Weak Convergence of Interacting Stochastic Systems.

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WEAK CONVERGENCE OF INTERACTING STOCHASTIC SYSTEMS

A Dissertation

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requirements for the degree of
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in

The Department of Mathematics

by
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Abstract

The aim of the dissertation is to establish the weak convergence of mean-field interacting particle systems driven by Poisson random measures and semimartingales. The limit of the stochastic systems is identified by the use of martingale problems and Picard iteration schemes. The interacting systems driven by Poisson random measures are shown to be stable with respect to the coefficients of the system as well as the driving terms. The same results can be achieved when a random interaction term independent of the driving terms is introduced into the coefficients of the system. Equations driven by semimartingales do not necessarily possess the Markov property. In such a case martingale problems are no longer available, and hence identification of the limit is established by suitable approximation schemes.
Introduction

Let us picture a collection of \( n \) particles moving through \( \mathbb{R} \) such that the velocity of each particle \( p^{n,i} \) is determined by random factors particular to that particle, the position of \( p^{n,i} \) in \( \mathbb{R} \), and the position of \( p^{n,i} \) relative to the other \( n - 1 \) particles, where this last factor, which we can call the interaction factor is the average of the pairwise interactions between \( p^{n,i} \) and each of the other particles. This sort of interacting system is called a mean field interacting system and can be modeled by the following type of stochastic differential equation.

\[
X_t^{n,i} = X_0^{n,i} + \int_0^t \frac{1}{n} \sum_{j=1}^{n} b(X_s^{n,i}, X_s^{n,j}) dA_s^{n,i} + \int_0^t \frac{1}{n} \sum_{j=1}^{n} \sigma(X_s^{n,i}, X_s^{n,j}) dM_s^{n,i}
\]

where \( X_t^{n,i} \) is the location of the particle \( p^{n,i} \) at time \( t \), \( A_s^i \) is a stochastic process with paths of bounded variation on compacts, and \( M_s^i \) is a martingale.

From the solutions to (1) we obtain the random measure

\[
\eta^n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_t^{n,i}}
\]

In papers by [Sz 84], [CKS 91], and [Ch 94] it has been shown that if we set \( A_t^i = s \), and \( M_t^i \) equal to a Brownian motion \( B_s \) then for suitable \( b \), and \( \sigma \)

\[
\mathcal{L}(\eta^n) \Rightarrow \eta = \delta_X
\]

where \( X \) is the solution to the McKean-Vlasov equation

\[
X_t = X_0 + \int_0^t b(X_s, \mathcal{L}(X_s)) ds + \int_0^t \sigma(X_s, \mathcal{L}(X_s)) dBs
\]
We will investigate mean field interacting particle systems of two types. The first being of the form

\[ X_t^{n,i} = X_0^{n,i} + \int_0^t \frac{1}{n} \sum_{j=1}^n b_n(s, X_s^{n,i}, X_s^{n,j}) \, ds \]

(2)

\[ + \int_0^t \int_V \frac{1}{n} \sum_{j=1}^n \sigma_n(s, X_s^{n,i}, X_s^{n,j}, v) \, N^{n,i}(dv, ds) \]

where \( N^{n,i}(dv, ds) \) is a Poisson random measure, and \( b_n \) and \( \sigma_n \) converge to \( b \) and \( \sigma \) in a suitable way. The second being of the form

\[ X_t^{n,i} = X_0^{n,i} + \int_0^t \frac{1}{n} \sum_{j=1}^n b(X_s^{n,i}, X_s^{n,j}) \, dA_s^{n,i} + \int_0^t \dot{b}(X_s^{n,i}) \, ds \]

(3)

\[ + \int_0^t \frac{1}{n} \sum_{j=1}^n \sigma(X_s^{n,i}, X_s^{n,j}) \, dM_s^{n,i} + \int_0^t \dot{\sigma}(X_s^{n,i}) \, dN_s^{n,i} \]

where \( N_s^i \) is a martingale Lévy process with bounded jumps.

In chapter 2 we will examine systems of form (2). We will consider the cases where

(i) \( \nu \Rightarrow \nu \neq \delta_0 \)

(ii) \( \nu \Rightarrow \delta_0. \)

In case (i) we will find that \( \mathcal{L}(\eta^n) \Rightarrow \delta_{\mathcal{L}(X)} \) where \( \mathcal{L}(X) \) satisfies a McKean-Vlasov equation of the form

(4) \[ X_t = X_0 + \int_0^t b(s, X_s, \mathcal{L}(X_s)) \, ds + \int_0^t \int_V \sigma(s, X_s, \mathcal{L}(X_s), v) \, N(dv, ds) \]

In case (ii) we will find that \( \mathcal{L}(\eta^n) \Rightarrow \delta_{\mathcal{L}(X)} \) where \( \mathcal{L}(X) \) satisfies a McKean-Vlasov equation of the form

(5) \[ X_t = X_0 + \int_0^t b(s, X_s, \mathcal{L}(X_s)) \, ds + \int_0^t \sigma(s, X_s, \mathcal{L}(X_s)) \, dB_s \]

The novelties in this chapter are

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a) the equations are driven by Poisson random measures, and the coefficients $\sigma_n$ and $\sigma$ depend on $v$: the jump size,

b) the stability result for $\sigma_n$ and $b_n$,

c) the stability result for the random measures $N^n_i(dv, ds)$. The approach for these cases will be to use a martingale problem argument.

In chapter 3 we will continue to examine systems of form (2) with an added random interaction term unrelated to the driving term. This sort of problem has been examined by Ding [Di 92] but with the random interaction term only in the drift term. We include the random interaction in both the drift and diffusion terms. The results in chapter are in the same spirit as in chapter 2 except for this added term.

In chapter 4 we will look at systems of form (3). The usual method for showing weak convergence to a McKean-Vlasov equation is to use the martingale problem see [Sz 84], [CKS 91], and [Ch 94]. However in this case the martingale problem posed by the limit will not be well defined because a stochastic integral driven by a semimartingale need not be a Markov process. Instead the result will be proved using Picard iteration and topological properties of $D_E[0, T]$.

$$\tau^i = \inf\{t : s + \int_0^s | dA^i_s | + < M^i, M^i >_s > t\}$$

We will find that

$$\mathcal{L}(\eta^n) = \mathcal{L}\left(\frac{1}{n} \sum_{i=1}^{n} \delta_{X^i_{\tau^i(s)}}\right) \Rightarrow \delta_{\mathcal{L}(X^i_{\tau^i(s)})}$$

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where $\mathcal{L}(X^i_{r_i(t)})$ satisfies the McKean-Vlasov equation:

$$X^i_{r_i(t)} = X^i_0 + \int_0^{r_i(t)} \frac{1}{n} \sum_{j=1}^n b(X^j_{r_i}, \mathcal{L}(X^i_{r_i})) \, dA^j_s + \int_0^{r_i(t)} \tilde{b}(X^n_{r_i}) \, ds$$

$$+ \int_0^{r_i(t)} \frac{1}{n} \sum_{j=1}^n \sigma(X^n_{r_i}, \mathcal{L}(X^i_{r_i})) \, dM^j_s + \int_0^{r_i(t)} \tilde{\sigma}(X^n_{r_i}) \, dN^i_s.$$  

Thus our primary goal of establishing weak convergence of stochastic systems as well as identifying the limit is achieved under a Markovian set up as well as a general semimartingale set up.
Chapter 1

Weak Convergence and the Martingale Problem

1.1: Some Results And Concepts Concerning Stochastic Analysis

A stochastic process is a measurable function $X(t, \omega)$ defined on $I \times \Omega$ where $(\Omega, \mathcal{F}, P)$ is a complete probability space and $I$ is an interval on the line. Usually we will denote a stochastic process by $X_t$.

Given a complete probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t), 0 < t < \infty$ is a family of $\sigma$-algebras which is increasing i.e. $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s < t$. In general instead of $(\mathcal{F}_t)$ we will simply write $\mathcal{F}_t$. A stochastic process $X$ on $(\Omega, \mathcal{F}, P)$ is said to be $\mathcal{F}_t$ adapted if $X_t$ is $\mathcal{F}_t$-measurable for all $t$.

Definition 1.1

An $\mathcal{F}_t$-adapted stochastic process is a martingale if $E[M_t | \mathcal{F}_s] = M_s$ for all $s \leq t$.

Definition 1.2

A random variable $T; \Omega \to [0, \infty]$ is a stopping time if the event $\{T \leq t\} \in \mathcal{F}_t$ for all $0 \leq t \leq \infty$.

An example of a stopping time is a hitting time. Let $X_t$ be a stochastic process and let $\Lambda$ be a Borel set in $\mathbb{R}$. Define $T(\omega) = \inf\{t > 0 : X_t \in \Lambda\}$. then $T$ is called a hitting time of $\Lambda$ for $X$.

Definition 1.3

Let $T$ be a stopping time. The stopping time $\sigma$-algebra $\mathcal{F}_T$ is defined to be
\{ \Lambda \in \mathcal{F} : \Lambda \cap \{ T \leq t \} \in \mathcal{F}_t \text{ for all } t \geq 0 \}.

**Definition 1.4**

A martingale \( M \) is said to be closed by a random variable \( Y \) if \( E[|Y|] < \infty \) and \( M_t = E[Y \mid \mathcal{F}_t], \ 0 \leq t < \infty \).

**Theorem 1.5 (Doob's Optional Sampling Theorem) [Pr 90]**

Let \( M \) be a right continuous martingale which is closed by a random variable \( X_\infty \). Let \( S \) and \( T \) be two stopping times such that \( S \leq T \) a.s. Then \( M_S \) and \( M_T \) are integrable and

\[
M_S = E[M_T \mid \mathcal{F}_S].
\]

**Theorem 1.6 [Pr 90]**

Let \( M \) be a right continuous martingale which is uniformly integral, then \( Y = \lim_{t \to \infty} M_t \) exists a.s., \( E[|Y|] < \infty \), and \( Y \) closes \( M \) as a martingale.

**Theorem 1.7 (Doob's Maximal Inequality) [Pr 90]**

Let \( M \) be a martingale with \( M_\infty \in L^2 \) then

\[
E[\sup_{s \leq \infty} (M_s)^2] \leq 4E[M_\infty^2].
\]

**Definition 1.8**

An adapted cadlag process \( Y \) is a classical semimartingale if there exists processes \( M \) and \( A \) with \( M_0 = A_0 = 0 \) such that

\[
Y_t = Y_0 + M_t + A_t
\]

where \( M_t \) is a local martingale and \( A_t \) has trajectories with finite variation on compacts with probability one.
Semimartingales do not in general have paths of finite variation on compacts. Thus the fundamental theorem of calculus tells us that a semimartingale cannot be written as an integral with respect to Lebesgue measure. In order to express a semimartingale as an integral one must use stochastic calculus. In this paper we will use the Itô integral which is defined as follows. Let

\[ H_t = H_0 1_{\{0\}}(t) + \sum_{i=1}^{n} H^i 1_{(T_i, T_{i+1})}(t) \]

where \( 0 = T_1 \leq \ldots \leq T_{n+1} < \infty \) is a finite sequence of stopping times, and \( H_i \) is a finite-valued \( \mathcal{F}_t \)-adapted random variable. We define the stochastic integral of \( H_t \) with respect to the semimartingale \( X_t \) by

\[
\int_0^t H_s dX_s \equiv H_0 X_0 + \sum_{i=1}^{n} H^i (X_{t_{i+1}} - X_{t_i}).
\]

This definition can be extended to the integral of a process with cadlag trajectories by taking the \( L^2 \) limit for partitions with mesh going to zero.

Let \( X \) be a semimartingale. We define the quadratic variation of \( X \) by

\[
[X, X]_t = X_t^2 - 2 \int_0^t X_s^{-} dX_s.
\]

It turns out that \( [X, X] \) has finite variation on compacts. We define \( < X, X > \) to be the unique finite variation process such that \( [X, X]_t - < X, X >_t \) is a martingale for all \( t \geq 0 \).

**Definition 1.9**

A martingale \( M \) is said to be purely discontinuous if and only if \( [M, M]_t; \) (the continuous part of \( [M, M]_t \)) is equal to zero for all \( 0 \leq t \leq \infty \).
Theorem 1.10 [Pr 90]

A martingale $M$ is said to be purely discontinuous if and only if

$$[M, M]_t = \sum_{s \leq t} (\Delta M_s)^2$$

Theorem 1.11 [Pr 90]

Let $M$ be a local martingale. Then $M$ is a martingale with $E[M_t^2] < \infty$, for all $t \leq 0$ if and only if $E\{[M, M]_t\} < \infty$ for all $t \leq 0$. If $E[M, M]_t < \infty$ then $E[M_t^2] = E\{[M, M]_t\}$.

This leads to a useful restatement Doob's maximal inequality Let $M$ be a martingale with $M_\infty \in L^2$ then

$$E[\sup_{s \leq \infty} M_s^2] \leq 4E\{[M, M]_\infty\}$$

Theorem 1.12 [Pr 90]

Let $X$ be a semimartingale. Let $H$ be a stochastic process which is right continuous with left limits. Then

$$[\int H_s^- dX_s, \int H_s^- dX_s]_t = \int H_s^2 d[X, X]_s.$$  

1.2: Lévy Processes

In this paper we are interested in martingale Levy processes with bounded jumps. The main purpose of this section is to provide a characterization for the quadratic variation of such a process.

Definition 1.13

An adapted process $X$ with $X_0 = 0$ a.s. is a Levy process if
i) \( X \) has increments independent of the past: i.e. \( X_t - X_s \) is independent of \( \mathcal{F}_s \) \( 0 \leq s < t < \infty \).

ii) \( X \) has stationary increments: i.e. \( X_t - X_s \) has the same distribution as \( X_{t-s} \) \( 0 \leq s < t < \infty \)

iii) \( X_t \) is continuous in probability: i.e. \( \lim_{t \to s} X_t = X_s \) where the limit is taken in probability.

Two well known examples of Levy processes are the Brownian motion, and the Poisson process.

**Theorem 1.14** [Pr 90]

Let \( X \) be a Levy process. There exists a unique modification \( Y \) of \( X \) which is cadlag (right continuous with left limits) and which is also a Levy process.

In this paper we will always assume we are dealing with the cadlag version of a Levy Process.

We will first turn our attention to the jumps in Levy processes. Let \( \Delta X_t = X_t - X_{t-} \) If \( \sup_t |\Delta X_t| \leq C < \infty \) a.s. where \( C \) is a nonrandom constant we say that \( X \) has bounded jumps.

**Theorem 1.15** [Pr 90]

Let \( X \) be a Levy process with bounded jumps. Then \( E\{|X_t|^n\} < \infty \) \( n = 1, 2, \ldots \)

Let \( \Lambda \) be a Borel set in \( \mathbb{R} \) bounded away from 0 (that is \( 0 \notin \bar{\Lambda} \) where \( \bar{\Lambda} \) is the closure of \( \Lambda \)). Define \( N^\Lambda_t = \sum_{0 < s \leq t} 1_{\Lambda}(\Delta X_s) \). \( N^\Lambda \) is a counting process with stationary and independent increments and so it is a Poisson process.
**Theorem 1.16** [Pr 90]

The function \( \Lambda \to N_t^\Lambda(\omega) \) defines a \( \sigma \)-finite measure on \((0, \infty)\) for each fixed \((t, \omega)\). The set function \( \nu(\Lambda) = E[N_t^\Lambda] \) also defines a \( \sigma \)-finite measure on \((0, \infty)\).

The measure \( \nu \) defined by \( \nu(\Lambda) = E[N_t^\Lambda] = E[\sum_{0 < s \leq 1} 1_{\Lambda}(\Delta X_s)] \) is called the Levy measure of the Levy process \( X \). The for any given \( t \) the random measure given by \( \Lambda \to N_t^\Lambda(\omega) \) satisfies the following definition:

**Definition 1.17**

Let \((X, \mathcal{B}_X)\) be a measurable space. Let \( M \) be the collection of non-negative (possibly infinite) integer-valued measures on \((X, \mathcal{B}_X)\) and \( \mathcal{B}_M \) be the smallest \( \sigma \)-field on \( M \) with respect to which all \( \mu \in M \to \mu(B) \in \mathbb{Z}^+ \cup \{\infty\}, \ b \in \mathcal{B}_M \) are measurable. An \((M, \mathcal{B}_M)\)-valued random variable is called a Poisson random measure if

1. for each \( B \in \mathcal{B}_X \), \( \mu(B) \) is Poisson distributed.

2. if \( B_1, B_2, \ldots, B_n \in \mathcal{B}_X \) are disjoint then \( \mu(B_1), \mu(B_2), \ldots, \mu(B_n) \) are mutually independent.

**Theorem 1.18** [Pr 90]

Let \( \Lambda \) be a Borel set with \( 0 \notin \bar{\Lambda} \). Let \( \nu \) be the Levy measure of \( X \), and let \( f(x)1_\Lambda(x) \in L^2(\,d\nu) \). Then

\[
E[\int_\Lambda f(x)N_t(\cdot, \,dx)] = t \int_\Lambda f(x)\nu(\,dx),
\]

and also

\[
E[(\int_\Lambda f(x)N_t(\cdot, \,dx) - t \int_\Lambda f(x)\nu(\,dx))^2] = t \int_\Lambda f(x)^2\nu(\,dx).
\]
Theorem 1.19 [Pr 90]

Let $X$ be a levy process with jumps bounded by $a : \sup_s |\Delta X_s| \leq a < \infty$ a.s. Let $Z_t = X_t - E[X_t]$. Then $Z$ is a martingale and $Z_t = Z^c_t + Z^d_t$ where $Z^c_t$ is a martingale with continuous paths, $Z^d_t$ is a martingale

$$Z^d_t = \int_{\{|x| \leq a\}} x(N_t(\cdot, dx) - t\nu(dx))$$

and $Z^c_t$ and $Z^d_t$ are independent Levy processes.

Theorem 1.20 [Br 73]

Let $X$ be a Levy process. Then $X$ has a decomposition

$$X_t = B_t + \int_{\{|x| < 1\}} x(N_t(\cdot, dx) - t\nu(dx))$$

$$= tE[X_1 - \int_{\{|x| < 1\}} xN_1(\cdot, dx)] + \int_{\{|x| \geq 1\}} xN_t(\cdot, dx)$$

where $B$ is a Brownian motion; for any set $\Lambda, 0 \notin \Lambda, N^\Lambda_t = \int_\Lambda N_t(\cdot, dx)$ is a Poisson process independent of $B$; $N^\Lambda_t$ is independent of $N^\Gamma_t$ if $\Lambda$ and $\Gamma$ are disjoint: $N^\Lambda_t$ has parameter $\nu(\Lambda)$; and $\nu(dx)$ is a measure on $\mathbb{R}\setminus\{0\}$ such that

$$\int \min(1, x^2)\nu(dx) < \infty.$$ 

Remark

From theorem 1.19, if $X_t$ is a martingale Levy process with bounded jumps, then

$$X_t = X^c_t + \int_{\{|x| \leq a\}} x(N_t(\cdot, dx) - t\nu(dx)).$$

From theorem 20 we can see that

$$X^c_t = B_t + tE[X_1 - \int_{\{|x| < 1\}} xN_1(\cdot, dx)].$$
Since $X_t^c$ and $B_t$ are martingales, the second summand must be zero, so $X_t^c = B_t$.

Then every martingale Levy process with bounded jumps has a decomposition

$$X_t = B_t + \int_{\{|x| \leq a\}} x(N_t(\cdot, dx) - t\nu(dx)).$$

$$< X, X>_t = < B, B>_t + 2 < B_t, \int_{\{|x| \leq a\}} x(N_t(\cdot, dx) - t\nu(dx))>_t$$

$$+ < \int_{\{|x| \leq a\}} x(N_t(\cdot, dx) - t\nu(dx)), \int_{\{|x| \leq a\}} x(N_t(\cdot, dx) - t\nu(dx))>_t$$

$< B, B>_t$ is well known to equal $t$, and the quadratic variation of independent martingales is equal to zero, so

$$< \int_{\{|x| \leq a\}} x(N_t(\cdot, dx) - t\nu(dx)), \int_{\{|x| \leq a\}} x(N_t(\cdot, dx) - t\nu(dx))>_t$$

is the unique process such that

$$X_t^2 - t - < \int_{\{|x| \leq a\}} x(N_t(\cdot, dx) - t\nu(dx)), \int_{\{|x| \leq a\}} x(N_t(\cdot, dx) - t\nu(dx))>_t$$

is a martingale and

$$< M, \int_{\{|x| \leq a\}} x(N_t(\cdot, dx) - t\nu(dx))>_t = 0$$

for all martingales $M$. From theorem 1.18

$$< \int_{\{|x| \leq a\}} x(N_t(\cdot, dx) - t\nu(dx)), \int_{\{|x| \leq a\}} x(N_t(\cdot, dx) - t\nu(dx))>_t$$

$$= t \int_\Lambda x^2 \nu(dx).$$

so

$$< X, X>_t = t(1 + \int_\Lambda x^2 \nu(dx)).$$
1.3: Weak Convergence

Let $C_b(S)$ be the set of real-valued bounded continuous functions on the metric space $(S, d)$ with norm $\|f\|_\infty = \inf_{x \in S} |f(x)|$.

**Definition 1.21**

A sequence $\{Q_n\} \subset P(S)$ is said to converge weakly to $Q \in P(S)$ if

$$\lim_{n \to \infty} \int f \, dP_n = \int f \, dP, \quad f \in C_b(S)$$

Weak convergence generates a metrizable topology on $P(S)$. One metric which yields a topology equivalent to the topology of weak convergence is the Prohorov metric.

**Definition 1.22**

The Prohorov metric is defined by

$$\pi(V, Q) = \inf\{\epsilon > 0 : V(F) \leq Q(F^\epsilon) \quad \text{for all} \quad F \in C\}$$

where $C$ is the collection of closed subsets of $S$ and

$$F^\epsilon = \{x \in S : \inf_{y \in F} d(x, y) < \epsilon\}.$$

For details about the Prohorov metric see [EK 85].

**Theorem 1.23**

Let $(S, d)$ be arbitrary, and let $\{P_n\} \subset P(S)$ and $Q \in P(S)$. Then $a)$ implies $b)$. If $S$ is separable then $a)$ and $b)$ are equivalent.

- $a)$ $\lim_{n \to \infty} \pi(Q_n, Q) = 0$
- $b)$ $Q_n \Rightarrow Q$
Theorem 1.24

If $S$ is separable, then $P(S)$ is separable. If in addition $(S, d)$ is complete then $(P(S), \pi)$ is complete.

An important concept related to weak convergence is tightness.

Definition 1.25

A family of probability measures $M$ is tight if for each $\epsilon > 0$ there exists a compact set $K \subseteq S$ such that

$$\inf_{Q \in \mathcal{M}} Q(K) \geq 1 - \epsilon.$$  

Theorem 1.26 Let $(S, d)$ be separable and complete, and let $\mathcal{M} \subset P(S)$. Then $\mathcal{M}$ is tight if and only if $M$ is relatively compact in $P(S), \pi$.

We will use the following result by Sznitman to show tightness of measure-valued random variables.

Theorem 1.27 [Sz 82]

Let $(S, d)$ be complete and separable and let $\{\eta^n\}$ be a sequence of random variables defined on the spaces $(\Omega_n, F_n, P_n)$ with values in $P(S)$

The sequence of measures in $\mathcal{P}(P(S)) \{\mathcal{L}(\eta^n)\}$ is tight if and only if the sequence $I(\eta^n)$ of probability measures on $S$ defined by:

$$< I(\eta^n), f > = \int_{\Omega_n} < \eta^n, f > dP_n(\omega)$$

is also tight for all continuous bounded functions $f$. 

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1.4: Poisson Random Measure

**Definition 1.28**

Let $(V, \mathcal{V})$ be a measurable space.

$$N : \Omega \times (\mathcal{B}(\mathbb{R}) \times \mathcal{V}) \to \mathbb{R}$$

is a random measure if $N(\omega, \cdot)$ is a measure on $(\mathbb{R}^+ \times V)$ for all $\omega \in \Omega$ and $N(\cdot, B)$ is a random variable for all $B \in \mathcal{B} \times V$. $N$ is adapted if $N(\cdot, B) \in \mathcal{F}_t$ for all $B \subset [0, T] \times V$.

**Definition 1.29**

$N$ is $\sigma$-finite if there exists $\{V_n\} \uparrow V$ such that

$$E[||N(\cdot, [0, T] \times V_n)||] < \infty$$

for all $N$ and $t > 0$.

**Definition 1.30**

$N$ is a martingale random measure if for any $D \in \mathcal{V}$ such that

$$E(||N([0, T] \times E)||) < \infty$$

for all $t$, we have

$$\{N([0, T] \times E)\}_{t \leq 0}$$

is a martingale.
Definition 1.31

A $\sigma$-finite adapted random measure $N$ is said to be of class $(QL)$ if there exists a $\sigma$-finite random measure $\tilde{N}$ such that

$$\tilde{N} = N - \tilde{N}$$

is a martingale random measure and for all $D$ as in Definition 1.30, $\tilde{N}([0, T] \times E)$ is constant in $t$. $\tilde{N}$ is called the compensator of $n$.

Theorem 1.32 [IW 81]

Let $N$ be an integer valued adapted random measure on $\mathbb{R}^+ \times V$. Then there exists a sequence of stopping times $\{\tau_n\}$ and a $V$-valued optional process $p$ such that

$$N(\omega, A) = \sum_{s \geq 0} 1_D(\omega, s)1_A(s.p_s(\omega))$$

for all $A \in B(\mathbb{R}^+ \times V)$ where

$$D = \cup_n\{(\omega, \tau_n(\omega)) : \omega \in \Omega\}.$$ 

$D$ is called the jump set. $p$ is called the point process corresponding to the integer-valued random measure $N$.

Definition 1.33

A random measure $N$ is a Poisson random measure if

a) for each $B \in V$ such that $E[N(t, B)] < \infty$ for all $0 \leq t \leq T$ $N(\cdot, B)$ is a Poisson process.

b) if $B_1, \ldots, B_n$ are mutually disjoint, then $N(\cdot, B_1), N(\cdot, B_2), \ldots, N(\cdot, B_n)$ are mutually independent.
Theorem 1.34 [IW 81]

Let $V$ be a Borel set in $\mathbb{R}$ bounded away from zero (i.e. $0 \notin \overset{\frown}{V}$). Let $X$ be a Lévy process and define

$$N(t, V) = \sum_{0 < s \leq t} \Delta X_s 1_V(\Delta X_s).$$

Then $N$ is a Poisson random measure, and $\nu(\cdot) \times \lambda = E[N(\cdot, \cdot)] \times \lambda$ is the compensator of $N$ where $\lambda$ denotes Lebesgue measure.

Theorem 1.35 [IW 81]

Let $N(dt, dv)$ be the random measure defined by

$$N(t, V) = \int \int \nu N(ds, dv) = \sum_{0 < s \leq t} \Delta X_s 1_V(\Delta X_s).$$

Then

$$\int_0^t \int_V f(v) N(ds, dv) = \sum_{0 < s \leq t} f(\Delta X_s) 1_V(\Delta X_s).$$

Theorem 1.36 [IW 81]

Let

$$F^1 = \{ f(t, v, \omega) : f \text{ is } \mathcal{F}_t \text{ - predictable and for every } t > 0 \}

E[\int_0^t \int_V | f(s, v, \omega) | \nu(dv) ds] < \infty \}$$

and

$$F^2 = \{ f(t, v, \omega) : f \text{ is } \mathcal{F}_t \text{ - predictable and for every } t > 0 \}

E[\int_0^t \int_V f(s, v, \omega)^2 \nu(dv) ds] < \infty \}. $$

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If \( f \in F^1 \) then \( \int_0^t \int_V f(s, v, \omega)N(ds, dv) \) is a Lévy process and

\[
\int_0^t \int_V f(s, v, \omega)[N(ds, dv) - \nu(dv) ds]
\]

is a martingale. If \( f \in F^1 \cap F^2 \) then

(1) \( \langle \int_0^t \int_V f(s, v, \omega)[N(ds, dv) - \nu(dv) ds] \rangle_t = \int_0^t \int_V f(s, v, \omega)^2 \nu(dv) ds \)

If \( f \in F^2 \) then if we let

\[ f_n(s, v, \omega) = 1_{(-\infty, \eta)}(f(s, v, \omega))f(s, v, \omega) \]

then \( f_n \in F^1 \cap F^2 \) and we can define

\[
\int_0^t \int_V f(s, v, \omega)[N(ds, dv) - \nu(dv) ds]
\]

to be the limit in \( \mathcal{M}_2 \) of

\[
\int_0^t \int_V f_n(s, v, \omega)[N(ds, dv) - \nu(dv) ds]
\]

and (1) holds.

**Theorem 1.37** [IW 81]

Let

(2) \[ X_t = X_0 + \int_0^t b(x, X_s) \, ds + \int_0^t \int_V \sigma(s, x, X_s, v) \tilde{N}(dv, ds) \]

where \( X_t \) is an \( \mathbb{R}^n \)-valued random variable, \( X_0 \) has finite second moment, \( \tilde{N} \) is a compensated \( \mathbb{R}^n \)-valued Poisson random measure on \( \mathbb{R}^+ \times V \), and \( b, \sigma \) satisfy the following assumptions:
a) for all \( t \in [0, T] \), \( b(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n \) is continuous,

b) for all \( t \in [0, T] \) and \( x \in \mathbb{R}^n \), \( \sigma(t, x, \cdot) \in L^2(V, \nu : \mathbb{R}^n) \),

c) for each fixed \( t \), \( x \to \sigma(t, x, \cdot) \) from \( \mathbb{R}^n \to L^2(V, \nu : \mathbb{R}^n) \) is continuous,

d) 
\[
|b(t, x)|^2 \leq K(1 + |x|^2)
\]
for all \( t \in [0, T] \) and \( x \in \mathbb{R}^n \),

e) 
\[
\int_V |\sigma(t, x, \nu)|^2 \nu(d\nu) \leq K(1 + |x|^2)
\]
f) 
\[
|b(t, x) - b(t, y)|^2 + \int_V |\sigma(t, x, \nu) - \sigma(t, y, \nu)|^2 \nu(d\nu) \leq L |x - y|^2
\]
for all \( t \in [0, T] \) and \( x, y \in \mathbb{R}^n \). Then there exists a strong, pathwise unique solution of (2) such that

\[
E[\sup_{0 \leq t \leq T} |X_t|^2] < \infty.
\]

**Theorem 1.38 (Itô's Lemma) [IW 81]**

Let \( X_0 \) be a random variable with finite second moment. Let \( M \) be a continuous square integral local martingale, and \( A \) be a continuous process with finite variation on compacts, and Let \( \tilde{N} \) be a compensated Poisson random measure. Define

\[
X_t^i = X_0^i + M_t^i + A_t^i + \int_0^t \int_V f^i(s, X_{s-}^i, \nu) \tilde{N}(dv, ds)
\]
Let \( F \in C^2 \). Then \( F(X_t) \) is a semimartingale and

\[
F(X_t) - F(X_0) = \int_0^t F'(X_s) dM_s + \int_0^t F'(X_s) dA_s + \int_0^t \frac{1}{2} F''(X_s) d[M, M]_s \\
+ \int_0^t \int_V [F(X_{s-} + f(s, X_{s-}, v)) - F(X_{s-})] N(dv, dt) \\
+ \int_0^t \int_V [F(X_{s-} + f(s, X_{s-}, v)) - F(X_s) - F'(X_{s-})f(s, X_{s-}, v)] \nu(dv) dt
\]

**Theorem 1.39 [IW 81]**

Let \( B \) be a \( d \)-dimensional Brownian motion on \((\Omega, \mathcal{F}, P)\) and let \( \mathcal{F}_t \) be the completion of \( \mathcal{F}_t \) under \( P \) of the filtration \( \sigma(B_s)_{s \leq t} \). Then for any square-integrable cadlag martingale \( M \), adapted to \( \mathcal{F}_t \) with \( M_0 = 0 \) there exists adapted processes \( Y^j, 1 \leq j \leq d \) such that

\[
E[\int_0^T (Y^j_t)^2 dt < \infty; 1 \leq j \leq d]
\]

for every \( 0 < T < \infty \), and

\[
M_t = \sum_{j=1}^d \int_0^t Y^j_s dB^j_s.
\]

In particular, \( M \) is a.s. continuous. Furthermore, if \( \tilde{Y}^j; 1 \leq j \leq d \) are any other adapted processes satisfying (2) and (4), then

\[
\sum_{j=1}^d \int_0^\infty (Y^j_t - \tilde{Y}^j_t)^2 dt = 0, \text{ a.s.}
\]

**Theorem 1.40 [IW 81]**

Let \((\Omega, \mathcal{F}, P)\) be a probability space with filtration \( \mathcal{F}_t \). Let \((S, B_S)\) be a measurable space and \( p \) be an adapted point process of class (QL) on \( S \) with the compensator \( \tilde{N}(dt, dv) = \nu(t, v, \omega) dt \). Define \( S^* \) to be \( S \cup \Delta \) where \( \Delta \) is and
extra point to be attached to $S$, and let $B_S$ be the $\sigma$-field generated by $B_S$ and $\Delta$. Suppose that there exists a $\sigma$-finite measure $m$ on a standard measurable space $(Z, B_Z)$ and a predictable $S^*$-valued process

$$\Theta(t, z, \omega) : [0, \infty) \times Z \times \Omega \to S^*$$

such that

$$m(\{z; \Theta(t, z, \omega) \in E\}) = q(t, E, \omega)$$

for every $E \in B_S$. Then, on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ of $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t$, there exists a stationary $\tilde{\mathcal{F}}_t$ Poisson point process $q$ on $Z$ with Lévy measure $m$ such that

$$N_p((0, t] \times E) = \int_0^t \int_Z 1_E(\Theta(s, z, \omega))N_q(ds, dz)$$

1.5: The Martingale Problem

Let $\{X_t : t \geq 0\}$ be an $\mathcal{F}_t$-adapted stochastic process defined on $(\Omega, \mathcal{F})$ with values in a separable complete metric space $E$. Let $\{P_x\}_{x \in E}$ be a collection of probability measures on $(\Omega, \mathcal{F})$.

**Definition 1.41**

$(\Omega, \mathcal{F}, (\mathcal{F}_t), X_t, \{P_x\})$ is a time homogeneous Markov process with state space $(E, B(E))$ if:

1. For all $t \geq 0$ and $\Gamma \in B(E)$, $P_x(X_t \in \Gamma) = P(t, x, \Gamma)$ is $B(E)$-measurable

2. $P(0, x, E \setminus \{x\}) = 0.$
(7) \[ P_x(X_t \in \Gamma | \mathcal{F}_s) = P(t-s, X_s, \Gamma) \] holds a.s. with respect to \( P_x \).

For the purposes of this paper we will assume that \( X_t \) takes paths in \( D(E)[0, T] \).

\( P(t, x, \Gamma) \) is called the transition function of the Markov process. Let \( B(E) \) be the space of bounded real-valued functions on \( E \). For \( f \in B(E) \) let \( ||f|| = \sup_{x \in E} |f(x)| \). On \( (B(E), || \cdot ||) \) define the non-negative contraction semigroup \( \{ T_t : t \geq 0 \} \) by

\[ (T_t f)(x) = \int_E f(y) p(t, x, dy) = E[f(X_t) | X_0 = x]. \]

The infinitesimal generator \( A \) of the semigroup \( T_t \) is defined by

\[ Af = \lim_{t \downarrow 0} \frac{T_t f - f}{t} \]

where the limit is in the sense of convergence in \( || \cdot || \).

If \( P(t, x, \Gamma) \) is stochastically continuous in the sense that

\[ \lim_{t \downarrow 0} P(t, x, E\setminus N) = 0 \]

for every \( x \in E \) and every neighborhood \( N \) of \( f \), then \( A \) uniquely determines all finite-dimensional distributions of the Markov process.

**Lemma 1.42 [EK 85]**

Let \( X \) be an \( E \)-valued Markov process with transition function \( P(t, x, \Gamma) \) with associated semigroup \( \{ T_t \} \) and generator \( A \). Then

\[ M_t = f(X_t) - \int_0^t Af(X_s) \, ds \]

is a martingale.
Definition 1.43

By a solution to the martingale problem for $A$ we mean a measurable stochastic process $X$ with values in $E$ defined on some probability space $(\Omega, \mathcal{F}, P)$ such that

$$f(X_t) - \int_0^t Af(X_s) \, ds$$

is a martingale with respect to the filtration $\mathcal{F}^X = \sigma(X_s, 0 \leq s \leq t)$.

When an initial distribution $\mu \in P(E)$ is specified we say that a solution $X$ of the martingale problem for $A$ is a solution of the martingale problem for $(A, \mu)$ if $P(X_0) = \mu$. For convenience we will say that a probability measure $P \in P(D_E[0,T])$ is a solution of the martingale problem for $A$ if the coordinate process defined on $(D_E[0,T], B(D_E[0,T]), P)$ by $X(t, \omega) = \omega(t)$ $\omega \in D_E[0,T]$, $t \geq 0$ is a solution of the martingale problem for $A$.

Lemma 1.44 [EK 85]

A measurable process $X$ is a solution of the martingale problem for $A$ if and only if

$$0 = E[(f(X_t) - f(X_s) - \int_s^t Af(X_s) \, ds) \prod_{k=1}^n g_k(X_{s_k})]$$

for all $n \in \mathbb{Z}^+$, $s = s_1 < \ldots < s_n = t$, and $g_1 \ldots, g_n \in B(E)$.

We are particularly interested in the martingale problem posed by Markov processes of the form

$$X_t = X_0 + \int \sigma(s, X_s) \, dB_s + \int_0^t b(s, X_s) \, ds$$

(8)
where $B_s$ is a Brownian motion and

\begin{align}
X_t = X_0 + \int \int V \sigma(s, X_s, v) \tilde{N}(dv, ds) + \int_0^t b(s, X_s) \, ds
\end{align}

where $\tilde{N}(dv, ds)$ is an adapted Poisson random measure.

There are two types of solutions to stochastic differential equations.

**Definition 1.45**

A weak solution of a stochastic differential equation exists if there exists a probability space $(\Omega, \mathcal{F}, P)$ and filtration $\mathcal{F}_t$ which will support the driving terms of the equation and a solution to the equation.

**Definition 1.46**

A strong solution to a stochastic differential equation is a solution which exists on a given probability space, with a given filtration, and given driving terms.

There are two types of uniqueness to a the solution of a stochastic differential equation which we consider.

**Definition 1.47**

Pathwise uniqueness is said to hold for solutions of a stochastic integral equation if given two solutions $X$ and $X'$ on the same probability space

$P(X_t = X'_t, 0 \leq t \leq T) = 1$.

**Definition 1.48**

Distribution uniqueness is said to hold for solutions of a stochastic integral equation if whenever $X$ is a solution to the equation on a probability space
\((\Omega, \mathcal{F}, P)\) and \(X'\) is a solution to the equation on a probability space \((\Omega', \mathcal{F}', P')\) then \(X\) and \(X'\) share the same law.

Let

\begin{equation}
A_1(f)(x) = f'(x)b(s, x) + \frac{1}{2} f''\sigma(s, x)^2
\end{equation}

and

\begin{equation}
A_2(f)(x) = f'(x)b(s, x) + \int_V f(x + \sigma(s, x, v)) - f(x) - f'(x)\sigma(s, x, v) \nu(dv)
\end{equation}

Observe that if \(X^1\) is a solution of the stochastic integral equation (8) and \(X^2\) is a solution of the stochastic integral equation (9), then by Itô's lemma

\[
f(X^1_t) - f(X_0) - \int_0^t A_1(f)(X^1_s) \, ds = \int_0^t f'(X^1_s)\sigma(s, X^1_s) \, dB_s
\]

and

\[
f(X^2_t) - f(X_0) - \int_0^t A_2(f)(X^2_s) \, ds = \int_0^t \int_V f(X^2_s + \sigma(s, X^2_s, v)) - f(X^2_s) \, \tilde{N}(dv, ds)
\]

for all \(t \geq 0\) and \(f \in C^2_c(\mathbb{R})\) so \(X^1\) is a solution of the martingale problem posed by (8) and \(X^2\) is a solution of the martingale problem posed by (9). The next theorem tells us that in the case of \(A_1\) the converse of this observation holds as well.
Theorem 1.49 [EK 85]

Let $X$ be a stochastic process on $(\Omega, \mathcal{F}, P)$ such that

$$f(X_t) - f(X_0) - \int_0^t A_1(f)(X_s) \, ds$$

is a continuous local martingale for all $t \geq 0$ and $f \in C^2_c(\mathbb{R})$ then there exists a Brownian motion $B_s$ defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ of $(\Omega, \mathcal{F}, P)$ such that $X$ is a weak solution to (8).

Suppose there is a sequence of processes whose martingale problem solutions converge to a solution of the martingale problem posed by (9). We show that a martingale problem solution of the limit yields a weak solution of the stochastic differential equation.

Theorem 1.50

Let $\eta \in P(\mathbb{R})$ be a solution of the martingale problem posed by (2). Then $\eta$ is a weak solution on $[0,T]$ of (9).

Proof

Let

$$M(t, x) = x_t - x_0 - \int_0^t b(s, x_s) \, ds$$

$$- \int_0^t \int_V f(x + \sigma(s, x, v)) - f(x) - f'(x)\sigma(s, x, v) \nu(du) \, ds$$

where $x_s(\omega) = \omega(s)$ for all $\omega \in D[0,T]$. Then $M(t, x)$ is a $\eta$-square integrable martingale. and

$$< M > (t, x) = \int_0^t \int_V \sigma^2(s, X_s, v) \nu(du) \, ds.$$

Besides it can be shown that $M(t, x)$ is a purely discontinuous martingale. See [KX 96] or [Ja 79]. We can also identify the compensator of the point process.
$\Delta X_t$. That is if

$$A = \{ A \in \mathcal{B}(\mathbb{R}\setminus\{0\}) : E_\eta[\sum_{0 \leq s \leq t} 1_A(\Delta X_s)] < \infty \text{ for all } 0 < t \leq T \}$$

then for each $A \in A$.

$$\sum_{0 \leq s \leq T} 1_A(\Delta X_s) - \int_0^t \int V 1_A(\sigma(s, X_s, v)) \nu(dv) ds$$

is a $\eta$-martingale. Thus the point process $\Delta M(t) = \Delta X_t$ is a point process with compensator $q(t, dv, \omega) dt$ where

$$q(t, A, \omega) = \nu\{v : \sigma(t, X_{t-}, v) \in A\}.$$ 

Therefore by the martingale representation theorem (see [IW 81] or [Ja 79] for a proof) on an extension $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \hat{F}_t)$ of the space $(D[0, T], \mathcal{B}(D[0, T]), \eta, \mathcal{B}_t)$ there exists a Poisson random measure $N$ with compensator $\nu dt$ such that

$$M_t = \int_0^t \int V \sigma(s, X_{s-}, v) \hat{N}(dv, ds).$$

Therefore

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \int V \sigma(S, X_{s-}, v) \hat{N}(dv, ds)$$

and we are done.

**Theorem 1.51** [YW 71]

Pathwise uniqueness and weak existence implies a unique strong solution.
1.6: The Skorohod Metric

Let $D_E[0, T]$ denote the space of $E$-valued right continuous functions with left limits. A suitable topology on $D_E[0, T]$ is the one due to Skorohod. We give below the definition of the Skorohod topology and a few pertinent results.

Definition 1.52

Let $(E, r)$ denote a metric space. Let $\Lambda$ be the collection of strictly increasing functions $\lambda$ mapping $[0, T]$ onto $[0, T]$. There is a metrizable topology on $D_E[0, T]$, called the Skorohod topology which is characterized as follows: A sequence $x_n(t)$ converges to $x(t)$ if and only if there is a sequence $\{\lambda_n\} \subset \Lambda$ such that

\begin{align*}
&\text{a)} \quad \sup_{0 \leq s \leq T} | \lambda_n(s) - s | \to 0 \\
&\text{b)} \quad \sup_{0 \leq s \leq T} | x_n(\lambda_n(s)) - x(s) | \to 0.
\end{align*}

Theorem 1.53 [Bi 68]

Let

$$
\gamma(\lambda) = \sup_{s > t \leq 0} | \log \left( \frac{\lambda(s) - \lambda(t)}{s - t} \right) | < \infty,
$$

then

$$
d(x, y) = \inf_{\lambda \in \Lambda} \gamma(\lambda) \lor \sup_{0 \leq t \leq T} r(x(t), y(\lambda(t))).
$$

is a metric which generates the Skorohod topology.

Remarks

(i) If $x \in D_E[0, T]$ then $x$ has at most countably many points of discontinuity.
(ii) Suppose $x_n(\cdot) \rightarrow x(\cdot)$ in the Skorohod topology. Let $t$ be a continuity point of $x(\cdot)$. Then $x_n(t) \rightarrow x(t)$

**Theorem 1.54 [Bi 68]**

If $E$ is separable, then $D_E[0, T]$ is separable. If $E$ is complete then $D_E[0, T]$ is complete.

**Theorem 1.55 (Aldous Tightness Criterion) [Al 78]**

Let $\{ (\tau^n, \delta^n) \}$ be such that for each $n$ $\tau^n$ is a bounded stopping time, and $\delta^n$ is a constant $0 \leq \delta^n + \tau^n \leq T$ and $\delta^n \rightarrow 0$ as $n \rightarrow \infty$. Suppose $\{ X_n \}$ is a sequence of $D_E[0, T]$ valued random variables such that:

a) $\{X_n(t)\}$ is tight for all $0 \leq t \leq T$,

b) $X_n(\tau^n + \delta^n) - X_n(\tau^n) \rightarrow 0$ for all sequences $\{ (\tau^n, \delta^n) \}$. Then $\{X_n\}$ is tight in $D_E[0, T]$.

The Aldous tightness criterion is easy to verify. In dealing with a sequence of continuous stochastic processes, the Aldous tightness criterion can still be used to establish tightness. However the tightness will be in $D_E[0, T]$. The following results will allow us to treat continuous processes as processes in $D_E[0, T]$ and then translate our results into one where the underlying space is $C_E[0, T]$.

**Proposition 1.56**

$C_E[0, T]$ is closed in $D_E[0, T]$.

**Proof**

Let $x$ be discontinuous with at least one jump of magnitude greater than $\epsilon$. Let the jump occur at $t = t_0$. Then for all $\lambda \in \Lambda$, and for all $y \in C_E[0, T]$, either
\[ |x(t_0) - y(\lambda(t_0))| > \frac{\varepsilon}{2}, \]

or

\[ \lim_{t \to t_0} |x(t) - y(\lambda(t))| > \frac{\varepsilon}{2}, \]

so

\[ \sup_{0 \leq t \leq T} |x(t) - y(\lambda(t))| > \frac{\varepsilon}{2}. \]

So \( C[0, T] \subset D[0, T] \) contains all of its limit points and is therefore closed.

**Proposition 1.57**

Let \( \tilde{C}_E[0, T] \) denote the space \( C_E[0, T] \) equipped with the Skorohod topology relativized to the subset \( C_E[0, T] \). By \( \tilde{C}_E[0, T] \) we will mean the topology of uniform convergence on \( C_E[0, T] \). Then

\[ \tilde{C}_E[0, T] = C_E[0, T] \]

**Proof**

Let \( \{x_n\} \subset C_E[0, T] \), \( x \in C_E[0, T] \), such that \( x_n \to x \) uniformly. Then \( x_n \to x \) in the Skorohod topology (by taking \( \lambda_n(t) = t \) for all \( n \)).

Conversely, let \( x_n \to x \) in the Skorohod topology. Then

\[ \sup_{0 \leq t \leq T} |x_n(t) - x(t)| \leq \gamma(\lambda_n) \vee \left[ \sup_{0 \leq t \leq T} (|x_n(t) - x(\lambda_n(t))| + |x(\lambda_n(t)) - x(t)|) \right] \]

for all \( \lambda_n \in \Lambda \). Using the continuity of \( x \) and the definition of the Skorohod topology, the right hand side of the inequality converges to zero for a suitable choice of \( \{\lambda_n\} \). So \( x_n \to x \) uniformly and we are done.
From now on we need make no differentiation between \( \mathcal{C}_E[0,T] \) and \( \tilde{\mathcal{C}}_E[0,T] \) but may simply write \( C_E[0,T] \). In fact, since \( C_E[0,T] \) is closed in \( D_E[0,T] \), it is easy to see that the Borel \( \sigma \)-algebra on \( C_E[0,T] \) formed by the topology of uniform convergence is the same as the Borel \( \sigma \)-algebra on \( D_E[0,T] \) restricted to \( C_E[0,T] \).

**Proposition 1.58**

Let \( \pi(\cdot, \cdot) \) be the Prohorov metric on \( P(D_E[0,T]) \), and let \( \pi^c(\cdot, \cdot) \) be the Prohorov metric on \( P(C_E[0,T]) \). Suppose that \( S, Q \in P(D_E[0,T]) \), and \( S(C_E[0,T]) = Q(C_E[0,T]) = 1 \).

Define \( K, L \in P(C_E[0,T]) \) by \( K = S |_B \), and \( L = Q |_B \), where \( B \) is the Borel \( \sigma \)-algebra on \( C_E[0,T] \). Then \( \pi(S, Q) = \pi^c(K, L) \).

**Proof**

\[
\pi(S, Q) = \inf \{ \epsilon : S(F) < Q(F^\epsilon) + \epsilon \text{ for all } D_E[0,T] - \text{closed sets } F \}
\]

\[
= \inf \{ \epsilon : S(F \cap C_E[0,Y]) < Q(F^\epsilon \cap C_E[0,Y]) + \epsilon \text{ for all } D_E[0,T] - \text{closed sets } F \}
\]

\[
= \pi^c(K, L),
\]

where the second to last equality is due to the fact that \( Q(C_E[0,T]) = 1 \).

**Proposition 1.59**

Let \( \mathcal{P}(P(D_E[0,T])) \) be the set of probability measures on \( P(D_E[0,T]) \) where
$P(D_E[0, T])$ is topologized by the Prohorov metric. Topologize $\mathcal{P}(P(D_E[0, T]))$ again with the Prohorov metric $\pi(\cdot, \cdot)$. Let $\mathcal{P}(P(C_E[0, T]))$ be the elements of $\mathcal{P}(P(D_E[0, T]))$ which have their mass concentrated in $P(C_E[0, T])$. Then $\mathcal{P}(P(C_E[0, T]))$ is closed in $\mathcal{P}(P(D_E[0, T]))$

**Proof**

Suppose $Q \in \mathcal{P}(P(D_E[0, T])) \setminus \mathcal{P}(P(C_E[0, T]))$, $S \in \mathcal{P}(P(C_E[0, T]))$. Then for some $\delta > 0$,

$$Q(P(D_E[0, T]) \setminus P(C_E[0, T])) = \delta$$

$$\lim_{\mu \to 0} Q(P(C_E[0, T])^{(\mu)}) = 1 - \delta.$$ 

So for some $\mu > 0$

$$1 - \delta \leq Q(P(C_E[0, T])^{(\mu)}) < 1 - \frac{\delta}{2}.$$ 

If $\frac{\delta}{2} \geq \mu$ then

$$S[P(C_E[0, T])] = 1 > Q(P(C_E[0, T])^{(\mu)}) + \mu.$$ 

If $\frac{\delta}{2} \leq \mu$ then

$$S[P(C_E[0, T])] = 1 > Q(P(C_E[0, T])^{\frac{\delta}{2}}) + \frac{\delta}{2},$$

so $\pi^P(S, Q) > \frac{\delta}{2}$. Thus $\mathcal{P}(P(D_E[0, T])) \setminus \mathcal{P}(P(C_E[0, T]))$ is open, so $\mathcal{P}(P(C_E[0, T]))$ is closed.
Proposition 1.60

Suppose that \( \{Q_n\} \subset \mathcal{P}(P(D_E[0, T])), Q \in \mathcal{P}(P(D_E[0, T])) \), and

\[
Q_n(P(C_E[0, T])) = Q(P(C_E[0, T])) = 1
\]

for all \( n \). Define \( \{S_n\} \subset \mathcal{P}(P(D_E[0, T])), S \in \mathcal{P}(P(D_E[0, T])) \) by \( S_n = Q_n |_{B^P} \), and \( S = Q |_{B^P} \), where \( B^P \) is the Borel \( \sigma \)-algebra on \( \mathcal{P}(P(C_E[0, T])) \). Then \( Q_n \Rightarrow Q \) implies \( S_n \Rightarrow S \).

Proof

\[
\pi(Q_n, Q) = \inf\{ \epsilon : Q_n(F) < Q(F^c) + \epsilon \text{ for all } D_E[0, T] - \text{closed sets } F \}
\]

\[
= \inf\{ \epsilon : Q_n(F \cap P(C_E[0, T])) < Q(F^c \cap P(C_E[0, T])) + \epsilon \text{ for all } P(D_E[0, T]) - \text{closed sets } F \}
\]

\[
= \pi^P(Q_n, Q),
\]

where the second to last equality is due to the facts that \( Q(C_E[0, T]) = 1 \). Since \( \pi(S_n, S) \to 0 \), we are done.

Finally here are two useful results concerning finite dimensional projections of \( D_E[0, T] \)-valued random variables.

Lemma 1.61

If \( X \) is a process with sample paths in \( D_E[0, T] \), then the complement in \([0, T]\) of

\[
D(X) = \{ t \geq 0 : p\{X_t = X_{t-}\} = 1 \}
\]

is at most countable.
Lemma 1.62

Let $E$ be separable, and let $X^n$, $X$ be processes with sample paths in $D_{E}[0,T]$. Then if $X^n \Rightarrow X$, then

$$(X^n_{t_1}, \ldots, X^n_{t_k}) \Rightarrow (X_{t_1}, \ldots, X_{t_k})$$

for all $\{t_1, \ldots, t_k\} \subset D(X)$.

Lemma 1.63 Let $\{X^{n,i}\}$ be a collection of $D_{E}[0,T]$-valued random variables such for fixed $n \{X^{n,i}\}, 1 \leq i \leq n$ is a set of identically distributed random variables, and for fixed $i$, the sequence $\{X^{n,i}\}$ is tight. Then $\{\eta^n\} = \{\frac{1}{n} \sum_{j=1}^{n} \delta_{X^{n,i}}\}$ is tight as a $P(D_{E}[0,T])$-valued random variable.

Proof

Since $\{X^{n,i}, \ldots, X^{n,n}\}$ is exchangeable and $\{X^{n,i}\}$ is tight for all $i$. For any $\epsilon > 0$ we may pick a compact subset $A_{\epsilon} \in D([0,T])$ such that

$$P(X^{n,i}_{t} \notin A_{\epsilon}^c) < \epsilon$$

for all $i \leq n \in \mathbb{Z}^+$. 

$$I(\eta^n)(A_{\epsilon}^c) = I(\eta^n), 1_{A_{\epsilon}^c} >$$

$$= E[\frac{1}{n} \sum_{i=1}^{n} 1_{A_{\epsilon}^c}(X^{n,i})]$$

$$= \frac{1}{n} \sum_{i=1}^{n} P(X^{n,i} \in A_{\epsilon}^c)$$

$$= P(X^{n,i} \in A_{\epsilon}^c) < \epsilon.$$ 

So by Theorem 1.27 we are done.

Theorem 1.64 Let $\{X^{n,i}\}$ be as above. Fix $t \geq 0$, $\{\eta^n_t\} = \{\frac{1}{n} \sum_{j=1}^{n} \delta_{X^{n,i}_t}\}$ is tight as a $P(\mathbb{R})$-valued random variable.
Proof

Since \( \{X_t^{n,1}, \ldots, X_t^{n,n}\} \) is exchangeable and \( \{X_t^{n,i}\} \) is tight for all \( i \). For any \( \epsilon > 0 \) we may pick a compact subset \( A_\epsilon \in D([0,T]) \) such that

\[
P(X_t^{n,i} \in A_\epsilon) < \epsilon
\]

for all \( i \leq n \in \mathbb{Z}^+ \).

\[
I(\eta_t^n)(A_\epsilon) = \langle I(\eta_t^n), 1_{A_\epsilon} \rangle
\]

\[
= E\left[ \frac{1}{n} \sum_{i=1}^{n} 1_{A_\epsilon}(X_t^{n,i}) \right]
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} P(X_t^{n,i} \in A_\epsilon)
\]

\[
= P\{X_t^{n,i} \in A_\epsilon\} < \epsilon.
\]

So by Theorem 1.27 we are done

1.7: The Vasserstein Metric

We will need to make use of a metric on the space of all probability distributions on \((S, B)\) where \( S \) is a metric space with metric \( r(x_1, x_2) \) and \( B \) is the Borel \( \sigma \)-algebra on \( S \). Let \( P \) and \( Q \) be a pair of probability distributions on \((S, B)\). We will define the Vassershtein metric by

\[
\rho(P, Q) = \inf E[r(X, Y)]
\]

Where the infimum is taken over all pairs of random variables \( X \) and \( Y \) with values in \((S, B)\) such that \( X \) and \( Y \) have distributions \( Q \) and \( P \) respectively.

We will use the following theorem which is a slightly altered version of the one appearing in [Do 70].
Theorem 1.65

Let \( \mathcal{B} \) be the \( \sigma \)-algebra of Borel sets on \( S \). Let \( P_0 \) be a probability distribution on \( (S, \mathcal{B}) \) and \( U(P_0) \) the collection of all probability distributions \( P \) on \( (S, \mathcal{B}) \) such that \( \rho(P_0, P) < \infty \). Then \( U(P_0) \) with the metric \( \rho(\cdot, \cdot) \) forms a metric space. If the metric space \( S \) is separable, then the metric space \( U(P_0) \) is complete. If the sequence \( P_n \) converges to \( \bar{P} \) in the Vasserstein metric, then \( P_n \) converges to \( \bar{P} \) in the topology of weak convergence. Let \( W \subset U(P_0) \) be such that for some \( x_0 \in S \), for all \( \varepsilon > 0 \), there exists a compact set \( K_\varepsilon \) such that

\[
\int_{(K_\varepsilon)^c} r(x, x_0) P(dx) < \varepsilon \text{ for all } P \in W.
\]

Then in the case of separable \( S \) the Vasserstein metric on \( W \) is topologically equivalent to the topology of weak convergence.

Proof

First we compare the Prohorov metric \( \pi(P, Q) \) with the Vasserstein metric \( \rho(P, Q) \). Suppose \( \pi(P, Q) > \varepsilon \), i.e. there exists a closed set \( F \) such that \( P(F) - Q(F^c) > \varepsilon \). Let \( (\xi, \eta) \) be a random variable on a probability space \( (\Omega, \mathcal{F}, M) \) with partial distributions \( P \) and \( Q \) respectively. Let \( \omega \in \xi^{-1}(F) \), but \( \omega \notin \eta^{-1}(F^c) \).

Then \( |\xi(\omega) - \eta(\omega)| > \varepsilon \). Then

\[
M(\xi^{-1}(F) \setminus \eta^{-1}(F^c)) = M(\xi^{-1}(F)) - M(\xi^{-1}(F) \cap \eta^{-1}(F^c)) \\
\geq M(\xi^{-1}(F)) - M(\eta^{-1}(F^c)) \\
= P(F) - Q(F^c) > \varepsilon
\]

so \( |\xi(\omega) - \eta(\omega)| > \varepsilon \) on a set of measure greater than equal to \( \varepsilon \). Thus

\[
E[|\xi(\omega) - \eta(\omega)|] > \varepsilon^2
\]
and

\[ \rho(P, Q) \geq \pi(P, Q)^2. \]

To see that \( \rho(\cdot, \cdot) \) is a metric, we note that

a) \( \rho(Q, P) = \rho(P, Q) \)

b) If \( P = Q \) then \( \rho(P, Q) = 0 \). If \( \rho(P, Q) = 0 \) then \( \pi(P, Q) = 0 \) so \( P = Q \).

So we need only prove the triangle inequality. Suppose \( Q, R, T \) are probability distributions on \((S, B)\). Let \((\xi, \eta)\) be a \( S^2 \) valued random variable with marginal distributions \( Q \), and \( R \), respectively. Let \((\eta', \zeta)\) be a \( S^2 \) valued random variable with marginal distributions \( R \), and \( T \), respectively. Let \( P[\xi \in A \mid \eta = x] \) be the conditional probability of \( \xi \) given \( \eta = x \) i.e.

\[ P(\xi \in A, \eta \in B) = \int_B P[\xi \in A \mid \eta = x] R(\, dx) \]

for all \( A, B \) in \( S \). Similarly let \( P[\zeta \in A \mid \eta = x] \) be the conditional probability of \( \zeta \) given \( \eta = x \). Let

\[ \tilde{P}(A, B, C) = \int_B P[\xi \in A \mid \eta = x] P[\zeta \in C \mid \eta = x] Q(\, dx) \]

\( \tilde{P}(S^3) = 1 \) so \( \tilde{P} \) is a probability measure. The martingale distributions with respect to each coordinate are \( Q \), \( R \), and \( T \), respectively. This probability measure clearly satisfies Kolmogorov's consistency conditions so there exists a random vector \((\xi, \hat{\eta}, \hat{\zeta})\) which has \( \tilde{P} \) as its law. Then

\[ \rho(Q, T) \leq E[r(\hat{\xi}, \hat{\zeta})] \leq E[r(\xi, \hat{\eta})] + E[r(\hat{\eta}, \zeta)]. \]

The right side of the above equation is arbitrarily close to \( \rho(Q, R) + \rho(R, T) \), so the triangle inequality is satisfied and \( \rho(\cdot, \cdot) \) is a metric space.
From the relationship between $\rho(\cdot, \cdot)$ and $\pi(\cdot, \cdot)$ we can see that any sequence $\{Q_n\}$ which is Cauchy in the Vasserstein metric is also Cauchy in the Prohorov metric, and hence has a limit $\bar{Q}$ in the topology of weak convergence. What is needed is to show that

$$\lim_{n \to \infty} \rho(Q_n, Q) = 0.$$ 

Let the random variables $\eta^{n,m}$ and $\bar{\eta}^{n,m}$ have distributions $Q_n$, and $Q_m$ respectively, and be such that

$$E[r(\eta^{n,m}, \bar{\eta}^{m,n})] \leq 2\rho(Q_n, Q_m).$$

Since $\{Q_n\}$ is weakly sequentially compact, $\{(\mathcal{L}(\eta^{n,m}), \mathcal{L}(\bar{\eta}^{n,m}))\}$ must be compact in the topology of weak convergence of distributions on $\mathcal{S} \times \mathcal{S}$ given the taxicab metric: $r((x_1, x_2), (y_1, y_2)) = r(x_1, y_1) + r(x_2, y_2)$. Fix $n$. Let the distribution of the pair $(\eta^n, \bar{\eta}^n)$ be the limit in the topology of weak convergence on $((\mathcal{S} \times \mathcal{S}), \hat{r})$ of the random variables $(\eta^{n,m_i}, \bar{\eta}^{n,m_i})$ for some sequence $\{m_i\} \to \infty$. Then $\eta^n$ must have distribution $Q_n$ for all $n$, and we will say that $\eta^n$ has distribution $\bar{Q}_n$. We wish to show that $\bar{Q}_n = Q$. To see this remember that $\lim_{i \to \infty} \pi(\mathcal{L}(\eta^{n,m_i}, \bar{\eta}^{n,m_i}), \mathcal{L}(\eta^n, \bar{\eta}^n)) = 0$ if and only if, $\mathcal{L}(\mathcal{L}(\eta^{n,m_i}, \bar{\eta}^{n,m_i})) \Rightarrow \mathcal{L}(\eta^n, \bar{\eta}^n)$. Then

$$\lim_{i \to \infty} \inf\{\varepsilon > 0 : \mathcal{L}(\eta^{n,m_i}, \bar{\eta}^{n,m_i})(F) \leq \mathcal{L}(\eta^n, \bar{\eta}^n)(F^c) + \varepsilon$$

for all $F$ closed in $\mathcal{S} \times \mathcal{S}$} = 0.

In particular, if $\mathcal{G}$ is the collection of all sets of the form $\mathcal{S} \times F$ where $F$ is closed in $\mathcal{S}$, then

$$\lim_{i \to \infty} \inf\{\varepsilon > 0 : \mathcal{L}(\eta^{n,m_i}, \bar{\eta}^{n,m_i})(G) \leq \mathcal{L}(\eta^n, \bar{\eta}^n)(G^c) + \varepsilon \text{ for all } G \in \mathcal{G} \} = 0.$$
But this is equal to

\[
\lim_{i \to \infty} \inf \{ \epsilon > 0 : \mathcal{L}(q_n^m)(F) \leq \mathcal{L}(\tilde{\eta}^n)(F^\epsilon) + \epsilon \text{ for all } F \text{ closed in } S \} = 0.
\]

By the uniqueness of the weak limit this is only satisfied if \( \tilde{Q}_n = Q \).

From Fatou's Lemma

\[
E[r(\eta^n, \tilde{\eta}^n)] \leq 2\lim_{i \to \infty} \rho(Q_n, Q_m).
\]

From the fact that \( \{Q_n\} \) is Cauchy in the Wasserstein metric it follows that \( E[r(\eta^n, \tilde{\eta}^n)] = 0 \), and therefore \( \rho(Q_n, \tilde{Q}) \to 0 \). So the Wasserstein metric is complete.

The fact that if \( \{Q_n\} \) converges to \( Q \) in the Wasserstein metric, then \( \{Q_n\} \) converges to \( Q \) in the topology of weak convergence, follows immediately from the inequality

\[
\rho(P, Q) \geq \pi(P, Q)^2,
\]

and the equivalence of the Prohorov topology to the topology of weak convergence.

Now we show that \( Q_n \Rightarrow Q \) implies that \( \rho(Q_n, P) \to 0 \). From the hypothesis, for all \( \epsilon > 0 \) there exists \( K_\epsilon \) such that

\[
\int_{(K_\epsilon)^c} r(x, x_0) P_m(dx) < \epsilon
\]

for all \( m \). Additionally we will insist that \( Q(K_\epsilon) \geq 1 - \epsilon \) Then there exists a finite set \( \{x_1, \ldots, x_n\} \subset K_\epsilon \) such that

\[
K_\epsilon \subset \bigcup_{i=1}^{n} N_\epsilon(x_i).
\]
For any positive $\lambda_1, \ldots, \lambda_{n\varepsilon}$, and $i = 1, \ldots, n\varepsilon$, let

$$A_i^{\lambda_1, \ldots, \lambda_{n\varepsilon}} = \{x : x \in S, \lambda_i r(x, x_k) < \min_{i=1,\ldots,n\varepsilon, i \neq k} \lambda_i r(x, x_i), \lambda_k r(x, x_k) < \varepsilon\}.$$ 

$Q(A_i^{\lambda_1, \ldots, \lambda_{n\varepsilon}})$ is a monotone function of $\lambda_i$. Therefore for almost all $n$-tuples $(\lambda_1, \ldots, \lambda_{n\varepsilon})$, $Q(A_i^{\lambda_1, \ldots, \lambda_{n\varepsilon}})$ is continuous with respect to $\lambda_i$. Then we may choose $\{\lambda_1, \ldots, \lambda_{n\varepsilon}\}$ such that $1 - \varepsilon \leq \lambda_i \leq \varepsilon$ for all $i$, and the boundary of $A_i^{\lambda_1, \ldots, \lambda_{n\varepsilon}}$ has measure zero. Let $B_i^\varepsilon = A_i^{\lambda_1, \ldots, \lambda_{n\varepsilon}}$. $K^\varepsilon \subset \bigcup_{i=1}^{n\varepsilon} B_i^\varepsilon$, so we have constructed pairwise disjoint sets $B_1^\varepsilon, \ldots, B_i^\varepsilon$ whose diameter is not more that $\frac{2\varepsilon}{1-\varepsilon}$, and for which

$$\sum_{i=1}^{n\varepsilon} Q(B_i^\varepsilon) \geq 1 - \varepsilon.$$

By virtue of the weak convergence of $\{Q_n\}$ to $Q$, $Q_n(b_i^\varepsilon) \rightarrow Q(B_i^\varepsilon)$. Let

$$q_i^{n\varepsilon} = \min(Q_n(B_i^\varepsilon), Q(B_i^\varepsilon))$$

$$Q_n^\varepsilon(A) = Q_n(A) - \sum_{i=1}^{n\varepsilon} q_i^{n\varepsilon} Q_n(A \cap B_i^\varepsilon) [Q_n(B_i^\varepsilon)]^{-1},$$

$$\bar{Q}_n^\varepsilon(A) = Q_n(A) - \sum_{i=1}^{n\varepsilon} q_i^{n\varepsilon} Q_n(A \cap B_i^\varepsilon) [Q(B_i^\varepsilon)]^{-1},$$

with the convention that $0/0 = 0$. $Q_n^\varepsilon$ and $\bar{Q}_n^\varepsilon$ are measures, also, $Q_n^\varepsilon(S), \bar{Q}_n^\varepsilon < 2\varepsilon$ for sufficiently large $n$. To see this, first note that it can be seen that $Q_n^\varepsilon(S) = \bar{Q}_n^\varepsilon(S)$ by setting $A = S$, so we need only show that $Q_n^\varepsilon(S) < 2\varepsilon$. 

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\[ Q_n^\epsilon(S) = Q_n^\epsilon(\bigcup_{l=1}^{n^\epsilon} B_l^\epsilon) + Q_n^\epsilon([\bigcup_{l=1}^{n^\epsilon} B_l^\epsilon]^c) \]

\[ Q_n^\epsilon([\bigcup_{l=1}^{n^\epsilon} B_l^\epsilon]^c) = Q_n([\bigcup_{l=1}^{n^\epsilon} B_l^\epsilon])^c < \epsilon. \]

\[ Q_n^\epsilon(\bigcup_{l=1}^{n^\epsilon} B_l^\epsilon) = Q_n(\bigcup_{l=1}^{n^\epsilon} B_l^\epsilon) - \sum_{l=1}^{n^\epsilon} q_l^{n^\epsilon} Q_n(B_l^\epsilon) [Q_n(B_l^\epsilon)]^{-1} \]

\[ = \sum_{l=1}^{n^\epsilon} Q_n(B_l^\epsilon)(1 - q_l^{n^\epsilon} [Q_n(B_l^\epsilon)]^{-1}). \]

By the weak convergence and the definition of \( q_l^{n^\epsilon} \cdot 1 - q_l^{n^\epsilon} \to 0 \) so \( Q_n^\epsilon(\bigcup_{l=1}^{n^\epsilon} B_l^\epsilon) < \epsilon \) for sufficiently large \( n \). Thus we have \( Q_n^\epsilon(S) < 2\epsilon \) for sufficiently large \( n \). So \( \tilde{Q}_n^\epsilon < 2\epsilon \) for sufficiently large \( n \).

Let \( \eta_{n,\epsilon} \), \( \zeta_{n,\epsilon} \) be random variables such that

\[ P\{\eta_{n,\epsilon} \in A, \zeta_{n,\epsilon} \in B\} = \sum_{l=1}^{n^\epsilon} q_l^{n^\epsilon} Q_n(A \cap B_l^\epsilon) Q(B \cap B_l^\epsilon) [Q_n(B_l^\epsilon) Q(B_l^\epsilon)]^{-1} \]

\[ + [Q_n^\epsilon]^{-1} Q_n^\epsilon(A) \tilde{Q}_n^\epsilon(B). \]

Clearly \( \eta_{n,\epsilon} \) has distribution \( Q_n(\cdot) \), and \( \zeta_{n,\epsilon} \) has distribution \( Q(\cdot) \). Further

\[ E[r(\eta_{n,\epsilon}, \zeta_{n,\epsilon})] = \sum_{l=1}^{n^\epsilon} q_l^{n^\epsilon} [Q_n(B_l^\epsilon) Q(B_l^\epsilon)]^{-1} \int_{B_l^\epsilon} \int_{B_l^\epsilon} r(x, \tilde{x}) Q_n(dx) Q(d\tilde{x}) \]

\[ + [Q_n^\epsilon(S)]^{-1} \int_S \int_S r(x, \tilde{x}) \tilde{Q}_n^\epsilon(dx) \tilde{Q}_n^\epsilon(d\tilde{x}). \]

If \( x, \tilde{x} \in B_l^\epsilon \), then \( r(x, \tilde{x}) < 2\epsilon(1 - \epsilon)^{-1} \) as noted when we created \( B_l^\epsilon \). Thus
$$E[r(\eta^{n,\epsilon}, \zeta^{n,\epsilon})] \leq 2\epsilon(1-\epsilon)^{-1} \sum_{i=1}^{n} q_{l_{i}}^{n,\epsilon}$$

+ $[Q_{n}^{\epsilon}(S)]^{-1} \int_{S} \int_{S} r(x, x_{0}) + r(x_{0}, \tilde{x})Q_{n}^{\epsilon}(dx)\bar{Q}_{n}^{\epsilon}(d\tilde{x})$

= $2\epsilon(1-\epsilon)^{-1} \sum_{i=1}^{n} q_{l_{i}}^{n,\epsilon}$

+ $[Q_{n}^{\epsilon}(S)]^{-1}[\bar{Q}_{n}^{\epsilon}(S) \int_{S} r(x, x_{0})Q_{n}^{\epsilon}(dx) + Q_{n}^{\epsilon}(S) \int_{S} r(x_{0}, \tilde{x})\bar{Q}_{n}^{\epsilon}(d\tilde{x})]$}

= $2\epsilon(1-\epsilon)^{-1} \sum_{i=1}^{n} q_{l_{i}}^{n,\epsilon} + \int_{S} r(x, x_{0})Q_{n}^{\epsilon}(dx) + \int_{S} r(x_{0}, \tilde{x})\bar{Q}_{n}^{\epsilon}(d\tilde{x}).$

$$\sum_{i=1}^{n} q_{l_{i}}^{n,\epsilon} \leq \sum_{i=1}^{n} Q(B_{i}^{\epsilon}) < 1,$$ so

$$E[r(\eta^{n,\epsilon}, \zeta^{n,\epsilon})] < 2\epsilon(1-\epsilon)^{-1} + \int_{S} r(x, x_{0})Q_{n}^{\epsilon}(dx) + \int_{S} r(x_{0}, \tilde{x})\bar{Q}_{n}^{\epsilon}(d\tilde{x}).$$

$$\int_{(K_{\epsilon})^{\epsilon}} r(x, x_{0}) dQ_{n} < \epsilon,$$ by hypothesis. Also there exists $M$ such that

$$\max_{L} r_{l_{i}}^{n,\epsilon} < \frac{\epsilon}{\max\{r(x, x_{0}) : x \in K_{\epsilon}\}}$$

so

$$\int_{K_{\epsilon}} r(x, x_{0}) dQ_{n} < \epsilon,$$

for large enough $n$. Then

$$\int_{S} r(x, x_{0}) dQ_{n} < 2\epsilon.$$

Similarly for large enough $n$

$$\int_{S} r(x_{0}, \tilde{x}) d\bar{Q}_{n} < 2\epsilon.$$ 

Since $\epsilon$ is arbitrary, we are done.
Remark Let $P_0(D_E[0, T])$ be the collection of probability distributions on $D_E[0, T]$ such that $\rho(Q, \delta_0) < \infty$ for all $Q \in P([0, T])$, where $\rho(\cdot, \cdot)$ is the Vasserstein metric on $P(D_E[0, T])$. Let $\mathcal{P}(P_0(D_E[0, T]))$ be the collection of probability distributions on $P(E[0, T])$ such that $\rho(Q, \delta_0) < \infty$, where $\rho(\cdot, \cdot)$ is the Vasserstein metric on $P(D_E[0, T]), (\hat{\rho})$. We want to use the Vasserstein metric in connection with the weak convergence of a sequence $\{\eta^n\} \subset \mathcal{P}(P_0(D_E[0, T]))$. In order to make full use of the above theorem, then it is necessary that the topology induced on $P(C[0, T])$ by $\hat{\rho}$ be separable. This fact follows from the topological equivalence of the Vasserstein metric to the topology of weak convergence and Theorem 1.24.

Lemma 1.66

Let $Y_s$ be a stochastic process with finite second moment, and let $\{Y^i_s\}$ be a collection of real-valued i.i.d stochastic processes with the same law as $Y_s$.

Then

$$\rho(\mathcal{L}(\delta_{Y_s}), \mathcal{L}(\frac{1}{n} \sum_{i=1}^{n} \delta_{Y^i_s})) \to 0$$

Proof

Let $f$ be a real-valued bounded continuous function. By the strong law of large numbers $<f, \frac{1}{n} \sum_{i=1}^{n} \delta_{Y^i_s} > = \frac{1}{n} \sum_{i=1}^{n} f(Y^i_s) \to E[f(Y_s)]$, so

$$\frac{1}{n} \sum_{i=1}^{n} f(Y^i_s) \Rightarrow \mathcal{L}(Y_s).$$

Then by the Vasserstein metric theorem we need only prove that $\frac{1}{n} \sum_{i=1}^{n} \delta_{Y^i_s}$ is uniformly integrable. i.e that

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\[
\lim_{c \to \infty} \sup_n E[1_{\{ \hat{\phi}(\frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i}, \delta_0) > c \}} \hat{\phi}(\frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i}, \delta_0)] = 0
\]

\[
\hat{\phi}(\frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i}, \delta_0) = \frac{1}{n} \sum_{i=1}^{n} \sup_{s \leq T} |Y_i^s|.
\]

So

\[
\lim_{c \to \infty} \sup_n E[1_{\{ \hat{\phi}(\frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i}, \delta_0) > c \}} \hat{\phi}(\frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i}, \delta_0)]
\]

\[
= \lim_{c \to \infty} \sup_n E[1_{\{ \frac{1}{n} \sum_{i=1}^{n} \sup_{s \leq T} |Y_i^s| > c \}} \frac{1}{n} \sum_{i=1}^{n} \sup_{s \leq T} |Y_i^s|]
\]

which is equal to zero by Chebychev's inequality.
Chapter 2

Poisson Random Measure Driven Processes

In this chapter we first consider a system of equations of the form

\[ X_t^{n,i} = X_0^{n,i} + \int_0^t b_n(s, X_s^{n,i}, \eta_s^{n,i}) \, ds + \int_0^t \int_{\mathcal{V}} \sigma_n(s, X_s^{n,i}, \eta_s^{n,i}, v) \tilde{N}^{n,i}(dv, ds) \]

where \( \tilde{N}^{n,i}(dv, ds) \) is the compensated Poisson random measure

\[ \tilde{N}^{n,i}(dv, ds) = N^{n,i}(dv, ds) - \nu_n(dv) \, dt. \]

We assume there exists \( b \) and \( \sigma \) such that \( b_n \) converges to \( b \) and \( \sigma_n \) converges to \( \sigma \) in an appropriate manner, and there exists a compensated Poisson random measure

\[ \tilde{N}(dv, ds) = N(dv, ds) - \nu(dv) \, dt \]

such that \( \nu_n \Rightarrow \nu \). We show that

\[ \mathcal{L}(\eta^n) = \mathcal{L}(\frac{1}{n} \sum_{i=1}^n \delta_{X_n^{n,i}}) \Rightarrow \delta_{\mathcal{L}(X)} \]

where \( \mathcal{L}(X) \) is the solution to the McKean-Vlasov equation

\[ X_t = X_0 + \int_0^t b(s, X_s, \mathcal{L}(X_s)) \, ds + \int_0^t \int_{\mathcal{V}} \sigma(s, X_s, \mathcal{L}(X_s), v) \tilde{N}(dv, ds) \]

The novelties are

a) the equations are driven by Poisson random measures, and the coefficients \( \sigma_n \) and \( \sigma \) depend on \( \nu \): the jump size,

b) the stability result for \( \sigma_n \) and \( b_n \).
c) the stability result for the random measures $N^{n,i}(dv, ds)$.

Secondly we consider the case where $\nu_n \Rightarrow \delta_0$. This is a generalization of the hydrodynamic limit. We find that

$$\eta^n \Rightarrow \delta_{L(X^n,1)}$$

where $L(X)$ is the solution to the McKean-Vlasov equation

$$X_t = X_0 + \int_0^t b(s, X_s, L(X_s)) \, ds + \int_0^t \sigma(s, X_s, L(X_s)) \, dB_s$$

2.1: Tightness of the System

Let $(\Omega^n, \mathcal{F}^n, P^n)$ be a collection of probability spaces with filtration $\mathcal{F}^n_t$, for each $n$ let $N^{n,i}(dv, ds)$ be a collection of i.i.d. Poisson random measures for $1 \leq i \leq n$. Let $\nu_n$ be the compensator of $N^{n,i}(dv, ds)$. Let $V \in \mathbb{R}$ such that $0 \in \bar{V}$ and $\{v : v = \Delta N_t \neq 0 \text{ for some } \omega \} \subset V$ and assume

$$\int_V (1 \wedge v^2) \nu_n(dv) < \infty.$$ 

Then

$$\tilde{N}^{n,i}_t = \int_0^t \int_V v(N^{n,i}(dv, ds) - \nu_n(dv)) \, ds$$

is a martingale.

Let

$$\hat{\sigma}_n : ([0, T], \mathbb{R}, \mathbb{R}, \mathbb{R}) \to \mathbb{R}, \hat{\sigma}_n(s, x, y, \cdot) \in L^2(V, \nu_n)$$

$$\hat{b}_n : ([0, T], \mathbb{R}, \mathbb{R}) \to \mathbb{R}$$

satisfy the following Lipschitz conditions
(1) \[ \int_V [\hat{\sigma}_n(s, x_1, y_1, u)^2 - \hat{\sigma}_n(s, x_2, y_2, u)^2] \nu_n(du) \leq K[(x_1 - x_2)^2 + (y_1 - y_2)^2] \]

(2) \[ \int_V (\hat{\sigma}_n(s, x_1, y_1, u) - \hat{\sigma}_n(s, x_2, y_2, u))^2 \nu_n(du) \leq K[(x_1 - x_2)^2 + (y_1 - y_2)^2] \]

(3) \[ |\hat{b}_n(s, x_1, y_1) - \hat{b}_n(s, x_2, y_2)| \leq K[|x_1 - x_2| + |y_1 - y_2|] \]

and growth conditions

(4) \[ \int_V [\hat{\sigma}_n(s, x, y, u)^2 \nu_n(du) \leq K[1 + x^2 + y^2] \]

(4') \[ |\hat{\sigma}_n(s, x, y, u)| \leq K[1 + |x| + |y|] \]

and

(5) \[ |\hat{b}_n(s, x, y)| \leq K[1 + |x| + |y|]. \]

We can define

\[ \hat{\sigma}_n : ([0, T], \mathbb{R}, \mathbb{R}^n, \mathbb{R}) \rightarrow \mathbb{R} \]

\[ \hat{b}_n : ([0, T], \mathbb{R}, \mathbb{R}^n) \rightarrow \mathbb{R} \]

by

\[ \hat{\sigma}_n(s, x, y, u) = \frac{1}{n} \sum_{i=1}^{n} \hat{\sigma}_n(s, x, y_i, u) \]

\[ \hat{b}_n(s, x, y) = \frac{1}{n} \sum_{i=1}^{n} \hat{b}_n(s, x, y_i). \]

from the Lipschitz conditions (1),(2) and (3), we get
\[
\int_V [\tilde{\sigma}_n(s, x_1, y, u)^2 - \overline{\sigma}_n(s, x_2, z, u)^2] \nu_n(du) \leq K[(x_1 - x_2)^2 + \frac{1}{n} \sum_{i=1}^{n} (y_i - z_i)^2]
\]

\[
\int_V [\tilde{\sigma}_n(s, x_1, y, u) - \overline{\sigma}_n(s, x_2, z, u)]^2 \nu_n(du) \leq K[(x_1 - x_2)^2 + \frac{1}{n} \sum_{i=1}^{n} (y_i - z_i)^2]
\]

and

\[
| \tilde{b}_n(s, x_1, y) - \overline{b}_n(s, x_2, z) | \leq K[| x_1 - x_2 | + \frac{1}{n} \sum_{i=1}^{n} | y_i - z_i |].
\]

Note that if \( \gamma = \frac{1}{n} \sum_{i=1}^{n} \delta_{y_i} \), then

\[
\tilde{\sigma}_n(s, x, y, u) = \int_{\mathbb{R}} \tilde{\sigma}_n(s, x, y, u) \gamma(dy)
\]

and

\[
\tilde{b}_n(s, x, y) = \int_{\mathbb{R}} \tilde{b}_n(s, x, y) \gamma(dy).
\]

Let \( \tilde{P}^n(\mathbb{R}) \) be the set of probability measures on \( \mathbb{R} \) of the form \( \gamma = \frac{1}{n} \sum_{i=1}^{n} \delta_{y_i} \), \( y_i \in \mathbb{R} \). We can define

\[
\sigma_n : ([0, T], \mathbb{R}, \tilde{P}^n(\mathbb{R}), \mathbb{R}) \rightarrow \mathbb{R}
\]

\[
b_n : ([0, T], \mathbb{R}, \tilde{P}^n(\mathbb{R})) \rightarrow \mathbb{R}
\]

by

\[
\sigma_n(s, x, \gamma, u) = \tilde{\sigma}_n(s, x, y, u)
\]

\[
b_n(s, x, \gamma) = \tilde{b}_n(s, x, y).
\]

Then \( \sigma_n, b_n \) satisfy the Lipschitz conditions.
and the growth conditions

\[
\int_V \sigma_n(s, x, \gamma, u)^2 - \sigma_n(s, x, \gamma, u)^2 \nu_n(dv) \leq K[(x_1 - x_2)^2 + \frac{1}{n} \sum_{i=1}^{n}(y_i - z_i)^2]
\]

and

\[
| b_n(s, x_1, \gamma_1) - b_n(s, x_2, \gamma_2) | \leq K[| x_1 - x_2 | + \frac{1}{n} \sum_{i=1}^{n} | y_i - z_i |],
\]

and the growth conditions

\[
\int_V \sigma_n(s, x, \gamma, u)^2 \nu_n(dv) \leq K(1 + x^2 + \int y^2 \gamma(dy))
\]

\[
| \sigma_n(s, x, \gamma, u) | \leq K(1 + | x | + \int | y | \gamma(dy))
\]

and

\[
| b_n(s, x, \gamma) | \leq K(1 + | x | + \int | y | \gamma(dy)).
\]

For \(1 \leq i \leq n\), the interacting system of stochastic differential equations is given by

\[
X_t^{n,i} = X_0^{n,i} + \int_0^t b_n(s, X_s^{n,i}, \eta_s^{n,i}) \, ds + \int_0^t \int_V \sigma_n(s, X_s^{n,i}, \eta_s^{n,i}, v) \tilde{N}^{n,i}(dv, ds).
\]

where \(X_0^{n,i}\) are i.i.d. and have finite fourth moment, and \(\eta_s^n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_s^{n,i}}\).
The following Lemma will be of use throughout this paper. The $A_s$ and $M_s$ mentioned in it will not appear until chapter 4.

**Lemma 2.1**

Let $A_s$ be a stochastic process which takes paths of finite variation on compacts, let $M_s$ be a square integrable martingale, and $N(du, ds)$ be a Poisson random measure. Let $Z_0, Z^1_t, Z^2_t, Z_t(v), Z^4_t$ be $\mathcal{F}_t$-adapted square integrable stochastic processes. By $Z^2_t(v)$ we mean that $Z^2_t(v)$ is a stochastic process indexed by both $t$ and $v$. Let

\begin{equation}
Z_T = Z_0 + \int_0^T Z^1_s ds + \int_0^T Z^2_s dA_s + \int_0^T \int_V Z^3_s(v) N(du, ds) + \int_0^T Z^4_s dM_s
\end{equation}

Then

$$E[\sup_{s \leq T}(Z_s)^2] \leq C \left[ E[(Z_0)^2] + (\int_0^T |Z^1_s| ds)^2 + (\int_0^T |Z^2_s| d|A_s|)^2 + (\int_0^T (Z^3_s(v))^2 \nu(du) ds + (\int_0^T (Z^4_s)^2 d < M, M >_s)\right]$$

for some $C < \infty$.

**Remark** For ease in reading we will use the notation $[X, X]_t = [X]_t$ and $<X, X>_t=<X>_t$. Also throughout this chapter we will use $M$ to denote a constant which doesn't depend on $n$ and may change from line to line.

**Proof**

Let $S^{n,i}_J = \inf\{t : Z^2_t \vee \int_V Z^3_t(v) N(du, dt) \vee Z^4_t \vee Z^1_t < J\}$ By squaring both sides of (10) and using the inequality $(\sum_{i=1}^n a_i)^2 = n \sum_{i=1}^n (a_i)^2$

$$(Z_T)^2 = 5[Z_0^2 + (\int_0^T Z^1_s ds)^2 + (\int_0^T Z^2_s dA_s)^2 + (\int_0^T \int_V Z^3_s(v) N(du, ds))^2 + (\int_0^T Z^4_s dM_s)^2]$$

Taking the sup on both sides and then the expectation

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\[ E[ \sup_{s \leq \tau \wedge S_{j}^{n,i}} (Z_s)^2 ] \leq 5[E[(Z_0)^2] \\
+ E[(\int_0^\tau 1_{\{u \leq S_{j}^{n,i}\} | Z_u^1 | \, du)^2] + E[(\int_0^\tau 1_{\{u \leq S_{j}^{n,i}\} | Z_u^2 | \, d | A_u^i |)^2] \\
+ E[ \sup_{s \leq \tau \wedge S_{j}^{n,i}} (\int_0^s Z_u^3(v) N(dv, du))^2] + E[ \sup_{s \leq \tau \wedge S_{j}^{n,i}} (\int_0^s Z_u^4 dM_u)^2]]. \]

Now we may use Doob’s maximal inequality to obtain

\[ E[ \sup_{s \leq \tau \wedge S_{j}^{n,i}} (Z_s)^2 ] \leq C[E[(Z_0)^2] \\
+ E[(\int_0^\tau 1_{\{u \leq S_{j}^{n,i}\} | Z_u^1 | \, du)^2] + E[(\int_0^\tau 1_{\{u \leq S_{j}^{n,i}\} | Z_u^2 | \, d | A_u^i |)^2] \\
+ E[\int_V (Z_u^3(v))^2 dN(dv, du)]_{\tau \wedge S_{j}^{n,i}} + E[\int_0^\tau 1_{\{u \leq S_{j}^{n,i}\}}(Z_u^4)^2 d[M, M]_u]]. \]

Using the fact that
\[ \int_0^\tau (Z_u^3(v))^2 dN(dv, du)_{\tau \wedge S_{j}^{n,i}} - \int_0^\tau (Z_u^3(v))^2 dN(dv, du)_{\tau \wedge S_{j}^{n,i}} \]

and \([M^i, M^i] - < M^i, M^i > are martingales\)

\[ E[ \sup_{s \leq \tau \wedge S_{j}^{n,i}} (Z_s)^2 ] \leq C[E[(Z_0)^2] \\
+ C E[(\int_0^\tau 1_{\{u \leq S_{j}^{n,i}\} | Z_u^1 | \, du)^2] + E[(\int_0^\tau 1_{\{u \leq S_{j}^{n,i}\} | Z_u^2 | \, d | A_u^i |)^2] \\
+ E[\int_0^\tau 1_{\{u \leq S_{j}^{n,i}\}} \int_V (Z_u^3(v))^2 \nu(dv) \, du] + E[\int_0^\tau 1_{\{u \leq S_{j}^{n,i}\}}(Z_u^4)^2 d < M, M >_u]] \]

where \( C \) is a constant. Finally we use the monotone convergence theorem to obtain

\[ E[\sup_{s \leq \tau} (Z_s)^2 ] \leq C[E[(Z_0)^2] \\
+ E[(\int_0^\tau | Z_u^1 | \, du)^2] + E[(\int_0^\tau | Z_u^2 | \, d | A_u^i |)^2] \\
+ E[\int_0^\tau \int_V (Z_u^3(v))^2 \nu(dv) \, du] + E[\int_0^\tau (Z_u^4)^2 d < M, M >_u]]. \]
Theorem 2.2

There exists a unique solution to equation (9) for all \( n \), and \( 1 \leq i \leq n \). Also we have the moment estimate

\[
E[\sup_{t \leq T}(X_t^{n,i})^4] \leq M[E[(X_0^{n,i})^4] + 1]e^{MT}
\]

for some \( M > 0 \).

Proof of existence and uniqueness

To show existence and uniqueness of a solution to (9) we need only verify that the conditions in Theorem 1.37 are met. We can write the system \( \{X^{n,1}, \ldots, X^{n,n}\} \) as the \( n \)-dimensional process

\[
X_t^n = X_0^{n,i} + \int_0^t \tilde{b}_n(s, X_s^n) \, ds + \int_0^t \int_V \tilde{\sigma}_n(s, X_s^n, u) \tilde{N}^n(du, ds)
\]

where \( \tilde{N}^n \) is the \( \mathbb{R}^n \)-valued compensated Poisson random measure

\[
(\tilde{N}^{n,1}, \ldots, \tilde{N}^{n,n}),
\]

\[
\tilde{b}_n(s, X_s^n) = \left( \frac{1}{n} \sum_{i=1}^n \hat{b}_n(s, X_s^{n,i}X_s^{n,i}), \ldots, \frac{1}{n} \sum_{i=1}^n \hat{b}_n(s, X_s^{n,n}, X_s^{n,i}) \right)
\]

and \( \tilde{\sigma}_n(s, X_s^n, u) \) is the \( n \times n \) matrix which has \( \frac{1}{n} \sum_{j=1}^n \hat{\sigma}_n(s, X_s^{n,i}, X_s^{n,j}, u) \) as it's \( (i, i) \)th coordinate and is equal to zero off the diagonal. From conditions (1) and (2) we see that \( \tilde{b}_n \) and \( \tilde{\sigma}_n \) are continuous. Next we verify the growth condition
for \( \tilde{b}_n \):

\[
| \tilde{b}_n(t, x) |^2 = \sum_{i=1}^{n} \left( \frac{1}{n} \sum_{j=1}^{n} \hat{b}_n(s, X_s^{n,i}, X_s^{n,j}) \right)^2 \\
\leq K^2 \sum_{i=1}^{n} \left( 1 + |X_s^{n,i}| + \frac{1}{n} \sum_{j=1}^{n} |X_s^{n,j}| \right)^2 \\
\leq 2K^2 \left[ 1 + \sum_{i=1}^{n} (X_s^{n,i})^2 + \sum_{i=1}^{n} \frac{1}{n} \sum_{j=1}^{n} (X_s^{n,j})^2 \right] \\
= 4K^2 \left[ 1 + \sum_{i=1}^{n} (X_s^{n,i})^2 \right] = 4K^2 \left[ 1 + |X_s^n|^2 \right].
\]

We verify the growth condition for \( \bar{\sigma}_n \) in the same manner using (3). Similarly the Lipschitz condition follows from (2) and (3).

**Proof of the moment estimate**

We begin by truncating the function \( x^2 \) in order to be able to take advantage of some martingale properties. Fix \( J > 0 \). Let

\[
S^{J,n} = \inf \{ s : |X_s^{n,i}| > J, 1 \leq i \leq n \}
\]

Let \( H_J(x) : \mathbb{R} \to \mathbb{R} \), such that \( H_J \in C_b^2(\mathbb{R}) \) with \( H_J(x) = x^2 \) if \( x |< J \). It is of no importance what \( H_J(x) \) does for \( |x| > J \). Then using Itô's Lemma

\[
H_J(X_t^{n,i}) = H_J(X_0^{n,i}) + \int_0^t H_J'(X_s^{n,i}) b_n(s, X_s^{n,i}, \eta_s^n) \, ds \\
+ \int_0^t \int_V (H_J(X_s^{n,i} + \sigma_n(s, X_s^{n,i}, \eta_s^n, v)) - H_J(X_s^{n,i}) \tilde{N}^{n,i}(dv, ds) \\
+ \int_0^t \int_V (H_J(X_s^{n,i} + \sigma_n(s, X_s^{n,i}, \eta_s^n, v)) - H_J(X_s^{n,i}) \\
+ H_J'(X_s^{n,i}) \sigma_n(s, X_s^{n,i}, \eta_s^n, v)) \nu_n(v) \, ds.
\]
Then

\[
E\left[ \sup_{t \leq T \wedge S_J.n} (H^J(X^n_t)) \right]^2 \leq E\left[ \sup_{t \leq T \wedge S_J.n} (X^n_t)^4 \right] \\
\leq M[E[(X^n_0)^4] + E[(\int_0^T \mathbf{1}_{\{s < S_J.n\}} | (X^n_s) b_n(s, X^n_s, \eta^n_s) | \, ds)^2] \\
+ E[\sup_{t \leq T} (\int_0^T \int_V \mathbf{1}_{\{s < S_J.n\}} [(X^n_s + \sigma_n(s, X^n_s, \eta^n_s, v))^2 - (X^n_s)^2] \, ds)^2]] \\
+ E[(\int_0^T \int_V \mathbf{1}_{\{s < S_J.n\}} [(X^n_s + \sigma_n(s, X^n_s, \eta^n_s, v))^2 - (X^n_s)^2] \, ds)^2] \\
+ E[|X^n_s\sigma_n(s, X^n_s, \eta^n_s, v)| |\nu_n(du) \, ds)^2] \\
\]

Using Doob's maximal inequality

\[
E\left[ \sup_{t \leq T \wedge S_J.n} (X^n_t)^4 \right] \\
\leq M[(X^n_0)^4] + E[(\int_0^T \mathbf{1}_{\{s < S_J.n\}} | X^n_s b_n(s, X^n_s, \eta^n_s) | ds)^2] \\
+ E[\int_0^T \int_V \mathbf{1}_{\{s < S_J.n\}} [(X^n_s + \sigma_n(s, X^n_s, \eta^n_s, v))^2 - (X^n_s)^2] \, ds)^2] \\
+ E[\int_0^T \int_V \mathbf{1}_{\{s < S_J.n\}} [(X^n_s + \sigma_n(s, X^n_s, \eta^n_s, v))^2 - (X^n_s)^2] \, ds)^2] \\
+ E[|X^n_s\sigma_n(s, X^n_s, \eta^n_s, v)| |\nu_n(du) \, ds)^2] \\
\]

Since \(1_{\{s < S_J.n\}}[(X^n_s + \sigma_n(s, X^n_s, \eta^n_s, v))^2 - (X^n_s)^2] \) is bounded

\[
[\int_0^T \int_V \mathbf{1}_{\{s < S_J.n\}} [(X^n_s + \sigma_n(s, X^n_s, \eta^n_s, v))^2 - (X^n_s)^2] \, ds)^2]_T \\
< [\int_0^T \int_V \mathbf{1}_{\{s < S_J.n\}} [(X^n_s + \sigma_n(s, X^n_s, \eta^n_s, v))^2 - (X^n_s)^2] \, ds)^2]_T \\
\]

is a martingale. So
\[ E[\sup_{t \leq T} (X^{n,i}_t)^4] \]
\[ \leq M[(X^{0,i}_0)^4 + E[\int_0^T \mathbf{1}_{\{s \leq S^{f,n}\}} | X^{n,i}_s b_n(s, X^{n,i}_s, \eta^{n}_s) \, ds]^2] \]
\[ + E[\mathbf{1}_{\{s \leq S^{f,n}\}} [(X^{n,i}_s + \sigma_n(s, X^{n,i}_s, \eta^{n}_s, v))^2 - (X^{n,i}_s)^2] \mathbf{1}_{\{\tilde{W}^{n,i}(dv, ds) > T\}}] \]
\[ + E[\int_0^T \int_V \mathbf{1}_{\{s \leq S^{f,n}\}} [(X^{n,i}_s + \sigma_n(s, X^{n,i}_s, \eta^{n}_s, v))^2 - (X^{n,i}_s)^2] \mathbf{1}_{\{\tilde{W}^{n,i}(dv, ds) > T\}} \, ds] \]
\[ + |X^{n,i}_s \sigma_n(s, X^{n,i}_s, \eta^{n}_s, v)| |\nu_n(dv)\, ds|^2] \]
\[ = M[(X^{0,i}_0)^4 + E[\int_0^T \mathbf{1}_{\{s \leq S^{f,n}\}} | X^{n,i}_s b_n(s, X^{n,i}_s, \eta^{n}_s) \, ds]^2] \]
\[ + E[\int_0^T \int_V \mathbf{1}_{\{s \leq S^{f,n}\}} [(X^{n,i}_s + \sigma_n(s, X^{n,i}_s, \eta^{n}_s, v))^2 - (X^{n,i}_s)^2] \mathbf{1}_{\{\tilde{W}^{n,i}(dv, ds) > T\}} \, ds] \]
\[ + E[\int_0^T \int_V \mathbf{1}_{\{s \leq S^{f,n}\}} [(X^{n,i}_s + \sigma_n(s, X^{n,i}_s, \eta^{n}_s, v))^2 - (X^{n,i}_s)^2] \mathbf{1}_{\{\tilde{W}^{n,i}(dv, ds) > T\}} \, ds] \]
\[ + |X^{n,i}_s \sigma_n(s, X^{n,i}_s, \eta^{n}_s, v)| |\nu_n(dv)\, ds|^2] \]

Now we use the monotone convergence theorem

\[ E[\sup_{t \leq T} (X^{n,i}_t)^4] \]
\[ \leq M[(X^{0,i}_0)^4 + E[\int_0^T | X^{n,i}_s b_n(s, X^{n,i}_s, \eta^{n}_s) \, ds]^2] \]
\[ + E[\int_0^T \int_V [(X^{n,i}_s + \sigma_n(s, X^{n,i}_s, \eta^{n}_s, v))^2 - (X^{n,i}_s)^2] \mathbf{1}_{\{\tilde{W}^{n,i}(dv, ds) > T\}} \, ds] \]
\[ + E[\int_0^T \int_V [(X^{n,i}_s + \sigma_n(s, X^{n,i}_s, \eta^{n}_s, v))^2 - (X^{n,i}_s)^2] \mathbf{1}_{\{\tilde{W}^{n,i}(dv, ds) > T\}} \, ds] \]
\[ + |X^{n,i}_s \sigma_n(s, X^{n,i}_s, \eta^{n}_s, v)| |\nu_n(dv)\, ds|^2] \]

Using Hölder's inequality
\[ E[\sup_{t \leq T}(X_t^{n,i})^4] \]
\[ \leq M[(X_0^{n,i})^4 + E[(\int_0^T \left| (X_{s_+}^{n,i})^2 b_n(s, X_{s_+}^{n,i}, \eta_{s_+}^n)^2 ds\right])] \]
\[ + E[(\int_0^T \int_V \left[ (X_{s_+}^{n,i} + \sigma_n(s, X_{s_+}^{n,i}, \eta_s^n, v))^2 - (X_{s_+}^{n,i})^2 \right] \nu_n(dv) ds] \]
\[ + E[(\int_0^T \int_V \left[ (X_{s_+}^{n,i} + \sigma_n(s, X_{s_+}^{n,i}, \eta_s^n, v))^2 - (X_{s_+}^{n,i})^2 \right]^2] \]
\[ + (X_{s_+}^{n,i})^2 \sigma_n(s, X_{s_+}^{n,i}, \eta_s^n, v)^2 \nu_n(dv) ds)]. \]

Using the identity \((a - b)^2 = (a + b)(a - b)\)

\[ E[\sup_{t \leq T}(X_t^{n,i})^4] \]
\[ \leq M[(X_0^{n,i})^4 + E[(\int_0^T \left| (X_{s_+}^{n,i})^2 b_n(s, X_{s_+}^{n,i}, \eta_{s_+}^n)^2 ds\right])] \]
\[ + E[(\int_0^T \int_V \left[ (2X_{s_+}^{n,i} + \sigma_n(s, X_{s_+}^{n,i}, \eta_s^n, v)))^2 \sigma_n(s, X_{s_+}^{n,i}, \eta_s^n, v))^2 \right] \nu_n(dv) ds] \]
\[ + (X_{s_+}^{n,i})^2 \sigma_n(s, X_{s_+}^{n,i}, \eta_s^n, v)^2 \nu_n(dv) ds)]. \]

Using growth condition (8')

\[ E[\sup_{t \leq T}(X_t^{n,i})^4] \]
\[ \leq M[(X_0^{n,i})^4 + E[(\int_0^T \left| (X_{s_+}^{n,i})^2 b_n(s, X_{s_+}^{n,i}, \eta_{s_+}^n)^2 ds\right])] \]
\[ + E[(\int_0^T \int_V \left[ (1 + 3X_{s_+}^{n,i})^2 + \left( \frac{1}{n} \sum_{j=1}^n X_{s_+}^{n,j} \right) \sigma_n(s, X_{s_+}^{n,i}, \eta_s^n, v))^2 \right] \nu_n(dv) ds] \]
\[ + (X_{s_+}^{n,i})^2 \sigma_n(s, X_{s_+}^{n,i}, \eta_s^n, v)^2 \nu_n(dv) ds)]. \]

Using the growth conditions (8) and (9)
\[ E[\sup_{t \leq T} (X_t^{n,i})^4] \]

\[ \leq M[(X_0^{n,i})^4 + E(\int_0^T |(X_{s-}^{n,i})^2 b_n(s, X_{s-}^{n,i}, \eta_{s-})^2 ds)] \]

\[ + E(\int_0^T \int_V [1 + |X_s^{n,i}| + \frac{1}{n} \sum_{j=1}^n X_s^{n,j}]^2 [1 + |X_s^{n,i}| + \frac{1}{n} \sum_{j=1}^n X_s^{n,j}]^2 \]

\[ + (X_s^{n,i})^2 [1 + |X_s^{n,i}| + \frac{1}{n} \sum_{j=1}^n X_s^{n,j}]^2 \nu_n(du) ds] \]

Using the identity \((\frac{1}{n} \sum_{i=1}^n a_i)^2 \leq n \sum_{i=1}^n a_i^2\) and combining like terms

\[ E[\sup_{t \leq T} (X_t^{n,i})^4] \]

\[ \leq M[(X_0^{n,i})^4 + 1 + E(\int_0^T (X_s^{n,i})^4 + \frac{1}{n} \sum_{j=1}^n (X_s^{n,j})^4 \]

\[ + \frac{1}{n} \sum_{j=1}^n (X_s^{n,i})^2 (X_s^{n,j})^2 ds] \]

\[ \leq M[(X_0^{n,i})^4 + 1 + \int_0^T E[(X_s^{n,i})^4] + \frac{1}{n} \sum_{j=1}^n E[(X_s^{n,i})^4] \frac{1}{2} E[X_s^{n,j})^4] \frac{1}{2} \]

\[ + \frac{1}{n} \sum_{j=1}^n E[(X_s^{n,i})^4] + E[(X_s^{n,j})^4] ds] \]

which by the exchangeability of the \(X_s^{n,j}\) yields

\[ E[\sup_{t \leq T} (X_t^{n,i})^4] \leq M[E(X_0^{n,i})^4 + 1 + \int_0^T E[(X_s^{n,i})^4] ds] \]

\[ \leq M[(X_0^{n,i})^4 + 1 + \int_0^T E[\sup_{t \leq s} (X_t^{n,i})^4] ds]. \]

Using Gronwall's inequality

\[ E[\sup_{t \leq T} (X_t^{n,i})^4] \leq M[E[(X_0^{n,i})^4] + 1] e^{MT}. \]
Theorem 2.3

1) \( X^{n,i} \) is a tight \( D[0,T] \)-valued random variable

2) \( \{\eta^n\} = \{\frac{1}{n} \sum_{j=1}^{n} \delta_{X^{n,i}}\} \) is tight as a \( P(D_E[0,T]) \)-valued random variable.

Proof

For part 1) we will use the Aldous tightness criterion. From Lemma 2.1

\[
E[(X^{n,i}_{(\tau_n+\delta_n)} - X^{n,i}_{\tau_n})^2] \\
\leq M[E[\int_{\tau_n}^{\tau_n+\delta_n} b_n(s, X^{n,i}_s, \eta^n_s) \, ds]^2] \\
+ E[\int_{\tau_n}^{\tau_n+\delta_n} \int_{\mathcal{V}} \sigma_n(s, X^{n,i}_s, \eta^n_s, v)^2 \nu_n(dv) \, ds]
\]

From the Growth condition and Hölder's inequality

\[
E[(X^{n,i}_{(\tau_n+\delta_n)} - X^{n,i}_{\tau_n})^2] \\
\leq M[E[\int_{\tau_n}^{\tau_n+\delta_n} 1 + (X^{n,i}_s)^2 + \frac{1}{n} \sum_{j=1}^{n} (X^{n,i}_s)^2 \, ds]] \\
= M[E[\int_{\tau_n}^{\tau_n+\delta_n} 1 + (X^{n,i}_s)^2 \, ds]] \\
\leq \delta_n M[1 + E[\sup_{\tau_n \leq s \leq \tau_n+\delta_n} (X^{n,i}_s)^2]]
\]

From the moment estimate of Theorem 2.2 we know that this goes to zero as \( \delta_n \) goes to zero. From the calculation which we have just finished, the moment estimate in Theorem 2.2, and the Aldous tightness criterion we conclude that \( X^{n,i} \) is tight.

Part 2) follows from Lemma 1.63.

Let \( \mu \in P(D[0,T]) \) and \( m \in P(\mathcal{R}) \). Define

\[
|\mu^k| = \int_{D[0,T]} \sup_{0 \leq s \leq T} |x_s|^k \, \mu(dx)
\]
and

\[ |m^k| = \int_{\mathbb{R}} |x|^k \, m(dx). \]

**Lemma 2.4**

\[ \int_{\mathcal{P}(D[0,T])} \int_{D[0,T]} \sup_{0 \leq s \leq T} |x_s|^4 \, d\mu(dx) \mathcal{L}(\eta^n(d\mu)) \]

is uniformly bounded for all \( n \). If \( \mathcal{L}(\eta^n) \Rightarrow \mathcal{L}(\eta) \) then

\[ \int_{\mathcal{P}(D[0,T])} \int_{D[0,T]} \sup_{0 \leq s \leq T} |x_s|^4 \, d\mu(dx) \mathcal{L}(\eta(d\mu)) \]

shares the same bound.

**Proof**

\[ \int_{\mathcal{P}(D[0,T])} \int_{D[0,T]} \sup_{0 \leq s \leq T} |x_s|^4 \, d\mu(dx) \mathcal{L}(\eta^n(d\mu)) = \frac{1}{n} \sum_{i=1}^{n} E[\sup_{s \leq t} (X^{n,i}_s)^4] \]

which is uniformly bounded for all \( n \) by Theorem 2.2.

If \( \mathcal{L}(\eta^n) \Rightarrow \mathcal{L}(\eta) \) then the Skorohod representation theorem tells us there is a probability space for which there are random variables \( X^n \) and \( X \) such that \( X^n \) has the same law as \( \eta^n \), \( X \) has the same law as \( \eta \), and \( X^n \to X \) a.s.. Then by Fatou's Lemma

\[ \int_{\mathcal{P}(D[0,T])} \int_{D[0,T]} \sup_{0 \leq s \leq T} |x_s|^k \, d\mu(dx) \mathcal{L}(\eta(d\mu)) \]

shares the same bound.

**2.2: Identification of the limit when \( \nu_n \Rightarrow \nu \)**

We need the following assumptions in order to identify the limit. There exist functions
\[
\dot{\sigma} : ([0, T], \mathbb{R}, \mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}
\]

and

\[
\hat{b} : ([0, T], \mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}
\]

such that

\[
\int_{\mathcal{V}} \hat{\sigma}_n(s, x, y, v)^2 \nu_n(dv) \rightarrow \int_{\mathcal{V}} \dot{\sigma}(s, x, y, v)^2 \nu(dv)
\]

uniformly on \([0, T] \otimes \mathbb{R} \otimes \mathbb{R}\).

(13)

\[
\hat{b}_n(s, x, y) \rightarrow \hat{b}(s, x, y)
\]

uniformly on \([0, T] \otimes \mathbb{R} \otimes \mathbb{R}\).

Now define

\[
\sigma : ([0, T], \mathbb{R}, P(\mathbb{R}), \mathbb{R}) \rightarrow \mathbb{R}
\]

and

\[
b : ([0, T], \mathbb{R}, P(\mathbb{R})) \rightarrow \mathbb{R}
\]

by
\begin{align}
\sigma(s, x, \mu, u) &= \int_{\mathbb{R}} \delta(s, x, y, u) \mu(dy) \\
\text{and} \\
b(s, x, \mu) &= \int_{\mathbb{R}} \tilde{b}(s, x, y) \mu(dy)
\end{align}

and further assume that

\begin{align}
\int_{\mathcal{V}} \sigma(s, x, \mu, u)^2 &\leq K(1 + x^2 + E\mu[y^2]) \\
|b(s, x, \mu)| &\leq K(1 + |x| + E\mu[|y|]) \\
\int_{\mathcal{V}} (\sigma(s, x_1, \mu_1, v)^2 - \sigma(s, x_2, \mu_2, v)^2) \nu_n(dv) &\leq K[(x_1 - x_2)^2 + \rho(\mu_1, \mu_2)^2] \\
|b(s, x_1, \mu_1) - b(s, x_2, \mu_2)| &\leq K[(x_1 - x_2) + \rho(\mu_1, \mu_2)]
\end{align}

where $\rho(\cdot, \cdot)$ is the Vasserstein metric on $P(\mathbb{R})$.

We wish to show that $\lim_{n \to \infty} \eta^n = \mathcal{L}(X_.)$ where

\begin{align}
X_t = X_0 + \int_0^t b(s, X_s, \mathcal{L}(X_s)) \, ds + \int_0^t \int_{\mathcal{V}} \sigma(s, X_s, \mathcal{L}(X_s), v) \tilde{N}(dv, ds)
\end{align}

and $X_0$ has the same distribution as $X_0^{i,n}$. Towards this we first establish the convergence of the random measures $\eta^n = \frac{1}{n} \sum_{i=1}^n \delta_{X^{n,i}}$ to solution of the martingale problem posed by (20).
For any $f \in C^2_b[0,T]$ and $g_1, \ldots, g_d \in C_b[0,T]$, $0 \leq s_1, \ldots, s_d \leq s < t, d \in \mathbb{Z}^+$, define

$$F(\mu) = F_{f,g_1,\ldots,g_d}^{s,t} : P(D[0,T]) \to \mathbb{R}$$

by

$$(21) \quad F(\mu) = \int_{D[0,T]} (f(x_t) - f(x_s) - \int_s^t L_{\mu,r,f}(x_r) \, dr) \prod_{k=1}^d g_k(x_{s_k}) \mu(dx)$$

where for $x \in D[0,T]$, $x_s = x(s)$, $0 \leq s \leq T$, and for all $a \in \mathbb{R}$

$$(22) \quad L_{\mu,r,f}(a) = f'(a)b(r,a,\mu) + \int_V [f(a + \sigma(s,a,\mu,v)) - f(a) - f'(a)\sigma(s,a,\mu,v)]\nu(dv).$$

Then $\eta \in P(D[0,T])$ is a solution to the martingale problem posed by (20) providing $F(\eta) = 0$ for all choices of $f, g_1, \ldots, g_d, s_1, \ldots, s_d, s, t,$ and $d$.

**Remark**

Since conditions (16) and (18) involve $\sigma^2$ term it will be convenient to write

$$\int_V [f(a + \sigma(s,a,\mu,v)) - f(a) - f'(a)\sigma(s,a,\mu,v)]\nu(dv)$$

in a form which contains a $\sigma^2$ term so that the conditions may be applied. The following calculation will be used repeatedly. Fix $x$ and $c$

$$f(x + c) - f(x) - f'(x)c = \int_x^{x+c} f'(u) \, du - f'(x)c.$$

Using the substitution $u = cz_1 - x$ we obtain
\[ f(x + c) - f(x) - f'(x)c = c \int_0^1 f'(x + cz_1) \, dz_1 - f'(x)c \]
\[ = c \int_0^1 f'(x + cz_1) - f'(x) \, dz_1. \]

Repeating the process we obtain

\[
(23) \quad f(x + c) - f(x) - f'(x)c = c^2 \int_0^1 z_1 \int_0^1 f''(x + cz_1 z_2) \, dz_2 \, dz_1. 
\]

Lemma 2.5

\[
F(\mu) \leq M(1 + |\mu^2|) 
\]

for all \( \mu \in P(D[0,T]) \).

Proof

Since \( f \) and \( \prod_{i=1}^d g_i \) are bounded

\[
| F(\mu) | = | \int_{D[0,T]} (f(x_t) - f(x_s) - \int_s^t L_{\mu,r,f}(x_r) \, dr) \prod_{i=1}^d g_i(x_{s_i}) \mu(dx) |
\]
\[ \leq M(1 + \int_{D[0,T]} \int_s^t |(f'(x_r)b(r,x_r,\mu_r)|
\]
\[ + \int_{V} | f(x_r + \sigma(r,x_r,\mu_r,v)) - f(x_r) |
\]
\[ - f'(x_r) \sigma(r,x_r,\mu_r,v) | \nu(dv) | \, dr \, \mu(dx). \]

Using (23)

\[
| F(\mu) | = | \int_{D[0,T]} (f(x_t) - f(x_s) - \int_s^t L_{\mu,r,f}(x_r) \, dr) \prod_{i=1}^d g_i(x_{s_i}) \mu(dx) |
\]
\[ \leq M(1 + \int_{D[0,T]} \int_s^t |(f'(x_r)b(r,x_r,\mu_r)|
\]
\[ + \int_{V} \sigma(r,x_r,\mu_r,v)^2 \int_0^1 z_1 
\]
\[ \times \int_0^1 f''(x + \sigma(r,x_r,\mu_r,v)z_1 z_2) \, dz_2 \, dz_1 \, \nu(dv) \, dr \, \mu(dx). \]
Since $f'$, $f''$ are bounded

$$| F(\mu) | \leq M \left( 1 + \int_{D[0,T]} \int_s^t (| b(r, x_r, \mu_r) | + \int_V | \sigma(r, x_r, \mu_r, v) |^2 \, \nu(du) \, dr \, \mu(dx)) \right)$$

$$\leq M \left( 1 + \int_{D[0,T]} \int_s^t (b(r, x_r, \mu_r)^2 + \int_V | \sigma(r, x_r, \mu_r, v) |^2 \, \nu(du) \, dr \, \mu(dx)) \right)$$

$$\leq M \left( 1 + \int_{D[0,T]} \int_s^t (b^2 + | \mu^2 |) \mu(dx) \right)$$

$$\leq M (1 + | \mu^2 |).$$

**Lemma 2.6**

Let $\eta$ be the weak limit of a subsequence $\eta^{n'}$ of $\eta^n$.

$$\lim_{n' \to n} \int_{P(D[0,T])} F^2(\mu) \mathcal{L}(\eta^{n'})(d\mu) = \int_{P(D[0,T])} F^2(\mu) \mathcal{L}(\eta)(d\mu).$$

**Proof**

Since $\eta^n$ has a subsequence which converges to $\eta$, we will write $\eta^n \Rightarrow \eta$ for simplicity of notation.

In this proof we will make repeated use of the inequality

$$(24) \quad | h(x+y_1) - h(x) - h'(x)y_1 - h(x+y_2) + h(x) + h'(x)y_2 | \leq \| h'' \|_{\infty} (y_1 - y_2)^2$$

where $h \in C^2_0[0,T]$. Recall that for any $a \in \mathbb{R}$

$$\sigma(s, a, \gamma, v) = \int_{\mathbb{R}} \hat{\sigma}(s, a, y, v) \gamma(dy)$$

and

$$b(s, a, \gamma) = \int_{\mathbb{R}} \hat{b}(s, a, y) \gamma(dy).$$
Define

\[ \sigma_R : [0, T] \times \mathbb{R} \times P(\mathbb{R}) \times \mathbb{R} \to \mathbb{R} \]

and

\[ b_R : [0, T] \times \mathbb{R} \times P(\mathbb{R}) \times \mathbb{R} \to \mathbb{R} \]

by

\[
\sigma_R(s, a, \gamma, v) =
\begin{cases}
\int_R \sigma(s, a, y, v) \gamma(dy) & \text{if } |a| \leq R \text{ and } |y| \leq R \\
\int_R \sigma(s, a, \frac{aR}{|a|}, y, v) \gamma(dy) & \text{if } |a| > R \text{ and } |y| \leq R \\
\int_R \sigma(s, a, \frac{aR}{|y|}, y, v) \gamma(dy) & \text{if } |a| \leq R \text{ and } |y| > R \\
\int_R \sigma(s, a, \frac{aR}{|a|}, \frac{aR}{|y|}, v) \gamma(dy) & \text{if } |a| > R \text{ and } |y| > R.
\end{cases}
\]

and

\[
b_R(s, a, \gamma) =
\begin{cases}
\int_R \tilde{b}(s, a, y) \gamma(dy) & \text{if } |a| \leq R \text{ and } |y| \leq R \\
\int_R \tilde{b}(s, a, \frac{aR}{|a|}, y) \gamma(dy) & \text{if } |a| > R \text{ and } |y| \leq R \\
\int_R \tilde{b}(s, a, \frac{aR}{|y|}, y) \gamma(dy) & \text{if } |a| \leq R \text{ and } |y| > R \\
\int_R \tilde{b}(s, a, \frac{aR}{|a|}, \frac{aR}{|y|}) \gamma(dy) & \text{if } |a| > R \text{ and } |y| > R.
\end{cases}
\]

Let

\[
L_{\gamma, r, f}^R(a) = f'(a)b_R(r, a, \gamma)
\]

(25)

\[
+ \int_V [f(a + \sigma_R(r, a, \gamma, v)) - f(a) - f'(a)\sigma_R(s, a, \gamma, v)]\nu(dv)
\]

and

(26) \[ F_R(\mu) = \int_{D[0, T]} (f(x_r) - f(x_s)) - \int_s^t L_{\mu_r, r, f}^R(x_r) dr \prod_{k=1}^d g_k(x_{s_k}) \mu(dx). \]

First we must show that

\[
\lim_{n \to \infty} \int_{P(D[0, T])} F_R(\mu)^2 \mathcal{L}(\eta^n)(d\mu) = \int_{P(D[0, T])} F_R(\mu)^2 \mathcal{L}(\eta)(d\mu).
\]

\(\tilde{b}\) is bounded and continuous in \(a\) and \(y\) so \(b_R(s, a, \gamma)\) is bounded and continuous in \(x\) and \(\gamma\) where continuity in \(\gamma\) is with respect to the topology of weak
convergence. Now we wish to show that

\[
\int_V \left[ f(a + \sigma_R(r, a, \gamma, v)) - f(a) - f'(a)\sigma_R(s, a, \gamma, v) \right] \nu(dv)
\]

is bounded and continuous with respect to \( a \) and \( \gamma \). For any probability measure \( \gamma \) we can define the probability measure \( \gamma^R \) by

\[
(27) \quad \gamma^R(A) = \int_{1_A \cap \{|y| < R\}} \gamma(dy) + \gamma(y \leq -R)1_A(-R) + \gamma(y \geq R)1_A(R).
\]

Then

\[
\sigma_R(s, a, \gamma, v) = \sigma(s, a, \gamma^R, v).
\]

Using (23)

\[
\int_V \left[ f(a + \sigma_R(r, a, \gamma, v)) - f(a) - f'(a)\sigma_R(s, a, \gamma, v) \right] \nu(dv)
\]

\[
= \int_V \left[ \sigma_R(r, a, \gamma, v)^2 \int z_1 \int f''(a + z_1z_2\sigma_R(r, a, \gamma, v)) dz_2 dz_1 \right] \nu(dv)
\]

\[
\leq 2R\|f''\|_\infty
\]

so \( \int_V \left[ f(a + \sigma_R(r, a, \gamma, v)) - f(a) - f'(a)\sigma_R(s, a, \gamma, v) \right] \nu(dv) \) is bounded. Suppose \( a_n \to a \) and \( \gamma_n \Rightarrow \gamma \). Using (24)

\[
\int_V \left[ f(a + \sigma_R(r, a_n, \gamma_n, v)) - f(a_n) - f'(a_n)\sigma_R(s, a_n, \gamma_n, v) \right]
\]

\[
- [f(a + \sigma_R(r, a, \gamma, v)) - f(a) - f'(a)\sigma_R(s, a, \gamma, v)] \nu(dv)
\]

\[
\leq \int_V \|f''\|_\infty (\sigma_R(s, a_n, \gamma_n, v) - \sigma_R(s, a, \gamma, v))^2 \nu(dv)
\]

\[
= \int_V \|f''\|_\infty (\sigma(s, a_n, \gamma_n^R, v) - \sigma(s, a, \gamma^R, v))^2 \nu(dv)
\]

\[
\leq M(a_n - a)^2 + \rho(\gamma_n^R, \gamma^R)^2.
\]

\( \gamma_n \Rightarrow \gamma \) implies \( \gamma_n^R \Rightarrow \gamma^R \) so we may conclude that \( \int_V [f(a + \sigma_R(r, a, \gamma, v)) - f(a) - f'(a)\sigma_R(s, a, \gamma, v)] \nu(dv) \) is bounded and continuous with respect to \( a \) and
\( \gamma \). If \( \mu^n \Rightarrow \mu \) then for almost all \( r \) \( \mu_r \Rightarrow \mu_r \), and there exists \( \{ Z^n_r \} \) and \( Z_r \) such that \( Z^n_r \) has \( \mu^n_r \) as its law, and \( Z^n_r \to Z_r \) for almost all \( r \). Then

\[
F_R(\mu^n) = E[f(Z^n_r) - f(Z^n_s) - \int_s^t f'(Z^n_r) b_R(r, Z^n_r, \mu^n_r) \, dr + \int_V [f(Z^n_r + \sigma_R(r, Z^n_r, \mu^n_r, v)) - f(Z^n_r)] - f'(Z^n_r) \sigma_R(s, Z^n_r, \mu^n_r, v)] \nu(du) \, dr
\]

which is bounded and converges to

\[
E[f(Z_t) - f(Z_s) - \int_s^t f'(Z_r) b_R(r, Z_r, \mu_r) \, dr + \int_V [f(Z_t + \sigma_R(r, Z_r, \mu_r, v)) - f(Z_t)] - f'(Z_t) \sigma_R(s, Z_r, \mu_r, v)] \nu(du) \, dr.
\]

Thus \( F_R(\cdot) \) is continuous. Clearly it is bounded, from which we may conclude that

\[
\lim_{n \to \infty} \int_{P(D[0,T])} F_R(\mu)^2 \mathcal{L}(\eta^n(d\mu)) = \int_{P(D[0,T])} F_R(\mu)^2 \mathcal{L}(\eta(d\mu)).
\]

Now we wish to show that

\[
\int_{P(D[0,T])} [F(\mu)^2 - F_R(\mu)^2] \mathcal{L}(\eta^n)(d\mu) \to 0
\]

\[
[F(\mu)^2 - F_R(\mu)^2] = [F(\mu) + F_R(\mu)][F(\mu) - F_R(\mu)]
\]

\[
= [F(\mu) + F_R(\mu)] \left[ \int_{D[0,T]} \int_s^t f'(x_r)[b_R(r, x_r, \mu_r) - b(r, x_r, \mu_r)] \, dr \right.
\]

\[
\left. + \int_V [f(x_r + \sigma_R(r, x_r, \mu_r, v)) - f(x_r + \sigma_R(r, x_r, \mu_r, v))] - f'(x_r)[\sigma_R(r, x_r, \mu_r, v) - \sigma_R(r, x_r, \mu_r, v)] \nu(du) \, dr \mu(dx). \right)
\]

Using (24)

\[
[F(\mu)^2 - F_R(\mu)^2] = [F(\mu) - F_R(\mu)]
\]

\[
\times \left[ \int_{D[0,T]} \int_s^t f'(x_r)[b_R(r, x_r, \mu_r) - b(r, x_r, \mu_r)] \, dr \right.
\]

\[
\left. + \int_V \|f''\|_\infty (\sigma_R(r, x_r, \mu_r, v) - \sigma_R(r, x_r, \mu_r, v))^2 \nu(du) \, dr \mu(dx). \right)
\]
\[ \int_{P(D[0,T])} \int_{D[0,T]} \int_{s}^{t} f'(x_r) \left[ b_R(r, x_r, \mu_r) - b(r, x_r, \mu_r) \right] dr \mu(dx) \mathcal{L}(\eta^n)(d\mu) = \int_{P(D[0,T])} \int_{D[0,T]} \int_{s}^{t} f'(x_r) \times \int_{\mathbb{R}} \left[ 1_{\{|x_r| > R\}} 1_{\{|y| > R\}} \left[ b(r, x_r, y) - b(r, x_r, y) \right] \cdot \frac{x_r R}{x_r} \cdot \frac{y R}{y} \right] + 1_{\{|x_r| \leq R\}} 1_{\{|y| > R\}} \left[ b(r, x_r, y) - b(r, x_r, y) \right] + 1_{\{|x_r| > R\}} 1_{\{|y| \leq R\}} \left[ b(r, x_r, y) - b(r, x_r, y) \right] \mu_r(dy) dr \mu(dx) \mathcal{L}(\eta^n)(d\mu). \]

Using the growth conditions

\[ \leq \int_{P(D[0,T])} \int_{D[0,T]} \int_{s}^{t} f'(x_r) \int_{\mathbb{R}} \left[ 1_{\{|x_r| > R\}} 1_{\{|y| > R\}} \left[ x_r - R + \frac{|y| - R}{} \right] + 1_{\{|y| > R\}} \left[ x_r - R \right] + 1_{\{|x_r| > R\}} \left[ y \right] \right] dr \mu(dx) \mathcal{L}(\eta^n)(d\mu) = 2 \int_{P(D[0,T])} \int_{D[0,T]} \int_{s}^{t} f'(x_r) \int_{\mathbb{R}} \left[ 1_{\{|y| > R\}} \left[ y \right] \right] + 1_{\{|x_r| > R\}} \left[ y \right] \mu_r(dy) dr \mu(dx) \mathcal{L}(\eta^n)(d\mu). \]

Using Chebychev’s inequality twice and the fact that \( f' \) is bounded

\[ \leq M \int_{P(D[0,T])} \int_{D[0,T]} \int_{s}^{t} \int_{\mathbb{R}} \left| \frac{\mu_r^2}{R^2} \right| dr \mu(dx) \mathcal{L}(\eta^n)(d\mu) \leq M \frac{\mu^2}{R^2}. \]

Now

\[ \int_{D[0,T]} \int_{s}^{t} \int_{V} \left\| f'' \right\|_\infty \left( \sigma_R(r, x_r, \mu_r, v) - \sigma(r, x_r, \mu_r, v) \right)^2 \nu(dv) dr \mu(dx) = \int_{D[0,T]} \int_{s}^{t} \int_{V} \left\| f'' \right\|_\infty \left[ 1_{\{|x_r| \leq R\}} \left( \sigma(r, x_r, \mu_r, v) - \sigma(r, x_r, \mu_r, v) \right)^2 + 1_{\{|x_r| > R\}} \left( \sigma(r, x_r, \mu_r, v) - \sigma(r, x_r, \mu_r, v) \right)^2 \right] \nu(dv) dr \mu(dx). \]

Using the Lipschitz condition

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\[
\leq M \int_{D[0,T]} \int_s^t 1_{\{|x_r| > R\}} \left[ \left( |x_r| - R \right)^2 + \rho(\mu_r^R, \mu_r)^2 \right] \, dr \, \mu(dx).
\]

Then from Chebychev's inequality
\[
\leq M \left[ \frac{\mu^2}{R^2} + \int_{D[0,T]} \int_s^t \rho(\mu_r^R, \mu_r)^2 \, dr \, \mu(dx) \right]
\]

Using the definition of the Vasserstien metric
\[
\rho(\mu_r^R, \mu_r)^2 \leq (E[|Y - Y^R|]^2)
\]
where \(Y\) has law \(\mu_r\), and
\[
Y^R = \begin{cases} Y, & \text{if } |Y| \leq R \\ \frac{Y}{|Y|}, & \text{if } |Y| > R. \end{cases}
\]
So
\[
\rho(\mu_r^R, \mu_r)^2 \leq (E[1_{\{|Y| > R\}} \cdot |Y|]^2)
\]
\[
\leq \frac{(E[|Y|^2])^2}{R^2}
\]
\[
\leq \frac{|\mu_r|}{R^2}
\]
From this we see that
\[
\int_{P(D[0,T])} [F(\mu)^2 - F_R(\mu)^2] \mathcal{L}(\eta^n)(d\mu) = M \int_{P(D[0,T])} [F(\mu) + F_R(\mu)]
\]
\[
\times \left[ \frac{\mu^2}{R^2} + \int_{D[0,T]} \int_s^t \frac{\mu_r}{R^2} \, dr \, \mu(dx) \mathcal{L}(\eta^n)(d\mu) \right].
\]
Using Lemma 2.5
\[
\int_{P(D[0,T])} [F(\mu)^2 - F_R(\mu)^2] \mathcal{L}(\eta^n)(d\mu) = M \int_{P(D[0,T])} [1 + |\mu^2|]
\]
\[
\times \left[ \frac{\mu^2}{R^2} \mathcal{L}(\eta^n)(d\mu) \right]
\]
\[
\leq M \int_{P(D[0,T])} \left[ \frac{1 + |\mu^4|}{R^2} \mathcal{L}(\eta^n)(d\mu) \right]
\]
\[
= \frac{1 + (\eta^n)^4}{R^2}
\]
which from Theorem 2.2 converges uniformly in \(n\) to 0 as \(R\) tends towards infinity.
In the same way we see that
\[ \int_{P(D[0,T])} [F(\mu)^2 - FR(\mu)^2] \mathcal{L}(\eta^n) (d\mu) \]
converges uniformly to zero.

Now
\[ \int_{P(D[0,T])} F(\mu)^2 \mathcal{L}(\eta^n) (d\mu) - \int_{P(D[0,T])} F(\mu)^2 \mathcal{L}(\eta) (d\mu) \]
\[ = \int_{P(D[0,T])} F(\mu)^2 \mathcal{L}(\eta^n) (d\mu) - \int_{P(D[0,T])} FR(\mu)^2 \mathcal{L}(\eta^n) (d\mu) \]
\[ + \int_{P(D[0,T])} FR(\mu)^2 \mathcal{L}(\eta^n) (d\mu) - \int_{P(D[0,T])} FR(\mu)^2 \mathcal{L}(\eta) (d\mu) \]
\[ + \int_{P(D[0,T])} FR(\mu)^2 \mathcal{L}(\eta) (d\mu) - \int_{P(D[0,T])} F(\mu)^2 \mathcal{L}(\eta) (d\mu). \]

We have just shown that the first and last of these summands converge uniformly to zero as \( R \to \infty \), and that the middle summand tends towards zero as \( n \) tends towards infinity, so we are done.

**Lemma 2.7**

\[ \lim_{n \to \infty} \int_{P(D[0,T])} F^2 \mathcal{L}(\eta^n) (d\mu) = 0. \]

**Proof**

Since \( \eta^n \) has a subsequence which converges to \( \eta \), we will write \( \eta^n \Rightarrow \eta \) for simplicity of notation.

\[ E_{\mathcal{L}(\eta^n)}[F^2] = E[F^2(\frac{1}{n} \sum_{j=1}^{n} \delta_{X_{t,j}})] \]
\[ = E[\frac{1}{n} \sum_{j=1}^{n} (f(X_{t,j}^{n,j}) - f(X_{s,j}^{n,j}) \]
\[ - \int_{s}^{t} L_{\eta^n,ir} f(X_{r,j}^{n,j}) dr \prod_{i=1}^{d} g_{s}(X_{s_i}^{n,j})^{2}] \]
is equal to

\[
E\left[\frac{1}{n} \sum_{j=1}^{n} (f(X_t^{n,j}) - f(X_s^{n,j})) - \int_s^t \left[ f'(X_r^{n,j})b(r, X_r^{n,j}, \eta_r^n) \right. \right.
\]
\[
\left. \left. + \int_V \left[ f(X_r^{n,j} + \sigma(r, X_r^{n,j}, \eta_r^n, v)) - f(X_r^{n,j}) \right. \right. \right.
\]
\[
\left. \left. - f'(x_r)\sigma(r, x_r, \mu_r, v)\nu(\mu, v) \right] dr \prod_{i=1}^{d} g_i(X_{s_i}^{n,j}) \right)^2 \right].
\]

Adding and subtracting

\[
E\left[\frac{1}{n} \sum_{j=1}^{n} (f(X_t^{n,j}) - f(X_s^{n,j})) - \int_s^t \left[ f'(X_r^{n,j})b(r, X_r^{n,j}, \eta_r^n) \right. \right.
\]
\[
\left. \left. + \int_V \left[ f(X_r^{n,j} + \sigma(r, X_r^{n,j}, \eta_r^n, v)) - f(X_r^{n,j}) \right. \right. \right.
\]
\[
\left. \left. - f'(X_r^{n,j})\sigma(r, X_r^{n,j}, \eta_r^n, v)\nu(\mu, v) \right] dr \prod_{i=1}^{d} g_i(X_{s_i}^{n,j}) \right)^2 \right]
\]
\[
\frac{1}{n} \sum_{j=1}^{n} \left( f(X_t^{n,j}) - f(X_s^{n,j}) \right)
\]
\[
- \int_s^t \left[ f'(X_r^{n,j})b(r, X_r^{n,j}, \eta_r^n) \right. \right.
\]
\[
\left. \left. + \int_V \left[ f(X_r^{n,j} + \sigma(r, X_r^{n,j}, \eta_r^n, v)) - f(X_r^{n,j}) \right. \right. \right.
\]
\[
\left. \left. - f'(X_r^{n,j})\sigma(r, X_r^{n,j}, \eta_r^n, v)\nu(\mu, v) \right] \right. \right.
\]
\[
\left. \left. dr \prod_{i=1}^{d} g_i(X_{s_i}^{n,j}) \right)^2 \right]
\]

we obtain

\[
E_{c(\eta^n)}[F^2] = I_1 + I_2
\]

where

\[
I_1 = E\left[\frac{1}{n} \sum_{j=1}^{n} (f(X_t^{n,j}) - f(X_s^{n,j})) - \int_s^t \left[ f'(X_r^{n,j})b(r, X_r^{n,j}, \eta_r^n) \right. \right.
\]
\[
\left. \left. + \int_V \left[ f(X_r^{n,j} + \sigma(r, X_r^{n,j}, \eta_r^n, v)) - f(X_r^{n,j}) \right. \right. \right.
\]
\[
\left. \left. - f'(X_r^{n,j})\sigma(r, X_r^{n,j}, \eta_r^n, v)\nu(\mu, v) \right] \right. \right.
\]
\[
\left. \left. dr \prod_{i=1}^{d} g_i(X_{s_i}^{n,j}) \right)^2 \right]
\]
and

$$I_2 = E[\frac{1}{n} \sum_{j=1}^{n} \int_{s}^{t} [f'(X_r^{n,j})b_n(r, X_r^{n,j}, \eta_r^n)$$

$$+ \int_{V} [f(X_r^{n,j} + \sigma_n(r, X_r^{n,j}, \eta_r^n, v)) - f(X_r^{n,j})$$

$$- f'(X_r^{n,j})\sigma_n(r, X_r^{n,j}, \eta_r^n, v)]\nu_n(du)] dr \prod_{i=1}^{d} g_i(X_{s_i}^{n,j})]$$

$$- \frac{1}{n} \sum_{j=1}^{n} (\int_{s}^{t} [f'(X_r)[b(r, X_r^{n,j}, \eta_r^n)$$

$$+ \int_{V} [f(X_r^{n,j} + \sigma(r, X_r^{n,j}, \eta_n)) - F'(X_r^{n,j})$$

$$- f'(X_r^{n,j})[\sigma(r, X_r^{n,j}, \eta_r^n, v)]\nu(du)] dr \prod_{i=1}^{d} g_i(X_{s_i}^{n,j}))^2].$$

Using Doob's maximal inequality and the fact that $\prod_{i=1}^{d} g_i(X_{s_i}^{n,i})^2$ is bounded

$$E[\frac{1}{n} \sum_{j=1}^{n} (f(X_t^{n,j}) - f(X_s^{n,j}) - \int_{s}^{t} [f'(X_r^{n,j})b_n(r, X_r^{n,j}, \eta_r^n)$$

$$+ \int_{V} [f(X_r^{n,j} + \sigma_n(r, X_r^{n,j}, \eta_r^n, v)) - f(X_r^{n,j})$$

$$- f'(X_t^{n,j})\sigma_n(r, X_t^{n,j}, \eta_r^n, v)]\nu_n(du)] dr \prod_{i=1}^{d} g_i(X_{s_i}^{n,j})^2]$$

is less than or equal to

$$\frac{M}{n^2} E[\sum_{j=1}^{n} (f(X_t^{n,j}) - f(X_s^{n,j}) - \int_{s}^{t} f'(X_r^{n,j})b_n(r, X_r^{n,j}, \eta_r^n)$$

$$+ \int_{V} [f(X_r^{n,j} + \sigma_n(r, X_r^{n,j}, \eta_r^n, v)) - f(X_r^{n,j})$$

$$- f'(X_t^{n,j})\sigma_n(r, X_t^{n,j}, \eta_r^n, v)]\nu_n(du) dr]_t].$$

But

$$-f(X_s^{n,j}) - \int_{s}^{t} f'(X_r^{n,j})b(r, X_r^{n,j}, \eta_r^n)$$

$$+ \int_{V} [f(X_r^{n,j} + \sigma_n(r, X_r^{n,j}, \eta_r^n, v)) - f(X_r^{n,j})$$

$$- f'(X_r^{n,j})\sigma_n(r, X_r^{n,j}, \eta_r^n, v)]\nu_n(du) dr$$
has finite variation on compacts so we may drop it from the quadratic variation

to get

\[ E_{\mathcal{L}(\eta^n)}[F^2] = \frac{M}{n^2} E \left[ \sum_{j=1}^{n} (f(X_{0,j}^n), \sum_{j=1}^{n} (f(X_{r,j}^n))] \right]. \]

We apply Itô's Lemma so that we can write

\[ E_{\mathcal{L}(\eta^n)}[F^2] \]

as the expected value of a stochastic differential equation.

\[
E_{\mathcal{L}(\eta^n)}[F^2] = \frac{M}{n^2} E \left[ \sum_{j=1}^{n} (f(X_{0,j}^n))
+ \int_0^T \left[ \sum_{j=1}^{n} \left( f'(X_{r,j}^n) b_n(r, X_{r,j}^n, \eta_{r,j}^n) + \int_0^T \left[ f'(X_{r,j}^n) + \sigma_n(r, X_{r,j}^n, \eta_{r,j}^n, \nu) - f(X_{r,j}^n) \right] N_{r,j}^n(dt, d\nu) \right) 
+ \int_0^T \left[ f'(X_{r,j}^n) + \sigma_n(r, X_{r,j}^n, \eta_{r,j}^n, \nu) - f(X_{r,j}^n) \right] - f(X_{r,j}^n) \sigma_n(r, X_{r,j}^n, \eta_{r,j}^n, \nu)] \nu(d\nu) dt \right] \right] \right].
\]

Once again we may eliminate those terms which have finite variation on compacts
to greatly simplify our equation.

\[
E_{\mathcal{L}(\eta^n)}[F^2] = \frac{M}{n^2} E \left[ \sum_{j=1}^{n} \left( \int_0^T \left[ f'(X_{r,j}^n) + \sigma_n(s, X_{r,j}^n, \eta_{r,j}^n, \nu) \right] N_{r,j}^n(dt, d\nu) \right) \right].
\]

Since \( \{N_{r,j}^n\} \) is pairwise independent the quadratic variation of terms driven by
different Poisson random measures will be equal to zero. Thus
\[ E_L(\eta^n)[F^2] = \frac{M}{n^2} \sum_{j=1}^{n} E\left[ \int_{0}^{t} \int_{V} \left( f'(X_r^{n,j} + \sigma_n(s, X_r^{n,j}, \eta_r^{n,j}, v)) - f(X_r^{n,j}) \right) \right. \\
\left. \cdot N^{n,j}(dr, dv) \right] \\
= \frac{M}{n^2} \sum_{j=1}^{n} E\left[ \int_{0}^{t} \int_{V} \left( f'(X_r^{n,j} + \sigma_n(s, X_r^{n,j}, \eta_r^{n,j}, v)) - f(X_r^{n,j}) \right)^2 \nu_n(dv) \right] dr \\
\leq \frac{M}{n^2} \sum_{j=1}^{n} E\left[ \int_{0}^{t} \int_{V} \sigma_n(s, X_r^{n,j}, \eta_r^{n,j}, v)^2 \nu_n(dv) \right] dr \\
\leq \frac{M}{n^2} \sum_{j=1}^{n} E\left[ \int_{0}^{t} \left( X_r^{n,j})^2 + E[| (\eta_r^{n,j})^2 ] \right) dr \right]. \\
\]

which goes to zero as \( n \) goes to infinity by Lemma 2.2. Then it remains to show that

\[ E\left[ \frac{1}{n} \sum_{j=1}^{n} \left( \int_{s}^{t} [f'(X_r) b_n(r, X_r^{n,j}, \eta_r^{n,j}) \\
+ \int_{V} [f(X_r^{n,j} + \sigma_n(r, X_r^{n,j}, \eta_r^{n,j}, v)) - f(X_r^{n,j}) \\
- f'(X_r^{n,j}) \sigma_n(r, X_r^{n,j}, \eta_r^{n,j}, v)] \nu_n(dv) \right) dr \prod_{i=1}^{d} g_i(X_{s_i}^{n,j}) \\
- \frac{1}{n} \sum_{j=1}^{n} \left( \int_{s}^{t} [f'(X_r) b(r, X_r^{n,j}, \eta_r^{n,j}) \\
+ \int_{V} [f(X_r^{n,j} + \sigma(r, X_r^{n,j}, \eta_r^{n,j})) - f(X_r^{n,j}) \\
- f'(X_r^{n,j}) \sigma(r, X_r^{n,j}, \eta_r^{n,j}, v)] \nu(dv) \right) dr \prod_{i=1}^{d} g_i(X_{s_i}^{n,j})^2 \\
\right] \]

goesto zero as \( n \to \infty \). We may replace \( \prod_{i=1}^{d} g_i(X_{s_i}^{n,j}) \) by it’s upper bound. By hypothesis

\[ E\left[ \int_{s}^{t} [f'(X_r) b_n(r, X_r^{n,j}, \eta_r^{n,j}) - b(r, X_r^{n,j}, \eta_r^{n,j}) dr)^2 \right] \to 0. \]

Using (23)

\[ E\left[\int_{s}^{t} [f'(X_r) b_n(r, X_r^{n,j}, \eta_r^{n,j}) - b(r, X_r^{n,j}, \eta_r^{n,j}) dr)^2 \right] \to 0. \]
\[
\begin{align*}
E\left[\frac{1}{n} \sum_{j=1}^{n} \left( \int_{s}^{t} \left[ f(X_t^{n,j} + \sigma_n(r, X_t^{n,j}, \eta_t^{n}, v)) - f(X_t^{n,j}) ight. ight. \\
&\left. \left. - f'(X_t^{n,j}) \sigma_n(r, X_t^{n,j}, \eta_t^{n}, v) \big| \nu_n(du) \big] dr \right) \right. \\
&\left. \left. - \frac{1}{n} \sum_{j=1}^{n} \int_{s}^{t} \left[ f(X_t^{n,j} + \sigma(r, X_t^{n,j}, \eta_n)) - f(X_t^{n,j}) ight. \\
&\left. \left. - f'(X_t^{n,j}) \sigma(r, X_t^{n,j}, \eta_n, v) \big| \nu(du) \big] dr \right) \right)^2 \right) \\
&= E\left[\frac{1}{n} \sum_{j=1}^{n} \left( \int_{s}^{t} \left[ \sigma_R(r, X_t^{n,j}, \eta_t^{n}, v) \right. \\
&\left. \left. \times \int_{0}^{1} z_1 \int_{0}^{1} f''(X_t^{n,j}) + \int_{R} \sigma_R(r, X_t^{n,j}, \eta_t^{n}, v) z_1 z_2 \big| \nu(du) \big] dr \right) \right. \\
&\left. \left. + \frac{1}{n} \sum_{j=1}^{n} \left( - \int_{s}^{t} \left[ \sigma_R(r, X_t^{n,j}, \eta_t^{n}, v) \right. \\
&\left. \left. \times \int_{0}^{1} z_1 \int_{0}^{1} f''(X_t^{n,j}) + \int_{R} \sigma_R(r, X_t^{n,j}, \eta_t^{n}, v) z_1 z_2 \big| \nu_n(du) \big] dr \right) \right) \right)^2 \right] \\
&\text{which goes to zero by assumption.}
\end{align*}
\]

**Theorem 2.8**

There is a unique solution to the McKean-Vlasov equation

\[
X_t = X_0 + \int_{0}^{t} b(s, X_s, \mathcal{L}(X_s)) \, ds + \int_{0}^{t} \int_{V} \sigma(s, X_s, \mathcal{L}(X_s), v) \tilde{N}(dv, ds).
\]

**Remark**

By existence of a solution we mean there is a measure $\mathcal{L}(X) \in \mathcal{P}(D[0,T])$ and a $D[0,T]$-valued square integral random variable $X$ satisfying the above equation. By uniqueness we mean pathwise uniqueness.
Proof

Let

\[ X^0_t = X_0 \]

and

\[ X^{n+1}_t = X_0 + \int_0^t b(s, X^n_s, \mathcal{L}(X^n_s)) \, ds + \int_0^t \int_\mathcal{V} \sigma(s, X^n_s, \mathcal{L}(X^n_s), v) \tilde{N}(dv, ds) \]

for \( n \geq 0 \). From Lemma 2.1

\[
E[\sup_{t \leq T} (X^{n+1}_t - X^n_t)^2] \\
\leq M \left[ E\left[ \left( \int_0^T | b(s, X^n_s, \mathcal{L}(X^n_s)) - b(s, X^{n-1}_s, \mathcal{L}(X^{n-1}_s)) | \, ds \right)^2 \right] \\
+ E\left[ \int_0^T \int_\mathcal{V} (\sigma(s, X^n_s, \mathcal{L}(X^n_s)) - \sigma(s, X^{n-1}_s, \mathcal{L}(X^{n-1}_s)))^2 \nu(dv) \, ds \right] \right].
\]

From the Lipschitz condition and Hölder's inequality

\[
E[\sup_{t \leq T} (X^{n+1}_t - X^n_t)^2] \\
\leq M \left[ E\left[ \int_0^T (X^n_s - X^{n-1}_s)^2 + \rho(X^n_s, X^{n-1}_s)^2 \, ds \right] \right] \\
\leq M \left[ \int_0^T E[\sup_{u \leq s} (X^n_u - X^{n-1}_u)^2] \, ds \right].
\]

Now for convenience let \( g_n(s) = E[\sup_{u \leq s} (X^n_u - X^{n-1}_u)^2] \). Then we have

\[
g_n(T) \leq C \int_0^T g_{n-1}(s) \, ds \\
\leq C^2 \int_0^T \int_0^s g_{n-2}(u) \, du \, ds \\
= C^2 \int_0^T (T - s) g_{n-2}(s) \, ds.
\]

Where the equality is due to integration by parts. Continuing inductively we obtain
\[ g_n(t) \leq C^{n-1} \int_0^T \frac{(T-s)^{n-2}}{(n-2)!} g_1(s) \, du \]
\[ = E[(X_0)^2] \frac{(CT)^{n-1}}{(n-1)!}. \]

This term is part of a convergent series so we may conclude that \( \{X^n\} \) is Cauchy in \( L^2 \). If we let \( X \) be the limit of \( \{X^n\} \) by construction \( X \) is square integrable.

What remains is to show that \( X \) is the unique solution to the equation

\[ X_t = X_0 + \int_0^t b(s, X_s, \mathcal{L}(X_s)) \, ds + \int_0^t \int_{\mathcal{V}} \sigma(s, X_s, \mathcal{L}(X_s), v) \tilde{N}(dv, ds). \]

From Lemma 2.1

\[
E[\sup_{t \leq T} (X_t^{n+1} - [X_0 + \int_0^t b(s, X_s, \mathcal{L}(X_s)) \, ds + \int_0^t \int_{\mathcal{V}} \sigma(s, X_s, \mathcal{L}(X_s), v) \tilde{N}(dv, ds)])^2]
\leq M[E[\int_0^T | b(s, X_s^n, \mathcal{L}(X_s^n)) - b(s, X_s, \mathcal{L}(X_s)) | \, ds]^2]
+ E[\int_0^T \int_{\mathcal{V}} (\sigma(s, X_s^n, \mathcal{L}(X_s^n)) - \sigma(s, X_s, \mathcal{L}(X_s)))^2 \nu(dv) \, ds].
\]

Using the growth condition and Hölder's inequality

\[
E[\sup_{t \leq T} (X_t^{n+1} - X_0) \bigg|^2]
\leq M\left[ E[\int_0^T (X_s^n - X_s)^2 + \rho(\mathcal{L}(X_s^n), \mathcal{L}(X_s))^2 \, ds]ight]
\leq M\left[ \int_0^T E[\sup_{u \leq s} (X_u^n - X_u)^2] + ds \right].
\]

Which we have just shown goes to zero.
So now we need only show the uniqueness of the solution. Suppose there is another square integrable solution to the equation

\[ Y_t = X_0 + \int_0^t b(s, Y_s, \mathcal{L}(Y_s)) \, ds + \int_0^t \int_{\mathcal{V}} \sigma(s, Y_s, \mathcal{L}(Y_s), v) \tilde{N}(dv, ds). \]

From Lemma 2.1

\[
E[\sup_{t \leq T} (X_t - Y_t)^2] \\
\leq M[E[\left( \int_0^T | b(s, X_s, \mathcal{L}(X_s)) - b(s, Y_s, \mathcal{L}(Y_s)) | \, ds \right)^2] \\
+ E[\int_0^T \int_{\mathcal{V}} (\sigma(s, X_s, \mathcal{L}(X_s)) - \sigma(s, Y_s, \mathcal{L}(Y_s)))^2 \nu(dv) \, ds].
\]

Using the growth condition and Hölder's inequality

\[
E[\sup_{t \leq T} (X_t - Y_t)^2] \\
\leq M[E[\int_0^T (X_s - Y_s)^2 + \rho(\mathcal{L}(X_s), \mathcal{L}(Y_s))^2 \, ds] \\
\leq M[\int_0^T E[\sup_{u \leq s} (X_u - Y_u)^2] + ds]
\]

which is equal to zero by Gronwall's inequality so we are done.

**Theorem 2.9**

Any weak limit of \( \{\eta^n\} \) is a solution of the McKean-Vlasov equation (20).

**Proof**

An element \( \mu \in P(D[0, T]) \) is a solution of the martingale problem corresponding to (20) if and only if \( F(\mu) = 0 \) for all \( f, g_1, \ldots, g_d, s_1, \ldots, s_d, s, t, d \).
From Lemma 2.5, and Lemma 2.6

\[ \int_{P(D[0,T])} F(\mu)^2 \eta(d\mu) = 0 \]

So \( F(\mu) = 0 \) a.s. Then by Theorem 1.48 \( \eta \) is a solution to (20). By the pathwise uniqueness of the solution to (20), \( \eta = \mathcal{L}(X) \) is the solution to the McKean-Vlasov equation.

2.3: Diffusion approximation

In this case we make the following assumptions about \( \{\nu_n\}, \{b_n\} \) and \( \{\sigma_n\} \):

\[(28) \quad \nu_n \Rightarrow \delta_0.\]

For all \( s, x, y, v \in \mathbb{R} \) there exists \( \hat{\sigma}(s, x, y) \) and \( \hat{b}(s, x, y) \) such that

\[(29) \quad \int_V \hat{\sigma}_n(s, x, y, v)^2 \nu_n(dv) \to \hat{\sigma}(s, x, y)^2 \]

uniformly on \([0, T] \otimes \mathbb{R} \otimes \mathbb{R}\).

\[(30) \quad \hat{b}_n(s, x, y) \to \hat{b}(s, x, y) \]

uniformly on \([0, T] \otimes \mathbb{R} \otimes \mathbb{R}\).

\[(31) \quad \lim_{n \to \infty} \sup_{x, y, t} \nu_n(\{\sigma_n(t, x, y, v) > N\}) = 0 \]

for all \( N > 0 \).

\[(32) \quad \lim_{n \to \infty} \sup_{x, y, t} \int_V \sigma_n(t, x, y, v)^2 1_{\{\sigma_n(t,x,y,v) > M\}} \nu_n(dv) = 0 \]
for all $M > 0$. Define $\sigma$ and $b$ as in the previous section. We wish to show that

$$\lim_{n \to \infty} \eta^n = \mathcal{L}(X)$$

where

$$X_t = X_0 + \int_0^t b(s, X_s, \mathcal{L}(X_s)) \, ds + \int_0^t \sigma(s, X_s, \mathcal{L}(X_s)) \, dB_s$$

where $X_0$ has the same distributions as $X_0^n$ and $B_s$ is a Brownian motion. Towards this we first establish the convergence of the random measures $\eta^n = \frac{1}{n} \sum_{i=1}^n \delta_{X^n_i}$ to solution of the martingale problem associated with (33). For any $f \in C^2_0[0,T]$ and $g_1, \ldots, g_d \in C^2_0[0,T], 0 \leq s_1, \ldots, s_d \leq s, s < t, d \geq 0$. Define

$$F(\mu) = F_{f,g_1,\ldots,g_d}^{s,t} : P(D[0,T]) \to \mathbb{R}$$

by

$$F(\mu) = \int_{D[0,T]} (f(x_t) - f(x_s) - \int_s^t L_{\mu,r,f}(x_r) \, dr) \prod_{k=1}^d g_k(x_{s_k}) \mu(dx)$$

where for $x \in D[0,T], x_s = x(s), 0 \leq s \leq T,$ and

$$L_{\mu,r,f}(x) = f'(x)b(r,x,\mu) + \frac{1}{2} f''(x)\sigma(s,x,\mu)^2$$

Then $\eta \in P(D[0,T])$ is a solution to the martingale problem corresponding to (30) providing $F(\eta) = 0$ for all choices of $f, g_1, \ldots, g_d, s_1, \ldots, s_d, s, t,$ and $d$.

**Lemma 2.10**

$$F(\mu) \leq M(1 + |\mu|^2)$$

for all $\mu \in P(D[0,T])$.

**Proof**
Since $f$ and $\prod_{i=1}^{d} g_i$ are bounded

$$| F(\mu) | = | \int_{D[0,T]} (f(x_t) - f(x_s) - \int_{s}^{t} L_{\mu, r, f}(x_r) \, dr) \prod_{i=1}^{d} g_i(x_{s_i}) \mu(dx) |$$

$$\leq M (1 + \int_{D[0,T]} \int_{s}^{t} | f'(x_r) b(r, x_r, \mu_r) | dr \mu(dx))$$

Since $f', f''$ are bounded

$$| F(\mu) | \leq M (1 + \int_{D[0,T]} \int_{s}^{t} | b(r, x_r, \mu_r) |$$

$$\quad + | \sigma(r, x_r, \mu_r) |^2 \, dr \mu(dx))$$

$$\leq M (1 + \int_{D[0,T]} \int_{s}^{t} (b(r, x_r, \mu_r) |$$

$$\quad + | \sigma(r, x_r, \mu_r) |^2 \, dr \mu(dx))$$

$$\leq M (1 + \int_{D[0,T]} \int_{s}^{t} (x_r^2 + | \mu_r^2 |) \mu(dx))$$

$$\leq M (1 + | \mu_r^2 |).$$

**Lemma 2.11**

Let $\eta$ be the weak limit of a subsequence $\eta^{n'}$ of $\eta^n$.

$$\lim_{n'} \int_{P(D[0,T])} F^2(\mu) \mathcal{L}(\eta^{n'})(d\mu) = \int_{P(D[0,T])} F^2(\mu) \mathcal{L}(\eta)(d\mu).$$

**Proof**

Since $\eta^n$ has a subsequence which converges to $\eta$, we will write $\eta^n \Rightarrow \eta$ for simplicity of notation.

Recall that for any $a \in \mathbb{R}$

$$\sigma(s, a, \mu_s) = \int_{\mathbb{R}} \tilde{\sigma}(s, a, y) \mu_s(dy)$$
and

\[ b(s, a, \mu_a) = \int_{\mathbb{R}} \hat{b}(s, a, y)\mu_a(dy). \]

Define

\[ \sigma_R : [0, T] \times \mathbb{R} \times P(\mathbb{R}) \times \mathbb{R} \to \mathbb{R} \]

and

\[ b_R : [0, T] \times \mathbb{R} \times P(\mathbb{R}) \times \mathbb{R} \to \mathbb{R} \]

by

\[
\sigma_R(s, a, \gamma) = \begin{cases} 
\int_{\mathbb{R}} \hat{\sigma}(s, a, y)\gamma(dy). & \text{if } |a| < R \text{ and } |y| < R \\
\int_{\mathbb{R}} \hat{\sigma}(s, a, \frac{aR}{|a|}, y)\gamma(dy). & \text{if } |a| > R \text{ and } |y| < R \\
\int_{\mathbb{R}} \hat{\sigma}(s, a, \frac{aR}{|a|}, \frac{yR}{|y|})\gamma(dy). & \text{if } |a| < R \text{ and } |y| > R \\
\int_{\mathbb{R}} \hat{\sigma}(s, a, \frac{aR}{|a|}, \frac{yR}{|y|})\gamma(dy). & \text{if } |a| > R \text{ and } |y| > R.
\end{cases}
\]

and

\[
b_R(s, a, \gamma) = \begin{cases} 
\int_{\mathbb{R}} \hat{b}(s, a, y)\gamma(dy). & \text{if } |a| < R \text{ and } |y| < R \\
\int_{\mathbb{R}} \hat{b}(s, a, \frac{aR}{|a|}, y)\gamma(dy). & \text{if } |a| > R \text{ and } |y| < R \\
\int_{\mathbb{R}} \hat{b}(s, a, \frac{aR}{|a|}, \frac{yR}{|y|})\gamma(dy). & \text{if } |a| < R \text{ and } |y| > R \\
\int_{\mathbb{R}} \hat{b}(s, a, \frac{aR}{|a|}, \frac{yR}{|y|})\gamma(dy). & \text{if } |a| > R \text{ and } |y| > R.
\end{cases}
\]

Let

\[ L^R_{\mu,r,f}(a) = f'(a)b_R(r, a, \mu) + \frac{1}{2}f''(a)\sigma_R(s, a, \mu)^2. \]

Define

\[ F_R = \int_{D[0,T]} (f(x_t) - f(x_s) - \int_s^t L^R_{\mu,r,f}(x_r) \, dr) \prod_{k=1}^d g_k(x_{s_k})\mu(dx). \]

First we show that \( F_R(\cdot) \) is continuous on \( P(D[0,T]) \) topologized by weak convergence. Towards this end we note that \( \hat{b} \) is bounded and continuous in \( a \) and \( y \) so \( b_R(s, a, \gamma) \) is bounded and continuous in \( x \) and \( \gamma \) where continuity in

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\( \gamma \) is with respect to the topology of weak convergence. Similarly \( \sigma_R(r, a, \gamma) \) is continuous in \( x \) and \( \gamma \). If \( \mu^n \Rightarrow \mu \) then for almost all \( r \mu_r \Rightarrow \mu_r \), and there exists \( \{Z^n_r\} \) and \( Z_r \) such that \( Z^n_r \) has \( \mu^n_r \) as its law, and \( Z^n_r \rightarrow Z_r \) for almost all \( r \).

Then

\[
F_R(\mu^n) = E[f(Z^n_t) - f(Z^n_s) - \int_s^t f'(Z^n_r)b_R(r, Z^n_r, \mu^n_r) \, dr + \frac{1}{2} f''(Z^n_r)\sigma_R(r, Z^n_r, \mu^n_r))^2 \, dr
\]

which is bounded and converges to

\[
E[f(Z_t) - f(Z_s) - \int_s^t f'(Z_r)b(r, Z_r, \mu_r) \, dr + \frac{1}{2} f''(Z^n_r)\sigma(r, Z^n_r, \mu^n_r))^2 \, dr.
\]

Thus \( F_R(\cdot) \) is continuous. Clearly it is bounded, from which we may conclude that

\[
\lim_{n \to \infty} \int_{P([0,T])} F_R(\mu)^2 \mathcal{L}(\eta^n(d\mu)) = \int_{P([0,T])} F_R(\mu)^2 \mathcal{L}(\eta(d\mu)).
\]

Now we wish to show that

\[
\int_{P([0,T])} [F(\mu)^2 - F_R(\mu)^2] \mathcal{L}(\eta^n(d\mu)) \to 0
\]

\[
[F(\mu)^2 - F_R(\mu)^2] = [F(\mu) + F_R(\mu)] [F(\mu) - F_R(\mu)]
\]

\[
= [F(\mu) + F_R(\mu)] \int_{[0,T]} \int_s^t f'(x_r)[b_R(r, x_r, \mu_r) - b(r, x_r, \mu_r)] \, dr, \mu(dx).
\]

Using (26)

\[
[F(\mu)^2 - F_R(\mu)^2]
\]

\[
= [F(\mu) + F_R(\mu)] \int_{[0,T]} \int_s^t f'(x_r)[b(r, x_r, \mu^R) - b(r, x_r, \mu_r)] \, dr, \mu(dx).
\]

\[
+ \frac{1}{2} f''(x_r)[\sigma(r, x_r, \mu^R)^2 - \sigma(r, x_r, \mu_r)^2 \, dr, \mu(dx).
\]

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\[
\int_{p(D[0,T] \cap [0,T] \int_s^t f'(x_r)[b_R(r, x_r, \mu_r) - b(r, x_r, \mu_r)] \,dr, \mu(dx)\mathcal{L}(\eta^n)(d\mu)
= \int_{p(D[0,T] \cap [0,T] \int_s^t f'(x_r)
\times \int_{\mathcal{R}} [1_{\{|x_r|>R\}} 1_{\{|y|>R\}} [b(r, x_r, y) - b(r, x_r, y/R)]
+ 1_{\{|x_r| \leq R\}} 1_{\{|y| > R\}} [b(r, x_r, y) - b(r, x_r, y/R)]
+ 1_{\{|x_r|>R\}} 1_{\{|y| \leq R\}} [b(r, x_r, y) - b(r, x_r, y/R)]\mu_r(dy) \,dr, \mu(dx)\mathcal{L}(\eta^n)(d\mu).
\]

Using the growth conditions
\[
\leq \int_{p(D[0,T] \cap [0,T] \int_s^t f'(x_r) \int_{x_r} [1_{\{|x|>R\}} 1_{\{|y|>R\}} |x|r |-R + |y|-R]
+ 1_{\{|y|>R\}} |y| |-R + 1_{\{|x|>R\}} |x| |-R| \mu_r(dy) \,dr, \mu(dx)\mathcal{L}(\eta^n)(d\mu)
= 2 \int_{D(D[0,T] \cap [0,T] \int_s^t f'(x_r) \int_{x_r} [1_{\{|y|>R\}} |y|]
+ 1_{\{|x|>R\}} |x| \mu_r(dy) \,dr, \mu(dx)\mathcal{L}(\eta^n)(d\mu).
\]

Using Chebychev's inequality twice and the fact that \(f'\) is bounded
\[
\leq M \int_{D(D[0,T] \cap [0,T] \int_s^t \int_{x_r} [\mu_r^2/R^2]
+ 1_{\{|\sup_{s \leq t} x_r|>R\}} |x| \mu_r(dy) \,dr, \mu(dx)\mathcal{L}(\eta^n)(d\mu)
\leq M |\mu_r^2|/R^2.
\]

Now
\[
\int_{D[0,T]} \int_{x_r} \frac{1}{2} f''(x_r)\sigma_r(r, x_r, \mu_r^R)^2 - \sigma_r(r, x_r, \mu_r)^2 \nu(dv) \,dr, \mu(dx)
\leq M \int_{D[0,T]} \int_{x_r} \int_{x_r^R} [1_{\{|x| \leq R\}} (\sigma_r(r, x_r, \mu_r^R) - \sigma_r(r, x_r, \mu_r))^2]
+ 1_{\{|x|>R\}} (\sigma_r(r, x_r, \mu_r^R, \mu_r^R, \nu) - \sigma(r, x_r, \mu_r, \nu))^2 \nu(dv) \,dr, \mu(dx).
\]

Using the Lipschitz condition
\[
\leq M \int_{D[0,T]} \int_{s}^{t} 1_{\{|x_r|>R\}} (|x_r| - R)^2 + \rho(\mu_r^R, \mu_r)^2 \, dr, \mu(dx).
\]

Then from Chebychev's inequality

\[
\leq M \left[ \frac{\mu^2}{R^2} \right] + \int_{D[0,T]} \int_{s}^{t} \rho(\mu_r^R, \mu_r)^2 \, dr, \mu(dx)
\]

Using the definition of the Vassershtien metric

\[
\rho(\mu_r^R, \mu_r)^2 \leq (E[|Y - Y^R|])^2
\]

where \( Y \) has law \( \mu_r \), and

\[
Y^R = \begin{cases} 
Y, & \text{if } |Y| \leq R \\
\frac{Y}{|Y|}, & \text{if } |Y| > R.
\end{cases}
\]

So

\[
\rho(\mu_r^R, \mu_r)^2 \leq (E[1_{\{|Y|>R\}} |Y|])^2
\]

\[
\leq \left( \frac{E[|Y|]}{R^2} \right)^2 \leq \frac{\mu_r}{R^2}.
\]

From this we see that

\[
\int_{P(D[0,T])} [F(\mu)^2 - F_R(\mu)^2] \mathcal{L}(\eta^n)(d\mu) = M \int_{P(D[0,T])} [F(\mu) + F_R(\mu)]
\]

\[
\times \left[ \frac{\mu^2}{R^2} \right] + \int_{D[0,T]} \int_{s}^{t} \frac{\mu_r}{R^2} \, dr, \mu(dx) \mathcal{L}(\eta^n)(d\mu).
\]

Using Lemma 2.5

\[
\int_{P(D[0,T])} [F(\mu)^2 - F_R(\mu)^2] \mathcal{L}(\eta^n)(d\mu) = M \int_{P(D[0,T])} [1+ |\mu^2|]
\]

\[
\times \left[ \frac{\mu^2}{R^2} \right] \mathcal{L}(\eta^n)(d\mu)
\]

\[
\leq M \int_{P(D[0,T])} \left[ 1+ \frac{\mu^4}{R^2} \right] \mathcal{L}(\eta^n)(d\mu) = \frac{1+ (\eta^n)^4}{R^2}
\]
which from Lemma 2.4 converges uniformly in \( n \) to 0 as \( R \) tends towards infinity.

In the same way we see that

\[
\int_{P(D[0,T])} [F(\mu)^2 - F_R(\mu)^2] \mathcal{L}(\eta^n)(d\mu)
\]

converges uniformly to zero. Now

\[
\int_{P(D[0,T])} F(\mu)^2 \mathcal{L}(\eta^n)(d\mu) - \int_{P(D[0,T])} F_R(\mu)^2 \mathcal{L}(\eta^n)(d\mu)
\]

\[
= \int_{P(D[0,T])} F(\mu)^2 \mathcal{L}(\eta^n)(d\mu) - \int_{P(D[0,T])} F_R(\mu)^2 \mathcal{L}(\eta^n)(d\mu)
\]

\[
+ \int_{P(D[0,T])} F_R(\mu)^2 \mathcal{L}(\eta^n)(d\mu) - \int_{P(D[0,T])} F_R(\mu)^2 \mathcal{L}(\eta^n)(d\mu)
\]

\[
+ \int_{P(D[0,T])} F_R(\mu)^2 \mathcal{L}(\eta^n)(d\mu) - \int_{P(D[0,T])} F(\mu)^2 \mathcal{L}(\eta^n)(d\mu).
\]

We have just shown that the first and last of these summands converge uniformly to zero as \( R \to \infty \), and that the middle summand tends towards zero as \( n \) tends towards infinity, so we are done.

**Lemma 2.12**

\[
\lim_{n \to \infty} \int_{P(D[0,T])} F^2(\mu) \mathcal{L}(\eta^n)(d\mu) = 0.
\]

**Proof**

Since \( \eta^n \) has a subsequence which converges to \( \eta \), we will write \( \eta^n \Rightarrow \eta \) for simplicity of notation.

\[
\int_{P(D[0,T])} [F^2] \mathcal{L}(\eta)(d\mu) = E[F^2(\frac{1}{n} \sum_{i=1}^{n} \delta_{X^{n,i}})]
\]

\[
= E[\frac{1}{n} \sum_{i=1}^{n} (f(X_t^{n,i}) - f(X_s^{n,i})) - \int_s^t [f'(X_r^{n,i})b(r, X_r^{n,i}, \eta^n_r) + \frac{1}{2} f''(X_r^{n,i})\sigma(r, X_r^{n,i}, \eta^n_r)^2] dr \prod_{i=1}^{d} g_i(X_{s_i}^{n,i})^2].
\]
Adding and subtracting the local martingale part of $f(X_t^{n,j})$ which we can obtain from Itô's lemma:

$$\frac{1}{n} \sum_{j=1}^{n} (f(X_t^{n,j}) - f(X_s^{n,j}) - \int_s^t [f'(X_r^{n,j})b_n(r, X_r^{n,j}, \eta_r^n)$$

$$+ \int_V [f(X_r^{n,j} + \sigma_n(r, X_r^{n,j}, \eta_r^n, v))$$

$$- f(X_r^{n,j}) - f'(X_r^{n,j})\sigma_n(r, X_r^{n,j}, \eta_r^n, v)]\nu_n(dv)] dr \prod_{i=1}^{d} g_i(X_{s_i}^{n,j})),$$

we obtain

$$E_{E(\eta^n)}[F^2] = I_1 + I_2$$

where

$$I_1 = E[\left(\frac{1}{n} \sum_{j=1}^{n} (f(X_t^{n,j}) - f(X_s^{n,j})$$

$$- \int_s^t [f'(X_r^{n,j})b_n(r, X_r^{n,j}, \eta_r^n)$$

$$+ \int_V [f(X_r^{n,j} + \sigma_n(r, X_r^{n,j}, \eta_r^n, v)) - f(X_r^{n,j})$$

$$- f'(X_r^{n,j})\sigma_n(r, X_r^{n,j}, \eta_r^n, v)]\nu_n(dv)] dr \prod_{i=1}^{d} g_i(X_{s_i}^{n,j}))$$

and

$$I_2 = E[\left(\frac{1}{n} \sum_{j=1}^{n} \left(\int_s^t [f'(X_r^{n,j})b_n(r, X_r^{n,j}, \eta_r^n)$$

$$+ \int_V [f(X_r^{n,j} + \sigma_n(r, X_r^{n,j}, \eta_r^n, v))$$

$$- f(X_r^{n,j}) - f'(X_r^{n,j})\sigma_n(r, X_r^{n,j}, \eta_r^n, v)]\nu_n(dv)] dr \prod_{i=1}^{d} g_i(X_{s_i}^{n,j}))$$

$$- \frac{1}{n} \sum_{j=1}^{n} \left(\int_s^t [f'(X_r)b(r, X_r^{n,j}, \eta_r^n)$$

$$\frac{1}{2} f''(X_r^{n,j})\sigma(r, X_r^{n,j}, \eta_r^n)^2 \right) dr \prod_{i=1}^{d} g_i(X_{s_i}^{n,j}))^2).$$

Using Doob's maximal inequality and the fact that $(\prod_{i=1}^{d} g_i(X_{s_i}^{n,i}))^2$ is bounded
\[ E\left[ \frac{1}{\theta} \sum_{j=1}^{n} \left( f(X_{t}^{n,j}) - f(X_{s}^{n,j}) - \int_{s}^{t} \left[ f'(X_{r}^{n,j})b_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}) + \int_{V} \left[ f(X_{r}^{n,j} + \sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v)) - f(X_{r}^{n,j}) \right] \nu_{n}(dv) \right] dr \right] \right] \]

\[ \leq \frac{M}{n^{2}} E\left[ \sum_{j=1}^{n} \left( f(X_{s}^{n,j}) - f(X_{t}^{n,j}) - \int_{s}^{t} f'(X_{r}^{n,j})b_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}) + \int_{V} \left[ f(X_{r}^{n,j} + \sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v)) - f(X_{r}^{n,j}) \right] \nu_{n}(dv) \right] dr \right] \]

But

\[ -f(X_{s}^{n,j}) - \int_{s}^{t} f'(X_{r}^{n,j})b_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}) + \int_{V} \left[ f(X_{r}^{n,j} + \sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v)) - f(X_{r}^{n,j}) \right] \nu_{n}(dv) \] has finite variation on compacts so we may drop it from the quadratic variation to get

\[ E_{\mathcal{L}(\eta_{n})}[F^{2}] = \frac{M}{n^{2}} E\left[ \sum_{j=1}^{n} \left( f(X_{0}^{n,j}) \right) \right] \]

From Itô's Lemma we obtain

\[ E_{\mathcal{L}(\eta_{n})}[F^{2}] = \frac{M}{n^{2}} E\left[ \sum_{j=1}^{n} \left( f(X_{0}^{n,j}) \right) \right] \]

Once again we may eliminate those terms which have finite variation on compacts.
Since \( \{N^n_j\} \) is pairwise independent

\[
E_{\mathcal{L}(\eta^n)}[F^2] = \frac{M}{n^2} E\left[ \sum_{j=1}^{n} \left( \int_{0}^{t} \int_\mathcal{V} \left[ f(X_{r-}^{n,j}) + \sigma_n(s, X_{r-}^{n,j}, \eta_{r-}^{n}, v) \right] 
- f(X_{r-}^{n,j})\right] N^n_j(\mathcal{V}, dv) \right]
\]

which goes to zero as \( n \) goes to infinity by Theorem 2.2. Then it remains to show that

\[
E\left[ \frac{1}{n} \sum_{j=1}^{n} \left( \int_{s}^{t} \left[ f'(X_{r}) b_n(r, X_{r}^{n,j}, \eta_{r}^{n}) + \int_\mathcal{V} \left[ f(X_{r}^{n,j}) + \sigma_n(s, X_{r}^{n,j}, \eta_{r}^{n}, v) \right] - f(X_{r}^{n,j}) \right] \nu_n(dv) \right] \n- \frac{1}{n} \sum_{j=1}^{n} \left( \int_{s}^{t} [f'(X_{r})[b(r, X_{r}^{n,j}, \eta_{r}^{n})] \n+ \frac{1}{2} f''(X_{r}^{n,j}) \sigma(r, X_{r}^{n,j}, \eta_{r}^{n})^2 \right] dr \prod_{i=1}^{d} g_i(X_{s_i}^{n,j})^2 \right)
\]

goes to zero as \( n \to \infty \). We may replace \( \prod_{i=1}^{d} g_i(X_{s_i}^{n,j}) \) by its upper bound. By hypothesis

\[
E[(\int_{s}^{t} \left[ f'(X_{r}) b_n(r, X_{r}^{n,j}, \eta_{r}^{n}) - b(r, X_{r}^{n,j}, \eta_{r}^{n}) \right] dr)^2] \to 0.
\]

Using (23)
\begin{align*}
E[-( & \frac{1}{n} \sum_{j=1}^{n} (\int_{s}^{t} \left[ f(X_{r}^{n,j} + \sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v)) - f(X_{r}^{n,j}) ight. \\
& \left. - f'(X_{r}^{n,j})\sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v)]\nu_{n}(dv)\right) \\
& + \frac{1}{2} f''(X_{r}^{n,j})\sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n})^{2}] dr)^{2} \\
& = E[(\frac{1}{n} \sum_{j=1}^{n} (\int_{s}^{t} \left[ \frac{1}{2} f''(X_{r}^{n,j})\sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n})^{2} \\
& - \int_{V} \sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v)^{2} \int_{0}^{1} z_{1} \\
& \times \int_{0}^{1} f''(X_{r}^{n,j} + \sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v)z_{1}z_{2}) \nu_{n}(dv) dr)^{2}] \\
& \text{Now we add and subtract} \\
& \int_{V} \frac{1}{2} \sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v)^{2} f''(X_{r}^{n,j})\nu_{n}(dv) \\
& \text{inside the integral to obtain} \\
& = E[(\frac{1}{n} \sum_{j=1}^{n} (\int_{s}^{t} \left[ \frac{1}{2} f''(X_{r}^{n,j})\sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n})^{2} \\
& - \int_{V} \frac{1}{2} \sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v)^{2} f''(X_{r}^{n,j})\nu_{n}(dv) \\
& + \int_{V} \sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v)^{2} \\
& \int_{0}^{1} z_{1} \int_{0}^{1} \frac{1}{2} f''(X_{r}^{n,j}) - f''(X_{r}^{n,j} + \sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v)z_{1}z_{2}) \nu_{n}(dv) dr)^{2}] \\
& \text{Condition (27) tells us that} \\
& E[(\frac{1}{n} \sum_{j=1}^{n} (\int_{s}^{t} \left[ \frac{1}{2} f''(X_{r}^{n,j})\sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n})^{2} \\
& - \int_{V} \frac{1}{2} \sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v)^{2} f''(X_{r}^{n,j})\nu_{n}(dv) dr)^{2}] \rightarrow 0
\end{align*}
Let $C > 0$

\[
E\left(\frac{1}{n} \sum_{j=1}^{n} \left( \int_{s}^{t} \sigma_{j}(r, X_{r}^{n,j}, \eta_{r}^{n}, v)^{2} \right. \right. \\
\times \left. \left. \int_{0}^{1} z_{1} \int_{0}^{1} \frac{1}{2} f''(X_{r}^{n,j}) - f''(X_{r}^{n,j} + \sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v) z_{1} z_{2}) \nu_{n}(dv) \, dr \right) \right)
\]

\[
= E\left(\frac{1}{n} \sum_{j=1}^{n} \left( \int_{s}^{t} \left[ \int_{V} \mathbf{1}_{\{ \sigma(r, X_{r}^{n,j}, \eta_{r}^{n}, v) > C \}} \sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v)^{2} \right. \right. \\
\times \left. \left. \int_{0}^{1} z_{1} \int_{0}^{1} \frac{1}{2} f''(X_{r}^{n,j}) - f''(X_{r}^{n,j} + \sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v) z_{1} z_{2}) \nu_{n}(dv) \, dr \right) \right) \right)
\]

Assumption (30) tells us that

\[
E\left(\frac{1}{n} \sum_{j=1}^{n} \left( \int_{s}^{t} \left[ \int_{V} \mathbf{1}_{\{ \sigma(r, X_{r}^{n,j}, \eta_{r}^{n}, v) \leq C \}} \sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v)^{2} \right. \right. \\
\times \left. \left. \int_{0}^{1} z_{1} \int_{0}^{1} \frac{1}{2} f''(X_{r}^{n,j}) - f''(X_{r}^{n,j} + \sigma_{n}(r, X_{r}^{n,j}, \eta_{r}^{n}, v) z_{1} z_{2}) \nu_{n}(dv) \, dr \right) \right) \right)
\]

goes to zero as $n$ approaches infinity.

Since $C$ can be arbitrarily small, and $E[(X_{r}^{n,j})^{4}]$ and $E[|\eta^{n}|^{4}]$ are bounded we are done.

**Remark**

The proof of pathwise uniqueness of the solution to (33) is a special case of Theorem 4.3. The proof is left in chapter 4 because of its importance in the identification of the limit in that chapter.
Theorem 2.13

Any weak limit of $\{\eta^n\}$ is a solution of the McKean-Vlasov equation.

Proof

An element $\mu \in P(D[0,T])$ is a solution of the martingale problem corresponding to (33) if and only if $F(\mu) = 0$ for all $f, g_1, \ldots, g_d, s_1, \ldots, s_d, s, t, d$. From Lemma 2.9, and Lemma 2.10

$$\int_{P(D[0,T])} F(\mu) \eta(d\mu) = 0$$

So $F(\mu) = 0$ a.s., so $\eta$ is a weak solution of (33). Then by the pathwise uniqueness of the solution to (30) and the Yamada-Watanabe argument $\eta = \mathcal{L}(X)$ is the solution to the McKean-Vlasov equation.
Chapter 3

Weak Convergence of Random Mean-Field Interaction Systems

In this chapter we generalize the results of chapter 2 to a system with a random interaction term independent of the driving terms. Let $(\Omega^n, \mathcal{F}^n, P^n)$ with filtration $\mathcal{F}^n_t$, for each $n$ let $N^{n,i}(dv, ds)$ be a collection of i.i.d. Poisson random measures for $1 \leq i \leq n$. Let $\nu_n$ be the associated Lévy measure. Let $V \in \mathbb{R}$ such that $0 \in \bar{V}$ and $\{u : u = \Delta N_t \neq 0 \text{ for some } \omega \} \subset V$ and assume

$$\int_V (1 + u^2) \nu_n(dv) < \infty.$$  

Then

$$\tilde{N}^{n,i}_t = \int_0^t \int_V v(N^{n,i}(dv, dt) - \nu_n(dv) dt)$$

is a martingale. Let

$$\hat{\sigma}_n : ([0, T], \mathbb{R}, \mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}, \hat{\sigma}_n(s, x, y, \cdot) \in L^2(V, \nu_n)$$

$$\hat{b}_n : ([0, T], \mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$$

have the following Lipschitz conditions

1. $$\int_V [\hat{\sigma}_n(s, x_1, y_1, u)^2 - \hat{\sigma}_n(s, x_2, y_2, u)^2] \nu_n(dv) \leq K[(x_1 - x_2)^2 + (y_1 - y_2)^2]$$

2. $$| \hat{b}_n(s, x_1, y_1) - \hat{b}_n(s, x_2, y_2) | \leq K[|x_1 - x_2| + |y_1 - y_2|]$$
and growth conditions

\[ \int_V [\tilde{\sigma}_n(s, x, y, u)^2 \nu_n(dv) \leq K[1 + x^2 + y^2] \]  

(4)  

\[ |\tilde{b}_n(s, x, y)| \leq K[1 + |x| + |y|]. \]

Let \( \{Z^{n,i}\} \) be a collection of i.i.d \( \mathcal{F}_0^n \)-measurable random variables such that \( Z^{n,i} \) assumes only \( l \) values \( \beta_1, \ldots, \beta_l \) and \( P(Z^{n,i} = \beta_k) = p_k, 1 \leq k \leq l \) for all \( n, i \). Let

\[ \alpha(w, z) : \mathbb{R}^2 \to \mathbb{R} \]

and

\[ \gamma(w, z) : \mathbb{R}^2 \to \mathbb{R} \]

be bounded and have Lipschitz condition

(5)

\[ |\alpha(z_1, w_1) - \alpha(z_2, w_2)| + |\gamma(z_1, w_1) - \gamma(z_2, w_2)| \leq K[|z_1 - z_2| + |w_1 - w_2|]. \]

Let \( \{X_{0}^{n,i}\} \) be a collection of i.i.d \( \mathcal{F}_0^n \)-measurable random variables with finite fourth moment, and let

\[ X_t^{n,i} = X_0^{n,i} \]

(6)

\[ + \int_0^t \frac{1}{n} \sum_{j=1}^n \alpha(Z_j^{n,i}, Z_j^{n,j}) \tilde{b}_n(X_s^{n,i}, X_s^{n,j}) ds \]

\[ + \int_0^t \int_V \frac{1}{n} \sum_{j=1}^n \gamma(Z_j^{n,i}, Z_j^{n,j}) \tilde{\sigma}_n(X_s^{n,i}, X_s^{n,j}) N(dv, ds). \]

We would like to know if the sequence of random measures \( \eta^{n,z} = \frac{1}{n} \sum_{i=1}^n \delta_{X^{n,i}} \) is convergent and if so what it converges to.
If we define $I^{k,n} = \{i : Z^{n,i} = \beta_k\}$, and denote the cardinality of $I^{k,n}$ by $\mid I^{k,n} \mid$ then we can decompose $\eta^{n,x}$ as follows:

$$\eta^{n,x} = \frac{1}{n} \sum_{k=1}^{l} \sum_{i \in I^{n,k}} \delta_{X^{n,i}}$$

$$= \sum_{k=1}^{l} \frac{\mid I^{n,k} \mid}{n} \eta^{n,x,k}$$

where $\eta^{n,x,k} = \frac{1}{\mid I^{n,k} \mid} \sum_{i \in I^{n,k}} \delta_{X^{n,i}}$. Since $\frac{\mid I^{n,k} \mid}{n}$ converges to $p_k$ a.s. it would be reasonable to guess that

$$\mathcal{L}(\eta^{n,x}) \Rightarrow \delta \sum_{k=1}^{l} p_k \mathcal{L}(X^k)$$

where $\eta^{n,x,k}$ converges to $\delta_{\mathcal{L}(X^k)}$.

In chapter 2 we defined $\sigma_n$ and $b_n$ as integrals so that we could write the limiting equation as a McKean-Vlasov equation. We will do the same thing here, but since $X^{n,i}$ and $Z^{n,i}$ are not independent it is important that in integrating we preserve the pairing between $X^{n,i}$ and $Z^{n,i}$. In order to accomplish this we let $\eta^n = \frac{1}{n} \sum_{i=1}^{n} \delta_{(X^{n,i},Z^{n,i})}$, then

$$(7) \quad (X_t^{n,i}, Z_t^{n,i}) = (X_0^{n,i}, Z_0^{n,i})$$

$$+ (\int_0^t \int_{\mathbb{R}^2} \alpha(Z^{n,i}, w) \hat{b}_n(s, X_s^{n,i}, y) \eta^{n,i}(\omega)(dy, dw) ds, 0)$$

$$+ (\int_0^t \int_{\mathbb{R}^2} \gamma(Z^{n,i}, w) \hat{d}_n(s, X_s^{n,i}, y) \eta^{n,i}(\omega)(dy, dw) N^{n,i}(dv, ds), 0).$$

Let $\mathcal{P}^n(\mathbb{R}^2)$ be the set of probability measures on $\mathbb{R}$ of the form $\mu = \frac{1}{n} \sum_{i=1}^{n} \delta_{(x_i, z_i)}$. Define

$$b_n : [0, T] \times \mathbb{R}^2 \times \mathcal{P}^n(\mathbb{R}^2) \rightarrow \mathbb{R}$$
and

\[ \sigma_n : [0, T] \times \mathbb{R}^2 \times P^n(\mathbb{R}^2) \to \mathbb{R} \]

by

\[ b_n(s, z, x, \mu) = \int_{\mathbb{R}^2} \alpha(z, w) \hat{b}_n(s, x, y) \mu(dy, dw) \]

and

\[ \sigma_n(s, z, x, \mu) = \int_{\mathbb{R}^2} \gamma(z, w) \hat{\sigma}_n(s, x, y) \mu(dx, dz) \]

We will be interested in two cases analogous to the cases studied in Chapter 2. In the first case we assume that \( \nu_n \Rightarrow \nu \) for all \( n \) where \( \nu \) is the Lévy measure for the Poisson random measure \( N(dv, ds) \). Also

\[ \int_V \hat{\sigma}_n(s, x, y, v)^2 \nu_n(dv) \to \int_V \hat{\sigma}(s, x, y, v)^2 \nu(dv) \]

uniformly on \([0, T] \times \mathbb{R} \times \mathbb{R}\).

\[ \hat{b}_n(s, x, y) \to \hat{b}(s, x, y) \]

uniformly on \([0, T] \times \mathbb{R} \times \mathbb{R}\). Define

\[ b : [0, T] \times \mathbb{R}^2 \times P(\mathbb{R}^2) \to \mathbb{R} \]

and

\[ \sigma : [0, T] \times \mathbb{R}^2 \times P^n(\mathbb{R}^2) \to \mathbb{R} \]
by

\[(12) \quad b(s, z, x, \mu) = \int_{\mathbb{R}^2} \alpha(z, w) b(s, x, w) \mu(dy, dw)\]

and

\[(13) \quad \sigma(s, z, x, \mu) = \int_{\mathbb{R}^2} \gamma(z, w) \sigma(s, x, y) \mu(dx, dz)\]

We wish to show that \(\lim_{n \to \infty} \eta^n = \mathcal{L}(X, Z)\) where

\[(14) \quad (X_t, Z) = (X_0, Z) + \left( \int_0^t b(s, Z, X_s, \mathcal{L}(X_s, Z)) \, ds \right) \quad \text{and} \quad X_0 \text{ has the same distributions as } X_0^{i_n}. \]

Towards this we first establish the convergence of the random measures \(\eta^n = \frac{1}{n} \sum_{i=1}^n \delta_{(X^n, Z^n, i)}\) to solution of the martingale problem associated with (14).

For any \(f \in C^2_b[0, T]\) and \(g_1, \ldots, g_d \in C_b[0, T], 0 \leq s_1, \ldots, s_d \leq s, s < t, d \in \mathbb{Z}^+.\) Define

\[F(\mu) = F_{f,g_1,\ldots,g_d}^{s,t} : P(D[0, T] \times D[0, T]) \to \mathbb{R}\]

by

\[(15) \quad F(\mu) = \int_{D[0, T] \times D[0, T]} \left( f(x_t, z_t) - f(x_s, z_s) \right) d\mu - \int_s^t L_{\mu,r,f}(x_r, z_r) \, dr \prod_{k=1}^d g_k(x_{s_k}, z_{s_k}) \mu(dx)\]

where for \(x \in D[0, T] \times D[0, T], x_s = x(s), 0 \leq s \leq T, \) and

\[(16) \quad L_{\mu,r,f}(x, z) = f'(x, z) b(r, z, x, \mu) \]

\[+ \int_V \left( f((x, z) + (\sigma(s, z, x, \mu, v), 0)) - f(x, z) \right. \]

\[\left. - f'(x, z) \sigma(s, z, x, \mu, v) \right) \nu(dv).\]
Then $\eta \in P(D[0, T] \times D[0, T])$ is a solution to the martingale problem corresponding to (9) providing $F(\eta) = 0$ for all choices of $f, g_1, \ldots, g_d, s_1, \ldots, s_d, s, t,$ and $d.$

We can establish the following result.

**Theorem 3.1**

Any weak limit of $\{\eta^n\}$ is a solution of the McKean-Vlasov equation (14).

The proof is omitted because it is essentially the same as the proof in chapter 2, section 2.

What we have done is answer a question about $\mathcal{L}(X, Z).$ But the question asked in the beginning of the chapter was about $\mathcal{L}(X).$ So far we do not know if $\mathcal{L}(X)$ has the structure hinted at in the beginning of the chapter.

**Theorem 3.2**

$$\mathcal{L}(\eta^{n,x}) = \mathcal{L}\left(\frac{1}{n} \sum_{i=1}^{n} \delta_{X^n,i}\right) \Rightarrow \delta \sum_{k=1}^{d} m^k$$

where $m^k$ is a measure on $D[0, T]$ which satisfies

$$(X_t, Z) = (X_0, Z)$$

$$+ \left( \int_0^t \int_{\mathbb{R}} \sum_{k=1}^{i} p_k \alpha(Z^{n,i}, \beta_k) \tilde{\sigma}(s, X_s, y) m^k_s(dy) ds, 0) \right.$$

$$+ \left( \int_0^t \int_{\mathbb{R}} \sum_{k=1}^{i} p_k \gamma(Z^{n,i}, \beta_k) \tilde{\sigma}(s, X_s, y) m^k_s(dy) N_n^{n,i}(dv, ds), 0) \right).$$

**Proof**

We know that

$$\mathcal{L}(\eta^n) \Rightarrow \delta_{\mathcal{L}(X,Z)}.$$
In order to conclude that

\[ \mathcal{L}(\eta^{n,x}) \Rightarrow \delta_{\mathcal{L}(X)} \]

where \( \mathcal{L}(X) \) is the marginal distribution of \( \mathcal{L}(X, Z) \) it will be sufficient to show that

\[ \pi^{x}(\mathcal{L}(\eta^{n,x}), \delta_{\mathcal{L}(X)}) \to 0 \]

where \( \pi^{x} \) is the Prohorov metric on \( \mathcal{P}(P(D[0, T])) \). This follows from the fact that

\[ \pi(\mathcal{L}(\eta^{n}), \delta_{\mathcal{L}(X,Z)}) \to 0 \]

where \( \pi^{x} \) is the Prohorov metric on \( \mathcal{P}(P(D[0, T]) \times \mathbb{R}) \) and the fact that the measure of each closed set in \( (P(D[0, T]) \times \mathbb{R}) \) under \( \mathcal{L}(\eta^{n,x}) \) is the same as the measure of the corresponding cylinder set in \( \mathcal{P}(P(D[0, T]) \times \mathbb{R}) \). Similarly

\[ \mathcal{L}\left(\frac{1}{N} \sum_{i=1}^{n} \delta_{Z^{n,i}}\right) \Rightarrow \delta_{\mathcal{L}(Z)}. \]

From Lemma 1.64 we see that \( \mathcal{L}(Z) = \mathcal{L}(Z^{n,i}) \) for \( 1 \leq i \leq n < \infty \). Thus \( Z \) takes on only the values \( \beta_{1}, \ldots, \beta_{l} \) and assigns a probability of \( p_{k} \) to \( \beta_{k} \) for \( 1 \leq k \leq l \).

Thus

\[ \eta = \mathcal{L}(X, Z)(E) = \sum_{k=1}^{l} \mathcal{L}(X, Z)(E \cap \{z = \beta_{k}\}) \]

for all \( E \in B(D[0, T], \mathbb{R}) \) Let \( m^{k} \) to be the measure on \( D[0, T] \) defined by

\[ m^{k}(A) = \frac{\mathcal{L}(X, Z)(A, \beta_{k})}{p_{k}} \]

for all \( A \in B(D[0, T]) \). Note that \( m^{k} \) is a probability
measure. Then for \( A \times \mathbb{R} \in \mathcal{B}(D[0, T], \mathbb{R}) \)

\[
\mathcal{L}(X)(A) = \mathcal{L}(X, Z)(A \times \mathbb{R}) = \sum_{k=1}^{l} \mathcal{L}(X, Z)(A, \beta_k) = \sum_{k=1}^{l} p_k m^k(A)
\]

where \( \Pi \) is the projection from \((D[0, T], \mathbb{R})\) onto \(D[0, T]\). Now we need only note that

\[
(X_t, Z) = (X_0, Z) + \left( \int_0^t b(s, Z, X_s, \mathcal{L}(X_s, Z)) \, ds, 0 \right) + \left( \int_0^t \int_{\mathbb{R}^2} \sigma(s, Z, X_s, \mathcal{L}(X_s, Z)), v) \tilde{N}(dv, ds), 0 \right)
\]

\[
= (X_0, Z) + \left( \int_0^t \int_{\mathbb{R}^2} \alpha(Z, w) \hat{b}(s, X_s, y) \mathcal{L}(X_s, Z)(dy, dw) \, ds, 0 \right) + \left( \int_0^t \int_{\mathbb{R}^2} \gamma(Z, w) \hat{\sigma}(s, X_s, y) \mathcal{L}(X_s, Z)(dy, dw) \tilde{N}(dv, ds), 0 \right)
\]

\[
= (X_0, Z) + \left( \int_0^t \int_{\mathbb{R}^2} \sum_{k=1}^{l} \alpha(Z, \beta_k) \hat{b}(s, X_s, y) m^k_s(dy) \, ds, 0 \right) + \left( \int_0^t \int_{\mathbb{R}^2} \sum_{k=1}^{l} \gamma(Z, \beta_k) \hat{\sigma}(s, X_s, y) m^k_s(dy) \tilde{N}(dv, ds), 0 \right)
\]

and we are done.

In the second case we make the following assumptions about \( \{\nu_n\}, \{b_n\} \) and \( \{\sigma_n\} \):

\[
(17) \quad \nu_n \Rightarrow \delta_0.
\]

There exists \( \hat{\sigma}(s, x, y) \) and \( \hat{b}(s, x, y) \) such that

\[
(18) \quad \int_{\mathbb{R}} \hat{\sigma}_n(s, x, y, v)^2 \nu_n(dv) \to \hat{\sigma}(s, x, y)^2
\]

uniformly on \([0, T] \times \mathbb{R} \times \mathbb{R}\).
(19) \[ \hat{b}_n(s, x, y) \to \hat{b}(s, x, y) \]

uniformly on \([0, T] \times \mathbb{R} \times \mathbb{R}\).

(20) \[ \lim_{n \to \infty} \sup_{s, x, y} \{ \nu_n(\{ |\hat{\sigma}_n(s, x, y, v)| > N \}) \} = 0 \]

for all \(N > 0\).

(21) \[ \lim_{n \to \infty} \sup_{s, x, y} \int \hat{\sigma}_n(x, y, v)^2 1_{\{|\hat{\sigma}_n(s, x, y, v)| > M\}} \nu_n(dv) = 0 \]

for all \(M > 0\). Define

\[ b : [0, T] \times \mathbb{R}^2 \times P(\mathbb{R}^2) \to \mathbb{R} \]

and

\[ \sigma : [0, T] \times \mathbb{R}^2 \times P^n(\mathbb{R}^2) \to \mathbb{R} \]

by

(22) \[ b(s, z, x, \mu) = \int_{\mathbb{R}^2} \alpha(z, w) \hat{b}(s, x, w) \mu(dy, dw) \]

and

(23) \[ \sigma(s, z, x, \mu) = \int_{\mathbb{R}^2} \gamma(z, w) \hat{\sigma}(s, x, y) \mu(dx, dz) \]

We wish to show that \( \lim_{n \to \infty} \eta^n = \mathcal{L}(X, Z) \) where

(24) \[ X_t = X_0 + \int_0^t b(s, Z, X_s, \mathcal{L}(X_s)) ds + \int_0^t \sigma(s, Z, X_s, \mathcal{L}(X_s)) dB_s \]

where \(X_0\) has the same distributions as \(X_0^{i,n}\) and \(B_s\) is a Brownian motion.

Towards this we first establish the convergence of the random measures \(\eta^n = \)
\[ \frac{1}{n} \sum_{i=1}^{n} \delta_{X^{n,i}} \to \text{solution of the martingale problem associated with (24).} \]

For any \( f \in C^2_0[0,T] \) and \( g_1, \ldots, g_d \in C_b[0,T], 0 \leq s_1, \ldots, s_d \leq s, s < t, d \geq 0. \) Define

\[ F(\mu) = F_{f,g_1,\ldots,g_d}^{s,t} : P(D[0,T]) \to \mathbb{R} \]

by

\[ (25) \quad F(\mu) = \int_{D[0,T]} (f(x_t) - f(x_s) - \int_s^t L_{\mu,r,f}(x_r) \, dr) \prod_{k=1}^d g_k(x_{s_k}) \mu(dx) \]

where for \( x \in D[0,T], x_s = x(s), 0 \leq s \leq T, \) and

\[ (26) \quad L_{\mu,r,f}(x) = f'(x)b(r, z, x, \mu) + \frac{1}{2} f''(x)\sigma(s, z, x, \mu)^2 \]

Then \( \eta \in P(D[0,T]) \) is a solution to the martingale problem corresponding to (30) providing \( F(\eta) = 0 \) for all choices of \( f, g_1, \ldots, g_d, s_1, \ldots, s_d, s, t, \) and \( d. \)

Then we have the following result.

**Theorem 3.3**

Any weak limit of \( \{\eta^n\} \) is a solution of the McKean-Vlasov equation.

The proof is omitted because it is essentially the same as the proof in chapter 2, section 2.

**Theorem 3.4**

\[ \mathcal{L}(\eta^{n,x}) = \mathcal{L}(\frac{1}{n} \sum_{i=1}^{n} \delta_{X^{n,i}}) \Rightarrow \delta_{\sum_{k=1}^{n} p_k m^k} \]
where $m^k$ is a measure on $D[0, T]$ which satisfies

$$(X_t, Z) = (X_0, Z)$$

$$+ \left( \int_0^t \int_{\mathbb{R}} \sum_{k=1}^I p_k \alpha(Z_{n,i}^k, \beta_k) \tilde{b}(s, X_s, y) m^k_s(dy) \, ds, 0 \right)$$

$$+ \left( \int_0^t \int_{\mathbb{R}} \sum_{k=1}^I p_k \gamma(Z_{n,i}^k, \beta_k) \tilde{\sigma}(s, X_s, y) m^k_s(dy) \, N^{n,i}(dv, ds), 0 \right).$$

The proof is similar in spirit to the proof of Theorem 3.2.
Chapter 4

Semimartingale Driven Interaction Systems

4.1: Tightness of the System

In this chapter the novelty is that the system we investigate is driven by semimartingales rather than Levy processes or Poisson random measures. In earlier work the martingale problem has always been utilized to show weak convergence. In the case we now study such an approach is unavailable to us because stochastic integrals with semimartingales as their driving terms do not in general possess the Markov property. Instead we will use a Picard iteration approach. There is a price to be paid for not using the martingale problem approach.

a) We must fix the $i$th driving terms for all $n$ (i.e. we have $\{A^i\}$ instead of $\{A^{n,i}\}$).

b) The interaction coefficients must be bounded.

c) The interaction coefficients are functions of the random measure $\eta^n$ rather than the measure $\eta^n(\omega)$.

While this price may seem steep one should bear in mind that by casting aside the Markov property, and thus the martingale problem, we lose a very powerful tool. Identification of the limit in the semimartingale case should be regarded as a very hard problem, and it should not be a surprise that there is a price to pay.

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Given
\[ X^n_{t,i} = X^i_0 + \int_0^t \hat{b}(X^n_{s,i}) \, ds + \int_0^t b(X^n_{s,i}; \eta^n_s) \, dA^i_s \]
\[ + \int_0^t \hat{\sigma}(X^n_{s,i}) \, dN^i_s + \int_0^t \sigma(X^n_{s,i}; \eta^n_s) \, dM^i_s \]
\[ , 0 \leq t \leq T, 1 \leq i \leq n. \]

We first show that \( \eta^n = \frac{1}{n} \sum_{j=1}^n \delta_{X^n_{\tau^i(j)}} \) converges to \( \mathcal{L}(X^i_{\tau^i(.)}) \) where \( \tau^i(s) \) is a stopping time to be defined later and \( \mathcal{L}(X^i) \) satisfies the McKean-Vlasov equation
\[ X^i_t = X^i_0 + \int_0^t \hat{b}(X^i_s) \, ds + \int_0^t b(X^i_s; \mathcal{L}(X)) \, dA^i_s \]
\[ + \int_0^t \hat{\sigma}(X^i_s) \, dN^i_s + \int_0^t \sigma(X^i_s; \mathcal{L}(X^i_s)) \, dM^i_s \]

Of course this result holds whether \( A^i \) and \( M^i \) are continuous or not as long as we regard \( \eta^n \) as a \( P(D[0,T]) \)-valued random variable. However it is more natural, in the continuous case, to regard \( \eta^n \) and \( \eta \) as being \( P(C[0,T]) \)-valued.

We apply results from Chapter 1 section 6 in order to be able to do so.

Given a probability space \((\Omega, \mathcal{F}, P)\) with a filtration \(\mathcal{F}_t\) satisfying the usual conditions. Let \(\{A^n\}\) be a sequence of continuous independent identically distributed processes with paths of finite variation on compact sets and, \(\{M^n\}\) a sequence of independent identically distributed local martingales with bounded jumps and finite second moments. Let \(\{N^n\}\) be a sequence of adapted independent Lévy martingale processes with bounded jumps. Thus there is a constant \(c > 0\) such that \(\langle N^i, N^i \rangle = ct\) for all \(i\). Let \(\{X^n_0\}\) be a vector of \(\mathcal{F}_t\)-adapted exchangeable random variables, independent of the driving terms with finite second moments. We will define \(P(\mathbb{R})\) to be the space of probability measures on \(\mathbb{R}\), and \(\mathcal{P}(P(\mathbb{R}))\) to be the space of probability measures on \(P(\mathbb{R})\). Let \(U\) be
the set random variables taking values in $P(\mathbb{R})$ such that $ho(\mathcal{L}(\eta), \mathcal{L}(\delta_0)) < \infty$ where $\rho(\cdot, \cdot)$ is the Vasserstein metric on $\mathcal{P}(P(\mathbb{R}))$:

$$
\rho(\eta, \xi) = \inf E[\hat{\rho}(E, F)]
$$

where the inf is taken over all random measures $(E, F)$ with marginals $\eta$ and $\xi$ respectively, and $\hat{\rho}(\cdot, \cdot)$ is the Vasserstein metric on $\mathbb{R}$:

$$
\hat{\rho}(E, F) = \inf E[\|X - Y\|],
$$

where the infimum is taken over all random variables $(X, Y)$ such that the marginal distribution of $X$ is $E$, and the marginal distribution of $Y$ is $F$.

Let $b : \mathbb{R} \times U \times \Omega \to \mathbb{R}, \sigma : \mathbb{R} \times U \times \Omega \to \mathbb{R}$ obey the Lipschitz condition:

$$
|b(x, \eta_s^1, \omega) - b(z, \eta_s^2, \omega)| + |\sigma(x, \eta_s^1, \omega) - \sigma(z, \eta_s^2, \omega)| \\
\leq K_1(1 + |x - z| \wedge L + \rho(\mathcal{L}(\eta_s^1), \mathcal{L}(\eta_s^2)) \wedge L)
$$

for some $K_1, L \in (0, \infty)$. The above Lipschitz condition leads to the following growth conditions:

$$
|b(x, \eta_s, \omega)| \leq C_1(1 + |x| \wedge L + \rho(\eta_s, 0) \wedge L)
$$

$$
|\sigma(x, \eta_s, \omega)| \leq C_1(1 + |x| \wedge L + \rho(\eta_s, 0) \wedge L)
$$

for some $C_1 \in (0, \infty)$ and where $0$ denotes $\mathcal{L}(\delta_0)$. In what follows we will suppress the $\omega$ for notational convenience, and simply write $\sigma(x, \eta_s), b(x, \eta_s)$ Let $\hat{\delta} : \mathbb{R} \to \mathbb{R}, \hat{\sigma} : \mathbb{R} \to \mathbb{R}$ obey the Lipschitz condition:
\[ | \hat{b}(x) - \hat{b}(z) | + | \hat{\sigma}(x) - \hat{\sigma}(z) | \leq K_2 | x - z | \]

for some \( K_2, L \in (0, \infty) \). The Lipschitz condition on \( \hat{b}(x) \) and \( \hat{\sigma}(x) \) leads to the following growth condition:

\[ | \hat{b}(x) | \leq C_2 (1 + | x |) \]

\[ | \hat{\sigma}(x) | \leq C_2 (1 + | x |) \]

for some \( C_2 \in (0, \infty) \). Let

\[
X^{n,i}_t = X^{i}_0 + \int_0^t \hat{b}(X^{n,i}_s) \, ds + \int_0^t b(X^{n,i}_s, \eta^n_s) \, dA^i_s \\
+ \int_0^t \hat{\sigma}(X^{n,i}_s) \, dN^i_s + \int_0^t \sigma(X^{n,i}_s, \eta^n_s) \, dM^i_s
\]

(1)

\[ , 0 \leq t \leq T, 1 \leq i \leq n \]

where \( \eta^n = \frac{1}{n} \sum_{j=1}^{n} \delta_{X^{n,i}_s} \).

Note that \( \{X^{n,i}_s\} \) is a set of exchangeable processes.

We leave the proof of uniqueness of a solution to (1) until Theorem 4.3 because of the importance the proof has in the identification of the limit.

**Theorem 4.1**

Let \( \{X^{n,i}_t\} \) \( 1 \leq i \leq n \) be the solutions of (1). Then \( X^{n,i} \) is a tight sequence of \( D[0,T] \)-valued processes.
Proof

We will use the Aldous tightness criterion. To do so we must prove the following:

\[ E[\sup_{s \leq T}(X^n_{s,i})^2] \]

a) \[ \leq C[E[(X^n_{0,i})^2] + 1 + E[(\int_0^T d | A_i^i |)^2] + E[< M^i, M^i >_T]e^{Ct} \]

for some \( C > 0 \), and if \( \{\tau_n\} \) be a sequence of stopping times bounded by \( T \), and \( \delta_n > 0 \) such that \( \delta_n \downarrow 0 \). Then

b) \[ E[(X^n_{\tau_n + \delta_n} - X^n_{\tau_n})^2] \to 0 \]

Proof of part a)

From Lemma 2.1

\[ E[\sup_{s \leq t}(X^n_{s,i})^2] \leq C[E[(X^n_{0,i})^2] \]

\[ + E[(\int_0^t (| b(X^n_{u,i}) | + du)^2)] \]

\[ + E[(\int_0^t | b(X^n_{u,i}, \eta^n_{u,i}) | d | A_i^i |)^2] \]

\[ + E[\int_0^t \sigma(X^n_{u,i})^2 ds] \]

\[ + E[\int_0^t \sigma(X^n_{u,i}, \eta^n_{u,i})^2 d < M^i, M^i >_u]]. \]

We now use the growth condition to obtain

\[ E[\sup_{s \leq t}(X^n_{s,i})^2] \leq C[E[(X^n_{0,i})^2] \]

\[ + E[(\int_0^t (1+ | X^n_{u,i} |) du)^2] + E[(\int_0^t d | A_i^i |)^2] \]

\[ + 2E[\int_0^t (1+ | X^n_{u,i} |)^2 du] + E[\int_0^t d < M^i, M^i >_u]]. \]
where C is a constant which may change from statement to statement. Then using Hölder’s inequality and Fubini’s theorem,

\[
E[\sup_{s \leq t} (X_{s}^{n,i})^2] \leq C [E[(X_{0}^{n,i})^2] + 1 + E[(\int_{0}^{t} d | A_{u}^{i}|)^2] + E[< M^i, M^i>_t] + \int_{0}^{t} E[\sup_{v \leq u} (X_{v}^{n,i})^2] du].
\]

Finally Gronwall’s inequality yields

\[
E[\sup_{s \leq t} (X_{s}^{n,i})^2] \\
\leq C [E[(X_{0}^{n,i})^2] + 1 + E[(\int_{0}^{t} d | A_{u}^{i}|)^2] + E[< M^i, M^i>_t] e^{Ct}.
\]

Proof of part b)

Let \{\tau_n\} be a sequence of stopping times, and let \{\delta_n\} be a sequence of positive constants such that \delta_n \downarrow 0 and \tau_n + \delta_n \leq T for all n. From Lemma 2.1,

\[
E[(X_{\tau_n + \delta_n}^{n,i} - X_{\tau_n}^{n,i})^2] \\
\leq C [E[\int_{\tau_n}^{\tau_n + \delta_n} \tilde{b}(X_{s}^{n,i}) ds]^2] \\
+ E[(\int_{\tau_n}^{\tau_n + \delta_n} | b(X_{s}^{n,i}; \eta_{s}^{n}) | d | A_{s}^{i}|)^2] \\
+ E[\int_{\tau_n}^{\tau_n + \delta_n} (\tilde{\sigma}(X_{s}^{n,i}))^2 ds] \\
+ E[\int_{\tau_n}^{\tau_n + \delta_n} (\sigma(X_{s}^{n,i}; \eta_{s}^{n}))^2 d < M^i, M^i>_s]].
\]

Using the growth condition,
\[ E[(X_{\tau_n+\delta_n}^{n,i} - X_{\tau_n}^{n,i})^2] \]
\[ \leq C[E[(\int_{\tau_n}^{\tau_n+\delta_n} 1 + |X_{s}^{n,i}| \, ds)^2] + E[(\int_{\tau_n}^{\tau_n+\delta_n} d |A_{s}^{i}|)^2] \]
\[ + E[\int_{\tau_n}^{\tau_n+\delta_n} 1 + (X_{s}^{n,i})^2 \, ds] + E[\int_{\tau_n}^{\tau_n+\delta_n} d < M^{i}, M^{i} >_s]]. \]

Hölder's inequality yields

\[ E[(X_{\tau_n+\delta_n}^{n,i} - X_{\tau_n}^{n,i})^2] \]
\[ \leq C[E[(\int_{\tau_n}^{\tau_n+\delta_n} 1 + (X_{s}^{n,i})^2 \, ds] + E[(\int_{\tau_n}^{\tau_n+\delta_n} d |A_{s}^{i}|)^2] \]
\[ + E[\int_{\tau_n}^{\tau_n+\delta_n} d < M^{i}, M^{i} >_s]] \]
\[ \leq C[\delta_n + E[(\int_{\tau_n}^{\tau_n+\delta_n} (X_{s}^{n,i})^2 \, ds] + E[(\int_{\tau_n}^{\tau_n+\delta_n} d |A_{s}^{i}|)^2] \]
\[ + E[\int_{\tau_n}^{\tau_n+\delta_n} d < M^{i}, M^{i} >_s]] \]
\[ \leq C[\delta_n + E[\delta_n \sup_{s \leq T} (X_{s}^{n,i})^2] + E[(\int_{\tau_n}^{\tau_n+\delta_n} d |A_{s}^{i}|)^2] \]
\[ + E[\int_{\tau_n}^{\tau_n+\delta_n} d < M^{i}, M^{i} >_s]]. \]

From part a) this goes to zero as \( n \to \infty \), and \( \delta_n \downarrow 0 \). The tightness of \( \{X_{t}^{n,i}\} \)
then follows immediately from The Aldous tightness criterion.

**Corollary 4.2**

a) \( \{\eta^{n}\} = \{\frac{1}{n} \sum_{j=1}^{n} \delta_{X_{t}^{n,j}}\} \) is tight as a \( P(D[0, T]) \)-valued random variable.

b) Fix \( t \geq 0 \), \( \{\eta^{n}_{t}\} = \{\frac{1}{n} \sum_{j=1}^{n} \delta_{X_{t}^{n,j}}\} \) is tight as a \( P(\mathbb{R}) \)-valued random variable.

**Proof**

Part a) follows as a result of Lemma 1.63.

Part b) follows as a result of Lemma 1.64.
Now pick any \( n \). Let \( X^{n,i,0}_t = X^{n,i}_0 \) and \( \eta^{n,0}_t = \frac{1}{n} \sum_{j=1}^{n} \delta_{X^{n,j}_0} \), and let

\[
X^{n,i,m+1}_t = X^{n,i}_0 + \int_0^t \hat{b}(X^{n,i}_s, \eta^{n,m}_s) \, ds + \int_0^t b(X^{n,i}_s, \eta^{n,m}_s) \, dA^i_s + \int_0^t \hat{\sigma}(X^{n,i}_s, \eta^{n,m}_s) \, dN^i_s + \int_0^t \sigma(X^{n,i}_s, \eta^{n,m}_s) \, dM^i_s,
\]

where \( \eta^{n,m}_t = \frac{1}{n} \sum_{j=1}^{n} \delta_{X^{n,j}_t} \).

**4.2: Identification of the Limit**

**Theorem 4.3**

Let

\[ \tau^i(t) = \inf \{ s : \int_0^s | A^i_s | + < M^i_s, M^i_s >_s + s > t \}. \]

Let \( d \) be the Skorohod metric. Then

\[
\lim_{m \to \infty} E[d(X^{n,i,m}_{\tau^i(t)_m}, X^{n,i}_{\tau^i(t)})^2] = 0
\]

where \( X^{n,i}_t \) is the unique solution of

\[
X^{n,i}_t = X^{n,i}_0 + \int_0^t \hat{b}(X^{n,i}_s) \, ds + \int_0^t b(X^{n,i}_s, \eta^{n}_s) \, dA^i_s + \int_0^t \hat{\sigma}(X^{n,i}_s, \eta^{n}_s) \, dN^i_s + \int_0^t \sigma(X^{n,i}_s, \eta^{n}_s) \, dM^i_s.
\]

Also \( (E[d(X^{n,i,m}_{\tau^i(t)}, X^{n,i}_{\tau^i(t)})^2])^{1/2} \leq \sum_{j=m}^{\infty} (E[(X_0)^2])^{(M(t))^{j-1}}(j-1)!^{1/2} \).

**Proof**

From Lemma 2.1 and the fact that \( d(x, y) \leq \sup_{0 \leq s \leq T} | x(s) - y(s) | \)

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We now use the Lipschitz condition

\[
E[\sup_{s \leq T} (X_{\tau^i(s)}^{n,i,m+1} - X_{\tau^i(s)}^{n,i,m})^2] \\
\leq C[E[\int_0^{\tau^i(T)} | \hat{b}(X_u^{n,i,m}) - \hat{b}(X_u^{n,i,m-1}) | \, du]^2] \\
+ E[\int_0^{\tau^i(T)} (\hat{\sigma}(X_u^{n,i,m}, \eta_u^{n,m}) - \hat{\sigma}(X_u^{n,i,m-1}))^2 \, du] \\
+ E[\int_0^{\tau^i(T)} (\hat{\delta}(X_u^{n,i,m}, \eta_u^{n,m}) - \hat{\delta}(X_u^{n,i,m-1}))^2 \, du] \\
+ E[\int_0^{\tau^i(T)} (\sigma(X_u^{n,i,m}, \eta_u^{n,m}) - \sigma(X_u^{n,i,m-1}, \eta_u^{n,m-1}))^2 \, d < M^i, M^i >_u].
\]

Using the definition of the Vasserstein metric

\[
E[\sup_{s \leq T} (X_{\tau^i(s)}^{n,i,m+1} - X_{\tau^i(s)}^{n,i,m})^2] \\
\leq C[E[\int_0^{\tau^i(T)} | X_u^{n,i,m} - X_u^{n,i,m-1} | \, du]^2] \\
+ E[\int_0^{\tau^i(T)} (X_u^{n,i,m} - X_u^{n,i,m-1})^2] \\
+ \rho(\mathcal{L}(\eta_u^{n,m}), \mathcal{L}(\eta_u^{n,m-1}))^2 \, d < M^i, M^i >_u].
\]

Using the definition of the Vasserstein metric

\[
\leq C[E[\int_0^{\tau^i(T)} | X_u^{n,i,m} - X_u^{n,i,m-1} | \, du]^2] \\
+ E[\int_0^{\tau^i(T)} (\frac{1}{n} \sum_{j=1}^{n} \sup_{v \leq u} | X_v^{n,j,m} - X_v^{n,j,m-1} |) \, d | A_u^i |]^2] \\
+ E[\int_0^{\tau^i(T)} (X_u^{n,i,m} - X_u^{n,i,m-1})^2] \\
+ \rho(\mathcal{L}(\eta_u^{n,m}), \mathcal{L}(\eta_u^{n,m-1}))^2 \, d < M^i, M^i >_u].
\]
From the exchangeability of our processes

\[
\leq C[E[\int_0^{r'(T)} | X_u^{n,i,m} - X_u^{n,i,m-1} | \, du]^2 \\
+ E[\int_0^{r'(T)} | X_v^{n,i,m} - X_v^{n,i,m-1} | ] \\
+ E[\sup_{v \leq u} | X_v^{n,i,m} - X_v^{n,i,m-1} | d \, A_u] ]^2 \\
+ E[\int_0^{r'(T)} (X_u^{n,i,m} - X_u^{n,i,m-1})^2 \\
+ E[\sup_{v \leq u} (X_v^{n,i,m} - X_v^{n,i,m-1})^2] \, d < M^i, M^i > u].
\]

We may now use the Lebesgue's change of time theorem to obtain

\[
E[\sup_{s \leq T} (X_{r'(s)}^{n,i,m+1} - X_{r'(s)}^{n,i,m})^2] \\
\leq C[E[\int_0^T | X_{r'(u)}^{n,i,m} - X_{r'(u)}^{n,i,m-1} | \, ds]^2 + E[\int_0^T | X_{r'(u)}^{n,i,m} - X_{r'(u)}^{n,i,m-1} | ] \\
+ E[\sup_{v \leq r'(u)} | X_v^{n,i,m} - X_v^{n,i,m-1} | \, du]^2] + E[\int_0^T (X_{r'(u)}^{n,i,m} - X_{r'(u)}^{n,i,m-1})^2 \\
+ E[\sup_{v \leq r'(u)} (X_v^{n,i,m} - X_v^{n,i,m-1})^2] \, du].
\]

Now we use Fubini's theorem, and Hölder's inequality

\[
E[\sup_{s \leq T} (X_{r'(s)}^{n,i,m+1} - X_{r'(s)}^{n,i,m})^2] \\
\leq C \int_0^T E[(X_{r'(u)}^{n,i,m} - X_{r'(u)}^{n,i,m-1})^2] + E[\sup_{v \leq r'(u)} (X_v^{n,i,m} - X_v^{n,i,m-1})^2] \, du \\
\leq C \int_0^T E[\sup_{v \leq r'(u)} (X_v^{n,i,m} - X_v^{n,i,m-1})^2] \, du \\
= C \int_0^T E[\sup_{v \leq u} (X_{r'(u)}^{n,i,m} - X_{r'(u)}^{n,i,m-1})^2] \, du.
\]
Now for convenience let $g_m(u) = E[\sup_{v \leq r^+}(X_v^{n,i,m+1} - X_v^{n,i,m})^2]$. Then we have

$$g_m(t) \leq C \int_0^T g_{m-1}(u) \, du$$

$$\leq C^2 \int_0^T \int_0^u g_{m-2}(v) \, dv \, du$$

$$= C^2 \int_0^T (T-u)g_{m-2}(u) \, du.$$

Where the equality is due to integration by parts. Continuing inductively we obtain

$$g_m(t) \leq C^{m-1} \int_0^T \frac{(T-u)^{n-2}}{(m-2)!} \, g_1(u) \, du$$

$$= E[(X_0)^2] \frac{(CT)^{m-1}}{(m-1)!}.$$

Since this term is part of a convergent series we may conclude that $\{X_{r^+(s)}^{n,i,m}\}$ is a Cauchy sequence in $L^2(D[0,T])$ and that if

$$\lim_{m \to \infty} X_{r^+(s)}^{n,i,m} = Y_{r^+(s)}^{n,i}$$

then

$$(E[d(X_{r^+(s)}^{n,i,m}, Y_{r^+(s)}^{n,i})^2])^{\frac{1}{2}} \leq \sum_{p=m}^{\infty} (E[d(X_{r^+(s)}^{n,i,p}, X_{r^+(s)}^{n,i,p+1})^2])^{\frac{1}{2}}$$

$$\leq \sum_{j=m}^{\infty} (E[(X_0)^2] \frac{(Mt)^{j-1}}{(j-1)!})^{\frac{1}{2}}.$$

Now we need only identify the limit and verify that it is unique. Let $\tilde{X}_{r^+(s)}^{n,i}$ be the limit of $\{X_{r^+(s)}^{n,i,m}\}$ and define $\tilde{\eta}_{r^+(s)}^n$ to be $\frac{1}{n} \sum_{j=1}^{n} \delta X_{s,j}^{n,i}.

It remains to show that $\tilde{X}_{r^+(s)}^{n,i} = X_{r^+(s)}^{n,i}$ and $\tilde{\eta}_{r^+(s)}^n = \eta_{r^+(s)}^n$. First we show that $\tilde{X}_{r^+(s)}^{n,i}$ has the representation:

$$\tilde{X}_{r^+(s)}^{n,i} = X_{r^+(s)}^{n,i} + \int_0^{r^+(s)} \hat{b}(\tilde{X}_{s}^{n,i}) \, ds + \int_0^{r^+(s)} \hat{b}(\tilde{X}_{s}^{n,i}, \tilde{\eta}_{s}^{n}) \, dA_{s}$$

$$+ \int_0^{r^+(s)} \hat{\sigma}(\tilde{X}_{s}^{n,i}) \, dN_{s} + \int_0^{r^+(s)} \sigma(\tilde{X}_{s}^{n,i}, \tilde{\eta}_{s}^{n}) \, dM_{s},$$

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To do this we will show that \( \{X_{r_i}^{n,i,m}\} \) converges to

\[
X_{0}^{n,i} + \int_{0}^{r'(t)} \hat{b}(\tilde{X}_{s}^{n,i}) \, ds + \int_{0}^{r'(t)} b(\tilde{X}_{s}^{n,i}, \tilde{\eta}_{s}^{n}) \, dA_{s}^{i} \\
+ \int_{0}^{r'(t)} \hat{\sigma}(\tilde{X}_{s}^{n,i}) \, dN_{s}^{i} + \int_{0}^{r'(t)} \sigma(\tilde{X}_{s}^{n,i}, \tilde{\eta}_{s}^{n}) \, dM_{s}^{i}.
\]

Now using the relationship between the Skorohod metric and the uniform convergence metric: \( d(x, y) \leq \sup_{0 \leq s \leq T} |x_{s} - y_{s}| \)

\[
E[d(X_{r_i}^{n,i,m}, X_{0}^{n,i} + \int_{0}^{r'(t)} \hat{b}(\tilde{X}_{u}^{n,i}) \, du \\
+ \int_{0}^{r'(t)} b(\tilde{X}_{u}^{n,i}, \tilde{\eta}_{u}^{n}) \, dA_{u}^{i} \\
+ \int_{0}^{r'(t)} \hat{\sigma}(\tilde{X}_{u}^{n,i}) \, dN_{u}^{i} + \int_{0}^{r'(t)} \sigma(\tilde{X}_{u}^{n,i}, \tilde{\eta}_{u}^{n}) \, dM_{u}^{i})^{2}]
\]

is less than or equal to

\[
E[\sup_{s \leq T} (X_{r_i}^{n,i,m} - (X_{0}^{n,i} + \int_{0}^{r'(s)} \hat{b}(\tilde{X}_{u}^{n,i}) \, du \\
+ \int_{0}^{r'(s)} b(\tilde{X}_{u}^{n,i}, \tilde{\eta}_{u}^{n}) \, dA_{u}^{i} \\
+ \int_{0}^{r'(s)} \hat{\sigma}(\tilde{X}_{u}^{n,i}) \, dN_{u}^{i} + \int_{0}^{r'(s)} \sigma(\tilde{X}_{u}^{n,i}, \tilde{\eta}_{u}^{n}) \, dM_{u}^{i}))^{2}].
\]

From Lemma 2.1

\[
E[\sup_{s \leq T} (X_{r_i}^{n,i,m} - (X_{0}^{n,i} + \int_{0}^{r'(s)} \hat{b}(\tilde{X}_{u}^{n,i}) \, du \\
+ \int_{0}^{r'(s)} b(\tilde{X}_{u}^{n,i}, \tilde{\eta}_{u}^{n}) \, dA_{u}^{i} \\
+ \int_{0}^{r'(s)} \hat{\sigma}(\tilde{X}_{u}^{n,i}) \, dN_{u}^{i} + \int_{0}^{r'(s)} \sigma(\tilde{X}_{u}^{n,i}, \tilde{\eta}_{u}^{n}) \, dM_{u}^{i}))^{2}]
\]
is less than or equal to

\[
K[E[(\int_0^{\tau(T)} | \hat{b}(\tilde{X}_{u_i}^{n,i}) - \hat{b}(X_{u_i}^{n,i,m}) | \, du)^2] \\
+ E[(\int_0^{\tau(T)} | b(\tilde{X}_{u_i}^{n,i}, \eta_{u_i}^n) - b(X_{u_i}^{n,i,m}, \eta_{u_i}^n) | \, dA_u^i | )^2] \\
+ E[\int_0^{\tau(T)} (\hat{\sigma}(\tilde{X}_{u_i}^{n,i}) - \hat{\sigma}(X_{u_i}^{n,i,m}))^2 \, du] \\
+ E[\int_0^{\tau(T)} (\sigma(\tilde{X}_{u_i}^{n,i}, \eta_{u_i}^n) - \sigma(X_{u_i}^{n,i,m}, \eta_{u_i}^n))^2 \, d < M^i, M^i >_u]].
\]

Using the Lipschitz condition on the coefficients of integration: \( b \), and \( \sigma \)

\[
E[\sup_{s \leq T}(X_{\tau(s)}^{n,i,m} - \tau(s)) + (X_{\tau(s)}^{n,i} + \int_0^{\tau(s)} \hat{b}(\tilde{X}_{u_i}^{n,i}) \, du) \\
+ \int_0^{\tau(s)} b(\tilde{X}_{u_i}^{n,i}, \eta_{u_i}^n) \, dA_u^i \\
+ \int_0^{\tau(s)} \hat{\sigma}(\tilde{X}_{u_i}^{n,i}) \, dN_u^i + \int_0^{\tau(s)} \sigma(\tilde{X}_{u_i}^{n,i}, \eta_{u_i}^n) \, dM_u^i)]
\]

is less than or equal to

\[
C[E[(\int_0^{\tau(T)} | \tilde{X}_{u_i}^{n,i} - X_{u_i}^{n,i,m} | \, du)^2] \\
+ E[(\int_0^{\tau(T)} | \tilde{X}_{u_i}^{n,i} - X_{u_i}^{n,i,m} | + \rho(\mathcal{L}(\eta_{u_i}^n), \mathcal{L}(\eta_{u_i}^{n,m})) \, dA_u^i | )^2] \\
+ E[\int_0^{\tau(T)} (\tilde{X}_{u_i}^{n,i} - X_{u_i}^{n,i,m})^2 \, du] \\
+ E[\int_0^{\tau(T)} (\tilde{X}_{u_i}^{n,i} - X_{u_i}^{n,i,m})^2 \\
+ \rho(\mathcal{L}(\eta_{u_i}^n), \mathcal{L}(\eta_{u_i}^{n,m}))^2 \, d < M^i, M^i >_u]]
\]
Using the definition of the Vasserstein metric

\[ \leq C[E[(\int_0^{\tau_i(T)} | \tilde{X}^{n,i}_{u} - X^{n,i,m}_{u} | \, du)^2]] + E[(\int_0^{\tau_i(T)} | \tilde{X}^{n,i}_{u} - X^{n,i,m}_{u} | + E[\sup_{v \leq u} | \tilde{X}^{n,i}_{v} - X^{n,m,j}_{v} | \, du | A^i_u |)^2]] + E[\int_0^{\tau_i(T)} (\tilde{X}^{n,i}_{u} - X^{n,i,m}_{u})^2 \, du] + E[\int_0^{\tau_i(T)} (\tilde{X}^{n,i}_{u} - X^{n,i,m})^2 \, du] + E[(\frac{1}{n} \sum_{j=1}^{n} \sup_{v \leq u} | \tilde{X}^{n,j}_{v} - X^{n,m,j}_{v} |)^2 \, du < M^i, M^i > u)]] \]

Using the exchangeability of \( \{X_i^i\} \) and \( \{X_v^{n,i,m}\} \)

\[ \leq C[E[(\int_0^{\tau_i(T)} | \tilde{X}^{n,i}_{u} - X^{n,i,m}_{u} | \, du)^2]] + E[(\int_0^{\tau_i(T)} | \tilde{X}^{n,i}_{u} - X^{n,i,m}_{u} | + E[\sup_{v \leq u} | \tilde{X}^{n,i}_{v} - X^{n,m,i}_{v} | \, du | A^i_u |)^2]] + E[\int_0^{\tau_i(T)} (\tilde{X}^{n,i}_{u} - X^{n,i,m}_{u})^2 \, du] + E[\int_0^{\tau_i(T)} (\tilde{X}^{n,i}_{u} - X^{n,i,m})^2 + E[\sup_{v \leq u} (\tilde{X}^{n,i}_{v} - X^{n,m,i}_{v})^2 \, du < M^i, M^i > u)]] \]

Now we use the Lebesgue change of time theorem and Hölder's inequality

\[ \leq C[E[(\int_0^{\tau_i(T)} (\tilde{X}^{n,i}_{\tau_i(u)} - X^{n,i,m}_{\tau_i(u)})^2 \, du)] + E[\int_0^{\tau_i(T)} (\tilde{X}^{n,i}_{\tau_i(u)} - X^{n,i,m}_{\tau_i(u)})^2 \, du] + E[\sup_{v \leq \tau_i(u)} (\tilde{X}^{n,i}_{v} - X^{n,m,i}_{v})^2 \, du)] \]

\[ \leq T E[\sup_{v \leq \tau_i(T)} (\tilde{X}^{n,i}_{v} - X^{n,m,i})^2] \]

which we have already determined converges to zero.
Now we proceed to show that \( X_t^{n,i} \) is the unique solution of the stochastic differential equation

\[
X_t^{n,i} = X_0^{n,i} + \int_0^t \hat{b}(X_s^{n,i}) \, ds + \int_0^t b(X_s^{n,i}, \eta_s^n) \, dA_s^i \\
+ \int_0^t \hat{\sigma}(X_s^{n,i}) \, dN_s^i + \int_0^t \sigma(X_s^{n,i}, \eta_s^n) \, dM_s^i.
\]

To see this we suppose there is some other solution to the stochastic differential equation

\[
Z_t^{n,i} = X_0^{n,i} + \int_0^t \hat{b}(Z_s^{n,i}) \, ds + \int_0^t b(Z_s^{n,i}, \gamma_s^n) \, dA_s^i \\
+ \int_0^t \hat{\sigma}(Z_s^{n,i}) \, dN_s^i + \int_0^t \sigma(Z_s^{n,i}, \gamma_s^n) \, dM_s^i
\]

where \( \gamma_s^n = \frac{1}{n} \sum_{j=1}^n \delta_{Z_s^{n,j}} \), and show that the two solutions must be equal to one another. From Lemma 2.1

\[
E[d(Z_{r^T}^{n,i} - X_{r^T}^{n,i})^2] \\
\leq E[\sup_{s \leq T} (Z_{r^T(s)}^{n,i} - X_{r^T(s)}^{n,i})^2] \\
\leq C[E[(\int_0^{r^T} | \hat{b}(Z_u^{n,i}) - \hat{b}(X_u^{n,i}) | \, du)^2] \\
+ E[(\int_0^{r^T} | b(Z_u^{n,i}, \gamma_u^n) - b(X_u^{n,i}, \eta_u^n) | \, d | A_u^i |)^2] \\
+ E[\int_0^{r^T} (\hat{\sigma}(Z_u^{n,i}) - \hat{\sigma}(X_u^{n,i}))^2 \, du] \\
+ E[\int_0^{r^T} (\sigma(Z_u^{n,i}, \gamma_u^n) - \sigma(X_u^{n,i}, \eta_u^n))^2 \, d < M^i, M^i >_u]].
\]

Now we use the Lipschitz condition
Now we use the Lebesgue change of time theorem and Hölder's inequality

\[
E[\sup_{s \leq T} (Z_{\tau^i(s)}^n - X_{\tau^i(s)}^n)^2] \\
\leq C[E[\int_0^{\tau^i(T)} | Z_u^n - X_u^n | \, du]^2] \\
+ E[\int_0^{\tau^i(T)} (Z_u^n - X_u^n)^2 \, du] \\
+ E[\int_0^{\tau^i(T)} (Z_u^n - X_u^n)^2 + \rho(\mathcal{L}(\gamma_u^n), \mathcal{L}(\eta_u^n)) \, d < M^i, M^i > u]] \\
\leq C[E[\int_0^{\tau^i(T)} | Z_u^n - X_u^n | \, du]^2] \\
+ E[\int_0^{\tau^i(T)} | Z_u^n - X_u^n | \, E[\sup_{u \leq v} (Z_v^n - X_v^n)^2] \, du] \\
+ E[\int_0^{\tau^i(T)} (Z_u^n - X_u^n)^2 \, du] \\
+ E[\int_0^{\tau^i(T)} (Z_u^n - X_u^n)^2 + E[\sup_{u \leq v} (Z_v^n - X_v^n)^2] \, d < M^i, M^i > u]].
\]

This is equal to zero by Gronwall's inequality so we may conclude that \(X_t^n\) is the unique solution.

**Theorem 4.4**

Let \(\{\eta^m\}\) be a subsequence of \(\{\eta^n\}\) such that \(\eta^m \Rightarrow \eta\) for some \(P(D_E[0, T])\)-valued random variable \(\eta\). Let
Then there is a further subsequence \( \{ \eta_{m'} \} \) such that \( E[d(X_{m'}^{m, i}, X^i_{m'}(s))^2] \to 0 \)

where \( X^i_t \) is the unique solution to

\[
X^i_t = X_0 + \int_0^t \hat{b}(X^i_s) \, ds + \int_0^t b(X^i_s, \eta_s) \, dA^i_s
+ \int_0^t \hat{\sigma}(X^i_s) \, dN^i_s + \int_0^t \sigma(X^i_s, \eta_s) \, dM^i_s.
\]

**Proof** Our strategy will be to show that

\[
E[d(X_{m', i}^{p, i} - X_{m', i}^{q, i})^2]
\]

goes to zero as the integers \( p, q \) go to infinity. Let \( p > q \) be two fixed integers.

From Lemma 2.1

\[
E[d(X_{m', i}^{p, i} - X_{m', i}^{q, i})^2] \leq E[\sup_{s \leq T} (X_{m', i}^{p, i} - X_{m', i}^{q, i})^2]
\]

\[
\leq C[E[\int_0^{T} (\bar{b}(X_{u}^{p, i} - \hat{b}(X_{u}^{q, i}) | du)^2]
+ E[\int_0^{T} b(X_{u}^{p, i}, \eta_u) - b(X_{u}^{q, i}, \eta_u) | dA_u^i)^2]
+ E[\int_0^{T} (\hat{\sigma}(X_{u}^{p, i}) - \hat{\sigma}(X_{u}^{q, i}))^2 du]
+ E[\int_0^{T} (\sigma(X_{u}^{p, i}, \eta_u) - \sigma(X_{u}^{q, i}, \eta_u))^2 d < M^i, M^i > u]].
\]

From the Lipschitz condition
\[
E[\sup_{s \leq T}(X_{\tau^i(s)}^p - X_{\tau^i(s)}^q)^2] \\
\leq C[E[(\int_0^{\tau^i(T)} |X_u^p - X_u^q|^2 \, du)^2]] \\
+ E[(\int_0^{\tau^i(T)} (|X_u^p - X_u^q| + \rho(\mathcal{L}(\eta_u^p), \mathcal{L}(\eta_u^q)) \wedge L) \, du)^2] \\
+ E[\int_0^{\tau^i(T)} (X_u^p - X_u^q)^2 \, du] \\
+ E[\int_0^{\tau^i(T)} ((X_u^p - X_u^q)^2 + \rho(\mathcal{L}(\eta_u^p), \mathcal{L}(\eta_u^q))^2 \wedge L^2) \, du < M^i, M^i > u]].
\]

Now we use the Lebesgue change of time theorem and Hölder's inequality

\[
E[\sup_{s \leq T}(X_{\tau^i(s)}^p - X_{\tau^i(s)}^q)^2] \leq \\
C[E[(\int_0^{\tau^i(T)} (X_{\tau^i(u)}^p - X_{\tau^i(u)}^q)^2 \, du)] \\
+ E[(\int_0^{\tau^i(T)} \rho(\mathcal{L}(\eta_{\tau^i(u)}^p), \mathcal{L}(\eta_{\tau^i(u)}^q))^2 \wedge L^2 \, du].
\]

Now we use Gronwall's inequality

\[
E[\sup_{s \leq T}(X_{\tau^i(s)}^p - X_{\tau^i(s)}^q)^2] \leq C e^{C T} \int_0^{\tau^i(T)} \rho(\mathcal{L}(\eta_{\tau^i(u)}^p), \mathcal{L}(\eta_{\tau^i(u)}^q))^2 \wedge L^2 \, du.
\]

Now we need to show that the left side goes to zero as \( p, q \to \infty \). Since \( \{\eta^m\} \Rightarrow \eta \), there is a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\), and \( P(D_E[0,T])\)-valued random variables \( \tilde{\eta}^m \), \( \tilde{\eta} \), such that \( \tilde{\eta}^m \to \tilde{\eta} \) a.s. \( \tilde{P} \) in the topology of weak convergence on \( D_E[0,T] \). Then for each \( \tilde{\omega} \in \tilde{\Omega} \) there exists a probability space \((\Omega^{\tilde{\omega}}, \mathcal{F}^{\tilde{\omega}}, \tilde{P}^{\tilde{\omega}})\) and \( D_E[0,T]\)-valued random variables \( \{Z^m_{\tilde{\omega}}\}, Z^{\tilde{\omega}} \) such that \( d(Z^m_{\tilde{\omega}}, Z^{\tilde{\omega}}) \to 0 \) a.s. \( \tilde{P}^{\tilde{\omega}} \) Let \( \Delta Z^{\tilde{\omega}}_t = Z^{\tilde{\omega}}_t - Z^{\tilde{\omega}}_{t^-} \). From Lemma 1.61 there are at most countable \( t \) such that \( \tilde{P}^{\tilde{\omega}}(\Delta Z^{\tilde{\omega}}_t > 0) > 0 \). From Lemma 1.62 , for \( t \) such that \( \tilde{P}^{\tilde{\omega}}(\Delta Z^{\tilde{\omega}}_t > 0) = 0 \),

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Since $D_E[0, T]$ is separable and complete the Vasserstein metric is equivalent to the topology of weak convergence on the space of measures with finite first moment, so

$$\hat{\rho}(\mathcal{L}(Z_t^{m, \omega}), \mathcal{L}(Z_t^{\omega})) \to 0,$$

a.s. $\bar{P}$, a.e. $t$. Now fix $t$ such that $P^{\omega}(|\Delta Z_t^{\omega}| > 0) = 0$, then $\hat{\rho}(\mathcal{L}(Z_t^{m, \omega}), \mathcal{L}(Z_t^{\omega}))$ converges to zero a.s. $\bar{P}$.

$$E_P[\hat{\rho}(\mathcal{L}(Z_t^{m, \omega}), \mathcal{L}(Z_t^{\omega}))] \leq E_P[\hat{\rho}(\mathcal{L}(Z_t^{m, \omega}), \delta_0)] + E_P[\hat{\rho}(\delta_0, \mathcal{L}(Z_t^{\omega}))].$$

$$E_P[\hat{\rho}(\mathcal{L}(Z_t^{m, \omega}), \delta_0)] = E[\frac{1}{m} \sum_{i=1}^{m} X_t^{m, i}]$$

$$\leq C[E[(X_t^{m, i})^2] + 1 + E[(\int_{0}^{t} d |A_u^i|)^2] + E[<M^i, M^i>_t]e^{CT}].$$

Since $Z_t^{m, \omega} \Rightarrow Z_t^{\omega}$,

$$E_P[\hat{\rho}(\delta_0, \mathcal{L}(Z_t^{\omega}))] = E_P[Z_t^{\omega}] \leq \limsup E_P[Z_t^{m, \omega}] = \limsup E_P[\hat{\rho}(\mathcal{L}(Z_t^{m, \omega}), \delta_0)].$$

So from the Chebychev inequality, $E_P[\hat{\rho}(\mathcal{L}(Z_t^{m, \omega}), \mathcal{L}(Z_t^{\omega}))]$ is tight, and, possibly taking a subsequence,

$$E_P[\hat{\rho}(\mathcal{L}(Z_t^{m, \omega}), \mathcal{L}(Z_t^{\omega}))] \Rightarrow 0.$$
Thus $\rho(\eta^m, \eta) \to 0$ for all but countably many $t$, and, since $\rho(\eta^m, \eta)$ is bounded for all $m$, the dominated convergence theorem tells us that

$$\int_0^{\tau^i(T)} \rho(\mathcal{L}^{\eta^P_{\tau^i(t)}}, \mathcal{L}^{\eta^{q1}_{\tau^i(t)}})^2 \land L^2 \, ds \to 0.$$  

We then know that $\{X_{\tau^i(t)}^{m,i}\}$ is Cauchy sequence in $L^2(D[0, T])$. Suppose $\{X_{\tau^i(t)}^{m,i}\}$ converges to $X_{\tau^i(t)}^i$. We now want to show that $X_t^i$ has the following representation:

$$X_t^i = X_0 + \int_0^t \hat{b}(X_s) \, ds + \int_0^t b(X_s, \eta_s) \, dA^i_s + \int_0^t \hat{\sigma}(X_s) \, dN^i_s + \int_0^t \sigma(X_s, \eta_s) \, dM^i_s.$$  

Using Lemma 2.1

$$E[d(X_{\tau^i(t)}^{n,i}, X_0 + \int_0^{\tau^i(T)} \hat{b}(X_u) \, du + \int_0^{\tau^i(T)} b(X_u, \eta_u) \, dA_u + \int_0^{\tau^i(T)} \hat{\sigma}(X_u) \, dN^i_u + \int_0^{\tau^i(T)} \sigma(X_u, \eta_u) \, dM^i_u)^2]$$  

$$\leq E[\sup_{s \leq T} (X_{\tau^i(t)}^{n,i} - X_0 - \int_0^{\tau^i(s)} \hat{b}(X_u) \, du - \int_0^{\tau^i(s)} b(X_u, \eta_u) \, dA_u - \int_0^{\tau^i(s)} \hat{\sigma}(X_u) \, dN^i_u - \int_0^{\tau^i(s)} \sigma(X_u, \eta_u) \, dM^i_u)^2]$$  

$$\leq C[E[\int_0^{\tau^i(T)} | \hat{b}(X_u^{n,i}) - \hat{b}(X_u)| \, du]^2] + E[\int_0^{\tau^i(T)} | b(X_u^{n,i}, \eta_u^n)b(X_u, \eta_u)| \, d|A^i_u|][^2] + E[\int_0^{\tau^i(T)} (\hat{\sigma}(X_u^{n,i}) - \hat{\sigma}(X_u))^2 \, du] + E[\int_0^{\tau^i(T)} (\sigma(X_u^{n,i}, \eta_u^n) - \sigma(X_u, \eta_u))^2 \, d < M^i, M^i >_u].$$
Now we use the Lebesgue change of time theorem, the growth condition and Fubini's theorem

\[
E[\sup_{s \leq T} (X^n_{\tau^i(s)} - X_0) - \int_0^{\tau^i(s)} \hat{b}(X_u) \, du - \int_0^{\tau^i(s)} b(X_u, \eta_u) \, dA_u^i \\
- \int_0^{\tau^i(s)} \hat{\sigma}(X_u) \, dN_u^i - \int_0^{s} \sigma(X_u, \eta_u) \, dM_u^i] \right] \\
\leq CE[\int_0^T E[(X^n_{\tau^i(u)} - X_{\tau^i(u)})^2 + \rho(\mathcal{L}(\eta^n_{\tau^i(u)}), \mathcal{L}(\eta_{\tau^i(u)}))^2 \, du]^2].
\]

and we already know that this goes to zero.

To show uniqueness consider

\[
Z_i^i = X_0 + \int_0^t \hat{b}(Z_s^i) \, ds + \int_0^t b(Z_s^i, \eta_s) \, dA_s^i + \int_0^t \hat{\sigma}(Z_s^i) \, dN_s^i + \int_0^t \sigma(Z_s^i, \eta_s) \, dM_s^i.
\]

Using the Lemma 2.1

\[
E[d(X^n_i(\cdot), Z^i_i(\cdot))] \leq E[\sup_{s \leq T} (X^n_{\tau^i(s)} - Z^i_{\tau^i(s)})^2] \\
\leq C[E(\int_0^{\tau^i(T)} | \hat{b}(X_u^i) - \hat{b}(Z_u^i) | \, du)^2] \\
+ E[| b(X_u^i, \eta_u) - b(Z_u^i, \eta_u) | \, d | A_u^i |)^2] \\
+ E[| \hat{\sigma}(X_u^i) - \hat{\sigma}(Z_u^i) |^2 \, du] \\
+ E[| \sigma(X_u^i, \eta_u) - \sigma(Z_u^i, \eta_u) |^2 \, d < M^i, M^i > u)].
\]

Now we use the Lipschitz condition and the Lebesgue change of time theorem
$$E[\sup_{s \leq T} (X^i_\tau(s) - Z^i_\tau(s))^2]$$
\[ \leq C[\int_0^T E(|X^i_{\tau(u)} - Z^i_{\tau(u)}|^2) du] + E[\int_0^T (X^i_{\tau(u)} - Z^i_{\tau(u)})^2 du]].$$

Now we use Hölder's inequality and Fubini's theorem

$$E[\sup_{s \leq T} (X^i_\tau(s) - Z^i_\tau(s))^2]$$
\[ \leq C[\int_0^T E((X^i_{\tau(u)} - Z^i_{\tau(u)})^2) du].$$

and by Gronwall's inequality this is equal to zero.

**Theorem 4.5**

Let $E[X^2_0] < \infty$, and $Y^0_t \equiv 0, Y^1_t \equiv X^1_0$

$$Y^m,i_t = X^i_0 + \int_0^t \hat{b}(Y^{m-1,i}_s) ds + \int_0^t b(Y^{m-1,i}_s, \mathcal{L}(Y^{m-1,i}_s)) dA^i_s$$
\[ + \int_0^t \hat{\sigma}(Y^{m-1,i}_s) dN^i_s + \int_0^t \sigma(Y^{m-1,i}_s, \mathcal{L}(Y^{m-1,i}_s)) dM^i_s.\]

Let

$$X^{n,i,m+1}_t = X^i_t + \int_0^t \hat{b}(X^{n,i,m}_s) ds + \int_0^t b(X^{n,i,m}_s, \eta^{n,m}_s) dA^i_s$$
\[ + \int_0^t \hat{\sigma}(X^{n,i,m}_s) dN^i_s + \int_0^t \sigma(X^{n,i,m}_s, \eta^{n,m}_s) dM^i_s,\]

where $\eta^{n,m}_t = \frac{1}{n} \sum_{j=1}^n \delta_{X^{n,j,m}_t}$. Then

$$\lim_{n \to \infty} E[d(Y^{m,i}_{\tau(i)}, X^{n,i,m}_{\tau(i)})^2] = 0$$

for all nonnegative integers $m$. 

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Proof

The claim is true for \( m = 1 \). Suppose the claim is true for \( m = k - 1 \). Using Lemma 2.1

\[
E[d(Y_{\tau(s)}^{m,i}, X_{\tau(s)}^{n,i,m})^2]
\leq E[\sup_{s \leq T}(Y_{\tau(s)}^{m,i} - X_{\tau(s)}^{n,i,m})^2]
\leq C[E[(\int_0^{\tau(T)} | \hat{b}(Y_u^{m-1,i}) - \hat{b}(X_u^{n,i,m-1}) | du)^2]
+ E[(\int_0^{\tau(T)} b(Y_u^{m-1,i}, L(Y_u^{m-1,i})) - b(X_u^{n,i,m-1}, \eta_u^{n,m-1}) | d | A_u^i |)^2]
+ E[(\int_0^{\tau(T)} (\hat{\sigma}(Y_u^{m-1,i}) - \hat{\sigma}(X_u^{n,i,m-1}))^2 du]
+ E[(\int_0^{\tau(T)} (\sigma(Y_u^{m-1,i}, L(Y_u^{m-1,i})) - \sigma(X_u^{n,i,m-1}, \eta_u^{n,m-1}))^2 d < M, M >^i_u]].
\]

Now we use the growth condition

\[
E[\sup_{s \leq T}(Y_{\tau(s)}^{m,i} - X_{\tau(s)}^{n,i,m})^2]
\leq C[E[(\int_0^{\tau(T)} | Y_u^{m-1,i} - X_u^{n,i,m-1} | du)^2]
+ E[(\int_0^{\tau(T)} | Y_u^{m-1,i} - X_u^{n,i,m-1} |
+ \rho(\delta_{L(Y_u^{m-1,i}), L(\eta_u^{n,m-1})}) d | A_u^i |)^2]
+ E[(\int_0^{\tau(T)} (Y_u^{m-1,i} - X_u^{n,i,m-1})^2 du]
+ E[(\int_0^{\tau(T)} (Y_u^{m-1,i} - X_u^{n,i,m-1})^2
+ \rho(\delta_{L(Y_u^{m-1,i}), L(\eta_u^{n,m-1})})^2 d < M, M >^i_u]].
\]

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Using the triangle inequality

\[
E[\sup_{s \leq \tau}(Y_{\tau(s)}^{m,i} - X_{\tau(s)}^{n,i,m})^2]
\]

\[
+ E[(\int_0^{\tau(T)} | Y_{u}^{m-1,i} - X_{u}^{n,i,m-1} | \, du)
+ \rho(\delta_{\mathcal{L}(Y_{u}^{m-1,i})}, \mathcal{L}(\frac{1}{n} \sum_{j=1}^{n} \delta_{Y_{u}^{m-1,i}}))
+ \rho(\mathcal{L}(\frac{1}{n} \sum_{j=1}^{n} \delta_{Y_{u}^{m-1,i}}), \mathcal{L}(\eta_{u}^{n,m-1})) \, d(\mathcal{A}_{u}^i)^2]
\]

\[
+ E[\int_0^{\tau(T)} (Y_{u}^{m-1,i} - X_{u}^{n,i,m-1})^2 \, du]
+ E[\int_0^{\tau(T)} (Y_{u}^{m-1,i} - X_{u}^{n,i,m-1})^2
+ \rho(\delta_{\mathcal{L}(Y_{u}^{m-1,i})}, \mathcal{L}(\frac{1}{n} \sum_{j=1}^{n} \delta_{Y_{u}^{m-1,i}}))^2)
+ \rho(\mathcal{L}(\frac{1}{n} \sum_{j=1}^{n} \delta_{Y_{u}^{m-1,i}}), \mathcal{L}(\eta_{u}^{n,m-1}))^2 \, d < M, M > u)].
\]

Using the definition of the Vasserstein metric

\[
E[\sup_{s \leq \tau}(Y_{\tau(s)}^{m,i} - X_{\tau(s)}^{n,i,m})^2]
\]

\[
\leq C[E[(\int_0^{\tau(T)} | Y_{u}^{m-1,i} - X_{u}^{n,i,m-1} | \, du)^2]
+ E[(\int_0^{\tau(T)} | Y_{u}^{m-1,i} - X_{u}^{n,i,m-1} | \, du)^2] + E[\sup_{v \leq u} | Y_{u}^{m-1,i} - X_{u}^{n,i,m-1} |]
+ \rho(\delta_{\mathcal{L}(Y_{u}^{m-1,i})}, \mathcal{L}(\frac{1}{n} \sum_{j=1}^{n} \delta_{Y_{u}^{m-1,i}})) \, d(\mathcal{A}_{u}^i)^2]
+ E[\int_0^{\tau(T)} (Y_{u}^{m-1,i} - X_{u}^{n,i,m-1})^2 \, du]
+ E[\int_0^{\tau(T)} (Y_{u}^{m-1,i} - X_{u}^{n,i,m-1})^2 + E[\sup_{v \leq u} (Y_{u}^{m-1,i} - X_{u}^{n,i,m-1})^2]
+ \rho(\delta_{\mathcal{L}(Y_{u}^{m-1,i})}, \mathcal{L}(\frac{1}{n} \sum_{j=1}^{n} \delta_{Y_{u}^{m-1,i}}))^2 \, d < M, M > u)].
\]
Using the Lebesque change of time theorem and Hölder's theorem

$$
E[\sup_{s \leq T}(Y_{m,i}^{n,i,m})^2] 
\leq C[E[\int_0^T (Y_{m-1,i}^{n,i,m-1} - X_{m,i,m-1}^{n,i,m-1})^2 + E[\sup_{u \leq u}(Y_{m-1,i}^{n,i,m-1} - X_{m,i,m-1}^{n,i,m-1})^2] 
+ \rho(\delta_{\mathcal{L}(Y_{m-1,i}^{n,i,m-1}), \mathcal{L}(\sum_{j=1}^n \delta_{Y_{m-1,i}^{n,i,m-1}}))^2] \right)] 
\leq C[E[\sup_{v \leq u}(Y_{m-1,i}^{n,i,m-1} - X_{m,i,m-1}^{n,i,m-1})^2] + \rho(\delta_{\mathcal{L}(Y_{m-1,i}^{n,i,m-1}), \mathcal{L}(\sum_{j=1}^n \delta_{Y_{m-1,i}^{n,i,m-1}}))^2].
$$

The first summand goes to zero by the induction hypothesis. The second summand goes to zero by Lemma 1.66

**Theorem 4.6**

Let $E[X_0^2] < \infty$, and $Y_0^{0,i} \equiv 0, Y_0^{1,i} \equiv X_0^i$

$$
Y_t^{m,i} = X_0^i + \int_0^t \tilde{b}(Y_s^{m-1,i}) ds + \int_0^t b(Y_s^{m-1,i}, \mathcal{L}(Y_s^{m-1,i})) dA_s^i + \int_0^t \tilde{\sigma}(Y_s^{m-1,i}) dN_s^i + \int_0^t \sigma(Y_s^{m-1,i}, \mathcal{L}(Y_s^{m-1,i})) dM_s^i.
$$

Then

$$
E[d(Y_{m,i}^{n,i}, Y_{m,i}^{n,i})^2] \rightarrow 0
$$

where $Y_t^i$ is the unique solution to

$$
Y_t^i = X_0^i + \int_0^t \tilde{b}(Y_s^i) ds + \int_0^t b(Y_s^i, \mathcal{L}(Y_s^i)) dA_s^i + \int_0^t \tilde{\sigma}(Y_s^i) dN_s^i + \int_0^t \sigma(Y_s^i, \mathcal{L}(Y_s^i)) dM_s^i.
$$

**Proof**

From Lemma 2.1
\[
E[d(Y_{\tau_1}^{n+1,i} - Y_{\tau_1}^{n,i})^2] \leq C[E(\int_0^{\tau_1(T)} | \hat{b}(Y_{u}^{n,i}) - \hat{b}(Y_{u}^{n-1,i}) | du)^2] + E[\int_0^{\tau_1(T)} b(Y_{u}^{n,i}, L(Y_{u}^{n,i})) - b(Y_{u}^{n-1,i}, L(Y_{u}^{n-1,i})) | du | A_{u}^i]^2] + E[\int_0^{\tau_1(T)} (\hat{\sigma}(Y_{u}^{n,i}) - \hat{\sigma}(Y_{u}^{n-1,i}))^2 du] + E[\int_0^{\tau_1(T)} (\sigma(Y_{u}^{n,i}, L(Y_{u}^{n,i})) - \sigma(Y_{u}^{n-1,i}, L(Y_{u}^{n-1,i}))^2 d < M^i, M^i >_u]].
\]

Now we use the growth condition and the monotone convergence theorem

\[
E[\sup_{s \leq T} (Y_{\tau_1(s)}^{n+1,i} - Y_{\tau_1(s)}^{n,i})^2] \leq C[E(\int_0^{\tau_1(T)} | Y_{u}^{n,i} - Y_{u}^{n-1,i} | du)^2] + E[\int_0^{\tau_1(T)} | Y_{u}^{n,i} - Y_{u}^{n-1,i} | \rho(\delta_{L(Y_{u}^{n,i})}, \delta_{L(Y_{u}^{n-1,i})}) du | A_{u}^i]^2] + E[\int_0^{\tau_1(T)} (Y_{u}^{n,i} - Y_{u}^{n-1,i})^2 du] + E[\int_0^{\tau_1(T)} ((Y_{u}^{n,i} - Y_{u}^{n-1,i})^2 + \rho(\delta_{L(Y_{u}^{n,i})}, \delta_{L(Y_{u}^{n-1,i})})^2) d < M^i, M^i >_u]].
\]

Now we use the Lebesgue change of time theorem and Hölder’s inequality

\[
E[\sup_{s \leq T} (Y_{\tau_1(s)}^{n+1,i} - Y_{\tau_1(s)}^{n,i})^2] \leq C[E(\int_0^{\tau_1(T)} [(Y_{\tau_1(u)}^{n,i} - Y_{\tau_1(u)}^{n-1,i})^2 + \rho(\delta_{L(Y_{\tau_1(u)}^{n,i})}, \delta_{L(Y_{\tau_1(u)}^{n-1,i})})^2] du < M^i, M^i >_u] + E[\int_0^{\tau_1(T)} [(Y_{\tau_1(u)}^{n,i} - Y_{\tau_1(u)}^{n-1,i})^2 + E[\sup_{v \leq u} (Y_{\tau_1(u)}^{n,i} - Y_{\tau_1(u)}^{n-1,i})^2] d < M^i, M^i >_u]].
\]

Using Fubini’s Theorem we have
\[ E[\sup_{s \leq T} (Y_{t(s)}^{n+1,i} - Y_{t(s)}^{n,i})^2] \]
\[ \leq C \int_0^T E[\sup_{v \leq u} (Y_{t(v)}^{n,i} - Y_{t(v)}^{n-1,i})^2] \, dv < M^i, M^i > u]. \]

Now for convenience let \( g_n(u) = E[\sup_{v \leq u} (Y_{t(v)}^{n,i} - Y_{t(v)}^{n-1,i})^2] \). Then we have
\[
g_n(T) \leq C \int_0^T g_{n-1}(u) \, du \]
\[ \leq C^2 \int_0^T \int_0^u g_{n-2}(v) \, dv \, du = C^2 \int_0^T (T - u) g_{n-2}(u) \, du. \]

Where the equality is due to integration by parts. Continuing inductively we obtain
\[
g_n(t) \leq C^{n-1} \int_0^T \frac{(T - u)^{n-2}}{(n - 2)!} g_1(u) \, du = E[(Y_0)^2] \frac{(CT)^{n-1}}{(n - 1)!}. \]

This term is part of a convergent series so we may conclude that \( \{Y^{n,i}\} \) is Cauchy. If we let \( Y^i \) be the limit of \( \{Y^{n,i}\} \) then what remains is to show that \( Y^i \) is the unique solution to
\[
Y_t^i = X_0^i + \int_0^t b(Y_s^i) \, ds + \int_0^t b(Y_s^i, L(Y_s^i)) \, dA_s^i \]
\[ + \int_0^t \sigma(Y_s^i) \, dN_s^i + \int_0^t \sigma(Y_s^i, L(Y_s^i)) \, dM_s^i. \]

From the Lemma 2.1
\[
E[d(Y_{t(s)}^{n,i}, X_0^i + \int_0^{r_i(s)} b(Y_u^i) \, du + \int_0^{r_i(s)} b(Y_u^i, L(Y_u^i)) \, dA_u^i \]
\[ + \int_0^{r_i(s)} \sigma(Y_u^i) \, dN_u^i + \int_0^{r_i(s)} \sigma(Y_u^i, L(Y_u^i)) \, dM_u^i)] \]
\[ \leq E[\sup_{s \leq T} (Y_{t(s)}^{n,i} - (X_0^i + \int_0^{r_i(s)} b(Y_u^i) \, du + \int_0^{r_i(s)} b(Y_u^i, L(Y_u^i)) \, dA_u^i \]
\[ + \int_0^{r_i(s)} \sigma(Y_u^i) \, dB_u^i + \int_0^{r_i(s)} 1_{\{u \leq S_j\}} \sigma(Y_u^i, L(Y_u^i)) \, dM_u^i)]^2] \]

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Using the inequality \((\sum_{i=1}^{n} a_i)^2 \leq n \sum_{i=1}^{n} (a_i)^2\)

\[
E[d(Y_{\tau_i(u)}^n, X_i^i)] + \int_0^{\tau_i(u)} b(Y_i^i) \, du + \int_0^{\tau_i(u)} b(Y_i^i, \mathcal{L}(Y_i^i)) \, dA_i^i
\]

\[+
\int_0^{\tau_i(u)} \tilde{\sigma}(Y_i^i) \, dN_i^i + \int_0^{\tau_i(u)} \sigma(Y_i^i, \mathcal{L}(Y_i^i)) \, dM_i^i \] \]

\[\leq C[E(\int_0^{\tau_i(u)} | \tilde{b}(Y_{\tau_i(u)}^{n-1,i}) - \tilde{b}(Y_i^i) | \, du)^2]
\]

\[+ E[\int_0^{\tau_i(u)} | b(Y_{\tau_i(u)}^{n-1,i}, \mathcal{L}(Y_{\tau_i(u)}^{n-1,i})) - b(Y_i^i, \mathcal{L}(Y_i^i)) | \, du | A_i^i |)^2]
\]

\[+ E[\int_0^{\tau_i(u)} (\tilde{\sigma}(Y_{\tau_i(u)}^{n-1,i}) - \tilde{\sigma}(Y_i^i))^2 \, du]
\]

\[+ E[\int_0^{\tau_i(u)} (\sigma(Y_{\tau_i(u)}^{n-1,i}, \mathcal{L}(Y_{\tau_i(u)}^{n-1,i})) - \sigma(Y_i^i, \mathcal{L}(Y_i^i))|^2 \, d \leq M^i, M^i > u]].
\]

Now we use the Lebesque change of time theorem and the growth condition

\[
\leq C[E(\int_0^{T} | Y_{\tau_i(u)}^{n-1,i} - Y_i^i | \, du)^2]
\]

\[+ E[\int_0^{T} \left[ | Y_{\tau_i(u)}^{n-1,i} - Y_i^i | \right] + \rho(\delta_{\mathcal{L}(Y_{\tau_i(u)}^{n-1,i})}, \delta_{\mathcal{L}(Y_i^i)} \right) \, du)^2]
\]

\[+ E[\int_0^{T} (Y_{\tau_i(u)}^{n-1,i} - Y_i^i)^2 \, du]
\]

\[+ E[\int_0^{T} ((Y_{\tau_i(u)}^{n-1,i} - Y_i^i)^2 + \rho(\delta_{\mathcal{L}(Y_{\tau_i(u)}^{n-1,i})}, \delta_{\mathcal{L}(Y_i^i)}))^2 \, du]
\]

From Hölder’s inequality and Fubini’s theorem

\[
\leq C[E(\int_0^{T} | Y_{\tau_i(u)}^{n-1,i} - Y_i^i | \, du)^2]
\]

\[+ E[\int_0^{T} \left[ | Y_{\tau_i(u)}^{n-1,i} - Y_i^i | \right] + E[\sup \left[ Y_{\tau_i(u)}^{n-1,i} - Y_i^i \right] \right] \, du)^2]
\]

\[+ E[\int_0^{T} (Y_{\tau_i(u)}^{n-1,i} - Y_i^i)^2 \, du]
\]

\[+ E[\int_0^{T} ((Y_{\tau_i(u)}^{n-1,i} - Y_i^i)^2 + E[\sup \left[ Y_{\tau_i(u)}^{n-1,i} - Y_i^i \right] \right) \, du]
\]

\[
\leq C[\int_0^{T} E[\sup_{v \leq u} (Y_{\tau_i(u)}^{n-1,i} - Y_i^i)^2] \, du.
\]

But we already know that this goes to zero.
Now we need only prove uniqueness. Suppose $Z^i_t$ is a solution of

$$Z^i_t = X^i_0 + \int_0^t \dot{b}(Z^i_s) \, ds + \int_0^t b(Z^i_s, \mathcal{L}(Z^i_s)) \, dA^i_s + \int_0^t \dot{\sigma}(Z^i_s) \, dN^i_s + \int_0^t \sigma(Z^i_s, \mathcal{L}(Z^i_s)) \, dM^i_s.$$ 

From the Lemma 2.1

$$E[d(Y^i_{\tau^i(t)} - Z^i_{\tau^i(t)})^2] \leq E[\sup_{s \leq T}(Y^i_s - Z^i_s)^2]$$

$$\leq C[E[\int_0^{\tau^i(T)} |\dot{b}(X^i_u) - \dot{b}(Z^i_u)| \, du]^2]$$

$$+ E[\int_0^{\tau^i(T)} |b(Y^i_u, \mathcal{L}(Y^i_u)) - b(Z^i_u, \mathcal{L}(Z^i_u))| \, dA^i_u|^2]$$

$$+ E[\int_0^{\tau^i(T)} (\dot{\sigma}(Y^i_u) - \dot{\sigma}(Z^i_u))^2 \, du]$$

$$+ E[\int_0^{\tau^i(T)} (\sigma(Y^i_u, \mathcal{L}(Y^i_u)) - \sigma(Z^i_u, \mathcal{L}(Z^i_u)))^2 \, d < M^i, M^i > u].$$

Now we use the Lebesgue change of time theorem and the growth condition

$$E[\sup_{s \leq T}(Y^i_s - Z^i_s)^2]$$

$$\leq C[E[\int_0^{T} |X^i_{\tau^i(u)} - Z^i_{\tau^i(u)}| \, du]^2]$$

$$+ E[\int_0^{T} (|Y^i_{\tau^i(u)} - Z^i_{\tau^i(u)}| + \rho(\delta\mathcal{L}(Y^i_{\tau^i(u)}), \delta\mathcal{L}(Z^i_{\tau^i(u)})) \, du)^2]$$

$$+ E[\int_0^{T} (Y^i_{\tau^i(u)} - Z^i_{\tau^i(u)})^2 \, du]$$

$$+ E[\int_0^{T} ((Y^i_{\tau^i(u)} - Z^i_{\tau^i(u)})^2 + \rho(\delta\mathcal{L}(Y^i_{\tau^i(u)}), \delta\mathcal{L}(Z^i_{\tau^i(u)}))^2) \, du]].$$
Using the definition of the Vasserstein metric

\[ C[E[(\int_0^T | X^{i}_{\tau(u)} - Z^{i}_{\tau(u)} | du)^2]] \]

\[ + E[\int_0^T [I_{Y^{i}_{\tau(u)} - Z^{i}_{\tau(u)}} + \sup_{v \leq u} | Y^{i}_{\tau(v)} - Z^{i}_{\tau(v)}|] du]^2 ] \]

\[ + E[\int_0^T (Y^{i}_{\tau(u)} - Z^{i}_{\tau(u)})^2 du] \]

\[ + E[\int_0^T [(Y^{i}_{\tau(u)} - Z^{i}_{\tau(u)})^2 + \sup_{v \leq u} (Y^{i}_{\tau(v)} - Z^{i}_{\tau(v)})^2] du]]. \]

Now we use Hölder's inequality and Fubini's theorem

\[ E[\sup_{s \leq T} (Y^{i}_{\tau(s)} - Z^{i}_{\tau(s)})^2] \leq K[\int_0^T E[\sup_{v \leq u} (Y^{i}_{\tau(v)} - Z^{i}_{\tau(v)})^2] du]]. \]

And by Gronwall's inequality we see that

\[ E[\sup_{s \leq T} (Y^{i}_{\tau(s)} - Z^{i}_{\tau(s)})^2] = 0 \]

so we are done.

**Lemma 4.7**

\[ \mathcal{L}(\frac{1}{n} \sum_{j=1}^{n} \delta_{X^{n,i}_{\tau(j)}}) \Rightarrow \mathcal{L}(Y^{i}_{\tau(\cdot)}) \]

where the underlying path space is \( D_E[0, T] \) equipped with the Skorohod topology.

**Proof**

From the previous four theorems for every subsequence of \( \{X^{n,i}\} \) there is a further subsequence \( \{X^{n',i}\} \) such that we can draw the following diagram
Where the upward convergence of \(X^{n',i,m}\) is uniform in \(n'\). Then one can construct a sequence \(\{X^{n'_i,i,l}\}\) with \(\{n_i\}\) strictly increasing, such that

\[
E[d(Y^{i_i}_{n'i}, X^{n'_i,i,l}_{n'i})^2] \to 0.
\]

By the triangle inequality \(\{X^{n'_i,i,l}\}\) converges to \(Y^{i_i}_{n'i}\). By the uniformity of the upward convergence \(\{X^{n'_i,i,l}\}\) converges to \(X^{i_i}_{n'i}\) as well. So we have \(X^{i_i}_{n'i} = Y^{i_i}_{n'i}\). Then

\[
\rho(\mathcal{L}(Y^{i_i}_{n'i}), \mathcal{L}(\frac{1}{n'} \sum_{j=1}^{n'} \delta_{X^{n'_i,j}_{n'i}})) \leq \rho(\mathcal{L}(Y^{i_i}_{n'i})), \mathcal{L}(\frac{1}{n'} \sum_{j=1}^{n'} \delta_{Y^{i_i}_{j}})) + \rho(\mathcal{L}(\frac{1}{n'} \sum_{j=1}^{n'} \delta_{Y^{i_i}_{j}}), \mathcal{L}(\frac{1}{n'} \sum_{j=1}^{n'} \delta_{X^{n'_i,j}_{n'i}})).
\]

The first summand converges to zero by lemma 1.66. The second summand goes to zero because

\[
\rho(\mathcal{L}(\frac{1}{n'} \sum_{j=1}^{n'} \delta_{Y^{i_i}_{j}}), \mathcal{L}(\frac{1}{n'} \sum_{j=1}^{n'} \delta_{X^{n'_i,j}_{n'i}})) \leq E \sup_{s \leq t} (X^{n'_i,i}_{n'i} - Y^{i_i}_{n'i})^2
\]

which we have already proved goes to zero. Thus for every subsequence of \(\{\frac{1}{n} \sum_{j=1}^{n} \delta_{X^{j}_{rj}}\}\) there is a further subsequence \(\{\frac{1}{n'} \sum_{j=1}^{n'} \delta_{X^{j}_{rj}}\}\) such that

\[
\frac{1}{n'} \sum_{j=1}^{n'} \delta_{X^{j}_{rj}} \Rightarrow Y^{i_i}_{r}.\]

So

\[
\frac{1}{n} \sum_{j=1}^{n} \delta_{X^{j}_{rj}} \Rightarrow Y^{i_i}_{r}.\]
**Theorem 4.8**

In the case where \( \{N^i\} \) is a collection of independent Brownian motions and \( \{M^i\} \) is a collection of independent identically distributed continuous martingales with finite second moment,

\[
\frac{1}{n} \sum_{j=1}^{n} \delta_{X_{\tau^i(\cdot)}} \Rightarrow \mathcal{L}(Y^i_{\tau^i(\cdot)})
\]

where the underlying path space is in the uniform topology on \( C_E[0,T] \).

**Proof**

We have just proved that

\[
\frac{1}{n} \sum_{j=1}^{n} \delta_{X_{\tau^i(\cdot)}} \Rightarrow \mathcal{L}(Y^i_{\tau^i(\cdot)})
\]

where the underlying path space is in the Skorohod topology on \( D_E[0,T] \).

\( \mathcal{L}(\frac{1}{n} \sum_{j=1}^{n} \delta_{X_{\tau^i(\cdot)}} ) \), and \( \mathcal{L}(Y^i_{\tau^i(\cdot)}) \) assign a probability of one to \( P(C_E[0,T]) \)

so we need only apply Proposition 1.60.
Summary

We have shown that a system of mean-field interacting equations driven by Poisson random measures converges weakly to the solution of a McKean-Vlasov equation, and that such systems are stable both in terms of their coefficients and in their driving terms. Diffusion approximation is established when the magnitude of the jumps of the Poisson random measures converges to 0 in an appropriate sense. We have also shown that these results continue to hold when a random interaction term is introduced which is independent of the driving terms. The identification of the limit is obtained by employing the method of martingale problems. When the driving terms are general semimartingales (with possible jumps) then the Markov property of solutions is no longer guaranteed. Hence, analytical tools based on the infinitesimal generator of the process are no longer available. In spite of this, we have obtained weak convergence and identified the limit when the driving semimartingales for the n-system do not depend on n. The tools we have employed to tackle the problem consist in suitable Picard approximation schemes and the connection between Vasserstein and Prohorov metrics on the Skorohod space.

Mean-field interacting systems are important because they provide a way of modeling systems of identical interacting objects or states. Some applications of mean-field interaction include: systems of interacting neurons [CKS 92], communication networks [Ku 94], and the spin glass problem [Di 92, SK 75]. An interacting system driven by Poisson random measure could be used to model
a system wherein the jumps of the particles are a function of , but not necessarily proportional to, the jumps in the underlying driving terms. If we include a random interaction term then we could model systems wherein the particles are the same on the average but among which there is random variation which may cause a significant difference between the way two given particles might interact with one another. If we have a system driven by semimartingales then we may model the particles which have properties which for some reason cannot be modeled by Lévy driven processes. In modeling a system of particles which remember the past and which choose their paths accordingly, then the Markov property is not guaranteed. There are also systems which are Markov but the driving terms may not be time homogeneous. Results for such models are provided in Chapter 4.

In the future I would like to study stochastic systems which have random perturbations that may possibly depend on the driving terms. Also I am interested in studying systems which are driven by Markov processes which are not of the Lévy type.
References


Vita

George Paslaski grew up in Glen Ellyn Illinois and graduated from Glenbard West High School in 1980. He then attended the University of Idaho and graduated with a Bachelor of Science degree in Mathematics in 1985. After a brief stint in graduate school George pursued a career in restaurant management until the fall of 1990 when he returned to University of Idaho to pursue a Master of Science degree in Mathematics. He graduated in 1992. and in the Fall of 1992 entered Louisiana State University to pursue his Doctor of Science degree which will be conferred in December 1997.
DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: George Paslaski

Major Field: Mathematics

Title of Dissertation: Weak Convergence of Interacting Stochastic Systems

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Date of Examination:

September 9, 1997