General Riemann Integrals and Their Computation via Domain.

Bin Lu

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GENERAL RIEMANN INTEGRALS
AND THEIR COMPUTATION VIA DOMAIN

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ABSTRACT

In this work we extend the domain-theoretic approach of the generalized Riemann integral introduced by A. Edalat in 1995. We begin by laying down a related theory of general Riemann integration for bounded real-valued functions on an arbitrary set \( X \) with a finitely additive measure on an algebra of subsets of \( X \). Based on the theory developed we obtain a formula to calculate integral of a bounded function in terms of the regular Riemann integral. By the classical extension theorems on set functions we can further extend this generalized Riemann integral to more general set functions such as valuations on lattices of subsets. For the setting of bounded functions defined on a continuous domain \( D \) with a Borel measure for the Scott topology, we can compute the Riemann integral of a function effectively and so the value of the integral can be obtained up to a given accuracy. By invoking the results of J. Lawson we can extend this type of Riemann integral to maximal point spaces, as a special case of the Polish spaces. Furthermore we show that when \( X \) is a compact metric space, our approach of Riemann integration is equivalent to the generalized Riemann integration introduced by A. Edalat in the sense that the two integrals yield the same value. Finally we prove that the approach of integration taken by R. Heckmann is equivalent to our approach, and the values of the integrals are the same.
CHAPTER 1. INTRODUCTION

The theory of Riemann integration of real-valued functions was developed by Fourier, Cauchy, Dirichlet, Riemann and Darboux, along with other mathematicians of the last century. With its simple and constructive features it became as it is today a foundation of calculus with applications in all fields of sciences. Even shortly after this theory was introduced, mathematicians of the era realized that the integral had serious shortcomings. In particular the integral was only defined for bounded functions defined on bounded intervals. In order to define integrals for unbounded functions or functions defined on unbounded intervals, a special integral called the Cauchy-Riemann integral was introduced. In addition to this deficiency, the theory has other limitations in the following areas:

(1) It treats only integration of functions defined on \( \mathbb{R}^n \).

(2) It covers only integration with respect to the usual measure on \( \mathbb{R}^n \).

(3) It has no regularity, i.e., it lacks suitable convergence properties.

(4) The theory lacks completeness properties.

(5) A function whose set of discontinuities has non-zero measure does not have a Riemann integral.

In the early years of this century, Lebesgue and Borel, among other mathematicians, laid the foundation of a new theory of integration which is now called the Lebesgue integral. This integral overcame most of the limitations of the Riemann integration theory. The practical advantages of the Lebesgue integral have mostly to do with the interchange of the integral with other limit operations, such as sums,
other integrals, differentiation, and the like. Since its inception the Lebesgue integral has continued to evolve and is recognized as basic for integration theory. The construction of the Lebesgue integral is in sharp contrast to that of the Riemann integral. In the theory of the Riemann integral, the domain of a given function is partitioned and the integral of the function is approximated by the lower and upper Darboux sums with respect to the partition, while in the theory of the Lebesgue integral, the range of a given function is partitioned, and the integral is defined as the least upper bound of integrals of the simple functions induced by the partition. Compared with the theory of Riemann integration, the theory of the Lebesgue integral is much less constructive. From the computational point of view, the theory of the Lebesgue integral is not satisfactory. This theory is usually viewed as a research or advanced theory of integration by many mathematicians. Therefore the Riemann theory of integration remains the preferred one in everyday practice.

There have been several attempts to generalize the theory of the Riemann integration. The most successful one is the Riemann-Stieltjes integral. This integral has found many applications in probability theory and stochastic processes. In addition to the Riemann-Stieltjes integral, there are two other types of generalized Riemann integrals in the literature, namely, the McShane and the Henstock integrals [16, 28]. The constructions of these two integrals are similar to the construction of the classical Riemann integral, i.e., dividing the domain of a given function, but with a tagged partition. The McShane integral turns out to be equivalent to the Lebesgue integral, while the Henstock integral (also called Henstock-Kurzweil integral) is a
generalization in the sense that the class of the Lebesgue integrable functions is a proper subset of the class of the Henstock integrable functions. The Henstock integral has the property that every continuous, nearly everywhere differentiable function can be recovered by integration from its derivative. This property, which does not hold for the Riemann or Lebesgue integrals, was historically the main motivation behind the definition of these types of integrals. As pointed out in [14], this theory, like the theory of the Lebesgue integral, lacks the effective framework and constructive features needed for computation.

The domain-theoretic approach of the generalized Riemann integral was introduced by Edalat [7, 14]. As its name indicates, this approach is based on the theory and techniques of the domain theory and its applications. Domain theory was introduced by Dana Scott in 1970 [31] as a mathematical theory of computation in the semantics of programming languages. It has been intensively investigated for the last two decades. In the last fews years a new direction for applications of domains in computation on classical spaces in mathematics has emerged. The main idea of this approach is the following: given a topological space $X$, we need to find a computational environment, i.e., some appropriate domain such that $X$ can be embedded onto some subset of this domain $D$, specifically, onto the subset of maximal points of the domain $D$. Once this embedding is established, we obtain a nice presentation of $X$ and a simple computational model for space $X$. This embedding also provides an effective structure on $X$ and in this case we can discuss computability of elements in $X$. One specific application of the idea above was de-
veloped by Edalat over the last few years [6, 7]. He showed that when a topological Hausdorff space is embedded onto the subset of maximal points of its upper space of compact subsets, i.e., the subset of singletons, then the Borel measures on the space can be embedded onto the maximal elements in the probabilistic powerdomain of the upper space. By this embedding it is possible to approximate Borel measures; hence this provides an order-theoretic version of approximation of Borel measures. Furthermore, the probabilistic powerdomain contains completely "new measures" and in many circumstances allows the construction of an increasing chain of approximating measures to a desired Borel measure. With this very effective and powerful tool Edalat in [8] introduced a novel approach to integration theory, the R-integral as it is called. This domain-theoretic approach to integration defines integrals of bounded functions on a compact metric space with respect to a Borel measure. With approximating measures for a given Borel measure he introduced generalized lower and upper Darboux sums, and then the R-integral. This theory overcomes the deficiencies we indicated earlier. More importantly, this theory preserves the constructive features of the Riemann integration. In addition, Edalat applied the theory in a wide variety of settings such as dynamical systems, image compression, neural networks, and stochastic processes [7, 8, 9, 13].

Our purpose in this dissertation is to extend, generalize, and develop this approach of the generalized Riemann integration introduced by Edalat, as briefly sketched above. We extend the theory to a general framework when X is a separable complete metric space. Furthermore, by applying a nice result by Lawson
[25, 26], which indicates the connection between the Polish spaces and spaces of maximal points, we can even generalize this theory for $X$ being a space of maximal points.

In Chapter 2 we first introduce the concept of the generalized Riemann integral, called the Riemann-like integral, for a bounded function $f : X \to \mathbb{R}$, where $X$ is an arbitrary set with an algebra $\mathcal{A}$ of subsets of $X$ and $\mu_\mathcal{A}$ is given as a finitely additive measure on $\mathcal{A}$. Parallel to the theory of the classical Riemann integral, we lay down foundation for the Riemann-like integral. Then we can derive a method to compute the Riemann-like integral of a given function with respect to $\mu_\mathcal{A}$ on $X$.

In Chapter 3 we extend the previous chapter; we use different classes of subsets of $X$ to partition $X$ and correspondingly we have different set functions. Consequently we obtain different variants of integrals. We consider cases such as $\mu_\mathcal{S}$ is defined on a semi-algebra of subsets $\mathcal{S}$ of $X$ and $f$ is a bounded and $\mathcal{S}$-measurable function, or $\mu_\mathcal{L}$ is defined on a lattice $\mathcal{L}$ of subsets of $X$ as a valuation on $\mathcal{L}$. Furthermore when $X$ is a topological space then $\mathcal{L}$ can be chosen as a lattice of open subsets of $X$, or even crescent subsets. Then from the classical extension theorems we can extend $\mu_\mathcal{S}, \mu_\mathcal{L}$ to the algebra $\mathcal{A}$ generated by these classes of subsets. Thus we can extend the integration theory for these cases to that defined in Chapter 2. Consequently we obtain similar formulas to calculate integrals.

In Chapter 4 we look at the case when the space $X$ is a continuous domain $D$ with the Scott topology, $f$ is an order-preserving function and $\mu$ is a continuous valuation. Then by the theory developed in previous two chapters, we can effectively
calculate integrals of order-preserving functions on \( D \). By Jones' result in [19] that
the probabilistic power domain of a continuous domain is also continuous with
simple valuations as a basis, we are able to calculate effectively the integral of a
Scott continuous function with respect to a Borel measure of the Scott open subsets.
In this way we can approximate the integral within its \( \epsilon \)-neighborhood. Hence
generally we are able to compute approximately the expected value for continuous
functions within this range.

In Chapter 5 we first review the important concept of a maximal point space
introduced by Lawson [25, 26]. Then with help of previous chapters, we can calculate
integrals of functions defined on maximal point spaces. By the result by Lawson
[25] giving the equivalence of maximal point spaces and the Polish spaces, we are
able to develop a theory of integration on complete separable metric spaces, i.e.,
the Polish spaces. Thus we extend successfully the Riemann integral to Polish
spaces. Consequently, when \( X \) is a compact metric space, we show our approach of
calculating the integral is equivalent to the approach of Edalat [7] and two integrals
yield the same value.

In Chapter 6 we look at approach to integration theory taken by Heckmann
[18]. Consider that \( X \) is a topological space and \( f \in [X \rightarrow \overline{\mathbb{R}}_+]_i \). Integration is
defined as a function \( \int_X : [X \rightarrow \overline{\mathbb{R}}_+]_i \otimes VX \rightarrow \overline{\mathbb{R}}_+ \) which is continuous in the
two arguments separately, where \( VX \) is the set of all Scott continuous valuations
and \([X \rightarrow \overline{\mathbb{R}}_+]_i \) is the Isbell function space. Then by the results from Section 2.3,
Section 3.3 and Section 4.3 we can show that our approach and his approach are
essentially equivalent. Moreover we can prove that the values of the two integrals yield the same value.

Finally we need to mention that there are still many questions remaining, especially in the areas of application of the theory to probability and stochastic processes. Because our approach is more general than Edalat's, from the many applications he mentioned in [7, 8, 9, 13], we have a glimps of possible applications of this approach of the Riemann integration via domain.
CHAPTER 2. RIEMANN-LIKE INTEGRATION

In this chapter we first give standard definitions of an algebra $A$ of subsets of $X$ and a finitely additive measure $\mu$ on the algebra. Then we develop the theory of the Riemann-like integration of function $f : X \to \mathbb{R}$, where $X$ is a set and $\mathbb{R}$ is the real line. We show that any function $f$ which is bounded and measurable with respect to $A$ is integrable with respect to the measure $\mu$. We also derive a formula to compute the integral of $f$.

2.1 Algebras of Sets and Measures

Let $X$ be a non-empty set and $A$ be a non-empty collection of subsets of $X$, i.e., $A \subseteq P(X)$, where $P(X)$ is the power set of $X$.

**Definition 1.** A non-empty collection $A$ of subsets of $X$ is a ring if and only if
1. $\emptyset \in A$;
2. $A, B \in A$ implies $A \cap B \in A$;
3. $A, B \in A$ implies $A \setminus B \in A$.

If in addition to (i)-(iii), $A$ satisfies:
4. $X \in A$, then $A$ is called an algebra.

It is clear from the definition that an algebra $A$ is closed under finite unions and finite intersections, and also closed under complementation.

**Example 1.** (i) It is clear that $A = P(X)$ is an algebra, where $P(X)$ is the powerset of $X$; (ii) Let $A = \{\emptyset, X\}$. Then $A$ is an algebra.

**Definition 2.** A function $\mu : A \to [0, \infty)$ is called a finitely additive measure if (i) $\mu(\emptyset) = 0$ and (ii) for all $A, B \in A$ with $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

**Proposition 1.** Let $A$ be an algebra of subsets of $X$ and $\mu$ be a finitely additive measure on $A$. Then (i) $\mu$ is monotone, i.e., $A \subseteq B$ implies $\mu(A) \leq \mu(B)$; (ii) $A \subseteq B$
implies $\mu(B \setminus A) = \mu(B) - \mu(A)$.

**Proof:** (i) Since $\mathcal{A}$ is an algebra, then $B = (B \setminus A) \cup A$, $B \setminus A$ is in $\mathcal{A}$. Thus, $\mu(B) = \mu(B \setminus A) + \mu(A) \geq \mu(A)$ from the definition.

(ii) Obvious, from part (i). ■

2.2 Definitions and Basic Results

In this section we extend the theory of Riemann integration for functions defined on a closed interval of the real line $\mathbb{R}$ to functions defined on an arbitrary set $X$. We develop this extension along the lines of the classical Riemann integration theory. Traditionally there are two equivalent ways to introduce Riemann integration. Here we choose the method using “Darboux sums”.

Let $X$ be a non-empty set, $\mathcal{A}$ an algebra of subsets of $X$, and $\mu$ a finitely additive measure on $\mathcal{A}$.

**Definition 1.** Let $\mathcal{P} \subseteq \mathcal{A}$. $\mathcal{P} = \{A_1, \ldots, A_n\}$ is a finite $\mu$-partition of $X$ if and only if (i) $\bigcup_{A \in \mathcal{P}} A = X$; (ii) for all $A, B \in \mathcal{P}$, $\mu(A \cap B) = 0$ if $A \neq B$.

**Definition 2.** Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be be two finite $\mu$-partitions of $X$. $\mathcal{P}_1$ is said to be finer than $\mathcal{P}_2$ if and only if for each subset $A \in \mathcal{P}_1$, there exists $B \in \mathcal{P}_2$ such that $A \subseteq B$. It is denoted by denoted by $\mathcal{P}_1 \prec \mathcal{P}_2$.

For a bounded function $f : X \to \mathbb{R}$ we have the following definition:

**Definition 3.** The Darboux upper sum of $f$ with respect to $\mu$, a finitely additive measure on an algebra of subsets of $X$, and $\mathcal{P}$, a finite $\mu$-partition of $X$ is defined
Similarly, the Darboux lower sum of \( f \) with respect to \( \mu \) and \( \mathcal{P} \) is defined by

\[
S^l(f, \mu, \mathcal{P}) = \sum_{P \in \mathcal{P}} \inf(f(P)) \mu(P).
\]

**Lemma 1.** For \( \mu \), \( \mathcal{P} \), and \( f \) as above,

\[
S^l(f, \mu, \mathcal{P}) \leq S^u(f, \mu, \mathcal{P}).
\]

**Proof:** This is clear from the definition. \( \square \)

**Lemma 2.** Let \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) be two finite partitions of \( X \). Assume that \( \mathcal{P}_1 \) is finer than \( \mathcal{P}_2 \). Then

\[
S^l(f, \mu, \mathcal{P}_2) \leq S^l(f, \mu, \mathcal{P}_1) \leq S^u(f, \mu, \mathcal{P}_1) \leq S^u(f, \mu, \mathcal{P}_2).
\]

**Proof:** By assumption \( \mathcal{P}_1 \ll \mathcal{P}_2 \), i.e., for each \( P \in \mathcal{P}_1 \), there exists \( P' \in \mathcal{P}_2 \) such that \( P \subseteq P' \). Let \( \mathcal{P}_1 = \{P_1k\}_{k=1}^m \), \( \mathcal{P}_2 = \{P_2k\}_{k=1}^n \). Then we have

\[
S^u(f, \mu, \mathcal{P}_1) = \sum_{k=1}^m \sup(f(P_1k)) \mu(P_1k).
\]

Suppose that \( m = n + 1 \) and \( P_{21} = P_{11} \cup P_{12} \). Then

\[
S^u(f, \mu, \mathcal{P}_1) = \sum_{k=1}^m \sup(f(P_1k)) \mu(P_1k)
\]

\[
= \sup f(P_{11}) \mu(P_{11}) + \sup f(P_{12}) \mu(P_{12}) + \sum_{k=3}^m \sup(f(P_{1k})) \mu(P_{1k})
\]

\[
\leq \sup f(P_{21})[\mu(P_{11}) + \mu(P_{12})] + \sum_{k=3}^m \sup(f(P_{2k})) \mu(P_{2k})
\]

\[
= \sup f(P_{21}) \mu(P_{21}) + \sum_{k=3}^m \sup(f(P_{2k})) \mu(P_{2k})
\]

\[
= S^u(f, \mu, \mathcal{P}_2).
\]
Similarly, consider

\[ S^t(f, \mu, \mathcal{P}_1) = \sum_{k=1}^{m} \inf (f(P_{1k})\mu(P_{1k})) \]

\[ = \inf f(P_{11})\mu(P_{11}) + \inf f(P_{12})\mu(P_{12}) + \sum_{k=3}^{m} \inf (f(P_{1k})\mu(P_{1k})) \]

\[ \geq \inf f(P_{21})[\mu(P_{11}) + \mu(P_{12})] + \sum_{k=3}^{m} \inf (f(P_{2k})\mu(P_{2k})) \]

\[ = \inf f(P_{21})\mu(P_{21}) + \sum_{k=3}^{m} \inf (f(P_{2k})\mu(P_{2k})) \]

\[ = S^t(f, \mu, \mathcal{P}_2). \]

The general case follows in a similar way by induction on \( k \), where \( m = n + k \). \( \blacksquare \)

**Lemma 3.** Let \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) be two finite partitions of \( X \). Then

\[ S^t(f, \mathcal{P}_1) \leq S^u(f, \mathcal{P}_2). \]

**Proof:** Consider a new partition \( \mathcal{P}_1 \cap \mathcal{P}_2 = \{ A \cap B : A \in \mathcal{P}_1, B \in \mathcal{P}_2, A \cap B \neq \emptyset \} \).

Clearly \( \mathcal{P}_1 \cap \mathcal{P}_2 \) is a partition of \( X \), because \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are partitions of \( X \). We have

\[ \bigcup_{i,j}(A_i \cap B_j) = \bigcup_{j}(\bigcup_{i}(A_i) \cap B_j) \]

\[ = \bigcup_{j}(X \cap B_j) \]

\[ = \bigcup_{j}B_j = X. \]

It is clear that \( \mathcal{P}_1 \cap \mathcal{P}_2 \prec \mathcal{P}_1, \mathcal{P}_2 \). Then from Lemma 2, we have

\[ S^t(f, \mu, \mathcal{P}_1 \cap \mathcal{P}_2) \leq S^u(f, \mu, \mathcal{P}_1 \cap \mathcal{P}_2) \leq S^u(f, \mu, \mathcal{P}_1 \cap \mathcal{P}_2) \leq S^u(f, \mu, \mathcal{P}_2). \]

Hence we are done. \( \blacksquare \)

**Definition 4.** The upper integral of \( f \) with respect to \( \mu \) is defined as follows:

\[ \int f d\mu = \inf_{\mathcal{P}} S^u(f, \mu, \mathcal{P}). \]
Similarly, the lower integral of \( f \) with respect to \( \mu \) is defined as

\[
\int_-^\mu \leq \sup_{\mathcal{P}} S^l(f, \mu, \mathcal{P}).
\]

We say \( f \) is (Riemann) integrable if

\[
\int_-^\mu = \int_\mu.
\]

Let \( I \) be this number, so \( I \) is the integral of \( f \) on \( X \), i.e., \( \int f \mu = I \).

It is easy to see that the integral of \( f \) on \( X \) is unique, for this is the direct consequence from the definition and lemmas above.

From the lemmas above we can easily obtain the following result:

**Proposition 1.** \( \int_-^\mu \leq \int_\mu \).

**Proof:** By the definition above, for each \( \epsilon > 0 \), there exists a finite partition \( \mathcal{P}_1 \) of \( X \), such that

\[
S^u(f, \mu, \mathcal{P}_1) < \int_\mu + \epsilon.
\]

So by the preceding lemma, \( \int_-^\mu + \epsilon \) is an upper bound of all \( S^l(f, \mu, \mathcal{P}) \). Hence

\[
\int_-^\mu = \sup_{\mathcal{P}} \{S^l(f, \mu, \mathcal{P})\} \leq \int_\mu + \epsilon.
\]

Letting \( \epsilon \to 0 \), we have the conclusion. 

Similar to the results from the classical Riemann integral, we have the following:

**Proposition 2.** \( f \) is integrable if and only if for any given \( \epsilon > 0 \), there exists \( \mathcal{P} \), a finite partition of \( X \), such that

\[
S^u(f, \mu, \mathcal{P}) - S^l(f, \mu, \mathcal{P}) < \epsilon.
\]

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Proof: Suppose first that for all \( \epsilon > 0 \) there exists \( \mathcal{P} \) a partition of \( X \), such that

\[
S^u(f, \mathcal{P}) < S^l(f, \mathcal{P}) + \epsilon.
\]

But we know

\[
\int f \, d\mu \leq S^u(f, \mathcal{P}) < S^l(f, \mathcal{P}) + \epsilon.
\]

So we have

\[
\int f \, d\mu < \int f \, d\mu + \epsilon.
\]

Letting \( \epsilon \to 0 \), we have

\[
\int f \, d\mu \leq \int f \, d\mu.
\]

We have shown that the following is always true:

\[
\int f \, d\mu \geq \int f \, d\mu.
\]

Hence we have

\[
\int f \, d\mu = \int f \, d\mu = \int f \, d\mu. \tag{1}
\]

Suppose now that we have (1), i.e., \( f \) is integrable. Let \( I = \int f \, d\mu \). Then for each \( \epsilon > 0 \), there exist \( \mathcal{P}_1, \mathcal{P}_2 \), partitions of \( X \), satisfying the following:

\[
S^l(f, \mu, \mathcal{P}_1) > \int f \, d\mu - \frac{\epsilon}{2} = I - \frac{\epsilon}{2}.
\]

\[
S^u(f, \mu, \mathcal{P}_2) < \int f \, d\mu + \frac{\epsilon}{2} = I + \frac{\epsilon}{2}.
\]

Let \( \mathcal{P} = \mathcal{P}_1 \wedge \mathcal{P}_2 \). From the definition in Lemma 3 that \( \mathcal{P} \preceq \mathcal{P}_1, \mathcal{P}_2 \). Then by Lemma 2, we have \( S^u(f, \mu, \mathcal{P}) \leq S^u(f, \mathcal{P}_2), S^l(f, \mathcal{P}) \geq S^l(f, \mathcal{P}_1) \). Thus,

\[
S^u(f, \mu, \mathcal{P}) - S^l(f, \mu, \mathcal{P}) < S^u(f, \mu, \mathcal{P}_2) - S^l(f, \mu, \mathcal{P}_1) < \epsilon.
\]
Then we are done. ■

Next we show an important alternate characterization of this type integration, which holds in the classical Riemann integration theory.

**Theorem 1.** Suppose that \( f \) is integrable. Then for all \( \epsilon > 0 \), there exists \( \mathcal{P} \) a partition of \( X \), such that for every \( \mathcal{P}' \ll \mathcal{P} \) and \( x_i \in A_i \in \mathcal{P}' \), we have

\[
I - \epsilon < \sum_{A_i \in \mathcal{P}'} f(x_i) \mu(A_i) < I + \epsilon,
\]

where \( I = \int f \, d\mu \).

**Proof:** From the definition, for every \( \epsilon > 0 \) there exists partitions \( \mathcal{P}_i, i = 1, 2 \) of \( X \), such that \( I - \epsilon < S^l(f, \mu, \mathcal{P}_1) \) and \( S^u(f, \mu, \mathcal{P}_2) < I + \epsilon \). Now let \( \mathcal{P} = \mathcal{P}_1 \land \mathcal{P}_2 \). So \( \mathcal{P} \ll \mathcal{P}_i, i = 1, 2 \). From Lemma 2, we have

\[
I - \epsilon < S^l(f, \mu, \mathcal{P}_1) \leq S^l(f, \mu, \mathcal{P}) \leq S^u(f, \mu, \mathcal{P}) \leq S^u(f, \mu, \mathcal{P}_2) < I + \epsilon.
\]

Now let \( \mathcal{P}' \ll \mathcal{P} \); then again by Lemma 2 we have

\[
I - \epsilon < \sum_{A_i \in \mathcal{P}'} \inf f(A_i) \mu(A_i) \leq \sum_{A_i \in \mathcal{P}'} \sup f(A_i) \mu(A_i) < I + \epsilon.
\]

Since for any \( x_i \in A_i \), \( \inf f(A_i) \leq f(x_i) \leq \sup f(A_i) \), therefore

\[
I - \epsilon < \sum_{A_i \in \mathcal{P}'} f(x_i) \mu(A_i) < I + \epsilon.
\]

Moreover just like the theory of the classical Riemann integration, the converse of this theorem above is also true.

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Theorem 2. If for all $\epsilon > 0$, there exists $\mathcal{P}$ a partition of $X$, such that for every $\mathcal{P}' < \mathcal{P}$ and $x_i \in A_i \in \mathcal{P}'$, we have

$$I - \epsilon < \sum_{A_i \in \mathcal{P}'} f(x_i) \mu(A_i) < I + \epsilon,$$

for some number $I$, then $f$ is integrable with integral $I$.

Proof: From the given conditions for $\epsilon > 0$ there exists $x_i \in A_i \in \mathcal{P}$ and $y_i \in A_i$ such that $\sup f(A_i) < f(x_i) + \epsilon$, $\inf f(A_i) + \epsilon > f(y_i)$, where $A_i \in \mathcal{P}$, for $i = 1, 2, \ldots, N$. Now consider the following

$$S^u(f; \mu; \mathcal{P}) = \sum_{A_i \in \mathcal{P}} \sup f(A_i) \mu(A_i),$$

$$< \sum f(x_i) \mu(A_i) + \sum \epsilon \mu(A_i),$$

$$< I + \epsilon + \epsilon \mu(X).$$

$$S^l(f; \mu; \mathcal{P}) = \sum_{A_i \in \mathcal{P}} \inf f(A_i) \mu(A_i),$$

$$> \sum f(y_i) \mu(A_i) - \sum \epsilon \mu(A_i),$$

$$> I - \epsilon - \epsilon \mu(X).$$

Hence we have

$$S^u(f; \mu; \mathcal{P}) - S^l(f; \mu; \mathcal{P}) < 2(1 + \mu(X))\epsilon.$$

Since $\epsilon$ is arbitrary, by the previous proposition, $f$ is integrable. It is clear that the integral of $f$ is $I$.

Next we have familiar properties of the integration.

Proposition 3. (i) Suppose that $f$ is integrable, $c$ is a constant. Then $cf$ is integrable, and

$$\int cf \, d\mu = c \int f \, d\mu.$$
(ii) Suppose that \( f, g \) are integrable. Then \( f + g \) is also integrable and

\[
\int (f + g)d\mu = \int f\,d\mu + \int g\,d\mu.
\] (2)

**Proof:** (i) This follows easily using Proposition 2.

(ii) Note that \( \sup_{x \in A} \{f(x) + g(x)\} \leq \sup_{x \in A} f(x) + \sup_{x \in A} g(x) \) and \( \inf_{x \in A} \{f(x) + g(x)\} \geq \inf_{x \in A} f(x) + \inf_{x \in A} g(x) \). Then the integrability of \( f + g \) is easily established. Now we show equation (2). Let \( f, g \) be integrable with integrals \( \int f\,d\mu \) and \( \int g\,d\mu \). Then by definition, for each \( \epsilon > 0 \), there exist \( P_1 \) and \( P_2 \), partitions of \( X \), such that

\[
\sum_{A \in P_1} \sup f(A)\mu(A) < \int f\,d\mu + \frac{\epsilon}{2},
\]

\[
\sum_{A \in P_2} \sup f(A)\mu(A) < \int f\,d\mu + \frac{\epsilon}{2}.
\]

Then we define a new partition \( P = P_1 \land P_2 \) which is finer than \( P_1, P_2 \), as before. Thus the above two inequalities still hold for \( A \in P \). Therefore we have

\[
\sum_{A \in P} \sup(f + g)(A)\mu(A) < \int f\,d\mu + \int g\,d\mu + \epsilon.
\]

Similarly we have

\[
\sum_{A \in P} \inf(f + g)(A)\mu(A) > \int f\,d\mu + \int g\,d\mu - \epsilon.
\]

Thus, from the Theorem 2 we are done.

**Remark.** From the previous definitions we observe the following: suppose that \( \mu \) is a finitely additive measure on an algebra \( A \) of subsets of \( X \). From the Carathéodory Extension Theorem in [29], we can extend \( \mu \) to a \( \sigma \)-algebra containing \( A \) if \( \mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) \) where \( \{A_i\} \subseteq A \) is a pairwise disjoint sequence.
of sets whose union is in $A$. Then if $f$ is a measurable function with respect to this
$\sigma$-algebra, then we can obtain the Lebesgue integral of the function $f$ (see in details
[29]). We conclude that the Riemann integral in Definition 4 in this case is exactly
same as the Lebesgue integral, since the Darboux sums are the Lebesgue integrals
of the simple functions which approximate the measurable function $f$.

2.3 Integrability of $A$-measurable Functions

In this section, we consider a special class of functions called measurable functions
with respect to an algebra $A$, and study the integrability of this class of functions.
We show that every bounded $A$-measurable function is integrable. We also obtain
a formula to compute the integral of a given $A$-measurable function.

Let $X$ be a non-empty set, and $\mu$ a finitely additive measure on $A$, where $A$ is an
algebra of subsets of $X$.

Definition 1. A function $f : X \to \mathbb{R}$ is called measurable with respect to $A$ or $f$
is an $A$-measurable function if for all $x \in \mathbb{R}$,

\[ f^{-1}(x, \infty) \in A. \]

Theorem 1. Let $\mu$ be a finitely additive measure on $X$. Let $f : X \to \mathbb{R}$ be a
bounded, measurable function with respect to $A$. Then $f$ is integrable.

Proof: First define a new function $g : [a, b] \to \mathbb{R}$, where $a = \inf_{x \in X} f(x)$, and
$b = \sup_{x \in X} f(x)$, as follows: $g(x) = \mu(f^{-1}(x, \infty))$, for each $x \in [a, b]$. Then it is easy
to see that $g(x)$ is non-increasing on $[a, b]$, because $\mu$ is a finitely additive measure.
Since every monotone function is Riemann integrable, $g(x)$ is Riemann integrable.

Let $I = \int_a^b g(x)dx$. From the definition of the Riemann integral, for each $\epsilon > 0,$
there exists a partition of \([a, b]\), say \(a = y_0 < y_1 < \ldots < y_{n-1} < y_n = b\), satisfying:

\[ I - \epsilon < \sum_{i=1}^{n} g(x_i) \Delta y_i < I + \epsilon, \]

where \(x_i \in [y_{i-1}, y_i]\). Alternately \(\sum_{i=1}^{n} \omega_i \Delta y_i < \epsilon\), where \(\Delta y_i = y_i - y_{i-1}\) and \(\omega_i = \sup_{y \in [y_{i-1}, y_i]} g(y) - \inf_{y \in [y_{i-1}, y_i]} g(y)\). Since \(g(y)\) is non-increasing, so we have \(\omega_i = \sup_{y \in [y_{i-1}, y_i]} g(y) - \inf_{y \in [y_{i-1}, y_i]} g(y) = g(y_{i-1}) - g(y_i)\). Now let \(\mathcal{P} = \{A_i : i = 0, 1, \ldots, n-1, n\}\), where \(A_0 = f^{-1}(y_0), A_i = f^{-1}(y_{i-1}, y_i], i = 1, 2, \ldots, n\). It is clear that this collection of sets is a partition of \(X\) with respect to \(\mathcal{A}\), since \(f\) is measurable and \(\mathcal{A}\) is an algebra. Note \(A_0 = f^{-1}(y_0) = f^{-1}(y_0 - \epsilon, \infty) \setminus f^{-1}(y_0, \infty) \in \mathcal{A}\). Now we consider

\[
S^u(f, \mu, \mathcal{P}) = \sum_{i=1}^{n} \sup \{f(A_i)\} \mu(A_i) + y_0 \mu(A_0) \\
\leq \sum_{i=1}^{n} y_i \mu(A_i) + y_0 \mu(A_0).
\]

Similarly,

\[
S^l(f, \mu, \mathcal{P}) = \sum_{i=1}^{n} \inf \{f(A_i)\} \mu(A_i) + y_0 \mu(A_0) \\
\geq \sum_{i=1}^{n} y_{i-1} \mu(A_i) + y_0 \mu(A_0).
\]

Then we have the following:

\[
S^u(f, \mu, \mathcal{P}) - S^l(f, \mu, \mathcal{P}) \leq \sum_{i=1}^{n} \mu(A_i) \Delta y_i \\
\leq \sum_{i=1}^{n} \mu(f^{-1}(y_{i-1}, y_i]) \Delta y_i \\
= \sum_{i=1}^{n} (\mu(f^{-1}(y_{i-1}, \infty)) - \mu(f^{-1}(y_i, \infty))) \Delta y_i \\
= \sum_{i=1}^{n} (g(y_{i-1}) - g(y_i)) \Delta y_i < \epsilon.
\]

Then from the definition of the integrability and Proposition 2 of Section 1.2, we have \(f\) is integrable on \(X\). \(\blacksquare\)
Furthermore we can obtain two more useful results from the proof in Theorem 1 above.

**Theorem 2.** Let \( f \) and \( g \) be defined as above, \( \mu \) a finitely additive measure. Then

\[
\int f(x) \, d\mu = \text{(Riemann)} \int_a^b g(x) \, dx + a\mu(X),
\]

where \( a = \inf_{x \in X} f(x) \) and \( b = \sup_{x \in X} f(x) \).

**Proof:** We preserve the notation above. Then for each \( \epsilon > 0 \), there exists a partition of \([a, b]\), such that

\[
I - \epsilon < \sum_{i=1}^{n} g(x_i) \Delta y < I + \epsilon,
\]

where \( x_i \in [y_{i-1}, y_i] \).

Let \( x_i = y_{i-1} \). Then we have

\[
\sum_{i=1}^{n} g(y_{i-1}) \Delta y_i = \sum_{i=1}^{n} g(y_{i-1})(y_i - y_{i-1})
\]

\[
= \sum_{i=1}^{n} g(y_{i-1})y_i - \sum_{i=1}^{n} g(y_{i-1})y_{i-1}
\]

\[
= \sum_{i=1}^{n} (g(y_{i-1}) - g(y_i))y_i + y_ng(y_{n-1}) - g(y_0)y_0,
\]

and

\[
I - \epsilon < \sum_{i=1}^{n} y_i\mu(A_i) + y_0\mu(A_0) - \mu(X)a < I + \epsilon,
\]

I.e.,

\[
I + \mu(X)a - \epsilon < \sum_{i=0}^{n} y_i\mu(A_i) < I + \mu(X)a + \epsilon.
\]

Thus

\[
I + a\mu(X) - \epsilon < S^n(f, \mu, \mathcal{P}) < I + a\mu(X) + \epsilon.
\]
It follows that
\[
I + a\mu(X) - \epsilon \leq \int fd\mu < I + a\mu(X) + \epsilon.
\]
Since \(\epsilon\) is arbitrary and since \(\int fd\mu = \int fd\mu\) by the previous theorem, we conclude that
\[
\int f\ d\mu = \int_a^b g\ dx + a\mu(X).
\]

**Theorem 3.** Let \(f\) and \(g\) be defined as above, \(\mu\) a finitely additive measure. If \(f \geq 0\) then
\[
\int_X f(x)\ d\mu = (\text{Riemann}) \int_{\mathbb{R}^+} g(x)\ dx.
\]

**Proof:** We can directly obtain this result by following computation:
\[
\int_{\mathbb{R}^+} g(x)dx = \int_0^a g(x)dx + \int_a^b g(x)dx + \int_b^\infty g(x)dx,
\]
where \(a = \inf_{x \in X} f(x)\) and \(b = \sup_{x \in X} f(x)\). But \(g(x) = \mu(f^{-1}(x, \infty))\), so on for each \(x \in (0, a), f^{-1}(x, \infty) = X\), hence we have \(\int_0^a g(x)dx = a\mu(X)\). On the interval \((b, \infty), f^{-1}(x, \infty) = \emptyset\). Thus \(\int_b^\infty g(x)dx = 0\). Therefore we have
\[
\int_X f(x)\ d\mu = \int_a^b g(x)\ dx + a\mu(X) = \int_{\mathbb{R}^+} g(x)\ dx.
\]

From Theorem 1, Theorem 2 and Theorem 3 above we can obtain a version of the Monotone Convergence Theorem:
Theorem 4. Let $f_n : X \to \mathbb{R}$ be a sequence of bounded functions. Let $f_n \leq f_{n+1}$ and $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in X$. Then we have

$$\lim_{n \to \infty} \int_X f_n(x) \, d\mu = \int_X \lim_{n \to \infty} f_n(x) \, d\mu.$$ 

In order to prove this theorem we need to invoke one classical result for the Riemann integration.

Lemma 1. Let $f_n(x) \leq f_{n+1}(x)$ and $\lim_{n \to \infty} f_n(x) = f(x)$, where $f_n(x)$ and $f(x)$ are both monotone. Then we have

$$\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b \lim_{n \to \infty} f_n(x) \, dx.$$ 

Proof: Since both $f_n$ and $f$ are monotone, so they are Riemann integrable. Thus for a finite partition $\mathcal{P} = \{x_i\}$ of $[a, b]$ consider

$$\sum_{i=1}^n \sup f_m(x) \Delta x_i = \sum_{i=1}^n f_m(x_i) \Delta x_i.$$ 

Because $f_m \to f$ as $m \to \infty$, then we can choose $N$ such that $m > N$ for each $x_i$ $f(x_i) < f_m(x_i) + \epsilon$. Therefore, we have

$$\sum_{i=1}^n \sup f(x) \Delta x_i - \epsilon < \sum_{i=1}^n \sup f_m(x) \Delta x_i.$$ 

But we know that

$$\sum_{i=1}^n \sup f_m(x) \Delta x_i \leq \sum_{i=1}^n \sup f(x) \Delta x_i,$$

therefore as $n, m$ go to the infinity, we obtain the result. □

Proof of Theorem 2: We can define as before $g_n(x) = \mu f_n^{-1}(x, \infty)$ for each $n$ and $g(x) = \mu f^{-1}(x, \infty)$. Then since $f_n \to f$ we have $g_n \to g$ as $n \to \infty$. Therefore by
the Lemma above and theorem before, we have
\[
\lim_{n \to \infty} \int f_n \, d\mu = \lim_{n \to \infty} \left[ \int g_n \, dx + a_n \mu(X) \right]
\]

The right hand side is equal to
\[
\int g \, dx + a \mu(X) = \int f \, d\mu.
\]
Hence we are done.

2.4 Integration on Subsets

In this section given a function \( f : X \to \mathbb{R} \) integrable with respect to an algebra \( A \) and \( E \in A \) we study integrability of \( f \) on \( E \).

First we have the following result:

**Proposition 1.** Suppose that \( f \) is \( A \)-integrable on \( X \). Let \( X \supseteq E \in A \). Then \( f \) is also \( A \)-integrable on \( E \).

**Proof:** Since \( f \) is integrable on \( X \), then for all \( \varepsilon > 0 \) there exists an \( A \)-partition \( \mathcal{P} = \{ A_i : i = 1, \ldots, n \} \) of \( X \) such that \( S^u(f, \mathcal{P}) - S^l(f, \mathcal{P}) < \varepsilon \). Consider \( \{ A_i \cap E : i = 1, \ldots, n \} \) which is \( A \)-partition of \( E \). Then the collection \( \mathcal{P}' = \{ A_i \in \mathcal{P} : A_i \cap E = \emptyset \} \cup \{ A_i \setminus E : A_i \cap E \neq \emptyset \} \cup \{ A_i \cap E : A_i \cap E \neq \emptyset \} \) is an \( A \)-partition of \( X \) which refines the partition of \( \mathcal{P} \). So by Lemma 2 of Section 1.2 we have
\[
S^u(f, \mathcal{P}') - S^l(f, \mathcal{P}') \leq S^u(f, \mathcal{P}) - S^l(f, \mathcal{P}) < \varepsilon.
\]
Hence the following is true:
\[
S^u(f, \mathcal{P}|_E) - S^l(f, \mathcal{P}|_E) < \varepsilon.
\]
Then $f$ is $\mathcal{A}$-integrable on $E$. 

From this result we can obtain a result which is analogous to the one in the Lebesgue integral.

**Proposition 2.** Let $f, g : X \to \mathbb{R}$ be bounded. Suppose that for all $\epsilon > 0$ there exists a set $A$ in $\mathcal{A}$ with $\{x \in X : f(x) \neq g(x)\} \subseteq A$ such that $\mu(A) < \epsilon$. If $f$ is $\mathcal{A}$-integrable then $g$ is $\mathcal{A}$-integrable and two integrals are equal.

**Proof:** By assumption we may pick $A \in \mathcal{A}$ such that $\mu(A) < \frac{\epsilon}{2M}$, where $|g(x)| < M$ for all $x \in X$. Since $f$ is $\mathcal{A}$-integrable on $X$, $f$ is also $\mathcal{A}$-integrable on $X \setminus A$ from preceding proposition. Then for all $\epsilon > 0$ there exists an $\mathcal{A}$-partition $\mathcal{P} = \{\mathcal{A}_i \in \mathcal{A} : i = 1, \ldots, n\}$ of $X \setminus A$ such that

$$\sum_{i=1}^{n} (\sup f(\mathcal{A}_i) - \inf f(\mathcal{A}_i))\mu(\mathcal{A}_i) < \frac{\epsilon}{2}.$$ 

But the collection $\{\mathcal{A}_i \in \mathcal{P} : i = 1, \ldots, n\} \cup \{A\}$ is an $\mathcal{A}$-partition of $X$. Then we have

$$\sum_{i=1}^{n} (\sup g(\mathcal{A}_i) - \inf g(\mathcal{A}_i))\mu(\mathcal{A}_i) + (\sup g(A) - \inf g(A))\mu(A) < \epsilon,$$

Hence $g$ is integrable on $X$.

Now let $I = \int g \, d\mu$ and $I' = \int f \, d\mu$. From the definition for all $\epsilon > 0$ there exists an $\mathcal{A}$-partition $\mathcal{P} = \{\mathcal{A}_i : i = 1, \ldots, n\}$ of $X$ such that $I < \sum_i \inf g(\mathcal{A}_i)\mu(\mathcal{A}_i) + \epsilon$. Without loss of generality assume that $\sum_i \sup f(\mathcal{A}_i)\mu(\mathcal{A}_i) < I' + \epsilon$. Then consider the collection $\{\mathcal{A}_i : \mathcal{A}_i \cap A = \emptyset\} \cup \{\mathcal{A}_i \cap A : \mathcal{A}_i \cap A \neq \emptyset\} \cup \{\mathcal{A}_i \setminus A : \mathcal{A}_i \cap A \neq \emptyset\}$. This is a partition of $X$ which refines $\{\mathcal{A}_i\}$. So by definition, we have

$$I < \sum_i \inf g(\mathcal{A}_i)\mu(\mathcal{A}_i) + \epsilon.$$
But from Lemma 2 of Section 2.2, \( \sum \inf g(A_i) \mu(A_i) \) is less than or equal to

\[
\sum_{A_i \cap A \neq \emptyset} \inf g(A_i) \mu(A_i) + \sum_{A_i \cap A \neq \emptyset} \inf g(A_i \cap A) \mu(A \cap A_i) + \\
+ \sum_{A_i \cap A \neq \emptyset} \inf g(A_i \setminus A) \mu(A \setminus A_i) = \\
\sum_{A_i \cap A = \emptyset} \inf f(A_i) \mu(A) + \sum_{A_i \cap A \neq \emptyset} \inf g(A_i \cap A) \mu(A \cap A_i) + \\
+ \sum_{A_i \cap A \neq \emptyset} \inf f(A_i \setminus A) \mu(A \setminus A_i).
\] (1)

Then the middle term above

\[
\sum_{A_i \cap A \neq \emptyset} \inf g(A_i \cap A) \mu(A \cap A_i) < \epsilon,
\]

since \( g \) is bounded and \( \mu(A) < \epsilon \). Thus (1) is less than \( \sum \sup f(A_i) \mu(A_i) \) by calculation. Therefore we have

\[
I < \sum \inf g(A_i) \mu(A_i) + \epsilon \leq \sum \sup f(A_i) \mu(A_i) + \epsilon < I' + 2\epsilon.
\]

Then \( I \leq I' \) by letting \( \epsilon \to 0 \). By interchange \( f \) and \( g \) we obtain \( I' \leq I \). Thus \( I = I' \).

By Corollary 1 of Section 2.3 we can easily obtain the following proposition:

**Proposition 3.** Let \( X = A_1 \cup A_2 \) and \( A_1, A_2 \in \mathcal{A} \) with \( A_1 \cap A_2 = \emptyset \). If \( f \) is integrable on \( X \), then \( f \) is integrable on \( A_1, A_2 \). Moreover we have

\[
\int_X f d\mu = \int_{A_1} f d\mu + \int_{A_2} f d\mu.
\]

**Proof:** From Proposition 1, \( f \) is integrable on \( A_1 \) and \( A_2 \). Without loss of generality we assume \( f \) is nonnegative since \( f = f^+ - f^- \) where \( f^+ \) and \( f^- \) are positive and negative part of \( f \) respectively. Then by Corollary 2 of Section 2.2, \( g(x) = \mu f^{-1}(x, \infty) \)
for each \( x \in \mathbb{R}_+ \). But we know that \( f^{-1}(t, \infty) = (f^{-1}(t, \infty) \cap A_1) \cup (f^{-1}(t, \infty) \cap A_2) \).

Therefore

\[
\mu f^{-1}(x, \infty) = \mu (f^{-1}(x, \infty) \cap A_1) + \mu (f^{-1}(x, \infty) \cap A_2).
\]

Thus we have

\[
\int_X f \, d\mu = \int_{\mathbb{R}_+} \mu (f^{-1}(t, \infty) \cap A_1) \, dt + \int_{\mathbb{R}_+} \mu (f^{-1}(t, \infty) \cap A_2) \, dt,
\]

i.e.,

\[
\int_X f \, d\mu = \int_{A_1} f \, d\mu + \int_{A_2} f \, d\mu.
\]
CHAPTER 3. EXTENSIONS OF THE INTEGRATION

In this chapter we extend the theory of integration developed in the last chapter. In Section 3.1, given $\mu$, a finitely additive measure defined on a semi-algebra $S$ of subsets of $X$, based on the results of previous chapter, we can extend the integral of a bounded function with respect to $\mu$. In Section 3.2, $\mu$ is a valuation defined on a lattice of subsets of $X$. By using the Smiley-Horn-Tarski Theorem, $\mu$ can be extended to a finitely additive measure on the algebra generated by the lattice. Thus we can have a variant integration theory based on this approach. In Section 3.3, $X$ is given as a topological space, and $\mu$ is a valuation defined on the lattice of open sets. Then $\mu$ is defined on the semi-algebra of crescent sets. We study the integration of a class of functions called crescent continuous functions.

3.1 Integration of $S$-measurable Functions

In this section, based on the theory we developed in the last chapter, we study the integration theory of functions $f$ with respect to a finitely additive measure on a semi-algebra.

Let $X$ be a non-empty set and $S \subseteq P(X)$, where $P(X)$ is the power set of $X$.

Definition 1. A non-empty collection $S$ of subsets of $X$ is a semi-algebra if and only if (i) $\emptyset, X \in S$; (ii) $A, B \in S$ implies $A \cap B \in S$; (iii) $A, B \in S$ implies $A \setminus B = \bigcup_{i=1}^{n} C_i$ for some $n$, where $\{C_i \in S : i = 1, \ldots, n\} \subseteq S$ is pairwise disjoint family.
Example 1. (i) Let $S = \{[c,d) : a < c < d < b, a < b\}$, and $[c,c) = \emptyset$ where $a, b \in \mathbb{R}$. Then $S$ is a semi-algebra.

(ii) Let $(X, \Omega(X))$ be a topological space. Then $\mathcal{C} = \{U \backslash V : U, V \in \Omega(X), V \subseteq U\}$ is a semi-algebra.

**Proposition 1.** Let $S$ be a semi-algebra. Then if $A \in S$ and $\{B_i \in S : i = 1, \ldots, n\}$, then $A \setminus \bigcup_{i=1}^{n} B_i$ is a finite pairwise disjoint union of subsets in $S$.

**Proof:** This can be easily obtained by induction on $n$. ■

**Proposition 2.** Let $S$ be a semi-algebra. The algebra $A$ generated by $S$ consists of all the pairwise disjoint finite unions of subsets of $S$.

**Proof:** We first note that the collection all the pairwise disjoint unions of subsets of $S$ is an algebra because the intersection of any two such sets is again a finite pairwise disjoint union of subsets of $S$. Any algebra containing $S$ clearly contains this collection. ■

**Definition 2.** Let $\mu : S \to [0, \infty)$. $\mu$ is called a finitely additive measure on $S$ if (i) $\mu(\emptyset) = 0$; (ii) For all $A, B \in S$, $A \cup B \in S$, and $A \cap B = \emptyset$ implies $\mu(A \cup B) = \mu(A) + \mu(B)$.

There is a standard result about extension of measures from a semi-algebra to an algebra [29]:

**Proposition 3.** Let $S$ be a semi-algebra of sets, $\mu$ a finitely additive measure on $S$. Then $\mu$ has a unique extension to a measure on the algebra $A$ generated by $S$. 27
if for all \( A = \bigcup_{i=1}^{\infty} B_i \), where \( A \in S \), and \( \{B_i : i = 1, \ldots, \infty\} \subseteq S \) is a pairwise disjoint family of \( S \), \( \mu(A) \leq \sum_{i=1}^{\infty} \mu(B_i) \).

Now consider a bounded function \( f : X \to \mathbb{R} \), a semi-algebra \( S \) of sets of \( X \), and a finitely additive measure \( \mu \) on \( S \) which satisfies the semiadditive condition. Then from Proposition 3, \( \mu \) can be extended to a finitely additive measure on the algebra generated by \( S \). We use the same notation \( \mu \) for the extension. Parallel to the theory in the previous chapter, we can study integration with respect to a semi-algebra.

**Definition 3.** Let \( P \subseteq S \). \( P \) is a finite \( \mu \)-partition if and only if (i) \( \bigcup_{A \in P} A = X \); (ii) for all \( A, B \in \mathcal{P} \), \( \mu(A \cap B) = 0 \) if \( A \neq B \).

By replacing the algebra \( A \) with the semi-algebra \( S \), we have the corresponding theory of integration of a bounded function \( f : X \to \mathbb{R} \), with respect to \( \mu \) on \( S \). Moreover we still have the similar results about the integrability of \( S \)-measurable functions. For convenience we call a partition \( \mathcal{P} \) a \( \mathcal{A} \)-partition if \( \mathcal{P} \subseteq \mathcal{A} \) and a \( S \)-partition if \( \mathcal{P} \subseteq S \). Similarly, we have \( \mathcal{A} \)-integrable and \( S \)-function according to the partition of \( X \) and set function \( \mu \).

**Lemma 1.** Let \( S \) be a semi-algebra of subsets on \( X \) and \( A \) be the algebra generated by \( S \). Then for each \( A \)-partition \( \mathcal{P} \) of \( X \) there exists a \( S \)-partition \( \mathcal{P}' \) of \( X \) which refines \( \mathcal{P} \).

**Proof:** Let \( \mathcal{P} = \{A_i : i = 1, \ldots, n\} \) be the \( A \) partition of \( X \). Then from Proposition 2 each \( A_i \in \mathcal{P} \) can be represented by finite disjoint union of sets in \( S \), i.e.,

\[
A_i = \bigcup_{i,j} C_{ij}, C_{ij} \in S,
\]
for each $i$ and finite $j$. Then the collection of $C_{ij}$ is a partition of $X$ with respect to $S$, which is clearly refines the $\mathcal{A}$-partition $\mathcal{P}$ of $X$.

By using this lemma we can prove the following theorem.

**Theorem 1.** Let $S$ be a semi-algebra and $\mathcal{A}$ be the algebra generated by $S$. Let $f : X \to \mathbb{R}$ be a bounded function and $\mu$ be a finitely additive measure on $\mathcal{A}$. Then $f$ is $S$-integrable if and only if $f$ is $\mathcal{A}$-integrable and in this case two integrals are equal.

**Proof:** Suppose that $f$ is $S$-integrable. Then from Proposition 3 in Section 1.2, for each $\epsilon > 0$, there exists an $S$ partition $\mathcal{P}$ of $X$ such that

$$S^u(f, \mathcal{P}, \mu) - S^l(f, \mathcal{P}, \mu) < \epsilon.$$ 

But $\mathcal{P}$ is also $\mathcal{A}$-partition. We use the proposition again to obtain that $f$ is $\mathcal{A}$-integrable.

On the other hand suppose that $f$ is $\mathcal{A}$-integrable. Then for every $\epsilon > 0$, there exists an $\mathcal{A}$-partition $\mathcal{P}$ of $X$ such that

$$S^u(f, \mathcal{P}, \mu) - S^l(f, \mathcal{P}, \mu) < \epsilon.$$ 

But from Lemma 1 above, we can find a $S$-partition $\mathcal{P}'$ of $X$ which refines $\mathcal{P}$. Thus from Proposition 2 in Section 2.2, we have

$$S^u(f, \mathcal{P}', \mu) - S^l(f, \mathcal{P}', \mu) \leq S^u(f, \mathcal{P}, \mu) - S^l(f, \mathcal{P}, \mu) < \epsilon.$$ 

Hence $f$ is $S$-integrable.
Let $I, I'$ be the $\mathcal{A}$ and $S$-integrals of $f$ on $X$ respectively. Then for each $\epsilon > 0$, there exists a $S$-partition $\mathcal{P}'$ such that

$$S^l(f, \mu, \mathcal{P}') - \epsilon < I',$$

and

$$S^u(f, \mu, \mathcal{P}') > I.$$

We can also choose $S$-partition $\mathcal{P}'$ to satisfy the Cauchy condition:

$$S^u(f, \mu, \mathcal{P}') - S^u(f, \mu, \mathcal{P}') < \epsilon.$$

Therefore

$$I - I' < S^u(f, \mu, \mathcal{P}') - (S^l(f, \mu, \mathcal{P}') - \epsilon) < 2\epsilon.$$

Thus we obtain $I = I'$ by letting $\epsilon \to 0$. ■

**Definition 4.** A function $f : X \to \mathbb{R}$ is called $S$-measurable if and only if for all $x \in \mathbb{R}$ we have $f^{-1}(x, \infty) \in S$.

It is clear that $f$ is $S$-measurable function implies that $f$ is $\mathcal{A}$-measurable function, where $\mathcal{A}$ is the algebra generated by $S$.

**Theorem 1.** Let $f$ be a bounded $S$-measurable function on $X$. Then $f$ is integrable and we have

$$\int f \, d\mu = \int_a^b g \, dx + a\mu(X),$$

where $a = \inf_{x \in X} f(x)$, $b = \sup_{x \in X} f(x)$, and $g(x) = \mu(f^{-1}(x, \infty))$.

**Proof:** We note first that $f$ is $S$-measurable implies $f$ is also $\mathcal{A}$-measurable, where $\mathcal{A}$ is the algebra generated by $S$. Then by Theorem 1 in Section 2.3, $f$ is integrable.
with respect to \( \mathcal{A} \). Then by applying Theorem 2 of Section 2.3 and the previous theorem we are done. 

\[ \square \]

3.2 Integration of \( \mathcal{L} \)-measurable Functions

In this section we study the integration of function \( f : X \to \mathbb{R} \) with respect to a valuation \( \mu \) defined on lattice of sets in \( X \). We proceed as before to obtain a similar theory of the Riemann integration.

Definition 1. Let \( \mathcal{L} \subseteq P(X) \). \( \mathcal{L} \) is called a lattice if and only if \( \mathcal{L} \) is closed under finite union and intersection.

If \( \mathcal{L} \) is given a partial order, say, \( A \leq B \) if and only if \( A \subseteq B \), then \( \mathcal{L} \) is a lattice means that for each pair of sets in \( \mathcal{L} \), there exists a least upper bound and a greatest lower bound in \( \mathcal{L} \).

Definition 2. Let \( \mathcal{L} \) be a lattice with \( \emptyset, X \in \mathcal{L} \). A function \( \mu : \mathcal{L} \to [0, \infty] \) is called a valuation if and only if \( \mu \) satisfies: (i) (strictness) \( \mu(\emptyset) = 0 \); (ii) (monotonicity) for all \( U, V \in \mathcal{L}, U \subseteq V \) implies \( \mu(U) \leq \mu(V) \); (iii) (modularity) for all \( U, V \in \mathcal{L} \),

\[ \mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V). \]

By the classical extension theorem there exists an extension of a valuation \( \mu \) on \( \mathcal{L} \) to the algebra generated by \( \mathcal{L} \): (See the details in [21])

Theorem 1. (Smiley-Horn-Tarski) Let \( \mu \) be a valuation on \( \mathcal{L} \). Then \( \mu \) can be uniquely extended to a finitely additive measure on the algebra generated by \( \mathcal{L} \).
Definition 3. Let $f : X \to \mathbb{R}$ be bounded function, $\mu$ a valuation on the lattice $\mathcal{L}$. Then $f$ is said to be integrable with respect to $\mu$ if and only if $f$ is integrable with respect to $\overline{\mu}$, the extension of $\mu$ and $\int f \, d\mu := \int f \, d\overline{\mu}$.

From this definition, we can have similar theory of integration discussed in last chapter. Here the algebra $\mathcal{A}$ is generated by a lattice $\mathcal{L}$.

Definition 4. $f$ is called $\mathcal{L}$-measurable if and only if $f^{-1}(x, \infty) \in \mathcal{L}$ for all $x \in \mathbb{R}$.

With the definitions above and results from the previous section, we have the following theorem.

Theorem 2. Let $f : X \to \mathbb{R}$ be a bounded $\mathcal{L}$-measurable function, $\mu$ a valuation on the lattice $\mathcal{L}$. Then $f$ is integrable. Moreover if $\mu$ is finite, then

$$\int f \, d\mu = \int f \, d\overline{\mu} = \int_a^b g \, dx + a\mu(X),$$

where $a = \inf_{x \in X} f(x)$, $b = \sup_{x \in X} f(x)$, and $g(x) = \mu(f^{-1}(x, \infty))$.

Proof: The proof can be carried out as the proof of Theorem 1 and Theorem 2 in Section 2.3. Here we just need to remember $\overline{\mu}$ is the extension of $\mu$ on the algebra generated by the lattice $\mathcal{L}$. ■

3.3 Integration of the Crescent Continuous Functions

In this section we study the Riemann integration of a function $f : X \to \mathbb{R}$, where $X$ is a topological space and $f$ is a crescent continuous. In a sense this is a further extension of the theory in earlier sections.
Let $X$ be a topological space and $\Omega(X)$ be the collection of all open sets of $X$. It is easy to see that $\Omega(X)$ is a lattice with the inclusion order. Hence as in the Section 2.2, we introduce the definition first.

**Definition 1.** Let $(X, \Omega(X))$ be a topological space. A crescent subset is a subset of the form $U \setminus V$ where $U, V \in \Omega(X)$ such that $V \subseteq U$. We denote the set of all crescents of $(X, \Omega(X))$ by $C(X)$.

**Example:** Let $(\mathbb{R}, \sigma)$ be the real line with the Scott topology, i.e., the topology with $\{(a, \infty) : a \in \mathbb{R}\}$ as a basis. Then the collection of subsets $(a, b]$, where $a \leq b \in \mathbb{R}$ is a crescent of $\mathbb{R}$.

The following are some properties of $C(X)$.

**Proposition 1.** (1) $C(X)$ is a semi-algebra. (2) $\Omega(X) \subseteq C(X)$. (3) Each crescent is the intersection of an open subset and a closed subset.

**Proof:** Since $(U_1 \setminus V_1) \cap (U_2 \setminus V_2) = (U_1 \cap U_2) \setminus (V_1 \cup V_2)$, we can see that $C$ is closed under finite intersections. Also for each crescent set $U \setminus V$, the complement is $(U \setminus V)^c = U^c \cup V = (X \setminus U) \cup (U \cap V) \setminus \emptyset$, which is the disjoint union of crescent sets. By the definition of the semi-algebra, $C(X)$ is a semi-algebra. We can obtain (2) and (3) directly by the definition. ■

**Proposition 2.** $C(X)$ is a subbasis for a topology, which is called the crescent topology.

**Proof:** Consider the collection $C(X)$ of crescent subsets of $X$. Then from above, we know that $\emptyset \in C(X)$ and $X \in C(X)$. Also $(U_1 \setminus V_1) \cap (U_2 \setminus V_2) = (U_1 \cap U_2) \setminus (V_1 \cup V_2)$. 

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Then by the standard construction from a collection of subsets, we can obtain a topology which contains $\emptyset, X$ and all finite intersections of crescents as a basis. It is not difficult to see that the crescent topology is finer that the original topology, since each open subset is a crescent. ■

Definition 2. Suppose that $f : X \to \mathbb{R}$. Let $x \in X$. If for each $(a, b) \subset X$ such that $f(x) \in (a, b)$, there exists $\{U_i, V_i : i = 1, \ldots, n\} \subset \Omega(X)$, such that $f^{-1}(a, b) = \bigcup_{i=1}^{n} (U_i \setminus V_i)$, then $f$ is said to be crescent continuous at $x$. $f$ is crescent continuous on $X$ if $f$ is crescent continuous at each point of $x$.

Recall that $f : X \to \mathbb{R}$ is Scott continuous if it is continuous with respect to the Scott topology on $\mathbb{R}$, i.e., with $\{(a, \infty) : a \in \mathbb{R}\}$ as a basis.

Proposition 3. $f$ is Scott continuous implies $f$ is crescent continuous.

Proof: Let $f$ be Scott continuous at $x$. Then for each $(a, b) \subset \mathbb{R}$ with $f(x) \in (a, b)$. We can define $U = f^{-1}(a, \infty)$ and $V = f^{-1}(b, \infty)$. Since $f$ is Scott continuous, $U$ and $V$ are open subsets of $X$. It is clear that $f^{-1}(a, b) = f^{-1}(a, \infty) \setminus f^{-1}(b, \infty) = U \setminus V$. By definition, $f$ is crescent continous. ■

In order to study the integrability of the crescent continuous functions, we need the following lemma.

Lemma 2. Let $X$ be a set, and let $S$ be a semi-algebra of subsets of $X$. Then for each finite cover of $X$ by elements of $S$, there exists a refinement of this cover which is also a finite partition.
Proof: Let \( C \subseteq S \) be a cover of \( X \), \( C = \{ C_i : i = 1, \ldots, m \} \). Then we construct a partition from \( C \) as follows: let \( A_1 = C_1 \), and \( A_2 = C_2 \setminus A_1 \). Since \( S \) is a semialgebra, then \( A_2 = \bigcup_i B_i \), where \( \{ B_i \in S \} \) is finite pairwise disjoint collection of subsets. Then we have some disjoint subsets from \( S \), say \( \{ A_i : i = 1, \ldots, k \} \). Next let \( A_{k+1} = C_3 \setminus (\bigcup_1^k A_i) \). Then from the Lemma 1, we know that \( A_{k+1} \) is a finite pairwise disjoint union of subsets in \( S \). We add these to the previous \( A_i \)'s. This process terminates because we have a finite cover. Hence we construct a finite partition of \( X \) from a finite cover of \( X \). 

Theorem 1. Let \( f : X \to \mathbb{R} \) be bounded and crescent continuous. Then \( f \) is integrable. Furthermore, if \( \mu \) is finite, then the integration can be computed by the following formula

\[
\int f \ d\mu = \int_a^b g \ dx + a\mu(X),
\]

where \( a = \inf_{x \in X} f(x) \), \( b = \sup_{x \in X} f(x) \), and \( g(x) = \mu(f^{-1}(x, \infty)) \).

Proof: We can show this along the lines of the proof of Theorem 1 and Theorem 2 in the Section 3.1.
CHAPTER 4. ORDER-PRESERVING FUNCTIONS

In this chapter, the integration of order-preserving functions is studied. Given a dcpo $D$, let $f : D \rightarrow \mathbb{R}$ be a bounded, order-preserving function and $\mu$ a valuation on the lattice of the open subsets of $D$ with the Scott topology. By the results of Chapter 3 we can define the integral of $f$ on $D$ with respect to $\mu$. Furthermore we obtain a method to compute the integral of $f$ effectively in the sense that the integral of $f$ with respect to $\mu$ is the supremum of finite sums. In Section 4.1 we study integration of functions with respect to simple valuations and give a method to compute the integrals. In Section 4.2 the integration of order-preserving functions with respect to valuations is discussed.

4.1 Integrals with Respect to Simple Valuations

4.1.1 Integrability with Respect to Simple Valuations

Let $X$ be a topological space. Let $\mu = \eta_a : \Omega X \rightarrow \mathbb{R}_+$ be defined as $\eta_a(O) = 1$ if $a \in O$, $\eta_a(O) = 0$ if $a \notin O$ where $a \in X$, $\eta_a$ is called point valuation, and $\Omega X$ is the lattice of open sets of $X$. Let $S$ be the semi-algebra of crescents of open sets of space $X$.

**Proposition 1.** If $f$ is continuous at a point $a$ then $f$ is integrable with respect to $\eta_a$, where $\eta_a$ is a point valuation at $a$.

**Proof:** Since $f$ is continuous at $a$ then from the definition, for every $\epsilon > 0$ there exists an open set $U$ with $a \in U$ such that $|f(x) - f(a)| < \epsilon$ for all $x \in U$. 

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Therefore we have \( \sup f(U) - \inf f(U) < \epsilon \). Since \( U \in S, X \setminus U = \bigcup_{i=1}^{m} A_i \), where \( \{A_i : i = 1, \ldots, m\} \) is finite collection of pairwise disjoint sets in \( S \). Then we choose a partition \( \mathcal{P} \) of \( X \) as \( \{U\} \cup \{A_i : i = 1, \ldots, m\} \). Then \( S^u(f, \mathcal{P}, \eta_a) - S^l(f, \mathcal{P}, \eta_a) \) becomes

\[
\sum_{i=1}^{m} (\sup f(A_i) - \inf f(A_i)) \eta_a(A_i) + (\sup f(U) - \inf f(U)) \eta_a(U);
\]

this is equal to

\[
(\sup f(U) - \inf f(U)) \eta_a(U) = \sup f(U) - \inf f(U) < \epsilon.
\]

Thus by the Cauchy condition \( f \) is integrable.

\[\blacksquare\]

**Corollary.** If \( f \) is continuous on the support \( |\mu| \) of \( \mu \), where \( \mu = \sum_{a \in |\mu|} r_a \eta_a \) and is called simple valuation, then \( f \) is integrable with respect to \( \mu \).

**Proof:** It follows directly by induction on the cardinality of the support of \( \mu \).

\[\blacksquare\]

**Definition 1.** A function \( f : X \to \mathbb{R} \) is \( S \)-continuous at \( a \in X \) if for each \( \epsilon > 0 \) there exists \( A \in S \) with \( a \in A \) such that \( \sup f(A) - \inf f(A) < \epsilon \). Here \( S \) is the semi-algebra of the crescents of open sets in \( X \).

It is easy to see that if \( f \) is continuous at \( a \) then \( f \) is \( S \)-continuous at \( a \).

**Proposition 2.** Let \( f \) be \( S \)-continuous on \( X \), then \( f \) is integrable with respect to any simple valuation.

**Proof:** Let \( \mu = \sum_{a \in |\mu|} r_a \eta_a \) be a simple valuation. Then since \( f \) is \( S \)-continuous, then for each \( a \in |\mu| \) there exists \( A_a \in S \) such that \( \sup f(A_a) - \inf f(A_a) < \epsilon \). Then
we choose \(\{A_a : a \in |\mu|\}\) and \(X \setminus \bigcup A_a = \bigcup B_j\). Then we have a partition of \(X\). Since \(|\mu|\) is finite, the Cauchy condition is satisfied.

4.1.2 Integrals with Respect to Simple Valuations

In this section let \(f : X \to \mathbb{R}\) be a function where \(X\) is a topological space. Let \(\eta_a\) be a point valuation on \(X\), where \(a \in X\).

**Proposition 1.** If \(f\) is integrable with respect to \(\eta_a\), then \(\int f \, d\eta_a = f(a)\).

**Proof:** Since \(f\) is integrable, let \(I\) be its integral. Then from the definition, for all \(\epsilon > 0\), there exists \(\mathcal{P}_1 = \{A_i : i = 1, \ldots, n\}\) as a partition of \(X\) with the crescents, such that

\[
|I - \sum_i f(x_i)\eta_a(U_i \setminus V_i)| < \epsilon,
\]

for all finer partition \(\{U_i \setminus V_i\}\) than \(\mathcal{P}_1\) and all \(x_i \in U_i \setminus V_i\). But \(\sum_i f(x_i)\eta_a(U_i \setminus V_i) = f(x_a)\), where \(x_a \in U_a \setminus V_a\) and \(a \in U_a \setminus V_a\). We choose \(x_a = a\); then we have \(|I - f(a)| < \epsilon\). Since \(\epsilon\) is arbitrary, we have \(I = f(a)\). 

We can generalize this result to simple valuations.

**Proposition 2.** Let \(f : X \to \mathbb{R}\) be integrable with respect to a simple valuation \(\mu = \sum_{b \in |\mu|} r_b \eta_b\), where \(X\) is a topological space. Then

\[
\int f \, d\mu = \sum_{b \in |\mu|} r_b f(b).
\]

**Proof:** We prove this by an argument similar to the above. Let \(I\) be the integral of the function \(f\). Then for each \(\epsilon > 0\), there exists a partition \(\mathcal{P}_0\) of \(X\), with respect to the crescents of the topology, such that \(|I - \sum_i f(x_i)\mu(A_i)| < \epsilon\), for all the finer
partitions $\mathcal{P} = \{A_i : i = 1, \ldots, n\}$ and all $x_i \in A_i$. There also exists a partition $\mathcal{P}_1$ such that different members of $|\mu|$ belong to different member of $\mathcal{P}_1$. Hence

$$\sum_i f(x_i) \mu(A_i) = \sum_i f(x_i) \sum_{b \in |\mu|} r_b \eta_b(A_i)$$

$$= \sum_{b \in |\mu|} \sum_i f(x_i) r_b \eta_b(A_i)$$

$$= \sum_{b \in |\mu|} r_b f(x_b) \eta_b(A_b)$$

$$= \sum_{b \in |\mu|} r_b f(x_b),$$

where $x_b \in A_b \cap \{x_1, \ldots, x_n\}$. Hence if we choose $x_b = b$, then we have

$$|I - \sum_{b \in |\mu|} r_b f(b)| < \epsilon.$$ 

Thus by letting $\epsilon \to 0$, we are done. ■

4.2 Integrals of the Order-Preserving Functions

4.2.1 Basics of the Domain Theory

In this subsection the basic concepts of the dcpo and continuous domain theory are presented. One can see [1] for details.

Definition 1. A partial order on $P$ is a relation $\subseteq$ on $P$ satisfies: (i) $a \subseteq a$ for each $a \in P$; (ii) $a \subseteq b$ and $b \subseteq a$ $\Rightarrow$ $a = b$; (iii) $a \subseteq b$ and $b \subseteq c$ $\Rightarrow$ $a \subseteq c$.

Definition 2. Let $(P, \subseteq)$ be a partial order set or poset. A subset $A$ of $P$ is directed if for all $a, b \in A$ there exists $c \in A$ such that $a \subseteq c$ and $b \subseteq c$.

Definition 3. Let $(P, \subseteq)$ be a partial order set or poset. $P$ is a directed complete partial order set (dcpo) if for every directed subset $A \subseteq P$ has a least upper bound (lub), denoted by $\bigcup^\uparrow A$.
We always assume that directed sets are nonempty. If the empty set is also required to have a supremum, then $P$ must have a least or bottom element, denoted by $\bot$. A pointed dcpo is one which has a bottom element.

**Definition 4.** Let $P$ be a partially ordered set. For $x \leq y \in P$, we say that $x$ approximates $y$ or $x$ is way-below $y$, denoted by $x \ll y$, if for every $D$-directed subset of $P$ with $y \subseteq \bigcup^\uparrow D$, then there exists some element $d \in D$ such that $x \subseteq d$.

**Definition 5.** A continuous poset is a partially ordered set $P$ in which each element is the directed supremum of all elements which approximate it, i.e., for all $x \in P$

$$x = \bigsqcup \{y \in P : y \ll x\}.$$ 

A continuous poset which is also a dcpo is called a continuous domain.

The following is a list of basic properties of this approximation relation [1]:

**Proposition 1.** Let $D$ be a continuous domain. The following properties hold:
(i) $x \ll y$ implies $x \subseteq y$; (ii) $x \subseteq y \ll u \subseteq v$ implies $x \ll v$; (iii) (interpolation property) $x \ll z$ implies that there exists $y \in D$ such that $x \ll y \ll z$.

**Definition 6.** The Scott topology on $P$ is a pair $(P, \tau)$, where $\tau$ is the collection of the open subsets of $P$, which is defined as follows: (1) for each $O \in \tau$, $O$ is upward closed, i.e. $x \in O$ and $x \subseteq y \Rightarrow y \in O$; (2) $O$ is inaccessible by lubs of directed subsets of $P$, i.e. if $A$ is a directed subset with $\bigsqcup^\uparrow A$, then $\bigsqcup^\uparrow A \in O$ implies there exists $a \in A$, such that $a \in O$.

Dually, a subset $A$ is Scott closed if (i) $A$ is lower subset, i.e., $x \in A$ and $y \subseteq x$ implies $y \in A$; (ii) if $D \subseteq A$ is directed, then $\bigsqcup^\uparrow D \in A$. 

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Definition 7. Given a topology on a dcpo $P$, a directed set $D$ is said to converge to $x \in P$ if given any open set $U$ containing $x$, there exists $b \in D$ such that $d \in U$ if $b \subseteq d$.

From this definition the following result is easily proved.

Proposition 2. Let $P$ be a dcpo. A directed set $D$ converges to $x \in P$ in the Scott topology if and only if $x \subseteq \bigcup^\uparrow D$.

Proposition 3. The Scott topology is $T_0$-space.

Proof: Suppose that $x \neq y$. Then $x \not\leq y$ or $y \not\leq x$. Without loss of generality, suppose that $x \not\leq y$. Then $x \not\in \downarrow y$, i.e., $x \in (\downarrow y)^c$. But $(\downarrow y)$ is Scott closed. Hence there is a Scott open subset which contains $x$ but not $y$. Therefore $(P, \tau)$ is a $T_0$-space.

Definition 8. A subset $B \subseteq P$ is a basis for $P$ if for each $x \in P$ the set $A$ of elements of $B$ which approximate $x$ is directed and $x = \bigcup^\uparrow A$, i.e., $x = \bigcup^\uparrow \{y : y \ll x\} \cap B$.

Remark: Sometimes we use the notation $\downarrow x = \{y : y \ll x\}$, and $\uparrow x = \{y : x \ll y\}$. There is an equivalent definition of a continuous poset as follows:

Definition 9. A poset is continuous if it has a basis; it is an $\omega$-continuous if it has a countable basis.

In a continuous domain there are close connections between the Scott open sets and the approximation relation.
Proposition 4. Let $D$ be a continuous domain equipped with the Scott topology.

(i) A subset $U$ of $D$ is open if and only if $U$ is upper set and for each $y \in U$
there exists $x \in U$ such that $x \ll y$.

(ii) $x \ll y$ if and only if $\uparrow x$ is a neighborhood of $y$.

(iii) The collection of subset $\{\uparrow b : b \in B\}$ forms a basis of the Scott topology.

(iv) The Scott topology has a countable base if and only if $D$ is $\omega$-continuous.

In this case, if $B$ is a countable base for $D$, then $\uparrow b, b \in B$ is a countable base for the Scott topology.

Proposition 5. A function $f : (D, \subseteq) \to (E, \subseteq)$ between two dcpos is continuous
with respect to the Scott-topologies on $D$ and $E$ if and only if it is monotone and
preserves suprema of directed subsets: $x \subseteq y \Rightarrow f(x) \subseteq f(y)$ and $f(\bigcup \uparrow(A)) = \bigcup \uparrow f(A)$, for each directed subset $A$ of $D$.

4.2.2 Integrals of the Order-Preserving Functions

Let $D$ be a dcpo with the Scott topology.

Definition 1. A function $f : D \to \mathbb{R}$ is called order-preserving if and if $x \subseteq y$
implies $f(x) \leq f(y)$ for all $x, y \in D$.

From this definition we can prove the following lemma:

Lemma 1. Suppose that $f : D \to \mathbb{R}$ is an order-preserving function. $\mu_1$, and $\mu_2$
are two simple valuations on $D$, with $\mu_1 \subseteq \mu_2$. Then we have

$$S^t(f, \mu_1, \mathcal{P}_1) \leq S^u(f, \mu_2, \mathcal{P}_2),$$
where $\mathcal{P}_1, \mathcal{P}_2$ are two partitions of $D$, where $S^l(f, \mu_1, \mathcal{P}_1)$ and $S^u(f, \mu_2, \mathcal{P}_2)$ are the lower sum and upper sum respectively.

**Proof:** Consider the definition of the Darboux upper sum and lower sum of $f$ with respect to the valuations: $\mu_1 = \sum_{b \in |\mu_1|} r_b \eta_b$ and $\mu_2 = \sum_{b \in |\mu_2|} s_b \eta_b$. For the simple valuation $\mu_1$, let $\mathcal{P}_1$ be a partition of $D$, which is subcollection of some algebra of subsets of $D$, say $\{A_i : i = 1, \ldots, n\}$. Then

$$S^l(f, \mu_1, \mathcal{P}_1) = \sum_{i=1}^{n} \inf f(A_i) \mu_1(A_i)$$

$$= \sum_{i=1}^{n} \inf f(A_i) \sum_{b \in |\mu_1|} r_b \eta_b(A_i)$$

$$= \sum_{b \in |\mu_1|} r_b \sum_{i=1}^{n} \inf f(A_i) \eta_b(A_i)$$

$$= \sum_{b \in |\mu_1|} r_b \inf f(A_b) \eta_b(A_b)$$

$$= \sum_{b \in |\mu_1|} r_b \inf f(A_b),$$

where $A_b$ is one of the partition sets such that $b \in A_b$. Similarly, corresponding to $\mu_2$ let $\mathcal{P}_2 = \{B_j : j = 1, \ldots, m\}$ be a partition of $D$, related to the algebra of subsets of $D$. Then we have the following

$$S^u(f, \mathcal{P}_2, \mu_2) = \sum_{c \in |\mu_2|} s_c \sup f(B_c),$$

where $B_c$ is one of the partition sets $\mathcal{P}_2 = \{B_j : j = 1, \ldots, m\}$ such that $c \in B_c$. Now let $\mathcal{P}_3 = \mathcal{P}_1 \wedge \mathcal{P}_2$, which is the refinement of $\mathcal{P}_1$, and $\mathcal{P}_2$. Let $\mathcal{P}_3 = \{C_k\}$. Similar to the computation above, and the property of Darboux upper and lower sum, we have the following

$$S^l(f, \mu_1, \mathcal{P}_1) \leq S^l(f, \mu_1, \mathcal{P}_3),$$

(1)
Thus,

\[ S^l(f, \mu_1, P_1) = \sum_{b \in |\mu_1|} r_b \inf f(A_b) \leq \sum_{b \in |\mu_1|} r_b f(b). \]  

(3)

But since \( f \) is order-preserving function, we have \( f(b) \leq f(c) \) if \( b \subseteq c \). On the other hand, by the Splitting Lemma (see [19]), there exists \( t_{b,c} \geq 0 \), such that \( r_b = \sum_{c \in |\mu_2|} t_{b,c} \) and \( \sum_{b \in |\mu_1|} t_{b,c} \leq s_c \), and for \( t_{b,c} > 0 \) if \( b \subseteq c \). Therefore from equation (1)

\[
\sum_{b \in |\mu_1|} r_b f(b) = \sum_{b \in |\mu_1|} \sum_{c \in |\mu_2|} t_{b,c} f(b) \\
= \sum_{c \in |\mu_2|} \sum_{b \in |\mu_1|} t_{b,c} f(b) \\
\leq \sum_{c \in |\mu_2|} f(c) \sum_{b \in |\mu_1|} t_{b,c} \\
\leq \sum_{c \in |\mu_2|} f(c)s_c.
\]

But we know that \( \sum_{c \in |\mu_2|} f(c)s_c \leq \sum_{c \in |\mu_2|} s_c \sup f(C_c) \), where \( C_c \in \{C_k\}\) such that \( c \in C_c \). Hence we obtain the following:

\[ S^l(f, \mu_1, P_3) \leq \sum_{b \in |\mu_1|} r_b f(b) \leq \sum_{c \in |\mu_2|} f(c) \leq S^u(f, \mu_2, P_3). \]  

(4)

Thus, combining with equations (1) and (2), we have

\[ S^l(f, \mu_1, P_1) \leq S^u(f, \mu_2, P_2). \]

Corollary. If \( f : D \to \mathbb{R} \) is order-preserving, and integrable with respect to \( \mu_1 \) and \( \mu_2 \), where \( \mu_1, \mu_2 \) are two simple valuations satisfying \( \mu_1 \leq \mu_2 \), then

\[ \int f \, d\mu_1 \leq \int f \, d\mu_2. \]
Proof: This can be done directly from the definition of the Riemann-like integral for \( f \): since \( f \) is integrable with respect to \( \mu_1, \mu_2 \), then for each \( \epsilon > 0 \), there exist partitions \( \mathcal{P}_1, \mathcal{P}_2 \) of \( D \), such that

\[
\int f \, d\mu_1 - \epsilon < S^t(f, \mu_1, \mathcal{P}_1),
\]

\[
\int f \, d\mu_2 + \epsilon > S^u(f, \mu_2, \mathcal{P}_2).
\]

From Lemma 1 we have

\[
\int f \, d\mu_1 - \epsilon < \int f \, d\mu_2 + \epsilon.
\]

Thus,

\[
\int f \, d\mu_1 < \int f \, d\mu_2 + 2\epsilon.
\]

By letting \( \epsilon \to 0 \), we are done. \( \square \)

From the recent results of the extension of continuous valuations by Alvarez-Manilla, Edalat, and Djahromi [27], we consider the valuation on the lattice of open sets of a continuous domain \( D \). This valuation can be extended to a measure \( \sigma \)-algebra generated by the open sets of the Scott topology. With these results, we can prove the following theorem.

Theorem 1. Let \( \mu \) be a continuous valuation on a dcpo \( D \) and \( \mu = \bigsqcup \uparrow \eta_i \), where \( \{\eta_i\} \) is a directed family of simple valuations on \( D \). If \( f : D \to \mathbb{R} \) is integrable with respect to \( \mu \) and \( \{\eta_i\} \) for each \( i \), and \( f \) is a bounded order-preserving, then

\[
\int f \, d\mu = \bigsqcup \int f \, d\eta_i.
\]
Proof: Notice that \( \{ \int f \, d\eta_i \} \) is directed from the Lemma 1 and Corollary above. We need to establish two side inequalities. Since \( f \) is integrable with respect to \( \mu \) and \( \eta_i \) for each \( i \). Fix some index \( i_0 \). Then from the definition of integral, for all \( \epsilon > 0 \), there exist \( P_1, P_2 \), partitions of \( D \) such that

\[
\int f \, d\eta_{i_0} - \epsilon < S_l(f, \eta_{i_0}, P_1) \leq \int f \, d\eta_{i_0} \leq S^u(f, \eta_{i_0}, P_1) < \int f \, d\eta_{i_0} + \epsilon.
\]

Similarly,

\[
\int f \, d\mu - \epsilon < S_l(f, \mu, P_2) \leq \int f \, d\mu \leq S^u(f, \mu, P_2) < \int f \, d\mu + \epsilon.
\]

Then define \( P = P_1 \land P_2 \), the refinement of \( P_1, P_2 \). Then

\[
S^u(f, \mu, P) = \sum_{i=1}^{n} \sup f(U_i \setminus V_i)(\mu(U_i) - \mu(V_i)),
\]

where \( \{U_i \setminus V_i \in P : i = 1, \ldots, n\} \). Since we know that \( \mu(V_i) = \bigsqcup \eta_i(V_i) \) and \( \mu(U_i) = \bigsqcup \eta_i(U_i) \), there exists \( k_i \) for each \( i \), such that

\[
\mu(U_i) < \eta_{k_i}(U_i) + \frac{\epsilon}{Mn},
\]

where \( |f(x)| \leq M \). Thus we obtain a finite set of simple valuations \( \{ \eta_{k_i} : i = 1, \ldots, n\} \). By the directedness of the simple valuation family, there exists \( \eta_k \) such that \( \eta_k \geq \eta_{k_i} \) for \( i = 1, \ldots, n \), and \( \eta_k \geq \eta_{i_0} \). Let \( I' = \{ i : \sup f(U_i \setminus V_i) \geq 0 \} \), and \( I'' = \{ i : \sup f(U_i \setminus V_i) < 0 \} \). Then we have the following:
\[
\sum \sup_{i} f(U_i \setminus V_i)(\mu(U_i) - \mu(V_i)) = \sum_{i \in I'} + \sum_{i \in I''} \sup_{i} f(U_i \setminus V_i)(\mu(U_i) - \mu(V_i)) \\
\leq \sum_{i \in I'} \sup_{i} f(U_i \setminus V_i)(\eta_k(U_i) + \frac{\epsilon}{Mn} - \eta_k(V_i)) \\
+ \sum_{i \in I''} \sup_{i} f(U_i \setminus V_i)(\eta_k(U_i) - \frac{\epsilon}{Mn} - \eta_k(V_i)) \\
< \sum_{i} \sup_{i} f(U_i \setminus V_i)(\eta_k(U_i) - \eta_k(V_i)) + \epsilon \\
= S^u(f, \eta_k, \mathcal{P}) + \epsilon.
\]

Then from the inequalities above we have

\[
\int f \, d\mu - \epsilon < S^u(f, \mu, \mathcal{P}_3) < S^u(f, \eta_k, \mathcal{P}_3) + \epsilon < \int f \, d\eta_k + 2\epsilon,
\]

thus \( \int f \, d\mu < \int f \, d\eta_k + 3\epsilon \). Hence we have one side inequality.

Similarly, we can show the other direction of inequality.

\[
S^u(f, \mu, \mathcal{P}) = \sum \sup_{i} f(U_i \setminus V_i)(\mu(U_i) - \mu(V_i)) \\
= \sum_{i \in I'} + \sum_{i \in I''} \sup_{i} f(U_i \setminus V_i)(\mu(U_i) - \mu(V_i)) \\
> \sum_{i \in I'} \sup_{i} f(U_i \setminus V_i)(\eta_k(U_i) - \eta_k(V_i)) - \frac{\epsilon}{Mn} \\
+ \sum_{i \in I''} \sup_{i} f(U_i \setminus V_i)(\eta_k(U_i) - \eta_k(V_i)) + \frac{\epsilon}{Mn} \\
> \sum_{i} \sup_{i} f(U_i \setminus V_i)(\eta_k(U_i) - \eta_k(V_i)) - \epsilon \\
\geq \sum_{i} \inf_{i} f(U_i \setminus V_i)(\eta_{i_0}(U_i) - \eta_{i_0}(V_i)) - \epsilon \\
> \int f \, d\eta_{i_0} - 2\epsilon.
\]

Hence we have \( \int f \, d\mu + \epsilon > \int f \, d\eta_{i_0} - 2\epsilon \). So by letting \( \epsilon \to 0 \), we have \( \int f \, d\mu \geq \int f \, d\eta_{i_0} \). Since \( i_0 \) is chosen arbitrarily, we have showed the other part of inequality.

\[\blacksquare\]
Proposition 1. Let $\mu_1$ and $\mu_2$ be two continuous valuations on a continuous domain $D$ with $\mu_1 \leq \mu_2$. $\mu_1 = \bigcup \eta_{1i}$, $\mu_2 = \bigcup \eta_{2i}$, where $\{\eta_{1i}\}$ and $\{\eta_{1i}\}$ are families of simple valuations, which approximate $\mu_1$ and $\mu_2$ respectively. If $f : D \to \mathbb{R}$ is integrable with respect to all these valuations and order-preserving, then we have

$$\int f \, d\mu_1 \leq \int f \, d\mu_2.$$ 

**Proof:** Since we know that $\mu_1 \leq \mu_2$. Then for each $i$, $\eta_{1i} \ll \mu_1 \leq \mu_2 = \bigcup \eta_{2i}$. This implies that $\eta_{1i} \ll \mu_2$. Then from the definition of approximation, for each $i$ there exists $k(i)$ such that $\eta_{1i} \leq \eta_{2k(i)}$. Hence by the Lemma 1, we have

$$\int f \, d\eta_{1i} \leq \int f \, d\eta_{2k(i)} \leq \int f \, d\mu_2,$$

for each $i$. Hence again by using Lemma 1 and Theorem 1, we are done. ■

The following result is more general than the results above.

**Theorem 2.** Let $f : D \to \mathbb{R}$ be order-preserving, bounded, and integrable with respect to $\mu$ and $\{\mu_i : i \in J\}$, where $J$ is an index set, and $\mu_i$ is simple valuation for each $i$ with $\mu = \bigcup \mu_i$. Then for all $\epsilon > 0$ there exist a partition $\mathcal{P}$ of $D$ with the crescents of the Scott topology and an index $k$ such that for all $l \geq k$

$$\int f \, d\mu - \epsilon < S^l(f, \mu, \mathcal{P}) \leq S^u(f, \mu, \mathcal{P}) < \int f \, d\mu + \epsilon.$$ 

**Proof:** Since $f$ is integrable, then for each $\epsilon > 0$, there exists $\mathcal{P}$-a partition of $D$ such that

$$\int f \, d\mu - \epsilon < S^l(f, \mu, \mathcal{P}) \leq S^u(f, \mu, \mathcal{P}) < \int f \, d\mu + \epsilon.$$  \hspace{1cm} (1)
From the definition of the Darboux sums, we have:

$$S^u(f, \mu, \mathcal{P}) = \sum_{i=1}^{n} \sup_{U_i \setminus V_i}(\mu(U_i) - \mu(V_i)),$$

and similarly

$$S^l(f, \mu, \mathcal{P}) = \sum_{i=1}^{n} \inf_{U_i \setminus V_i}(\mu(U_i) - \mu(V_i)),$$

where $\mathcal{P} = \{U_i \setminus V_i : i = 1, \ldots, n\}$. For each $i$, we have $\mu(V_i) = \bigcup^+ \mu_i(V_i)$ and $\mu(U_i) = \bigcup^+ \mu_i(U_i)$. Then there exists $k_i$ for each $i$ such that

$$\mu(U_i) < \mu_{k_i}(U_i) + \frac{\epsilon}{Mn},$$

$$\mu(V_i) < \mu_{k_i}(V_i) + \frac{\epsilon}{Mn},$$

where $M$ is a bound of $f(x)$. Thus we obtain a finite set of simple valuations $\{\mu_{k_i} : i = 1, \ldots, n\}$. By the directedness of the simple valuation family, there exists $\mu_k$ such that $\mu_k \geq \mu_{k_i}$ for $i = 1, \ldots, n$. Then, we have

$$\mu(U_i) < \mu_l(U_i) + \frac{\epsilon}{Mn},$$

$$\mu(V_i) < \mu_l(V_i) + \frac{\epsilon}{Mn},$$

for $l \geq k$. Let $I' = \{i : \inf_{U_i \setminus V_i} f(U_i) \geq 0\}$, and $I'' = \{i : \inf_{U_i \setminus V_i} f(U_i) < 0\}$. Then we
have the following:

\[
S^l(f, \mu, \mathcal{P}) = \sum_{i=1}^{n} \inf_{i \in I'} f(U_i \setminus V_i)(\mu(U_i) - \mu(V_i)) + \sum_{i \in I''} \inf_{i \in I'} f(U_i \setminus V_i)(\mu(U_i) - \mu(V_i)) \\
\leq \sum_{i=1}^{n} \inf_{i \in I'} f(U_i \setminus V_i)(\mu(U_i) + \frac{\epsilon}{M} - \mu(V_i)) + \sum_{i \in I''} \inf_{i \in I'} f(U_i \setminus V_i)(\mu(U_i) - \frac{\epsilon}{M} - \mu(V_i)) \\
\leq \sum_{i=1}^{n} \inf_{i \in I'} f(U_i \setminus V_i)(\mu(U_i) - \mu(V_i) + \epsilon) + \sum_{i \in I''} \inf_{i \in I'} f(U_i \setminus V_i)(\mu(U_i) - \mu(V_i)) + \epsilon \\
= S^l(f, \mu, \mathcal{P}) + \epsilon.
\]

Hence we have

\[
S^l(f, \mu, \mathcal{P}) \leq S^l(f, \mu, \mathcal{P}) + \epsilon,
\]

i.e.,

\[
S^l(f, \mu, \mathcal{P}) \leq S^l(f, \mu, \mathcal{P}).
\]

Next similar to the above, we can compute as follows:

\[
S^u(f, \mu, \mathcal{P}) = \sum_{i=1}^{n} \sup_{i \in I'} f(U_i \setminus V_i)(\mu(U_i) - \mu(V_i)) + \sum_{i \in I''} \sup_{i \in I'} f(U_i \setminus V_i)(\mu(U_i) - \mu(V_i)) \\
= (\sum_{i \in I'} + \sum_{i \in I''}) \sup_{i \in I'} f(U_i \setminus V_i)(\mu(U_i) - \mu(V_i)) \\
\geq \sum_{i \in I'} \sup_{i \in I'} f(U_i \setminus V_i)(\mu(U_i) - \frac{\epsilon}{nM} - \mu(V_i)) + \sum_{i \in I''} \sup_{i \in I'} f(U_i \setminus V_i)(\mu(U_i) + \frac{\epsilon}{nM} - \mu(V_i)) \\
\geq \sum_{i=1}^{n} \sup_{i \in I'} f(U_i \setminus V_i)(\mu(U_i) - \mu(V_i)) - \epsilon.
\]
Hence we have

\[ S^u(f, \mu_1, \mathcal{P}) \leq S^u(f, \mu, \mathcal{P}). \]

Then, together with (1) we are done.
CHAPTER 5. ON MAXIMAL POINT SPACES

In this chapter we first introduce the concept of a maximal point space defined by J. Lawson in his papers [25, 26]. Then we introduce briefly an integration theory developed by A. Edalat [7, 14]. When $X$ is given as a locally compact separable metric space, $X$ can be embedded in its upper space $UX$ which turns out to be an $\omega$-continuous domain. To study the integration of $f : X \rightarrow \mathbb{R}$ with respect to a Borel measure we can first extend $f$ to $\hat{f}$ on $UX$ and $\mu$ to $\bar{\mu}$, then study the integration of $\hat{f}$ with respect to $\bar{\mu}$ on $UX$. We prove that if a Scott continuous function $f$ is integrable as defined by Edalat, then $f$ is integrable in the sense we defined in Chapter 2 and in this case two the integrals are equal.

5.1 Maximal Point Space

In this section we introduce the notion of space of maximal points defined by J. Lawson in his papers [25, 26]. We collect some properties of maximal point spaces. We refer details to his papers.

Definition 1. A separable metric space $X$ is called a maximal point space if there exists an $\omega$-continuous domain $P$ satisfying the condition

$$p \in P \Rightarrow \exists A \text{ Scott closed in } P, \uparrow p \cap \text{Max}(P) = A \cap \text{Max}(P)$$

\hspace{1cm} (†)

or equivalently

$$\text{Scott topology}|\text{Max}(P) = \text{Lawson topology}|\text{Max}(P)$$

\hspace{1cm} (‡)
such that $X$ is homeomorphic to $\text{Max}(P)$ equipped with the relative Scott topology. In this case the embedding $X \hookrightarrow \text{Max}(P) \hookrightarrow P$ is called a domain hull for $X$.

The equations (†) and ($) say that the Scott and Lawson topology restricted to the set of maximal elements agree. In this case, $\text{Max}(P)$ is a separable metric space, since the Lawson topology is separable metric for an $\omega$-continuous domain.

Example 1 (The upper space). For a locally compact separable metric space $X$, there exists a standard domain hull for $X$, i.e., its upper space $UX$. As mentioned in a previous section, $UX$ is an $\omega$-continuous domain and the homeomorphic injection $x \to \{x\}: X \to UX$ is a domain hull for $X$. The Scott topology on $UX$ is the upper topology and the Lawson topology is the usual Vietoris topology, or equivalently the topology on $UX$ induced by the Hausdorff metric (see Section 5.2).

Example 2 (The Cantor tree). Let $P$ consist of all finite and infinite strings of $\{0,1\}$ (including the empty string $\bot$) with the prefix order. Then $P$ is an $\omega$-continuous domain and the set of maximal points consists of all infinite strings. We can check directly from the definition that the Cantor tree is a domain hull for the Cantor set.

Definition 2. A Polish space is a separable metric space for which the topology is given by a complete metric.

The following is an important characterization of maximal point spaces given by J. Lawson [25]:

**Theorem 1.** A metric space $X$ is a maximal point space if and only if it is a Polish space.
Corresponding to the upper space construction for the locally compact separable metric space, we have a more general result due to J. Lawson [26]:

**Theorem 2.** Let $X$ be a maximal point space and let $X \leftrightarrow \text{Max}(P) \leftrightarrow P$ be a domain hull for $X$. Then $\text{Prob}(X) \leftrightarrow \text{Max}(M_{\mu}^{1}(P)) \leftrightarrow M_{\mu}^{1}(P)$ is a domain hull for the space of probability measures $\text{Prob}(X)$ on $X$ endowed with the weak topology, where $M_{\mu}^{1}(P)$ is the probabilistic power domain of $P$. In particular, $\text{Prob}(X)$ is a maximal point space.

Another very useful result is listed here:

**Lemma 1.** Let $P$ is an $\omega$-continuous domain satisfying $(\dagger)$. Then there exists a descending sequence $\{U_n\}$ of Scott open sets such that

$$\text{Max}(P) = \bigcap_n U_n.$$ 

This lemma says a maximal point space is a $G_{\delta}$ set and hence is a Borel set.

We conclude this section by giving another example which gives directly the domain hull for complete separable metric space (see details in [12]).

**Example 3** (The domain of closed formal balls). Let $(X, d)$ be a metric space. The domain of the closed formal balls is given by $BX := X \times \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$. A partial order $\sqsubseteq$ is defined on $BX$ by

$$(x, r) \sqsubseteq (y, s) \text{ if } d(x, y) \leq r - s.$$ 

From the definition, it is not too difficult to derive the following:

1. $(BX, \sqsubseteq)$ is continuous poset with $(x, r) \ll (y, s) \iff d(x, y) < r - s.$
2. \((X, d)\) is complete metric \(\iff\) \((BX, \subseteq)\) is a dcpo.
3. \((X, d)\) is separable \(\iff\) \((BX, \subseteq)\) is \(\omega\)-continuous.
4. \(\text{Max}(BX) = \{(x, 0) : x \in X\}\).
5. If \((X, d)\) is complete separable metric space, then \((BX, \subseteq)\) is a domain hull for \(X\) via
\[
x \rightarrow (x, 0) : X \rightarrow BX.
\]

5.2 Probabilistic Power Domains on the Upper Space

In this section we introduce some concepts which are important to study the integration theory defined and developed by Edalat. One of the important concepts is the notion of the probabilistic power domain, which plays a crucial role in studying this integral and other problems (see in [7, 14]).

Definition 1. Let \(X\) be any Hausdorff topological space. The upper space \(UX\) is the set of all non-empty compact subsets of \(X\) with the upper topology, i.e., the topology generated by basic open sets of the form: \(O' = \{C \in UX : C \subseteq O\}\) for each \(O \in \Omega(X)\), where \(\Omega(X)\) is the lattice of open sets of \(X\).

The specialization order is given as reverse inclusion, and \((UX, \supseteq)\) is a bounded complete dcpo, in which the least upper bound of a directed set of elements in \(UX\) is the intersection of these elements.

Proposition 1. [6] Let \(X\) be a second countable locally compact Hausdorff space.

(i) The dcpo \((UX, \supseteq)\) is \(\omega\)-continuous.

(ii) The Scott topology on \((UX, \supseteq)\) coincides with the upper topology.
(iii) The approximation relation \( B \ll C \) holds in \((UX, \supseteq)\) if and only if \( C \) is contained in the interior of \( B \) as subsets of \( X \).

**Definition 2.** [4, 30, 23, 19] A valuation on a topological space \( X \) is a map

\[ \nu : \Omega X \rightarrow [0, \infty) \]

which, for all \( U, V \in \Omega X \), where \( \Omega X \) is the lattice of open sets, satisfies:

1. (modularity) \( \nu(U) + \nu(V) = \nu(U \cap V) + \nu(U \cup V) \);
2. (strictness) \( \nu(\emptyset) = 0 \);
3. (monotonicity) \( U \subseteq V \Rightarrow \nu(U) \leq \nu(V) \).

A continuous valuation [23, 20, 19] is a valuation such that whenever \( A \) is a directed family with respect to \( \subseteq \) in \( \Omega(X) \) then

\[ \nu(\bigcup_{O \in A} O) = \sup_{O \in A} \nu(O). \]

**Definition 3.** The point valuation \( \mu_a \) based at \( a \) is the valuation

\[ \mu_a : \Omega X \rightarrow [0, \infty) \]

defined by

\[ \mu_a(O) = \begin{cases} 1 & \text{if } a \in O \\ 0 & \text{if otherwise.} \end{cases} \]

**Definition 4.** Any finite linear combination

\[ \sum_{i=1}^{n} \tau_i \mu_{a_i}(\tau_i \in [0, \infty), i = 1, \ldots, n) \]

of point valuations is called a simple valuation.

Next we introduce the important concept of the probabilistic power domain of a continuous domain. This concept had its origins in the work of Saheb-Djahromi [30] and was further studied by Graham [17], Jones [19], and Jones and Plotkin [20].
Definition 5. The probabilistic power domain, $PX$, of a topological space $X$ consists of the set of continuous valuations $\nu$ on $X$ with $\nu(X) \leq 1$ and is ordered as follows:

$$\nu_1 \subseteq \nu_2 \iff \forall O \in \Omega X, \ \nu_1(O) \leq \nu_2(O).$$

The normalized probabilistic power domain is that subset of $PX$ consisting of all valuations such that $\nu(X) = 1$.

For simple valuations there is very useful characterization on the order relation called the Splitting Lemma by Jones [19].

Proposition 2. Let $Y$ be a dcpo. Consider two given simple valuations $\mu_1 = \sum_{b \in |\mu_1|} r_b \eta_b$ and $\mu_2 = \sum_{c \in |\mu_2|} s_c \xi_c$ in $PY$ (or $P^1Y$). Then $\mu_1 \subseteq \mu_2$ if and only if for all $b \in |\mu_1|$ and all $c \in |\mu_2|$ there exists a nonnegative number $t_{b,c}$ such that $\sum_{c \in |\mu_2|} t_{b,c} = r_b$ and $\sum_{b \in |\mu_1|} t_{b,c} \leq s_c$ (or $\sum_{b \in |\mu_1|} t_{b,c} = s_c$) and $t_{b,c} \neq 0$ implies $b \subseteq c$.

Theorem 1. [19] If $X$ is an $(\omega)$-continuous dcpo, then the (normalized) probabilistic power domain $PX$ is also $(\omega)$-continuous and has a basis consisting of simple valuations.

By the result in [6], the upper space $(UX, \supseteq)$ of any locally compact Hausdorff space $X$ is continuous. Thus by Theorem 1, $PUX$ and $P^1UX$ are both continuous.

5.3. The Generalized Riemann Integral

In this section we will review briefly the definition of the generalized Riemann integration defined by A. Edalat. Some relevant results are included in this section.
All the details can be found in [7].

Let $f : X \to \mathbb{R}$ be a bounded function and $(X, d)$ a compact metric space. Let $\mu$ be a normalized Borel measure on $X$. Then from the result in [7], $\mu$ corresponds to a unique valuation $e(\mu) = \mu \circ s^{-1} \in S^1(X) \subseteq P^1UX$ with support $s(X)$, which is also a maximal element of $P^1UX$. Then $\mu$ is identified with $e(\mu)$.

**Definition 1.** For any simple valuation $\nu = \sum_{b \in [\nu]} r_b \eta_b \in PUX$, the generalized Darboux lower sum of $f$ with respect to $\nu$ is

$$S^l(f, \nu) = \sum_{b \in [\nu]} r_b \inf f[b].$$

Similarly, the Darboux upper sum of $f$ with respect to $\nu$ is

$$S^u(f, \nu) = \sum_{b \in [\nu]} r_b \sup f[b].$$

**Remark:** Because of the boundedness of $f$, the Darboux lower sum and upper sum are both well-defined.

**Proposition 1.** (1) Let $\mu_1, \mu_2 \in P^1UX$ be simple valuations with $\mu_1 \subseteq \mu_2$, then

$$S^l(f, \mu_1) \leq S^l(f, \mu_2) \quad \text{and} \quad S^u(f, \mu_1) \geq S^u(f, \mu_2).$$

(2) If $\mu_1, \mu_2 \in P^1UX$ are simple valuations with $\mu_1, \mu_2 \ll \mu$, then $S^l(f, \mu_1) \leq S^u(f, \mu_2)$.

**Definition 5.** The lower integral of $f$ with respect to $\mu$ on $X$ is

$$\int_{-} f \, d\mu = \sup_{\nu \ll \mu} S^l(f, \nu).$$
The upper integral of \( f \) with respect to \( \mu \) on \( X \) is

\[
\int f \, d\mu = \inf_{\nu \ll_{1} \mu} S^{u}(f, \nu).
\]

**Definition 6.** \( f \) is \( R \)-integrable with respect to \( \mu \) on \( X \) if the lower integral equals the upper integral, i.e.,

\[
\int f \, d\mu = \int f \, d\mu = \int f \, d\mu.
\]

Comparing this theory with the classical theory of the Riemann integration, one obtains similar properties and characterization of this type of integration. We collect some results, details can be found in [7, 14].

**Proposition 2.** (Cauchy Condition) A function \( f : X \to \mathbb{R} \) is \( R \)-integrable if and only if for all \( \epsilon > 0 \) there exists a simple valuation \( \nu \in P^{1}UX \) with \( \nu \ll_{1} \mu \) such that

\[
S^{u}(f, \nu) - S^{l}(f, \nu) < \epsilon.
\]

**Proposition 3.** A function \( f \) is \( R \)-integrable with integral \( I \) if and only if for all \( \epsilon > 0 \) there exists a simple valuation \( \nu = \sum_{b \in [\nu]} \tau_{b} \eta_{b} \in P^{1}UX \) with \( \nu \ll_{1} \mu \) such that

\[
|I - \sum_{b \in [\nu]} \tau_{b} f(c_{b})| < \epsilon,
\]

for \( c_{b} \in b \) for each \( b \in [\nu] \).

**Proposition 4.** (1) If \( f \) is \( R \)-integrable and \( \mu = \bigcup^{\uparrow} \mu_{i} \), where \( \{\mu_{i}\} \) is an \( \omega \)-chain in \( PUX \), then

\[
\int f \, d\mu = \lim_{i \to \infty} S^{l}(f, \mu_{i}) = \lim_{i \to \infty} S^{u}(f, \mu_{i}).
\]
(2) $S^i(f, \mu_i)$ increases to $\int f \, d\mu$ and $S^u(f, \mu_i)$ decreases to $\int f \, d\mu$.

**Proposition 5.** (i) If $f, g$ are $R$-integrable with respect to $\mu$ then $f + g$ is $R$-integrable with respect to $\mu$ and $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$. (ii) If $f$ is $R$-integrable with respect to $\mu$ and $c \in \mathbb{R}$, then $cf$ is also $R$-integrable with respect to $\mu$ and $\int cf \, d\mu = c \int f \, d\mu$.

**Theorem 1.** (i) Any continuous function $f : X \to \mathbb{R}$ is $R$-integrable with respect to $\mu$. (ii) A bounded function $f$ is $R$-integrable with respect to a bounded Borel measure if and only if its set of discontinuities has measure zero.

The next two theorems illustrate the relations between this type of integral and the classical Riemann integral and Lebesgue integral [7]:

**Theorem 2.** A bounded real-valued function on a compact real interval is Riemann integrable if and only if it is $R$-integrable with respect to $\lambda$ the Lebesgue measure. Furthermore the two integrals are equal when they exist.

**Theorem 3.** If a bounded real-valued function $f$ is $R$-integrable with respect to a Borel measure $\mu$ on a compact metric space $X$, then it is also Lebesgue integrable and the two integrals coincide.

### 5.4. Integration on Maximal Point Space

Let $f : X \to \mathbb{R}$ be a bounded function, where $X$ is a maximal point space. Let $\mu$ be a normalized Borel measure defined on $X$. From the definition there exists an $\omega$-continuous domain $P$ satisfying condition (†) of Section 5.1 and there is a domain
hull for $X$: $X \leftrightarrow \text{Max}(P) \leftrightarrow P$. Then we extend function $f$ to $\hat{f}: P \to \mathbb{R}$ and extend $\mu$ on the Borel algebra on $X$ to $\overline{\mu}$ on the Borel algebra on $P$ with respect to the Scott topology. In this way integration of $f$ on $X$ with respect to $\mu$ can be computed through computation integral of $\hat{f}$ on $P$ with respect to $\overline{\mu}$. By the results of last chapter we can compute the integral of $f$ on $X$ in a constructive way.

**Lemma 1.** Suppose that $f: X \to \mathbb{R}$ is bounded. We define a function $\hat{f}: P \to \mathbb{R}$ where $X \leftrightarrow \text{Max}(P) \leftrightarrow P$ is the domain hull as follows:

$$\hat{f}(x) = \sup \{ \inf \{ f(y) : y \in (X \cap \uparrow z) \} \}.$$  

Then $\hat{f}(x) \leq f(x)$ for each $x \in X$.

**Proof:** $\hat{f}(x)$ is well-defined, since the function $f$ is bounded, and supremum and infimum exist uniquely. Let $x$ be in $X$. We claim that $\hat{f}(x) \leq f(x)$. Since $x \in X$ and $z \ll x$, we have $x \in (X \cap \uparrow z)$. Then $\inf \{ f(y) : y \in X, y \geq z \} \leq f(x)$. Therefore $\hat{f}(x) \leq f(x)$.

**Lemma 2.** The function defined above $\hat{f}: P \to \mathbb{R}$ is an extension of $f$ provided $f$ is Scott continuous, i.e., $\hat{f}(x) = f(x)$ for each $x \in X$.

**Proof:** From the above we need only to establish that for all $\epsilon > 0$, $f(x) - \epsilon \leq \hat{f}(x)$ when $x \in X$. Since $f$ is Scott continuous at $x$, there exists $U$ open in $X$ with $x \in U$ such that $f(U) \subseteq (f(x) - \epsilon, \infty)$. Then there exists a Scott open subset $W$ such that $x \in W$ and $W \cap X \subseteq U$. Then there exists $z \in W$ such that $z \ll x$. So $\inf f(\uparrow z \cap X) \geq \inf f(W \cap X) \geq \inf f(U) > f(x) - \epsilon$. Then by taking supremum, we have $\hat{f}(x) \geq f(x)$. Thus we have $\hat{f}(x) = f(x)$ for each $x \in X$ from Lemma 1. 

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Proposition 1. The extension $\hat{f}(x)$ of $f$ defined above is Scott continuous.

Proof: First we need to show that $\hat{f}$ is monotone. Let $u \leq v$. Then from the definition, we know that $z \ll u \leq v \Rightarrow z \ll v$. Hence we have

$$\{z : z \ll u\} \subseteq \{z : z \ll v\}.$$ 

Hence $\hat{f}(u) \leq \hat{f}(v)$ from the definition of $\hat{f}$.

Next we need to show that $\hat{f}$ preserve the supremum, i.e.,

$$\hat{f}(\bigcup u_i) = \bigcup \hat{f}(u_i),$$

where $\{u_i\}$ is a directed family in $P$. But since we know that $\hat{f}$ is monotone, and $\bigcup u_i \geq u_i$ for each $i$, so we have $\hat{f}(\bigcup u_i) \geq \bigcup \hat{f}(u_i)$. To show the other direction, notice $\{z : z \ll \bigcup u_i\} \subseteq \bigcup_i \{z : z \ll u_i\}$; this is because $P$ is a continuous domain. So we have the other inequality:

$$\hat{f}(\bigcup u_i) = \sup_{z \ll \bigcup u_i} \{\inf\{f(y) : y \in (X \cap \uparrow z)\}\}$$

$$\leq \bigcup \sup_{z \ll u_i} \{\inf\{f(y) : y \in (X \cap \uparrow z)\}\}$$

$$= \bigcup \hat{f}(u_i).$$

Then $\hat{f}$ is Scott continuous. ■

Corollary 1. The extension $\hat{f}$ defined above is the largest continuous extension of $f$ if and only if $f$ is Scott continuous.

Proof: Let $g : P \rightarrow \mathbb{R}$ be an extension of $f$, where $g$ is Scott continuous. Then for each $x \in P$ we have $x = \bigcup z \ll x$. Hence

$$g(x) = \bigcup \bigcup z \ll x g(z) \leq \bigcup \bigcup z \ll x \inf\{g(y) : y \in (X \cap \uparrow z)\},$$

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by definition above this is the same as
\[ g(x) \leq \sup_{z \leq x} \{ \inf \{ g(y) : y \in (X \cap z) \} \} . \]

But \( g \) is an extension of \( f \), so \( g(y) = f(y) \). Thus from the definition we have \( g(x) \leq \hat{f}(x) \). \( \blacksquare \)

Next is a standard result in measure theory (see details in [3]):

Let \( A \) be a class of sets in \( Y \) and \( X \subseteq Y \).

**Theorem 1.** (i) If \( \mathcal{F} \) is a \( \sigma \)-algebra in \( Y \), then \( \mathcal{F} \cap X \) is a \( \sigma \)-algebra in \( X \). (ii) If \( A \) generates a \( \sigma \)-algebra in \( Y \), then \( A \cap X \) generates a \( \sigma \)-algebra in \( X \): \( \sigma(X \cap A) = \sigma(A) \cap X \).

**Corollary 2.** Then Borel \( \sigma \)-algebra \( B_X \) in \( X \) equals the Borel \( \sigma \)-algebra \( B_Y \cap X \), where \( B_Y \) is Borel \( \sigma \)-algebra in \( Y \).

From the above corollary for a given Borel measure on the Borel algebra \( B_X \) we can define
\[ \overline{\mu}(B) = \mu(B \cap X) , \]
where \( B \in B_Y \) as an extension of \( \mu \) to \( B_Y \).

**Proposition 2.** Suppose that \( X \in B_Y \). Then
\[ \overline{\mu}(B) = \mu(B \cap X) , \]
defines a measure on \( B_Y \). Moreover a bounded function \( f : Y \to \mathbb{R} \) is integrable with respect to \( \overline{\mu} \) if and only if \( f|_X \) is integrable with respect to \( \mu \) and in this case
\[ \int_Y f \, d\overline{\mu} = \int_X f|_X \, d\mu . \]
Proof: Suppose that $f$ is integrable with respect to $\mu$. Then for each $\epsilon > 0$, there exists a partition $P \subseteq \mathcal{B}_Y$ such that

$$S^u(f, \mu, P) - S^l(f, \mu, P) < \epsilon.$$  

Let $P = \{B_i \in \mathcal{B}_Y : i = 1, \ldots, n\}$. Then we have

$$S^u(f, \mu, P) - S^l(f, \mu, P) = \sum_{i=1}^n (\sup f(B_i) - \inf f(B_i))\mu(B_i) < \epsilon.$$  

But $P' = \{B_i \cap X \in \mathcal{B}_X : i = 1, \ldots, n\}$ is a partition of $X$. Hence

$$\sum_{i=1}^n (\sup f(B_i \cap X) - \inf f(B_i \cap X))\mu(B_i \cap X) \leq \sum_{i=1}^n (\sup f(B_i) - \inf f(B_i))\mu(B_i).$$  

Then $S^u(f_\mu, P') - S^l(f_\mu, P') < \epsilon$. Thus $f|_X$ is integrable with respect to $\mu$.

Suppose that $f|_X$ is integrable with respect to $\mu$. Then for all $\epsilon > 0$ there exists a partition $P'$ with respect to $\mathcal{B}_X$ such that

$$S^u(f|_X, \mu, P') - S^l(f|_X, \mu, P') < \epsilon.$$  

But from Theorem 1 above for each $C_i \in P'$ there exists $B_i \in \mathcal{B}_Y$ such that $C_i = B_i \cap X$ for $i = 1, \ldots, m$. Then $\{C_i \cap X : i = 1, \ldots, m\} \cup \{Y \setminus X\}$ is a partition of $Y$ with respect to $\mathcal{B}_Y$. So

$$\sum_{i=1}^m (\sup f(B_i \cap X) - \inf f(B_i \cap X))\mu(B_i \cap X) +$$(\sup f(Y \setminus X) - \inf f(Y \setminus X))\mu(Y \setminus X) =$$

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\[ \sum_{i=1}^{m} (\sup f(C_i) - \inf f(C_i))\mu(C_i) < \epsilon. \]

Hence \( f \) is integrable with respect to \( \bar{\mu} \).

Let \( I = \int_Y f \, d\bar{\mu} \) and \( I' = \int_X f \, d\mu \). From the definition, for all \( \epsilon > 0 \), there exists \( \{C_i \in B_X : i = 1, \ldots, m\} \) such that

\[ I' < \sum_{i=1}^{m} \inf f(C_i)\mu(C_i) + \epsilon. \]

From the hypothesis, \( C_i = B_i \cap X \) for some \( B_i \in B_Y \). Hence \( \{B_i \cap X\} \) and \( Y \setminus X \) is a partition of \( Y \). Hence we have

\[ I' < \sum_{i=1}^{m} \inf f(C_i)\mu(C_i) + \epsilon = \sum_{i=1}^{m} \inf f(C_i)\mu(C_i) + \inf f(Y \setminus X)\bar{\mu}(Y \setminus X) + \epsilon < I + \epsilon. \]

Hence we have \( I' - I < \epsilon \). Similarly we can obtain \( I - I' < \epsilon \). Therefore by letting \( \epsilon \to 0 \) we obtain the result.

With above preparations we can give the following theorem:

**Theorem 2.** Let \( X \) be maximal point space and \( f : X \to \mathbb{R} \) a bounded Scott continuous function. Then \( f \) is integrable with respect to \( \mu \) and

\[ \int_X f \, d\mu = \int_P \hat{f} \, d\bar{\mu}. \]

Since \( P \) is \( \omega \)-continuous domain, its probabilistic power domain is also \( \omega \)-continuous with basis of simple valuations. Therefore \( \bar{\mu} = \bigcup \mu_i \). Then from the result of Section 4.2, we have

\[ \int_X f \, d\mu = \int_P \hat{f} \, d\bar{\mu} = \bigcup \int_P \hat{f} \, d\mu_i. \]

Then we obtain a constructive way of computing integral of \( f \) on \( X \).
Corollary 3. Let $X$ be a Polish space and $f : X \to \mathbb{R}$ a bounded Scott continuous function. Then we have
\[ \int_X f \, d\mu = \int_{BX} \hat{f} \, d\bar{\mu}, \]
where $(BX, \sqsubseteq)$ is the domain of formal balls.

Proof: From Theorem 1 in Section 5.1, we know $X$ is a maximal point space. Furthermore, $(BX, \sqsubseteq)$ is the domain hull. Hence by applying Theorem 2 we obtain the result. ■

Now consider $X$ is compact separable metric space. From integration theory developed by A. Edalat and introduced in Section 5.2, we can compute the integral of $f$ in this sense. In the following we show this integral is the same as the integral developed in Section 3.3.

Theorem 3. Let $f : X \to \mathbb{R}$ be a bounded Scott continuous, where $X$ is a compact separable metric space. If $f$ is $R$-integrable in a sense defined by A. Edalat, then we have
\[ (\text{Edalat}) \int_X f \, d\mu = \int_X f \, d\mu. \]

Proof: From Section 5.1, we know that $UX$ is domain hull of $X$. Hence we can carry out the computations as described above. First we extend the Borel measure $\mu$ on $X$ to $\bar{\mu}$ on $UX$. Since $P^1 UX$ is continuous, so $\mu = \bigcup^f \eta_i$, where $\eta_i$ is a sequence of simple valuations satisfying $\eta_i \ll \mu$. Hence from the definition given in Section 5.2, we know the upper integral and lower integral of $f$ are the same, i.e.,
\[ \int f \, d\mu = \sup_{\eta \ll \mu} S^U(f, \eta) = \sup_{\eta \ll \mu} \sum_{b \in |\eta|} \tau_b \inf f(b), \]
equals to
\[ \int f \, d\mu = \inf_{\eta \ll \mu} S^w(f, \eta) = \sup_{\eta \ll \mu} \sum r_b \sup f(b), \]
where \( \eta = \sum_{b \in |\eta|} r_b \delta_b \) with \( \delta_b \) being a point valuation.

By Proposition 1 above, we know that \( \hat{f} \) is a Scott continuous extension of \( f \).

From results in Secton 3.3 and Section 4.1, we know that \( \hat{f} \) is integrable with respect to \( \mu \) and all simple valuations. Hence we can apply the computation theorem in Section 4.2 and Theorem 2 above to find the integral of \( \hat{f} \) on \( UX \), i.e.,
\[ \int_X f \, d\mu = \int_{UX} \hat{f} \, d\mu = \bigcup_{\mu = \bigcup \eta_i} \int_{UX} \hat{f} \, d\eta_i. \]

But from Proposition 1 of Section 4.1.2 we know that for each simple valuation \( \eta = \sum_{b \in |\eta|} r_b \delta_b \)
\[ \int_{UX} \hat{f} \, d\eta = \sum_{b \in |\eta|} r_b \hat{f}(b). \]

However, we can calculate \( \hat{f} \) from the definition:
\[ \hat{f}(b) = \sup_c \inf \{ f(z) : z \in (\uparrow c \cap X) \} = \sup_c \inf f(c) = \inf f(b). \]

Hence
\[ \int_{UX} \hat{f} \, d\mu = \bigcup_{\mu = \bigcup \eta_i} \int_{UX} \hat{f} \, d\eta_i = \bigcup_{\mu = \bigcup \eta_i} \sum_{b \in |\eta_i|} r_b \inf f(b_i). \]
This is the lower integral of \( f \) on \( X \) defined by Edalat (see Section 5.2) and is the integral of \( f \) on \( X \) since \( f \) is integrable. Hence we have
\[ \int_{UX} \hat{f} \, d\mu = \int_X \hat{f}1_X \, d\mu = (\text{Edalat}) \int_X f \, d\mu. \]

\[ \blacksquare \]
Corollary 4. (i) Let $X = [a, b]$ and $f$ a bounded Scott continuous function. If $f$ is $R$-integrable then

$$(\text{Riemann}) \int_a^b f \, d\lambda = \int_{B[a,b]} \hat{f} \, d\lambda.$$ 

(ii) Let $X$ be compact Polish space and $f : X \to \mathbb{R}$ be bounded Scott continuous. Then if $f$ is $R$-integrable, then we have

$$(\text{Lebesgue}) \int_X f \, d\mu = \int_{B_X} \hat{f} \, d\mu.$$ 

Proof: By applying Theorem 2, Theorem 3 in Section 5.3 and Theorem 3 above, we can obtain the results as special cases. ■
CHAPTER 6. HECKMANN'S APPROACH

In this Chapter we first introduce some basic concepts and theory, and then briefly give an introduction of theory of the integration defined by Heckmann in [18]. Based on these preparations we establish the equivalence of the integral introduced by Heckmann and our approach of integration defined in Chapter 2.

6.1 Some Basics

In this section we introduce some basic theory which will be useful to study the integration defined by Heckmann. One can see the details in [18].

Let $X$ be a topological space. Let $\mathcal{O}X$ be the set of open sets of $X$. Then $\mathcal{O}X$ can be topologized in the following two ways:

First consider $(\mathcal{O}X, \subseteq)$ as a poset. Clearly it is a dcpo. Then it can be endowed with the Scott topology. We call this space $\Omega_\mathcal{O}X$. In another way $\mathcal{O}X$ can be given the point topology with subbasis $\{O(x) : x \in X\}$, this topology is denoted as $\Omega_pX$. A set $O$ of open sets is open in the point topology if and only if for every $O \in O$ there is a finite set $F$ such that $O \in O(F) \subseteq O$. It is clear that $\Omega_pX \subseteq \Omega_\mathcal{O}X$ since $\{O(x) : x \in X\}$ is Scott open in $\Omega_\mathcal{O}X$.

For two spaces $X$ and $Y$, the pointwise function space $[X \to Y]_p$ consists of all continuous functions $f : X \to Y$ with subbase $\{(x \to V) : x \in X, V \in \mathcal{O}Y\}$ where $\langle x \to V \rangle = \{f : f : X \to Y, fx \in V\}$. The tensor product $X \otimes Y$ has the topology as follows: a set $W$ is open in $X \otimes Y$ if for every $(x, y)$ in $W$, there are open set $U$ of $X$ and $V$ of $Y$ such that $(x, y) \in \{x\} \times V \subseteq W$ and $(x, y) \in U \times \{y\} \subseteq W$. A function $f : X \otimes Y \to Z$ is continuous if and only if all the functions $f_x : Y \to Z$
with \( f_{x y} = f(x, y) \) and \( f^v : X \to Z \) with \( f^v x = f(x, y) \) are continuous if and only if \( g : X \to [Y \to Z]_p \) with \( gxy = f(x, y) \) is well-defined and continuous. Often a continuous function \( f : X \otimes Y \to Z \) is called continuous in the two arguments separately. Composition \( o \) is defined as: \([Y \to Z]_p \otimes [X \to Y]_p \to [X \to Z]_p \) with \((g \circ f)(x) = g(fx)\) is continuous.

For every two spaces \( X \) and \( Y \) the function \( \Omega_p : [X \to Y]_p \to [\Omega_p Y \to \Omega_p X]_p \) with \( \Omega_p f(V) = f^-V \) is well-defined and continuous.

The Isbell function space \([X \to Y]_i\) consists of all continuous functions \( f : X \to Y \) with subbase \( \{ \langle U \leftarrow V \rangle : U \in \Omega(\Omega_\ast X), V \in \Omega Y \} \) where \( \langle U \leftarrow V \rangle = \{ f : X \to Y | f^-V \in U \}. \) Since \( \langle x \to V \rangle = \langle \mathcal{O}(x) \leftarrow V \rangle \), the topology of \([X \to Y]_i\) includes that \([X \to Y]_p\).

Similar to the pointwise function space, \( f : X \times Y \to Z \) is continuous then the curried variant \( g : X \to [Y \to Z]_i \) is well-defined and continuous. Composition \( o : [Y \to Z]_i \otimes [X \to Y]_i \to [X \to Z]_i \) with \((g \circ f)x = g(fx)\) is continuous.

For every two \( T_0 \)-spaces \( X \) and \( Y \) the function \( \Omega_\ast : [X \to Y] \to [\Omega Y \to \Omega X] \) with \( \Omega_\ast f = f^- \) is injective no matter which topology is chosen for the two function spaces. If \([\Omega_\ast Y \to \Omega_\ast X] \) is equipped with the pointwise topology, it is just the Isbell topology on \([X \to Y] \) which makes \( \Omega_\ast \) into an embedding. Hence we obtain a continuous function \( \Omega_\ast : [X \to Y]_i \to [\Omega_\ast Y \to \Omega_\ast X]_p \), whose type differs from that of the continuous function \( \Omega_p \).

Because there are different topologies on the space \( \Omega X \) hence there are different continuity notions as defined in the following: (1) A valuation \( \mu \) is Scott continuous if and only if \( \mu : \Omega_\ast X \to \overline{\mathbb{R}}_+ \) is continuous; (2) A valuation \( \mu \) is point continuous if
and only if $\mu : \Omega_pX \to \overline{\mathbb{R}}_+$ is continuous, where $\mathbb{R}_+ = [0, +\infty)$ and $\overline{\mathbb{R}}_+ = [0, +\infty]$.

From the definition it is easy to obtain that every point continuous valuation is Scott continuous. The converse is true in the situation when $X$ is a locally finitary. We list the results in the following. See details in [18].

**Theorem 1.** Every Scott continuous valuation on a locally finitary space is point continuous.

Since every continuous dcpo is locally finitary then we have

**Corollary 1.** On a continuous dcpo every Scott continuous valuation is point continuous.

**Theorem 2.** Every Scott or point continuous valuation can be obtained as a directed join of bounded Scott continuous or point continuous valuations.

Let $VX$ be the set of all Scott continuous valuations on $X$, and $V_pX$ the set of all point continuous valuations. We topologize these sets as subspaces of the pointwise function space $[\Omega_pX \to \overline{\mathbb{R}}_+]_p$.

**Theorem 3.** Let $X$ and $Y$ be topological spaces. Then every continuous linear function from $V_pX$ to a $\tau_0$-cone is uniquely determined by its values on point valuations.

Next theorem is very important in introducing the integral by Heckmann:

**Theorem 4.** $V_pX$ is the free locally convex sober cone over $X$ in $\mathcal{TOP}$.

This theorem says that $V_pX$ is a locally convex-based sober cone, and for every continuous function $f : X \to M$ from a topological space $X$ to a locally convex
sober cone $M$, there is a unique continuous linear function $f : V_pX \rightarrow M$ with $f \circ s = f$, i.e., $f(x) = fx$ for all $x \in X$, where $s : X \rightarrow VX$ with $sx = \hat{x}$, a point valuation.

6.2 Heckmann's Approach

In this section we introduce briefly the integration defined by Heckmann [14] and also some properties of this approach of integration are discussed.

Let $X$ be a topological space and $f \in [X \rightarrow \mathbb{R}^+]_i$.

**Definition 1. Integration is defined as a function** $\int_X : [X \rightarrow \mathbb{R}^+]_i \otimes VX \rightarrow \mathbb{R}^+$, **which is continuous in the two arguments separately.**

The function $\int_X$ is constructed from the following processes:

1. The function $\Omega_s : [X \rightarrow \mathbb{R}^+]_i \rightarrow [\Omega_s \mathbb{R}_+ \rightarrow \Omega_s X]_p$ with $\Omega_sf = f^-$ is continuous. This is so because the Isbell topology is chosen.

2. Using $\Omega_s$ we map from $[X \rightarrow \mathbb{R}^+]_i \otimes VX \rightarrow [\Omega_s \mathbb{R}_+ \rightarrow \Omega_s X]_p \otimes [\Omega_s X \text{ mod } \mathbb{R}_+]_p$.

Then we can use function composition to have $[\Omega_s \mathbb{R}_+ \text{ mod } \mathbb{R}_+]_p = V\mathbb{R}_+$. This composition is continuous in its two arguments separately.

3. Since $\mathbb{R}_+$ is a continuous dcpo so $V\mathbb{R}_+ = V_p\mathbb{R}_+$.

4. $\mathbb{R}_+$ is a locally convex sober cone. By the Theorem 4 identity $id : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ can be extended to a continuous linear function $\overline{id} : V_p\mathbb{R}_+ \rightarrow \mathbb{R}_+$ with the property $\overline{id}(r) = r$ for all $r \in \mathbb{R}$.

By putting all these together we have $\int_X : [X \rightarrow \mathbb{R}^+]_i \otimes VX \rightarrow \mathbb{R}^+$

$$\int_X (f, \mu) = \overline{id}(\mu \circ f^-),$$

which is continuous in its two arguments separately.
Theorem 1. (i). Integration \( \int_X : [X \to \mathbb{R}_+] ; \odot V X \to \mathbb{R}_+ \) is continuous in its two arguments separately; (ii). Integration is Scott continuous in its two arguments; (iii). Integration is linear in both arguments; (iv). For \( f : X \to \mathbb{R}_+ \) and \( x \in X \), 
\[
\int_X (f, \hat{x}) = f(x).
\]

6.3 Equivalence of Integrations

In this section consider \( f : X \to \mathbb{R}_+ \) be a bounded Scott continuous function. Then from Section 3.3, then \( f \) is integrable and we also have a formula to compute this integral. On the other hand from the theory of integration developed by Heckmann in the previous section we can obtain another integral of \( f \). We will prove in this section these two integrals are the same.

Let \( \text{id} : \mathbb{R}_+ \to \mathbb{R}_+ \) be the identity. Since \( V_p \mathbb{R}_+ \) is convex-based sober cone, then \( \text{id} \) has a linear extension from \( V_p \mathbb{R}_+ \) to \( \mathbb{R}_+ \) such that \( \text{id}(\hat{x}) = x \). The following lemma provide a concrete form for this extension of \( \text{id} \).

Lemma 1. The extension \( \overline{\text{id}} : V_p \mathbb{R}_+ \to \mathbb{R}_+ \) is given by

\[
\overline{\text{id}}(\mu) = \int_{\mathbb{R}_+} \mu(x, \infty) \, dx.
\]

Proof: By Theorem 3 in Section 6.1, the continuous linear function from \( V_p X \) to a cone is uniquely determined by its values on point valuations. Consider a function \( g : V_p X \to [\mathbb{R}_+ \to \mathbb{R}_+] \) defined as \( g(\mu)(t) = \mu(t, \infty) \). Then it is clear that \( g \) is Scott continuous. Then since \( \mu(t, \infty) \) is monotone, so it is Riemann integrable. Then we define another function \( F : [\mathbb{R}_+ \to \mathbb{R}_+] \to \mathbb{R}_+ \) defined as \( F(g) = \int_{\mathbb{R}_+} g \, dx \). This
function is also Scott continuous since we can apply monotone convergence theorem. Therefore the composition $F \circ g : V_p X \to [\mathbb{R}_+ \to \mathbb{R}_+] \to \mathbb{R}_+$ is Scott continuous. Hence

$$F \circ g(\mu) = \int_{\mathbb{R}_+} \mu(t, \infty) \, dt.$$  

It is obvious to see that this function is linear. Now let $\eta_r$ be a point valuation where $r \in \mathbb{R}_+$. Then

$$F \circ g(\eta_r) = \int_{\mathbb{R}_+} \eta_r(t, \infty) \, dt = \int_r^\infty \eta_r(t, \infty) \, dt + \int_0^r \eta_r(t, \infty) \, dt = r.$$  

But we know that $\overline{id}(\eta_r) = r$. Hence by the uniqueness theorem in Section 6.1 we conclude that

$$\overline{id}(\mu) = F \circ g(\mu) = \int_{\mathbb{R}_+} \mu(t, \infty) \, dt.$$

\[\blacksquare\]

**Theorem 1.** Let $f \in [X \to \mathbb{R}_+]$ be bounded function and $\mu \in V X$. Then we have

$$\int_X (f, \mu) = \int_X f \, d\mu.$$  

**Proof:** By the definition of the integral introduce in Section 6.2 we know

$$\int_X (f, \mu) = \overline{id}(\mu \circ f^{-}).$$

But from the lemma above we have

$$\overline{id}(\mu \circ f^{-}) = \int_{\mathbb{R}_+} (\mu \circ f^{-})(t, \infty) \, dt.$$  

Then from the Corollary 2 of Section 2.3,

$$\int_{\mathbb{R}_+} (\mu \circ f^{-})(t, \infty) \, dt = \int_X f \, d\mu.$$  

\[\blacksquare\]

When $f : D \to \mathbb{R}_+$ is Scott continuous and $\mu = \bigsqcup^\dagger \mu_i$ where $D$ is a dcpo and $\mu_i$ is simple valuation for each $i$, the proof of equivalence is more direct.
Theorem 2. Let $f : D \to \mathbb{R}_+$ be Scott continuous and $\mu = \bigcup^\dagger \mu_i$ where $\{\mu_i : i \in I\}$ is directed family of simple valuations and $D$ is a dcpo. Then $f$ is integrable with respect to $\mu$. Moreover we have

$$\int_X f \, d\mu = \int_X (f, \mu).$$

Proof: Since $f$ is Scott continuous then $f$ is integrable with respect to $S$ where $S$ is semi-algebra of crescents of Scott topology. Hence from Theorem 1 in Section 4.2.2 we know that

$$\int_X f \, d\mu = \bigcup \int_X f \, d\mu_i.$$

But we know that

$$\int_X f \, d\mu_i = \sum_{a \in [\mu_i]} r_a f(a).$$

On the other hand, by linearity of the integral defined in Section 6.2 and Scott continuity, we have

$$\int_X (f, \mu) = \int_X (f, \bigcup \mu_i) = \bigcup \int_X (f, \mu_i).$$

But from Theorem 1 in section 6.2,

$$\int (f, \mu_i) = \sum_{a \in [\mu_i]} r_a \int_X (f, \eta_a) = \sum_{a \in [\mu_i]} r_a f(a).$$

Hence we are done. $\blacksquare$

Corollary. Let $D$ be a continuous domain and $\mu$ a continuous valuation on $D$. Then if $f : D \to \mathbb{R}_+$ is Scott continuous, then $f$ is integrable with respect to $\mu$. Moreover we have

$$\int_X f \, d\mu = \int_X (f, \mu).$$
Proof: Just notice that since $D$ is continuous domain then $PD$ is also continuous domain where $PD$ is probabilistic power domain. Then there exists a collection of simple valuations $\{\mu_i : i \in I\}$ such that $\mu = \bigcup^\dagger \mu_i$. Then by applying theorem above we are done. ■
REFERENCES


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