dots, x_n(t)).$$

This is the standard way of writing the system (2.1) in coordinates. It follows from the existence and uniqueness theorem for ordinary differential equations that

on a given interval J , for every initial time $t_0 \in J$ and initial point $x^0 = (x^0_1, \dots, x^0_n)$ belonging to $\phi(U)$, there is a unique maximally defined solution $x(t; x^0, t_0)$ to the initial value problem

$$\frac{dx}{dt}(t) = W(x(t)), \quad x(t_0) = x^0.$$

By patching together coordinate neighborhoods, the following concept follows.

Definition 2.9. Let x be a point of M . Let W be a vector field on M . An *integral curve of W through x* is an absolutely continuous curve $x(t)$ defined in some maximal neighborhood J of 0, that satisfies almost everywhere in J

$$\frac{dx}{dt}(t) = W(x(t)), \quad x(0) = x.$$

This integral curve will be denoted by $(\exp tW)x$.

For each fixed t in some neighborhood of 0, the map $I_t: M \rightarrow M$ given by $I_t x = (\exp tW)x$ for each $x \in M$ is a diffeomorphism. We will denote by $\{\exp tW\}$ the *one-parameter family of diffeomorphisms generated by W* .

If f is a smooth function on M and W is any vector field on M , the map $x \rightarrow W(x)(f)$ yields a smooth function on M which will be denoted by Wf . An important operation among vector fields on M is defined next.

Definition 2.10. If W and Z are vector fields on M , and f is a smooth function on M . The *Lie bracket* $[W, Z]$ is the vector field on M given by

$$[W, Z]f = Z(Wf) - W(Zf).$$

The collection $X^\infty(M)$ of all vector fields forms a Lie algebra under the operation of Lie bracket.

2.3 Optimal Control on Manifolds

From now on, let U be an arbitrary subset of \mathbb{R}^m .

Definition 2.11. A *smooth control system* on M with control set U is a map $F : M \times U \rightarrow TM$ such that

- (i) for each $u \in U$, the map $F_u : x \rightarrow F(x, u)$ is a vector field on M ; and
- (ii) for every coordinate chart (V, ϕ) of M , the map

$$F^\phi : \phi(V) \times U \rightarrow \mathbb{R}^n,$$

defined by

$$F^\phi(y, u) = \phi_{*}(\phi^{-1}(y))F(\phi^{-1}(y), u),$$

has the property that F^ϕ and all of its partial derivatives with respect to its first variable are jointly continuous on $\phi(V) \times U$.

Example 2.12. Let W_0, W_1, \dots, W_m be a finite set of vector fields on M . Then the map $A : M \times U \rightarrow M$ given by $A(x, u) = W_0(x) + \sum_{i=1}^m u_i W_i(x)$ determines a smooth control system on M . Any control system of this type is said to be *affine* in the controls. In this case, the vector field W_0 is called the *drift* and the remaining vector fields W_1, \dots, W_m are called *controlled* vector fields.

Definition 2.13. A *Lagrangian* is a function $c : M \times U \rightarrow \mathbb{R}$ such that for every coordinate chart (V, ϕ) of M , the map

$$c^\phi : \phi(V) \times U \rightarrow \mathbb{R},$$

defined by

$$c^\phi(y, u) = c(\phi^{-1}(y), u),$$

is C^1 .

Let F be a smooth control system on M and let c be a Lagrangian. The class of *admissible controls* \mathcal{U} for (c, F) is the set of all locally bounded Lebesgue measurable functions u which take values in U . Observe that on a given compact subset $I \subset \mathbb{R}$ an admissible control $u(t)$ takes values in a compact subset of U almost everywhere on I . Thus, for every coordinate chart (V, ϕ) of M the maps

$$h(y, t) = F^\phi(y, u(t)),$$

$$L(y, t) = c^\phi(y, u(t))$$

satisfy the C^1 -Carathéodory conditions. Therefore, if u is admissible for (c, F) , then it follows by patching together coordinate neighborhoods and Theorem 2.3, that on a given interval J , for every initial condition $(x_0, t_0) \in M \times J$, a unique solution $x(t; x_0, t_0)$ to the initial value problem

$$\dot{x} = F(x, u),$$

$$x(t_0) = x_0$$

on M exists, and is defined in a maximal subinterval of J containing t_0 . Moreover, this solution is continuously differentiable with respect to x_0 .

Definition 2.14. Let F be a control system on M with control set U . Let $x_0 \in M$. A *trajectory starting at x_0* generated by the admissible control $u(t)$ is the absolutely continuous curve $x(t)$ in M , defined on some interval $[0, T]$, which satisfies

$$\dot{x}(t) = F(x(t), u(t)),$$

$$x(0) = x_0.$$

The pair $(x(\cdot), u(\cdot))$ will be referred to as an *admissible input trajectory pair* for the control system F .

Definition 2.15. Let x and y be points of M . We say an admissible control $u(t)$ *steers* x to y if the corresponding trajectory starting at x has $x(T) = y$ for some nonnegative time T . We say y is *reachable* from x if there is an admissible control steering x to y . Let $\epsilon > 0$. The set of all points of M which are reachable from x in time less than or equal to ϵ will be denoted by \mathcal{R}_x^ϵ . The *reachable set* from x is defined by $\mathcal{R}_x = \cup_{\epsilon > 0} \mathcal{R}_x^\epsilon$

A control system F has the *strong accessibility property* from x if the reachable set from x has nonempty interior. Let \mathcal{F} be the collection of vector fields F_u induced by U . The *accessibility algebra* $\mathcal{L}(\mathcal{F})$ of the system F is the smallest subalgebra of $X^\infty(M)$ that contains \mathcal{F} . Define $\mathcal{L}(\mathcal{F})(x)$ to be the subspace of $T_x M$ spanned by the vectors $X(x)$, where $X \in \mathcal{L}(\mathcal{F})$. The following well known result yields a sufficient condition for strong accesibility. Its proof may be found in [10].

Proposition 2.16. If $\mathcal{L}(\mathcal{F})(x) = T_x M$, then \mathcal{R}_x^ϵ has nonempty interior for every $\epsilon > 0$. In particular, the control system F has the strong accesibility property from x .

Let $F : M \times U \rightarrow M$ be a smooth control system on M , $c : M \times U \rightarrow \mathbb{R}$ be a Lagrangian and \mathcal{U} be the set of admissible controls for (c, F) . For every admissible input trajectory pair for (c, F) defined on some interval $[0, T]$ consider

$$\Phi(x(\cdot), u(\cdot)) = \int_0^T c(x(t), u(t)) dt. \quad (2.2)$$

The functional Φ is called a *cost function*. Let S be a predetermined immersed submanifold of M which intercepts nontrivially the reachable set from x_0 . The problem

$$\min_{u(\cdot) \in \mathcal{U}} \Phi(x(\cdot), u(\cdot)) \quad (2.3)$$

subject to

$$\begin{aligned}\dot{x} &= F(x, u), \\ x(0) &= x_0, \\ x(T) &\in S,\end{aligned}\tag{2.4}$$

is called a *free-time optimal control problem with terminal conditions in S*. An optimal solution for (2.3)–(2.4) is an admissible trajectory pair $(x^*(\cdot), u^*(\cdot))$ for (c, F) defined in some interval $[0, T^*]$ such that $x^*(T^*) \in S$ and

$$\Phi(x^*(\cdot), u^*(\cdot)) \leq \Phi(x(\cdot), u(\cdot)),$$

for all admissible input trajectory pairs $(x(\cdot), u(\cdot))$ having $x(T) \in S$ for some $T > 0$.

2.4 The Hamiltonian Framework

The Pontryagin maximum principle provides a first order necessary condition for optimality for the problem (2.3)–(2.4). To formulate the principle in a geometric coordinate free setting, additional notions must be introduced.

A linear functional on $T_x M$ is called a *covector* to M at x . The space of covectors on $T_x M$ is called the *cotangent space* of M at x and is denoted by $T_x^* M$. If (x_1, \dots, x_n) is a local coordinate system at x , then dx_1, \dots, dx_n is the dual basis for $T_x^* M$ with respect to the basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ of $T_x M$. Thus, locally a covector ξ at x is written as $\xi = \sum \xi_i dx_i$.

The *cotangent bundle* $T^* M$ is the union of all the cotangent spaces to M at all of its points. It is a smooth manifold of dimension $2n$. A point on $T^* M$ is a covector ξ on the tangent space to M at some point x of M . The local coordinates of a point of $T^* M$ are $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$, where (x_1, \dots, x_n) indicate the coordinates of x and (ξ_1, \dots, ξ_n) are the coordinates of the covector ξ at x with respect to the

basis dx_1, \dots, dx_n of the cotangent space T_x^*M . The *natural projection on T^*M* is the mapping $\tilde{\pi} : T^*M \rightarrow M$ which takes a covector $\xi \in T_x^*M$ to its base point x .

Let $r > 0$ be an integer. A *differential r -form* on M is a smooth map that assigns to each $x \in M$ an r -linear antisymmetric mapping $\omega_x : \underbrace{T_x M \times \dots \times T_x M}_r \rightarrow \mathbb{R}$. A 0-form is a smooth real valued function on M . Denote by $\bigwedge^r(M)$ the collection of all r -forms on M . Let ω be an r -form and σ be an s -form. Let v_1, \dots, v_{r+s} be vector fields on M . The *exterior product* $\omega \wedge \sigma$ is the $(r + s)$ -form given by

$$\omega \wedge \sigma(v_1, \dots, v_{r+s}) = \sum_{i_1 < \dots < i_r; j_1 < \dots < j_s} (-1)^\nu \omega(v_{i_1}, \dots, v_{i_r}) \sigma(v_{j_1}, \dots, v_{j_s}),$$

where $(i_1, \dots, i_r, j_1, \dots, j_s)$ is a permutation of $(1, 2, \dots, r + s)$ and

$$\nu = \begin{cases} 1 & \text{if the permutation is odd;} \\ 0 & \text{if the permutation is even.} \end{cases}$$

In particular if ω and σ are 1-forms $\omega \wedge \sigma(v_1, v_2) = \omega(v_1)\sigma(v_2) - \sigma(v_1)\omega(v_2)$. Define $\bigwedge(M) = \cup_{r \geq 0} \bigwedge^r(M)$.

The *exterior derivative* is the map $d: \bigwedge(M) \rightarrow \bigwedge(M)$ such that

- (i) if f is a 0-form, df is the differential of f ;
- (ii) if ω is an r -form, then $d\omega$ is an $(r + 1)$ -form;
- (iii) for $\omega \in \bigwedge^r(M)$ and $\sigma \in \bigwedge^s(M)$,

$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^r \omega \wedge d\sigma; \text{ and}$$

- (iv) $d^2 \equiv 0$.

A form is said to be *closed* if its exterior derivative is zero. A form ω is *exact* if it can be written as $\omega = d\varpi$ for some form ϖ . Thus, if ω is exact it must also be closed since then $d\omega = d^2\varpi = 0$.

We refer the reader to [2] for a proof of the next proposition.

Proposition 2.17. Let θ be a 1-form. For any vector fields W_1 and W_2 on M the exterior derivative $d\theta$ is given by

$$d\theta(W_1, W_2) = W_1(\theta(W_2)) - W_2(\theta(W_1)) + \theta([W_1, W_2]).$$

A 2-form ζ is *nondegenerate* if for each x , $\zeta_x(v, w) = 0$ for all w only if $v = 0$.

Definition 2.18. A *symplectic manifold* is a smooth manifold on which a closed nondegenerate 2-form can be defined. This form is called a *symplectic form*.

The cotangent bundle has a natural symplectic structure. The natural projection $\tilde{\pi} : T^*M \rightarrow M$ defines a canonical differential 1-form θ on T^*M , given by the dual of the tangent map of $\tilde{\pi}$. Namely, for a vector $\nu \in T_\xi(T^*M)$ tangent to the cotangent bundle at the point $\xi \in T_x^*M$, define $\theta(\nu) = \xi\tilde{\pi}_*(\nu)$. To express θ in local coordinates let x_1, \dots, x_n denote the coordinates of a point x on M and let p_1, \dots, p_n denote the coordinates of a covector ξ in T_x^*M relative to the basis dx_1, \dots, dx_n . The vector ν is given by $\nu = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial p_i}$. Since $\tilde{\pi}_*(\nu) = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$, we have

$$\begin{aligned} \theta(\nu) &= \sum_{i=1}^n p_i dx_i \left(\sum_{j=1}^n a_j \frac{\partial}{\partial x_j} \right) \\ &= \sum_{i=1}^n p_i a_i. \end{aligned}$$

Let $\omega = -d\theta$. Then ω is a closed 2-form. To see that ω is nondegenerate and thus a symplectic form for T^*M we first express ω in terms of canonical coordinates.

Let

$$X_1, \dots, X_n, P_1, \dots, P_n \text{ and } Y_1, \dots, Y_n, Q_1, \dots, Q_n$$

be functions which are constant in a neighborhood of the point ξ in T_x^*M . Then the vector fields

$$X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i} + P_i \frac{\partial}{\partial p_i}$$

and

$$Y = \sum_{i=1}^n Y_i \frac{\partial}{\partial x_i} + Q_i \frac{\partial}{\partial p_i}$$

are constant vector fields on T_x^*M . It follows that $\theta(X) = \sum_{i=1}^n p_i X_i$ and $Y(\theta(X)) = \sum_{i=1}^n Q_i X_i$. Since X_i, P_i, Y_i and Q_i are constant functions $[X, Y] = 0$. So, by Proposition (2.17) we have

$$\omega(X, Y) = \sum_{i=1}^n Q_i X_i - P_i Y_i. \quad (2.5)$$

The nondegeneracy of ω follows from seeing that if X is such that $\omega(X, Y) = 0$ for all Y , then $P_i = \omega(X, \frac{\partial}{\partial x_i}) = 0$ and $Q_i = \omega(X, \frac{\partial}{\partial p_i}) = 0$. Thus, we must have that $X = 0$. In what follows, $\omega = -d\theta$ will be the symplectic form we consider on T^*M .

Vector fields on M induce functions on T^*M as follows.

Definition 2.19. Let W be a vector field on M . The *Hamiltonian lift of the vector field W* is the function $H_W : T^*M \rightarrow \mathbb{R}$ defined by $H_W(\xi) = \xi W(x)$ for each $\xi \in T_x M$.

By the nondegeneracy of the symplectic form ω , there is a one-to-one correspondence between functions on T^*M and vector fields on T^*M . This is obtained as follows.

Definition 2.20. Let H be a smooth function on T^*M . The *Hamiltonian vector field \vec{H} induced by H* assigns to each $\xi \in T^*M$ the unique $\vec{H}(\xi) \in T_\xi(T^*M)$ that satisfies $dH_\xi(\nu) = \omega(\vec{H}(\xi), \nu)$ for all $\nu \in T_\xi(T^*M)$.

Let H be a smooth function on T^*M . To obtain an expression for the components $X_1, \dots, X_n, P_1, \dots, P_n$ of the vector field \vec{H} in terms of the canonical coordinates

for T^*M , we write

$$\vec{H} = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i} + P_i \frac{\partial}{\partial p_i},$$

and let

$$Y = \sum_{i=1}^n Y_i \frac{\partial}{\partial x_i} + Q_i \frac{\partial}{\partial p_i}$$

be a vector in $T_\xi(T^*M)$. We also take $\frac{\partial H}{\partial x_1}, \dots, \frac{\partial H}{\partial x_n}, \frac{\partial H}{\partial p_1}, \dots, \frac{\partial H}{\partial p_n}$ as canonical coordinates of dH . From the definition of the Hamiltonian vector field \vec{H} and equation (2.5), we get

$$\sum_{i=1}^n \frac{\partial H}{\partial x_i} Y_i + \frac{\partial H}{\partial p_i} Q_i = \sum_{i=1}^n Q_i X_i - P_i Y_i,$$

from which it follows that

$$\begin{aligned} X_i &= \frac{\partial H}{\partial p_i}, \\ P_i &= -\frac{\partial H}{\partial x_i}. \end{aligned} \tag{2.6}$$

In particular, this means that if an integral curve $\xi(t)$ of \vec{H} is expressed in canonical coordinates, say $(x(t), p(t))$, we must have

$$\dot{x}(t) = H_p(x(t), p(t)), \tag{2.7}$$

$$\dot{p}(t) = -H_x(x(t), p(t)). \tag{2.8}$$

It is customary to call $p(t)$ the *adjoint variable* and to refer to the system of equations obtained from (2.8) as the adjoint system. An important operation among functions on T^*M is defined next.

Definition 2.21. Let F and H be smooth functions on T^*M . Denote by $\{\exp t\vec{H}\}$ the one-parameter group of diffeomorphisms generated by the Hamiltonian vector field \vec{H} . The *Poisson bracket* $\{F, H\}$ is the function on T^*M defined by $\{F, H\}(\xi) = \frac{d}{dt} F \circ \exp t\vec{H}(\xi)|_{t=0}$ for each $\xi \in T^*M$.

The following propositions provide important properties of the Poisson bracket that will be useful to us later. We refer the reader to [10] for a proof of these statements.

Proposition 2.22. The Lie bracket $[\vec{F}, \vec{H}]$ of two Hamiltonian vector fields \vec{F} and \vec{H} is a Hamiltonian vector field and $[\vec{F}, \vec{H}] = \{F, H\}$.

Proposition 2.23. If H_W and H_Z are the Hamiltonian functions on T^*M which correspond to vector fields W and Z on M then $\{H_W, H_Z\} = H_{[W, Z]}$.

As a consequence of the preceding propositions the Hamiltonian lifts of vector fields on M form an algebra under the Poisson bracket operation isomorphic to the Lie algebra of vector fields on M .

2.5 First Order Necessary Conditions

We are now almost ready to state the first order optimality conditions for the optimal control problem (2.3)–(2.4). The final step is to look at an extended system on $\mathbb{R} \times M$ obtained by considering the cost as an additional state variable

$$\dot{x}_0 = c(x, u) \quad \dot{x} = F(x, u), \quad (2.9)$$

and to carry out the following construction. Each u in the control set U defines a vector field $W_u = (c(x, u), F(x, u))$ on $\mathbb{R} \times M$. This vector field defines a Hamiltonian function H_u on the cotangent bundle $T^*(\mathbb{R} \times M)$ of $\mathbb{R} \times M$. The family of Hamiltonians $\mathcal{H} = \{H_u : u \in U\}$ is called the Hamiltonian lift of (2.9). An admissible control $u(\cdot)$ defines a time varying Hamiltonian lift $\mathcal{H}(\xi, u(t))$ with $\xi \in T^*(\mathbb{R} \times M)$. Denote by $\vec{\mathcal{H}}(\xi, u(t))$ the corresponding time varying Hamiltonian vector field. The projection of an integral curve $\xi(t)$ of $\vec{\mathcal{H}}(\xi, u(t))$ onto M is a solution to (2.9).

Notice that the cotangent bundle of $\mathbb{R} \times M$ equals $T^*\mathbb{R} \times T^*M$. Moreover, $T^*(\mathbb{R}) = \mathbb{R} \times \mathbb{R}$, so each point of $T^*\mathbb{R}$ is represented by the coordinates (x_0, p_0) relative to the basis dx_0 in $T_{x_0}^*\mathbb{R}$. Since the vector fields in (2.9) do not depend explicitly on the variable x_0 , we have that for each $\mathcal{H}(\cdot, u(t))$, the coordinate p_0 must remain constant along the integral curves of $\vec{\mathcal{H}}(\xi, u(t))$. Therefore, regarding p_0 as a parameter, the domain of \mathcal{H} can be reduced to the cotangent bundle of M . Indeed, we define

$$\mathcal{H}_{p_0}(\xi, u) = p_0 c(x, u) + \xi F(x, u), \quad \xi \in T_x^*M.$$

We refer the reader to [21] for a proof of the next theorem.

Theorem 2.24 (*The Pontryagin maximum principle*). Suppose that $x(t)$ is an optimal trajectory of the optimal control problem (2.3)–(2.4) on $[0, T]$ and let $u(t)$ be an admissible control function that generates $x(t)$. Then, there is a constant $p_0 \leq 0$ and an integral curve $\xi(t)$ of the Hamiltonian vector field $\vec{\mathcal{H}}_{p_0}(\cdot, u(t))$ defined on $[0, T]$ such that

- (i) the trajectory $x(t)$ is the projection of $\xi(t)$ onto M for all t in $[0, T]$.
Moreover, if $p_0 = 0$ then $\xi(t) \neq 0$ for all t .
- (ii) $\mathcal{H}_{p_0}(\xi(t), u(t)) = \sup_{u \in U} \mathcal{H}_{p_0}(\xi(t), u)$ for almost all $t \in [0, T]$,
- (iii) $\sup_{u \in U} \mathcal{H}_{p_0}(\xi(t), u) = 0$ for all $t \in [0, T]$, and
- (iv) the transversality condition $\xi(T)(v) = 0$ holds for all $v \in T_{x(T)}S$.

A pair $(\xi(t), u(t))$ satisfying the Pontryagin conditions is said to be an *extremal pair*. The trajectory $\xi(t)$ of an extremal pair is called an *extremal*. The extremals that are independent of the cost functional associated to the problem correspond to $p_0 = 0$ and are called *abnormal*. Extremals with $p_0 \neq 0$ are called *regular*. For regular extremals it suffices to consider only the case $p_0 = -1$.

2.6 Necessary Conditions for Optimality of Singular Extremals

Assume that our control system F is affine with a control set U which is a subset of \mathbb{R} with nonempty interior. Further assume that the Lagrangian function is linear in u . Therefore, the extended control system on $\mathbb{R} \times M$ leading to the Pontryagin maximum principle is also affine of the form $A(x, u) = W_0(x) + uW_1(x)$. Consider an extremal pair $(\xi(t), u(t))$ defined on $[0, T]$ for the optimal control problem (2.3)–(2.4). Let $x(t)$ be the projection of $\xi(t)$ onto M . In the case that $\frac{\partial}{\partial u} \mathcal{H}(\xi(t), u(t))$ vanishes identically on a subinterval (t_1, t_2) , the value of the control $u(t)$ that generates $\xi(t)$ is not completely determined by the maximization condition (Theorem 2.24, ii) of the Pontryagin principle. Extremals of this type are said to be *singular on* (t_1, t_2) . Accordingly, necessary conditions have been developed to test the optimality of singular controls. The generalized Legendre–Clebsch condition is one of them. To state it we need to introduce some additional concepts.

For vector fields W and Z defined on a manifold N , let $\text{ad}^0(W)Z = Z$ and $\text{ad}^k(W)Z = [W, \text{ad}^{k-1}(W)Z]$. Let $D^1(x(t))$ be the span of the set

$$\{\text{ad}^k(W_0)W_1(x(t)) : k = 0, \dots, \infty\}.$$

The *degree of singularity of the control* $u(t)$ is $h + 1$ if h is the smallest nonnegative integer such that $[W_1, \text{ad}^h(W_0)W_1](x(t))$ does not belong to $D^1(x(t))$. Let H_1 and $H_{\text{ad}^h(W_0)W_1}$ be the Hamiltonian lifts of the vector fields W_1 and $\text{ad}^h(W_0)W_1$ respectively. The following result has been shown by Krener in [12] under the assumption that the optimal control is C^∞ .

Theorem 2.25 (The generalized Legendre–Clebsch condition for scalar controls).

Suppose that the extended control system on $\mathbb{R} \times M$ for the optimal control problem (2.3)–(2.4) is affine of the form $A(x, u) = W_0(x) + uW_1(x)$. Let $(\xi(t), u(t))$ be a

singular extremal pair defined on (t_1, t_2) , such that $u(t)$ is in the interior of U and has degree of singularity $h + 1$ for some nonnegative integer h . If $u(t)$ is optimal, then

$$(-1)^{\frac{h+1}{2}} \{H_1, H_{\text{ad}^h(w_0)w_1}\}(\xi(t)) \leq 0 \text{ for all } t \in (t_1, t_2). \quad (2.10)$$

To consider the situation of vector valued controls let us look again at a general optimal control problem where F is not necessarily affine and the control set is a subset of \mathbb{R}^m . By restricting our attention to a coordinate neighborhood (V, ϕ) we may consider the optimal control problem defined on $\phi(V) \subset \mathbb{R}^n$ by the extended control system $f = (c^\phi, F^\phi)^T$. We will abuse notation and denote the $n + 1$ vector of state variables $(x_0, x)^T$ by x . We will also assume that the terminal condition is written as $y_i(x(T)) = 0$ where $i = 1, \dots, m$. Let p be an $n + 1$ vector and define the Hamiltonian function

$$H(x, p, u) = \langle p, f(x, u) \rangle .$$

In this context, the optimality conditions given by the maximum principle for an extremal trajectory amount to the existence of a nontrivial *adjoint* vector $p(t) = (p_0, p_1(t), \dots, p_n(t))$ with $p_0 \leq 0$ such that for almost all $t \in [0, T]$, the equations

$$\begin{aligned} \dot{x}(t) &= H_p(x(t), p(t), u(t)), \\ \dot{p}(t) &= -H_x(x(t), p(t), u(t)), \end{aligned}$$

as well as the maximization condition for the Hamiltonian

$$H(x(t), p(t), u(t)) = \sup_{u \in U} H(x(t), p(t), u) \equiv 0,$$

and the transversality conditions

$$p(T) = \sum a_i \frac{\partial}{\partial x} y_i(x(T)) + (p_0, 0, \dots, 0)$$

hold. The following concept was introduced by Krener in [12].

Definition 2.26. The control u_i is *singular* of degree $h_i + 1$ on $[t_1, t_2]$ if h_i is the smallest integer such that for some $t \in (t_1, t_2)$ there exists a $p(t)$ satisfying the adjoint equations as well as the constant and linear necessary conditions

$$H(x(t), u(t), p(t)) = 0, \quad (2.11)$$

and

$$\frac{d^j}{dt^j} \frac{\partial}{\partial u_i} H(x(t), u(t), p(t)) = 0. \quad (2.12)$$

for $j = 0, 1, \dots, \infty$ on any nontrivial subinterval of $[t_1, t_2]$ such that for some t in this interval

$$\frac{\partial}{\partial u_i} \frac{d^{h_i+1}}{dt^{h_i+1}} \frac{\partial}{\partial u_i} H(x(t), u(t), p(t)) \neq 0. \quad (2.13)$$

We end this chapter by quoting a general theorem due to Krener, [12] which provides necessary conditions for vector valued singular controls u to be optimal. We will refer to it as the generalized Legendre–Clebsch condition for vector valued controls.

Theorem 2.27. Suppose $(x(t), u(t))$ is an input trajectory pair for the optimal control problem on $[0, T]$ such that $u(t)$ is in the interior of U and each u_i is C^∞ and singular of degree $h_i + 1$ on (t_1, t_2) . If $u(t)$ is optimal, then there is an adjoint vector satisfying the Pontryagin maximum principle on $[0, T]$ such that on the interval $[t_1, t_2]$

$$\frac{\partial}{\partial u_i} \frac{d^k}{dt^k} \frac{\partial}{\partial u_j} H(x(t), u(t), p(t)) = 0. \quad (2.14)$$

for $k = 0, \dots, \frac{(h_i+h_j)}{2}; 1 \leq i, j \leq m$. Moreover, if $h_i < \infty$ for $i = 1, \dots, k \leq m$, then the $k \times k$ matrix whose i, j entry is

$$(-1)^{\frac{h_j+1}{2}} \frac{\partial}{\partial u_i} \frac{d^{\frac{h_i+h_j}{2}+1}}{dt^{\frac{h_i+h_j}{2}+1}} \frac{\partial}{\partial u_j} H(x(t), u(t), p(t)) \quad (2.15)$$

must be symmetric and nonpositive definite.

Chapter 3. Differential Invariants for Curves in the Centroaffine Plane

The arclength s and the curvature $\kappa(s)$ are familiar geometric differential invariants of a smooth curve in an Euclidean plane. Since arclength remains unchanged under reparametrization and distance preserving transformations, the natural equation $\kappa = \kappa(s)$ provides a signature for the curve from which the curve can be recovered up to an Euclidean motion. Under the arclength parametrization of the curve, the tangent and normal vectors to the curve yield a moving frame of linearly independent vectors along the curve. The evolution of this frame along the curve is governed by a system of differential equations called a Serret–Frenet system. Of course, a similar construction can be done in higher dimensional Euclidean spaces with the added complication of having to deal with more curvature functions.

The aim of this chapter is to provide a general introduction to the planar centroaffine geometry and its differential invariants. We begin with a brief general exposition on affine spaces and n dimensional centroaffine geometry. Then the class of substantial planar curves are discussed. These are the curves in the centroaffine plane for which the centroaffine differential invariants are well defined. These invariants play the same role as the arclength and curvature functions do for smooth curves in the Euclidean plane, in the sense that they characterize a smooth centroaffine curve up to a centroaffine motion. Thus, by analogy they will be referred to as centroaffine arclength and centroaffine curvature. Substantial curves can be described via a Serret–Frenet type system by parametrizing the curves with respect to this centroaffine arclength and by choosing moving frames appropriately. We conclude the chapter with a description of the curves of constant centroaffine

curvature as it appears in [17]. This well known result will be useful to us in later chapters.

The theory of invariants for the geometry of the centroaffine plane goes back to the work of Mayer and Myller [17]. Some sections of the classical book by Schirokow [22] on affine differential geometry deal with the centroaffine differential geometry of curves and surfaces in two and three dimensions. Laugwitz produced some global results for ovals on the centroaffine plane [15] as well as a monograph on the centroaffine geometry of curves and surfaces [14]. An application of the centroaffine curve theory to differential equations may be found in the book by Boruvka [3].

Recently, Gardner and Wilkens [6] developed a complete theory of differential invariants for centroaffine curves in arbitrary dimensions by using Cartan's method of moving frames. Wilkens [26] related these centroaffine invariants to feedback invariants for control systems with two states and one control.

The following survey synthesizes the basic results on these invariants in a manner agreeable to our ultimate goal, the study of variational problems for this geometry using the techniques of optimal control.

In this chapter we will regard \mathbb{R}^n as a set of column vectors. However, we will abuse notation and write $x = (x_1, \dots, x_n)$ to denote elements of \mathbb{R}^n .

Definition 3.1. A set \mathcal{A} is an n dimensional affine space, if there is an n dimensional vector space V and a function $F : \mathcal{A} \times \mathcal{A} \rightarrow V$ given by $(A, B) \xrightarrow{F} \vec{AB}$, satisfying the following conditions:

- (i) there is a $P \in \mathcal{A}$ such that for every $v \in V$, there is a unique $A \in \mathcal{A}$ with $\vec{PA} = v$;
- (ii) if $\vec{PA} = a$ and $\vec{PB} = b$, then $\vec{AB} = b - a$.

In the following, the n dimensional affine space \mathcal{A} is related to the vector space V by the function F .

Proposition 3.2. If $A \in \mathcal{A}$ and $\gamma \in V$, there is a point $B \in \mathcal{A}$ such that $\vec{AB} = \gamma$.

Proof. Let $v = \vec{PA}$. Then $\gamma + v$ is a vector in V . By the definition, there is a unique B in \mathcal{A} , such that $\vec{PB} = \gamma + v$. Therefore,

$$\vec{AB} = \vec{PB} - \vec{PA} = \gamma + v - v = \gamma. \quad (3.1)$$

□

The preceding proposition shows that any point of \mathcal{A} may play the role P plays in the definition of an affine space.

If a point \mathbf{O} is chosen as the origin for \mathcal{A} , then there is a unique correspondence between each point $M \in \mathcal{A}$ and the vector $\vec{\mathbf{O}M}$. The vector $\vec{\mathbf{O}M}$ is known as the position vector of M . A choice of a basis $\beta = \{e_1, \dots, e_n\}$ for V determines a coordinate system $\{\mathbf{O}; e_1, \dots, e_n\}$ for \mathcal{A} . Thus, we may refer to the point M by the coordinates $x = (x_1, \dots, x_n)$ of its position vector with respect to β . Since there is no preferred origin, a different choice of origin P and basis $\beta' = \{e'_1, \dots, e'_n\}$ determines a new coordinate system $\{P; e'_1, \dots, e'_n\}$ for \mathcal{A} .

Let us suppose that P has coordinates $b = (b_1, \dots, b_n)$ with respect to $\{\mathbf{O}; e_1, \dots, e_n\}$, and let A be the change of basis matrix from β' to β . Suppose the coordinates of a point \tilde{M} with respect to the coordinate system $\{P; e'_1, \dots, e'_n\}$ are $x = (x_1, \dots, x_n)$. It follows that the coordinates $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$ of \tilde{M} with respect to $\{\mathbf{O}; e_1, \dots, e_n\}$ are

$$\tilde{x} = Ax + b.$$

A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of the form

$$T(x) = Ax + b, \quad A \in \mathbf{GL}(n, \mathbb{R}), \quad b \in \mathbb{R}^n$$

is called an n dimensional *affine transformation*. From the preceding observations we see that these transformations are precisely the change of coordinate transformations between two coordinate systems in an n dimensional affine space. The set \mathcal{T} of all n dimensional affine transformations is a group. Affine geometry studies the properties of an affine space which remain invariant under subgroups of \mathcal{T} . The elements of this subgroups are referred to as the *motions* of the geometry. In this framework, the content of Euclidean geometry is obtained by allowing translations and rotations as motions.

A centered affine space \mathcal{C} is an affine space in which a preferred origin \mathbf{O} is chosen. So every point of \mathcal{C} is identified with the coordinates of its position vector in the system of coordinates determined by the basis $\{e_1, \dots, e_n\}$ of V . The centroaffine geometry is obtained by considering the subgroup of \mathcal{T} that leaves the point \mathbf{O} fixed. Therefore, an element of the centroaffine group of motions has the form

$$T(x) = Ax, \quad A \in \mathbf{GL}(n, \mathbb{R}).$$

We will refer to any such transformation as a $\mathbf{GL}(n, \mathbb{R})$ motion. The planar centroaffine geometry is obtained by letting the dimension of \mathcal{C} be two.

In the following, let \mathbb{R}_0^2 denote the set of nonzero column vectors on \mathbb{R}^2 . For \mathbf{x} and \mathbf{y} in \mathbb{R}_0^2 denote the 2×2 matrix having \mathbf{x} as first column and \mathbf{y} as second column by (\mathbf{x}, \mathbf{y}) . Let I be an open interval.

Definition 3.3. An immersed curve $\mathbf{x} : I \rightarrow \mathbb{R}_0^2$, which is at least three times differentiable, is called a *substantial curve* if

- (i) $\mathbf{x}(u)$ and $\mathbf{x}'(u)$ are linearly independent for all u in I , and
- (ii) $\mathbf{x}'(u)$ and $\mathbf{x}''(u)$ are linearly independent for all u in I .

It follows immediately from the definition that no line through the origin may be tangent to a substantial curve. A little more work is needed to establish the next statement.

Proposition 3.4. Consider a substantial curve \mathbf{x} , defined on the open interval I . Let $u_0 \in I$ and let L be the tangent line to \mathbf{x} at u_0 . Define

$$\epsilon = - \operatorname{sign} \frac{\det(\mathbf{x}''(u), \mathbf{x}'(u))}{\det(\mathbf{x}(u), \mathbf{x}'(u))}.$$

- (i) If $\epsilon = 1$, then on a neighborhood of u_0 the curve \mathbf{x} lies to the side of L that contains the origin.
- (ii) If $\epsilon = -1$, then on a neighborhood of u_0 the curve \mathbf{x} lies to the side of L that does not contain the origin.

Proof. Let $\mathbf{x}(u) = (x(u), y(u))$ be a substantial curve defined on the open interval I . Since $\mathbf{x}(u)$ and $\mathbf{x}'(u)$ are linearly independent there are unique functions $a(u)$ and $b(u)$ such that

$$\mathbf{x}''(u) = a(u)\mathbf{x}(u) + b(u)\mathbf{x}'(u). \quad (3.2)$$

The linear independence of $\mathbf{x}'(u)$ and $\mathbf{x}''(u)$ implies that $a(u)$ is nonzero on I . Moreover, the linear independence of $\mathbf{x}(u)$ and $\mathbf{x}'(u)$ implies that

$$a(u) = \frac{\det(\mathbf{x}''(u), \mathbf{x}'(u))}{\det(\mathbf{x}(u), \mathbf{x}'(u))}, \quad (3.3)$$

so that a is continuous on I . Since $a(u)$ is a nonvanishing continuous function on I , its sign must remain the same along I . Thus, $\epsilon = -1$ if and only if a is positive on I and $\epsilon = 1$ if and only if a is negative.

Let $u_0 \in I$. Write $\mathbf{x}(u_0) = (x_0, y_0)$ and $\mathbf{x}'(u_0) = (x'_0, y'_0)$. The equation of the line L tangent to \mathbf{x} at u_0 is given by

$$xy'_0 - yx'_0 = x_0y'_0 - y_0x'_0.$$

Let

$$L^- = \{(x, y) : xy'_0 - yx'_0 < x_0y'_0 - y_0x'_0\}$$

and

$$L^+ = \{(x, y) : xy'_0 - yx'_0 > x_0y'_0 - y_0x'_0\}.$$

Since no line through the origin may be tangent to a substantial curve, it follows that the origin belongs to only one of these sets. Define

$$F(u) = x(u)y'_0 - y(u)x'_0. \quad (3.4)$$

The linear independence of $\mathbf{x}(u_0)$ and $\mathbf{x}'(u_0)$ makes $F(u_0) \neq 0$. In particular, the origin lies on L^- if and only if $F(u_0) > 0$. Also note that $F'(u_0) = 0$. Since

$$\begin{aligned} F''(u) &= x''(u)y'_0 - y''(u)x'_0 \\ &= [a(u)x(u) + b(u)x'(u)]y'_0 - [a(u)y(u) + b(u)y'(u)]x'_0 \\ &= a(u)F(u) + b(u)F'(u), \end{aligned}$$

it follows that $F''(u_0) = a(u_0)F(u_0)$. Assume first that $F(u_0) > 0$. If $\epsilon = 1$, then F has a local maximum at u_0 . Thus, there is a neighborhood N of u_0 where

$$x(u)y'_0 - y(u)x'_0 \leq x_0y'_0 - x'_0y_0 \quad (3.5)$$

holds. Hence $(x(u), y(u))$ lies on L^- as long as u belongs to $N \setminus \{u_0\}$. So the curve lies to the side of L that contains the origin.

If $\epsilon = -1$, then F has a local minimum at u_0 . It follows that on a neighborhood N of u_0 the inequality

$$x(u)y'_0 - y(u)x'_0 \geq x_0y'_0 - x'_0y_0 \quad (3.6)$$

holds, and $(x(u), y(u))$ lies on L^+ , the side of L that does not contain the origin, as long as u belongs to $N \setminus \{u_0\}$. To complete the proof observe that in the case

that $F(u_0) < 0$, the origin lies on L^+ and the function F attains a local minimum at u_0 if $\epsilon = 1$ and a local maximum if $\epsilon = -1$. \square

The substantial curves stand out as the class of planar curves for which the notions of centroaffine arclength and centroaffine curvature may be defined. The next theorem makes this explicit. A proof based on Cartan's method of moving frames appears in [26].

Theorem 3.5 (Mayer and Myller [17]). Let $\mathbf{x} : I \rightarrow \mathbb{R}_0^2$ be a substantial curve parametrized by u . The curve \mathbf{x} may be reparametrized with respect to a $\mathbf{GL}(2, \mathbb{R})$ invariant parameter s . Denote the reparametrized curve by $\mathbf{x}(s)$ and differentiation with respect to the parameter s by a dot. The vectors $\mathbf{x}(s)$ and $\dot{\mathbf{x}}(s)$ form a moving frame satisfying the Serret–Frenet type equations

$$\dot{\mathbf{x}}(s) = \mathbf{y}(s), \quad (3.7)$$

$$\dot{\mathbf{y}}(s) = -\epsilon \mathbf{x}(s) + \kappa(s) \mathbf{y}(s),$$

where

$$\kappa = \frac{d}{ds} \ln |\det(\mathbf{x}, \dot{\mathbf{x}})|. \quad (3.8)$$

If the curve is parametrized with respect to s , then the coefficients ϵ and $\kappa(s)$ are also $\mathbf{GL}(2, \mathbb{R})$ invariant and determine the curve up to a $\mathbf{GL}(2, \mathbb{R})$ motion.

Proof. Let $\mathbf{x} : I \rightarrow \mathbb{R}_0^2$ be a substantial curve. Let A be any element of $\mathbf{GL}(2, \mathbb{R})$. Define $\mathbf{x}_1(u) = A\mathbf{x}(u)$. We will use a prime to denote differentiation with respect to u . Since

$$(\mathbf{x}_1(u), \mathbf{x}'_1(u)) = (A\mathbf{x}(u), A\mathbf{x}'(u)) = A(\mathbf{x}(u), \mathbf{x}'(u)), \quad (3.9)$$

it follows that $\det(\mathbf{x}_1(u), \mathbf{x}'_1(u)) = \det A \det(\mathbf{x}(u), \mathbf{x}'(u))$. We also have

$$(\mathbf{x}'_1(u), \mathbf{x}''_1(u)) = A(\mathbf{x}'(u), \mathbf{x}''(u))$$

by differentiating both sides of (3.9) with respect to u . Therefore,

$$\det(\mathbf{x}'_1(u), \mathbf{x}''_1(u)) = \det A \det(\mathbf{x}'(u), \mathbf{x}''(u)).$$

Hence,

$$\frac{\det(\mathbf{x}'_1(u), \mathbf{x}''_1(u))}{\det(\mathbf{x}_1(u), \mathbf{x}'_1(u))} = \frac{\det(\mathbf{x}'(u), \mathbf{x}''(u))}{\det(\mathbf{x}(u), \mathbf{x}'(u))}. \quad (3.10)$$

Let J be an open interval and $\phi: J \rightarrow I$ a diffeomorphism. Also, define $\hat{\mathbf{x}}(t) = \mathbf{x}(\phi(t))$. Repeated differentiation with respect to t yields

$$\begin{aligned} \frac{d}{dt} \hat{\mathbf{x}}(t) &= \mathbf{x}'(\phi(t)) \frac{d}{dt} \phi(t), \\ \frac{d^2}{dt^2} \hat{\mathbf{x}}(t) &= \mathbf{x}'(\phi(t)) \frac{d^2}{dt^2} \phi(t) + \mathbf{x}''(\phi(t)) \left(\frac{d}{dt} \phi(t) \right)^2. \end{aligned}$$

It follows that $\hat{\mathbf{x}}(t)$ is substantial since

$$\begin{aligned} \det(\hat{\mathbf{x}}(t), \frac{d}{dt} \hat{\mathbf{x}}(t)) &= \frac{d}{dt} \phi(t) \det(\mathbf{x}(\phi(t)), \mathbf{x}'(\phi(t))), \text{ and} \\ \det\left(\frac{d}{dt} \hat{\mathbf{x}}(t), \frac{d^2}{dt^2} \hat{\mathbf{x}}(t)\right) &= \left(\frac{d}{dt} \phi(t)\right)^3 \det(\mathbf{x}'(\phi(t)), \mathbf{x}''(\phi(t))) \end{aligned}$$

are both nonzero. Moreover,

$$\frac{\det\left(\frac{d}{dt} \hat{\mathbf{x}}(t), \frac{d^2}{dt^2} \hat{\mathbf{x}}(t)\right)}{\det(\hat{\mathbf{x}}(t), \frac{d}{dt} \hat{\mathbf{x}}(t))} = \frac{\left(\frac{d}{dt} \phi(t)\right)^2 \det(\mathbf{x}'(\phi(t)), \mathbf{x}''(\phi(t)))}{\det(\mathbf{x}(\phi(t)), \mathbf{x}'(\phi(t)))}. \quad (3.11)$$

As a consequence of equations (3.10) and (3.11) the value of ϵ is invariant under $\mathbf{GL}(2, \mathbb{R})$ motions as well as under reparametrizations of the curve. Moreover, for any $u_0, u_1 \in I$, the change of variables formula yields

$$\begin{aligned} \int_{u_0}^{u_1} \sqrt{\epsilon \frac{\det(\mathbf{x}'(u), \mathbf{x}''(u))}{\det(\mathbf{x}(u), \mathbf{x}'(u))}} du &= \int_{t_0=\phi^{-1}(u_0)}^{t_1=\phi^{-1}(u_1)} \sqrt{\epsilon \frac{\det(\mathbf{x}'(\phi(t)), \mathbf{x}''(\phi(t)))}{\det(\mathbf{x}(\phi(t)), \mathbf{x}'(\phi(t)))}} \frac{d}{dt} \phi(t) dt \\ &= \int_{t_0}^{t_1} \sqrt{\epsilon \frac{\det\left(\frac{d}{dt} \hat{\mathbf{x}}(t), \frac{d^2}{dt^2} \hat{\mathbf{x}}(t)\right)}{\det(\hat{\mathbf{x}}(t), \frac{d}{dt} \hat{\mathbf{x}}(t))}} dt. \end{aligned} \quad (3.12)$$

Equations (3.10) and (3.12) imply that

$$s = s(u) = \int_{u_0}^u \sqrt{\epsilon \frac{\det(\mathbf{x}'(v), \mathbf{x}''(v))}{\det(\mathbf{x}(v), \mathbf{x}'(v))}} dv \quad (3.13)$$

is also an invariant with respect to $\mathbf{GL}(2, \mathbb{R})$ motions as well as to reparametrization.

The parameter s is precisely the parameter needed for the coefficient of \mathbf{x} to be ± 1 when $\ddot{\mathbf{x}}$ is written as a linear combination of the independent vectors \mathbf{x} and $\dot{\mathbf{x}}$. This can be seen as follows. Since $\mathbf{x}(u)$ and $\mathbf{x}'(u)$ are linearly independent there are unique functions $a(u)$ and $b(u)$ such that

$$\mathbf{x}''(u) = a(u)\mathbf{x}(u) + b(u)\mathbf{x}'(u).$$

Consider a reparametrization $u = u(s)$ with $\dot{u} > 0$. Then

$$\begin{aligned} \ddot{\mathbf{x}}(s) &= \mathbf{x}'(u(s))\ddot{u} + \mathbf{x}''(u(s))\dot{u}^2 \\ &= \mathbf{x}'(u(s))\ddot{u} + [a(u(s))\mathbf{x}(u(s)) + b(u(s))\mathbf{x}'(u(s))]\dot{u}^2 \\ &= a(u(s))\dot{u}^2\mathbf{x}(u(s)) + [b(u(s))\dot{u}^2 + \ddot{u}]\mathbf{x}'(u(s)) \\ &= a(u(s))\dot{u}^2\mathbf{x}(s) + \left[\frac{b(u(s))\dot{u}^2 + \ddot{u}}{\dot{u}} \right] \dot{\mathbf{x}}(s). \end{aligned}$$

From equations (3.3) and (3.13) we obtain $\frac{ds}{du} = \sqrt{-\epsilon a(u(s))}$. So,

$$\dot{u}^2 = \frac{1}{-\epsilon a(u(s))}.$$

It is clear now that $a(u(s))\dot{u}^2 = -\epsilon$, since $|\epsilon| = 1$. Define

$$\kappa(s) = \frac{b(u(s))\dot{u}^2 + \ddot{u}}{\dot{u}}$$

and write

$$\ddot{\mathbf{x}}(s) = -\epsilon\mathbf{x}(s) + \kappa(s)\dot{\mathbf{x}}(s). \quad (3.14)$$

Since $\mathbf{x}(s)$ is substantial the vectors $\mathbf{x}(s)$ and $\mathbf{y}(s) = \dot{\mathbf{x}}(s)$ are linearly independent, so they determine a moving frame for the curve. Writing (3.14) as a first order system we obtain the Serret–Frenet system (3.7).

From equation (3.14) we now see that (3.8) follows

$$\begin{aligned}\kappa &= \frac{\det(\mathbf{x}, \ddot{\mathbf{x}})}{\det(\mathbf{x}, \dot{\mathbf{x}})} \\ &= \frac{d}{ds} \ln |\det(\mathbf{x}, \dot{\mathbf{x}})|.\end{aligned}\tag{3.15}$$

The invariance with respect to $\mathbf{GL}(2, \mathbb{R})$ motions of κ follows from (3.15) by an argument similar to the one used to establish the $\mathbf{GL}(2, \mathbb{R})$ invariance of ϵ .

To see that ϵ and κ determine a substantial curve up to a $\mathbf{GL}(2, \mathbb{R})$ motion consider an arbitrary element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathbf{GL}(2, \mathbb{R})$. The existence and uniqueness theorem guarantees that the scalar differential equation

$$\ddot{x}(s) = -\epsilon x(s) + \kappa(s)\dot{x}(s)$$

has solutions $x_1(s)$ and $x_2(s)$ satisfying the initial conditions $x_1(0) = a$, $\dot{x}_1(0) = c$ and $x_2(0) = b$, $\dot{x}_2(0) = d$ respectively. Let I_1 and I_2 the maximal neighborhoods of zero on which each solution is defined. Let $J = I_1 \cap I_2$. Since the Wronskian at $s = 0$ is $ad - bc \neq 0$, the solutions $x_1(s)$ and $x_2(s)$ are linearly independent on J . Define $\mathbf{x} = (x_1, x_2)$. Clearly, \mathbf{x} is determined up to a $\mathbf{GL}(2, \mathbb{R})$ motion and satisfies (3.14). Since $\det(\mathbf{x}(s), \dot{\mathbf{x}}(s)) = x_1(s)\dot{x}_2(s) - x_2(s)\dot{x}_1(s) \neq 0$ and $\det(\dot{\mathbf{x}}(s), \ddot{\mathbf{x}}(s)) = -\epsilon \det(\mathbf{x}(s), \dot{\mathbf{x}}(s))$ it follows that $\mathbf{x}(s)$ is substantial on J . \square

The *order of a differential invariant* is the order of the highest derivative that occurs in the local expression for it. The fact that ds is an invariant of the lowest possible order enables us to refer to s as the *centroaffine arclength* parameter. Since given ϵ the coefficient $\kappa(s)$ determines the curve up to a centroaffine motion we will refer to $\kappa(s)$ as the *centroaffine curvature*.

The following representation of substantial curves will be crucial for our work in later chapters, where we formulate several variational geometric problems as control problems over Lie groups.

Corollary 3.6. *The frames $\mathbf{x}(s)$ and $\dot{\mathbf{x}}(s)$ of a substantial curve parametrized by centroaffine arclength determine a curve $g(s) = (\mathbf{x}(s), \dot{\mathbf{x}}(s))$ in $\mathbf{GL}(2, \mathbb{R})$ which satisfies the matrix differential equation*

$$\dot{g}(s) = g(s) \begin{pmatrix} 0 & -\epsilon \\ 1 & \kappa(s) \end{pmatrix}. \quad (3.16)$$

Proof. The result follows immediately by differentiating $g(s)$ and using the Serret-Frenet type equations (3.7). \square

To conclude this introductory chapter we review the work of Mayer and Myller [17] who classified the centroaffine curves of constant centroaffine curvature $\kappa(s) \equiv \kappa$ into the following four types.

(i) If $\epsilon = -1$, then

$$x(s) = \exp(ms), \quad y(s) = \exp\left(-\frac{s}{m}\right),$$

where $m^2 - \kappa m - 1 = 0$. Up to a centroaffine motion the curve $\mathbf{x}(s) = (x(s), y(s))$ lies on the graph of

$$xy^{m^2} = 1.$$

Note that when $\kappa = 0$ this is a hyperbola.

(ii) If $\epsilon = +1$ and $|\kappa| > 2$, then

$$x(s) = \exp(ms), \quad y(s) = \exp\left(\frac{s}{m}\right),$$

where $m^2 - \kappa m - 1 = 0$. Up to a centroaffine motion the curve $\mathbf{x}(s) = (x(s), y(s))$ lies on the graph of

$$x = y^{m^2}.$$

(iii) If $\epsilon = +1$ and $|\kappa| < 2$, then

$$x(s) = \exp(ms) \cos(ns), \quad y(s) = \exp(ms) \sin(ns), \quad \text{where}$$

$$m = \frac{\kappa}{2} \quad \text{and} \quad n = \sqrt{1 - \frac{\kappa^2}{4}}.$$

So up to a centroaffine motion the curve $\mathbf{x}(s) = (x(s), y(s))$ lies on the graph of the logarithmic spiral with center at the origin and equation

$$\log(x^2 + y^2) = \frac{m}{n} \tan^{-1} \frac{y}{x}.$$

Note that when $\kappa = 0$ this is a circle with center at the origin.

(iv) If $\epsilon = +1$ and $\kappa = \pm 2$, then

$$x(s) = \exp s, \quad y(s) = s \exp s.$$

Up to a centroaffine motion the curve $\mathbf{x}(s) = (x(s), y(s))$ lies on the graph of

$$y = x \log x.$$

These curves are obtained by solving the second order equation (3.14) with $\kappa(s) \equiv \kappa$ and choosing the initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ such as to simplify the equations of the curve.

Chapter 4. Two Variational Problems for Substantial Curves

In this chapter we deal with the problem of finding the substantial curves with predetermined position and velocity at the initial and terminal points, with minimal centroaffine arclength. Under the same data for initial and terminal points we are also interested in the centroaffine elastic problem. This is the problem of searching for the substantial curves minimizing the total square curvature functional $\int_0^T \frac{\kappa^2(s)}{2} ds$, as κ varies over all square summable functions on each interval $[0, T]$.

Our first crucial observation is that $\mathbf{GL}(2, \mathbb{R})$ serves as a group of isometries for the centroaffine plane, and that Corollary 3.6 naturally suggests an affine control system on this group. This allows us to study the above variational geometric problems as optimal control problems over the group of isometries on the centroaffine plane with the curvature $\kappa(s)$ as the control function. Once the variational problems are set up as problems on the Lie group $\mathbf{GL}(2, \mathbb{R})$ we apply the maximum principle as done by Jurdjevic in [9] for other geometric problems.

For the minimal centroaffine arclength problem this leads immediately to the realization that extremal trajectories are the curves of constant curvature. But to obtain further results we go further and use the generalized Legendre–Clebsch condition. Using these tools we obtain our first striking result. Unlike the analogue minimal arclength problem in the Euclidean plane, the minimal centroaffine arclength problem has no solution in the centroaffine plane. This result goes beyond the results by Mayer and Myller [17] who proved that the extremals for the variational problem of the centroaffine arclength are the curves of constant curvature

and that a minimum is not possible in the case that $\epsilon = -1$ or the case that $\epsilon = 1$ and $|\kappa| \geq 2$.

For the centroaffine elastic problem, aided by the maximum principle we prove that an optimizing control κ must satisfy the nonlinear third order differential equation

$$\frac{d^3}{ds^3}\kappa = \left(\frac{3}{2}\kappa^2 - 4\epsilon\right)\frac{d}{ds}\kappa. \quad (4.1)$$

It is then easy to see from this equation that the curves of constant curvature are also extremals for the centroaffine elastic problem. Unfortunately, it appears that this nonlinear ordinary differential equation cannot be solved in closed form. To study the nonconstant solutions of this equation we consider it as a second order conservative differential equation depending upon parameters. Then nonconstant solutions may be classified in terms of their qualitative behaviour.

We used numerical routines to produce solutions to (4.1). In principle, the graphs of the various types of extremals for the centroaffine elastic problem may be obtained by plugging these solutions into the Serret–Frenet centroaffine equations. We follow this program in the case $\epsilon = 1$ and obtain numerically the graphs of the extremals corresponding to each type of qualitative behaviour of the solution to (4.1).

4.1 Controlled Dynamics over $\mathbf{GL}(2, \mathbb{R})$

Let $\epsilon = \pm 1$ and let \vec{L}_1 and \vec{L}_3 be vector fields on $\mathbf{GL}(2, \mathbb{R})^+$ such that for $g \in \mathbf{GL}(2, \mathbb{R})^+$

$$\vec{L}_1(g) = g \begin{pmatrix} 0 & -\epsilon \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \vec{L}_3(g) = g \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{respectively.}$$

Define the vector fields

$$\begin{aligned}\vec{L}_2(g) = [\vec{L}_1, \vec{L}_3](g) &= g \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix} \text{ and} \\ \vec{L}_4(g) = [\vec{L}_1, \vec{L}_2](g) &= g \begin{pmatrix} 2\epsilon & 0 \\ 0 & -2\epsilon \end{pmatrix}.\end{aligned}$$

The Lie bracket relations among the vector fields \vec{L}_i can be summarized by Table 4.1 in which the first entry for $[\cdot, \cdot]$ is taken from the first row and the second entry for $[\cdot, \cdot]$ is taken from the first column.

TABLE 4.1. Lie bracket relations.

$[\cdot, \cdot]$	\vec{L}_1	\vec{L}_2	\vec{L}_3	\vec{L}_4
\vec{L}_1	0	\vec{L}_4	\vec{L}_2	$-4\epsilon\vec{L}_2$
\vec{L}_2	$-\vec{L}_4$	0	\vec{L}_1	$-4\epsilon\vec{L}_1$
\vec{L}_3	$-\vec{L}_2$	$-\vec{L}_1$	0	0
\vec{L}_4	$4\epsilon\vec{L}_2$	$4\epsilon\vec{L}_1$	0	0

Proposition 4.1. Consider the affine control system $F : \mathbf{GL}(2, \mathbb{R})^+ \times \mathbb{R} \rightarrow T\mathbf{GL}(2, \mathbb{R})^+$ given by

$$F(g, \kappa) = \vec{L}_1(g) + \kappa\vec{L}_3(g). \quad (4.2)$$

For every $g \in \mathbf{GL}(2, \mathbb{R})^+$ the control system F has the strong accessibility property from g .

Proof. Since the matrices

$$\begin{aligned}L_1 &= \begin{pmatrix} 0 & -\epsilon \\ 1 & 0 \end{pmatrix}, & L_2 &= \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}, \\ L_3 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & \text{and } L_4 &= \begin{pmatrix} 2\epsilon & 0 \\ 0 & -2\epsilon \end{pmatrix}\end{aligned}$$

form a basis for the vector space of 2×2 matrices over \mathbb{R} , the corresponding vector fields form a basis for $T_g \mathbf{GL}(2, \mathbb{R})^+$ for every $g \in \mathbf{GL}(2, \mathbb{R})$. It follows from Proposition 2.16 that the smooth control system F has the strong accessibility property. \square

4.2 The Optimal Control Problem on $\mathbf{GL}(2, \mathbb{R})$

Let A, B be arbitrary elements of $\mathbf{GL}(2, \mathbb{R})$. Assume that the smooth control system (4.2) has the property that B is reachable from A , and denote by $h(s)$ the trajectory joining A and B generated by a control $\kappa(s)$. Then $g(s) = h(0)^{-1}h(s)$ joins the identity element I to $A^{-1}B$ by means of the same control. This forces $A^{-1}B$ to lie in $\mathbf{GL}(2, \mathbb{R})^+$, the connected component of $\mathbf{GL}(2, \mathbb{R})$ containing the identity matrix I . Since the trajectories $g(s)$ and $h(s)$ are both generated by the same control, then for any Lagrangian f depending only on the control set, the cost functional $\int_0^T f(\kappa(s))ds$ will have the same value for both trajectories. So, without loss of generality we will assume that the initial data is the identity element $I \in \mathbf{GL}(2, \mathbb{R})$ and that the terminal point is a given matrix C with positive determinant that is reachable from the identity. Thus, we consider the optimal control problem [OCP] :

$$\min \int_0^T f(\kappa(s))ds,$$

over all locally bounded measurable functions κ , such that

$$\frac{dg}{ds} = \bar{L}_1(g(s)) + \kappa(s)\bar{L}_3(g(s)), \quad (4.3)$$

has a solution $g(s) \in \mathbf{GL}(2, \mathbb{R})^+$ satisfying $g(0) = I$ and $g(T) = C$, for some $T \geq 0$.

As an example, consider the case where $\epsilon = 1$, the Lagrangian is $f(\kappa) = \frac{\kappa^2}{2}$ and the terminal state is $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Notice that for this case, we can steer the system from the identity to $g(T) = C$ with no cost in time $T = \frac{\pi}{2}$ by taking $\kappa = 0$.

This leads to a substantial curve which minimizes the elastic energy. Integrating the Serret–Frenet equations with $\kappa = 0$, we see that the curve must be an arc lying on the unit circle. On the other hand, since $\mathbf{GL}(2, \mathbb{R})^+$ is a Lie group, the integral curve $h(s)$ through I of the vector field $\vec{L}_1 + \vec{L}_3$ is defined for all $s \in \mathbb{R}$. In other words $h(s)$ is a solution of the system (4.3) with $\kappa = 1$ and $h(0) = I$. Clearly $h(\frac{\pi}{2})$ is reachable from the identity. The elastic energy of this configuration is $\frac{\pi}{4}$. The natural question is: what necessary conditions must hold for a curve joining the states $h(0)$ and $h(\frac{\pi}{2})$ to have minimal elastic energy? As we will see in the next section, the maximum principle will be useful for this purpose.

However, let us first observe that any solution for the problem [OCP] that renders the optimal control to be almost everywhere continuous yields a substantial curve which minimizes the corresponding functional over the class of substantial curves. This is a consequence of the following:

Lemma 4.2. If $\kappa(s)$ is an admissible control for the optimal control problem [OCP] that agrees almost everywhere with a continuously differentiable function $\tilde{\kappa}$ on $[0, T]$, then the trajectory of (4.2) generated by $\kappa(s)$ determines a substantial curve on $[0, T]$.

Proof. Observe that a trajectory $g(s)$ generated by $\kappa(s)$ is an integral curve of the time varying vector field $F(\cdot, \kappa(s))$, thus it must be an absolutely continuous solution of

$$\begin{aligned} \frac{dg}{ds} &= \vec{L}_1(g(s)) + \kappa(s)\vec{L}_3(g(s)), \\ g(0) &= I. \end{aligned} \tag{4.4}$$

So, it must satisfy

$$\dot{g}(s) = g(s) \begin{pmatrix} 0 & -\epsilon \\ 1 & \kappa(s) \end{pmatrix}. \quad (4.5)$$

Write $g(s) = \begin{pmatrix} x(s) & v(s) \\ y(s) & w(s) \end{pmatrix}$. The functions x , v , y and w must be absolutely continuous with

$$\dot{x} = v \quad \dot{v} = -\epsilon x + \kappa v \quad (4.6)$$

$$\dot{y} = w \quad \dot{w} = -\epsilon y + \kappa w$$

almost everywhere on $[0, T]$ and must satisfy the linear independence condition

$$xw - yv > 0. \quad (4.7)$$

Since v is absolutely continuous, x is continuously differentiable. Moreover, the absolute continuity of v yields, that for any $s \in [0, T]$,

$$\begin{aligned} v(s) &= -\epsilon \int_0^s x(t) dt + \int_0^s \kappa(t) v(t) dt \\ &= -\epsilon \int_0^s x(t) dt + \int_0^s \bar{\kappa}(t) v(t) dt. \end{aligned}$$

Therefore, $\dot{v} = -\epsilon x + \bar{\kappa}v$ on $[0, T]$. A similar argument shows that the equations for y and w are satisfied everywhere on $[0, T]$, when we replace κ by $\bar{\kappa}$. It follows that the curve $\mathbf{x}(s) = (x(s), y(s))$ is C^2 and $\det(\mathbf{x}(s), \dot{\mathbf{x}}(s)) = \det g(s) > 0$. Moreover, \mathbf{x} satisfies the equation

$$\ddot{\mathbf{x}}(s) = -\epsilon \mathbf{x}(s) + \bar{\kappa}(s) \dot{\mathbf{x}}(s), \quad (4.8)$$

from which we see at once that $\ddot{\mathbf{x}}(s)$ is differentiable and that

$$\det(\ddot{\mathbf{x}}(s), \dot{\mathbf{x}}(s)) = -\epsilon \det(\mathbf{x}(s), \dot{\mathbf{x}}(s)) \neq 0.$$

Observe that if we write equation (4.8) as a first order system, we obtain our Serret–Frenet type equations. Thus, the curve is parametrized by centroaffine arclength and has centroaffine curvature $\bar{\kappa}$. \square

The following example illustrates that the choice of the function f is important for the nontriviality of the control problem. Let \mathbf{e}_1 and \mathbf{e}_2 be the standard basis vectors for \mathbb{R}^2 . Let κ be any admissible control that steers I into C in time T . Let $g(s)$ be its trajectory. Write $\mathbf{x}(s) = g(s)\mathbf{e}_1$. Then $\ddot{\mathbf{x}}(s) = -\epsilon\mathbf{x}(s) + \kappa(s)\dot{\mathbf{x}}(s)$ almost everywhere on $[0, T]$. It follows that

$$\begin{aligned}\kappa &= \frac{\det(\mathbf{x}, \ddot{\mathbf{x}})}{\det(\mathbf{x}, \dot{\mathbf{x}})} \\ &= \frac{d}{ds} \ln \det g\end{aligned}$$

almost everywhere in $[0, T]$. Thus,

$$\int_0^T \kappa(s) ds = \ln \det g(T) = \ln \det C. \quad (4.9)$$

So that, if we take $f(\kappa) = \kappa$ we see that the cost is the same for any trajectory that joins I to C .

4.3 Main Results

For each $p \in \mathbb{R}$ we define the Hamiltonian function

$$H^p(\xi, \kappa) = pf(\kappa) + H_1(\xi) + \kappa H_3(\xi).$$

Here, H_i is the Hamiltonian lift of the left invariant vector fields \vec{L}_i . The H_i 's satisfy a Poisson bracket table isomorphic to Table 4.1. The case where $f(\kappa) = 1$ corresponds to the minimal centroaffine arclength problem. The choice $f(\kappa) = \frac{\kappa^2}{2}$ yields the Hamiltonian for the centroaffine elastic problem. We now state the first consequence of the Pontryagin maximum principle for our problem.

Proposition 4.3. The problem [OCP] has no abnormal extremals.

Proof. Let $\xi(s)$ be an abnormal extremal. Then $\xi(s)$ is an integral curve of $\vec{H}^0 = \vec{H}_1 + \kappa(s)\vec{H}_3$ for some function $\kappa(s)$. By maximixing the Hamiltonian with respect to κ we have the condition $H_3(\xi(s)) = 0$. Since $\xi(s)$ is the integral curve of the Hamiltonian vector field \vec{H}^0 , the definition of the Poisson bracket yields $\frac{d}{ds}H_3(\xi(s)) = \{H_3, H^0\}(\xi(s)) = 0$. The Poisson bracket table for the H_i , yields $\{H_3, H^0\} = \{H_3, H_1\} = -H_2$, so it follows that $H_2(\xi(s)) = 0$. Differentiating this constraint along ξ , we get $\frac{d}{ds}H_2(\xi(s)) = \{H_2, H^0\}(\xi(s)) = \{H_2, H_1\}(\xi(s)) + \kappa(s)\{H_2, H_3\}(\xi(s)) = H_4(\xi(s)) + \kappa(s)H_1(\xi(s)) = 0$. But condition (iii) of the maximum principle (Theorem 2.24) implies $H_1(\xi(s)) = 0$, then $H_4(\xi(s)) = 0$. Since $H_i(\xi(s)) = \xi(s)\vec{L}_i = 0$ and all the L_i 's form a basis for the space of two by two matrices, we must have $\xi(s) = 0$, a contradiction to the Pontryagin maximum principle.

□

In [17] Mayer and Myller identified the curves of constant centroaffine curvature as the extremals for the variational problem of finding the substantial curve of minimal centroaffine arclength, given its initial and terminal position and velocities. By using the second variation they were able to show that a minimum is not possible in the case that $\epsilon = -1$ or the case that $\epsilon = 1$ and $|\kappa| \geq 2$. As an illustration of the power of the optimal control formulation we improve this result.

Proposition 4.4. The minimal centroaffine arclength problem has no solution.

Proof. Let $\xi(s)$ be a regular extremal defined on $[0, T]$ of the problem [OCP] with $f(\kappa) = 1$. The Hamiltonian function is $H = -1 + H_1 + \kappa H_3$. To maximize H with

respect to κ it is necessary that $H_3(\xi(s)) = 0$. Upon differentiation:

$$\begin{aligned} 0 = \frac{d}{ds}H_3(s) &= \{H_3, -1 + H_1 + \kappa H_3\}(\xi(s)) \\ &= -H_2(s), \end{aligned}$$

and similarly $0 = \frac{d}{ds}H_2(s) = H_4(\xi(s)) + \kappa(s)H_1(\xi(s))$. Since we have a free time problem $H_1(\xi(s)) = 1$, so that $\kappa(s) = -H_4(\xi(s))$. Now,

$$\begin{aligned} \dot{\kappa} &= \frac{d}{ds}H_4(\xi(s)) \\ &= \{-H_4, -1 + H_1 + \kappa H_3\}(\xi(s)) \\ &= -4\epsilon H_2(\xi(s)) = 0 \end{aligned}$$

and κ must be constant on $[0, T]$. Let $M = |\kappa|$. It follows from Lemma 4.2 that the projection of an extremal $\xi(s)$ for [OCP] onto $\mathbf{GL}(2, \mathbb{R})$ yields a substantial curve $\mathbf{x}(s)$ that must be an extremal for the variational problem of the centroaffine arclength. In particular, if $\mathbf{x}(s)$ yields an optimal solution, then it would also be an optimal solution of the problem of finding the substantial curve with minimal arclength having the absolute value of the curvature no greater than $M + 1$. Accordingly, $\xi(s)$ must be a singular extremal of the problem

$$\min \int_0^T ds,$$

subject to

$$\frac{dg}{ds} = \bar{L}_1(g(s)) + \kappa(s)\bar{L}_3(g(s)), \text{ where } |\kappa| \leq M + 1, \quad (4.10)$$

and $g(0) = I$ and $g(T) = C$, for some $T > 0$. In particular, if the projection of $\xi(s)$ onto $\mathbf{GL}(2, \mathbb{R})$ is optimal, then the generalized Legendre–Clebsch condition (Theorem 2.25) guarantees that $\{H_3, \{H_1, H_3\}\}(\xi(s)) = -H_1(\xi(s)) \geq 0$. This contradicts the fact that $H_1(\xi(s)) = 1$ on $[0, T]$. \square

Let us consider now the centroaffine elastic problem. It follows from Proposition 4.3 that this problem has no abnormal extremals. So, it is enough to look at regular extremals.

Proposition 4.5. Suppose that $\xi(s)$ is a regular extremal for the centroaffine elastic problem. Let $\kappa(s)$ be the curvature function that generates $\xi(s)$. Write $H_i(s)$ to denote the function $H_i(\xi(s))$. Then

(i) the Hamiltonian function is $H = \frac{-\kappa^2}{2} + H_1 + \kappa H_3$,

(ii) $\kappa(s) = H_3(s)$, and

(iii)

$$\begin{aligned}\frac{d}{ds}H_1(s) &= H_3(s)H_2(s), \\ \frac{d}{ds}H_2(s) &= H_3(s)H_1(s) - H_4(s), \\ \frac{d}{ds}H_3(s) &= -H_2(s), \\ \frac{d}{ds}H_4(s) &= 4\epsilon H_2(s).\end{aligned}$$

(iv) An optimal curvature function must satisfy the differential equation

$$\frac{d^3}{ds^3}\kappa = \left(\frac{3}{2}\kappa^2 - 4\epsilon\right)\frac{d}{ds}\kappa. \quad (4.11)$$

Proof. The first statement follows from the fact that $f(\kappa) = \frac{\kappa^2}{2}$ for the centroaffine elastic problem. The second condition of Theorem 2.24 yields $\kappa(s) = H_3(s)$. The equations in (iii) follow from

$$\frac{d}{ds}H_i = \{H_i, H\}(s) = \{H_i, H_1\}(s) + \kappa(s)\{H_i, H_3\}(s)$$

and the Poisson bracket relations among the H_i 's. Finally, to get (iv) differentiate (ii) twice to get $\ddot{\kappa} = -\dot{H}_2 = -H_3H_1 + H_4$. But the third condition of Theorem 2.24 implies $H_1(s) = \frac{-\kappa^2(s)}{2}$. Thus, $\ddot{\kappa} = \frac{\kappa^3}{2} + H_4$. Differentiating once more we obtain the differential equation (4.11). \square

4.4 Searching for Extremals of the Centroaffine Elastic Problem

Equation (4.11) describes the evolution of the optimal curvature for the centroaffine elastic problem. In principle, the graphs of the various types of extremals for the centroaffine elastic problem may be obtained by plugging its solutions into the Serret–Frenet centroaffine equations. Equation (4.11) is a stationary version of the modified KdV partial differential equation. Rather than trying to look for closed forms solutions to (4.11) we study the qualitative behaviour of its nonconstant solutions by considering it as a second order conservative differential equation depending upon parameters. Our results are summarized in Proposition 4.6. For the case $\epsilon = 1$, we use this information to identify all possible types of behaviour for the extremal curvature function. Finally, we use a numerical routine to generate the extremal curves for an example that includes all the possible qualitative behaviours of the optimal curvature. By writing (4.11) as the first order system

$$\begin{aligned}\dot{\kappa} &= l \\ \dot{l} &= m \\ \dot{m} &= \left(\frac{3}{2}\kappa^2 - 4\epsilon\right)l,\end{aligned}$$

it is easy to see that a point on phase space is stationary if and only if it lies on the κ axis. These fixed points correspond to constant solutions of (4.11). As noted earlier, the centroaffine substantial curves of constant curvature are the extremals for the problem of minimizing arclength between two nearby points. To look for nonconstant solutions of (4.11), we integrate it once and get

$$\ddot{\kappa} = \frac{\kappa^3}{2} - 4\epsilon\kappa + D, \quad D \in \mathbb{R}. \quad (4.12)$$

Since (4.12) is independent of $\dot{\kappa}$ it determines a one parameter family of cylindrical surfaces in phase space. Therefore, whenever the initial values of κ and its first two

derivatives belong to one of these surfaces, the triplet $(\kappa, \dot{\kappa}, \ddot{\kappa})$ remains in that surface. The curves in phase space are completely determined by their projection on the $\kappa\dot{\kappa}$ plane via

$$\begin{aligned}\dot{\kappa} &= L \\ \dot{L} &= \frac{\kappa^3}{2} - 4\epsilon\kappa + D.\end{aligned}\tag{4.13}$$

Figure (4.1) illustrates some of the phase diagrams in the case $\epsilon = 1$ as the value of D changes. The plots were produced using the MATLAB routine `pplane5.m` [19].

We can state the following

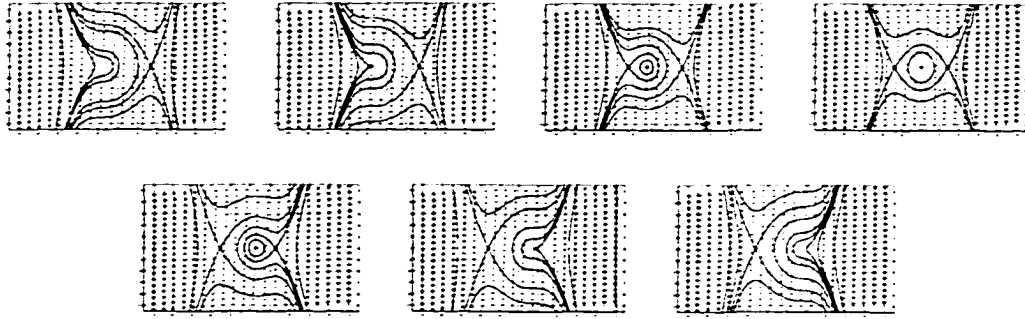


FIGURE 4.1. Phase diagrams for $D = \frac{-9}{8}\alpha, -\alpha, -\frac{\alpha}{2}, 0, \frac{\alpha}{2}, \alpha, \frac{9}{8}\alpha$.

Proposition 4.6. If $\epsilon = -1$, then for each D , the system (4.13) has only one fixed point of saddle type. If $\epsilon = 1$, let $\alpha = p(-\sqrt{\frac{8}{3}})$ where $p(k) = \frac{\kappa^3}{2} - 4\kappa\epsilon$, then for each D all the fixed points of (4.13) may be classified as follows:

- (i) Whenever $|D| < \alpha$, the system has three fixed points: a center at $(\kappa_D, 0)$ with $|\kappa_D| < \sqrt{\frac{8}{3}}$, and two saddle points located at

$$\left(\frac{-\kappa_D + \sqrt{32 - 3\kappa_D^2}}{2}, 0\right)$$

and

$$\left(\frac{-\kappa_D - \sqrt{32 - 3\kappa_D^2}}{2}, 0\right).$$

- (ii) For $|D| = \alpha$, the system has two fixed points: a cusp point at $(\kappa_D, 0)$ with $|\kappa_D| = \sqrt{\frac{8}{3}}$, and a saddle point at $(-2\kappa_D, 0)$.
- (iii) Finally, if $|D| > \alpha$, then the system has a unique fixed point of saddle type at $(\kappa_D, 0)$ with $|\kappa_D| > \sqrt{\frac{8}{3}}$.

Proof. Consider the function

$$F^\epsilon(\kappa, L) = L^2 - \frac{\kappa^4}{4} + 4\epsilon\kappa^2 - 2D\kappa. \quad (4.14)$$

Along any trajectory $(\kappa(s), L(s))$ of (4.13), we have

$$\frac{d}{ds}F^\epsilon(\kappa(s), L(s)) = 0.$$

Therefore, the level curves of F^ϵ are unions of trajectories of (4.13) on the phase plane. Since $\dot{\kappa} = L$, the trajectories move from left to right on the upper half of the κL plane and on the opposite direction below the κ axis.

Let $g(\kappa) = p(\kappa) + D$. Any fixed point of (4.13) must have the form $(\kappa_*, 0)$, where κ_* is a root of g . The local phase portrait of the system around a fixed point $(\kappa_*, 0)$, may be obtained by sketching the level curves of F^ϵ on a neighborhood of $(\kappa_*, 0)$. Note that the gradient

$$\nabla F^\epsilon(\kappa, L) = (-\kappa^3 + 8\epsilon\kappa - 2D, 2L) = (-2g(\kappa), 2L).$$

So the fixed points of the system are precisely the critical points of F^ϵ . Observe further that $F^\epsilon_{\kappa\kappa} = -3\kappa^2 + 8\epsilon$, $F^\epsilon_{LL} = 2$ and $F^\epsilon_{\kappa L} = 0$. Define

$$\Delta^\epsilon(\kappa_*) = F^\epsilon_{\kappa\kappa}(\kappa_*, 0)F^\epsilon_{LL}(\kappa_*, 0) - F^\epsilon_{\kappa L}^2(\kappa_*, 0).$$

When $\epsilon = -1$, the fact that $g'(k) > 0$ guarantees that the system has only one fixed point, say at $(\kappa_D, 0)$. Since we have $\Delta^\epsilon(\kappa_D) < 0$, the second derivative test shows that it must be a saddle point.

Assume now that $\epsilon = 1$. Observe that the polynomial p is an odd function. Also, the value of $\Delta^\epsilon(\kappa_*)$ is positive for $|\kappa_*| < \sqrt{\frac{8}{3}}$, and negative for $|\kappa_*| > \sqrt{\frac{8}{3}}$. To see (i) note that if $|D| < \alpha$, we have

$$g(-\sqrt{\frac{8}{3}}) = \alpha + D > 0 > -\alpha + D = g(\sqrt{\frac{8}{3}}).$$

Therefore, by the intermediate value theorem $g(\kappa_D) = 0$ for some $|\kappa_D| < \sqrt{\frac{8}{3}}$.

Since

$$g(\kappa) = \frac{1}{2}(\kappa - \kappa_D)(\kappa^2 - \kappa_D\kappa + \kappa_D^2 - 8),$$

the quadratic formula yields the other two fixed points. Since $|\kappa_D| < \sqrt{\frac{8}{3}}$, we have $F^{\epsilon}_{\kappa\kappa}(\kappa_D, 0) > 0$ and $\Delta^\epsilon(\kappa_D) > 0$. Thus, F^ϵ has a local minimum at $(\kappa_D, 0)$. The points where the gradient vanishes are isolated, so there is a neighborhood N of $(\kappa_D, 0)$ such that every $(\kappa, L) \neq (\kappa_D, 0)$ in N has $F^\epsilon(\kappa, L) < F^\epsilon(\kappa_D, 0)$. By the fact that F^ϵ has a strict local minimum at $(\kappa_D, 0)$, the level curve passing through any $(\kappa, L) \neq (\kappa_D, 0)$ in N , must enclose the point $(\kappa_D, 0)$. Therefore $(\kappa_D, 0)$ must be a center. Let $\kappa_1 = \frac{-\kappa_D + \sqrt{32 - 3\kappa_D^2}}{2}$ and denote the remaining root of g by κ_2 . Since $\kappa_1 > \sqrt{\frac{8}{3}}$ and $\kappa_2 < -\sqrt{\frac{8}{3}}$, we conclude that the other two fixed points must be saddle points.

To get (ii), observe that if $D = -\alpha$, then $g(-\sqrt{\frac{8}{3}}) = 0$. If $D = \alpha$, the fact that p is an odd function implies that $g(\sqrt{\frac{8}{3}}) = 0$. So, taking $\kappa_D = \sqrt{\frac{8}{3}} \text{ sign } D$ we have

$$g(\kappa) = \frac{1}{2}(\kappa - \kappa_D)^2(\kappa + 2\kappa_D).$$

Therefore, when $|D| = \alpha$ we have the required fixed points. The second derivative test shows that the fixed point at $(-2\kappa_D, 0)$ must be of saddle type. Therefore, we need only check that we have a cusp point at $(\kappa_D, 0)$. Now, if $\kappa_D = -\sqrt{\frac{8}{3}}$ then the point $(\kappa_D, 0)$ lies on the the level curve $F^\epsilon(\kappa, L) = \frac{-16}{3}$. It is sufficient to show that

this level curve has a cusp point at $(\kappa_D, 0)$. Let

$$f(\kappa) = \frac{\kappa^4}{4} - 4\kappa^2 + 2D\kappa - \frac{16}{3}.$$

The level curve containing $(\kappa_D, 0)$ must lie along $L^2 = f(\kappa)$. Since

$$f(\kappa) = \frac{1}{4}(\kappa - \kappa_D)^3(\kappa - 3\kappa_D),$$

we must have that if $D = \alpha$, then f is nonnegative when $\kappa \geq \kappa_D$ or $\kappa \leq -3\kappa_D$. Clearly $(\kappa_D, 0)$ belongs to the part of the curve having $\kappa \geq \kappa_D$. The result follows from the fact that $L = \pm\sqrt{f(\kappa)}$.

The same type of argument applies when $D = -\alpha$. However, in this case f is nonnegative when $\kappa \leq \kappa_D$ or $\kappa \geq -3\kappa_D$. So, $(\kappa_D, 0)$ belongs to the part of the curve having $\kappa \leq \kappa_D$.

To see (iii) let us consider first the case where $D > \alpha$. Note that since $p(\kappa) \geq -\alpha$ for $\kappa \geq -\sqrt{\frac{8}{3}}$, we must have $g(\kappa) > 0$ for $\kappa \geq -\sqrt{\frac{8}{3}}$. Being a cubic polynomial g must have at least one real root κ_D , with $\kappa_D \in (-\infty, -\sqrt{\frac{8}{3}})$. Since $g'(\kappa)$ is positive on this interval, $(\kappa_D, 0)$ must be the only fixed point of (4.13). Using the symmetry with respect to the origin of $p(\kappa)$, we see that if $D < -\alpha$, the cubic polynomial g has only one zero κ_D , which must lie on $(-\sqrt{\frac{8}{3}}, \infty)$. Since $|\kappa_D| > \sqrt{\frac{8}{3}}$, the second derivative test guarantees that $(\kappa_D, 0)$ must be a saddle point.

□

In the case $\epsilon = 1$, Proposition 4.6 leads to the following observations:

- (i) For any $|\kappa_0| \geq \sqrt{\frac{8}{3}}$, there are suitable initial conditions $\kappa(0), \dot{\kappa}(0), \ddot{\kappa}(0)$ such that equation (4.11) has a solution $\kappa(s)$ converging to κ_0 .
- (ii) Initial conditions may be chosen such that the solution to (4.11) is periodic.
- (iii) Initial conditions may be chosen such that a solution to (4.11) blows up.

MAPLE's RKF45 routine was used to generate numerical solutions for (4.11) via a fourth order Runge Kutta Fehlberg method. Initial conditions were chosen according to the preceding criteria. By feeding this solution to the second order equation (3.14), and using again the RKF45 routine we obtained graphs for some extremals corresponding to some nonconstant solutions of (4.11).

For example, in the case $\epsilon = 1$, choosing $\kappa(0) = 0$, $\dot{\kappa}(0) = b$, $\ddot{\kappa}(0) = 0$, the solution of (4.11): (i) goes to $-\infty$ for $b < -4$, (ii) converges to $-\sqrt{8}$ for $b = -4$, (iii) is periodic whenever $0 < |b| < 4$, (iv) converges to $\sqrt{8}$ for $b = 4$, and (v) goes to ∞ for $b > 4$. Figure (4.2) illustrates some of the possible extremal trajectories for values of b for each of the cases (i) through (v).

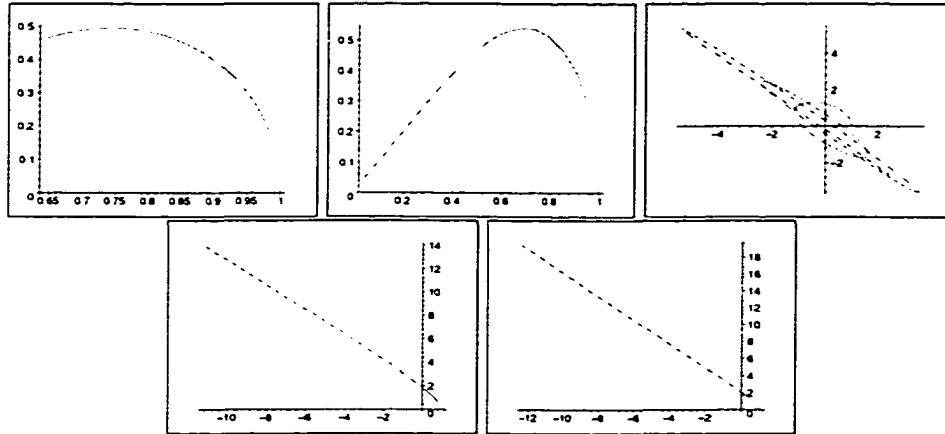


FIGURE 4.2. $b = -5, -4, 1, 4, 5$.

Chapter 5. A Problem with Impulses

In this chapter we consider the optimal control problem of minimizing

$$\Psi(\tau, \kappa) = \int_0^S \alpha\tau(s) + \beta\kappa(s)ds, \quad \text{with fixed } \alpha > 0 \text{ and } \beta \geq 0, \quad (5.1)$$

over the set \mathcal{U} of ordered pairs (τ, κ) of nonnegative locally bounded measurable functions on $[0, \infty)$ subject to

$$\begin{aligned} \dot{x} &= v\tau, & x(0) &= x_0, & x(S) &= x_f, \\ \dot{v} &= -\epsilon x\tau + \kappa v, & v(0) &= v_0, & v(S) &= v_f, \\ \dot{y} &= w\tau, & y(0) &= y_0, & y(S) &= y_f, \\ \dot{w} &= -\epsilon y\tau + \kappa w, & w(0) &= w_0, & w(S) &= w_f, \\ \dot{t} &= \tau, & t(0) &= 0, & S \text{ and } t(S) & \text{ free,} \end{aligned} \quad (5.2)$$

where $x_0w_0 - v_0y_0$ and $x_fw_f - v_fy_f$ are both nonzero and have the same sign. Here a dot denotes differentiation with respect to s . We show that in the absence of chattering controls, extremal trajectories for this problem are concatenations of trajectories determined by impulsive controls ($\tau = 0, \kappa = 1$) and null controls ($\tau = 1, \kappa = 0$). The main tool is the Pontryagin maximum principle along with the generalized Legendre–Clebsch condition for singular vector valued controls, a second order condition for optimality of singular extremals developed by Krener [12]. We also describe the trajectories associated to the null control and the impulsive controls for our dynamics and compute their costs.

Problem (5.1)–(5.2) is motivated by the free–terminal time problem of minimizing

$$\Phi(u) = \int_0^T \alpha + \beta u(t)dt, \quad \text{with fixed } \alpha > 0 \text{ and } \beta \geq 0, \quad (5.3)$$

over the set \mathcal{U}_0 of all the nonnegative locally bounded measurable functions u defined on $[0, \infty)$ subject to

$$\begin{aligned}
 x' &= v, & x(0) &= x_0, & x(T) &= x_f, & (5.4) \\
 v' &= -\epsilon x + uv, & v(0) &= v_0, & v(T) &= v_f, \\
 y' &= w, & y(0) &= y_0, & y(T) &= y_f, \\
 w' &= -\epsilon y + uw, & w(0) &= w_0, & w(T) &= w_f,
 \end{aligned}$$

where a prime denotes differentiation with respect to t and $x_0w_0 - v_0y_0$ and $x_fw_f - v_fy_f$ are both nonzero and have the same sign. Problem (5.3)–(5.4) is related to the minimal arclength problem of Chapter 4. The dynamics (5.4) correspond to writing the system (4.3) in coordinate form. The cost functional (5.3) reduces to the arclength functional in the case that $\alpha = 1$ and $\beta = 0$. In geometric terms, problem (5.3)–(5.4) may be thought of as a variational problem for substantial curves under the constraint that the curvature is nonnegative.

The fact that no locally bounded measurable control is optimal for the minimal centroaffine arclength problem, leads us to search for meaningful optimal control problems involving the centroaffine arclength functional. A possibility is to restrict the class of admissible controls so that the control functions satisfy a uniform bound, as in the classical Dubins problem. In such a case the so called bang–bang controls guaranteed by the Pontryagin conditions would yield a way to construct extremal trajectories. However, since bang–bang trajectories are generated by a bounded control, no part of this extremal would be optimal for the general minimum arclength problem. At the other extreme one may enlarge the class of admissible controls to allow for impulses. One may suspect then that by applying an impulsive control we may reach any given point at no cost. This would render the problem as hopelessly uninteresting. But, as we shall see, impulsive controls

are not cost free, and not every state can be reached by applying an impulse. By allowing only nonnegative controls, extremals may include trajectories generated by both impulses and bang controls.

As stated problem (5.3)–(5.4) has no solution, for it leads to impulsive optimal controls. Another complication lies in the fact that the Hamiltonian for this problem is linear in the control. This makes the existence of singular extremals a likely possibility. The more general problem (5.1)–(5.2) provides a model for the impulsive behaviour of the optimal controls of (5.3)–(5.4). It is posed by applying a time reparametrization technique used by Dorroh and Ferreyra to model the action of impulsive controls for some singular problems with unbounded controls in one [4] and two dimensional [5] Euclidean spaces.

To see that problem (5.1)–(5.2) is indeed a generalized version of the problem (5.3)–(5.4), let us consider a trajectory $(x(t), v(t), y(t), w(t))$ of (5.4) generated by a control u in \mathcal{U}_0 . We can write $\hat{\kappa}(t) = \int_0^t u(\xi)d\xi$. If we introduce a reparametrization of time, say $t = t(s)$ with $\dot{t} > 0$, and denote differentiation with respect to s with a dot, then the formula of change of variables for integrals gives

$$\begin{aligned} \dot{x} &= v\dot{t} \\ \dot{v} &= -\epsilon x\dot{t} + \hat{\kappa}'\dot{t}v \\ \dot{y} &= w\dot{t} \\ \dot{w} &= -\epsilon y\dot{t} + \hat{\kappa}'\dot{t}w, \end{aligned}$$

where the prime denotes differentiation with respect to t . Under the reparametrization, the change of variables formula implies that the cost of this trajectory is

$$\begin{aligned} \Phi(\hat{\kappa}') &= \int_0^T \alpha + \beta\hat{\kappa}'(\xi)d\xi \\ &= \int_0^S \alpha\dot{t}(\xi) + \beta\hat{\kappa}'(t(\xi))\dot{t}(\xi)d\xi, \end{aligned}$$

where $S = t^{-1}(T)$. Thus, any admissible trajectory for the problem (5.3)–(5.4) generated by u is a trajectory of (5.1)–(5.2) for which $\tau \equiv 1$ and $\kappa(s) = \hat{\kappa}'(t(s))$. In particular, $\Phi(u) = \Psi(1, \hat{\kappa}')$.

To model jumps in the trajectories of (5.3)–(5.4) due to impulsive controls, we can consider any trajectory of (5.1)–(5.2) generated by a pair $(\tau, \kappa) \in \mathcal{U}$ such that $\tau(s) = 0$ in a nontrivial interval (s_1, s_2) . In this case the components $t(s)$, $x(s)$ and $y(s)$ of the trajectory of (5.2) remain constant through the interval, but $v(s)$ and $w(s)$ can still change according to the equations

$$\dot{v} = \kappa v,$$

$$\dot{w} = \kappa w.$$

Indeed, if this is the case,

$$v(s_2) = v(s_1) \exp\left(\int_{s_1}^{s_2} \kappa(\xi) d\xi\right) \quad (5.5)$$

and

$$w(s_2) = w(s_1) \exp\left(\int_{s_1}^{s_2} \kappa(\xi) d\xi\right) \quad (5.6)$$

Since equations (5.5) and (5.6) involve $\int_{s_1}^{s_2} \kappa(\xi) d\xi$ they determine the cost of such a jump. The following proposition summarizes this discussion.

Proposition 5.1. If the system (5.4) starts at the point

$$(x(t_1), v(t_1), y(t_1), w(t_1)) = (x_1, v_1, y_1, w_1)$$

and an impulse is applied at time $t = t_1$, then the system jumps instantaneously to a point (x_1, v, y_1, w) , where the values of v and w satisfy the following relations:

(i) If v_1 and w_1 are nonzero, then $\frac{v}{v_1} = \frac{w}{w_1} > 0$.

(ii) If $v_1 = 0$ and $w_1 \neq 0$, then $v = 0$ and $\frac{w}{w_1} > 0$.

(iii) If $w_1 = 0$ and $v_1 \neq 0$, then $w = 0$ and $\frac{v}{v_1} > 0$.

The cost of such a jump is $\beta \ln \frac{w}{w_1}$ in the first two cases. In the case that $v_1 \neq 0$ the jump cost is $\beta \ln \frac{v}{v_1}$.

With this understanding of the problem (5.1)–(5.2) we proceed to obtain information about its extremals by using the maximum principle. Consider our parameter s and the cost c as state variables by adjoining two equations with corresponding initial data

$$\begin{aligned} \dot{s} &= 1, & s(0) &= 0, \\ \dot{c} &= \alpha\tau + \beta\kappa, & c(0) &= 0, \end{aligned}$$

to the system (5.2). Define $\bar{x} = (s, x, v, y, w, t, c)^T$ so that the extended control problem (5.1)–(5.2) is posed as

$$\min_{\tau, \kappa \in \mathcal{U}} c(S) \tag{5.7}$$

subject to

$$\dot{\bar{x}} = a_0(\bar{x}) + a_1(\bar{x})\tau + a_2(\bar{x})\kappa, \tag{5.8}$$

where

$$a_0(\bar{x}) = (1, 0, 0, 0, 0, 0, 0),$$

$$a_1(\bar{x}) = (0, v, -\epsilon x, w, -\epsilon y, 1, \alpha),$$

$$a_2(\bar{x}) = (0, 0, v, 0, w, 0, \beta).$$

Write $\bar{u} = (\tau, \kappa)$, $\bar{p} = (\lambda, \mu, \eta, \varphi, \rho, \sigma, p_0)$ and let the Hamiltonian $H(\bar{x}, \bar{u}, \bar{p})$ be defined by the dot product:

$$H(\bar{x}, \bar{u}, \bar{p}) = \bar{p}(a_0(\bar{x}) + a_1(\bar{x})\tau + a_2(\bar{x})\kappa) \tag{5.9}$$

$$= \lambda + H_\tau \tau + H_\kappa \kappa, \tag{5.10}$$

where

$$H_\kappa = \eta v + \rho w + p_0 \beta, \quad (5.11)$$

$$H_\tau = \mu v + \varphi w - \epsilon(\eta x + \rho y) + \sigma + \alpha p_0. \quad (5.12)$$

By the maximum principle there is a nontrivial adjoint vector $\bar{p}(s)$ such that its components satisfy the system

$$\begin{aligned} \dot{\lambda} &= 0 \\ \dot{\mu} &= \epsilon \eta \tau \\ \dot{\eta} &= -\mu \tau - \kappa \eta \\ \dot{\varphi} &= \epsilon \rho \tau \\ \dot{\rho} &= -\varphi \tau - \kappa \rho \\ \dot{\sigma} &= 0 \\ \dot{p}_0 &= 0, \end{aligned} \quad (5.13)$$

with $p_0(S) \leq 0$. The transversality conditions $\lambda(S) = \sigma(S) = 0$ force λ and σ to vanish identically on $[0, S]$.

Along an extremal $(\bar{x}(s), \bar{p}(s))$ defined on $[0, S]$ we must have

$$H(\bar{x}(s), \bar{u}(s), \bar{p}(s)) = \max_{\tau, \kappa \geq 0} \{H_\tau(s)\tau + H_\kappa(s)\kappa\} = 0.$$

This equation is satisfied in the following four cases, and no others:

- (i) $H_\tau(s) \leq 0, \quad H_\kappa(s) \leq 0, \quad \tau(s) = 0, \quad \kappa(s) = 0;$
- (ii) $H_\tau(s) < 0, \quad H_\kappa(s) = 0, \quad \tau(s) = 0, \quad \kappa(s) \in (0, \infty);$
- (iii) $H_\tau(s) = 0, \quad H_\kappa(s) < 0, \quad \kappa(s) = 0, \quad \tau(s) \in (0, \infty);$
- (iv) $H_\tau(s) = 0, \quad H_\kappa(s) = 0, \quad \tau(s), \kappa(s) \in (0, \infty).$

The first case is not interesting, since it only leads to trivial trajectories. Proposition 5.1 describes the trajectories associated with the second case. In case (iii) the value of $\tau(s)$ is not unique. Since we are interested in extremal trajectories of (5.1)–(5.2) which are also trajectories of (5.4) we solve the nonuniqueness by taking $\tau(s) = 1$. Let us now describe the trajectories associated with case (iii).

Proposition 5.2. Suppose that at time $s = s_1$, system (5.2) has

$$(x(s_1), v(s_1), y(s_1), w(s_1), t(s_1)) = (x_1, v_1, y_1, w_1, t_1),$$

with $x_1 w_1 - y_1 v_1 \neq 0$. Let $s_2 > s_1$ and apply the controls $\tau = 1$ and $\kappa = 0$ on the interval $[s_1, s_2]$.

(i) If $\epsilon = 1$, then

$$\begin{aligned} x(s) &= x_1 \cos(s - s_1) + v_1 \sin(s - s_1), & v(s) &= v_1 \cos(s - s_1) - x_1 \sin(s - s_1), \\ y(s) &= y_1 \cos(s - s_1) + w_1 \sin(s - s_1), & w(s) &= w_1 \cos(s - s_1) - y_1 \sin(s - s_1) \end{aligned}$$

on $[s_1, s_2]$. So that $\mathbf{x}(s) = (x(s), y(s))$ lies on the ellipse

$$(y_1^2 + w_1^2)x^2 - 2(x_1 y_1 + v_1 w_1)xy + (x_1^2 + v_1^2)y^2 = (x_1 w_1 - y_1 v_1)^2 \quad (5.14)$$

and $\dot{\mathbf{x}}(s) = (v(s), w(s))$ is tangent to the ellipse at $\mathbf{x}(s)$.

(ii) If $\epsilon = -1$, then on $[s_1, s_2]$,

$$\begin{aligned} x(s) &= x_1 \cosh(s - s_1) + v_1 \sinh(s - s_1), & v(s) &= v_1 \cosh(s - s_1) - x_1 \sinh(s - s_1), \\ y(s) &= y_1 \cosh(s - s_1) + w_1 \sinh(s - s_1), & w(s) &= w_1 \cosh(s - s_1) - y_1 \sinh(s - s_1). \end{aligned}$$

In this case, $\mathbf{x}(s)$ lies on the hyperbola

$$(w_1^2 - y_1^2)x^2 + 2(x_1 y_1 - v_1 w_1)xy + (v_1^2 - x_1^2)y^2 = (x_1 w_1 - y_1 v_1)^2, \quad (5.15)$$

and $\dot{\mathbf{x}}(s) = (v(s), w(s))$ is tangent to the hyperbola at $\mathbf{x}(s)$.

For both cases $t(s) = t_1 + s - s_1$. The cost of such a trajectory is $\alpha(s_2 - s_1)$.

Proof. Write $\mathbf{x}(s) = (x(s), y(s))$. The choice of $\kappa = 0$ and $\tau = 1$ in system (5.2), leads to

$$\ddot{\mathbf{x}} = -\epsilon\mathbf{x}, \quad (5.16)$$

$$\text{with } \mathbf{x}(s_1) = (x_1, y_1) \text{ and } \dot{\mathbf{x}}(s_1) = (v_1, w_1). \quad (5.17)$$

Thus, if $\epsilon = 1$ we have $\mathbf{x}(s) = \begin{pmatrix} x_1 & v_1 \\ y_1 & w_1 \end{pmatrix} \begin{pmatrix} \cos(s - s_1) \\ \sin(s - s_1) \end{pmatrix}$. Since the matrix of initial values is invertible, we can solve the last equation for $\begin{pmatrix} \cos(s - s_1) \\ \sin(s - s_1) \end{pmatrix}$. The equation for the ellipse follows from using the Pythagorean identity. Since $\cosh(s - s_1)$ and $\sinh(s - s_1)$ form a fundamental set of solutions for (5.16) when $\epsilon = -1$, a similar argument establishes the second statement. We get the expression $t(s) = t_1 + s - s_1$ by solving the initial value problem $\dot{t} = 1, t(s_1) = t_1$. The cost of this trajectory is $\int_{s_1}^{s_2} \alpha ds = \alpha(s_2 - s_1)$. \square

The next proposition ensures that controls \bar{u} with values in $(0, \infty) \times (0, \infty)$ are not optimal for the problem (5.7)–(5.8) whenever case (iv) holds. To prove this, we use Krener's higher order conditions for optimality for vector controls (Theorem 2.27).

Proposition 5.3. Let $(\bar{x}(s), \bar{p}(s), \bar{u}(s))$ be an extremal defined on $[0, S]$ for the optimal control problem (5.7)–(5.8). Let $J = (s_1, s_2)$ be a subinterval of $[0, S]$. If the control $\bar{u}(s)$ takes values on the interior of the set $[0, \infty) \times [0, \infty)$ and $H_\kappa(\bar{x}(s), \bar{p}(s), \bar{u}(s)) = H_\tau(\bar{x}(s), \bar{p}(s), \bar{u}(s)) = 0$ on the open subinterval J , then $\bar{x}(s)$ is not optimal for (5.7)–(5.8).

Proof. Let

$$L = \mu v + \varphi w + \epsilon(\eta x + \rho y). \quad (5.18)$$

Note that using the adjoint and system equations we obtain

$$\frac{d}{ds}H_\kappa = -\tau L, \quad (5.19)$$

$$\frac{d}{ds}H_\tau = \kappa L, \quad (5.20)$$

and

$$\dot{L} = \kappa(\mu v + \varphi w - \epsilon(\eta x + \rho y)) + 2\epsilon\tau[(\eta v + \rho w) - (\varphi y + \mu x)]. \quad (5.21)$$

Write $\bar{u} = (\tau, \kappa)$. Suppose $p_0 = 0$. Then $\eta v + \rho w = 0$ and $\mu v + \varphi w = \epsilon(\eta x + \rho y)$. Since H_κ vanishes and τ is positive, it follows from (5.19) that $L = 0$. In other words,

$$\mu v + \varphi w = -\epsilon(\eta x + \rho y).$$

Thus, $\mu v + \varphi w = \eta x + \rho y = 0$. Moreover, $\dot{L} = 0$ implies $2\epsilon\tau(\varphi y + \mu x) = 0$.

Therefore,

$$\begin{pmatrix} \mu & \varphi \\ \eta & \rho \end{pmatrix} \begin{pmatrix} x & v \\ y & w \end{pmatrix} = 0.$$

Since the matrix $\begin{pmatrix} x & v \\ y & w \end{pmatrix}$ is invertible, it follows that $\mu = \eta = \varphi = \rho = 0$. Hence, the adjoint vector is trivial. This is a contradiction.

Thus, without loss of generality we may take $p_0 = -1$. By hypothesis the extremal is singular. The degree of singularity of the controls is different from zero since the expressions for H_τ and H_κ do not depend on the controls. Equation (5.19) forces $L = 0$. It follows from equation (5.21) that

$$0 = \dot{L} = \alpha\kappa + 2\epsilon\tau[\beta - (\mu x + \varphi y)]. \quad (5.22)$$

Let $B = \beta - (\mu x + \varphi y)$. Since $\frac{d}{ds}(\mu x + \varphi y) = \tau L = 0$, B is a constant. Moreover, B is nonzero since $\kappa > 0$. This implies that the degrees of singularity of κ and τ are both 2 because $\frac{\partial}{\partial \tau} \frac{d^2}{ds^2} H_\tau = 2\epsilon\kappa B \neq 0$ and $\frac{\partial}{\partial \kappa} \frac{d^2}{ds^2} H_\kappa = \alpha\tau \neq 0$. Now suppose that $\bar{x}(s)$ is optimal. It follows from Theorem (2.27) that the matrix

$$M = \begin{pmatrix} -2\epsilon\kappa B & 2\epsilon\tau B \\ -\alpha\kappa & \alpha\tau \end{pmatrix}$$

is symmetric and nonpositive definite. The eigenvalues of M are $\lambda_1 = 0$ and $\lambda_2 = \alpha\tau - 2\epsilon\kappa B$. However, equation (5.22) yields $\kappa = \frac{-2\epsilon\tau B}{\alpha}$. Thus, $\lambda_2 = \frac{\tau(\alpha^2 + 4B^2)}{\alpha} > 0$. Contradicting the nonpositivity of the eigenvalues of M . \square

Corollary 5.4. In the absence of chattering controls an optimal trajectory for problem (5.1)–(5.2) is a concatenation of trajectories generated by either an impulsive control ($\tau = 0, \kappa = 1$) or a null control ($\tau = 1, \kappa = 0$).

Chapter 6. Conclusions and Open Problems

Variational problems analogous to the Euclidean minimal arclength and the elastic energy problem may be defined for the class of centroaffine substantial curves. After parametrizing with respect to centroaffine arclength, these problems may be posed as optimal control problems over the Lie group $\mathbf{GL}(2, \mathbb{R})$. The coordinate free formulation of the Pontryagin maximum principle provides an elegant way to obtain information about the optimizing control functions. We showed that the minimal centroaffine arclength problem has no solution in the class of substantial curves, by using the generalized Legendre–Clebsch condition. For the centroaffine elastica problem, the maximum principle leads to a differential equation describing the evolution of an optimizing control. The study of this nonlinear equation via phase space analysis provides the qualitative behaviour of the possible nonconstant extremal controls. Numerical solutions for the optimal control may be obtained, by studying the phase portraits. Plugging in this numerical solution for the control into the state equations, some extremal trajectories can be constructed numerically. A description of the optimal synthesis and the location of conjugate points remains an open question.

The fact that no locally bounded control is optimal for the minimal centroaffine arclength problem, leads us to search for meaningful optimal control problems involving the centroaffine arclength functional. Problem (5.3)–(5.4) is related to the minimum centroaffine arclength problem under the constraint that the centroaffine curvature is positive and leads to impulsive controls. Problem (5.1)–(5.2) is a more general problem which models the action of the impulsive controls. We show that in the absence of chattering controls, extremal trajectories for problem (5.1)–(5.2)

are concatenations of trajectories determined by impulsive controls ($\tau = 0, \kappa = 1$) and null controls ($\tau = 1, \kappa = 0$). The main tool is the generalized Legendre–Clebsch condition for singular vector valued controls, a second order condition for optimality of singular extremals developed by Krener [12]. We also describe the trajectories associated to the null control and the impulsive controls for our dynamics and compute their costs. Obtaining the optimal synthesis for this problems remains an open problem.

References

- [1] V. ARNOLD, *Mathematical Methods of Classical Mechanics, 2nd edition*, Springer Verlag, (1989).
- [2] W. BOOTHBY, *An introduction to Differentiable Manifolds and Riemannian Geometry, 2nd edition*, Academic Press, (1986).
- [3] O. BORUVKA, *Lineare Differentialtransformationen Zweiten Ordnung*, Veb Deutscher Verlag der Wissenschaften, Berlin, (1967).
- [4] J.R. DORROH AND G. FERREYRA, *Optimal Advertising in Exponentially Decaying Markets*, Journal of Optimization Theory and Applications, Vol. 79, No. 2, (1993), pp. 219-236.
- [5] J.R. DORROH AND G. FERREYRA, *A Multistate, Multicontrol Problem with Unbounded Controls*, SIAM J. Control and Optimization, 32, (1994), pp 1322-1331.
- [6] R. GARDNER AND G. WILKENS, *The Fundamental Theorems of Curves and Hypersurfaces in Centro-affine Geometry*, Bulletin of the Belgian Mathematical Society, Simon Stevin, Vol.4 , No. 3, (1997), pp. 379-401.
- [7] M. GREEN, *The Moving Frame, Differential Invariants and Rigidity Theorems for Curves in Homogeneous Spaces*, Duke Mathematical Journal, 4, (1978), pp. 735-779.
- [8] P. GRIFFITHS, *Exterior Differential Systems and the Calculus of Variations*, Birkhauser, Boston (1983).
- [9] V. JURDJEVIC, *Non-Euclidean Elastica*, American Journal of Mathematics 117, (1995), pp. 93-124.
- [10] V. JURDJEVIC, *Geometric Control Theory*, Cambridge Studies in Advanced Mathematics, Vol. 52, (1997).
- [11] V. JURDJEVIC, *Lie Determined Systems and Optimal Problems with Symmetries*, Conference on Geometric Control and Non-holonomic Mechanics, June 19-21, 1996, Mexico City, Canadian Mathematical Society Conference proceedings, Vol.25, ed. V. Jurdjevic, R.W. Sharpe, American Mathematical Society, (1998), pp. 1-28.
- [12] A. KRENER, *The High Order Maximal Principle and its Application to Singular Extremals*, SIAM Journal of Control and Optimization, Vol.15, No.2, (1977), pp. 256-293.

- [13] A.J. KRENER AND H. SCHÄTTLER, *The Structure of Small Time Reachable Sets in Low Dimensions*, SIAM J. Control and Optimization, 27, (1989), pp. 120–147.
- [14] D. LAUGWITZ, *Differentialgeometrie in Vectorräumen*, Friedr. Vieweg & Sohn, Braunschweig, (1965).
- [15] D. LAUGWITZ, *Über Eilinen in Großen in der Zentralaffinen Differentialgeometrie*, Mathematische Zeitschrift, 79, (1962), pp. 425–438.
- [16] J. LANGER AND D. SINGER, *The Total Squared Curvature of Closed Curves*, J. Differential Geom., 20, (1984), pp. 1–22.
- [17] O. MAYER AND A. MYLLER, *La géométrie centroaffine des courbes planes*, Annales Scientifiques de l'Université de Jassy, 18, (1933), pp. 234–280.
- [18] E. MCSHANE, *Integration*, Princeton University Press, (1944).
- [19] J. POLKING, *Ordinary Differential Equations using MATLAB*, Prentice Hall, (1995).
- [20] P. L. SACHDEV, *A Compendium on Nonlinear Ordinary Differential Equations*, Wiley, (1997).
- [21] H. SCHÄTTLER, *The Maximum Principle and Reachable Sets a Classical Perspective*, Lecture Notes, AMS Short Course on Nonlinear Control, San Antonio, (1999).
- [22] P. SCHIROKOW AND A. SCHIROKOW, *Affine Differentialgeometrie*, B. G. Teubner Verlagsgesellschaft, Leipzig, (1962).
- [23] H. SUSSMAN AND G. TANG, *Shortest Paths for the Reeds-Shepp Car: a Worked Out Example of the Use of Geometric Techniques in Nonlinear Optimal Control*, Rutgers University Systems and Control Center, Report SYCON-91-10, (1991).
- [24] G. TANG, *Regularity Properties of Time-Optimal Trajectories for Certain Classes of Multi Input Systems in Low Dimensions*, PhD. thesis, Rutgers University, (1992).
- [25] F.W. WARNER, *Foundations of Differentiable Manifolds and Lie Groups*, Scott, Foreman and Company, Glenview, (1971).
- [26] G. WILKENS, *Centro-affine Geometry in the Plane and Feedback Invariants of Two-State Scalar Control Systems*, in Differential Geometry and Control, Proc. Symp. Pure Math., Vol. 64, ed. G. Ferreyra, R. Gardner, H. Hermes, and H. Sussmann, American Mathematical Society, Providence, RI, (1999).

Vita

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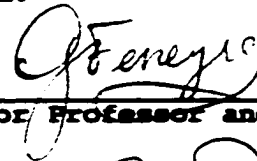
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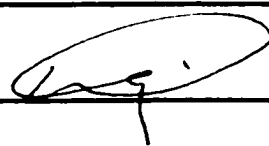
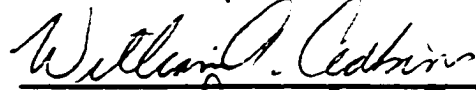
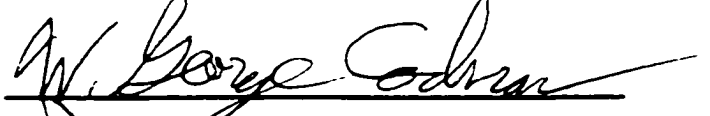


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