The Applications of the Method of Quasi Reversibility to Some Ill-Posed Problems for the Heat Equation.

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THE APPLICATIONS OF THE METHOD OF QUASIREVERSIBILITY TO SOME ILL-POSED PROBLEMS FOR THE HEAT EQUATION

A Dissertation
Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy
in
The Department of Mathematics

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ABSTRACT

In this work, we study the Cauchy problem for the heat equation, as well as the inverse heat conduction problem, both of which are ill-posed problems in the sense of Hadamard.

The first chapter provides the background material about the previous investigations on the ill-posed Cauchy problem for the heat equation and the inverse heat conduction problem by other mathematicians. The method of Quasi-Reversibility is also introduced.

In the second chapter we apply the method of Quasi-Reversibility to the Cauchy problem for the heat equation and obtain a formal approximate solution. We prove that the convergence of the approximate solution to the presumed exact solution holds under certain conditions. Based on this result, we propose a modified numerical scheme in Chapter 3. It is implemented using the finite difference method, and the computational results for some example problems are satisfactory.

For the inverse heat conduction problem, we first investigate a simple case of positive heat flux at the boundary of the object over a finite time. Some estimates are obtained for that special case. In considering the general case of the inverse heat conduction problem, we smooth the given temperature at the given interior location, then transform the problem into its frequency domain by Fourier transform. An approximate solution is then constructed in the form of an inverse Fourier transform. We can prove that the data of the approximate solution at the given location could be very close to the given temperature, which validates the effectiveness of the application of the method of Quasi-Reversibility to the inverse heat conduction problem.
CHAPTER 1
INTRODUCTION

1.1 Overview

At the beginning of this century, Hadamard defined the proper posedness of problems for partial differential equations. A problem is said to be well posed, or properly posed, if a unique solution exists which depends continuously on the data [20]. Clearly, there are various notions of continuous dependence. Generally, any initial or boundary values, prescribed values of the operator, coefficients of the equation and the geometry of the domain of definition in problems are considered as "data" in the consideration of continuous dependence. An ill-posed, non-well posed, or improperly posed problem refers to a problem that is not well posed. Some examples of ill-posed problems are the Cauchy problem and the backward problem for the heat equation, the Cauchy problem for the Laplace equation, and the Dirichlet problem for the wave equation. In this work, we shall be interested in the Cauchy problem for the heat equation as well as the inverse heat conduction problem, which is also an ill-posed problem.

One of the obvious troubles from an ill-posed problem is that it is almost impossible to compute the numerical solutions directly for the problem by currently available numerical methods, e.g., finite difference method. Since data errors, e.g., truncation error and approximation error, inevitably exist in numerical computation, without the continuous dependence of the solution on the given data, reliable results are not guaranteed.

The method of Quasi-Reversibility was proposed by R. Lattès and J. L. Lions [18] in the 1960's. Its aim is the numerical computation of solutions for classes of
boundary value problems that are ill-posed in the sense of Hadamard [14]. The general idea of this method is to suitably modify the partial differential operators arising in the problem by adding small higher order derivative terms (formally tending to zero) or by degenerating at the boundary for the purpose of making a well-posed problem whose solution is computable by usual numerical methods. Payne explained this method very well in [21] by applying it to the backward problem for the heat equation. In that example, the operator is perturbed to make a well posed problem, and then the solution of the altered problem is used to construct an approximate solution of the original ill-posed problem for the heat equation. In our work, the method of Quasi-Reversibility will be used to investigate both the Cauchy problem for the heat equation and inverse heat conduction problem.

1.2 The Cauchy Problem for the Heat Equation

There are many works already on the Cauchy problem for the heat equation, which is a problem of determining a function \( u(x,t) \) that satisfies

\[
\begin{align*}
  u_t &= u_{xx}, & 0 < t < \gamma_1, & 0 < x < \gamma_2, \\
  u(0,t) &= f(t), & 0 < t < \gamma_1, \\
  u_x(0,t) &= g(t), & 0 < t < \gamma_1,
\end{align*}
\]  

(1.1)

where \( \gamma_1 \) and \( \gamma_2 \) are positive constants, and \( f \) and \( g \) are given functions.

A uniformly and absolutely convergent power series solution of (1.1) exists, if both \( f \) and \( g \) belong to a suitable Holmgren class, which means that, for the positive constants \( \gamma_1, \gamma_2 \), there exists a positive constant \( C \), such that \( f(t) \) and \( g(t) \) are infinitely differentiable functions on \( 0 < t < \gamma_1 \) that satisfy

\[
|f^{(j)}(t)| \leq C \frac{(2j)!}{\gamma_2}^j 
\]

and

\[
|g^{(j)}(t)| \leq C \frac{(2j)!}{\gamma_2}^j 
\]
for all \( t \) in \( 0 < t < \gamma_1 \), and all non-negative integers \( j = 0, 1, 2, \ldots \). Then the solution of (1.1) can be written as

\[
u(x, t) = \sum_{j=0}^{\infty} \left\{ f^{(j)}(t) \frac{x^{2j}}{(2j)!} + g^{(j)}(t) \frac{x^{2j+1}}{(2j + 1)!} \right\}
\]

(1.2)

The theorem is stated in [25] and [6].

The uniqueness of the solution of (1.1) can be obtained by the Fritz John global Holmgren uniqueness theorem in [22]. But this problem is not well posed, since small data does not imply a small solution. Examples to show the ill-posedness are given in both [9] and [6].

Manselli and Miller made a very detailed explanation for the ill-posedness of this problem in [19]. They considered a transient temperature conduction problem with slab symmetry, in which the temperature and heat flux histories \( f(t) \) and \( g(t) \) are given at the left hand side \( (x = 0) \) of the slab, while the temperature and heat flux histories at the right hand side \( (x = 1) \) are desired. Mathematically, if the left hand surface is insulated, the unknown temperature \( u(x, t) \) in the slab satisfies:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, & 0 \leq x \leq 1, & -\infty < t < \infty, \\
u(0, t) &= f(t), & -\infty < t < \infty, \\
\frac{\partial u}{\partial x}(0, t) &= g(t) = 0, & -\infty < t < \infty.
\end{align*}
\]

(1.3)

The sinusoidal in time solutions of the heat equation are given by:

\[
\exp \left( (\mu + i\sigma \mu)x \right) \exp (i\omega t) \quad \text{and} \quad \exp \left( -(\mu + i\sigma \mu)x \right) \exp (i\omega t),
\]

where \( \mu = \sqrt{|\omega|}/2, \sigma = \text{sign}\omega. \) With the second boundary condition, we have

\[
u(x, t) = \cosh \left( (\mu + i\sigma \mu)x \right) \exp(i\omega t)
\]

Let \( u(0, t) = f(t) = \exp (i\omega t) \), then

\[
u(1, t) = \cosh (\mu + i\sigma \mu) \exp (i\omega t).
\]
For very large $\omega$,

$$u(1, t) \approx \frac{\exp \mu}{2} \exp \imath \omega (t + \sigma \mu / \omega).$$

Thus, $u(1, t)$ will magnify an error in a high frequency component from $u(0, t) = f(t) = \exp (\imath \omega t)$ by the factor of $\exp \mu$. Therefore, (1.3) is an ill-posed problem.

In order to stabilize (1.3) Manselli and Miller discussed two possible assumptions for (1.3) in [19], by which the logarithmic data dependence and Hölder type data dependence are obtained, respectively.

Bell considered the noncharacteristic Cauchy problems for a class of equations with time dependence

$$Lu = F(x, t, u, u_x), \text{ in } \Omega
\quad u = g, \quad \text{ on } \Sigma
\quad \frac{\partial u}{\partial n} = h, \quad \text{ on } \Sigma$$

(1.4)

where $Lu = a(x, t)u_{xx} + b(t)u_t, a(x, t) \geq c > 0, F(x, t, u, u_x)$ satisfies

$$|F(x, t, u_1, v_1) - F(x, t, u_2, v_2)| \leq c(|u_1 - u_2| + |v_1 - v_2|),$$

$\Omega = \{(x, t) : s_1(t) \leq x \leq s_2(t), t \geq 0\}$, $\Sigma = \{(s_1(t), t), t_0 \leq t \leq t_1\}$, and $s_1(t)$ and $s_2(t)$ are piecewise $C^1$ curves with $s_1(t) < s_2(t)$ for all $t$. This is the Cauchy problem for a more general equation than the heat equation. Using the method of weighted energy, he proved the uniqueness of the solution of this problem and weak logarithmic continuous dependence on the data within a restricted stabilization class, i.e., the function class with an a priori bound. (see [3] and [4])

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Cannon and Douglas studied the following Cauchy problem for the heat equation in [5],

\[ \begin{align*}
  u_t &= u_{xx}, & 0 < t < T, & 0 < x < s(t), \\
  u(x,0) &= \varphi(x), & 0 \leq x \leq s(0), \\
  u(s(t),t) &= f(t), & 0 < t < T, & f(0) = \varphi(s(0)), \\
  u_x(s(t),t) &= g(t), & 0 < t < T, & g(0) = \varphi'(s(0)), \\
  |u_x(0,t)| &< M, & 0 < t \leq T,
\end{align*} \]

where \( s(t) \in C^1([0,T]) \), \( \varphi \in C^1([0,s(0)]) \), \( f \in C^1([0,T]) \), \( g \in C^1([0,T]) \), and \( M \) is a positive constant. They estimated an \textit{a priori} bound for the solution \( u(x,t) \) of (1.2) and applied the idea to the pure Cauchy problem, which is

\[ \begin{align*}
  u_t &= u_{xx}, & (x,t) \in D = \{0 < t \leq T, & 0 < x < s(t)\}, \\
  u(s(t),t) &= f(t), & 0 < t < T, \\
  u_x(s(t),t) &= g(t), & 0 < t < T, \\
  |u(x,t)| &< M, & (x,t) \in D,
\end{align*} \]

to establish Hölder continuous dependence on the given data for the solutions that satisfy an \textit{a priori} bound.

Payne [20] investigated the Cauchy problem for the heat equation (1.4) with \( b(t) \equiv -1 \). He used a modified weighted energy method to obtain Hölder continuous dependence of the solution, which is of significant importance for computational purposes.

Dorroh [9] studied the nonnegative solutions of the problem, and (1.1) computed a bound for the solution, which depends on \( f \), \( g \) and \( f' \). He concluded that the admissible Cauchy data is closed under uniform convergence of the data and one first-order derivative of the data.

Ginsberg [12] gives a method of constructing the approximate solution of the same problem in \( L^2 \) norm. This method truncates the formal solution of Fourier
series in time variable $t$. The Fourier coefficients are determined by the Lagrange interpolation of the boundary function from given discrete data at the boundary $x = 0$, which contains measurement error. This approach is proven to be valid by the imposition of an \textit{a priori} bound on the solution.

The basic idea in [7] by Cannon and Ewing is to write the solution of (1.1) as two parts, i.e.,

$$u = v + w,$$

where $v$ satisfies initial-boundary problem for the heat equation

$$v_t = v_{xx}, \quad 0 < t \leq T, \quad 0 < x,$$

$$v_x(0,t) = g(t), \quad 0 < t \leq T,$$

$$v(x,0) = 0, \quad 0 < x,$$

which has bounded solution

$$v(x,t) = \int_0^t g(r)(4\pi(t - r))^{1/2}\exp\left\{-\frac{x^2}{4(t - r)}\right\} \, dr.$$  

and $w$ satisfies

$$w_t = w_{xx}, \quad 0 < t \leq T, \quad 0 < x < 1,$$

$$w(0,t) = f(t) - v(0,t), \quad 0 < t \leq T,$$

$$w_x(0,t) = 0, \quad 0 < t \leq T,$$

$$| w(x,0) | < M, \quad 0 < x < 1$$

which has a formal solution in the form of (1.2). Thus using computational solutions of $v$ and $w$, the approximate solution of $u$ is obtained and also the error in the computational result can be estimated.

Based on all the previous works on the Cauchy problem for the heat equation, we wish to develop a numerical method to compute the numerical solutions on
certain areas directly from the prescribed temperature and the heat flux data at the boundary.

In this work, by the idea of the method of Quasi-Reversibility, the heat equation is perturbed to a wave equation to make a well posed problem, in which we reasonably modify the given conditions from that of the original problem in order that this altered problem is solvable. The perturbed problem is solved in a certain region, of which the solution continuously depends on the modified data or given data. Then the data of this solution will be used to construct another well posed problem for the heat equation, for example, an initial value problem or a boundary-initial value problem for the heat equation. We take the solution of the second well posed problem as an approximate to the solution of Cauchy problem for the heat equation. This is our main scheme to numerically solve the ill-posed Cauchy problem.

By "approximate solution" we mean that the theoretical result at the boundary is close to the given data of the original problem in certain sense. Under some assumptions, the validity of the approximation by the method of Quasi-Reversibility is proved in Chapter 2. And the application of the method of Quasi-Reversibility to the Cauchy problem is also numerically implemented in Chapter 3 by using finite difference method. The computational result of two examples are also given in Chapter 3, which is satisfactory.

1.3 The Inverse Heat Conduction Problem

The inverse heat conduction problem (IHCP) is also studied in this work. A general description of inverse problems in mathematics can be found in [13]. IHCP is a problem to determine the heat flux or temperature at the surface or boundary of a body by using the temperature measurement at one or several interior locations of the body. In our work only the one dimensional case of the IHCP is considered.
Carasso summarized IHCP in [8] as follows. Let \( u(x, t) \) to be the temperature function that satisfies the initial boundary value problem in the quarter \( x, t \)-plane,

\[
\begin{align*}
    u_t &= a^2 u_{xx}, & 0 < x < \infty, & 0 < t < \infty, \\
    u(x, 0) &= 0, & 0 < x < \infty, \\
    u(0, t) &= f(t), & 0 < t < \infty,
\end{align*}
\]

(1.5)

where \( a > 0 \) is a constant. Then, \( u(x, t) \) can be expressed in terms of \( f(t) \) explicitly,

\[
u(x, t) = \frac{x}{2a\sqrt{\pi}} \int_0^t \frac{f(t)}{(t - \tau)^{3/2}} \exp \left[ -\frac{x^2}{4a^2(t - \tau)} \right] d\tau.\]

Here, we are especially interested in the solution \( u(x, t) \) in the area of \( 0 < x \leq 1 \) and \( 0 < t < \infty \).

If \( h(t) \), for \( 0 \leq t < \infty \), is the given temperature history at the fixed location \( x = l > 0 \), then \( f(t) \) can be obtained from the inverse problem

\[
(Kf)(t) = h(t), \quad 0 \leq t < \infty,
\]

(1.6)

where

\[
(Kf(t)) = \frac{l}{2a\sqrt{\pi}} \int_0^t \frac{f(t)}{(t - \tau)^{3/2}} \exp \left[ -\frac{x^2}{4a^2(t - \tau)} \right] d\tau, \quad 0 \leq t < \infty.
\]

The author showed that \( K^{-1} \), the inverse of the operator \( K \), exists but is unbounded. Thus, a direct computation method will lead to enormous errors in \( f(t) \) given small errors in \( h(t) \). Based on the Tikhonov regularization procedure, assuming that \( f(t) \) has an \( \text{a priori} \) bound in \( L^2 \)-norm, Carasso constructed an approximate solution, or regularized solution for the exact temperature \( u(x, t) \) as well as the temperature gradient \( u_x(x, t) \) and estimated the errors between the regularized solution and the exact solution.

Later, Levine [17] obtained the Hölder type stability estimates for the unique bounded solution of (1.5), which states that, for \( 0 < x < 1 \), \( u(x, \cdot) \) satisfies

\[
\|u(x, \cdot)\|_2 \leq \|u(0, \cdot)\|_2^{1-x}\|u(1, \cdot)\|_2^x,
\]
where the norm is defined as

\[ \| f \|_2 = \left( \int_{-\infty}^{\infty} |f(t)|^2 \, dt \right)^{1/2}. \]

Elden [?] observed the ill-posedness of IHCP. For the practical purpose of (1.5), he considered a stabilized problem with an a priori bound instead of (1.5)

\begin{align*}
    u_t &= a^2 u_{xx}, & x \geq 0, & t \geq 0, \\
    u(x, 0) &= 0, & x \geq 0, \\
    \|u(0, \cdot)\|_2 &\leq M, \\
    \|u(1, \cdot) - h_m(\cdot)\|_2 &\leq \epsilon,
\end{align*}

(1.7)

where \( h_m \) denotes the measured data of \( h \), and the difference \( \|h - h_m\|_2 \) is assumed to be small. Then the author made a modification of the heat equation from the point of view of approximating a solution of (1.7), and proved that the solution of the modified problem can be logarithmically close to the solution of the stabilized problem of IHCP (1.7) with exact data \( h(t) \).

Beck also investigated the IHCP and designed a numerical method called function specification method in [2].

The method of Quasi-Reversibility is also applied to IHCP in this work. In order to consider the problem in the frequency domain by Fourier transform, the given temperature at the interior location of the subject is mollified. In the first step of the method, the heat equation is perturbed by adding a higher order partial derivative term multiplied by a small parameter. We can find the Fourier transform of the solution of the perturbed problem. The second step is to solve the heat equation in the quarter plane, whose solution is taken as the approximate of the original problem. Our theoretical result in Chapter 4 shows that the value of this constructed approximate solution can be close to the given data in the original problem.

9
2.1 The Method of Quasireversibility

Considering the Cauchy problem for the heat equation

\[ u_t(x, t) = u_{xx}(x, t), \quad 0 < t < 2\pi, \quad 0 < x < \infty, \]
\[ u(0, t) = f(t), \quad 0 < t < 2\pi, \]
\[ u_x(0, t) = g(t), \quad 0 < t < 2\pi, \]

which is obtained from (1.1) by specifying \( \gamma_1 \) and \( \gamma_2 \) as \( 2\pi \) and \( \infty \), respectively, for the consideration of convenience of Fourier expansion. We suppose that \( f, g \in L^2[0, 2\pi] \), so \( f \) and \( g \) can be expanded in their Fourier series,

\[
f(t) = \sum_{n=-\infty}^{+\infty} C_n e^{int}, \quad g(t) = \sum_{n=-\infty}^{+\infty} D_n e^{int}
\]

with

\[
\sum_{n=-\infty}^{\infty} C_n^2 < \infty; \quad \sum_{n=-\infty}^{\infty} D_n^2 < \infty.
\]

Then, the (2.1) has the following formal solution

\[
u(x, t) = \sum_{n=-\infty}^{+\infty} \left( C_n e^{int} \cosh(\sqrt{n} x) + \frac{D_n}{\sqrt{n}} e^{int} \sinh(\sqrt{n} x) \right).
\]

But in most cases of \( f \) and \( g \), just like the explanation in the previous chapter, the above series does not converge for any positive \( x \), since

\[
\sqrt{n} = \begin{cases} 
\sqrt{\frac{\pi}{4}}(1 + i) & \text{when } n \geq 0, \\
\sqrt{\frac{\pi}{4}}(1 - i) & \text{when } n < 0,
\end{cases}
\]
and as \( n \to \infty \),

\[
e^{int} \cosh(\sqrt{in} \, x) \sim O(e^{\sqrt{n}/2}), \quad \text{and} \quad e^{int} \sinh(\sqrt{in} \, x) \sim O(e^{\sqrt{n}/2}).
\]

Using the idea of the method of Quasi-Reversibility, we shall perturbe the heat equation to obtain a well-posed problem with the same boundary conditions.

Let \( u^\varepsilon(x, t) \) be the solution of the following perturbed problem

\[
\begin{align*}
u^\varepsilon_{xx}(x, t) &= u^\varepsilon_t(x, t) + \varepsilon^2 u^\varepsilon_{tt}(x, t), \quad 0 < t < 2\pi, \quad -\infty < x < \infty, \\
u^\varepsilon(0, t) &= f(t), \quad 0 < t < 2\pi, \\
u^\varepsilon_x(0, t) &= g(t), \quad 0 < t < 2\pi,
\end{align*}
\]

for small \( \varepsilon > 0 \). Then

\[
u^\varepsilon(x, t) = \sum_{n=-\infty}^{+\infty} \left[ C_n e^{int} \cosh(\sqrt{in} - n^2\varepsilon^2 \, x) + D_n e^{int} \sinh(\sqrt{in} - n^2\varepsilon^2 \, x) \right] \tag{2.6}
\]

where,

\[
\sqrt{in} - n^2\varepsilon^2 = \begin{cases} 
\alpha_n + i\beta_n & \text{when } n \geq 0, \\
\alpha_n - i\beta_n & \text{when } n < 0.
\end{cases} \tag{2.7}
\]

with

\[
\alpha_n = \frac{1}{\sqrt{2}} \sqrt{\sqrt{n^2 + n^4\varepsilon^4} - n^2\varepsilon^2}, \quad \beta_n = \frac{1}{\sqrt{2}} \sqrt{\sqrt{n^2 + n^4\varepsilon^4} + n^2\varepsilon^2}. \tag{2.8}
\]

The real part \( \alpha_n \) has limit of \( 1/(2\varepsilon) \), as \( n \to \infty \). Thus, \( u^\varepsilon(x, \cdot) \) is a valid \( L^2[0, 2\pi] \) solution for any positive \( x \) value. This is the first step of the method.

Then, we use \( u^\varepsilon(x, 0) \) to solve the heat equation on the upper half-plane:

\[
\begin{align*}
u^\varepsilon_t(x, t) &= u^\varepsilon_{xx}(x, t), \quad -\infty < x < \infty, \quad t > 0, \\
u^\varepsilon(x, 0) &= u^\varepsilon(x, 0), \quad -\infty < x < \infty,
\end{align*}
\]

\[11\]

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where

\[ u^\varepsilon(x,0) = \sum_{n=-\infty}^{+\infty} [C_n \cosh(\sqrt{n\,2\varepsilon^2} \, x) + \frac{D_n}{\sqrt{n\,2\varepsilon^2}} \sinh(\sqrt{n\,2\varepsilon^2} \, x)]. \]

Here, the formal solution of this initial value problem for the heat equation is

\[ u^\varepsilon(x,t) = \sum_{n=-\infty}^{+\infty} C_n e^{(in-n^2\varepsilon^2)t} \cosh(\sqrt{n\,2\varepsilon^2} \, x) + \frac{D_n}{\sqrt{n\,2\varepsilon^2}} e^{(in-n^2\varepsilon^2)t} \sinh(\sqrt{n\,2\varepsilon^2} \, x). \]  \hspace{1cm} (2.10)

Before we can take (2.10) as an approximate solution of (2.1), we need to check the convergence of the solution (2.10) and its derivative on the boundary \( x = 0 \), that is the convergence of the following series,

\[ u^\varepsilon(0,t) = \sum_{n=-\infty}^{+\infty} C_n e^{(in-n^2\varepsilon^2)t}, \quad \text{and} \quad u_x^\varepsilon(0,t) = \sum_{n=-\infty}^{+\infty} D_n e^{(in-n^2\varepsilon^2)t}. \]

\( u(0,t) \) and \( u_x(0,t) \) are still \( L^2 \) functions on \([0,2\pi]\), since

\[ ||u^\varepsilon(0,t)||_2^2 = \int_0^{2\pi} \left| \sum_{n=-\infty}^{+\infty} C_n e^{(in-n^2\varepsilon^2)t} \right|^2 \, dt \]

\[ = \int_0^{2\pi} \sum_{n,m=-\infty}^{+\infty} C_n \bar{C}_m e^{(n-m)t} e^{-(n^2+m^2)\varepsilon^2 t} \, dt \]

\[ \leq \sum_{n,m=-\infty}^{+\infty} \int_0^{2\pi} |C_n| |C_m| e^{-(n^2+m^2)\varepsilon^2 t} \, dt \]

\[ = \sum_{n,m=-\infty}^{+\infty} |C_n| |C_m| \frac{1-e^{-2\pi(n^2+m^2)\varepsilon^2}}{(n^2+m^2)\varepsilon^2} \]

\[ \leq \sum_{n,m=-\infty}^{+\infty} \frac{|C_n| |C_m|}{2\pi |n| |m|} \]

\[ \leq \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} |C_n|^2 \sum_{n=-\infty}^{\infty} \frac{1}{n^4} \]

which is finite for any fixed small \( \varepsilon \). And similar arguments are applicable to \( u_x^\varepsilon(0,t) \).

But from the above argument, the uniform bound of \( u^\varepsilon(0,t) \) in terms of \( \varepsilon \) is not obtained.
Next, we will show that $u^e(0,t)$, and $u^g(0,t)$ are close to the given data $f(t)$, and $g(t)$, respectively, which is the main concern on the validity of the method of Quasi-Reversibility.

2.2 The Theoretical Estimates

For further discussion for the problem (2.1), we assume that the Fourier coefficients of $f$ satisfy

$$
\sum_{n=\infty}^{\infty} |C_n| < \infty
$$

(2.11)

One must note that this is not enough to guarantee that (2.1) has a solution. As a matter of fact, $f$ satisfies (2.11), if $f(t)$ is absolutely continuous and has derivative in $L^2$. Suppose that the derivative function $f'(t)$ has Fourier expansion $f'(t) = \sum_{n=\infty}^{\infty} C_n e^{int}$, then $f(t) = \sum_{n=-\infty}^{\infty} \frac{C_n'}{in} e^{int}$. Comparing this Fourier expansion of $f(t)$ with its expansion in (2.2), we may obtain that, for $n = 0, \pm 1, \pm 2, \cdots$,

$$
C_n = \frac{C_n'}{in},
$$

then,

$$
\sum_{n=-\infty}^{+\infty} |C_n| = \sum_{n=-\infty}^{+\infty} \frac{|C_n'|}{in} \leq \sqrt{\sum_{n=-\infty}^{+\infty} (C_n')^2} \sqrt{\sum_{n=-\infty}^{+\infty} \frac{1}{n^2}}.
$$

Proposition 2.1. Under the assumption (2.11), for fixed $x > 0$, and $\varepsilon > 0$, the series in (2.10) is uniformly convergent for $t$ in $[0, 2\pi]$.

Proof: It is enough to obtain the result, if the remainder of the series of (2.10) tends to zero, for fixed $x > 0$, and $\varepsilon > 0$, uniformly in $t$.

For arbitrary $\eta > 0$, there exists an $N > 0$ such that $\sum_{|n| > N} |C_n| < \eta$; also by (2.3), we have
\[ \sum_{|n|>N} \left| \frac{D_n}{\sqrt{|in - \epsilon^2 n^2 x|}} \right| = \sum_{|n|>N} \frac{|D_n|}{(n^2 + \epsilon^2 n^4)^{1/4}} \]
\[ \leq \sum_{|n|>N} \frac{|D_n|}{\epsilon n} \]
\[ \leq \frac{1}{\epsilon} \sum_{|n|>N} |D_n|^2 \sum_{|n|>N} \frac{1}{n^2} \]
\[ < \eta, \]

On the other hand, for any integer \( n \), from (2.7), \( \cosh(\sqrt{|in - \epsilon^2 n^2 x|}) \) and \( \sinh(\sqrt{|in - \epsilon^2 n^2 x|}) \) are bounded by \( e^{\sqrt{2\epsilon}/2} \), thus,
\[ \sum_{|n|>N} \left[ |C_n e^{(\sqrt{|in - \epsilon^2|} - n^2 \epsilon^2) t} \cosh(\sqrt{|in - \epsilon^2 n^2|} x)| \right. \]
\[ + \left. \left| \frac{D_n}{\sqrt{|in - \epsilon^2 n^2|} e^{(\sqrt{|in - \epsilon^2|} - n^2 \epsilon^2) t} \sinh(\sqrt{|in - \epsilon^2 n^2|} x)} \right| \right] \leq 2 \eta e^{\sqrt{2\epsilon}/2} \]

which gives the proof of the claim. \( \Box \)

To investigate the distance between \( u^\epsilon(0,t) \) and \( f(t) \), one looks at the difference \( u^\epsilon(0,t) - f(t) \); here is an estimate of it.

**Proposition 2.2.** Under the assumption (2.11), the following holds,
\[ \lim_{\epsilon \to 0} |u^\epsilon(0,t) - f(t)| = 0 \]

**Proof:** From (2.10),
\[ u^\epsilon(0,t) = \sum_{n=-\infty}^{\infty} C_n e^{int - \epsilon^2 n^2 t}, \]
then,
\[ |u^\epsilon(0,t) - f(t)| \leq \sum_{n=-\infty}^{\infty} |C_n| |1 - e^{-\epsilon^2 n^2 t}|. \]
For arbitrary $\eta > 0$, there is an $N > 0$ such that $\sum_{|n| > N} |C_n| < \eta$, thus

$$|u^\varepsilon(0,t) - f(t)| = \sum_{|n| \leq N} |C_n| \left|1 - e^{-\varepsilon^2 n^2 t}\right| + \sum_{|n| > N} |C_n| \left|1 - e^{-\varepsilon^2 n^2 t}\right|$$

$$\leq \sum_{|n| \leq N} |C_n| \left|1 - e^{-\varepsilon^2 n^2 t}\right| + \sum_{|n| \leq N} |C_n|$$

$$\leq \left(\sum_{n=-\infty}^{\infty} |C_n|\right) \eta + \eta$$

since, for each $n$ between $-N$ and $N$, $1 - e^{-\varepsilon^2 n^2 t}$ converges to 0, uniformly for $t \in [0, 2\pi]$, as $\varepsilon$ tends to 0. Thus, we prove this claim. \(\square\)

By a similar argument, we also have

**Proposition 2.3.** If

$$\sum_{n=-\infty}^{+\infty} |D_n| < \infty$$

then,

$$\lim_{\varepsilon \to 0} |u^\varepsilon_x(0,t) - g(t)| = 0$$

For simplicity, let $g(t) = 0$, i.e., $D_n = 0$ for all integer $n$ in the following theorem.

**Theorem 2.4.** Assume that (2.4) is the solution of (2.1) for $x > 0$, i.e., the series of the right hand side of (2.4) converges, then both the series in (2.10) and (2.6) converge for $x > 0$ and $\varepsilon > 0$; moreover,

$$\lim_{\varepsilon \to 0} u^\varepsilon(x,t) = u(x,t), \quad (2.13)$$

$$\lim_{\varepsilon \to 0} v^\varepsilon(x,t) = u(x,t). \quad (2.14)$$

holds for $x > 0$.  

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Proof: Rewrite the series of $u^f(x,t)$ in (2.10) and $v^f(x,t)$ in (2.6) as,

$$u^f(x,t) = \sum_{n=-\infty}^{+\infty} [C_n e^{int-\pi^2t} \cosh(\sqrt{in} x) \frac{\cosh(\sqrt{in} - n^2\varepsilon^2 x)}{\cosh(\sqrt{in} x)}]$$

$$v^f(x,t) = \sum_{n=-\infty}^{+\infty} [C_n e^{int} \cosh(\sqrt{in} x) \frac{\cosh(\sqrt{in} - n^2\varepsilon^2 x)}{\cosh(\sqrt{in} x)}]$$

To obtain the convergence of $u^f(x,t)$ and $v^f(x,t)$, we only need to show that for each integer $n$, $\cosh(\sqrt{in} - n^2\varepsilon^2 x)/\cosh(\sqrt{in} x)$ is uniformly bounded for positive $x$ and $\varepsilon$.

Actually, for arbitrary $n$, by (2.8)

$$\frac{|\cosh(\sqrt{in} - n^2\varepsilon^2 x)|}{|\cosh(\sqrt{in} x)|} = \frac{\sqrt{e^{2\alpha_n x} + e^{-2\alpha_n x} + 2 \cos 2\beta_n x}}{\sqrt{e^{2\alpha_n x} + e^{-2\alpha_n x} + 2 \cos 2\alpha_n x}}$$

$$= \sqrt{\frac{e^{-(\sqrt{2n} - 2\alpha_n)x} + e^{-(\sqrt{2n} + 2\alpha_n)x} + 2e^{-\sqrt{2n}\cos 2\beta_n x}}{1 + e^{-2\sqrt{2n}x} + 2e^{-2\sqrt{2n}x} \cos 2\sqrt{2n}x}}$$

$$\leq \frac{2}{\sqrt{1 + 2e^{-2\sqrt{2n}x} \cos 2\sqrt{2n}x}}$$

where $\alpha_n$ and $\beta_n$ from (2.8). Note that $\sqrt{2n} - 2\alpha_n > 0$ in the previous estimate and that $2e^{-\sqrt{2n}x} \cos 2\sqrt{2n}x$ has a minimum value of $-\sqrt{2}e^{-\frac{nx}{2}}$, which is a constant less than 1. Thus, we have that $\cosh(\sqrt{in} - n^2\varepsilon^2 x)/\cosh(\sqrt{in} x)$ is uniformly bounded for $x, \varepsilon > 0$.

(2.13) holds since for arbitrary small $\eta > 0$, there exists an $N > 0$, which is independent of $\varepsilon$ and $x$, such that,

$$|\sum_{|n|>N} C_n e^{int} \cosh(\sqrt{in} x)| < \eta,$$
and
\[ \left| \sum_{|n| > N} C_n e^{(\text{in-}n^2\varepsilon^2)t} \cosh(\sqrt{\text{in-}n^2\varepsilon^2} \ x) \right| < \eta. \]

On the other hand,
\[ \left| \sum_{-N}^{N} C_n e^{\text{int}} \cosh(\sqrt{\text{in}} \ x) - \sum_{-N}^{N} C_n e^{(\text{in-}n^2\varepsilon^2)t} \cosh(\sqrt{\text{in-}n^2\varepsilon^2} \ x) \right| < \eta \]

for sufficiently small \( \varepsilon \), since, \( C_n e^{(\text{in-}n^2\varepsilon^2)t} \cosh(\sqrt{\text{in-}n^2\varepsilon^2} \ x) \) has limit of \( C_n e^{\text{int}} \cosh(\sqrt{\text{in}} \ x) \)
for each \( n \) between \(-N\) and \( N \), as \( \varepsilon \to 0 \). Thus (2.13) is proved.

(2.14) can be proved by similar arguments.

**Remark 2.5.** When \( g(t) \neq 0 \), in order to have the same result in the above theorem, we need to assume that, in (2.4), both \( \sum_{n=-\infty}^{+\infty} C_n e^{\text{int}} \cosh(\sqrt{\text{in}} \ x) \) and \( \sum_{n=-\infty}^{+\infty} \frac{D_n}{\sqrt{\text{in}}} e^{\text{int}} \sinh(\sqrt{\text{in}} \ x) \) should converge.
CHAPTER 3
THE NUMERICAL IMPLEMENTATION

3.1 The Numerical Scheme

The general idea of the numerical implementation will follow the theoretical method in the previous section with some modification, which is to solve (2.5) numerically by a finite difference method, then using the data of the solution of (2.5) at the line of $t = t_0$ for certain $t_0 > 0$ and $x = \pm x_0$ for certain $x_0 > 0$, one can solve the initial-boundary value problem of the heat equation, which is a well-posed problem.

For the convenience of the implementation, one needs the following assumptions:

- In problem (2.5), letting $g(t) = 0$ makes the solution of (2.5) even in $x$.
- Instead of the time interval $[0, 2\pi]$ in (2.1) or (2.5), the time interval of $[0, 1]$ will ease the numerical computation.

Now considering the perturbed problem (2.5) for $x > 0$, we let $h$ and $k$ be the mesh steps for $x$ and $t$ variables, respectively, $(x_i, t_j)$ be the mesh points; i.e., $x_i = (i - 1)h$, $t_j = (j - 1)k$, since the time interval being considered is $[0, 1]$. Then, the number of mesh points in time is $m + 1$, where $m$ satisfies $k = 1/m$, and the number of mesh points in space is $2n + 1$, where $n$ satisfies $h = \bar{x}_0/n$, and $\bar{x}_0$ is the largest $x$-value for all the mesh points.

Let $v_{i,j} = v^\varepsilon(x_i, t_j)$ to represent the value of the numerical solution of (2.5) at the mesh point $(x_i, t_j)$. Then the perturbed equation in (2.5) is discretized as

$$\frac{1}{h^2} \delta_x^2 v_{i,j} = \frac{\varepsilon^2}{k^2} \left( \frac{1}{4} \delta_t^2 v_{i-1,j} + \frac{1}{2} \delta_t^2 v_{i,j} + \frac{1}{4} \delta_t^2 v_{i+1,j} \right) + \frac{1}{2k} \delta_t v_{i,j}$$

(3.1)

for $i = 2, 3, ..., n + 1$, and $j = i, i + 1, ..., m + 2 - i$, where $\delta_x$ and $\delta_t$ denote the central difference formula in half mesh step for the partial derivative $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$. 

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\[
\frac{\partial}{\partial t}, \text{ respectively; i.e., } \delta_x v_{t,j} = v_{i+\frac{1}{2},j} - v_{i-\frac{1}{2},j} \text{ and } \delta_t v_{t,j} = v_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}}. \text{ Thus, } \\
\delta_x^2 v_{t,j} = v_{t+1,j} - 2v_{t,j} - v_{t-1,j} \text{ and } \delta_t^2 v_{t,j} = v_{t,j+1} - 2v_{t,j} + v_{t,j-1}. \text{ While, } \delta_t \text{ denotes the central difference formula in } t \text{ variable; i.e., } \delta_t v_{t,j} = v_{t,j+1} - v_{t,j-1}. \text{ Actually, (3.1) is not used for the extreme values of } j.
\]

In (3.1), we use \( \frac{1}{h} \delta_x^2 v_{i,j} \) to approximate the second derivative term in \( x \), \( v_{xx}(x,t) \) at \((x_i, t_j)\), the weighted average of the approximations of the second derivative in \( t \) at the mesh points \((x_{i-1}, t_j)\), \((x_i, t_j)\), \((x_{i+1}, t_j)\) to approximate the value of the second derivative of \( v'(x,t) \) at the point \((x_i, t_j)\), and the central difference approximation to the first derivative in \( t \) of \( v'(x,t) \) at the point \((x_i, t_j)\).

The discretization (3.1) yields the following implicit scheme

\[
- \frac{\varepsilon^2 h^2}{4k^2} v_{i+1,j-1} + (1 + \frac{\varepsilon^2 h^2}{2k^2}) v_{i+1,j} - \frac{\varepsilon^2 h^2}{4k^2} v_{i+1,j+1} = \\
\left( \frac{\varepsilon^2 h^2}{2k^2} - \frac{h^2}{2k} \right) v_{i,j-1} + (2 - \frac{\varepsilon^2 h^2}{k^2}) v_{i,j} + \left( \frac{\varepsilon^2 h^2}{2k^2} + \frac{h^2}{2k} \right) v_{i,j+1},
\]

\[
+ \frac{\varepsilon^2 h^2}{4k^2} v_{i-1,j-1} + (-1 - \frac{\varepsilon^2 h^2}{2k^2}) v_{i-1,j} + \frac{\varepsilon^2 h^2}{4k^2} v_{i-1,j+1} \tag{3.2}
\]

The mesh points used in (3.2) are shown in Figure 3.1, where the value of the "x" points are used to compute the value of the "o" points in this implicit scheme.

From Figure 3.1, we can see that the column of the data of \( v_{i+1,j} \) is determined by the previous two columns, i.e., \( v_{i,j} \) and \( v_{i-1,j} \), which means that we need the
first two columns of data; i.e., the data at $x = 0$ ($i = 1$) and $x = h$ ($i = 2$) to do the computation for the solution.

By the given conditions $v^x(0, t) = f(t)$ and $v^x(0, t) = g(t) = 0$, we have $v^1$ from $f(t)$. The data for $v^x_{2,j}$ is approximated by the Taylor expansion of $v^x(x, t)$ at $x = h$

$$v^x(h, t) = v^x(0, t) + h \cdot v^x_2(0, t) + \frac{h^2}{2} \cdot v^x_{x2}(0, t) + O(h^3)$$

or

$$v^x(h, t) = v^x(0, t) + \frac{h^2}{2} \cdot (2 \cdot 2v^x(0, t) + v^x(0, t)) + O(h^3)$$

(3.3)

since $v^x(0, t) = g(t) = 0$ and by the equation (2.5). The discretization of (3.3) is

$$v^x_{2,j} = v^1_{2,j} + \frac{\varepsilon^2 h^2}{2k^2} \cdot (2v^x_{2,j+1} - 2v^x_{2,j} + v^x_{2,j-1}) + \frac{h^2}{4k} \cdot (v^1_{2,j+1} - v^1_{2,j-1})$$

(3.4)

for $j = 2, 3, ..., m$. Here, we may not get the data for $v^2_{2,1}$ and $v^2_{2,m+1}$ by the above approximation. As a matter of fact, we are losing two ending points at $i = 2$, see Figure 3.2.

Now, let us look into the system of equations to solve $v^x_{3,i}$. When (3.2) is used to solve for $v^x_{3,i}$, the $i$ has its value of 2, and $j$ goes from 4 to $m - 2$; i.e., we have $m - 5$ equations in the system. But the unknown variables involved in the system are $v^x_{3,3}, v^x_{3,4}, ..., v^x_{3,m-1}$, which is $m - 3$ in total, shown in Figure 3.2.

To make the system solvable, we will use an explicit scheme at $j = 3$ and $j = m - 1$, respectively, to obtain another two equationa. They are

$$v^x_{3,3} = \left(\frac{\varepsilon^2 h^2}{k^2} - \frac{h^2}{2k^2}\right)v^x_{3,2} + (2 - \frac{2\varepsilon^2 h^2}{k^2})v^x_{3,3} + \left(\frac{\varepsilon^2 h^2}{k^2} + \frac{h^2}{2k}\right)v^x_{3,4} - v^1_{3,3}$$

and

$$v^x_{3,m-1} = \left(\frac{\varepsilon^2 h^2}{k^2} - \frac{h^2}{2k}\right)v^x_{3,m-2} + (2 - \frac{2\varepsilon^2 h^2}{k^2})v^x_{3,m-1} + \left(\frac{\varepsilon^2 h^2}{k^2} + \frac{h^2}{2k}\right)v^x_{3,m} - v^1_{3,m-1}$$
Now the number of the unknown variables matches the number of the equations in the system. Generally, two mesh points are lost when we compute the data for a new column. Thus, the computing area looks like a picture shown in the Figure 3.3. Thus, the numerical computation reflects the well-known "domain of influence" of initial data for the wave equation; see [15].

But remember that only the data on the initial line of \( t = 0 \) is of interest.

Let \( V^\varepsilon_i = \{v_{i,i}^\varepsilon, v_{i,i+1}^\varepsilon, \ldots, v_{i,m-i+2}^\varepsilon\}^T \), \( V_{i-1}^\varepsilon = \{v_{i-1,i}^\varepsilon, v_{i-1,i+1}^\varepsilon, \ldots, v_{i-1,m-i+2}^\varepsilon\}^T \), and \( V_{i+1}^\varepsilon = \{v_{i+1,i+1}^\varepsilon, v_{i+1,i+2}^\varepsilon, \ldots, v_{i+1,m-i+1}^\varepsilon\}^T \), then (3.2) can be rewritten in matrix form

\[
A \cdot V_{i+1}^\varepsilon = B \cdot V_i^\varepsilon + C \cdot V_{i-1}^\varepsilon \tag{3.5}
\]
where

\[
A = \begin{pmatrix}
1 - \frac{r}{4} & \frac{5r}{4} & -r & \frac{r}{4} & \cdots & 0 & 0 \\
-\frac{r}{4} & 1 + \frac{r}{2} & -\frac{r}{4} & 0 & \cdots & 0 & 0 \\
0 & -\frac{r}{4} & 1 + \frac{r}{2} & -\frac{r}{4} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -\frac{r}{4} & 1 + \frac{r}{2} & -\frac{r}{4} & 0 \\
0 & 0 & \cdots & 0 & -\frac{r}{4} & 1 + \frac{r}{2} & -\frac{r}{4} \\
0 & 0 & \cdots & \frac{r}{4} & -r & \frac{5r}{4} & 1 - \frac{r}{2} \\
\end{pmatrix}_{m-2i+1,m-2i+1}
\]  

\[
B = \begin{pmatrix}
\frac{r-s}{2} & 2 - r & \frac{r+s}{2} & 0 & \cdots & 0 & 0 \\
0 & \frac{r-s}{2} & 2 - r & \frac{r+s}{2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{r-s}{2} & 2 - r & \frac{r+s}{2} & 0 \\
0 & 0 & \cdots & 0 & \frac{r-s}{2} & 2 - r & \frac{r+s}{2} \\
\end{pmatrix}_{m-2i+1,m-2i+3}
\]
with \( r = \frac{2h^2}{k^3} \), and \( s = \frac{h^2}{k} \).

Conceptually, (3.5) can be solved in the form

\[
V^{\varepsilon}_{i+1} = A^{-1}B \cdot V^{\varepsilon}_i + A^{-1}C \cdot V^{\varepsilon}_{i-1}
\]
or,

\[
V^{\varepsilon}_{i+1} = \begin{pmatrix} A^{-1} \cdot B & A^{-1} \cdot C \end{pmatrix} \begin{pmatrix} V^{\varepsilon}_i \\ V^{\varepsilon}_{i-1} \end{pmatrix}
\]

It is important here to consider the stability of the above linear equation, which is the stability of the matrix on the right side in the equation,

\[
L = \begin{pmatrix} A^{-1} \cdot B & A^{-1} \cdot C \end{pmatrix}
\]

So far, we are not able to use any theorem to prove that the matrix (3.9) has any kind of norm that is less than 1. But by numerical computing, if we fill several extra zero rows at the bottom of the matrix (3.9) to make it square, i.e.,

\[
L' = \begin{pmatrix} A^{-1} \cdot B & A^{-1} \cdot C \\ 0 & 0 \end{pmatrix}
\]

then the maximum absolute value of all the eigenvalues of \( L' \) will be less than 1. For example, if the size of (3.10) is about 400 × 400, the eigenvalue of (3.10) is
about 0.4—0.5, although we find that all the norms of both $A^{-1} \cdot B$ and $A^{-1} \cdot C$
are larger than 1.

In this way, we compute the data on the rightside of the computing area shown
in Figure 3.3. Using the even property of the solution we mentioned previously,
the computational results on the whole computing area are obtained. We will use
these data to form a boundary-initial value problem for the heat equation, which
is the second step of the computation.

Suppose that the right side boundary of the computing area in Figure 3.3 is
at $x = x_0$. In the second part of the computation, the numerical solution of the
following problem is considered,

$$
u^e(x, t) = \nu^e_T(x, t), \quad t_0 < t < T_0, \quad -x_0 < x < x_0,$$

$$
u^e(x, t_0) = \nu^e(x_0, t_0), \quad -x_0 < x < x_0,$$

$$
u^e(\pm x_0, t) = \nu^e(\pm x_0, t), \quad t_0 < t < T_0
$$

(3.11)

for some $t_0 \leq 0$ and $0 \leq x_0 \leq x_0$, where $\nu^e(x, t)$ is given as discretized data, which
is computed in the previous step.

Let $u^e_{i,j}$ denote the discretized data of $\nu^e(x_i, t_j)$, the heat equation can be dis­
cretized as

$$rac{1}{k} \delta_i u^e_{i,j+\frac{1}{2}} = \frac{1}{2h^2} (\delta_x^2 u^e_{i,j} + \delta_x^2 u^e_{i,j+1})$$

which yields the following implicit scheme, called Crank-Nicholson scheme,

$$-rac{k}{2h^2} u^e_{i-1,j+1} + (1 + \frac{k}{h^2}) u^e_{i,j+1} - \frac{k}{2h^2} u^e_{i+1,j+1} = \frac{k}{2h^2} u^e_{i-1,j} + (1 - \frac{k}{h^2}) u^e_{i,j} + \frac{k}{2h^2} u^e_{i+1,j}
$$

(3.12)

which is proved to be a unconditionally stable scheme. To show this, we can use
von Neumann's method; see [23].
Substitution of $u_{i,j} = e^{\sqrt{-1} \beta i h} \xi j$ into (3.12), we obtain that

$$-\frac{k}{2h^2} e^{\sqrt{-1} \beta (i-1)h} \xi j^{i+1} + (1 + \frac{k}{h^2}) e^{\sqrt{-1} \beta i h} \xi j^{i+1} - \frac{k}{2h^2} e^{\sqrt{-1} \beta (i+1)h} \xi j^{i+1}$$

$$= \frac{k}{2h^2} e^{\sqrt{-1} \beta (i-1)h} \xi j + (1 - \frac{k}{h^2}) e^{\sqrt{-1} \beta i h} \xi j + \frac{k}{2h^2} e^{\sqrt{-1} \beta (i+1)h} \xi j$$

Division by $e^{\sqrt{-1} \beta i h} \xi j$, then by the coefficient of $\xi$, leads to

$$\xi = \frac{(1 - \frac{k}{h^2} (1 - \cos \beta h))}{(1 + \frac{k}{h^2} (1 - \cos \beta h))}$$

$$= \frac{(1 - \frac{k}{h^2} \sin^2 \frac{\theta h}{2})}{(1 + \frac{2k}{h^2} \sin^2 \frac{\theta h}{2})}$$

It is sufficient to see that $| \xi | < 1$ is guaranteed, which means that the scheme (3.12) is always stable for any $h$ and $k$.

### 3.2 The Examples and Numerical Results

Using the numerical scheme we develop in the previous section, we compute the solutions of the following examples.

**Example 1** Let $f(t) = 1 - e^{-t}$, and $g(t) = 0$.

The exact solution $u(x, t)$ of the heat equation with $u(0, t) = f(t)$, and $u_x(0, t) = g(t)$ is $u(x, t) = 1 - e^{-t} \cos x$. In the computation, we will choose $k = 0.01$, $h = 0.01$. Let $x_0 = 1$ and $\varepsilon = 0.01$. Just as the previous explanation, we solve the damped wave equation (2.5). The area that we could obtain numerical data for (2.5) is the rombus among $(x, t) = (\pm 0.5, 0)$ and $(\pm 0.5, 1)$. As the second step to compute the solution of the initial-boundary value problem for the heat equation, we start at the line of $t = 0.3$ with boundary of $x$ at $x = \pm 0.3$. In this numerical experiment, relative random error of $[-0.5 \times 10^{-4}, 0.5 \times 10^{-4}]$ is added to $f(t)$, both of the following table and Figure 3.4 show the relative error for $0.30 \leq t \leq 0.70$. 

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TABLE 3.1: Maximum relative error for \(.30 \leq t \leq .70\) for Example 1.

<table>
<thead>
<tr>
<th>x</th>
<th>Max Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>.00</td>
<td>3.1047e-05</td>
</tr>
<tr>
<td>.05</td>
<td>3.0965e-05</td>
</tr>
<tr>
<td>.10</td>
<td>3.0143e-05</td>
</tr>
<tr>
<td>.15</td>
<td>4.3827e-05</td>
</tr>
<tr>
<td>.20</td>
<td>6.7706e-05</td>
</tr>
<tr>
<td>.25</td>
<td>1.2398e-04</td>
</tr>
<tr>
<td>.30</td>
<td>2.2326e-04</td>
</tr>
</tbody>
</table>

FIGURE 3.4. The relative error of the computed data for Example 1

An effort was also made to implement the theoretical method more directly by using finite Fourier transforms. However, this did not yield good results.

**Example 2** Let \(f(t) = 1 - \cos t\), and \(g(t) = 0\).

The exact solution \(u(x, t)\) of the heat equation with \(u(0, t) = f(t)\), and \(u_x(0, t) = g(t)\) is \(u(x, t) = 1 - \frac{1}{2}(e^{x/\sqrt{2}} \cos (\frac{x}{\sqrt{2}} + t) + e^{-x/\sqrt{2}} \cos (\frac{x}{\sqrt{2}} - t))\). By the same parameter

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value for $h$, $k$, $x_0$, and $\varepsilon$, and same range of perturbation on the given exact data of $f(t)$, similar good results are also obtained.

TABLE 3.2: Maximum relative error for $0.30 \leq t \leq 0.70$ for Example 2.

<table>
<thead>
<tr>
<th>x</th>
<th>Max Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>.00</td>
<td>7.9138e-05</td>
</tr>
<tr>
<td>.05</td>
<td>8.0244e-05</td>
</tr>
<tr>
<td>.10</td>
<td>8.1923e-05</td>
</tr>
<tr>
<td>.15</td>
<td>9.2092e-05</td>
</tr>
<tr>
<td>.20</td>
<td>1.1708e-04</td>
</tr>
<tr>
<td>.25</td>
<td>1.8058e-04</td>
</tr>
<tr>
<td>.30</td>
<td>2.3025e-04</td>
</tr>
</tbody>
</table>

FIGURE 3.5. The relative error of the computed data for Example 2

Cannon and Ewing presented a direct numerical method for the same problem in [6].
CHAPTER 4
THE INVERSE HEAT CONDUCTION PROBLEM

If the heat flux or temperature histories at the surface of a solid are known as function of time, then the temperature distribution can be found. This is called a direct problem. In many engineering situations, the surface heat flux and temperature histories of a solid must be determined from transient temperature measurements at one or more interior location(s); this is an inverse problem, or inverse heat conduct problem (IHCP). In one dimension case, IHCP is the problem to find the temperature function $u(x,t)$ which satisfies

$$u_t = u_{xx}, \quad 0 \leq x < \infty, \quad 0 \leq t < \infty,$$

(4.1)

$$u(x,0) = 0, \quad 0 \leq x < \infty,$$

(4.2)

$$u(1,t) = h(t), \quad 0 \leq t < \infty,$$

(4.3)

where $u(x,t)$ stays bounded as $x \to \infty$. We are especially interested in the temperature or heat flux at $x = 0$; i.e., $u(0,t)$ or $-u_x(0,t)$. The ill-posedness of this problem is discussed in Chapter 1. In the following section, we will observe the relation among $u(0,t)$, $-u_x(0,t)$ and $h(t)$ by considering a simple case of this problem.

4.1 IHCP with Positive Heat Flux at the Boundary

Suppose that the temperature increase of the object with zero initial temperature is caused by an unknown positive heat flux at the boundary over a finite time interval $t \in [0,1]$. 

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Let \( u(x,t) \) be the temperature of the object, \( q(t) \) the unknown positive heat flux at the boundary \( x = 0 \), and \( h(t) \) the temperature at the interior location, \( x = 1 \). The problem of finding \( u(x,t) \), especially \( -u_x(0,t) = q(t) > 0 \), can also be modeled by (4.1)-(4.3).

Under this assumption, \( h(t) \) and \( q(t) \) are related by the following formula

\[
h(t) = \int_{0}^{\infty} q(\tau) \frac{1}{\sqrt{\pi(t-\tau)}} e^{-\frac{4(1-\tau)}{t-\tau}} d\tau = \int_{0}^{1} q(\tau) \frac{1}{\sqrt{\pi(t-\tau)}} e^{-\frac{4(1-\tau)}{t-\tau}} d\tau \tag{4.4}
\]

Based on this formula, we have the following estimates.

**Proposition 4.1.**

\[
\| q \|_1 \leq C \| h \|_{\infty}
\]

where \( C \) a constant, for \( q(t) \geq 0 \).

**PROOF:** It is obvious that

\[
\| h \|_{\infty} \geq h(r_0)
\]

with arbitrary constant \( 0 < r_0 - 1 < \frac{1}{2} \), thus by (4.4),

\[
\| h \|_{\infty} \geq \int_{0}^{1} q(\tau) \frac{1}{\sqrt{\pi(r_0 - \tau)}} e^{-\frac{4(1-\tau)}{r_0-\tau}} d\tau.
\]

Since the function \( F(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \) for \( x \in (0, \infty) \) satisfies

\[
0 < \frac{1}{\sqrt{\pi x}} e^{-\frac{1}{4x}} \leq \sqrt{\frac{2}{\pi e}},
\]

and \( 0 < r_0 - 1 < r_0 - \tau < \frac{3}{2} \), thus we have

\[
\min \left\{ \sqrt{\frac{2}{3\pi}} e^{-\frac{1}{4}}, \frac{1}{\sqrt{\pi(r_0 - 1)}} e^{-\frac{4(1-\tau)}{r_0-\tau}} \right\} \leq \frac{1}{\sqrt{\pi(r_0 - \tau)}} e^{-\frac{4(1-\tau)}{r_0-\tau}} \leq \sqrt{\frac{2}{\pi e}},
\]

also \( q(\tau) \geq 0 \) on \( \tau \in [0,1] \), then

\[
\| h \|_{\infty} \geq \min \left\{ \sqrt{\frac{2}{3\pi}} e^{-\frac{1}{4}}, \frac{1}{\sqrt{\pi(r_0 - 1)}} e^{-\frac{4(1-\tau)}{r_0-\tau}} \right\} \int_{0}^{1} q(\tau) d\tau,
\]

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or

\[ \| h \|_\infty \geq C \| q \|_1, \]

with \( C = \min \left\{ \sqrt{\frac{2}{3\pi}} e^{-\frac{t}{4}}, \frac{1}{\sqrt{\pi}} \right\} \), which proves the proposition. \( \square \)

**Remark 4.2.** Suppose that \( h_1(t) \) and \( h_2(t) \) are the temperature at \( x = 1 \), which are caused by the unknown positive heat fluxes \( q_1(t) \) and \( q_2(t) \), respectively, if the difference of \( h_1(t) \) and \( h_2(t) \) is small in \( \infty \)-norm, then by the previous proposition the difference of \( q_1(t) \) and \( q_2(t) \) in \( L^1 \)-norm must be also small.

**Proposition 4.3.**

\[ u(x, t) \leq C \sqrt{\frac{2}{\pi e x}} \| h \|_\infty, \]

where \( C \) is a constant, for \( 0 < x < 1 \).

Since

\[ u(x, t) = \int_0^1 q(\tau) \frac{1}{\sqrt{\pi(t - \tau)}} e^{-\frac{\tau^2}{2(t - \tau)}} d\tau, \]

the function \( \frac{1}{\sqrt{\pi t}} e^{-\frac{\tau^2}{4t}} \) has maximum value of \( \sqrt{\frac{2}{\pi e x}} \) at \( t = \frac{x^2}{2} \) for each \( x > 1 \), and \( q(t) \) is positive on \([0, 1]\) by assumption, thus

\[ u(x, t) \leq \sqrt{\frac{2}{\pi e x}} \int_0^1 q(\tau) d\tau \]

or

\[ u(x, t) = \sqrt{\frac{2}{\pi e x}} \| q \|_1. \]

By the previous proposition, we have

\[ u(x, t) \leq \sqrt{\frac{2}{\pi e x}} C \| h \|_\infty. \]

\( \square \)
Proposition 4.4. $h(t)$ is an infinitely differentiable rapidly decreasing function; i.e.,

$$\sup_{t \in [0, \infty)} |t^m h^{(n)}(t)| < \infty,$$  \hspace{1cm} (4.5)

for all positive integers $m$ and $n$.

**Proof:** The function

$$F(x) = \begin{cases} \frac{1}{\sqrt{\pi x}} e^{-\frac{1}{x}}, & \text{for } x > 0, \\ 0, & \text{for } x < 0. \end{cases}$$

is infinitely differentiable, and for arbitrary positive integer $n$, by (4.4),

$$h^{(n)}(t) = \int_0^1 q(\tau) \frac{d^n}{dt^n} \left( \frac{1}{\sqrt{\pi(t-\tau)}} e^{-\frac{1}{\sqrt{\pi(t-\tau)}}} \right) d\tau.$$  

Thus, $h(t)$ is an infinitely differentiable.

To prove (4.5), it is equivalent to prove that

$$\lim_{t \to \infty} |t^m h^{(n)}(t)| = 0,$$  \hspace{1cm} (4.6)

holds for all positive integers $m$ and $n$. This is because

$$t^m h^{(n)}(t) = \int_0^1 q(\tau) t^m \frac{d^n}{dt^n} \left( \frac{1}{\sqrt{\pi(t-\tau)}} e^{-\frac{1}{\sqrt{\pi(t-\tau)}}} \right) d\tau,$$

and $\frac{d^n}{dt^n} \left( \frac{1}{\sqrt{\pi(t-\tau)}} e^{-\frac{1}{\sqrt{\pi(t-\tau)}}} \right)$ has the form of $P\left( \frac{1}{\sqrt{\pi(t-\tau)}} \right) \frac{1}{\sqrt{\pi(t-\tau)}} e^{-\frac{1}{\sqrt{\pi(t-\tau)}}}$, where $P(x)$ is a polynomial of order $2n$; it is obvious that

$$\lim_{t \to \infty} t^m P\left( \frac{1}{\sqrt{\pi(t-\tau)}} \right) \frac{1}{\sqrt{\pi(t-\tau)}} e^{-\frac{1}{\sqrt{\pi(t-\tau)}}} = 0,$$

thus (4.6) is true. \(\Box\)

However, it is not guaranteed that the heat flux $q(t)$ is positive in general case of the problem (4.1)-(4.3). Moreover, the temperature function $h(t)$ can not be exact,
although the error of the measurement may be small. In the following section, the method of Quasi-Reversibility is applied to the problem (4.1)-(4.3) to construct an approximate solution \( w^e(x, t) \) with \( w^e(1, t) \) being close to given \( h(t) \) in certain sense.

### 4.2 Smoothing the Given Temperature \( h(t) \)

From Proposition (4.4) in the previous section, we can see that it is reasonable to assume that

\[
h(t) \in L^1[0, \infty) \tag{4.7}
\]

In our later discussion, we wish the Fourier transform of \( h(t) \)

\[
\hat{h}(t)(s) = \int_{-\infty}^{\infty} h(t) e^{-ist} dt \tag{4.8}
\]

to be \( L^1 \) so that

\[
h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{h}(s) e^{ist} ds. \tag{4.9}
\]

However, we know that, for arbitrary \( L^1 \) function \( h(t) \), its Fourier series \( \hat{h}(s) \) is not necessarily \( L^1 \) in the frequency domain and the inversion (4.9) is not always satisfied. But we can make an approximation of \( h(t) \) that has \( L^1 \) Fourier transform and satisfies (4.9).

Let

\[
h^*_{\delta}(t) = \tilde{h}(t) * \varphi_{\delta}(t - \delta) = \int_{-\infty}^{\infty} \tilde{h}(t - \tau) \varphi_{\delta}(\tau - \delta) d\tau, \tag{4.10}
\]

where \( \tilde{h} \) is an extension of \( h \) for \( t \in (-\infty, \infty) \), which is defined as

\[
\tilde{h}(t) = \begin{cases} 
  h(t), & \text{for } t \geq 0, \\
  0, & \text{for } t < 0.
\end{cases}
\]
\( \varphi_{\delta} \) is the approximate identity, i.e.,
\[
\varphi(t) = \begin{cases} 
\exp\left(\frac{1}{|t|^2 - 1}\right) / \int_{|t|<1} \exp\left(\frac{1}{|t|^2 - 1}\right) dt, & \text{for } |t| < 1, \\
0, & \text{for } |t| \geq 1
\end{cases}
\]
and \( \varphi_{\delta}(t) = \delta^{-1} \varphi(t/\delta) \).

We can prove that, as \( \delta \to 0 \), \( h_{\delta}^{*} \) pointwise converges to \( h(t) \) and uniformly converges to \( h(t) \) on any set where \( h(t) \) is uniformly continuous.

Also, we can verify that the modification \( h_{\delta}^{*} \) of \( h \) still keeps the same initial value; i.e., \( h_{\delta}^{*}(0) = 0 \).

From (4.10), we have
\[
\int_{-\infty}^{\infty} \tilde{h}(-\tau) \varphi_{\delta}(\tau - \delta) d\tau = \int_{-\infty}^{\infty} \int_{0}^{\infty} h(s) \varphi_{\delta}(-s - \delta) ds d\tau.
\]

Let \( s = -\tau \), then
\[
\int_{-\infty}^{\infty} \tilde{h}(s) \varphi_{\delta}(-s - \delta) ds = \int_{0}^{\infty} h(s) \varphi_{\delta}(-s - \delta) ds.
\]

But in the above integral \( -\infty < -s - \delta < -\delta \), when \( s > 0 \); and \( \varphi_{\delta}(-s - \delta) = 0 \), for \( -\infty < -s - \delta < -\delta \). Thus the above integral equals 0, which guarantees the compatibility of the conditions in problem (4.13).

The modified function \( h_{\delta}^{*} \) is smooth enough so that its Fourier transform
\[
\hat{h}_{\delta}^{*}(s) = \int_{-\infty}^{\infty} h_{\delta}^{*}(t) e^{-ist} dt
\]
is an \( L^1(-\infty, \infty) \) function.

Note that \( \varphi_{\delta}(t) \) is a rapidly decreasing function and its Fourier transform is \( L^1 \).

More detailed material about rapidly decreasing function space and their Fourier transform can be seen in [1] and [26].

By (4.7), we have the Fourier transform of \( \tilde{h}(t) \) is only bounded over its frequency domain, since
\[
\hat{h}(s) = \int_{-\infty}^{\infty} \tilde{h}(t) e^{-ist} dt = \int_{0}^{\infty} h(t) e^{-ist} dt \leq \int_{0}^{\infty} |h(t)| dt.
\]
However, after $\tilde{h}$ being mollified in (4.10), the Fourier transform of $h^*_\delta$ is

$$h^*_\delta = \hat{\tilde{h}} \varphi_\delta e^{-is\delta},$$

and it is $L^1(-\infty, \infty)$ since

$$\int_{-\infty}^{\infty} |\hat{\tilde{h}}_\delta(s)| \, ds \leq \max_s |\hat{\tilde{h}}(s)| \int_{-\infty}^{\infty} |\varphi_\delta(s)| \, ds.$$

Hence, $\hat{\tilde{h}}_\delta$ has inverse Fourier transform.

Finally, we can claim that

$$h^*_\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\tilde{h}}_\delta(s)e^{ist} \, ds.$$

Apparently, $\varphi_\delta(t)$ is an infinitely differentiable rapidly decreasing function, thus, if $\varphi_\delta(s)$ is the Fourier transform, then we have

$$\varphi_\delta(t - \delta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_\delta(s)e^{-is\delta}e^{ist} \, ds. \quad (4.11)$$

By (4.10) and (4.11), we have

$$h^*_\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\tilde{h}} \varphi_\delta e^{-is\delta} \, ds,$$

(see Theorem VI.1.4 in [16], page 122).

For further consideration of problem (4.1)-(4.3), we simply make the following assumption:

1. $h(t)$ in (4.1)-(4.3) is extended to be 0 for negative $t$,
2. $h(t)$ is a both $L^1(-\infty, \infty)$ and $L^2(-\infty, \infty)$ function,
3. $h(t)$ has $L^1$ Fourier transform $\hat{h}(s)$, \hspace{1cm} (4.12)
4. $h(t)$ satisfies the inversion relation (4.9),
5. $\|h(t)\|_1, \|h(t)\|_2$ and $\|\hat{h}(s)\|_1$ are bounded by a constant $C$.  

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4.3 The Application of the Method of Quasi-Reversibility to IHCP

Applying the method of Quasi-Reversibility to IHCP, first we will consider a perturbation of the heat equation

\[ v_{xx}^\varepsilon(x,t) = v_t^\varepsilon(x,t) + \varepsilon^2 v_{ttxx}^\varepsilon(x,t), \]

where \( \varepsilon \) is a small positive parameter.

Since a higher order derivative term in \( t \) is added to the heat equation, one more conditions are needed to make the constructed problem solvable. Thus, a perturbed problem for IHCP may be as

\[ v_{xx}^\varepsilon(x,t) = v_t^\varepsilon(x,t) + \varepsilon^2 v_{ttxx}^\varepsilon(x,t), \quad x \geq 0, \quad t \geq 0, \]
\[ v^\varepsilon(x,0) = v_t^\varepsilon(x,0) = 0, \quad x \geq 0, \quad (4.13) \]
\[ v^\varepsilon(1,t) = h(t), \quad 0 < t < \infty, \]

The solution of (4.13) \( v^\varepsilon(x,t) \) should be also bounded as \( x \to \infty \).

In order to apply Fourier transforms to our problem (4.13), we extend \( v^\varepsilon(x,t) \) to \( -\infty < t < \infty \) to let \( v^\varepsilon(x,t) = 0 \) for \( t < 0 \), thus \( v_{xx}^\varepsilon(x,t) = 0 \) for \( t < 0 \). However, it is not necessary to consider the solution for \( t < 0 \). Also, we assume that \( v^\varepsilon(x, \cdot) \) is \( L^1 \) for all \( x > 0 \).

Fourier transform of \( v^\varepsilon(x,t) \) and \( v_{xx}^\varepsilon(x,t) \) are

\[ \hat{v}^\varepsilon(x,s) = \int_{-\infty}^{\infty} v^\varepsilon(x,t)e^{-ist}dt = \int_{0}^{\infty} v^\varepsilon(x,t)e^{-ist}dt. \]

\[ \hat{v}_{xx}^\varepsilon(x,s) = \int_{-\infty}^{\infty} v_{xx}^\varepsilon(x,t)e^{-ist}dt = \int_{0}^{\infty} v_{xx}^\varepsilon(x,t)e^{-ist}dt. \]

Note that

\[ \hat{v}_t^\varepsilon(x,s) = \int_{-\infty}^{\infty} v_t^\varepsilon(x,t)e^{-ist}dt = \int_{0}^{\infty} e^{-ist}dv^\varepsilon. \]
Integrating it by parts, we have

\[ \dot{v}_f(x, s) = v^f(x, t) \bigg|_{t=0}^{t=\infty} + is \int_0^\infty v^f(x, t)e^{-ist}dt, \]

by the initial condition of (4.13) and the assumption that \( v^f(x, \cdot) \) is \( L^1 \), we have \( v^f(x, t) \big|_{t=0}^{t=\infty} = \lim_{t \to \infty} v^f(x, t) - v^f(x, 0) = 0 \), thus

\[ \dot{v}_f(x, s) = is\dot{v}_f(x, t)e^{-ist}. \]

Similarly, we have

\[ \ddot{v}_f(x, s) = -s^2v^f_x(x, s) \]

Now we can write the perturbed problem (4.13) in the frequency domain as

\[ \ddot{v}_f(x, s) = \frac{is}{1+s^2} v^f(x, s), \quad 0 < x < \infty \]

\[ \ddot{v}_f(1, s) = \hat{h}(s), \]

for \(-\infty < s < \infty\), where \( \ddot{v}_f(x, s) \) must be bounded for large \( x \).

Considering the given conditions, the solution \( \ddot{v}_f(x, s) \) of (4.14) is

\[ \ddot{v}_f(x, s) = \hat{h}(s)e^{\sqrt{\frac{is}{1+s^2}}(1-x)}. \]

(4.15)

Formally, the inverse Fourier transform of \( \ddot{v}_f(x, s) \)

\[ v^f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \ddot{v}_f(x, s)e^{ist}ds \]

(4.16)

for \( t > 0 \) is taken as the solution of (4.13).

For an arbitrary positive number \( \epsilon \), \( v^f(x, t) \) exists in \([0, 1] \times [0, \infty)\), since, by (4.15) and (4.16), we have

\[ v^f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{h}(s)e^{\sqrt{\frac{is}{1+s^2}}(1-x)}e^{-ist}ds. \]

Under the assumption (4.12) that \( \hat{h}(s) \) is \( L^1 \), and for each \( \epsilon > 0 \),

\[ |e^{\sqrt{\frac{is}{1+s^2}}(1-x)}| \leq e^{\sqrt{\frac{1}{3(1+s^2)}}} \leq e^{\frac{1}{3s^2}}, \]

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which is bounded in terms of \( \varepsilon \). Thus, the left side of (4.16) is integrable. Moreover, it could be estimated that

\[
| v^\varepsilon(x,t) | \leq e^{\frac{1}{\varepsilon^2}} \| \hat{h}(s) \|_1,
\]

(4.17)

uniformly for all \( 0 \leq x \leq 1 \).

For the second step of the method of Quasi-Reversibility, we will consider the initial-boundary value problem for the heat equation in a quarter plane with data of \( \tilde{v}^\varepsilon(x,t) \),

\[
\begin{align*}
\frac{w^\varepsilon}{x} &= \frac{w^\varepsilon}{t}, & 0 < x < \infty, & 0 < t < \infty, \\
\frac{w^\varepsilon}{x}(x,0) &= 0, & 0 < x < \infty, \\
\frac{w^\varepsilon}{x}(0,t) &= v^\varepsilon(0,t), & 0 < t < \infty.
\end{align*}
\]

(4.18)

Similarly, let \( w^\varepsilon(t) = 0 \) for \( t < 0 \), then by Fourier transform, the problem (4.18) in the frequency domain can be written as the following

\[
\begin{align*}
\hat{w}^\varepsilon_{xx}(x,s) &= is\hat{w}^\varepsilon(x,s), & 0 < x < \infty \\
\hat{w}^\varepsilon(0,s) &= \hat{v}^\varepsilon(0,s),
\end{align*}
\]

(4.19)

for \(-\infty < s < \infty\). By considering the boundedness of the solution as \( x \to \infty \), the solution of (4.19) is

\[
\hat{w}^\varepsilon(x,s) = \hat{v}^\varepsilon(0,s) e^{-\sqrt{\frac{1}{4}(1+is\text{sign}(s))}x}. 
\]

(4.20)

By the solution (4.15), (4.20) can be written as the following

\[
\hat{w}^\varepsilon(x,s) = \hat{h}(s) e^{-\sqrt{\frac{1}{4}(1+is\text{sign}(s))}(x - \frac{1}{\sqrt{1+is\text{sign}(s)}})}, 
\]

and

\[
\hat{w}^\varepsilon(1,s) = \hat{h}(s) e^{-\sqrt{\frac{1}{4}(1+is\text{sign}(s))}(1 - \frac{1}{\sqrt{1+is\text{sign}(s)}})} 
\]

(4.21)
which is $L^1$, since $\hat{h}(s)$ is $L^1$ and $e^{-\sqrt{\frac{1}{4}((1+\text{sign}(s))(1-\frac{1}{\sqrt{1+s^2}}))}}$ is uniformly bounded for $s$. Thus we can have inverse Fourier transform of $\hat{w}^\varepsilon(1,s)$, which is

$$w^\varepsilon(1,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{w}^\varepsilon(1,s)e^{ist}ds$$

for $t > 0$.

By the estimate (4.17), for $0 < x \leq 1$, we have

$$w^\varepsilon(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(s)e^{-\sqrt{\frac{1}{4}((1+\text{sign}(s))(x-\frac{1}{\sqrt{1+s^2}}))}e^{ist}ds,}$$

and the similar estimate holds for $w^\varepsilon(x,t)$, i.e.,

$$|w^\varepsilon(x,t)| \leq e^{-\frac{1}{x}}\|\hat{h}(s)\|_1.$$  \hspace{1cm} (4.22)

uniformly for $0 < x \leq 1$.

**Proposition 4.5.** $\lim_{\varepsilon \to 0} \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{w}^\varepsilon(1,s)e^{ist}ds - h(t) \right| = 0$, uniformly for $t \in (0,+,\infty)$.

**Proof:** As assumed,

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{h}(s)e^{ist}ds.$$  

From (4.21), we also have

$$|\hat{w}^\varepsilon(1,s)| \leq |\hat{h}(s)|.$$ 

Thus, by the assumption (4.12), for all $\rho > 0$, $\exists M_\rho > 0$, such that,

$$\frac{1}{2\pi} \left( \int_{M_\rho}^{\infty} + \int_{-\infty}^{-M_\rho} \right) \hat{w}^\varepsilon(1,s)ds < \frac{\rho}{2}$$

and

$$\frac{1}{2\pi} \left( \int_{M_\rho}^{\infty} + \int_{-\infty}^{-M_\rho} \right) \hat{h}(s)ds < \frac{\rho}{2}$$

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Moreover,

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\eta}(s) \left( e^{-\sqrt{\frac{4}{\pi}} \left( 1 + i \text{sign}(\xi) \right) \left( 1 - \frac{1}{\sqrt{1 + s^2}} \right)} - 1 \right) e^{ist} ds \]

\[ \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \hat{\eta}(s) \right| \left| e^{-\sqrt{\frac{4}{\pi}} \left( 1 + i \text{sign}(\xi) \right) \left( 1 - \frac{1}{\sqrt{1 + s^2}} \right)} - 1 \right| ds \]

Since \( e^{-\sqrt{\frac{4}{\pi}} \left( 1 + i \text{sign}(\xi) \right) \left( 1 - \frac{1}{\sqrt{1 + s^2}} \right)} - 1 \) uniformly continuous on \( s \in [-M_p, M_p] \), and converges to 0 as \( \varepsilon \to 0 \), there exists \( \delta > 0 \), for all \( \varepsilon \in (0, \delta) \), we have

\[ \left| e^{-\sqrt{\frac{4}{\pi}} \left( 1 + i \text{sign}(\xi) \right) \left( 1 - \frac{1}{\sqrt{1 + s^2}} \right)} - 1 \right| < \rho. \]

Let

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \hat{\eta}(s) \right| ds \leq L, \]

then

\[ \left| w^\varepsilon(1, t) - h(t) \right| \leq \rho + L \rho, \]

which has proved that \( w^\varepsilon(1, t) \to h(t) \) uniformly, as \( \varepsilon \to 0. \)

In the similar argument, we can prove the uniform convergence of \( w^\varepsilon(1, t) \) to \( h(t) \) in \( L^2 \) space; i.e.,

**Proposition 4.6.** \( \lim_{\varepsilon \to 0} \| \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{w}(1, s) e^{ist} ds - h(t) \|_2 = 0. \)

To summarize the application of the method of Quasi-Reversibility, we have the following theorems.

**Theorem 4.7.** If \( h(t) \) in problem (4.1)-(4.3) is \( L^1 \) and has \( L^1 \) Fourier transform, then the method of Quasi-Reversibility constructs solutions \( w^\varepsilon(x, t) \) of problem (4.1)-(4.2) with \( w^\varepsilon(1, t) \) being uniformly convergent to \( h(t) \) as \( \varepsilon \to 0. \)
Theorem 4.8. If $h(t)$ in problem (4.1)-(4.3) is $L^1$, then the method of Quasi-Reversibility constructs solutions $w^*(x, t)$ of problem (4.1)-(4.2) with $w^*(1, t)$ being pointwise convergent to $h(t)$ and uniformly convergent to $h(t)$ on any set on which $h(t)$ is uniformly continuous.

Remark 4.9. Because of the ill-posedness of (4.1)-(4.3), no continuous dependence of the solutions on the given data $h(t)$ could be obtained. Although we have estimate (4.22), it is difficult to predict $h(t)$ by using $\hat{h}(s)$.

Considering the continuous dependence of the solution on the given data in problem (4.1) - (4.3), we have to discuss this problem in $L^2$ norm, since there are some related works in $L^2$ space already.

According to [11], it is necessary to impose an a priori bound on the solution, i.e.,

$$\|u(x, t)\|_2 \leq M \quad \text{or} \quad \|u(x, 0)\|_2 \leq M,$$  
(4.23)

where $M$ is a constant.

Levine proved in [17] that the unique bounded solution $u(x, t)$ of (4.1) - (4.3) satisfies

$$\|u(x, t)\|_2 \leq A\|u(0, t)\|^\frac{\beta(x)}{2}\|u(1, t)\|^{1-\beta(x)}$$

where $A$ is a constant and $\beta(x) = 1 - x$. His theorem is called Three Lines Theorem.

Actually, by (4.12), since $h$ is $L^2$ and its $L^2$ norm is bounded by a constant $C$, so does $\hat{h}(s)$. Thus, by (4.15),

$$\|\hat{w}^*(x, \cdot)\|_2 \leq e^{\frac{1}{\sqrt{\tau}}}\|\hat{h}(s)\|_2 \leq C e^{\frac{1}{\sqrt{\tau}}}$$

By (4.20), we have

$$\|w(x, \cdot)\|_2 = \|\hat{w}(x, \cdot)\|_2 \leq \|\hat{w}(0, \cdot)\|_2 \leq C e^{\frac{1}{\sqrt{\tau}}}$$
Applying Levine's Three Lines Theorem, we let $u(x, t)$ be the exact solution of the problem (4.1) - (4.3), which is bounded as in (4.23) with $M = C e^{\frac{1}{\sqrt{t}}}$, and $w^\varepsilon(x, t)$ is the approximation of (4.1) - (4.3) by the method of Quasi-Reversibility, which also satisfies (4.23). By Proposition 4.6, if the difference of $u(1, t)$ and $w^\varepsilon(1, t)$ is small in $L^2$ norm, say
\[
\|u(1, \cdot) - w^\varepsilon(1, \cdot)\|_2 < \rho,
\]
where $\rho$ is a small positive number. Then we have Hölder continuous dependence of the solution of problem (4.1) - (4.3) for $0 < x \leq 1$, i.e., there exists a positive constant $A$ such that
\[
\|u(x, \cdot) - w^\varepsilon(x, \cdot)\|_2 \leq A \rho^x M^{1-x}.
\]
REFERENCES


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Xueping Ru was born on July 24, 1964, in Shanghai, China. She earned her bachelor of science degree in Mathematics at East China Normal University in July 1985. She earned a master of science degree in mathematics from the same university in July 1988. From July 1988 to July 1994 she worked as a lecturer in Shanghai University. In August 1994 she came to the United States to pursue graduate studies in mathematics at Louisiana State University. She is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August, 1999.
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