On Harmonic Analysis for White Noise Distribution Theory.

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ON HARMONIC ANALYSIS FOR
WHITE NOISE DISTRIBUTION THEORY

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Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
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Doctor of Philosophy
in
The Department of Mathematics

by
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To Gabriela

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Abstract

This thesis is composed of two parts, each part treating a different problem from the theory of Harmonic Analysis. In the first part we present an inequality in White Noise Analysis similar to the classical Heisenberg Inequality for functions in $L^2(\mathbb{R}^n)$. To do this we replace the finite dimensional space $\mathbb{R}^n$ and its Lebesgue measure by the infinite dimensional space $\mathcal{E}'$, which is the dual of a nuclear space $\mathcal{E}$, and its Gaussian measure. Choosing an arbitrary element $\eta$ in $\mathcal{E}$, we may define the multiplication operator $\tilde{Q}_\eta$, which is the sum between the differentiation operator $\tilde{D}_\eta$ and its adjoint $D^*_\eta$. We then use $\tilde{Q}_\eta$ as a substitute for the multiplication operator by $x$ from the finite dimensional case. Because the Fourier transform is a unitary operator on $L^2(\mathbb{R}^n)$, whose eigenvalues are powers of $-i$ and the second quantization operator of $-iI$, denoted by $\Gamma(-iI)$, from $L^2$ into itself has the same properties, we replace the Fourier transform by $\Gamma(-iI)$. Here, $(L^2)$ is the space of all complex valued, square integrable functions on $\mathcal{E}'$. The proof of our Heisenberg Inequality relies on the Schwartz Inequality and the commutation relationships between any two of the following three operators: $\tilde{D}_\eta$, $D^*_\eta$, and $\Gamma(-iI)$.

In the second part of this work, we show some results in White Noise Analysis analogous to the classical Paley-Wiener Theorem that describes functions on $\mathbb{R}^n$ with compact support in terms of their Fourier transform. Because the Fourier transform for a Schwartz function is defined pointwise as an inner product between that function and an exponential function and the S-transform from White Noise Analysis is defined similarly, we have chosen this time the S-transform as the natural replacement for the Fourier transform. After giving a thorough description of the weakly and strongly compact subsets of $\mathcal{E}'$, we finally characterize some
classes of \((L^2)\) functions with compact support in terms of their \(S\)-transform. As in the classical Paley-Wiener theorem two conditions are essential, namely: an analyticity and a growth condition.
Chapter 1
Introduction to Heisenberg's Inequality

The well-known Heisenberg Uncertainty Principle [28] says that for any function \( f \in L^2(\mathbb{R}^n) \) with \( |f|_2 = 1 \), we have

\[
\int_{\mathbb{R}^n} |xf(x)|^2 dx \cdot \int_{\mathbb{R}^n} |\gamma \hat{f}(\gamma)|^2 d\gamma \geq \frac{n^2}{4(2\pi)^{n-1}},
\]

(1.1)

where \( \hat{f} \) is the Fourier transform of \( f \). Since \( \lim_{n \to \infty} \frac{n^2}{(2\pi)^{n-1}} = 0 \), it appears that there is no such uncertainty principle for the infinite dimensional case. This is reflected by the fact that the Lebesgue measure does not exist in an infinite dimensional space. Moreover, the Fourier transform needs to be generalized to such a space.

First we briefly describe the idea to obtain an infinite dimensional analogue of the above inequality. Take a basic Gel'fand triple \( \mathcal{E} \subset E \subset \mathcal{E}' \); for example \( \mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}) \), where \( \mathcal{S}(\mathbb{R}) \) is the Schwartz space of rapidly decreasing functions on \( \mathbb{R} \). Let \( |\cdot|_0 \) denote the norm on \( E \). The space \( \mathbb{R}^n \) is replaced by \( \mathcal{E}' \) and the Lebesgue measure on \( \mathbb{R}^n \) is replaced by the standard Gaussian measure \( \mu \) on \( \mathcal{E}' \). Let \( (L^2) \) denote the complex \( L^2(\mu) \) - space with norm \( |\cdot|_0 \). Let \( (\mathcal{E}) \subset (L^2) \subset (\mathcal{E})^* \) be the associated Gel'fand triple (see section 4.2 in [15] for details).

The multiplication by \( x \) in (1.1) is replaced by a multiplication operator \( \widetilde{Q}_\eta \) which is continuous from \( (\mathcal{E})^* \) into itself (Theorem 9.18 in [15]). The Fourier transform is replaced by the second quantization operator \( \Gamma(-iI) \) of \( -iI \). Thus the infinite dimensional analogue of the inequality in (1.1) takes the form

\[
|\langle \widetilde{Q}_\eta \varphi \rangle_0 \langle \widetilde{Q}_\eta \Gamma(-iI)\varphi \rangle_0 | \geq |\eta|_0^2 |\varphi|_0^2,
\]

(1.2)

for any \( \eta \in \mathcal{E} \) and \( \varphi \in (L^2) \).
In Chapter 2 we will provide a brief background concerning the Gel'fand triples \( \mathcal{E} \subset E \subset \mathcal{E}' \) and \((\mathcal{E}) \subset (L^2) \subset (\mathcal{E})^*\). The inequality in (1.2) will be proved in Chapter 3. We will discuss the equality in (1.2) in Chapter 4.
Chapter 2
Background

In this chapter we introduce some basic concepts and notations from White Noise Analysis that will be useful throughout our work. Most of the results are presented without proof since they are standard and easily found in any basic book of White Noise Analysis. Two good references are [15] and [18].

2.1 Concept and Notations
Let $E$ be a real separable Hilbert space with norm $|·|_0$. Let $A$ be a densely defined self-adjoint operator on $E$, whose eigenvalues $\{\lambda\}_{n\geq 1}$ satisfy the following conditions:

- $1 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$.
- $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$. (Hence $A^{-1}$ is a Hilbert-Schmidt operator.)

For any $p \geq 0$, we consider the space $\mathcal{E}_p := \{f \in E \mid |A^p f|_0 < \infty\}$. On the space $\mathcal{E}_p$ we introduce the norm $|f|_p = |A^p f|_0$. Each of these spaces is a Hilbert space and we have the inclusion $\mathcal{E}_q \subset \mathcal{E}_p$ for $p < q$. By the second condition the inclusion $i : \mathcal{E}_{p+1} \rightarrow \mathcal{E}_p$ is a Hilbert-Schmidt operator. Thus the space $\mathcal{E} = \bigcap_{p \geq 0} \mathcal{E}_p$, equipped with the topology given by the family $\{|·|_p\}_{p\geq 0}$ of seminorms, is a nuclear space.

It can be shown that for all $p \geq 0$, the dual space of $\mathcal{E}_p$ is isomorphic to $\mathcal{E}_{-p}$, which is the completion of the space $E$ with respect to the norm $|f|_{-p} = |A^{-p} f|_0$. Moreover, we have $\mathcal{E}' = \bigcup_{p \geq 0} \mathcal{E}_{-p}$ and for any $p < q$,

$$
\mathcal{E} \subset \mathcal{E}_q \subset \mathcal{E}_p \subset \mathcal{E}_0 \subset \mathcal{E}_{-p} \subset \mathcal{E}_{-q} \subset \mathcal{E}'.
$$
We can equip \( \mathcal{E}' \) with the inductive limit topology. The triple \( \mathcal{E} \subset E \subset \mathcal{E}' \) is a Gel'fand triple.

By Minlos' theorem, there exists a unique probability measure \( \mu \) on the Borel subsets of \( \mathcal{E}' \) such that for all \( f \in \mathcal{E} \), the random variable \( \langle \cdot , f \rangle \) is normally distributed with mean 0 and variance \( |f|^2 \). Here \( \langle , \rangle \) is the duality between \( \mathcal{E}' \) and \( \mathcal{E} \). That means, there exists a unique probability measure \( \mu \) on the Borel subsets of \( \mathcal{E}' \) such that

\[
\int_{\mathcal{E}'} e^{i \langle x, \xi \rangle} d\mu(x) = e^{-\frac{|\xi|^2}{2}}, \quad \forall \xi \in \mathcal{E}.
\]  

(2.1)

Because of the denseness of \( \mathcal{E} \) in \( E \), we can define for each \( f \in E \), a random variable \( \langle \cdot , f \rangle \) on \( \mathcal{E}' \) which is normally distributed with mean 0 and variance \( |f|^2 \).

For \( x \in \mathcal{E}' \), we define

\[
x^{\otimes n} := \sum_{k=0}^{n/2} \frac{(-1)^k n!}{(n-2k)! k! 2^k} \tau^{\otimes k} \otimes x^{\otimes (n-2k)},
\]

(2.2)

where \( \tau \in (\mathcal{E} \otimes \mathcal{E})' \) is defined by \( \langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle \). (Here we have used the following notation: if \( u \in \mathcal{E} \otimes \mathcal{E}' \) and \( v \in \mathcal{E} \otimes \mathcal{E}' \), then we denote by \( u \otimes v \) the symmetrization of the vector \( u \otimes v \).)

Let \( E_c \) denote the complexification of \( E \). Let

\[
(L^2) := \{ \varphi : \mathcal{E}' \to \mathbb{C} \mid \varphi \text{ is measurable and } \int_{\mathcal{E}'} |f(x)|^2 d\mu(x) < \infty \}.
\]

Hence \( (L^2) \) is the space of all complex valued square integrable functions on \( \mathcal{E}' \). We denote the norm in \( (L^2) \) by \( \| \cdot \|_0 \). That means, if \( \varphi \in (L^2) \), then

\[
\| \varphi \|_0^2 := \int_{\mathcal{E}'} |f(x)|^2 d\mu(x).
\]

It can be proved that for each \( \varphi \in (L^2) \), there exists a unique sequence \( \{f_n\}_{n \geq 0} \), \( f_n \in E_c^{\otimes n} \), such that:

\[
\varphi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle.
\]

(2.3)
(Here $E_{\mathcal{C}}^{\otimes n}$ means the space of all symmetric $n$-tensors.) Moreover we have

$$||\varphi||_0^2 = \sum_{n=0}^{\infty} n! |f_n|^2.$$ 

The second quantization operator $\Gamma(A)$ of $A$ is defined by

$$\Gamma(A)\varphi = \sum_{n=0}^{\infty} \langle x^{\otimes n} : A^{\otimes n} f_n \rangle.$$ \hspace{1cm} (2.4)

By using $(L^2)$ and $\Gamma(A)$ instead of $E$ and $A$, respectively, we can construct another Gel'fand triple $(E) \subset (L^2) \subset (E)^*$. For any $p \geq 0$, let

$$(E_p) := \{ \varphi \in (L^2) | ||\Gamma(A)^p \varphi||_0 < \infty \}.$$ 

On $(E_p)$ we define the norm $||\varphi||_p := ||\Gamma(A)^p \varphi||_0$. ($(E_p)$, $|| \cdot ||_p$) is a Hilbert space. If $0 < p < q$, then $(E_q) \subset (E_p)$. Let $(E) := \bigcap_{p \geq 0} (E_p)$.

The dual of $(E_p)$ is isomorphic to $(E_{-p})$, which is the completion of $(L^2)$ with respect to the norm $|| \cdot ||_{-p}$ defined by $||\varphi||_{-p} := ||\Gamma(A)^{-p} \varphi||_0$. If $0 < p < q$, then $(E_{-p}) \subset (E_{-q})$. The dual of $(E)$ is $(E)^* := \bigcup_{p \geq 0} (E_{-p})$.

The elements in $(E)$ are called test functions on $E'$. The elements in $(E)^*$ are called generalized functions on $E'$.

The bilinear pairing between $(E)^*$ and $(E)$ is denoted by $\langle \cdot, \cdot \rangle$. It must be mentioned that if $\varphi \in (L^2)$ and $\psi \in (E)$, then $\langle \langle \varphi, \psi \rangle \rangle = (\varphi, \overline{\psi})$, where $(\cdot, \cdot)$ is the inner product on the complex Hilbert space $(L^2)$.

Let $\varphi \in (L^2)$ be represented by $\varphi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n} : f_n \rangle$. It can be shown that $\varphi \in (E)$ if and only if for all $p \geq 0$, we have

$$||\varphi||_p^2 = \sum_{n=0}^{\infty} n! |f_n|^2 < \infty.$$ 

On the other hand each $\Phi \in (E)^*$ can be represented as

$$\Phi = \sum_{n=0}^{\infty} \langle x^{\otimes n} : F_n \rangle, \quad F_n \in (E')^{\otimes n},$$
and there exists a $p > 0$ depending on $\Phi$ such that

\[ ||\Phi||^2_{p} = \sum_{n=0}^{\infty} n! |F_n|^2_p < \infty. \]

For $\Phi \in (\mathcal{E})^*$ and $\varphi \in \mathcal{E}$ from above we have

\[ \langle \langle \Phi, \varphi \rangle \rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle. \]

### 2.2 A Reconstruction of the Schwartz Space

As an example of the previous construction we consider the real separable Hilbert space $E = L^2(\mathbb{R})$ and the operator $A = -d^2/dx^2 + x^2 + 1$. Let $\mathcal{S}(\mathbb{R})$ be the space of all infinitely many times differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $\forall n, k \in \mathbb{N}$, $\sup_{x \in \mathbb{R}} |x^n(d/dx)^k f(x)| < \infty$. This space is called "the Schwartz space".

The Schwartz space is contained in the domain of the operator $A$. Because $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ we conclude that $A$ is densely defined on $L^2(\mathbb{R})$.

The operator $A$ has the following properties:

1. Let $H_n(x) = (-1)^n e^{x^2}(d/dx)^n e^{-x^2}$ be the Hermite polynomial of degree $n$ and let

\[ e_n(x) = \frac{1}{\sqrt{\sqrt{n}2^n n!}} H_n(x) e^{-x^2} \]

be the corresponding Hermite function. Then the set $\{e_n \mid n \geq 0\}$ is an orthonormal basis for $L^2(\mathbb{R})$. The Hermite functions are also eigenfunctions of $A$, namely

\[ \forall n \geq 0, \quad Ae_n = (2n + 2)e_n. \]

2. $A^{-1}$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R})$ because:

\[ \sum_{n=0}^{\infty} \frac{1}{(2n + 2)^2} < \infty. \]
Now, for each \( p \geq 0 \), we define

\[
|f|_p = |A^p f|_0,
\]

where \( |\cdot|_0 \) is the \( L^2(\mathbb{R}) \)-norm. If we denote by \((\cdot,\cdot)\) the inner product of \( L^2(\mathbb{R}) \), then

\[
|f|_p^2 = \sum_{n=0}^{\infty} (2n+2)^{2p} (f, e_n)^2.
\]

Let

\[
S_p(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) | |f|_p < \infty \}.
\]

Then \( \bigcap_{p \geq 0} S_p(\mathbb{R}) = S(\mathbb{R}) \). Therefore, in this case the Gel'fand triple is

\[
S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'(\mathbb{R}).
\]

### 2.3 Differential Operators and the Adjoints

Consider a simple test function \( \varphi(x) = \langle x^\otimes n, f \rangle \in (\mathcal{E}) \). Let \( y \in \mathcal{E}' \). We can show that

\[
\lim_{t \to 0} \frac{\varphi(x + ty) - \varphi(x)}{t} = n \langle x^{\otimes(n-1)}, y \hat{\otimes}_1 f \rangle,
\]

where \( y \hat{\otimes}_1 : E_c^\otimes n \to E_c^{\otimes(n-1)} \) is the unique continuous and linear map such that

\[
y \hat{\otimes}_1 g^{\otimes n} = \langle y, g \rangle g^{\otimes(n-1)}, \quad g \in E_c.
\]

This shows that the function \( \varphi \) has Gâteaux derivative \( D_y \varphi \).

In general, for \( \varphi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, f_n \rangle \in (\mathcal{E}) \), we may define

\[
D_y \varphi(x) = \sum_{n=1}^{\infty} n \langle x^{\otimes(n-1)}, y \hat{\otimes}_1 f_n \rangle.
\]  \( \tag{2.5} \)

It can be checked that \( D_y \) is a continuous linear operator on \( (\mathcal{E}) \) (see [15], Theorem 9.1).

We can define the adjoint operator \( D_y^* \) of \( D_y \) by the duality between \( (\mathcal{E})^* \) and \( (\mathcal{E}) \), i.e.,

\[
\langle \langle D_y^* \Phi, \psi \rangle \rangle = \langle \langle \Phi, D_y \psi \rangle \rangle, \quad \Phi \in (\mathcal{E})^*, \quad \psi \in (\mathcal{E}). \]  \( \tag{2.6} \)
The adjoint $D_y^*$ is a continuous linear operator. For $\Phi(x) = \sum_{n=0}^{\infty}\langle : x^{(n+1)} : , F_n \rangle \in (\mathcal{E})^*$, we have

$$D_y^*\Phi(x) = \sum_{n=0}^{\infty}\langle : x^{(n+1)} : , y\otimes F_n \rangle.$$  \hfill (2.7)

For $y \in \mathcal{E}$, the differential operator $D_y$ extends by continuity to a continuous linear operator from $(\mathcal{E})^*$ into itself ([15], Theorem 9.10). The extension is denoted by $\widetilde{D}_y$. Moreover, for such $y \in \mathcal{E}$, the restriction of $D_y^*$ to $(\mathcal{E})$ is a continuous linear operator from $(\mathcal{E})$ into itself ([15], Corollary 9.14).

### 2.4 Multiplication Operators

If $\varphi, \psi \in (\mathcal{E})$, then the pointwise multiplication $\varphi \cdot \psi$ is also in $(\mathcal{E})$. Let $\Phi \in (\mathcal{E})^*$ be fixed. For $\varphi \in (\mathcal{E})$, define $\Phi \cdot \varphi \in (\mathcal{E})^*$ by

$$\langle \langle \Phi \cdot \varphi, \psi \rangle \rangle = \langle \langle \Phi, \varphi \cdot \psi \rangle \rangle, \quad \psi \in (\mathcal{E}).$$  \hfill (2.8)

This multiplication operator by $\Phi$ is a continuous linear map from $(\mathcal{E})$ into $(\mathcal{E})^*$.

In particular, if $\eta \in \mathcal{E}$, then the multiplication by $\langle \cdot , \eta \rangle$, denoted by $Q_\eta$, is a continuous linear operator from $(\mathcal{E})$ into itself and can be extended to a continuous linear operator $\widetilde{Q}_\eta$ from $(\mathcal{E})^*$ into itself. The operators $\widetilde{Q}_\eta, \widetilde{D}_\eta$, and $D_\eta^*$ are related by the formula

$$\widetilde{Q}_\eta = \widetilde{D}_\eta + D_\eta^*,$$

see [15], Theorem 9.18.

### 2.5 The Renormalized Exponential Functions

Let $x \in \mathcal{E}'_c$. We define the following renormalized exponential function

$$: e^{\langle \cdot , x \rangle} : = \sum_{n=0}^{\infty} \frac{1}{n!} \langle : x^{\otimes n} : , x^{\otimes n} \rangle.$$  \hfill (2.9)

It is easy to see that

$$\Vert : e^{\langle \cdot , x \rangle} : \Vert_p = e^{\frac{|x|^2}{2}}.$$  \hfill (2.10)
Thus for all \( x \in \mathcal{E}_c' \), we have \( : e^{(: x :)} : \in (\mathcal{E})^* \). Also \( : e^{(: x :)} : \in (L^2) \) if and only if \( x \in E_c \) and \( : e^{(: x :)} : \in (\mathcal{E}) \) if and only if \( x \in E_c \). If \( x \in \mathcal{E}_c' \), \( \xi \in \mathcal{E}_c \), then we have

\[
\langle (: e^{(: x :)} : , : e^{(: \xi :)} :) \rangle = e^{(x, \xi)}. \tag{2.11}
\]

If \( x \in \mathcal{E}_c' \), \( \xi \in \mathcal{E}_c \), then we have

\[
:e^{(: x, \xi :)} = e^{(: x, \xi :)} - \frac{1}{2} (x, \xi). \tag{2.12}
\]

The renormalized exponential functions \( \{ : e^{(: \xi :)} : \mid \xi \in \mathcal{E}_c \} \) are linearly independent and span a dense subspace of \( (\mathcal{E}) \).

### 2.6 The S-transform

For all \( \Phi \in (\mathcal{E})^* \), we define the S-transform of \( \Phi \) to be the function on \( \mathcal{E}_c \)

\[
S\Phi(\xi) = \langle \langle \Phi , : e^{(: \xi :)} : \rangle \rangle , \quad \xi \in \mathcal{E}_c. \tag{2.13}
\]

Because the renormalized exponential functions span a dense subspace of \( (\mathcal{E}) \), the S-transform is injective.

### 2.7 The Wick Product

The Wick product of two generalized functions \( \Phi \) and \( \Psi \) in \( (\mathcal{E})^* \), denoted by \( \Phi \circ \Psi \), is the unique generalized function in \( (\mathcal{E})^* \) such that:

\[
S(\Phi \circ \Psi) = (S\Phi)(S\Psi). \tag{2.14}
\]

The mapping \( (\phi, \psi) \mapsto \phi \circ \psi \) is continuous from \( (\mathcal{E}) \times (\mathcal{E}) \) into \( (\mathcal{E}) \). It is also continuous from \( (\mathcal{E})^* \times (\mathcal{E})^* \) into \( (\mathcal{E})^* \).

If \( \varphi = \sum_{n=0}^{\infty} \langle : \otimes^n : , f_n \rangle \in (\mathcal{E}) \) and \( \psi = \sum_{n=0}^{\infty} \langle : \otimes^n : , g_n \rangle \in (\mathcal{E}) \), then

\[
\varphi \circ \psi = \sum_{n=0}^{\infty} \langle : \otimes^n : , \sum_{p+q=n} f_p \otimes g_q \rangle. \tag{2.15}
\]
Chapter 3
Heisenberg Uncertainty Principle

3.1 Commutation Relationships

Theorem 3.1. [15, Thm 9.15] For all $\xi, \eta \in \mathcal{E}$ the commutator of $\tilde{D}_\xi$ and $D_\eta^*$ is given by

$$[\tilde{D}_\xi, D_\eta^*] = \langle \xi, \eta \rangle I.$$  \hspace{1cm} (3.1)

Proof. Since the operators $\tilde{D}_\xi$ and $D_\eta^*$ are both linear and continuous it is enough to check that for $\varphi(x) = \langle :x^{\otimes n} :, u^{\otimes n} \rangle$, where $u \in E_c$, we have:

$$[\tilde{D}_\xi, D_\eta^*] \varphi(x) = \langle \xi, \eta \rangle \varphi(x).$$

For such a function $\varphi$ we have:

$$[\tilde{D}_\xi, D_\eta^*] \varphi(x) = \tilde{D}_\xi D_\eta^* \varphi(x) - D_\eta^* D_\xi \varphi(x)$$

$$= \tilde{D}_\xi \left( \langle : x^{\otimes (n+1)} :, \eta \otimes u^{\otimes n} \rangle \right)$$

$$- D_\eta^* \left( n \langle : x^{\otimes (n-1)} :, \langle \xi, u \rangle u^{\otimes (n-1)} \rangle \right)$$

$$= \tilde{D}_\xi \left( \frac{1}{n+1} \sum_{i=0}^{n} \langle : x^{\otimes (n+1)} :, u^{\otimes i} \otimes \eta \otimes u^{\otimes (n-i)} \rangle \right)$$

$$- n \left( \langle : x^{\otimes n} :, \langle \xi, u \rangle \eta \otimes u^{\otimes (n-1)} \rangle \right)$$

$$= \frac{1}{n+1} (n+1) \langle : x^{\otimes n} :, \langle \xi, \eta \rangle u^{\otimes n} \rangle$$

$$+ \frac{1}{n+1} (n+1) \sum_{i=1}^{n} \langle : x^{\otimes n} :, \langle \xi, u \rangle u^{\otimes (i-1)} \otimes \eta \otimes u^{\otimes (n-i)} \rangle$$

$$- n \left( \langle : x^{\otimes n} :, \langle \xi, u \rangle \eta \otimes u^{\otimes (n-1)} \rangle \right)$$

$$= \langle \xi, \eta \rangle \langle : x^{\otimes n} :, u^{\otimes n} \rangle$$

$$+ \langle \xi, u \rangle \sum_{j=0}^{n-1} \langle : x^{\otimes n} :, u^{\otimes j} \otimes \eta \otimes u^{\otimes (n-j-1)} \rangle$$

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Thus we have proved that $[\tilde{D}_\xi, D_\eta^*] = \langle \xi, \eta \rangle I$. \hfill \Box

**Theorem 3.2.** For any operator $B$ on $E_c$ that is self-adjoint with respect to the bilinear pair $\langle \cdot, \cdot \rangle$ and for any $\eta \in \mathcal{E}$, we have

$$D_\eta \Gamma(B) = \Gamma(B) D_B \eta$$

(3.2)

**Proof.** Let $\varphi(x) = \langle x^{\otimes n} : u^{\otimes n} \rangle$, where $u \in E_c$. Then we have:

$$D_\eta \Gamma(B) \varphi(x) = D_\eta \langle x^{\otimes n} : (Bu)^{\otimes n} \rangle$$

$$= n \langle \eta, Bu \rangle \langle x^{\otimes(n-1)} : (Bu)^{\otimes(n-1)} \rangle$$

$$= n \langle B \eta, u \rangle \langle x^{\otimes(n-1)} : u^{\otimes(n-1)} \rangle$$

$$= \Gamma(B) \langle B \eta, u \rangle \langle x^{\otimes(n-1)} : u^{\otimes(n-1)} \rangle$$

$$= \Gamma(B) D_B \eta \langle x^{\otimes n} : u^{\otimes n} \rangle$$

$$= \Gamma(B) D_B \eta \varphi(x).$$

Hence $D_\eta \Gamma(B) = \Gamma(B) D_B \eta$. \hfill \Box

**Theorem 3.3.** For any operator $B$ on $E_c$ that is self-adjoint with respect to the bilinear pair $\langle \cdot, \cdot \rangle$ and for any $\eta \in \mathcal{E}$, we have

$$D_\eta^* \Gamma(B) = \Gamma(B) D_{B^{-1}} \eta$$

(3.3)
Proof 1. Let \( \varphi(x) = \langle x^{\otimes n} : u^{\otimes n} \rangle \), where \( u \in E_{\eta} \). Then we have:

\[
D^*_{\eta} \Gamma(B) \varphi(x) = D^*_{\eta} \langle x^{\otimes n} : (Bu)^{\otimes n} \rangle = \langle x^{\otimes(n+1)} : \eta \widehat{(Bu)^{\otimes n}} \rangle = \langle x^{\otimes(n+1)} : B(B^{-1}\eta) \widehat{u^{\otimes n}} \rangle = \Gamma(B) \langle x^{\otimes(n+1)} : (B^{-1}\eta) \widehat{u^{\otimes n}} \rangle = \Gamma(B) D^*_{B^{-1}\eta} (\langle x^{\otimes n} : u^{\otimes n} \rangle) = \Gamma(B) D^*_{B^{-1}\eta} \varphi(x).
\]

Thus \( D^*_{\eta} \Gamma(B) = \Gamma(B) D^*_{B^{-1}\eta} \). \( \square \)

Proof 2. Using Theorem 3.2 for \( D_{B^{-1}\eta} \) and \( \Gamma(B) \) we have \( D_{B^{-1}\eta} \Gamma(B) = \Gamma(B) D_{B^{-1}\eta} \).

Taking the adjoint of the operators from both sides of the last relation we get \( \Gamma(B) D^*_{B^{-1}\eta} = D^*_{\eta} \Gamma(B) \). \( \square \)

3.2 A Heisenberg Inequality for White Noise Analysis

Theorem 3.4. Let \( \eta \in E \setminus \{0\} \) and \( \varphi \in (L^2) \). Then

\[
||Q_\eta \varphi||_0 \cdot ||Q_\eta \Gamma(-iI) \varphi||_0 \geq |\eta|^2 \cdot ||\varphi||^2_0.
\]

Proof. We will see later that if \( Q_\eta \varphi = 0 \) or \( Q_\eta \Gamma(-iI) \varphi = 0 \), then \( \varphi = 0 \). If \( ||Q_\eta \varphi||_0 = \infty \) or \( ||Q_\eta \Gamma(-iI) \varphi||_0 = \infty \), then the inequality in the theorem is obvious. Hence without loss of generality we may assume that \( ||(\tilde{D}_\eta + D^*_\eta) \varphi||_0 < \infty \) and \( ||(\tilde{D}_\eta + D^*_\eta) \Gamma(-iI) \varphi||_0 < \infty \). This is equivalent to \( ||(\tilde{D}_\eta + D^*_\eta) \varphi||_0 < \infty \) and \( ||(\tilde{D}_\eta - D^*_\eta) \varphi||_0 < \infty \), which in turn is equivalent to \( ||\tilde{D}_\eta \varphi||_0 < \infty \) and \( ||D^*_\eta \varphi||_0 < \infty \).

The space of the functions satisfying the last two conditions is dense in \( (L^2) \). For example, all the functions with a "finite Fock decomposition" \( \varphi = \sum_{n=0}^{N} \langle \cdot^{\otimes n} : f_n \rangle, \) \( N \geq 0 \) are in this space.

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For such functions we have:

\[
\begin{align*}
| |\tilde{Q}_\eta \varphi||_0 \cdot | |\tilde{Q}_\eta \Gamma(-iI)\varphi||_0 & \\
= | |(\tilde{D}_\eta + D_{\eta}^*)\varphi||_0 \cdot | |(\tilde{D}_\eta + D_{\eta}^*)\Gamma(-iI)\varphi||_0 & \\
= | |(\tilde{D}_\eta + D_{\eta}^*)\varphi||_0 \cdot | |\Gamma(-i)\Gamma(-i)(\tilde{D}_{-i\eta} + D_{-i\eta}^*)\varphi||_0 & \\
= | |(\tilde{D}_\eta + D_{\eta}^*)\varphi||_0 \cdot | |(\tilde{D}_{-i\eta} + D_{\eta}^*)\varphi||_0 & \\
= | |(\tilde{D}_\eta + D_{\eta}^*)\varphi||_0 \cdot | |(\tilde{D}_\eta - D_{\eta}^*)\varphi||_0 & \\
& = \frac{1}{2} \left[ | |(\tilde{D}_\eta + D_{\eta}^*)\varphi||_0 \cdot | |(\tilde{D}_\eta - D_{\eta}^*)\varphi||_0 + | |(\tilde{D}_\eta - D_{\eta}^*)\varphi||_0 \cdot | |(\tilde{D}_\eta + D_{\eta}^*)\varphi||_0 \right] \\
& \geq \frac{1}{2} \left[ | |(\tilde{D}_\eta + D_{\eta}^*)\varphi, (\tilde{D}_\eta - D_{\eta}^*)\varphi|| + | |(\tilde{D}_\eta - D_{\eta}^*)\varphi, (\tilde{D}_\eta + D_{\eta}^*)\varphi|| \right] & (3.4) \\
& \geq \frac{1}{2} | |(\tilde{D}_\eta + D_{\eta}^*)\varphi, (\tilde{D}_\eta - D_{\eta}^*)\varphi + ((\tilde{D}_\eta - D_{\eta}^*)\varphi, (\tilde{D}_\eta + D_{\eta}^*)\varphi)| | & (3.5) \\
& = | |(\tilde{D}_\eta \varphi, \tilde{D}_\eta \varphi) - (D_{\eta}^* \varphi, D_{\eta}^* \varphi)| | \\
& = | |\langle \bar{\tilde{D}}_\eta \varphi, \tilde{D}_\eta \varphi \rangle - \langle \bar{D}_{\eta}^* \varphi, \bar{D}_{\eta}^* \varphi \rangle| | \\
& = | |\langle \bar{\tilde{D}}_\eta \varphi, \tilde{D}_\eta \varphi \rangle - \langle D_{\eta}^* \varphi, D_{\eta}^* \varphi \rangle| | \\
& = | |\langle D_{\eta}^* \tilde{D}_\eta \varphi, \varphi \rangle - \langle \bar{\tilde{D}}_\eta D_{\eta}^* \varphi, \varphi \rangle| | \\
& = | |\langle \langle D_{\eta}^* D_{\eta}^* \varphi, \varphi \rangle \rangle | | \\
& = | |\langle \langle \eta, \eta \rangle I \varphi, \varphi \rangle \rangle | | \\
& = \langle \eta, \eta \rangle \langle \varphi, \varphi \rangle \\
& = | |\eta||_0^2 \cdot | |\varphi||_0^2. \\
\end{align*}
\]

Therefore we have proved that \(| |\tilde{Q}_\eta \varphi||_0 \cdot | |\tilde{Q}_\eta \Gamma(-iI)\varphi||_0 \geq | |\eta||_0^2 \cdot | |\varphi||_0^2. \)
Chapter 4
Equality in the Heisenberg Uncertainty Principle

4.1 The Equality Case

Now we want to answer the question: When does the equality in Theorem 3.4 hold?

Lemma 4.1. Let \( \eta \in \mathcal{E}, \varphi = \sum_{n=0}^{\infty} \langle \cdot, x^{\otimes n} \cdot, f_n \rangle \in (L^2) \), and \( \alpha \in \mathbb{C} \). The following equations are equivalent:

\[
\tilde{D}_\eta \varphi = \alpha D^*_\eta \varphi, \quad (4.1)
\]
\[
\forall n \geq 0, \quad (n + 1)f_{n+1} \hat{\otimes} \eta = \alpha f_{n-1} \hat{\otimes} \eta, \quad (4.2)
\]
\[
\tilde{D}_\eta \varphi = \frac{\alpha}{\alpha + 1} \langle \cdot, \eta \rangle \varphi, \quad (4.3)
\]

where by convention \( f_{-1} = 0 \). In the case \( \alpha = -1 \), then the relation (4.3) means \( \langle \cdot, \eta \rangle \varphi = 0 \).

Proof. We have

\[
\tilde{D}_\eta \varphi = \sum_{n=0}^{\infty} n \langle \cdot, x^{\otimes(n-1)} \cdot, f_n \hat{\otimes} \eta \rangle = \sum_{n=0}^{\infty} (n + 1) \langle \cdot, x^{\otimes n} \cdot, f_{n+1} \hat{\otimes} \eta \rangle
\]
\[
\alpha D^*_\eta \varphi = \alpha \sum_{n=0}^{\infty} \langle \cdot, x^{\otimes(n+1)} \cdot, f_n \hat{\otimes} \eta \rangle = \alpha \sum_{n=0}^{\infty} \langle \cdot, x^{\otimes n} \cdot, f_{n-1} \hat{\otimes} \eta \rangle.
\]

This shows that (4.1) \( \iff \) (4.2). To see that (4.1) \( \iff \) (4.3), we make use of the formula \( \tilde{D}_\eta + D^*_\eta = \tilde{Q}_\eta \), where \( \tilde{Q}_\eta \) is the multiplication operator by \( \langle \cdot, \eta \rangle \), i.e.,

\[
\tilde{D}_\eta \varphi = \alpha D^*_\eta \varphi \iff \tilde{D}_\eta \varphi = \alpha (\langle \cdot, \eta \rangle - \tilde{D}_\eta) \varphi \iff \tilde{D}_\eta \varphi = \frac{\alpha}{\alpha + 1} \langle \cdot, \eta \rangle \varphi.
\]

Thus (4.1) \( \iff \) (4.3). \( \square \)

Lemma 4.2. For all \( y \in \mathcal{E}' \), \( \phi, \psi \in (\mathcal{E})^* \) we have:

\[
D^*_y (\phi \circ \psi) = (D^*_y \phi) \circ \psi = \phi \circ (D^*_y \psi).
\]
Proof. The equalities follow from [15], Corollary 9.14. On the other hand, they can be proved as follows:

Let \( \phi = \sum_{n=0}^{\infty} \langle : \Theta^n : , f_n \rangle \) and \( \psi = \sum_{n=0}^{\infty} \langle : \Theta^n : , g_n \rangle \). Then

\[
\phi \circ \psi = \sum_{n=0}^{\infty} \langle : \Theta^n : , h_n \rangle,
\]

where \( h_n = \sum_{p+q=n} f_p \hat{\otimes} g_q \). Hence

\[
D_y^* (\phi \circ \psi) = \sum_{n=0}^{\infty} \langle : \Theta^{(n+1)} : , \sum_{p+q=n} y \hat{\otimes} (f_p \hat{\otimes} g_q) \rangle.
\]

On the other hand, since \( D_y^* \phi = \sum_{n=0}^{\infty} \langle : \Theta^{(n+1)} : , y \hat{\otimes} f_n \rangle \), we have

\[
(D_y^* \phi) \circ \psi = \sum_{n=1}^{\infty} \langle : \Theta^n : , \sum_{p+q=n} (y \hat{\otimes} f_{p-1}) \hat{\otimes} g_q \rangle
\]

\[
= \sum_{n=0}^{\infty} \langle : \Theta^{(n+1)} : , \sum_{p+q=n} (y \hat{\otimes} f_p) \hat{\otimes} g_q \rangle.
\]

Hence we get \( D_y^* (\phi \circ \psi) = (D_y^* \phi) \circ \psi \). By the symmetry of the Wick product we also have \( D_y^* (\phi \circ \psi) = \phi \circ (D_y^* \psi) \).

\[\square\]

Notation. Let \( K \) be a closed subspace of \( E \). We use \( K_c \) to denote the complexification of \( K \). Let \( (L^2_K) \) be the subspace of \( (L^2) \) consisting of all functions \( \varphi = \sum_{n=0}^{\infty} \langle : \Theta^n : , f_n \rangle \) such that \( f_n \in K_c \hat{\otimes}^n \), for all \( n \geq 0 \). For \( p \geq 0 \), define

\[
(E_K)_p = \{ \varphi \in (L^2_K) \mid ||\varphi||_p < \infty \}.
\]

Let \( (E_K)_{-p} \) be the completion of \( (L^2_K) \) with respect to the norm \( || \cdot ||_{-p} \). Thus for all \( p \in \mathbb{R} \) we have \( (E_K)_p \subseteq (E)_p \). Let

\[
(E_K) = \bigcap_{p \geq 0} (E_K)_p \subseteq (E), \quad (E_K)^* = \bigcup_{p \geq 0} (E_K)_{-p} \subseteq (E)^*.
\]

Definition 4.3. A function \( \phi \in (E)^* \) is said to be supported by \( K \) if \( \phi \in (E_K)^* \).
Lemma 4.4. Let $H_1$ and $H_2$ be two orthogonal closed subspaces of $E$. Suppose $\phi$ and $\psi \in (E)^*$ are supported by $H_1$ and $H_2$, respectively. Then

$$||\phi \circ \psi||_0 = ||\phi||_0 \cdot ||\psi||_0,$$

where we use the convention that $\infty \cdot 0 = 0$.

Proof 1. Let $\phi = \sum_{n=0}^{\infty} (\cdot \otimes_n, f_n)$ and $\psi = \sum_{n=0}^{\infty} (\cdot \otimes_n, g_n)$. Then

$$\phi \circ \psi = \sum_{n=0}^{\infty} (\cdot \otimes_n, \sum_{p+q=n} f_p \otimes g_q).$$

Let $S_n$ denote the permutation group of the set $\{1, 2, \ldots, n\}$. For each $\sigma \in S_n$ we consider the continuous linear operator $\bar{\sigma}$ from $E^{\otimes n}$ into itself defined by

$$\bar{\sigma}(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}.$$

Then we have:

$$||\phi \circ \psi||_0^2 = \sum_{n=0}^{\infty} n! \left( \sum_{p+q=n} f_p \otimes g_q \right)^2$$

$$= \sum_{n=0}^{\infty} n! \left( \sum_{p+q=n} f_p \otimes g_q, \sum_{p'+q'=n} f_{p'} \otimes g_{q'} \right)$$

$$= \sum_{n=0}^{\infty} n! \sum_{p+q=n} \sum_{p'+q'=n} (f_p \otimes g_q, f_{p'} \otimes g_{q'}).$$

Because $\phi$ and $\psi$ are supported by orthogonal subspaces of $E$ we see that whenever $p \neq p'$, which is the same as $q \neq q'$, we have $(f_p \otimes g_q, f_{p'} \otimes g_{q'}) = 0$. Thus we have

$$||\phi \circ \psi||_0^2 = \sum_{n=0}^{\infty} n! \sum_{p+q=n} (f_p \otimes g_q, f_{p'} \otimes g_{q'})$$

$$= \sum_{n=0}^{\infty} n! \sum_{p+q=n} \frac{1}{n!} \sum_{\sigma \in S_n} \tilde{\sigma}(f_p \otimes g_q), \frac{1}{n!} \sum_{\sigma' \in S_n} \tilde{\sigma}'(f_{p'} \otimes g_{q'})$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{p+q=n} \sum_{\sigma \in S_n} \sum_{\sigma' \in S_n} (\tilde{\sigma}(f_p \otimes g_q), \tilde{\sigma}'(f_{p'} \otimes g_{q'})).$$

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Now let $p$ and $q$ be fixed. On exactly $p$ positions $\{\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(p)\} \subseteq \{1, 2, \ldots, n\}$ we have “components” of $f_p$, and on the other $q$ positions we have “components” of $g_q$. Again because $\phi$ and $\psi$ are supported by two orthogonal subspaces of $E$ we see that if

$$\{\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(p)\} \neq \{\sigma'^{-1}(1), \sigma'^{-1}(2), \ldots, \sigma'^{-1}(p)\},$$

then

$$(\bar{\sigma}(f_p \otimes g_q), \bar{\sigma}'(f_p \otimes g_q)) = 0,$$

and if $\{\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(p)\} = \{\sigma'^{-1}(1), \sigma'^{-1}(2), \ldots, \sigma'^{-1}(p)\}$, then

$$(\bar{\sigma}(f_p \otimes g_q), \bar{\sigma}'(f_p \otimes g_q)) = |f_p|_0^2 \cdot |g_q|_0^2.$$

The last relation happens just because $f_p$ and $g_q$ are symmetric vectors.

Now, we need to check the number of times the term $|f_p|_0^2 \cdot |g_q|_0^2$ appears in the sum of $||\phi \circ \psi||_0^2$. For this to happen we must pick $p$ positions $\{i_1, i_2, \ldots, i_p\}$ from $\{1, 2, \ldots, n\}$. This can be done in $\frac{n!}{p!q!}$ different ways. After choosing the $p$ positions, we consider all the pairs $(\tau, \tau')$ of possible permutations of these positions among themselves. There are exactly $p! \times p!$ such pairs. Any of these pairs gives us a $|f_p|_0^2$. We also consider all the pairs $(\nu, \nu')$ of possible permutations of the other $q$ positions among themselves. There are exactly $q! \times q!$ such pairs. Any of these pairs gives us a $|g_q|_0^2$. Hence the coefficient of $|f_p|_0^2 \cdot |g_q|_0^2$ in the sum that gives us $||\phi \circ \psi||_0^2$ is $\frac{1}{n!} \cdot \frac{n!}{p!q!} \cdot (p!)^2 \cdot (q!)^2$. Therefore, we have:

$$||\phi \circ \psi||_0^2 = \sum_{n=0}^{\infty} \sum_{p+q=n} p! \cdot q! \cdot |f_p|_0^2 \cdot |g_q|_0^2$$

$$= \sum_{p=0}^{\infty} p! |f_p|_0^2 \cdot \sum_{q=0}^{\infty} q! |g_q|_0^2.$$

Thus $||\phi \circ \psi||_0 = ||\phi||_0 \cdot ||\psi||_0.$
**Proof 2.** We can prove the last lemma much easier using the following:

**Remark 4.5.** If $\phi$ and $\psi$ are $(L^2)$ functions supported by two orthogonal subspaces $H_1$ and $H_2$, then

$$
\phi \odot \psi = \phi \cdot \psi. \quad (4.4)
$$

**Proof.** Let $P_1$ and $P_2$ be the orthogonal projections on $H_1$ and $H_2$ and let $P_3 = I - P_1 - P_2$, where $I$ is the identity operator of the Hilbert space $E$. Because $\phi$ and $\psi$ are supported by two orthogonal subspaces, they are independent random variables. Let $\xi \in \mathcal{E}$ be fix. We can also see that $e^{(\cdot , P_3 \xi)}$ : and $e^{(\cdot , P_3 \xi)}$ : are supported by the same orthogonal subspaces so they are also independent. Therefore

$$
S(\phi \cdot \psi)(\xi) = S(\phi)(P_1 \xi) \cdot S(\psi)(P_2 \xi) \cdot S(1)(P_3 \xi) \\
= S(\phi)(\xi) \cdot S(\psi)(\xi) \cdot 1 \\
= S(\phi \odot \psi)(\xi).
$$

Thus $S(\phi \cdot \psi) = S(\phi \odot \psi)$. Because the $S$-transform is injective, we conclude that $\phi \cdot \psi = \phi \odot \psi$. □

Since $\phi$ and $\psi$ are independent random variables we have:

$$
||\phi \odot \psi||_0 = ||\phi \cdot \psi||_0 = ||\phi||_0 \cdot ||\psi||_0.
$$

This finishes the proof. □

**Comments.**

(a) The equality in the 1 dimensional case holds for the functions

$$
f(x) = c \exp(-kx^2), \quad k > 0, \quad c \in \mathbb{C}.
$$

(b) The function $f(x) = c \exp(-kx^2)$ satisfies the differential equation

$$
f'(x) = -2kx \cdot f(x). \quad (4.5)
$$
(c) The white noise analogue of the functions in (a) is the additive renormalization 
\( e^{z\cdot y} : = e^{z\cdot y}^2 \) of \( e^{z\cdot y}^2 \) defined by

\[
e^{z\cdot y}^2 : = \sum_{n=0}^{\infty} \frac{z^n}{n!} (\cdot ^{2n} \cdot \cdot ^{2n}), \tag{4.6}
\]

where \( y \in \mathcal{E}' \) and \( z \in \mathbb{C} \).

**Lemma 4.6.** Let \( p \in \mathbb{R} \), \( z \in \mathbb{C} \), and \( y \in \mathcal{E}' \). Then \( \| e^{z\cdot y}^2 : \|_p < \infty \) if and only if \( 2|z| \cdot |y|^2_p < 1 \). In that case, we have

\[
\| : e^{z\cdot y}^2 : \|_p^2 = \frac{1}{\sqrt{1 - 4|z|^2 |y|^4_p}}.
\]

**Proof.** We have:

\[
\| : e^{z\cdot y}^2 : \|_p^2 = \sum_{n=0}^{\infty} (2n)! \cdot \frac{|z|^{2n}}{(n!)^2} \cdot |y|^{4n}_p
\]

\[
= \sum_{n=0}^{\infty} (2n)!! \cdot (2n-1)!! \cdot \frac{|z|^{2n}}{(n!)^2} \cdot |y|^{4n}_p
\]

\[
= \sum_{n=0}^{\infty} 2^n n! \cdot (2n-1)!! \cdot \frac{|z|^{2n}}{(n!)^2} \cdot |y|^{4n}_p
\]

\[
= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot 4^n |z|^{2n} |y|^{4n}_p
\]

\[
= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot (4|z|^2 |y|^{4}_p)^n.
\]

We recognize that the series \( \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} w^n \), by the binomial series, is the Taylor expansion of the analytic function \( f(w) = (1 - w)^{-1/2} \), \( |w| < 1 \). □

**Corollary 4.7.** \( e^{z\cdot y}^2 : \in (\mathcal{E})^* \) for any \( z \in \mathbb{C} \) and \( y \in \mathcal{E}' \).

**Proof.** Since \( y \in \mathcal{E}' \), there exists \( p \in \mathbb{R} \) such that \( |y|^p_p < \infty \). Then for all \( q > 0 \) we have \( 2|z||y|_{p-q}^2 \leq 2|z|\rho^q |y|_p^2 \to 0 \) as \( q \to \infty \), where \( \rho := \frac{1}{\lambda} \in (0,1) \). Thus for \( q \) sufficiently large we have \( 2|z||y|_{p-q}^2 < 1 \) and by Lemma 4.6 this implies that \( : e^{z\cdot y}^2 : \in (\mathcal{E}_{p-q}) \subset (\mathcal{E})^* \). □
Theorem 4.8. (Equality in Heisenberg Inequality) Let \( \eta \in \mathcal{E} \setminus \{0\} \). Then a function \( \varphi \in (L^2) \) satisfies the equation

\[
||\mathcal{Q}_\eta \varphi||_0 \cdot ||\mathcal{Q}_\eta \Gamma \varphi||_0 = ||\eta||_0^2 \cdot ||\varphi||_0^2.
\]

if and only if \( \varphi \) is of the form

\[
\varphi = e^{(\alpha/2) \cdot \alpha / |\eta|} : \alpha \psi,
\]

where \( \alpha \in \mathbb{R} \), \( |\alpha| < 1 \), and \( \psi \in (L^2) \) is supported by the orthogonal complement \( \eta^\perp \) of \( \eta \).

Proof. It is clear that for \( \varphi = 0 \) we have equality in Theorem 3.4, so we may assume that \( \varphi \neq 0 \). In order to have equality in Theorem 3.4, we must have equalities for all the inequalities used in the proof of this theorem. First of all, we must have equality in (3.4); that means \( \exists c \in \mathbb{C} \cup \{\infty\} \) such that \( (\bar{D}_\eta + D^*_\eta) \varphi = c(\bar{D}_\eta - D^*_\eta) \varphi \), which in turn is equivalent to \( \bar{D}_\eta \varphi = \alpha D^*_\eta \varphi \), where \( \alpha = \frac{c + 1}{c - 1} \). Then we must have equality in (3.5), which after simplifications becomes \( |c| + |\bar{c}| = |c + \bar{c}| \) or \( c \in \mathbb{R} \cup \{\infty\} \), that means \( \alpha \in \mathbb{R} \cup \{\infty\} \). Thus we obtain the equation:

\[
\bar{D}_\eta \varphi = \alpha D^*_\eta \varphi
\]

which is equivalent to

\[
(n + 1)f_{n+1} \otimes_1 \eta = \alpha f_{n-1} \otimes \eta, \quad \forall n \geq 0.
\]

The case \( \alpha = \infty \) means that \( D^*_\eta \varphi = 0 \).

Let's consider first the case \( d \in \mathbb{R} \). We first try to find a particular solution of (4.1) of the form

\[
\varphi_\alpha = \sum_{n=0}^{\infty} \langle : \otimes^n : , c_n \eta \otimes^n \rangle, \quad (4.7)
\]
where \( \{c_n\}_{n \geq 0} \subset \mathbb{C} \) and \( c_0 = 1 \). Since (4.1) is equivalent to (4.2), we must have

\[
\forall n \geq 0, \quad (n + 1)c_{n+1}|\eta|^2 = \alpha c_{n-1}. \tag{4.8}
\]

This is the same as

\[
c_{n+1} = \frac{\alpha}{(n + 1)|\eta|^2} c_{n-1}, \tag{4.9}
\]

where \( c_{-1} = 0 \). Because \( c_{-1} = 0 \) we have \( \forall n \geq 0, c_{2n+1} = 0 \). Moreover

\[
\forall n \geq 0, c_{2n} = \frac{\alpha}{2n|\eta|^2} c_{2n-2} = \frac{\alpha^2}{2n(2n-2)|\eta|^2} c_{2n-4} = \cdots = \frac{\alpha^n}{(2n)!!|\eta|^{2n} c_0}.
\]

Thus we obtain

\[
\varphi_\alpha = \sum_{n=0}^{\infty} \left( c_{2n} \cdot \frac{\alpha^n}{2n|\eta|^2 c_{2n-2}} = e^{(\alpha/2)(\cdot, |\eta|^2) + (\alpha/2)(\cdot, |\eta|^2)}
\]

So \( \varphi_\alpha \) is a Gaussian function as in Comment (c). Let \( u = \frac{\eta}{|\eta|^2} \). Then \( |u|_0 = 1 \). The \( S \) transform of \( \varphi_\alpha \) is

\[
S\varphi_\alpha(\xi) = e^{(\alpha/2)(\xi, u)^2}. 
\]

Now, we observe that \( \varphi_{\alpha_1} \ast \varphi_{\alpha_2} = \varphi_{\alpha_1 + \alpha_2} \). To see this let's use the \( S \) transform. By the definition of the Wick product we have:

\[
S(\varphi_{\alpha_1} \ast \varphi_{\alpha_2})(\xi) = S\varphi_{\alpha_1}(\xi) \cdot S\varphi_{\alpha_2}(\xi)
\]

\[
= e^{(\alpha_1/2)(\cdot, u)^2} \cdot e^{(\alpha_2/2)(\cdot, u)^2}
\]

\[
= e^{((\alpha_1 + \alpha_2)/2)(\xi, u)^2}
\]

\[
= S\varphi_{\alpha_1 + \alpha_2}(\xi)
\]

With this particular solution \( \varphi_\alpha \), we can produce the general solution of equation (4.1). Take the Wick product of equation (4.1) with \( \varphi_{-\alpha} \)

\[
(\bar{D}_\eta \varphi) \ast \varphi_{-\alpha} = \alpha(D^*_\eta \varphi) \ast \varphi_{-\alpha} = -\varphi \ast (-\alpha D^*_\eta \varphi_{-\alpha}) = -\varphi \ast \bar{D}_\eta \varphi_{-\alpha}.
\]
Thus we obtain

\[(\tilde{D}_\eta \varphi) \circ \varphi_{-\alpha} + \varphi \circ (\tilde{D}_\eta \varphi_{-\alpha}) = 0.\]  

(4.10)

This is equivalent to

\[\tilde{D}_\eta (\varphi \circ \varphi_{-\alpha}) = 0.\]  

(4.11)

Let \(\psi = \varphi \circ \varphi_{-\alpha} \). We have \(\tilde{D}_\eta \psi = 0\). But \(\varphi \circ \varphi_{-\alpha} = \psi\). Hence

\[\psi \circ \varphi_{-\alpha} = (\varphi \circ \varphi_{-\alpha}) \circ \varphi_{-\alpha} = \varphi \circ (\varphi_{-\alpha} \circ \varphi_{-\alpha}) = \varphi \circ \varphi_0 = \varphi \circ 1 = \varphi.\]

Let \(\{e_n\}_{n \geq 0}\) be an orthonormal basis for \(E\) with \(e_0 = u\). Let

\[\psi = \sum_{n=0}^{\infty} \langle ; g_{\otimes n} ;, g_n \rangle.\]

Since \(D_\eta \psi = 0\), for all \(n \geq 1\), we must have \(g_n \otimes_1 \gamma = 0\). Let

\[g_n = \sum_{i_1, \ldots, i_n} c_{i_1, \ldots, i_n} e_{i_1} \otimes \ldots \otimes e_{i_n}.\]

If at least one of the indexes \(i_k\) is 0, then \(c_{i_1, \ldots, i_n} = 0\). Hence \(\psi\) is supported by \(\eta^\perp\).

But \(||\varphi||_0 < \infty\), i.e. \(||\varphi||_0 \cdot ||\psi||_0 < \infty\). This is the same as

\[|\alpha| < 1 \quad \text{and} \quad \psi \in (L^2).\]

Thus \(g_n \in (\eta^\perp_c)_{\otimes n}\), where \(\eta^\perp_c\) is the complexification of the orthogonal complement of \(\mathbb{R}\eta\) in \(E\). Thus \((g_n)_{n \geq 0} \in \Gamma(\eta^\perp_c)\), where \(\Gamma(\eta^\perp_c)\) is the Fock space of \(\eta^\perp_c\). Hence the solution of equation (4.1) is:

\[\varphi = \psi \circ \varphi_{\alpha}.\]

It is easy to check that these functions satisfy equation (4.1).

Let's now discuss the case \(\alpha = \infty\), i.e. \(D_\eta^* \varphi = 0\). This equation is equivalent to

\[\forall n \geq 0, \quad \eta \otimes f_n = 0.\]
Choosing the same base \{e_n\}_{n \geq 0} as before, we can see that \(\forall n \geq 0, \ f_n = 0\). Thus \(\varphi = 0\). If \(\widetilde{Q}_n \varphi = 0\) or \(\widetilde{Q}_n \Gamma(-iI) \varphi = 0\), then \(\widetilde{D}_n \varphi = \alpha D_n^* \varphi\), where \(\alpha = 1\) or \(-1\), and in this case we saw that the only possibility is \(\varphi = 0\). This was used in the proof of Theorem 3.4. \(\square\)

Notation. If \(\{\phi_i\}_{i=1}^n \subset (E)^*\), then we denote \(\phi_1 \circ \phi_2 \circ \cdots \circ \phi_n\) by \(\bowtie_{i=1}^n \phi_i\).

Corollary 4.9. Let \(\{e_i\}_{i=1}^n \subset E\) be an orthonormal subset of \(E\). Then for all \(\varphi \in (L^2)\) we have

\[
\frac{1}{n} \sum_{i=1}^n ||\widetilde{Q}_i \varphi||_0 \cdot ||\widetilde{Q}_i \Gamma(-iI) \varphi||_0 \geq ||\varphi||^2_0.
\]

The equality holds if and only if

\[
\varphi = \bowtie_{i=1}^n : e^{(\alpha_i/2)\langle \cdot, e_i \rangle^2} \bowtie \psi,
\]

where for all \(i \in \{1, 2, \ldots, n\}\), \(\alpha_i \in (-1, 1)\) and \(\psi\) is supported by the orthogonal complement of the vector subspace of \(E\) spanned by \(\{e_i\}_{i=1}^n\).

Proof. Apply Theorem 3.4 to each \(\eta = e_i, 1 \leq i \leq n\). Then add the inequalities and divide the result by \(n\). To obtain equality apply Theorem 4.8 repeatedly. \(\square\)

Corollary 4.10. Let \(\{e_n\}_{n \geq 1} \subset E\) be an orthonormal base for \(E\). Let \(\{c_n\}_{n \geq 1}\) be a sequence of complex numbers such that \(\sum_{n=1}^\infty |c_n|^2 = 1\) and \(c_n \neq 0, \forall n \geq 0\). Then for all \(\varphi \in (L^2)\) we have

\[
\sum_{n=1}^\infty |c_n|^2 ||\widetilde{Q}_{e_n} \varphi||_0 \cdot ||\widetilde{Q}_{e_n} \Gamma(-iI) \varphi||_0 \geq ||\varphi||^2_0.
\]

The equality holds if and only if

\[
\varphi = k \bowtie_{n=1}^\infty : e^{(\alpha_n/2)\langle \cdot, e_n \rangle^2} ;,
\]

where for all \(n \geq 1\), \(\alpha_n \in (-1, 1)\), and \(\sum_{n=1}^\infty \alpha_n^2 < \infty\) and \(k \in \mathbb{C}\).
Proof. Apply Theorem 3.4 for each \( \eta = e_n, \ n \geq 1 \). Then multiply the inequalities obtained by \( |c_{n,0}^2| \). Finally add all the resulting inequalities.

Let's discuss now the equality case. Let \( \varphi \in (L^2) \) for which we have equality. Because

\[
||\tilde{Q}_e\varphi||_0 \cdot ||\tilde{Q}_e \Gamma(-iI)\varphi||_0 = ||\varphi||_0^2
\]
we have \( \varphi = e^{(\alpha_1/2)(\cdot,e_1)^2} : \psi_1 \) for some \( \alpha_1 \in (-1,1) \) and \( \psi_1 \) supported by \( e_1^\perp \).

Because

\[
||\tilde{Q}_e \varphi||_0 \cdot ||\tilde{Q}_e \Gamma(-iI)\varphi||_0 = ||\varphi||_0^2
\]
we have \( \tilde{D}_e \varphi = \alpha_2 D_e^* \varphi \) for some \( \alpha_2 \in (-1,1) \). This is the same as

\[
(\tilde{D}_e : e^{(\alpha_1/2)(\cdot,e_1)^2} : \psi_1) + e^{(\alpha_1/2)(\cdot,e_1)^2} : \tilde{D}_e \psi_1
= \alpha_2 : e^{(\alpha_1/2)(\cdot,e_1)^2} : D_e^* \psi_1
\]
or equivalently

\[
e^{(\alpha_1/2)(\cdot,e_1)^2} : \tilde{D}_e \psi_1 = e^{(\alpha_1/2)(\cdot,e_1)^2} : \alpha_2 D_e^* \psi_1.
\]

After taking the Wick product with \( e^{-(\alpha_1/2)(\cdot,e_1)^2} : \) we conclude that \( \tilde{D}_e \psi_1 = \alpha_2 D_e^* \psi_1 \). Therefore \( \psi_1 = e^{(\alpha_2/2)(\cdot,e_2)^2} : \psi_2 \), where \( \psi_2 \) is supported by \( e_2^\perp \), and because \( \psi_2 = e^{-(\alpha_2/2)(\cdot,e_2)^2} : \psi_1 \), we can see that \( \psi_2 \) is supported by \( e_1^\perp \), too. Hence \( \varphi = e^{(\alpha_1/2)(\cdot,e_1)^2} : \psi_2 \). where \( \psi_2 \) is supported by \( \{e_1, e_2\}^\perp \).

Repeating this argument we conclude that for all \( n \geq 1 \),

\[
\varphi = \diamond_{i=1}^n e^{(\alpha_i/2)(\cdot,e_i)^2} : \psi_n,
\]
where \( \alpha_i \in (-1,1) \) and \( \psi_n \) is supported by \( \{e_1, e_2, \ldots, e_n\}^\perp \). We distinguish between two cases:

Case 1 \( S\varphi(0) = 0 \). In this case we have \( \langle \langle \varphi, 1 \rangle \rangle = 0 \). Then for all \( n \in \mathbb{N} \)

\[
0 = \prod_{i=1}^n S : e^{(\alpha_i/2)(\cdot,e_i)^2} : (0) \cdot S\psi_n(0) = \prod_{i=1}^n 1 \cdot \langle \langle \psi_n, 1 \rangle \rangle = \langle \langle \psi_n, 1 \rangle \rangle.
\]
Thus for all \( u = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n \in \text{Span}_c \{ e_1, e_2, \cdots, e_n \} \), \((\text{Span}_c \) means the complexification of \( \text{Span} \)), we have

\[
S \varphi(u) = \prod_{i=1}^{n} e^{(\alpha_i/2) \cdot \langle \cdot, e_i \rangle^2} \cdot \langle \langle \psi_n, 1 \rangle \rangle = 0.
\]

Therefore \( \langle \langle \varphi, e^{\langle \cdot, u \rangle} \rangle \rangle = 0 \), for all \( u \in \text{Span}_c \{ e_i \mid i \in \mathbb{N} \} \). Since the subspace generated by the functions of the form : \( e^{\langle \cdot, u \rangle} ; u \in \text{Span}_c \{ e_i \mid i \in \mathbb{N} \} \), is dense in \((L^2, \langle \cdot, \cdot \rangle)\), we conclude that \( \varphi = 0 \).

**Case 2** \( S \varphi(0) \neq 0 \). In this case \( \langle \langle \varphi, 1 \rangle \rangle \neq 0 \). Let \( k := S \varphi(0) \) and \( \varphi' := \frac{1}{k} \varphi \). Then \( \varphi = k \varphi' \) and \( S \varphi'(0) = 1 \). Following the same argument as before we see that for all \( n \in \mathbb{N} \), we have:

\[
\varphi' = \bigcirc_{i=1}^{n} : e^{(\alpha_i/2) \cdot \langle \cdot, e_i \rangle^2} : \psi'_n,
\]

where \( \alpha_i \in (-1, 1) \) and \( \psi'_n \) is supported by \( \{ e_1, e_2, \cdots, e_n \}^\perp \). Because \( S \psi'_n(0) = 1 \) we have \( S \psi'_n(0) = 1 \). Therefore

\[
||\psi'_n||_0 = ||\psi'_n||_0 \cdot ||1||_0 \geq \langle \langle \psi'_n, 1 \rangle \rangle = 1.
\]

Hence we have the following estimation of the norm of \( \varphi' \).

\[
||\varphi'||_0 = \prod_{i=1}^{n} || e^{(\alpha_i/2) \cdot \langle \cdot, e_i \rangle^2} : ||_0 \cdot ||\psi'_n||_0
\geq \prod_{i=1}^{n} || e^{(\alpha_i/2) \cdot \langle \cdot, e_i \rangle^2} : ||_0
= \prod_{i=1}^{n} \frac{1}{\sqrt{1 - \alpha_i^2}}.
\]

Therefore the infinite product

\[
\prod_{i=1}^{\infty} \frac{1}{\sqrt{1 - \alpha_i^2}}
\]

is convergent and

\[
1 \leq \prod_{i=1}^{\infty} \frac{1}{\sqrt{1 - \alpha_i^2}} \leq ||\varphi'||_0.
\]
The convergence of this infinite product is equivalent to the convergence of the series \( \sum_{i=1}^{\infty} a_i^2 \).

For any \( n \in \mathbb{N} \), let \( \sigma_n := \bigotimes_{i=1}^{n} : e^{(\alpha_i/2)(\cdot, e_i)^2} : \).

**Claim** The sequence \( \{\sigma_n\}_{n \geq 1} \) is Cauchy in \((L^2)\).

Let \( n, p \in \mathbb{N} \). We have

\[
\|\sigma_{n+p} - \sigma_n\|_0^2 = \|\sigma_n\|_0^2 \cdot \bigotimes_{i=1}^{p} : e^{(\alpha_{n+i}/2)(\cdot, e_{n+i})^2} : - 1\|_0^2 \leq \|\varphi'\|_0^2 \cdot \bigotimes_{i=1}^{p} : e^{(\alpha_{n+i}/2)(\cdot, e_{n+i})^2} : - 1\|_0^2 = \|\varphi'\|_0^2 \cdot (\|\bigotimes_{i=1}^{p} : e^{(\alpha_{n+i}/2)(\cdot, e_{n+i})^2} : \|_0^2 - 1) \leq \|\varphi'\|_0 \left( \prod_{i=1}^{p} \frac{1}{\sqrt{1 - \alpha_{n+i}^2}} - 1 \right).
\]

(Here we use the equality:

\[
\|\bigotimes_{i=1}^{p} : e^{(\alpha_{n+i}/2)(\cdot, e_{n+i})^2} : - 1\|_0^2 = \|\bigotimes_{i=1}^{p} : e^{(\alpha_{n+i}/2)(\cdot, e_{n+i})^2} : \|_0^2 - 1,
\]

which is true because if \( \bigotimes_{i=1}^{p} : e^{(\alpha_{n+i}/2)(\cdot, e_{n+i})^2} : = \sum_{n=0}^{\infty} \langle : e_n \cdot f_n \rangle \), then \( f_0 = 1 \).)

But \( \prod_{i=1}^{p} \frac{1}{\sqrt{1 - \alpha_{n+i}^2}} - 1 \to 0 \) as \( n \to \infty \) since the product \( \prod_{i=1}^{\infty} \frac{1}{\sqrt{1 - \alpha_i^2}} \) is convergent. Therefore the sequence \( \{\sigma_n\}_{n \geq 1} \) is Cauchy in \((L^2)\). So it is convergent in \((L^2)\) and we denote its limit by \( \bigotimes_{n=1}^{\infty} : e^{(\alpha_n/2)(\cdot, e_n)^2} : \).

For all \( u = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n \in Span_c\{e_1, e_2, \cdots, e_n\} \) we have

\[
\langle \langle \varphi', : e^{(\cdot, u)^2} : \rangle \rangle = S\varphi'(u)
= \prod_{i=1}^{n} S : e^{(\alpha_i/2)(\cdot, e_i)^2} : (u) \cdot S\psi_n(u)
= \prod_{i=1}^{n} e^{(\alpha_i/2) \cdot c_i^2} \cdot 1
= S\bigotimes_{i=1}^{\infty} : e^{(\alpha_i/2)(\cdot, e_i)^2} : (u)
= \langle \langle \bigotimes_{i=1}^{\infty} : e^{(\alpha_i/2)(\cdot, e_i)^2} :, e^{(\cdot, u)^2} : \rangle \rangle.
\]

Since the subspace generated by \( : e^{(\cdot, u)^2} : , u \in Span_c\{e_i \mid i \in \mathbb{N}\} \), is dense in \((L^2)\), we have \( \varphi' = \bigotimes_{i=1}^{\infty} : e^{(\alpha_i/2)(\cdot, e_i)^2} : \). Hence \( \varphi = k \bigotimes_{i=1}^{\infty} : e^{(\alpha_i/2)(\cdot, e_i)^2} : \). ⊓⊔
Comments.

1. The commutation relation (3.1) that we have used may be called, according to [20], Heisenberg commutation relation. This commutation relation has appeared in many books; see for example [15] or [18].

2. The classical Heisenberg Inequality has appeared in many books. A good reference is [2]. Also the fact that the Heisenberg Uncertainty Principle relies on a commutation relation is a well-known fact; see, for example, [30]. In his book *Mathematical Foundations of Quantum Mechanics*, Von Neumann clearly stated that whenever we have two self-adjoint operators $P$ and $Q$ on a Hilbert space, satisfying a commutation relationship $[P, Q] = cI$, we have an uncertainty inequality. The proof of this fact is easy, and our proof is almost the same. It is Heisenberg’s merit to discover the power of this relationship and then to give the right physical interpretation to his celebrated inequality.
Chapter 5
An Extension of the Heisenberg Inequality

To use completely the commutation relationships between any two of the following three operators: the differentiation, the adjoint of the differentiation, and the second quantization operator, we will present below an extension of previous Heisenberg Inequality. We will replace the self-adjoint operator $\tilde{Q}_\eta$ by two non symmetric operators, that are dual one two another $\tilde{D}_\xi + D_\eta^*$ and $\tilde{D}_\eta + D_\xi^*$. Even though the self-adjointness is removed, the symmetry of the new inequality remains.

5.1 An Extension of the Heisenberg Inequality

Theorem 5.1. Let $\xi, \eta \in \mathcal{E} \setminus \{0\}$. Then for any $\varphi \in (L^2)$ we have

$$
\|\tilde{D}_\xi + D_\eta^*\varphi\|_0\|\tilde{D}_\xi + D_\eta^*\Gamma(-iI)\varphi\|_0 + \\
\|\tilde{D}_\eta + D_\xi^*\varphi\|_0\|\tilde{D}_\eta + D_\xi^*\Gamma(-iI)\varphi\|_0 \geq (|\xi|_0^2 + |\eta|_0^2)\|\varphi\|_0^2. \quad (5.1)
$$

Proof. For $\xi, \eta \in \mathcal{E}$, and $\varphi \in (L^2)$ we have

$$
\|\tilde{D}_\xi + D_\eta^*\varphi\|_0\|\tilde{D}_\xi + D_\eta^*\Gamma(-iI)\varphi\|_0 + \\
\|\tilde{D}_\eta + D_\xi^*\varphi\|_0\|\tilde{D}_\eta + D_\xi^*\Gamma(-iI)\varphi\|_0 = \\
\|\tilde{D}_\xi + D_\eta^*\varphi\|_0\|\Gamma(-iI)(\tilde{D}_\xi - D_\eta^*)\varphi\|_0 + \\
\|\tilde{D}_\eta + D_\xi^*\varphi\|_0\|\Gamma(-iI)(\tilde{D}_\eta - D_\xi^*)\varphi\|_0 = \\
\|\tilde{D}_\xi + D_\eta^*\varphi\|_0\|(-iI)\Gamma(-iI)(\tilde{D}_\xi - D_\eta^*)\varphi\|_0 + \\
\|\tilde{D}_\eta + D_\xi^*\varphi\|_0\|(-iI)\Gamma(-iI)(\tilde{D}_\eta - D_\xi^*)\varphi\|_0 = \\
\|\tilde{D}_\xi + D_\eta^*\varphi\|_0\|\tilde{D}_\xi - D_\eta^*\varphi\|_0 + \\
\|\tilde{D}_\eta + D_\xi^*\varphi\|_0\|\tilde{D}_\eta - D_\xi^*\varphi\|_0
$$

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\[
\begin{align*}
&= \frac{1}{2} \left[ \| (\tilde{D}_\xi + D^*_\eta) \varphi \|_{\alpha} \| (\tilde{D}_\xi - D^*_\eta) \varphi \|_{\alpha} + \| (\tilde{D}_\xi - D^*_\eta) \varphi \|_{\alpha} \| (\tilde{D}_\xi + D^*_\eta) \varphi \|_{\alpha} \\
&\quad + \| (\tilde{D}_\eta + D^*_\xi) \varphi \|_{\alpha} \| (\tilde{D}_\eta - D^*_\xi) \varphi \|_{\alpha} + \| (\tilde{D}_\eta - D^*_\xi) \varphi \|_{\alpha} \| (\tilde{D}_\eta + D^*_\xi) \varphi \|_{\alpha} \right] \\
&\geq \frac{1}{2} \left[ \| (\tilde{D}_\xi + D^*_\eta) \varphi, (\tilde{D}_\xi - D^*_\eta) \varphi \| + \| (\tilde{D}_\xi - D^*_\eta) \varphi, (\tilde{D}_\xi + D^*_\eta) \varphi \| \\
&\quad + \| (\tilde{D}_\eta + D^*_\xi) \varphi, (\tilde{D}_\eta - D^*_\xi) \varphi \| + \| (\tilde{D}_\eta - D^*_\xi) \varphi, (\tilde{D}_\eta + D^*_\xi) \varphi \| \right] \\
&\geq \frac{1}{2} \left[ \| (\tilde{D}_\xi + D^*_\eta) \varphi, (\tilde{D}_\xi - D^*_\eta) \varphi \| + \| (\tilde{D}_\xi - D^*_\eta) \varphi, (\tilde{D}_\xi + D^*_\eta) \varphi \| \\
&\quad + \| (\tilde{D}_\eta + D^*_\xi) \varphi, (\tilde{D}_\eta - D^*_\xi) \varphi \| + \| (\tilde{D}_\eta - D^*_\xi) \varphi, (\tilde{D}_\eta + D^*_\xi) \varphi \| \right] \\
&= \| (\tilde{D}_\xi \varphi, \tilde{D}_\xi \varphi) \| - \| (D^*_\eta \varphi, D^*_\eta \varphi) \| + \| (\tilde{D}_\eta \varphi, \tilde{D}_\eta \varphi) \| - \| (D^*_\eta \varphi, D^*_\eta \varphi) \| \\
&\geq |\langle (\tilde{D}_\xi \varphi, \tilde{D}_\xi \varphi) \rangle - \langle (D^*_\eta \varphi, D^*_\eta \varphi) \rangle + \langle (\tilde{D}_\eta \varphi, \tilde{D}_\eta \varphi) \rangle - \langle (D^*_\eta \varphi, D^*_\eta \varphi) \rangle | \\
&= |\langle (D^*_\xi \tilde{D}_\xi \varphi, \varphi) \rangle - \langle \tilde{D}_\eta D^*_\eta \varphi, \varphi \rangle \rangle + \langle (D^*_\eta \tilde{D}_\eta \varphi, \varphi) \rangle - \langle (D^*_\eta D^*_\eta \varphi, \varphi) \rangle | \\
&= |\langle (D^*_\eta \tilde{D}_\eta \varphi, \varphi) \rangle + \langle (D^*_\eta \tilde{D}_\eta \varphi, \varphi) \rangle | \\
&= |\langle (\tilde{D}_\xi \varphi, \varphi) \rangle + \langle (\tilde{D}_\xi \varphi, \varphi) \rangle | \\
&= |\langle (\tilde{D}_\eta \varphi, \varphi) \rangle + \langle (\tilde{D}_\eta \varphi, \varphi) \rangle | \\
&= |\|\varphi\|_0^2 + \|\eta\|_0^2 | |\varphi\|_0^2 \\
&= |\|\varphi\|_0^2 + |\|\eta\|_0^2 | |\varphi\|_0^2.}
\end{align*}
\]

Thus the proof is complete. \(\square\)

**Corollary 5.2.** (Theorem 3.4) Let \(\eta \in \mathcal{E} \setminus \{0\}\) and \(\varphi \in (L^2)\). Then

\[
\|\tilde{Q}_\eta \varphi\|_0 \cdot \|\tilde{Q}_\eta \Gamma(-iI) \varphi\|_0 \geq |\eta|_0^2 \cdot |\|\varphi\|_0^2.
\]

**Proof.** Apply Theorem 5.1 for \(\xi = \eta\) and use the formula \(\tilde{Q}_\eta = \tilde{D}_\xi + D^*_\eta\). \(\square\)
Chapter 6
Equality in the Extension of the Heisenberg Inequality

6.1 Equality in the Extension of the Heisenberg Inequality

Now we want to answer the question: When does the equality in Theorem 5.1 hold? We will first present some results that will be used to answer this question.

Lemma 6.1. Let \( \eta \in \mathcal{E} \) and \( \psi \in (L^2) \) such that \( \tilde{D}_\eta \psi = 0 \). Then \( \psi \) is supported by the orthogonal complement \( \eta^\perp \) of \( \eta \).

Proof. Let \( \psi = \sum_{n=0}^{\infty} \langle \cdot, \xi_n \rangle_{\mathcal{E}} \), where \( \forall n \geq 0, g_n \in E^{\otimes n} \). If \( \eta = 0 \) we have nothing to prove. If \( \eta \neq 0 \), then let \( u = \frac{1}{|\eta|} \eta \). Let \( \{e_i\}_{i \geq 0} \) be an orthonormal basis of \( E \) such that \( e_0 = u \). Since \( D_\eta \psi = 0 \), for all \( n \geq 1 \), we must have \( g_n \otimes \eta = 0 \). Let \( g_n = \sum_{i_1, \ldots, i_n} c_{i_1, \ldots, i_n} e_{i_1} \otimes \cdots \otimes e_{i_n} \).

If at least one of the indexes \( i_k \) is 0, then \( c_{i_1, \ldots, i_n} = 0 \). Hence \( \psi \) is supported by \( \eta^\perp \). Thus \( g_n \in (\eta^\perp_c)^{\otimes n} \), where \( \eta^\perp_c \) is the complexification of the orthogonal complement of \( \mathbb{R} \eta \) in \( E \). Thus \( (g_n)_{n \geq 0} \in \Gamma(\eta^\perp_c) \), where \( \Gamma(\eta^\perp_c) \) is the Fock space of \( \eta^\perp_c \). \( \square \)

Lemma 6.2. Let \( \eta \in \mathcal{E} \setminus \{0\} \) and \( \psi \in (L^2) \) such that \( D^*_\eta \psi = 0 \). Then \( \psi = 0 \).

Proof. Let \( \xi \in \mathcal{E}_c \). Then \( 0 = SD^*_\eta \psi(\xi) = \langle \xi, \eta \rangle S\psi(\xi) \). If \( \langle \xi, \eta \rangle \neq 0 \), then \( S\psi(\xi) = 0 \). If \( \langle \xi, \eta \rangle = 0 \), then \( \forall z \in \mathbb{C} \setminus \{0\}, \langle \xi + z\eta, \eta \rangle \neq 0 \); therefore \( S\psi(\xi + z\eta) = 0 \). Letting \( z \to 0 \), because the map \( z \mapsto S\psi(\xi + z\eta) \) is continuous (being analytic), we obtain \( S\psi(\xi) = 0 \). Hence the S-transform of \( \psi \) is identically zero and thus \( \psi = 0 \). \( \square \)
Lemma 6.3. Let \( \xi, \eta \in \mathcal{E} \), \( \varphi = \sum_{n=0}^{\infty} \langle x^{\otimes n} :, f_n \rangle \in (L^2) \), and \( \alpha \in \mathbb{C} \). The following equations are equivalent:

\[
\tilde{D}_\xi \varphi = \alpha D_{\eta}^* \varphi, \tag{6.1}
\]

\[
\forall n \geq 0, \quad (n + 1) f_{n+1} \hat{\otimes}_1 \xi = \alpha f_{n-1} \hat{\otimes} \eta, \tag{6.2}
\]

where by convention \( f_{-1} = 0 \).

Proof. We have

\[
\tilde{D}_\xi \varphi = \sum_{n=1}^{\infty} n \langle x^{\otimes (n-1)} :, f_n \hat{\otimes}_1 \xi \rangle = \sum_{n=0}^{\infty} (n + 1) \langle x^{\otimes n} :, f_{n+1} \hat{\otimes}_1 \xi \rangle
\]

\[
\alpha D_{\eta}^* \varphi = \alpha \sum_{n=0}^{\infty} \langle x^{\otimes (n+1)} :, f_n \hat{\otimes} \eta \rangle = \alpha \sum_{n=0}^{\infty} \langle x^{\otimes n} :, f_{n-1} \hat{\otimes} \eta \rangle.
\]

This shows that (6.1) \( \iff \) (6.2). \( \square \)

Lemma 6.4. Let \( \xi, \eta \in \mathcal{E} \) such that \( \langle \xi, \eta \rangle \neq 0 \). Let \( \varphi \in (L^2) \) satisfy the equation

\[
\tilde{D}_\xi \varphi = \alpha D_{\eta}^* \varphi,
\]

for some \( \alpha \in \mathbb{C} \). Then there exists \( \psi \in (\mathcal{E})^* \) supported by \( \xi^\perp \) such that

\[
\varphi = \psi^{\otimes \xi} (\eta)^2 : \varphi \psi.
\]

Proof. The equation

\[
\tilde{D}_\xi \varphi = \alpha D_{\eta}^* \varphi
\]

is equivalent to

\[
(n + 1) f_{n+1} \hat{\otimes}_1 \xi = \alpha f_{n-1} \hat{\otimes} \eta, \quad \forall n \geq 0.
\]

We first try to find a particular solution of (6.1) of the form

\[
\varphi_\alpha = \sum_{n=0}^{\infty} \langle :, x^{\otimes n} :, c_n \eta^{\otimes n} \rangle,
\]

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where \( \{c_n\}_{n \geq 0} \subset \mathbb{C} \) and \( c_0 = 1 \). Since (6.1) is equivalent to (6.2), we must have

\[
\forall n \geq 0, \quad (n + 1)c_{n+1}(\xi, \eta) = \alpha c_{n-1}.
\]

This is the same as

\[
c_{n+1} = \frac{\alpha}{(n + 1)(\xi, \eta)} c_{n-1},
\]

where \( c_{-1} = 0 \). Because \( c_{-1} = 0 \) we have \( \forall n \geq 0, \ c_{2n+1} = 0 \). Moreover

\[
\forall n \geq 0, \ c_{2n} = \frac{\alpha}{2n(\xi, \eta)} c_{2n-2} = \frac{\alpha^2}{2n(2n-2)(\xi, \eta)^2} c_{2n-4} = \cdots = \frac{\alpha^n}{(2n)!!(\xi, \eta)^n} c_0.
\]

Thus

\[
\varphi_\alpha = \sum_{n=0}^{\infty} \frac{\alpha^n}{2^n n!(\xi, \eta)^n} \eta^{2n} = e^\overline{\eta(\xi, \eta)}^2.
\]

The \( S \) transform of \( \varphi_\alpha \) is

\[
S\varphi_\alpha(\theta) = e^{\overline{\theta(\xi, \eta)}^2}.
\]

Now, we observe that \( \varphi_{\alpha_1} \circ \varphi_{\alpha_2} = \varphi_{\alpha_1 + \alpha_2} \). To see this let's use the \( S \) transform. For any \( \theta \in \mathcal{E}_c \) we have

\[
S(\varphi_{\alpha_1} \circ \varphi_{\alpha_2})(\theta) = S\varphi_{\alpha_1}(\theta) \cdot S\varphi_{\alpha_2}(\theta)
\]

\[
= e^{\overline{\alpha_1(\xi, \eta)}^2} \cdot e^{\overline{\alpha_2(\xi, \eta)}^2}
\]

\[
= e^{\overline{\alpha_1 + \alpha_2(\xi, \eta)}^2}
\]

\[
= S\varphi_{\alpha_1 + \alpha_2}(\theta).
\]

With this particular solution \( \varphi_\alpha \), we can produce the general solution of equation (6.1). Take the Wick product of equation (6.1) with \( \varphi_{-\alpha} \)

\[
(\tilde{D}_\xi \varphi) \circ \varphi_{-\alpha} = \alpha(D_\eta^* \varphi) \circ \varphi_{-\alpha} = -\varphi \circ (-\alpha D_\eta^* \varphi_{-\alpha}) = -\varphi \circ \tilde{D}_\xi \varphi_{-\alpha}.
\]

Thus we obtain

\[
(\tilde{D}_\xi \varphi) \circ \varphi_{-\alpha} + \varphi \circ (\tilde{D}_\xi \varphi_{-\alpha}) = 0.
\]
This is equivalent to
\[ \tilde{D}_\xi (\varphi \circ \varphi_{-\alpha}) = 0. \]

Let \( \psi = \varphi \circ \varphi_{-\alpha} \). We have \( \tilde{D}_\xi \psi = 0 \). Thus \( \psi \) is supported by \( \xi^\perp \). But \( \varphi \circ \varphi_{-\alpha} = \psi \).

Thus we have
\[ \psi \circ \varphi_{\alpha} = (\varphi \circ \varphi_{-\alpha}) \circ \varphi_{\alpha} = \varphi \circ (\varphi_{-\alpha} \circ \varphi_{\alpha}) = \varphi \circ \varphi_0 = \varphi \circ 1 = \varphi. \]

Hence the solution of equation (6.1) is
\[ \varphi = \psi \circ \varphi_{\alpha}. \]

It is easy to check that these functions satisfy equation (6.1). \( \square \)

Note that \( e^{2\pi i \langle \xi, \eta \rangle^2} \) and \( \psi \) are supported by \( \mathbf{R} \eta \) and \( \mathbf{R} \xi \), respectively, and \( \eta \) and \( \xi \) are not perpendicular. Hence we cannot say anything about the conditions that \( \alpha \) and \( \psi \) must satisfy for \( \varphi \) to be in \( (L^2) \).

Now we want to solve the equation
\[ \tilde{D}_\xi \varphi = \alpha D^*_\eta \varphi, \]
for the case \( \xi \perp \eta \), where \( \xi, \eta \in \mathcal{E} \setminus \{0\} \), and \( \alpha \in \mathbb{C} \). Replacing eventually \( \xi \) by \( \frac{1}{|\xi|_0} \xi \), \( \eta \) by \( \frac{1}{|\eta|_0} \eta \), and \( \alpha \) by \( \frac{\alpha_0}{|\xi|_0} \), we may assume that \( |\xi|_0 = |\eta|_0 = 1 \).

**Definition 6.5.** Let \( \xi, \eta \in \mathcal{E} \), with \( |\xi|_0 = |\eta|_0 = 1 \), such that \( \xi \perp \eta \), and let \( \alpha \in \mathbb{C} \).

We define the following function
\[ \rho^\xi_\eta := \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \langle \xi \otimes (2n) \otimes \eta \otimes \eta \rangle. \]

**Observation 6.6.** \( \rho^\xi_\eta = : e^{(\alpha/4)(\cdot, \xi + \eta)^2} : \circ : e^{-(\alpha/4)(\cdot, \xi - \eta)^2} :. \)
Proof. For any $\theta \in \mathcal{E}$ we have

$$S(\cdot e^{(\alpha/4)(\cdot \zeta + \eta)^2} : \cdot e^{-(\alpha/4)(\cdot \zeta - \eta)^2})(\theta) = S : e^{(\alpha/4)(\cdot \zeta + \eta)^2} : (\theta) \cdot S : e^{-(\alpha/4)(\cdot \zeta - \eta)^2} : (\theta) = e^{(\alpha/4)(\theta, \zeta + \eta)^2} \cdot e^{-(\alpha/4)(\theta, \zeta - \eta)^2} = e^{\alpha(\theta, \zeta)(\theta, \eta)} = S \rho_{\alpha}^{\xi, \eta}(\theta).$$

Thus $\rho_{\alpha}^{\xi, \eta} = : e^{(\alpha/4)(\cdot \zeta + \eta)^2} : \cdot e^{-(\alpha/4)(\cdot \zeta - \eta)^2}$.

Corollary 6.7. For any $\alpha \in \mathbb{C}$ and $\xi, \eta \in \mathcal{E}$, $\rho_{\alpha}^{\xi, \eta} \in (\mathcal{E})^\ast$.

Lemma 6.8. Let $\xi, \eta \in \mathcal{E}$ with $|\xi|_0 = |\eta|_0 = 1$, be such that $\xi \perp \eta$, and let $\alpha \in \mathbb{C}$. Then $||\rho_{\alpha}^{\xi, \eta}||_0^2 = \sum_{n=0}^{\infty} |\alpha|^{2n}$. In particular $\rho_{\alpha}^{\xi, \eta} \in (L^2)$ if and only if $|\alpha| < 1$ in which case $||\rho_{\alpha}^{\xi, \eta}||_0 = \frac{1}{\sqrt{1-|\alpha|^2}}$.

Proof. Let's consider the sequence $\{u_i\}_{i=1}^{2n}$, where $u_i = \xi$, if $1 \leq i \leq n$, and $u_i = \eta$, if $n + 1 \leq i \leq 2n$. Let's denote by $S_n$ the group of permutations of the set \{1, 2, \ldots, n\}. Because $\xi \perp \eta$ and $|\xi|_0 = |\eta|_0 = 1$ we have

$$||\rho_{\alpha}^{\xi, \eta}||_0^2 = \sum_{n=0}^{\infty} (2n)! \frac{|\alpha|^{2n}}{(n!)^2} \langle \xi^{\otimes n} \otimes \eta^{\otimes n}, \xi^{\otimes n} \otimes \eta^{\otimes n} \rangle$$

$$= \sum_{n=0}^{\infty} (2n)! \frac{|\alpha|^{2n}}{(n!)^2} \frac{1}{(\frac{2n}{n})^2} \sum_{\sigma \in S_n} |u_{\sigma(1)}|_0^2 |u_{\sigma(2)}|_0^2 \cdots |u_{\sigma(n)}|_0^2$$

$$= \sum_{n=0}^{\infty} |\alpha|^{2n} \frac{1}{(\frac{2n}{n})^2} \sum_{\sigma \in S_n} |\xi|_0^{2n} |\eta|_0^{2n}$$

$$= \sum_{n=0}^{\infty} |\alpha|^{2n} \frac{1}{(\frac{2n}{n})^2} \binom{2n}{n}$$

$$= \sum_{n=0}^{\infty} |\alpha|^{2n}.$$

This finishes the proof. \qed

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Lemma 6.9. Let $\xi, \eta \in \mathcal{E}$ with $|\xi|_0 = |\eta|_0 = 1$, be such that $\xi \perp \eta$, and let $\alpha \in \mathbb{C}$.

Let $\varphi \in (L^2)$ satisfy the equation

$$\tilde{D}_\xi \varphi = \alpha D^*_\eta \varphi.$$ 

Then there exists $\psi \in (\mathcal{E})^*$ supported by $\xi^\perp$ such that

$$\varphi = \rho^{\xi, \eta} \circ \psi.$$ 

Proof. Let's see first that $\rho^{\xi, \eta}$ satisfies equation (6.1). We have:

$$\tilde{D}_\xi \rho^{\xi, \eta} = \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} 2n \langle :\otimes(2n-1) :, \xi \hat{\otimes}_1 (\xi^{\otimes n} \otimes \eta^{\otimes n}) \rangle$$

$$= \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} 2n \langle \xi, \xi \rangle \left( \frac{2n-1}{n} \right) \langle :\otimes(2n-1) :, \xi^{\otimes (n-1)} \otimes \eta^{\otimes n} \rangle$$

$$= \sum_{n=1}^{\infty} \frac{\alpha^n}{(n-1)!} \langle :\otimes(2n-1) :, \xi^{\otimes (n-1)} \otimes \eta^{\otimes n} \rangle$$

$$= \alpha \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{(n-1)!} \langle :\otimes(2n-1) :, \eta \hat{\otimes} (\xi^{\otimes (n-1)} \otimes \eta^{\otimes (n-1)}) \rangle$$

$$= \alpha \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \langle :\otimes(2n+1) :, \eta \hat{\otimes} (\xi^{\otimes n} \otimes \eta^{\otimes n}) \rangle$$

$$= \alpha D^*_\eta \rho^{\xi, \eta}.$$ 

Using the S-transform as in Lemma 6.4 we can prove that $\rho^{\xi, \eta} \circ \rho^{\xi, \eta} = \rho^{\xi, \eta}$. After this, following the same way that we followed in Lemma 6.4 we can easily obtain the conclusion of Lemma 6.9. 

Lemma 6.10. Let $\xi, \eta \in \mathcal{E}$ be such that $|\xi|_0 = |\eta|_0 = 1$. Suppose that $\varphi \in (L^2)$ satisfies the system of equations:

$$\tilde{D}_\xi \varphi = \alpha D^*_\eta \varphi \quad (6.3)$$

$$\tilde{D}_\eta \varphi = \beta D^*_\xi \varphi, \quad (6.4)$$

where $\alpha, \beta \in \mathbb{C}$. Then either $\varphi = 0$ or $\alpha = \beta$. 

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First Proof. We have two cases:

Case 1 If \( \langle \xi, \eta \rangle \neq 0 \), then according to Lemma 6.4, there exist two generalized functions \( \psi_1 \) and \( \psi_2 \) supported by \( \xi^\perp \) and \( \eta^\perp \), respectively, such that:

\[
\varphi = e^{\gamma (\xi, \eta) \phi (\cdot, \cdot)^2} \cdot \psi_1
\]

(6.5)

\[
\varphi = e^{\delta (\xi, \eta) \phi (\cdot, \cdot)^2} \cdot \psi_2.
\]

(6.6)

Let \( \gamma = \frac{\alpha}{2 (\xi, \eta)} \) and \( \delta = \frac{\beta}{2 (\xi, \eta)} \). Let \( V \) be the subspace over the real field generated by \( \xi \) and \( \eta \). Let \( V^\perp := \{ \theta \in \mathbb{C} | \theta \perp V \} \). Because \( \psi_1 \) and \( \psi_2 \) are supported by \( \xi^\perp \) and \( \eta^\perp \), respectively, for any \( s, t \in \mathbb{C} \), and \( \theta \in V^\perp \), we have

\[
S \psi_1 (s \xi + t \eta + \theta) = S \psi_1 (t \eta + \theta)
\]

(6.7)

\[
S \psi_2 (s \xi + t \eta + \theta) = S \psi_2 (s \xi + \theta).
\]

(6.8)

Using the relations (6.5), (6.6), (6.7), and (6.8) we get:

\[
S \varphi (s \xi + t \eta + \theta) = S (e^{\gamma (\xi, \eta) \phi (\cdot, \cdot)^2} \cdot \psi_1) (s \xi + t \eta + \theta)
\]

\[
= e^{\gamma (s \xi + t \eta + \theta)} S \psi_1 (t \eta + \theta).
\]

and

\[
S \varphi (s \xi + t \eta + \theta) = S (e^{\delta (\xi, \eta) \phi (\cdot, \cdot)^2} \cdot \psi_2) (s \xi + t \eta + \theta)
\]

\[
= e^{\delta (s \xi + t \eta + \theta)} S \psi_2 (s \xi + \theta).
\]

Thus, for all \( s, t \in \mathbb{C} \), and \( \theta \in V^\perp \), we have:

\[
e^{\gamma (s (\xi, \eta) + t) \phi (\cdot, \cdot)^2} S \psi_1 (t \eta + \theta) = e^{\delta (s (\xi, \eta) + t) \phi (\cdot, \cdot)^2} S \psi_2 (s \xi + \theta).
\]

(6.9)

Choosing \( s = t = 0 \) in the last relation we get:

\[
S \psi_1 (\theta) = S \psi_2 (\theta).
\]

(6.10)
Choosing $s = 0$ in the relation (6.9) we get:

$$e^{\gamma t^2}S\psi_1(t\eta + \theta) = e^{\delta t^2(\xi,\eta)^2}S\psi_2(\theta).$$

(6.11)

From (6.11) we conclude that:

$$S\psi_1(t\eta + \theta) = e^{-\gamma t^2}e^{\delta t^2(\xi,\eta)^2}S\psi_2(\theta).$$

(6.12)

Choosing $t = 0$ in the relation (6.9) we get:

$$e^{\gamma s^2(\xi,\eta)^2}S\psi_1(\theta) = e^{\delta s^2}S\psi_2(s\xi + \theta).$$

(6.13)

From (6.13) we conclude that:

$$S\psi_2(s\xi + \theta) = e^{-\delta s^2}e^{\gamma s^2(\xi,\eta)^2}S\psi_1(\theta).$$

(6.14)

Replacing $S\psi_1(t\eta + \theta)$ from (6.12) and $S\psi_2(s\xi + \theta)$ from (6.14), into (6.9) we get

$$e^{\gamma(s(\xi,\eta)+t)^2}e^{-\gamma t^2}e^{\delta t^2(\xi,\eta)^2}S\psi_2(\theta) = e^{\delta(s+t(\xi,\eta))^2}e^{-\delta s^2}e^{\gamma s^2(\xi,\eta)^2}S\psi_1(\theta).$$

(6.15)

After simplifications the relation (6.15) becomes:

$$e^{2\gamma st(\xi,\eta)}S\psi_2(\theta) = e^{2\delta st(\xi,\eta)}S\psi_1(\theta).$$

(6.16)

We have two subcases:

**Subcase 1** If there exists $\theta \in V^\perp$ such that $S\psi_2(\theta) = S\psi_1(\theta) \neq 0$, then after cancelling by $S\psi_2(\theta)$ and $S\psi_1(\theta)$, relation (6.16) becomes:

$$\forall s, t \in C, \quad e^{2\gamma st(\xi,\eta)} = e^{2\delta st(\xi,\eta)}.$$

Because $(\xi, \eta) \neq 0$ we conclude that $\gamma = \delta$. Hence $\alpha = \beta$.

**Subcase 2** If for all $\theta \in V^\perp$ we have $S\psi_2(\theta) = S\psi_1(\theta) = 0$, then according to relation (6.12) we get $S\psi_1(t\eta + \theta) = 0$, for all $t \in C$ and $\theta \in V^\perp$. Because $\psi_1$ is supported by $\xi^\perp$ we conclude that $\psi_1 = 0$. Thus $\varphi = : e^{\gamma(\eta,\eta)^2} : \alpha \psi_1 = 0$. 

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Case 2 If $\xi \perp \eta$, then according to Lemma 6.9 there exist $\psi_1$ and $\psi_2$ supported by $\xi^\perp$ and $\eta^\perp$, respectively, such that:

\begin{align}
\varphi &= \rho^\xi_\eta \circ \psi_1 \\
\varphi &= \rho^\xi_\beta \circ \psi_2.
\end{align}

Using the relation (6.17) we have:

\begin{align}
S\varphi(s\xi + t\eta + \theta) &= S(\rho^\xi_\eta \circ \psi_1)(s\xi + t\eta + \theta) \\
&= e^{\alpha st}S\psi_1(t\eta + \theta).
\end{align}

Using the relation (6.18) we have:

\begin{align}
S\varphi(s\xi + t\eta + \theta) &= S(\rho^\xi_\beta \circ \psi_2)(s\xi + t\eta + \theta) \\
&= e^{\beta st}S\psi_2(s\xi + \theta).
\end{align}

Therefore, we get:

\begin{align}
e^{\alpha st}S\psi_1(t\eta + \theta) &= e^{\beta st}S\psi_2(s\xi + \theta).
\end{align}

Then following the same way that we followed in Case 1 we conclude finally that either $\alpha = \beta$ or $\varphi = 0$. \hfill \Box

There exists a nicer proof of this lemma which is based on the commutation relationship between the Differential Operator and its Adjoint.

Second Proof. Using first (6.3) and then (6.4) we have:

\begin{align}
\tilde{D}_\eta \tilde{D}_\xi \varphi &= \alpha \tilde{D}_\eta D^*_\eta \varphi \\
&= \alpha [\langle \eta, \eta \rangle I + D^*_\eta \tilde{D}_\eta] \varphi \\
&= \alpha \varphi + \alpha D^*_\eta \tilde{D}_\eta \varphi \\
&= \alpha \varphi + \alpha \beta D^*_\eta D^*_\xi \varphi.
\end{align}
Using first (6.4) and then (6.3) we have:

\[
\begin{align*}
\bar{D}_\xi \bar{D}_\eta \varphi &= \beta \bar{D}_\xi D_\xi^* \varphi \\
&= \beta [\langle \xi, \xi \rangle I + D_\xi^* \bar{D}_\xi] \varphi \\
&= \beta \varphi + \beta D_\xi^* \bar{D}_\xi \varphi \\
&= \beta \varphi + \beta \alpha D_\xi^* D_\eta^* \varphi.
\end{align*}
\]  

(6.21)

Since \( \bar{D}_\eta \bar{D}_\xi = \bar{D}_\xi \bar{D}_\eta \) and \( D_\eta^* D_\xi^* = D_\xi^* D_\eta^* \) according to relations (6.20) and (6.21) we conclude that \( \alpha \varphi = \beta \varphi \) and thus we have either \( \alpha = \beta \) or \( \varphi = 0 \). \( \square \)

**Lemma 6.11.** Let \( \xi, \eta \in \mathcal{E} \) be such that \( |\xi|_0 = |\eta|_0 = 1 \). Suppose that \( \varphi \in (L^2) \) satisfies the system of equations:

\[
\begin{align*}
\bar{D}_\xi \varphi &= \alpha D_\eta^* \varphi \\
\bar{D}_\eta \varphi &= \beta D_\xi^* \varphi,
\end{align*}
\]

where \( \alpha, \beta \in \mathbb{C} \). Then

1. If \( \alpha \neq \beta \), then \( \varphi = 0 \).

2. If \( \alpha = \beta \) and \( \max\{2|\alpha| \cdot |\xi + \eta|^2, 2|\alpha| \cdot |\xi - \eta|^2\} \geq 1 \), then \( \varphi = 0 \).

3. If \( \alpha = \beta \) and \( \max\{2|\alpha| \cdot |\xi + \eta|^2, 2|\alpha| \cdot |\xi - \eta|^2\} < 1 \), then there exits \( \psi \in (L^2) \), such that \( \psi \) is supported by the orthogonal complement of the subspace generated by \( \xi \) and \( \eta \), and

\[
\varphi = : e^{\alpha \langle \cdot, \xi + \eta \rangle^2} : \circ : e^{-\alpha \langle \cdot, \xi - \eta \rangle^2} : \circ \psi.
\]

**Proof.** If \( \alpha \neq \beta \), then according to Lemma 6.10, we have \( \varphi = 0 \).

Now, let's suppose that \( \alpha = \beta \). Adding relations (6.3) and (6.4) together we get:

\[
\bar{D}_{\xi+\eta} \varphi = \alpha D_{\xi+\eta}^* \varphi.
\]

(6.22)
Subtracting relation (6.4) from (6.3) we obtain:

\[ \tilde{D}_{\xi - \eta} \phi = -\alpha D_{\xi - \eta}^* \phi. \tag{6.23} \]

Let's observe that \( \langle \xi + \eta, \xi - \eta \rangle = 0 \), because \( |\xi|_0 = |\eta|_0 = 1 \). Let \( u := \xi + \eta \) and \( v := \xi - \eta \). Then \( u \perp v \). We also have the following relations:

\[ \tilde{D}_u \phi = \alpha D_u^* \phi \tag{6.24} \]
\[ \tilde{D}_v \phi = -\alpha D_v^* \phi. \tag{6.25} \]

Because of the relation (6.24), according to Lemma 6.4, there exists \( \psi_1 \in (\mathcal{E})^* \), such that \( \psi_1 \) is supported by \( u^\perp \) and

\[ \phi = e^{\alpha(l,w)^2} : \psi_1. \tag{6.26} \]

We know that \( || : e^{\alpha(l,w)^2} : ||_0 < \infty \) if and only if \( 2|\alpha| \cdot |u|_0^2 < 1 \). Also because \( : e^{\alpha(l,w)^2} : \) and \( \psi_1 \) are supported by two orthogonal subspaces we conclude that \( ||\phi||_0 = || : e^{\alpha(l,w)^2} : ||_0 \cdot ||\psi_1||_0 \). Hence, if \( 2|\alpha| \cdot |u|_0^2 \geq 1 \) and \( \psi_1 \neq 0 \), then \( ||\phi||_0 = \infty \), which contradicts the fact that \( \phi \in (L^2) \). Thus, if \( 2|\alpha| \cdot |u|_0^2 \geq 1 \), then \( \psi_1 = 0 \), which implies \( \phi = 0 \).

Let's assume now that \( 2|\alpha| \cdot |u|_0^2 < 1 \) and \( \phi \neq 0 \). Then \( \psi_1 \neq 0 \) and \( ||\psi_1||_0 = \sqrt{1 - 4\alpha^2|u|_0^4} \cdot ||\phi||_0 < \infty \). Therefore \( \psi_1 \in (L^2) \). We have

\[ \tilde{D}_v \phi = \tilde{D}_v ( : e^{\alpha(l,w)^2} : \psi_1) \]
\[ = (\tilde{D}_v : e^{\alpha(l,w)^2} : ) \cdot \psi_1 + : e^{\alpha(l,w)^2} : \tilde{D}_v \psi_1 \]
\[ = : e^{\alpha(l,w)^2} : \tilde{D}_v \psi_1. \]
Thus we get:

\[-\alpha : e^{\alpha(\cdot,u)^2} : \phi_v^*\psi_1 = -\alpha \phi_v^*(e^{\alpha(\cdot,u)^2} : \phi_1) = -\alpha \phi_v \varphi = \tilde{\phi}_v \varphi = e^{\alpha(\cdot,u)^2} : \phi_{\tilde{\phi}_v}\psi_1.\]

Hence we have obtained:

\[\alpha : e^{\alpha(\cdot,u)^2} : \phi_v^*\psi_1 = -e^{\alpha(\cdot,u)^2} : \phi_{\tilde{\phi}_v}\psi_1.\]

Doing the Wick product of the last relation with \(e^{-\alpha(\cdot,u)^2}:\) we get

\[\tilde{\phi}_v\psi_1 = -\alpha \phi_v^*\psi_1. \tag{6.27}\]

According to Lemma 6.4, there exists \(\psi \in (\mathcal{E})^*,\) such that \(\psi\) is supported by \(v^\perp\) and

\[\psi_1 = e^{-\alpha(\cdot,u)^2} : \phi \psi. \tag{6.28}\]

The same argument as before shows that if \(2|\alpha| \cdot |v_0|^2 \geq 1,\) then \(\psi_1 = 0\) which implies \(\varphi = 0.\)

Now, let's suppose that \(2|\alpha| \cdot |v_0|^2 < 1.\) Because \(\psi_1\) is supported by \(u^\perp, u \perp v,\) and

\[\psi = e^{\alpha(\cdot,u)^2} : \phi_1,\]

we conclude that \(\psi\) is supported by \(u^\perp,\) too. Thus \(\psi\) is supported by both \(u^\perp\) and \(v^\perp.\) Hence \(\psi\) is supported by the orthogonal complement of the subspace generated by \(u\) and \(v\) which is the same as the subspace generated by \(\xi\) and \(\eta.\)
Because $||\psi_1||_0 < \infty$ we also have $||\psi||_0 < \infty$. Hence $\psi \in (L^2)$. Finally

$$\varphi = :e^{\alpha(\cdot, \xi + \eta)^2} : \psi_1$$

$$= :e^{\alpha(\cdot, \xi + \eta)^2} : e^{-\alpha(\cdot, \xi - \eta)^2} : \psi.$$

Hence $\varphi = :e^{\alpha(\cdot, \xi + \eta)^2} : e^{-\alpha(\cdot, \xi - \eta)^2} : \psi$. \hfill \Box

**Theorem 6.12.** (Equality in the Extension of Heisenberg Inequality) Let $\xi, \eta \in \mathcal{E} \setminus \{0\}$. Then a function $\varphi \in (L^2)$ satisfies

$$||\check{D}_\xi + \check{D}_\eta^*\varphi||_0 ||\check{D}_\xi + \check{D}_\eta^*\Gamma(\cdot - iI)\varphi||_0 +$$

$$||\check{D}_\eta + \check{D}_\xi^*\varphi||_0 ||\check{D}_\eta + \check{D}_\xi^*\Gamma(\cdot - iI)\varphi||_0 = (||\xi||_0^2 + ||\eta||_0^2)||\varphi||_0^2$$

if and only if there exist $\alpha \in \mathbb{R}$ and $\psi \in (L^2)$, such that $2|\alpha| \cdot ||\xi' + \eta'||_0^2 < 1$, $2|\alpha| \cdot ||\xi' - \eta'||_0^2 < 1$, $\psi$ is supported by the orthogonal complement of the subspace generated by $\xi$ and $\eta$, and

$$\varphi = :e^{\alpha(\cdot, \xi' + \eta')^2} : e^{-\alpha(\cdot, \xi' - \eta')^2} : \psi,$$

where $\xi' := \frac{\xi}{||\xi||_0}$ and $\eta' := \frac{\eta}{||\eta||_0}$.

**Proof.** A trivial solution of equation (6.29) is $\varphi = 0$. Let's assume that $\varphi \in (L^2) \setminus \{0\}$ satisfies relation (6.29). To have equality in inequality (5.1) we must have equality in all the inequalities that we used in the proof of Theorem 5.1.

First, we must have equality in inequality (5.2). Because inequality (5.2) is based upon Schwartz inequality, it implies that there exist $c, d \in \mathbb{C} \cup \{\infty\}$ such that

$$(\check{D}_\xi + \check{D}_\eta^*)\varphi = c(\check{D}_\xi - \check{D}_\eta^*)\varphi$$

$$(\check{D}_\eta + \check{D}_\xi^*)\varphi = d(\check{D}_\eta - \check{D}_\xi^*)\varphi.$$
1. If $c = \infty$, then the relation (6.30) means $(\bar{D}_\xi - D^*_\eta)\varphi = 0$.

2. If $d = \infty$, then the relation (6.31) means $(\bar{D}_\eta - D^*_\xi)\varphi = 0$.

Coming back to the relations (6.30) and (6.31), we conclude that there exist $\gamma, \delta \in \mathbb{C} \cup \{\infty\}$ such that

$$\bar{D}_\xi \varphi = \gamma D^*_\eta \varphi \quad (6.32)$$
$$\bar{D}_\eta \varphi = \delta D^*_\xi \varphi. \quad (6.33)$$

Let $\xi' := \frac{1}{|\xi|_0} \xi, \eta' := \frac{1}{|\eta|_0} \eta$, $\alpha := \frac{\gamma|\eta|_0}{|\xi|_0}$, and $\beta := \frac{\delta|\xi|_0}{|\eta|_0}$. Then $|\xi'|_0 = |\eta'|_0 = 1$ and equations (6.32) and (6.33) become:

$$\bar{D}_{\xi'} \varphi = \alpha D^*_{\eta'} \varphi \quad (6.34)$$
$$\bar{D}_{\eta'} \varphi = \beta D^*_{\xi'} \varphi. \quad (6.35)$$

If $\alpha = \infty$, then $D^*_{\eta'} \varphi = 0$, and thus $\varphi = 0$, which is a contradiction. Therefore $\alpha \neq \infty$. Similarly $\beta \neq \infty$. Therefore $\alpha, \beta \in \mathbb{C}$.

Next we must have equality in inequality (5.3) which means

$$|c| + |\bar{c}| = |c + \bar{c}| \quad (6.36)$$
$$|d| + |\bar{d}| = |d + \bar{d}|. \quad (6.37)$$

The relations (6.36) and (6.37) are equivalent to the fact that $c, d \in \mathbb{R} \cup \{\infty\}$. Hence $\alpha, \beta \in \mathbb{R}$. Because $\varphi \neq 0$ satisfies equations (6.34) and (6.35), according to the previous lemma, we must have $\alpha = \beta, 2|\alpha| \cdot |\xi' + \eta'|_0^2 < 1$, and $2|\alpha| \cdot |\xi' - \eta'|_0^2 < 1$. Moreover, there exists $\psi \in (L^2)$ supported by the orthogonal complement of $\xi$ and $\eta$ such that

$$\varphi = : e^{\alpha \cdot (\xi' + \eta')^2} : \circ : e^{-\alpha \cdot (\xi' - \eta')^2} \circ \psi.$$
Finally we must have equality in inequality (5.4). This means

\[ |(\gamma^2 - 1)||D_\eta \varphi||_0| + |(\delta^2 - 1)||D_\xi \varphi||_0| \]
\[ = |(\gamma^2 - 1)||D_\eta \varphi||_0 + (\delta^2 - 1)||D_\xi \varphi||_0|. \quad (6.38) \]

This means that \((\gamma^2 - 1)\) and \((\delta^2 - 1)\) have the same sign. But this is already true since \(\gamma = \delta\), because \(\alpha = \beta\). \qed
Chapter 7
Introduction to Paley-Wiener Theorem

Before presenting some analogue results to Paley-Wiener Theorem for White Noise Analysis we state the classical Paley-Wiener theorem for Schwartz functions and distributions in $\mathbb{R}^n$.

The well-known Paley-Wiener theorem ([22], Theorem IX.11) for test functions says that an entire function $g(\tau)$ of $n$ complex variables is the Fourier transform of a $C^\infty_0(\mathbb{R}^n)$ function with support in the ball $\{x \in \mathbb{R}^n | |x| \leq R\}$ if and only if for each $N \in \mathbb{N}$ there exists a $C_N$ such that

$$|g(\tau)| \leq \frac{C_N e^{R|\text{Im} \tau|}}{(1 + |\tau|)^N}$$

for all $\tau \in \mathbb{C}^n$.

The Paley-Wiener theorem ([22], Theorem IX.12) for temperate distributions says that a distribution $T$ in $\mathbb{R}^n$ has compact support contained in the ball of radius $R$ if and only if its Fourier transform $\hat{T}$ has an analytic continuation to an entire function $\hat{T}(\tau)$ of $n$ complex variables satisfying

$$|\hat{T}(\tau)| \leq C(1 + |\tau|)^N e^{R|\text{Im} \tau|}$$

for all $\tau \in \mathbb{C}^n$ and some constants $C$ and $N$.

We observe that whenever we want to describe functions with compact support in terms of their S-transform we have to impose an analytic condition and a growth condition. The connection between the support properties of a function and the analyticity properties of its Fourier transform was first pointed out by R. Paley and N. Wiener in [19]. They focused on $L^2$ functions and $L^2$ boundary value. E.C. Titchmarsh continued the study of the relationship between analyticity and the
Fourier transform in [29]. L. Schwartz was the first one who found the connection between the distributions with compact support and their Fourier transform in [23]. In our study, as in the original work of Paley and Wiener, we will consider only \((L^2)\) functions.

In Chapter 8 we will give a complete description of weakly and strongly compact subsets of \(E'\) and in Chapter 9 we will characterize some classes of \((L^2)\) functions with compact support in terms of their S-transform.
Chapter 8
Compact Subsets of the Dual of a Nuclear Space

8.1 Compact Sets in $\mathcal{E}'$

To be able to give an analogue of the Paley-Wiener theorem the first thing that we have to do is to find a clear description of the compact subsets of $\mathcal{E}'$. There are two important topologies on $\mathcal{E}'$, namely: the weak topology and the strong topology. The later one is the same as the inductive limit topology.

The weak topology is the locally convex topology on $\mathcal{E}'$ given by the family of seminorms $\{||\cdot||_\xi \mid \xi \in \mathcal{E}\}$, where $||x||_\xi = |\langle x, \xi \rangle|$. The strong topology is the locally convex topology on $\mathcal{E}'$ given by the family of seminorms $\{||\cdot||_B \mid B \text{ bounded subset of } \mathcal{E}\}$, where $||x||_B = \sup\{ |\langle x, \xi \rangle| \mid \xi \in B\}$.

**Theorem 8.1.** A subset $K$ of $\mathcal{E}'$ is relatively weakly compact if and only if there exists $p > 0$ such that $K \subset \mathcal{E}_{-p}$ and $K$ is bounded with respect to the norm $|\cdot|_{-p}$.

**Proof.** ($\Rightarrow$) Let $K$ be a relatively weakly compact subset of $\mathcal{E}'$. For any $\xi \in \mathcal{E}$, the function $f_\xi : K \to \mathbb{R}$, defined by $f_\xi(x) = \langle x, \xi \rangle$ is weakly continuous and since $K$ is relatively weakly compact we conclude that $f_\xi$ is bounded. Therefore for each $\xi \in \mathcal{E}$, there exists $M(\xi) > 0$, such that $\forall x \in K$, $|\langle x, \xi \rangle| \leq M(\xi)$. Now for each $x \in K$ let's consider the bounded linear operator $T_x : \mathcal{E} \to \mathbb{R}$, defined by $T_x(\xi) = \langle x, \xi \rangle$. Then the family of operators $\{T_x \mid x \in K\}$ is pointwise bounded. Because $(\mathcal{E}, \{|\cdot|_p\}_{p \in \mathbb{N}})$ is a Fréchet space, by the Uniform Bounded Principle, we conclude that there exist $p > 0$ and $M > 0$ such that $\forall x \in K$ and $\forall \xi \in \mathcal{E}$, $|T_x(\xi)| \leq M \cdot |\xi|_p$. Therefore $\forall x \in K$, $x \in \mathcal{E}_{-p}$ and $|x|_{-p} \leq M$.

($\Leftarrow$) Let $K$ be a bounded subset of $\mathcal{E}_{-p}$, for some $p > 0$. There exists $r > 0$
such that \( K \subset B_{-p}[0, r] \), where \( B_{-p}[0, r] = \{ x \in \mathcal{E}_{-p} \mid |x|_{-p} \leq r \} \). Then by Alaoglu’s theorem \( B_{-p}[0, r] \) is weakly compact in \( \mathcal{E}_{-p} \), that means compact with respect to the topology given by the family of seminorms \( \{|x|_{-p} \mid \xi \in \mathcal{E}_p\} \). Because \( \mathcal{E} \subset \mathcal{E}_p \), \( B_{-p}[0, r] \) is also compact with respect to the topology given by the family of seminorms \( \{|x|_{-p} \mid \xi \in \mathcal{E}\} \). Since \( K \subset B_{-p}[0, r] \), we conclude that \( K \) is relatively weakly compact.

If \( K \) is a strongly compact subset of \( \mathcal{E}' \), then \( K \) is weakly compact and therefore there exists \( p > 0 \) such that \( K \subset \mathcal{E}_{-p} \). Using the fact that the strong topology on \( \mathcal{E}' \) is the same as the inductive limit topology, we conclude that \( K \) is compact in \( \mathcal{E}' \) if and only if \( K \) is compact in \( \mathcal{E}_{-p} \). Because \( (\mathcal{E}_{-p}, | \cdot |_{-p}) \) is a separable Hilbert space, we see that describing the strongly compact subsets of \( \mathcal{E}' \) is the same as describing the compact subsets of a separable Hilbert space.

Let \((H, (,))\) be a separable Hilbert space and let \( \{e_n\}_{n \geq 1} \) be an orthonormal basis in \( H \). Let’s consider the projections \( P_n : H \to H \), defined by

\[
P_n x = \sum_{i=n}^{\infty} (x, e_i)e_i.
\]

Let \( || \cdot || \) denote the norm of \( H \).

**Theorem 8.2.** A subset \( K \) of \( H \) is compact if and only if \( K \) is closed, bounded, and \( ||P_n x|| \searrow 0 \) as \( n \to \infty \), uniformly on \( K \).

**Proof.** (\(\Rightarrow\)) Let \( K \) be a compact subset of \( H \). Then \( K \) is closed and bounded. Since for all \( x \in K \), \( ||P_n x|| \searrow 0 \) as \( n \to \infty \), by Dini’s theorem we conclude that \( ||P_n x|| \searrow 0 \) as \( n \to \infty \), uniformly on \( K \).

(\(\Leftarrow\)) Let’s suppose that \( K \) is closed, bounded and \( ||P_n x|| \searrow 0 \) as \( n \to \infty \), uniformly on \( K \). We want to prove that \( K \) is compact. Since in a separable Hilbert space the compactness is equivalent to sequential compactness, it is enough to show that any sequence in \( K \) has a convergent subsequence. Let \( \{x_n\}_{n \geq 1} \subset K \). Since
$K$ is bounded, each of the sequences $\{(x_n,e_i)\}_{n \geq 1}$ is bounded. Using a diagonal procedure we can find a subsequence $\{x_{n_q}\}_{q \geq 1}$ of $\{x_n\}_{n \geq 1}$ such that for each $i \geq 1$, there exists $\alpha_i \in \mathbb{R}$ such that $(x_{n_q},e_i) \to \alpha_i$, as $q \to \infty$.

Let $M = \sup \{|x| \mid x \in K\}$. Since $\sum_{i=1}^{\infty} (x_{n_q}, e_i)^2 \leq M^2$ for all $q \geq 1$, considering the partial sums of these series and letting $q$ go to infinity, we can see that $
\sum_{i=1}^{\infty} \alpha_i^2 \leq M^2$. Thus we can consider the element $y := \sum_{i=1}^{\infty} \alpha_i e_i$ of $H$.

**Claim** $x_{n_q} \to y$, as $q \to \infty$.

Let $\epsilon > 0$. Since $||P_u x_{n_q}|| \searrow 0$ as $u \to \infty$ uniformly, there exists $U \in \mathbb{N}$, such that for all $u \geq U$, and $q \in \mathbb{N}$, we have $||P_u x_{n_q}|| \leq \epsilon/4$. Considering first the partial sums of $||P_u x_{n_q}||^2$ and passing to the limit as $q$ goes to infinity, we can see that for all $u \geq U$, $||P_u y|| \leq \epsilon/4$. Since for all $u \in \{1, 2, \cdots, U - 1\}$, $(x_{n_q}, e_u) \to \alpha_u$, as $q \to \infty$, there exists a $Q \in \mathbb{N}$ such that for all $q \geq Q$, $\sum_{u=1}^{U-1} [(x_{n_q}, e_u) - \alpha_u]^2 < (\epsilon/2)^2$. Therefore for all $q \geq Q$, we have:

$$||x_{n_q} - y|| \leq \sqrt{\sum_{u=1}^{U-1} [(x_{n_q}, e_u) - \alpha_u]^2 + ||P_U x_{n_q}|| + ||P_U y||} < \epsilon/2 + \epsilon/4 + \epsilon/4 = \epsilon.$$

So $K$ is compact. $\square$

Let $d_0$ denote the set of all nonnegative real sequences decreasing to 0. By the word “decreasing” we mean in fact “nonincreasing”. The above arguments allow us to find a nice “correspondence” between the compact subsets of $H$ and the elements of $d_0$. To each element $\alpha = \{\alpha_n\}_{n \geq 1} \in d_0$ we associate the following compact set

$$B_\alpha = \{x \in H \mid \forall n \geq 1, ||P_n x|| \leq \alpha_n\}.$$ 

This set is closed, bounded, and $||P_n x|| \to 0$, uniformly on $B_\alpha$ as $n \to \infty$. Therefore $B_\alpha$ is compact.

On the other hand any compact set is contained in a compact set of this type since if $K$ is compact we can consider the sequence $\alpha = \{\alpha_n\}_{n \geq 1} \in d_0$, where

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\[ \alpha_{n} := \sup\{||P_{n}x|| \mid x \in K\}. \]

We can see that \( \alpha_{n} \downarrow 0 \) as \( n \to \infty \) since \( ||P_{n}x|| \to 0 \)
as \( n \to \infty \), uniformly on \( K \). It is clear that \( K \subset B_{\alpha} \).

To emphasize the analogy with the finite dimensional case one may call \( B_{\alpha} \) as “the ball of center 0 and radius the sequence \( \alpha \)”.

Coming back to our Gel’fand triple \( E \subset E \subset E' \), we have seen that for a subset \( K \) of \( E' \) to be compact with respect to the strong topology or inductive limit topology, it is necessary and sufficient, that the set \( K \) be contained in some Hilbert space \( E_{p} \) and be compact with respect to the topology of \( E_{p} \). If we consider the basis \( \{e_{n}\}_{n \geq 1} \) of \( E \) composed of eigenvectors of the operator \( A \), then each compact subset of \( E_{p} \) is contained in some

\[ B_{\alpha}^{-p} := \{x \in E_{-p} \mid \forall n \geq 1, |P_{n}x|_{-p} \leq \alpha_{n}\}. \]

Here \( P_{n}x := \sum_{i=n}^{\infty} (x,e_{i})e_{i} \) and \( \alpha = \{\alpha_{n}\}_{n \geq 1} \subset d_{0} \) hence

\[ B_{\alpha}^{-p} = \{x \in E_{-p} \mid \forall n \geq 1, \sum_{i=n}^{\infty} \lambda_{i}^{-2p}(x,e_{i})^{2} \leq \alpha_{n}^{2}\}. \]

We have obtained the following result:

**Theorem 8.3.** A subset \( K \) of \( E' \) is strongly compact if and only if there exist \( p > 0 \) and \( \alpha \subset d_{0} \), such that \( K \subset E_{-p} \), \( K \) is closed in \( E_{-p} \), and \( K \subset B_{\alpha}^{-p} \).
Chapter 9
The Characterization of Certain Classes of Functions with Compact Support

9.1 Functions in \((\mathcal{E})^*\) with Compact Support

Proposition 9.1. If \(K\) is a compact subset of \(\mathcal{E}'\), then there are no functions in \((\mathcal{E})\), other than the identically zero function, that vanish outside of \(K\).

Proof. To see this let's consider a function \(\varphi \in (\mathcal{E})\) vanishing outside of \(K\). It is known, from [16], that \(\varphi\) has an analytic extension to the complexification \(\mathcal{E}'_c\) of \(\mathcal{E}'\). Then for a fix \(x \in \mathcal{E}' \setminus \{0\}\) we may consider the analytic function \(\varphi_x : \mathbb{C} \to \mathbb{C}\) defined by \(\varphi_x(z) = \varphi(zx)\). Because \(K\) is contained and bounded in some \(\mathcal{E} - \varnothing\) and \(\varphi\) vanishes outside of \(K\), we can see that \(\varphi_x(z) = 0\), for all real \(z\) of sufficiently large modulus. By the Identity Theorem for analytic functions, we conclude that \(\varphi_x(z) = 0\) for all \(z \in \mathbb{C}\), in particular for \(z = 1\) which means that \(\varphi(x) = 0\). Since \(x \in \mathcal{E}' \setminus \{0\}\) was chosen arbitrarily and \(\varphi\) is continuous, we conclude that \(\varphi\) is the identically zero function. \(\square\)

This proposition discourages us from trying to find an analogue result of the classical Paley-Wiener theorem that describes the \(C_{0}^{\infty}\) functions with compact support in terms of their Fourier transform.

Proposition 9.2. If \(\alpha = \{\alpha_n\}_{n \in \mathbb{N}} \in \mathcal{d}_{0}\) is strictly decreasing, then there are no functions in \((\mathcal{E})\), other than the identically zero function, that vanish on \(B_{\alpha}^{-p}\).

Proof. To see this let's consider a function \(\varphi \in (\mathcal{E})\) vanishing on \(B_{\alpha}^{-p}\). Let \(n \in \mathbb{N}\) be fixed. Let \(\beta_1 = \sqrt{\alpha_1^2 - \alpha_2^2}, \ \beta_2 = \sqrt{\alpha_2^2 - \alpha_3^2}, \ldots, \ \beta_{n-1} = \sqrt{\alpha_{n-1}^2 - \alpha_n^2},\) and \(\beta_n = \alpha_n\). Then \(\forall i \leq n, \ \beta_i > 0\), because the sequence \(\{\alpha_n\}_{n \in \mathbb{N}}\) is strictly decreasing.
to zero. Now, let $\beta = \min\{\beta_i \mid i \leq n\}$. Then $\beta > 0$. We can see that the set $M_\beta = \{x \in \Re_1 + \Re_2 + \cdots + \Re_n \mid \forall i \leq n, \ |\langle x, e_i \rangle| \leq \beta\}$ is contained in $B^{-p}_\alpha$. Let $x \in \Re_1 + \Re_2 + \cdots + \Re_n$ be fixed. Let's consider again the analytic function $\varphi_x : \mathbb{C} \to \mathbb{C}$ defined by $\varphi_x(z) = \varphi(zx)$. We can see that, for all real $z$ of sufficiently small modulus, $zx \in M_\beta \subset B^{-p}_\alpha$, and since $\varphi$ vanishes on $B^{-p}_\alpha$, we conclude that $\varphi_x(z) = 0$. By the Identity Theorem for analytic functions, we conclude that $\varphi_x(z) = 0$ for all $z \in \mathbb{C}$, in particular for $z = 1$ which means that $\varphi(x) = 0$. Thus we have proved that $\varphi$ vanishes on all finite dimensional subspaces of the form $\Re_1 + \Re_2 + \cdots + \Re_n$, $n \in \mathbb{N}$. Because of the density of the union of these subspaces in $\mathcal{E}'$ and the fact that $\varphi$ is continuous, we conclude that $\varphi$ is the identically zero function.

This proposition makes the definition of a generalized function $\phi \in (\mathcal{E})^*$ with support in $B^{-p}_\alpha$ be difficult to formulate.

For the reasons explained above we will try to describe only the $(L^2)$ functions, having compact support, in terms of their $S$ transform.

**Definition 9.3.** We say that a function in $(L^2)$ has the support contained in a Borel subset $K$ of $\mathcal{E}'$ if it vanishes almost everywhere outside of $K$.

First of all we will show that there exists a strongly compact set $K$, such that $\mu(K) > 0$. If we do this, then the characteristic function $\chi_K$ of $K$ will be a function in $(L^2)$ with compact support and not equal to zero almost everywhere.

**Lemma 9.4.** There exists a strongly compact subset $K$ of $\mathcal{E}'$ such that $\mu(K) > 0$.

**Proof.** Let $\{\beta_n\}_{n \geq 1} \subset (0, \infty)$, such that $\sum_{n=1}^{\infty} \beta_n^2 < \infty$. Let

$$K = \{x \in \mathcal{E}' \mid \forall n \geq 1, \lambda_n^{-p}|\langle x, e_n \rangle| \leq \beta_n\},$$

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for some \( p > 1 \). Then it is easy to see that \( K \subset \mathcal{E}_p \), \( K \) is closed and bounded in \( \mathcal{E}_p \), and \( |P_n x|_p \to 0 \) as \( n \to \infty \), uniformly on \( K \). Therefore \( K \) is a strongly compact subset of \( \mathcal{E}' \). We have:

\[
\mu(K) = \int_K 1 \mu(dx) = \prod_{n=1}^{\infty} \left( \frac{1}{\sqrt{2\pi}} \int_{-\lambda_n^p \beta_n}^{\lambda_n^p \beta_n} e^{-\frac{x^2}{2}} dx \right) = \prod_{n=1}^{\infty} \left( 1 - \frac{1}{\sqrt{2\pi}} \int_{\lambda_n^p \beta_n}^{\infty} e^{-\frac{x^2}{2}} dx \right)
\]

To have \( \mu(K) > 0 \) we need to show that the last product is convergent which is equivalent to the fact that the series \( \sum_{n=1}^{\infty} \int_{\lambda_n^p \beta_n}^{\infty} e^{-\frac{x^2}{2}} dx \) is convergent. But

\[
\sum_{n=1}^{\infty} \int_{\lambda_n^p \beta_n}^{\infty} e^{-\frac{x^2}{2}} dx \leq \sum_{n=1}^{\infty} \frac{1}{\lambda_n^p \beta_n} \int_{\lambda_n^p \beta_n}^{\infty} xe^{-\frac{x^2}{2}} dx = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^p \beta_n} e^{-\frac{\lambda_n^p \beta_n^2}{2}} \leq \sum_{n=1}^{\infty} \frac{1}{\lambda_n^p \beta_n}.
\]

We may choose \( \beta_n = \lambda_n^{2-p} \), for all \( n \), and \( p \geq 3 \). Using the fact that \( \sum_{n=1}^{\infty} \lambda_n^{-2} < \infty \), we conclude that \( \sum_{n=1}^{\infty} \beta_n^2 < \infty \) and \( \sum_{n=1}^{\infty} \frac{1}{\lambda_n^p \beta_n} \) is convergent. Thus \( \mu(K) > 0 \). □

Let \( \nu \) be the standard Gaussian measure on \( \mathbb{R}^n \), i.e the probability measure given by the density function \( g : \mathbb{R}^n \to \mathbb{R}, g(x) = \frac{1}{\sqrt{(2\pi)^n}} e^{-\frac{|x|^2}{2}} \). Let’s also consider the space \( \mathcal{H}L^2(\mathbb{C}^n, m) \) of all analytic functions on \( \mathbb{C}^n \) that are square integrable with respect to the measure \( m \) on the Borel subsets of \( \mathbb{C}^n \) given by the density function \( h : \mathbb{C}^n \to \mathbb{R}, h(z) = \frac{1}{\pi^n} e^{-|z|^2} \). The following result is known from [1] and [6].

**Lemma 9.5.** The linear map \( S : L^2(\mathbb{R}^n, \nu) \to \mathcal{H}L^2(\mathbb{C}^n, m) \) defined by

\[
(Sf)(z) = \int_{\mathbb{R}^n} f(u)e^{z^*u - \frac{1}{2}z^*z} \nu(du)
\]

is a surjective unitary operator.
Here if \( z = (x_1, x_2, \cdots, x_n) + i(y_1, y_2, \cdots, y_n) \) and \( z' = (x'_1, x'_2, \cdots, x'_n) + i(y'_1, y'_2, \cdots, y'_n) \), then
\[
z \cdot z' = \sum_{j=1}^{n} (x_j x'_j - y_j y'_j) + i \sum_{j=1}^{n} (x_j y'_j + y_j x'_j).
\]

Before characterizing the functions \( \varphi \) in \( L^2 \) with compact support in terms of their \( S \)-transform we introduce some notations.

Let \( \{e_n\}_{n \geq 1} \) be an orthonormal basis given by the eigenvalues \( \{\lambda_n\}_{n \geq 1} \) of the operator \( A \). Let \( \alpha = \{\alpha_n\}_{n \geq 1} \in d_0 \) be fixed.

For any two natural numbers \( r \) and \( n \) such that \( r \leq n \), we define the operators \( P_n, Q_{r,n} : \mathcal{E}' \rightarrow \mathcal{E}' \) by
\[
P_n x = \sum_{i=n}^{\infty} \langle x, e_i \rangle e_i
\]
and
\[
Q_{r,n} x = \sum_{i=r}^{n-1} \langle x, e_i \rangle e_i.
\]

When \( r = n \), \( Q_{r,n} \) is understood to be 0. If \( r = 1 \), then we denote \( Q_{1,n} \) simply by \( Q_n \). We also consider the vector space
\[
V_r^n = \mathbb{R} e_r + \mathbb{R} e_{r+1} + \cdots + \mathbb{R} e_{n-1}.
\]

If \( r = n \), \( V_r^n \) is understood to be the null space. If \( r = 1 \), then we denote \( V_1^n \) simply by \( V^n \). We denote by \( V_{r,c}^n \) the complexification of \( V_r^n \).

In the following theorem we will consider \( \frac{1}{\pi^{n-r}} \int_{V_{r,c}^n} |F(\xi_1 + \xi)|^2 e^{-|\xi|^2} d\xi \), for \( \xi_1 \perp V_{r,c}^n \).

If \( r = n \), then this integral is understood to be \( |F(\xi_1)|^2 \).

We denote by \( \mu_{r,n} \) the Gaussian measure on the finite dimensional space \( V_r^n \). If we consider the Hilbert space \( E_{r,n} = (I - Q_{r,n}) E \), and the restriction \( A_{r,n} \) of the operator \( A \) to this space, then we have that \( A_{r,n} \) is a densely defined, positive, and self-adjoint operator on \( E_{r,n} \), whose eigenvalues are \( \{\lambda_i \mid 1 \leq i \leq r - 1\} \cup \{\lambda_j \mid j \geq n\} \).
The eigenvalues of $A_{r,n}$ satisfy the same conditions as the eigenvalues of $A$, therefore we can do the same construction using $A_{r,n}$, as we did using $A$, and obtain a Gel'fand triple $\mathcal{E}_{r,n} \subset E_{r,n} \subset \mathcal{E}'_{r,n}$. Using again the Minlos' Theorem we can construct a Gaussian measure, which we will denote by $\mu_{r,n}$, on the Borel subsets of $\mathcal{E}'_{r,n}$. We will denote $\mu_{1,n} = \mu_n$ and $\mu_{1,n}^\perp = \mu_n^\perp$. Because $V_r^n \perp E_{r,n}$ and $V_r^n + E_{r,n} = E$ we can see that the measure $\mu$ on $\mathcal{E}$ is the product measure of $\mu_{r,n}$ and $\mu_{r,n}^\perp$. We will denote $K_{r,n} = \{ y \in \mathcal{E}'_{r,n} \mid |P_n y|_p \leq \alpha_n \}$. Then we denote $K_n = K_{1,n}$.

If $x, y \in \mathcal{E}'$, and $f \in (L^2)$, then we denote $f_y(x) = f(x + y)$.

**Theorem 9.6.** Let $\alpha = \{\alpha_n\}_{n \geq 1}$ be a sequence of real numbers decreasing to zero and let $p > 0$. A function $F : \mathcal{E} \to \mathbb{C}$ is the S-transform of a function $\varphi \in (L^2)$, with support in $B_{\alpha^{-p}}$ if and only if the following three conditions hold

1. $F$ is continuous.

2. For all $\xi, \eta \in \mathcal{E}$, the function $z \mapsto F(z\xi + \eta)$ is analytic.

3. There exists a constant $C > 0$ such that for all $r, n \in \mathbb{N}$ satisfying $r \leq n$, and for all $\xi_1 \in \mathcal{E}$, $\xi_1 \perp V_{r,c}^n$ we have

\[
\frac{1}{\pi^{n-r}} \int_{V_{r,c}^n} |F(\xi + \xi_1)|^2 e^{-|\xi_1|^2} d\xi \leq C e^{\frac{|Q_r \xi_1|^2}{2} + \frac{|R_{n} \xi_1|}{2} - \Re(P_n \xi_1, P_n \xi_1)}.\]

**Proof.** ($\Rightarrow$) Let $\varphi \in (L^2)$ be supported on $B_{\alpha^{-p}}$. It is easy to see that the S-transform of $\varphi$ satisfies conditions 1. and 2. from above. Let $r, n \in \mathbb{N}$, such that $r \leq n$, and $\xi_1 \in \mathcal{E}$, $\xi_1 \perp V_{r,c}^n$. Then we have:

\[
\frac{1}{\pi^{n-r}} \int_{V_{r,c}^n} |(S\varphi)(\xi + \xi_1)|^2 e^{-|\xi_1|^2} d\xi
\]

\[
= \frac{1}{\pi^{n-r}} \int_{V_{r,c}^n} \left| \int_{\mathcal{E}'} \varphi(x) e^{(x, \xi_1) - \frac{1}{2} (\xi, \xi_1)} \cdot e^{(x, \xi_1) - \frac{1}{2} (\xi, \xi_1)} \mu(dx) \right|^2 e^{-|\xi_1|^2} d\xi
\]

\[
= \frac{1}{\pi^{n-r}} \int_{V_{r,c}^n} \left| \int_{\mathcal{E}'} \varphi(x) : e^{(x, \xi_1)} : \cdots : e^{(x, \xi_1)} : \mu(dx) \right|^2 e^{-|\xi_1|^2} d\xi. \tag{9.1}
\]

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Now we use the fact that the measure $\mu$ is the product measure of $\mu_{r,n}$ and $\mu_{r,n}^\perp$. Because the support of $\varphi$ is contained in $B_{\alpha}^{-p}$, applying Fubini's Theorem in (9.1) we obtain:

$$\frac{1}{\pi^{n-r}} \int_{V_{r,n}^c} |(S\varphi)(\xi + \xi_1)|^2 e^{-|\xi_0|^2} d\xi = \frac{1}{\pi^{n-r}} \int_{V_{r,n}^c} \left( \int_{K_{r,n}} \varphi_y(x) : e^{(x,\xi)} : \mu_{r,n}(dx) \right) : e^{(y,\xi_1)} : \mu_{r,n}^\perp(dy) e^{-|\xi_0|^2} d\xi.$$

Applying the Schwartz inequality in the last expression we obtain:

$$\frac{1}{\pi^{n-r}} \int_{V_{r,n}^c} |(S\varphi)(\xi + \xi_1)|^2 e^{-|\xi_0|^2} d\xi \leq \frac{1}{\pi^{n-r}} \int_{V_{r,n}^c} \left\{ \int_{K_{r,n}} \left( \int_{V_{r,n}^c} \varphi_y(x) : e^{(x,\xi)} : \mu_{r,n}(dx) \right) : e^{(y,\xi_1)} : \mu_{r,n}^\perp(dy) \right\} e^{-|\xi_0|^2} d\xi. \quad (9.2)$$

Therefore we have:

$$\frac{1}{\pi^{n-r}} \int_{V_{r,n}^c} |(S\varphi)(\xi + \xi_1)|^2 e^{-|\xi_0|^2} d\xi \leq A \cdot B, \quad (9.3)$$

where

$$A := \frac{1}{\pi^{n-r}} \int_{V_{r,n}^c} \int_{K_{r,n}} \left( \int_{V_{r,n}^c} \varphi_y(x) : e^{(x,\xi)} : \mu_{r,n}(dx) \right) : e^{(y,\xi_1)} : \mu_{r,n}^\perp(dy) e^{-|\xi_0|^2} d\xi$$

and

$$B := \int_{K_{r,n}} \left( : e^{(y,\xi_1)} : \right) : e^{(x,\xi)} : \mu_{r,n}(dx) \mu_{r,n}^\perp(dy).$$

Because the support of $\varphi$ is contained in $B_{\alpha}^{-p}$ we have:

$$A = \frac{1}{\pi^{n-r}} \int_{V_{r,n}^c} \int_{\mathcal{E}_{r,n}^c} \left( \int_{V_{r,n}^c} \varphi_y(x) : e^{(x,\xi)} : \mu_{r,n}(dx) \right) : e^{(y,\xi_1)} : \mu_{r,n}^\perp(dy) e^{-|\xi_0|^2} d\xi.$$

Therefore, applying Fubini's theorem, we obtain:

$$A = \int_{\mathcal{E}_{r,n}^c} \left\{ \frac{1}{\pi^{n-r}} \int_{V_{r,n}^c} \left( \int_{V_{r,n}^c} \varphi_y(x) : e^{(x,\xi)} : \mu_{r,n}(dx) \right) : e^{(y,\xi_1)} : \mu_{r,n}^\perp(dy) \right\} : e^{(x,\xi_1)} : \mu_{r,n}^\perp(dy). \quad (9.4)$$
Using Lemma 9.5 we get:

\[ A = \int_{r,n} \| \varphi_y \|^2 \mu_{r,n}^\perp(dy) \]
\[ = \| \varphi \|^2_0. \]  \hspace{1cm} (9.5)

On the other hand:

\[ B = \int_{y,n} \left| e^{(y,Q,\xi)} \right|^2 \mu_{r,n}^\perp(dy) \]
\[ = \int_{y,n} \left| e^{(y,P_n \xi)} \right|^2 \mu_{r,n}^\perp(dy). \]  \hspace{1cm} (9.6)

Applying Fubini’s Theorem again, we obtain:

\[ B = \int_{y,n} \left| e^{(y,Q,T)} \right|^2 \mu_r(dy) \int_{K_n} \left| e^{(y,P_n \xi)} \right|^2 \mu_n^\perp(dy). \]  \hspace{1cm} (9.7)

Thus we have

\[ B \leq \left| e^{Q,T} \right|^2 \sup_{y \in K_n} \left| e^{(P_n \xi)} \right|^2 \]
\[ \leq e^{Q,T} \sup_{y \in K_n} e^{2|P_n \xi| - |ReP_n \xi| - Re(P_n \xi, P_n \xi)} \]
\[ = e^{Q,T} e^{2|P_n \xi|} Re(P_n \xi, P_n \xi). \]  \hspace{1cm} (9.8)

According to the relations (9.3), (9.5), and (9.8) we have:

\[ \frac{1}{\pi^{n-r}} \int_{V_{r,c}} |(S \varphi)(\xi + \xi_1)|^2 e^{-|\xi_1|^2} d\xi \leq \| \varphi \|^2_0 e^{Q,T} e^{2|P_n \xi|} Re(P_n \xi, P_n \xi). \]

(\Leftarrow) Let’s suppose that \( F : \mathcal{E}_c \to \mathbb{C} \) satisfies the conditions 1., 2., and 3. from Theorem 9.6. Therefore there exists a constant \( C > 0 \) such that for all \( r, n \in \mathbb{N} \) satisfying \( r \leq n \), and for all \( \xi_1 \in \mathcal{E}_c, \xi_1 \perp V_{r,c} \) we have

\[ \frac{1}{\pi^{n-r}} \int_{V_{r,c}} |F(\xi + \xi_1)|^2 e^{-|\xi_1|^2} d\xi \leq C e^{Q,T} e^{2|P_n \xi|} Re(P_n \xi, P_n \xi). \]  \hspace{1cm} (9.9)

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Let \( r, n \in \mathbb{N}, r < n, \) be fixed. Let \( \xi_1 \in V_{1,r}^n \) be also fixed. Now let's consider the analytic function \( F_{r,n} : V_{r,c}^n \to \mathbb{C} \) defined by

\[
F_{r,n}(\xi) = F(\xi_1 + \xi).
\]

Because

\[
\frac{1}{\pi^{n-r}} \int_{V_{r,c}^n} |F(\xi_1 + \xi)|^2 e^{-|\xi|^2} d\xi < \infty,
\]

according to Lemma 9.5 we conclude that there exists a function \( \psi_{r,n} \in L^2(V_{r,c}^n, \mu_{r,n}) \), such that

\[
\forall \xi \in V_{r,c}^n, \quad \int_{V_{r,c}^n} \psi_{r,n}(x) e^{\langle x, \xi \rangle} e^{-\frac{1}{2} \langle \xi, \xi \rangle} \mu_{r,n}(dx) = F_{r,n}(\xi).
\]

Now, let's consider the cylindric function \( \varphi_{r,n} : \mathcal{E}' \to \mathbb{C} \) defined by

\[
\varphi_{r,n}(x) = \psi_{r,n}(Q_{r,n}x).
\]

Then it is clear that \( \varphi_{r,n} \in (L^2) \) and \( S\varphi_{r,n}(\xi) = F_{r,n}(\xi_1 + Q_{r,n}\xi) \).

Let's consider the function \( f_{r,n} : \mathbb{R}^{n-r} \to \mathbb{C} \) defined by

\[
f_{r,n}(t_r, t_{r+1}, \cdots, t_{n-1}) = \left( \prod_{j=r}^{n-1} \lambda_j^p \right) \varphi_{r,n}(x) e^{-\frac{1}{2} \langle x, x \rangle},
\]

where \( x = \lambda_r^p t_re_r + \lambda_{r+1}^p t_{r+1}e_{r+1} + \cdots + \lambda_{n-1}^p t_{n-1}e_{n-1} \). Then it is easy to see that

\[
\frac{1}{\sqrt{(2\pi)^{n-r}}} \int_{\mathbb{R}^{n-r}} |f_{r,n}(t)| dt = \int_{\mathcal{E}'} |\varphi_{r,n}(x)| \mu(dx) < \infty.
\]

Therefore \( f_{r,n} \in L^1(\mathbb{R}^{n-r}, dt) \).

We can also see that for any \( \tau \in \mathbb{C}^{n-r} \), we have

\[
\frac{1}{\sqrt{(2\pi)^{n-r}}} \int_{\mathbb{R}^{n-r}} f_{r,n}(t)e^{-i\langle t, \tau \rangle} dt = S\varphi_{r,n}(-i\eta)e^{-\frac{1}{2} \langle \eta, \eta \rangle},
\]

where \( \eta = \lambda_r^p \tau_r e_r + \lambda_{r+1}^p \tau_{r+1}e_{r+1} + \cdots + \lambda_{n-1}^p \tau_{n-1}e_{n-1} \). Applying the inequality (9.9) for \( r' = n' = r \) and \( \xi'_1 = \xi_1 + \eta \), we have

\[
|F(\xi_1 + \eta)| \leq \sqrt{c} e^{\frac{1}{2} |\xi'_1|^2 e^{\alpha r} |\text{Re} \eta|^r} e^{-\frac{1}{2} \text{Re} \langle \eta, \eta \rangle}.
\]
Because $|S\varphi_{r,n}(\eta)| = |F(\xi_1 + \eta)| \leq \sqrt{C}e^{\frac{1}{2}|\xi_1|^2}e^{\alpha_r|\Re\eta|}e^{-\frac{1}{2}\Re(\eta,\eta)}$, we conclude that the Fourier transform $\hat{f}$ of $f$ is analytic and satisfies the following inequality:

$$\forall \tau \in C^{n-r}, \quad |\hat{f}(\tau)| \leq \sqrt{C}e^{\frac{1}{2}|\xi_1|^2}e^{\alpha_r|Im\tau|}.$$

According to the classical Paley-Wiener theorem, as a distribution in $\mathbb{R}^{n-r}$, $f$ has compact support contained in $\{t \in \mathbb{R}^{n-r} \mid |t| \leq \alpha_r\}$. Since $f$ is an $L^1$-function, we conclude that $f$ vanishes almost everywhere outside of the ball $\{t \in \mathbb{R}^{n-r} \mid |t| \leq \alpha_r\}$. Let $\Lambda = \prod_{j=r}^{n-1} \lambda_j^{-p}$. Because

$$\varphi_{r,n}(x) = \Lambda f(\lambda_r^{-p}(x,e_r), \lambda_{r+1}^{-p}(x,e_{r+1}), \cdots, \lambda_{n-1}^{-p}(x,e_{n-1}))e^{\frac{1}{2}(Q_r,x,Q_r,x)},$$

we can see that $\varphi_{r,n}$ has the support contained in $\{x \in \mathcal{E}' \mid |Q_{r,n}x|_{-p} \leq \alpha_r\}$.

Now let $n$ vary from $r + 1$ to $\infty$. We have:

$$||\varphi_{r,n}||_3^2 = \frac{1}{n^{n-r}} \int_{V_{r+c}^*} |F(\xi_1 + \xi)|^2 e^{-|\xi|^2} d\xi \leq Ce^{\xi_1^2}.$$

Therefore the sequence $\{\varphi_{r,n}\}_{n>r}$ is bounded in $(L^2)$ and thus by Alaoglu’s Theorem, it contains a weakly convergent subsequence. Let $\varphi_r(\xi_1) \in (L^2)$ be the weak limit of such a subsequence. Because $F$ is continuous we conclude that $S\varphi_r(\xi_1)(\xi) = F(\xi_1 + P_r\xi)$, and $\varphi_r(\xi_1)$ has support contained in the set $\{x \in \mathcal{E}' \mid |P_r x|_{-p} \leq \alpha_r\}$.

In particular for $r = 1$, $S\varphi_1(0)(\xi) = F(\xi)$. Let’s denote $\varphi_1(0)$ simply by $\varphi$. Therefore $F$ is the S-transform of the function $\varphi$ in $(L^2)$. We also know that $\varphi$ has support in $K_1$. It remains to prove that $\varphi$ has support in $K_{r,r}$, for any $r \geq 2$. To do so, let $r \geq 2$ be fixed, and let $\xi_1 \in V_{r, c}^*$ be arbitrary. It is easy to check that

$$\varphi_r(\xi_1)(x) = \int_{V_r} \varphi(y + P_r x) e^{(y,\xi_1) - \frac{1}{2}(\xi_1,\xi_1)} \mu_r(dy).$$

Let’s consider only those $\xi_1 \in V_{r, c}^*$ whose coordinates with respect to the basis $\{e_1, e_2, \cdots, e_{r-1}\}$ are all in $\mathbb{Q} + i\mathbb{Q}$. Therefore they form a countable set, which we
will call $D_r$. For a fixed $\xi_1$ in $D_r$ we denote

$$A(\xi_1) = \{ x \in \mathcal{E}' \setminus K_{r,r} \mid \varphi_r(\xi_1)(x) \neq 0 \}.$$ 

We know that $\mu(A(\xi_1)) = 0$. Let

$$A = \bigcup_{\xi_1 \in D_r} A(\xi_1).$$

Then $\mu(A) = 0$ and if $x \notin A$, then $\varphi(\cdot + P_{r} x) \perp e^{i(\cdot,\xi_1) - \frac{1}{2}(\xi_1,\xi_1)}$, for all $\xi_1 \in D_r$. Since the functions $\{ e^{i(\cdot,\xi_1) - \frac{1}{2}(\xi_1,\xi_1) \mid \xi_1 \in D_r \}$ span a dense subspace in $L^2(V^r, \mu_r)$ we conclude that $\varphi(x) = 0$. Therefore $\varphi$ vanishes almost everywhere outside of $K_{r,r}$, for all $r \in \mathbb{N}$. In this way we have proved that the function $\varphi$ vanishes almost everywhere outside of $B^{-p}_r = \bigcap_{r \geq 1} K_{r,r}$. □

Let $B_{-p}[0,R] = \{ x \in \mathcal{E}_- \mid |x|_{-p} \leq R \}$. We have seen that any weakly compact subset of $\mathcal{E}'$ is contained in $B_{-p}[0,R]$, for some $p$, and $R > 0$. Therefore it will be of interest to describe the $(L^2)$ functions with support contained in $B_{-p}[0,R]$. In exactly the same way that we proved Theorem 4, we can prove the following result:

**Theorem 9.7.** Let $p > 0$ and $R > 0$. A function $F : \mathcal{E} \rightarrow \mathbb{C}$ is the $S$-transform of a function $\varphi \in (L^2)$, with support in $B_{-p}[0,R]$ if and only if the following three conditions hold

1. $F$ is continuous.

2. For all $\xi, \eta \in \mathcal{E}$, the function $z \mapsto F(z\xi + \eta)$ is analytic.

3. There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and for all $\xi_1 \in \mathcal{E}$,

$$\xi_1 \perp V^o_{n,c}$$

we have

$$\frac{1}{\pi^{n-1}} \int_{V^o_{n,c}} |F(\xi_1)|^2 e^{-|\xi_1|^2} d\xi \leq C e^{2R|\text{Re}P_n\xi_1| - \text{Re}\langle P_n\xi_1, P_n\xi_1 \rangle}.$$
More Notations.

(1) For any compact subset $K$ of $\mathcal{E}'$, let's consider the space:

$$(L^2(K)) = \{ \varphi \in (L^2) \mid \varphi \text{ has support contained in } K \}.$$ 

(2) For any compact subset $K$ of $\mathcal{E}'$, let's consider the space:

$$C^\infty(K) = \{ \varphi \in (L^2(K)) \mid \forall n, i_1, i_2, \ldots, i_n \in \mathbb{N}, D_{e_{i_1}}^* D_{e_{i_2}}^* D_{e_{i_n}}^* \varphi \in (L^2) \}.$$ 

Because $\forall i \in \mathbb{N}, \tilde{D}_{e_i} \varphi = \langle x, e_i \rangle \varphi - D_{e_i}^* \varphi$, $[\tilde{D}_{\xi}, D_{\eta}^*] = \langle \xi, \eta \rangle I$, and the function $\langle \cdot, e_i \rangle$ is continuous and hence bounded on the compact $K$, it's easy to see that

$$C^\infty(K) = \{ \varphi \in (L^2(K)) \mid \forall n, i_1, i_2, \ldots, i_n \in \mathbb{N}, \tilde{D}_{e_{i_1}} \tilde{D}_{e_{i_2}} \cdots \tilde{D}_{e_{i_n}} \varphi \in (L^2) \}.$$ 

**Lemma 9.8.** There exists a compact subset $K$ of $\mathcal{E}'$ such that $C^\infty(K) \neq 0$.

**Proof.** Let's consider a sequence $\alpha = \{\alpha_n\}_{n \geq 1} \in \mathbb{d}_0$, such that $\mu(B^{-p}_\alpha) > 0$. Let $f : \mathbb{R} \to [0, 1]$ be such that $f$ is infinitely many times differentiable, vanishes outside of $(-1, 1)$, and is identically 1 on $(-\frac{1}{4}, \frac{1}{4})$. Then let's consider the function $\varphi : \mathcal{E}' \to \mathbb{R}$ defined by

$$\varphi(x) = \prod_{n=1}^\infty f \left( \frac{1}{4\alpha_n^2} |P_n x|^2_{-p} \right).$$

Then it is clear that

$$\forall x \in \mathcal{E}', \varphi(x) = \lim_{n \to \infty} \prod_{j=1}^n f \left( \frac{1}{4\alpha_j^2} |P_j x|^2_{-p} \right).$$

The limit exists for all $x \in \mathcal{E}'$ and is a number between 0 and 1, because the partial product is decreasing. For each $n \in \mathbb{N}$,

$$f \left( \frac{1}{4\alpha_n^2} |P_n x|^2_{-p} \right) = \lim_{r \to \infty} f \left( \frac{1}{4\alpha_n^2} |Q_{n,n+r} x|^2_{-p} \right).$$

The functions from the right hand side of the last equality are cylindric functions; therefore measurable. This implies that the limit function from the left hand side
is also measurable. Hence $\varphi$ is measurable, too. We can see that $\varphi$ is identically 1, on $B_\alpha^{-p}$ which is a set of strictly positive measure; therefore this function is not the zero function. Because $\varphi$ is bounded and $\mu$ is a probability measure, we conclude that $\varphi \in (L^2)$. We can also see that $\varphi$ has the support contained in $B_{2\alpha}^{-p}$. It remains to show that

$$\forall n, i_1, i_2, \ldots, i_n \in \mathbb{N}, \tilde{D}_{e_{i_1}} \tilde{D}_{e_{i_2}} \cdots \tilde{D}_{e_{i_n}} \varphi \in (L^2).$$

We will show only that for a given $k \in \mathbb{N}$, $\tilde{D}_{e_k} \varphi \in (L^2)$, which will be enough for our purpose. To see this, let's remark that

$$\tilde{D}_{e_k} \varphi(x) = \sum_{j=1}^{k} \left[ \prod_{i=1}^{\infty} f \left( \frac{1}{4\alpha_i^2} |P_i x|^2 - p \right) \right] \frac{\lambda_k \mu_-^{p}(x, e_k)}{2\alpha_j^2} f' \left( \frac{1}{4\alpha_j^2} |P_j x|^2 - p \right).$$

This relation shows us that $\tilde{D}_{e_k} \varphi \in (L^2)$. Therefore $\varphi \in C^\infty(B_{2\alpha}^{-p})$. \qed

Applying Theorem 9.6 repeatedly for $D_{e_1}^* \varphi$, $D_{e_2}^* \varphi$, $\ldots$, $D_{e_m}^* \varphi$, which have also the support in $B_\alpha^{-p}$ because $D_{e_i}^* \varphi = \langle \cdot, e_i \rangle \varphi - \tilde{D}_{e_i} \varphi$, and using the formula $(SD_{e_i}^* \varphi)(\xi) = \langle \xi, e_i \rangle S\varphi(\xi)$, then multiplying by suitable binomial coefficients and summing up all the inequalities obtained, we can easily derive the following:

**Theorem 9.9.** Let $\alpha = \{\alpha_n\}_{n \geq 1}$ be a sequence of real numbers decreasing to zero and let $p > 0$. A function $F : \mathcal{E}_c \to \mathbb{C}$ is the S-transform of a function in $C^\infty(B_\alpha^{-p})$ if and only if the following three conditions hold

1. $F$ is continuous.

2. For all $\xi, \eta \in \mathcal{E}_c$, the function $z \mapsto F(z\xi + \eta)$ is analytic.

3. For all $N, M \in \mathbb{N}$ there exists a constant $C > 0$ such that for all $r, n \in \mathbb{N}$ satisfying $r \leq n$ and for all $\xi_1 \in \mathcal{E}_c$, $\xi_1 \perp V^n_{r,c}$ we have

$$\frac{1}{\pi^{n-r}} \int_{V^n_{r,c}} (1 + |Q_M \xi_0|^2 + |Q_M \xi_1|^2)^N |F(\xi + \xi_1)|^2 e^{-|\xi_1|^2} d\xi \leq C e^{Q_M \xi_0|^2 + 2Re P_n \xi_1 - Re(P_n \xi_1, P_n \xi_1),} \quad (9.10)$$

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where, when \( r = n \), the left-hand side of the inequality (9.10) is understood to be \( (1 + |Q_M \xi_1|_0^2)^N |F(\xi_1)|^2 \).

**Theorem 9.10.** Let \( p > 0 \) and \( R > 0 \). A function \( F : E_c \to \mathbb{C} \) is the S-transform of a function in \( C^\infty(B_{-p}[0, R]) \) if and only if the following three conditions hold

1. \( F \) is continuous.

2. For all \( \xi, \eta \in E_c \), the function \( z \mapsto F(z \xi + \eta) \) is analytic.

3. For all \( N, M \in \mathbb{N} \) there exists a constant \( C > 0 \) such that for all \( n \in \mathbb{N} \) and for all \( \xi_1 \in E_c \), \( \xi_1 \perp V_{1,c}^n \) we have

\[
\frac{1}{\pi^{n-r}} \int_{V_{1,c}^n} (1 + |Q_M \xi_1|_0^2 + |Q_M \xi_1|_0^2)^N |F(\xi + \xi_1)|^2 e^{-|\xi_1|^2} d\xi \leq C e^{2R|\text{Re}(\xi_1)|_R - R\text{Re}(P_n \xi_1, P_n \xi_1)}. \tag{9.11}
\]

**Comments.**

The initial purposes of this work were to find analogues of the classical Paley-Wiener theorem for functions with compact support in (a) \( (E) \) and (b) \( (E)^* \). Because of the analytical property of the test functions, the only function in \( (E) \) that has compact support is the zero function. This fact makes the problem (a) not interesting. We have also observed that for "almost all" compact subsets \( K \) of \( E' \), the only test function that vanishes on \( K \) is again the zero function. This remark shows that it is hard to define what a generalized function in \( (E)^* \) with compact support means. For this reason we focused only on functions in \( (L^2) \). It is of interest to find a natural definition of generalized functions with compact support and then give a characterization of these functions in terms of their S-transform. In our work the description of the strongly compact sets and the characterization theorems obtained depend on the choice of the basis \( \{e_n\}_{n \in \mathbb{N}} \) of the separable
Hilbert space $E$. It is important to find analogue results that are base free. It is my belief that this work is the first step toward characterizing classes of functions, defined on infinite dimensional spaces, with compact support. It opens a gate for further research and a deeper understanding of the infinite dimensional spaces in the world of White Noise Analysis.
References


Vita

Aurel Iulian Stan was born on April 26, 1969, in the town Cisnădie, county Sibiu, România. He attended his high-school in Găiști, Dâmbovița, and studied mathematics at Universitatea din București, România, where he received a Diploma in mathematics, in June 1993. From October 1993 to November 1994, he worked as a “Preparator” (Lecturer) at Universitatea din București, Facultatea de Matematică. Since January 1995 Aurel Stan has been a graduate student at Louisiana State University, Department of Mathematics. In December 1996 he earned a master of science degree in mathematics. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August, 1999.
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Major Field: Mathematics

Title of Dissertation: On Harmonic Analysis for White Noise Distribution Theory

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Major Professor and Chairman

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Date of Examination:

2 July 1999