4-4-2024

The Modular Generalized Springer Correspondence for the Symplectic Group

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THE MODULAR GENERALIZED SPRINGER CORRESPONDENCE FOR THE SYMPLECTIC GROUP

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
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A.S., Georgia Perimeter College, 2012
B.S., University of Georgia, 2015
M.S., University of Georgia, 2018
May 2024

This thesis is dedicated to Pramod Achar, Tamanna Chatterjee, and all of those who helped me along the way.

So I lingered there, pretending, in
front of my own self, that I had
something to write
Nabokov–Despair

Acknowledgments

Partial funding was graciously provided by the National Science Foundation NSF ID: 1802241

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Abstract

The Modular Generalized Springer Correspondence (MGSC), as developed by Achar, Juteau, Henderson, and Riche, stands as a significant extension of the early groundwork laid by Lusztig's Springer Correspondence in characteristic zero which provided crucial insights into the representation theory of finite groups of Lie type. Building upon Lusztig's work, a generalized version of the Springer Correspondence was later formulated to encompass broader contexts.

In the realm of modular representation theory, Juteau's efforts gave rise to the Modular Springer Correspondence, offering a framework to explore the interplay between algebraic geometry and representation theory in positive characteristic. Achar, Juteau, Henderson, and Riche further extended this correspondence to encompass a more expansive range of phenomena, culminating in the development of the Generalized Modular Springer Correspondence. In their series of papers, they describe explicitly the correspondence for a number of linear algebraic groups in part. The goal of this paper is to finish their work for the case when $\Bbbk$ is a field of positive characteristic $\ell \neq 2$. The case when $\ell = 2$ was treated in [4].

Chapter 1. Introduction

1.1. Notation

Throughout $\Bbbk$ will denote a field either of characteristic zero or of characteristic $\ell > 0$. When a statement holds for only one of the cases, we will be sure to specify. We will consider sheaves of $\Bbbk$-vector spaces on varieties defined over $\mathbb{C}$. Given a complex algebraic group H acting on a variety X, we denote by $D^b_H(X, \Bbbk)$, the derived category of constructible H-equivariant sheaves on X and by $\mathbf{Perv}_H(X, \Bbbk)$ its subcategory of H-equivariant perverse $\Bbbk$-sheaves on X, and by $\mathbf{Loc}^H(X, \Bbbk)$ the category of H-equivariant local systems.

Let G denote a connected reductive algebraic group over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra and $\mathcal{N}_G \subset \mathfrak{g}$ its nilpotent cone. For $x \in \mathcal{N}_G$, let G^x be its stabilizer in G and define $A_G(x) = G^x/(G^x)^\circ$ its component group. Note that G has finitely many nilpotent orbits in $\mathcal{N}_G$ [7] and $A_G(x)$ is a finite group. We will consider G-equivariant perverse sheaves on the nilpotent cone $\mathcal{N}_G$ with coefficients in a field of positive characteristic ℓ. Every simple object in $\mathbf{Perv}_G(\mathcal{N}_G, \Bbbk)$ is of the form $\mathcal{IC}(\mathcal{O}, \mathcal{E})$, where $\mathcal{E}$ is an irreducible G-equivariant $\Bbbk$-local system defined on $\mathcal{O}$. By $\mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk))$ we denote the set of these pairs $(\mathcal{O}, \mathcal{E})$, where the local systems $\mathcal{E}$ on a given orbit $\mathcal{O}$ are taken up to isomorphism.

Springer first discovered that the cohomology groups of Springer fibers realized representations of Weyl groups; this is the so-called *Springer correspondence* [19].

Explicitly, the *Springer Correspondence* over a field $\Bbbk$ is an injective map from the set of isomorphism classes of irreducible $\Bbbk[W]$-modules (to wit, irreducible representations of W) to the set of isomorphism classes of simple G-equivariant perverse $\Bbbk$-sheaves on the nilpotent cone $\mathcal{N}_G$ for G:

$$\mathrm{Irr}(\Bbbk[W]) \hookrightarrow \mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk)).$$

For $\Bbbk$ a field of characteristic zero, this is the classical Springer correspondence. For $\Bbbk$ a field of positive characteristic, a correspondence was given by Juteau; namely, the *modular Springer correspondence* [11]. However, this correspondence fails to be surjective in general.

Lusztig and Spaltenstein gave a more uniform answer in the case $\Bbbk$ has characteristic zero resulting in the generalized Springer correspondence [15, 18], a bijection,

$$\bigsqcup_{\substack{L\subset G \text{ a Levi subgroup} \\ \mathcal{F}\in \mathrm{Irr}(\mathrm{Perv}_L(\mathcal{N}_L,\Bbbk)) \text{ cuspidal}}} \mathrm{Irr}(\Bbbk[N_G(L)/L]) \longleftrightarrow \mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk)).$$

where the disjoint union is over the G-conjugacy classes of cuspidal pairs $(L, \mathcal{F})$. The classical Springer correspondence is then a special case of the bijection with $L = T$ a maximal torus.

For positive characteristic, Achar, Juteau, Henderson, and Riche refined the generalized Springer correspondence [3, 1, 4]. Assume that $\Bbbk$ is large enough such that for every Levi subgroup L of G and pair $(\mathcal{O}_L, \mathcal{E}_L)$ the irreducible L-equivariant local system $\mathcal{E}_L$ is absolutely irreducible. Denote by $\mathfrak{M}_{G,\Bbbk}$ the collection of G-orbits of cuspidal data given by $g \cdot (L, \mathcal{O}_L, \mathcal{E}_L) = (gLg^{-1}, \mathrm{Ad}(g)(\mathcal{O}_L), \mathrm{Ad}(g^{-1})^*\mathcal{E}_L)$. We have a disjoint union

$$\mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk)) = \bigsqcup_{(L,\mathcal{O}_L,\mathcal{E}_L)\in\mathfrak{M}_{G,\Bbbk}} \mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk))^{(L,\mathcal{O}_L,\mathcal{E}_L)}. \tag{1.1}$$

For a given datum $(L, \mathcal{O}_L, \mathcal{E}_L) \in \mathfrak{M}_{G,\Bbbk}$ there is a bijection

$$\mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk))^{(L,\mathcal{O}_L,\mathcal{E}_L)} \longleftrightarrow \mathrm{Irr}(\Bbbk[N_G(L)/L]), \tag{1.2}$$

where $\mathrm{Irr}(\Bbbk[N_G(L)/L)]$ denotes the isomorphism classes of irreducible $\Bbbk$-representations of $N_G(L)/L$. Combining the two, we have the modular generalized Springer correspondence for G,

$$\mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk)) \longleftrightarrow \bigsqcup_{(L,\mathcal{O}_L,\mathcal{E}_L)\in\mathfrak{M}_{G,\Bbbk}} \mathrm{Irr}(\Bbbk[N_G(L)/L]). \tag{1.3}$$

In this dissertation we complete the determination of the generalized modular Springer correspondence for $\mathrm{Sp}(2n)$ in the case that $\Bbbk$ has arbitrary characteristic. In future work, we will complete the determination of the generalized modular Springer correspondence for the remaining classical groups in arbitrary characteristic.

1.2. Combinatorics

1.2.1. Notation

For $m \in \mathbb{N}_{\geq 0}$ let $\mathrm{Part}(m)$ be the set of *partitions* of p. We identify $\mathrm{Part}(m)$ with decreasing sequences $\mathbf{p} = (p_1, p_2, \ldots) \in \mathbb{N}^\infty$ subject to the condition that $\sum_{i\geq 0} p_i = m$. For $\lambda \in \mathrm{Part}(m)$, let $\mathbf{m}(\lambda) = (\mathbf{m}_1(\lambda), \mathbf{m}_2(\lambda), \ldots)$ be the sequence in which $\mathbf{m}_i(\lambda)$ is the multiplicity of i in λ. When we write λ^t, we refer to the transpose partition defined by the property that $\lambda^t_i - \lambda^t_{i+1} = \mathbf{m}_i(\lambda)$ for all i.

By $\mathrm{Part}_\ell(m) \subset \mathrm{Part}(m)$, we refer to the set of *ℓ-regular partitions*, i.e., partitions in which $\mathbf{m}_i(\lambda) < \ell$ for all i. Dually, we denote by $\mathrm{Part}(m, \ell) \subset \mathrm{Part}(m)$ the set of ℓ-powered partitions all of whose parts are of the form $\ell^{(i)}$ for some i. We say that a partition λ is ℓ-restricted if λ^t is ℓ-regular. Let $\mathrm{Bipart}(m)$ denote the set of bipartitions of m. That is, a pair $\binom{\lambda}{\mu}$ of partitions is such that $\sum \lambda_i + \mu_i = m$. We denote the conditions on bipartitions (e.g. $\text{Bipart}_\ell(m)$ and $\text{Bipart}(m, \ell)$) as we did with partitions where the conditions are applied to each component of the the pair.

1.2.2. Combinatorics of the Symplectic Group

It is well-known that $\text{Sp}(2n)$ has adjoint representation $\mathfrak{sp}_{2n}$ of Lie type C_n. We'll now give an explicit determination of the Weyl group W_n.

Let $n \geq 1$ and consider W_n to be the subgroup of $\text{GL}_n(\mathbb{R})$ consisting of signed permutation matrices. To wit, matrices which have exactly one nonzero entry in each column and row and each entry is ± 1.

Now, let $H_n \subseteq W_n$ be the subgroup of all (unsigned) permutation matrices. Thus, each $\sigma \in H_n$ acts on $\mathbb{R}^n$ by permuting coordinates in the standard basis. In fact, every permutation of the coordinates can be realized in this way; hence, $H_n \cong S_n$, the symmetric group on n letters.

Let $N_n \subseteq W_n$ be the subgroup of diagonal matrices (whose entries are ± 1, and, as such, can be identified with $\mathbb{Z}/2\mathbb{Z}$. Note that $H_n \bigcap N_n = \text{id}$. We can see that $|H_n| = n!$, $|N_n| = 2^n$, and $H_n \bigcup N_n = W_n$. Hence $|W_n| = 2^n n!$.

Conjugation of any element of N_n by and element of H_n again lies in N_n. We therefore have that N_n is a normal subgroup and notice that we can express W_n as a semi-direct product, $W_n \cong N_n \rtimes H_n$.

Let $s_i \in W_n$ for $1 \leq i \leq n-1$ be the matrix obtained from the identity by exchanging the

i^{th} and $i+1$st rows, so that $s_1, \ldots, s_{n-1}$ generate H_n.

For $0 \leq i \leq n-1$, let $t_i \in W_n$ be the diagonal matrix whose $i+1^{\text{st}}$ entry is -1 and all other entries 1. Then, $t_0, \ldots, t_{n-1}$ generate N_n and we can describe how elements t_i are conjugate in W_n:

$$t_i = s_i t_{i-1} s_i = s_i s_{i-1} \cdots s_1 t_0 s_1 \cdots s_{i-1} s_i, \tag{1.4}$$

for all $1 \leq i \leq n-1$. By letting $t := t_0$, a set of generators is given by $\{t, s_1, \ldots, s_{n-1}\}$. They are subject to the following relations:

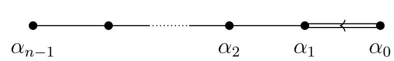
$$ts_1ts_1 = s_1ts_1t, \qquad ts_i = s_it \qquad \text{for } i > 1$$

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \qquad s_is_j = s_js_i, \qquad \text{for } |i-j| > 1.$$

For $n \geq 2$ and $W_n = \langle t, s_1, \ldots, s_{n-1} \rangle \subset \mathrm{GL}_n(\mathbb{R}^n)$, define a (Cartan) matrix $C = (c_{ij})_{0 \leq i,j \leq n-1}$ by $c_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$, where,

$$\alpha_0 := 2e_1, \quad \alpha_1 := e_2 - e_1, \quad \cdots, \quad \alpha_{n-1} := e_n - e_{n-1},$$

with $(\cdot, \cdot)$ is the standard dot product in $\mathbb{R}^n$. Thus, $(e_i, e_j) = \delta_{ij}$. Then C is the Cartan matrix given by the Dynkin diagram



where the elements $t, s_1, \ldots, s_{n-1}$ correspond to the reflections with roots $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ respectively.

Standard Representatives of Conjugacy Classes For Coxeter groups W_n of type C_n the conjugacy classes of W_n are parametrized by pairs of partitions–i.e., *bipartitions* $\binom{\alpha}{\beta}$ with $|\alpha| + |\beta| = n$.

Let $W_n = \langle s_1, \ldots, s_{n-1}, t \rangle \subset \mathrm{GL}_n(\mathbb{R})$ and consider the linear character $\epsilon'(s_i) : W_n \to \{\pm 1\}$ given by $\epsilon'(t) = -1$ and $\epsilon'(s_i) = 1$ for $1 \leq i \leq n-1$. Define $W'_n \subseteq W_n$ to be the kernel of ϵ'. Let $\binom{\lambda}{\mu} = \binom{\lambda_1, \ldots, \lambda_r}{\mu_1, \ldots, \mu_s}$ be a bipartition of of n. Define a new subgroup $W'_{\lambda,\mu} \subseteq W_n$. For every index i let $W'_{\lambda_i} \subseteq \mathrm{GL}_{\lambda_i}(\mathbb{R})$ and for every index j let$W'_{\mu_j} \subseteq \mathrm{GL}_{\mu_j}(\mathbb{R})$ and we establish the rule that $W_0 = W'_0 = \{1\}$. By considering each of the subgroups arising from λ_i and μ_j as blocks we embed the direct product as a subgroup of W_n. Denote this subgroup by

$$W'_{\lambda,\mu} := W'_{\lambda_1} \times \cdots \times W'_{\lambda_r} \times W_{\mu_1} \times \cdots \times W_{\mu_s} \tag{1.5}$$

Generators of $W'_{\lambda,\mu}$ are obtained by defining $t_0 := t$, and $t_i := s_i t_{i-1} s_i$ for $1 \leq i \leq n-1$ and similarly $t'_0 := t s_1 t$ and $t'_i := s_i t'_{i-1} s_i$ for $1 \leq i \leq n-1$. Define $n_i = \lambda_1 + \ldots + \lambda_{i-1}$ for $1 \leq i \leq r$ and similarly $m_j = |\lambda| + \mu_1 + \ldots + \mu_{j-1}$ for $1 \leq j \leq s$. Note that $n_1 = 0$ and $m_1 = |\lambda|$. Then we have the following

- $(W'_{\lambda_i}, \{t'_{n_i}, s_{n_i+1}, \ldots, s_{n_i+\lambda_i-1}\})$ is a Coxeter system of type D_{λ_i},
- $(W_{\mu_j}, \{t_{m_j}, s_{m_j+1}, \ldots, s_{m_j+\mu_j-1}\})$ is a Coxeter system of type B_{μ_j}.

The a-invariant of a partition Let $(\lambda_1, \ldots, \lambda_r) = \lambda \in \mathrm{Part}(n)$, and define an integer $a(\lambda)$ by

$$a(\lambda) := \sum_{1 \leq j < i \leq r} \min(\lambda_i, \lambda_j). \tag{1.6}$$

or equivalently $a(\lambda) = \sum_{i=1}^{r} (i-1)\lambda_i$.

The a^*-invariant Now let λ be a partition of n. We define another integer $a^*(\lambda)$ by

$$a^*(\lambda) := \frac{1}{2}\sum_{i=1}^{r} \lambda_i(\lambda_i - 1) = \sum_{i=1`}^{r} \binom{\lambda_i}{2}. \tag{1.7}$$

Table 1.1. The a and a^*-invariants for $\mathrm{Part}_{\mathrm{Sp}}(6)$

$\lambda \in \mathrm{Part}_{\mathrm{Sp}}(6)$	λ^*	$a-$ invariant	a^*- invariant
6	1^6	0	0
$4, 1^2$	$3, 1^3$	1	3
3^2	2^3	3	3
$2^2, 1^2$	$4, 2$	7	7
$2, 1^4$	$4, 1^2$	10	6
1^6	6	15	15

1.2.3. Irreducible Characters of W_n

Recall from 1.2.2 that $N_n := \langle t_0, t_1, \ldots, t_{n-1}$ is an abelian normal 2-group of order 2^n. Further, note that $W_n/N_n \cong S_n$ is a canonical isomorphism; to wit, $W_n \cong N_n \ltimes S_n$.

For non-negative integers a, b with $a + b = n$, let $\eta_{a,b} : N_n \to \{\pm 1\}$ be a linear character given by $\eta_{a,b}(t_i) = 1$ if $i < a$ and $\eta_{a,b}(t_j) = -1$ if $j \geq a$. When we write $\mathrm{Irr}(W_n \mid \eta_{a,b})$ we refer to the set of all characters $\chi \in \mathrm{Irr}(W_n)$ whose restriction to N_n contains $\eta_{a,b}$. 1.2.3 records the data for the case $n = 3$.

Table 1.2. Elements of $\mathrm{Irr}(W_3 \mid \eta_{a,b})$

(a, b)	$Bipart(3) \leftrightarrow \mathrm{Irr}(W_3 \mid \eta_{a,b})$
$(3, 0)$	$\binom{3}{-}, \binom{1,2}{-}, \binom{1,1,1}{-}$
$(0, 3)$	$\binom{-}{3}, \binom{-}{1,2}, \binom{-}{1,1,1}$
$(1, 2)$	$\binom{1}{2}, \binom{1}{1,1}$
$(2, 1)$	$\binom{2}{1}, \binom{1,1}{1}$

Note that W_n acts on N_n via permutation of the t_i, thus the characters $\eta_{a,b}$ form a complete set of representatives for the orbits of $\mathrm{Irr}(N_n)$ via the induced action of W_n. In fact, a theorem of Clifford ([9], Thm 6.5) shows that

$$\mathrm{Irr}(W_n) = \bigsqcup_{a+b=n} \mathrm{Irr}(W_n \mid \eta_{a,b}). \tag{1.8}$$

Fix a and b. Notice that the stabilizer of $\eta_{a,b}$ for a fixed a and b in W_n is precisely the subgroup $W_{a,b} := W_a \times W_b$ which again is achieved via the diagonal embedding $\mathrm{GL}_a(\mathbb{R}) \times \mathrm{GL}_b(\mathbb{R}) \subseteq \mathrm{GL}_N(\mathbb{R}$

Chapter 2. Macdonald-Lusztig-Spaltenstein Induction

We now consider the induced action of the reflection representation on the symmetric powers of a vector space V, (namely $\mathrm{Sym}(\mathfrak{h}^*)$ where $\mathfrak{h}$ denotes a Cartan subalgebra of $\mathfrak{g}$).

2.1. Symmetric Powers

Let V be a finite-dimensional vector space over $\mathbb{C}$ and $\mathcal{S}^d(V)$ the d^{th} symmetric power of V. That is, $\mathcal{S}^d(V)$ is the quotient of $\bigotimes_d V$ by the subspace generated by elements $(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}) - (v_1 \otimes \cdots \otimes v_d)$ with $v_i \in V$ and $\sigma \in S_n$. For short, denote by $v_1 \cdots v_d$ the image of $v_1 \otimes \cdots \otimes v_d$ in $\mathcal{S}^d(V)$.

This vector space has a well-defined product given by

$$\mathcal{S}^d(V) \times \mathcal{S}^{d'}(V) \to \mathcal{S}^{d+d'}(V), \quad (v_1 \cdots v_d) \cdot (v'_1 \cdots v'_{d'}) = v_1 \cdots v_d \cdot v'_1 \cdots v'_{d'}$$

so that $\mathcal{S}(V) = \bigoplus_{d \geq 0} \mathcal{S}^d(V)$ has the structure of a commutative $\mathbb{C}$-algebra which is termed the *symmetric algebra* of V.

Given a basis $\{v_1, \ldots, v_n\}$ of V, a $\mathbb{C}$-basis for $\mathcal{S}^d(V)$ is given by

$$\{v_1^{i_1} \cdots v_n^{i_n} | i_1, \ldots, i_n \geq 0, \ i_1 + \cdots + i_n = d\}.$$

Of note is that $\dim_{\mathbb{C}}(\mathcal{S}^d(V)) = \binom{n+d-1}{d}$. Given indeterminates $x_1, \ldots, x_n$, there is an isomorphism of $\mathbb{C}$-algebras $\mathbb{C}[x_1, \ldots, x_n] \to \mathcal{S}(V)$ given by $x_1^{i_1} \cdots x_n^{i_n} \mapsto v_1^{i_1} \cdots v_n^{i_n}$. Thus, we may consider $\mathcal{S}^d(V)$ as the collection of homogeneous degree d polynomials. And any $\mathbb{C}$-linear map $\phi : V \to V$ induces a map

$$\phi^{(d)} : \mathcal{S}^d(V) \to \mathcal{S}^d(V), \quad v_1 \cdots v_d \mapsto \phi(v_1) \cdots \phi(v_d).$$

Whenever ϕ is diagonalizable with eigenvalues $\epsilon_1, \ldots, \epsilon_n$, then so too is $\phi^{(d)}$ with eigenvalues $\epsilon_1^{i_1}, \cdots + \epsilon_{i_n}^{i_n}$ where the multi-index $(i_1, \ldots, i_n)$ exhausts all sequences with $i_1 + \cdots + i_n = d$. To wit,

$$\text{trace}(\phi^{(d)}) = \sum_{i_1 + \cdots + i_n = d} \epsilon_1^{i_1}, \cdots \epsilon_{i_n}^{i_n}.$$

2.1.1. b-invariants and The Molien Series

For G a finite group V a $\mathbb{C}G$-module where $\dim_{\mathbb{C}}(V) < \infty$ and for $d \geq 0$, the space $\mathcal{S}^d(V)$ can be considered as a G-module with the action of $g \in G$ given by $g \cdot (v_1 \cdots v_d) = (g \cdot v_1) \cdots (g \cdot v_d)$. Let $\rho_V^{(d)}$ be the character afforded by $\mathcal{S}^d(V)$, so that $\rho_V^{(1)}$ is the character afforded by V.

For $\chi \in \text{Irr}(G)$, define

$$\mathbf{n}_d(\chi) := \langle \rho_V^{(d)}, \chi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \rho_V^{(d)}(g)\chi(g^{-1}),$$

thus, $\mathbf{n}_d(\chi)$ records the multiplicity of χ in $\rho_V^{(d)}$. Now we consider the formal power series $\mathbb{C}[[q]]$ called the *Molien series* and elements of the form

$$P_\chi(q) := \sum_{d \geq 0} \mathbf{n}(\chi) q^d.$$

Assuming that $\mathbf{n}_d(\chi) \neq 0$ for at least one $d \geq 0$, we see that

$$P_\chi(q) = \gamma_\chi q^{b_\chi} + \text{higher degree terms},$$

where the lowest degree term is such that $b_\chi \geq 0$ and $\gamma_\chi \in \mathbb{N}$. As in [13] we refer to b_χ as the the b-invariant of χ. When needed, we specify the underlying module V by writing $P_\chi^V(q)$, b_χ^V, γ_χ^V.

Theorem 1. *(Macdonald-Lusztig-Spaltenstein) Let G be a finite group and V a $\mathbb{C}G$-module. Let $H \leq G$ and $U := V/Fix_H(V)$ where $Fix_H(V)$ is the $\mathbb{C}H$-invariant submodule. We may consider U as a $\mathbb{C}H-module$. Let $\psi \in Irr(H)$ and $0 \leq d < \infty$. Assume $b_\psi^U = d$ and $\gamma_\psi^U = 1$, so that*

$$P_\psi^U(q) = q^d + \text{ higher degree terms.}$$

Then $Ind_H^G(\psi)$ has unique constituent $\chi \in Irr(G)$ with $b_\psi^U = d$ and $\gamma_\psi^U = 1$, hence

$$P_\chi^V(q) = q^d + \text{higher degree terms.}$$

2.2. j-Induction

Let V be a $\mathbb{C}G$-module and $H \subseteq G$ a subgroup. For a given $d \geq 0$ denote by $\mathrm{Irr}(H, d)$ the set of all characters $\psi \in \mathrm{Irr}(H)$ with $P_\psi^U(q) = q^d +$ higher degree terms, where $U = V/\mathrm{Fix}_H(V)$. Then by 1 there exists a map

$$j_H^G : \mathrm{Irr}(H, d) \to \mathrm{Irr}(G, d)$$

the titular *j-induction* which is defined by the condition that $j_H^G(\psi)$, for $\psi \in \mathrm{Irr}(H, d)$, is the unique irreducible constituent of $\mathrm{Ind}_H^G(\psi)$ whose b-invariant is equal to d. Properties of j-induction:

- j-induction is compatible with direct products.
- j-induction is transitive on chains of subgroups ($G \supseteq H' \supseteq H$).
$$j_H^G(\psi) = j_{H'}^G(j_H^{H'}) \text{ for all } \psi \in \mathrm{Irr}(H, d).$$

Let $\binom{\lambda}{\mu} \in \mathrm{Bipart}(n)$, and denote their dual partitions λ^* and μ^* and let $W'_{\lambda^*,\mu^*} \subseteq W_n$ be the corresponding subgroup as defined in 1.5. Define

$$b(\lambda, \mu) := 2a^*(\lambda^*) + 2a^*(\mu^*) + |\mu^*| = 2a(\lambda) + 2a(\mu) + |\mu|. \tag{2.1}$$

By 1.6 and 1.7 we have that $0 \leq b(\lambda,\mu) \leq n^2$ where $b(\lambda,\mu) = 0$ if and only if $\binom{\lambda}{\mu} = \binom{n}{-}$ and $b(\lambda,\mu) = n^2$ if and only if $\binom{\lambda}{\mu} = \binom{-}{1^n}$. Let $\epsilon_{(\lambda^*,\mu^*)}$ be the restriction of the sign character ϵ to $W'_{(\lambda^*,\mu^*)}$ and set

$$\theta_{(\lambda^*,\mu^*)} := \mathrm{Ind}_{W'_{(\lambda^*,\mu^*)}}^{W_n}(\epsilon_{(\lambda^*,\mu^*)})$$

We can uniquely define an irreducible character $\chi_{(}\lambda,\mu)$ of W_n by the condition that $b_{\chi_{(\lambda,\mu)}} = b(\lambda,\mu)$, and

$$\theta_{(\lambda^*,\mu^*)} = \chi_{(\lambda,\mu)} + \text{ combination of } \chi \in \mathrm{Irr}(W_n) \text{ with } b_\chi > b(\lambda,\mu),$$

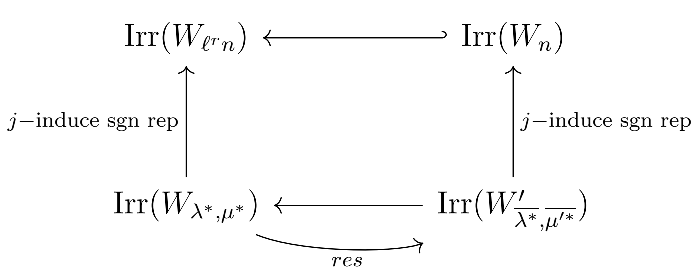
where the b-invariants are with respect to the natural representation of W_n

Lemma 1. *Given $W_n = N_n \rtimes H_n$ as above, for any $\gamma \in Part(n)$ we have*

$$b_{\tilde{\chi_\gamma}} = 2a(\gamma) \;\; \textit{and} \;\; b_{\eta' \otimes \tilde{\chi_\gamma}} = 2a(\gamma) + n,$$

where $\tilde{\chi_\gamma} \in Irr(W_n)$ is the pull-back of $\chi_\gamma \in Irr(\mathfrak{S}_n)$ via $W_n \to H_n \cong \mathfrak{S}_n$.

We then explore the commutativity of the following diagram where horizontal maps are the natural embeddings



Each (relative) Weyl group W acts on the associated symmmetric power. After a choice of embedding, this leads naturally to an action on the symmetric algebra $\mathrm{Sym}(\mathfrak{h}^*)$. Then, considering relative Weyl groups W', our aim is to find a copy of the sign representation $\epsilon_{W'}$ in as low a degree as possible of $\mathrm{Sym}(\mathfrak{h}^*)$. Let $\Delta_{W'}$ denote the product of all positive roots for W'.

Lemma 2. *W' acts on $\Delta_{W'}$ via the sign representation.*

Lemma 3. $\mathbb{C}\Delta_{W'} = \epsilon_{W'}$.

Now, given a bipartition $\binom{\lambda}{\mu}$ of n, which is not necessarily ℓ-restricted, we can take its ℓ-adic expansion $\binom{\lambda^*}{\mu^*}$ which is ℓ-restricted, and of the form

$$\binom{\lambda}{\mu} = \sum \ell^i \binom{\lambda^{(i)}}{\mu^{(i)}}.$$

In characteristic 0, the irreducible representations associated to the bipartition $\binom{\lambda}{\mu}$ of n are the irreducible representations induced from the group $B_{\lambda^{(1)}} \times B_{\lambda^{(2)}} \times \cdots \times D_{\mu^{(1)}} \times \cdots$. In particular, we j-induce the sign representation. Denote it by

$$\mathcal{E}^{\mathbb{F}_\ell} := j^{B_n}_{B_{\lambda^{(1)}} \times B_{\lambda^{(2)}} \times \cdots \times D_{\mu^{(1)}} \times \cdots}(\epsilon)$$

2.2.1. j-induction for type B components

Let $\binom{\lambda}{\mu} = \sum \ell^i \binom{\lambda^{(i)}}{\mu^{(i)}}$, and denote by $\lambda_j^{(i)}$ denote the i^{th} part of the ℓ-adic expansion of part λ_j of the partition our original λ. In characteristic ℓ, suppose $|\lambda^{(i)}| + |\mu^{(i)}| = m_i$ such that $\sum \ell^i m_i = n$. We look at irreducible representations of $B_{m_1} \times B_{m_2} \times \cdots$. For instance, the partition $\lambda = 2 + 11$ has 3-adic expansion $\lambda^* = 2 + (2 + 3^2(1))$, so that $\lambda_1^{(0)} = 2$, $\lambda_2^{(0)} = 2$ and $\lambda_2^{(2)} = 1$. Focusing on those components of type B, we consider the composition

$$B_{\lambda_1^{(0)}} \times B_{\lambda_1^{(1)}} \times B_{\lambda_1^{(2)}} \times \cdots \overset{\phi}{\hookrightarrow} B_{\lambda_1^{(0)}} \times B_{\ell\lambda_1^{(1)}} \times B_{\ell^2\lambda_1^{(2)}} \times \cdots \hookrightarrow B_{\lambda_1},$$

where $\lambda_1 = \lambda_1^{(0)} + \ell\lambda_1^{(1)} + \ell^2\lambda_1^{(2)} + \dots$, and $\phi|_{B_{\lambda_1^i}} : \underbrace{B_{\lambda_1^i} \to B_{\lambda_1^i} \times \cdots \times B_{\lambda_1^i}}_{\ell^i} \leq B_{\ell^i\lambda_1^i}$ is the diagonal embedding for each i.

It will be helpful to adopt the following notation for when we take progressively larger portions, denoted τ_i, of the expansion for λ_1 as follows:

$$\lambda_1 = \underbrace{\underbrace{\underbrace{\lambda_1^{(0)}}_{\tau_0} + \ell\lambda_1^{(1)}}_{\tau_1} + \ell^2\lambda_1^{(2)}}_{\tau_2} + \cdots \tag{2.2}$$

Recall that B_n is the group of permutations with sign changes for n elements and as such acts on $\mathfrak{h}^* = \langle e_1, \dots, e_n \rangle$. Using the root system as described in 1.2.2 we denote by $\Delta_{j,(i)}$ the product of all positive roots in the root system for $B_{\lambda_1^{(i)}}$ as it appears in the j^{th} entry of the diagonal embedding into $B_{\ell^i\lambda_1^{(i)}}$, explicitly.

$$\Delta_{j,(i)} = e_{\tau_{j-1}+(j-1)\ell^i+1} \cdots e_{\tau_{j-1}+j\ell^i} \prod_{\tau_{j-1}+(j-1)\ell^i+1 \leq r < s \leq \tau_{j-1}+j\ell^i} (e_r - e_s)(e_r + e_s). \tag{2.3}$$

which has total degree $(\lambda_1^{(i)})^2 = \lambda_1^{(i)} + 2\binom{\lambda_1^{(i)}}{2}$.

We now construct the corresponding symmetric algebras. Let $V_m = \langle e_1, \dots, e_m \rangle$ be a vector space with given basis. By $\mathcal{S}(V_m)$, we denote the symmetric algebra defined on V_m with respect to that basis. Then the associated product of symmetric algebras arising from part λ_1 of our ℓ-adic expansion of λ, is

$$\mathcal{S}(V_{\lambda_1^{(0)}}) \otimes \mathcal{S}(V_{\lambda_1^{(1)}}) \otimes \mathcal{S}(V_{\lambda_1^{(2)}}) \otimes \cdots$$

where the $B_{\lambda_1^{(i)}}$ acts on $\mathcal{S}(V_{\lambda_1^{(i)}})$. By viewing $\mathcal{S}(V_{\lambda_1^{(0)}})$ as a vector subspace of the tensor algebra, we say $B_{\lambda_1^{(i)}}$ acts on the "block" corresponding to $\mathcal{S}(V_{\lambda_1^{(0)}})$

In particular we examine the image of this algebra in the greater algebra of $\mathcal{S}(V_{\lambda_1})$, with map denoted Φ.

$$\mathcal{S}(V_{\lambda_1^{(0)}}) \otimes \mathcal{S}(V_{\lambda_1^{(1)}}) \otimes \mathcal{S}(V_{\lambda_1^{(2)}}) \otimes \cdots \xrightarrow{\quad\Phi\quad} \mathcal{S}(V_{\lambda_1})$$

By construction this Φ is compatible with the map ϕ on groups and is defined on basis elements as follows:

$\Phi(\mathcal{S}(V_{\lambda_1^{(0)}}))$	$\Phi(\mathcal{S}(V_{\lambda_1^{(1)}}))$	$\Phi(\mathcal{S}(V_{\lambda_1^{(2)}}))$
$e_1 \mapsto e_1$	$e_1 \mapsto \Delta_{1,(1)}$	$e_1 \mapsto \Delta_{(1),2}$
$e_2 \mapsto e_2$	$e_2 \mapsto \Delta_{2,(1)}$	$e_2 \mapsto \Delta_{2,(2)}$
$\vdots$	$\vdots$	$\vdots$
$e_{\lambda_1^{(0)}} \mapsto e_{\lambda_1^{(0)}}$	$e_{\lambda_1^{(1)}} \mapsto \Delta_{\lambda_1^{(1)},(1)}$	$e_{\lambda_1^{(2)}} \mapsto \Delta_{\lambda_1^{(2)},(2)}$

We see that $\Phi(\mathcal{S}(V_{\lambda_1^{(i)}}))$ maps each generator $e_i \in \mathcal{S}(V_{\lambda_1^{(i)}})$ to a generator of the symmetric algebra of dimension ℓ^{2i}. This is the space for which, $B_{\lambda_1^{(i)}}$ acts on $\mathcal{S}(V_{\lambda_1^{(i)}})$ by acting on the corresponding block of size ℓ^i. Similarly, $B_{\ell^i\lambda_1^{(i)}}$ acts on $\underbrace{\mathcal{S}(V_{\lambda_1^{(i)}}) \otimes \cdots \otimes \mathcal{S}(V_{\lambda_1^{(i)}})}_{\ell^i}$ as it sits in $\mathcal{S}(V_{\ell^i\lambda_1^{(i)}})$. We will show that the map Φ is equivariant for each component group $B_{\lambda_1}^{(i)}$, and so is compatible with the map on groups ϕ. Specifically, the sign representation for $B_{\lambda_1^{(i)}}$ will be sent to a product of sign representations, one for each component in its respective diagonal embedding into $B_{\ell^i\lambda_1^{(i)}}$. On the level of symmetric algebras, the sign representation in $\mathcal{S}(V_{\lambda_1^{(0)}}) \otimes \mathcal{S}(V_{\lambda_1^{(1)}}) \otimes \mathcal{S}(V_{\lambda_1^{(2)}}) \otimes \cdots$ is given by the tensor product of the corresponding sign representation in each component.

Lemma 4. *The map $\phi : B_{\lambda^{(i)}} \to B_{\ell^i\lambda_1^{(i)}}$ induces the map Φ on symmetric algebras. In particular, Φ is $B_{\lambda_1^{(i)}}$ equivariant.*

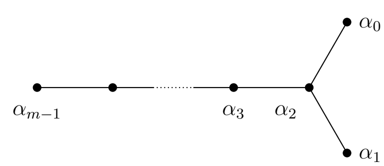
Proof. The first statement follows by construction. What remains to show is that Φ is $B_{\lambda_1^{(i)}}$-equivariant. Recall that $B_{\lambda_1^{(i)}}$ acts on $\mathcal{S}(V_{\lambda_1^{(i)}})$, then under ϕ we diagonally embed $B_{\lambda_1^{(i)}}$ into $B_{\ell^i\lambda_1^{(i)}}$ as ℓ^i copies. Thus, $\underbrace{B_{\lambda_1^{(i)}} \times \cdots \times B_{\lambda_1^{(i)}}}_{\ell^i}$ acts on $\underbrace{\mathcal{S}(V_{\lambda_1^{(i)}}) \otimes \cdots \otimes \mathcal{S}(V_{\lambda_1^{(i)}})}_{\ell^i} \subseteq \mathcal{S}(V_{\ell^i\lambda_1^{(i)}})$ componentwise. Hence Φ is $B_{\lambda_1^{(i)}}$-equivariant as desired. ◻

2.2.2. j-induction for type D components

Similar to the components of type B, we now consider those components of type D. Recall from 1.5 that $(W'_{\mu_i}, \{t'_{n_i}, s_{n_i+1}, \ldots, s_{n_i+\mu_i-1}\})$ is a Coxeter system of type D_{μ_i}. We define a matrix $C_D = (c_{ij})_{0\le i,j\le m-1}$ by $c_{ij} = 2(\alpha'_i, \alpha'_j)/(\alpha'_i, \alpha'_i)$, with

$$\alpha'_0 := e_1 + e_2,\ \alpha'_1 := e_2 - e_1, \ldots,\ \alpha'_{n-1} := e_m - e_{m-1}.$$

We note that C_D is the Cartan matrix with the given Dynkin diagram



The generators $\{t'_{m_i}, s_{m_i+1}, \ldots, s_{m_i+\mu_i-1}\})$ are then reflections with respect to this matrix and their corresponding simple roots are respectively $\alpha'_0, \alpha'_1, \ldots, \alpha'_{m-1}$. A positive (reduced) root system of type D_m is given by $\{e_j \pm e_i | 1 \le i < j \le m\}$. Hence, there are $n(n-1)$ positive roots. In general, given a Coxeter system of type D_n with generators $\{\tau, s_1, \ldots, s_{n-1}\}$, we may embed it into a coxeter system of type B_n with generators $\{t, s_1, \ldots, s_{n-1}\}$ via the map $u \mapsto ts_1t,\ s_i \mapsto s_i$.

We analogously construct the corresponding symmetric algebras on which components of type D act. Let $V_m = \langle e_1, \ldots, e_m\rangle$ be a vector space with given basis. By $\mathcal{S}(V_m)$, we denote the symmetric algebra defined on V_m with respect to that basis. Then the associated product of symmetric algebras arising from part μ_1 of our ℓ-adic expansion of μ, is

$$\mathcal{S}(V_{\mu_1^{(0)}}) \otimes \mathcal{S}(V_{\mu_1^{(1)}}) \otimes \mathcal{S}(V_{\mu_1^{(2)}}) \otimes \cdots$$

In particular we examine the image of this algebra in the greater algebra of $\mathcal{S}(V_{\mu_1})$, with map denoted Φ. We again define a product of all positive roots as we did before by letting $\Delta'_{[j,(i)]}$ denote the image of $e_j \in \mathcal{S}(V_{\mu_1^{(i)}})$ under Φ. Explicitly,

$$\Delta'_{[j,(i)]} = \prod_{\tau_{(i-1)}+(j-1)\ell^i \leq a \leq b \leq \tau_{(i-1)}+j\ell^i} (e_a - e_b)(e_a + e_b)$$

The map $\mathcal{S}(V_{\mu_1^{(0)}}) \otimes \mathcal{S}(V_{\mu_1^{(1)}}) \otimes \mathcal{S}(V_{\mu_1^{(2)}}) \otimes \cdots \to \mathcal{S}(V_{\mu_1})$ follows 2.2.1 by replacing all λ's with μ. Thus, basis elements of $\Phi(\mathcal{S}(V_{\mu_1^{(0)}})$ are degree 1 and there are $\mu_1^{(0)}$ of them; therefore, their product is of total degree $\mu_1^{(0)}((\mu_1^{(0)}) - 1)$, basis elements of $\Phi(\mathcal{S}(V_{\mu_1^{(1)}})$ have total degree $\ell(\ell - 1)$ and there are $\mu_1^{(1)}(\mu_1^{(1)} - 1)$ of them, hence their product has total degree $\mu_1^{(1)}(\mu_1^{(1)} - 1)\ell(\ell - 1)$. We deduce that the degree of the image of Φ in $\mathcal{S}(V_{\mu_1})$ is $\mu_1^{(0)}(\mu_1^{(0)} - 1) + \sum_{i>0} \ell^i(\ell^i - 1)\mu_1^{(i)}(\mu_1^{(i)} - 1)$.

Lemma 5. *The map $\phi : D_{\mu^{(i)}} \to D_{\ell^i \mu_1^{(i)}}$ induces the map Φ on symmetric algebras. In particular, Φ is $D_{\mu_1^{(i)}}$ equivariant.*

The proof is identical to that of Lemma 4 by replacing D with B and λ_1 with μ_1.

Then, combining this with the corresponding statement for type B, we are looking at maps,

$$
\begin{array}{ccc}
(B_{\lambda_1^{(0)}} \times B_{\lambda_1^{(1)}} \times B_{\lambda_1^{(2)}} \times \cdots) & \times & (D_{\mu_1^{(0)}} \times D_{\mu_1^{(1)}} \times D_{\mu_1^{(2)}} \times \cdots) \\
\big\downarrow & \big\downarrow{\scriptstyle \phi} & \big\downarrow \\
(B_{\lambda_1^{(0)}} \times B_{\ell\lambda_1^{(1)}} \times B_{\ell^2\lambda_1^{(2)}} \times \cdots) & \times & (D_{\mu_1^{(0)}} \times D_{\ell\mu_1^{(1)}} \times D_{\ell^2\mu_1^{(2)}} \times \cdots) \\
\searrow & \big\downarrow & \swarrow \\
 & B_{\binom{\lambda_1}{\mu_1}} &
\end{array}
$$

We define another helpful notation to denote the product of all positive roots that are constituents of the $\Delta'_{[j,(i)]}$. Namely,

$$\prod_{[j,(i)]} := \prod_{\tau_{(i-1)}+(j-1)\ell^i \leq a \leq \tau_{(i-1)}+j\ell^i} e_a.$$

2.3. Degree Considerations

Careful attention must be paid to the the degrees of the representations as we pass from one algebra/group representation to another. For instance, the degree of the sign representation of $B_{\lambda_1^{(0)}} \times B_{\lambda_1^{(1)}} \times B_{\lambda_1^{(2)}} \times \cdots$ in $B_{\lambda_1^{(0)}} \times B_{\ell^1\lambda_1^{(1)}} \times B_{\ell^2\lambda_1^{(2)}} \times \cdots$ is

$$\deg(\epsilon_{B_{\lambda_1^{(0)}} \times B_{\lambda_1^{(1)}} \times B_{\lambda_1^{(2)}} \times \cdots}) = (\lambda_1^{(0)})^2 + (\lambda_1^{(1)})^2 + \ldots,$$

whereas the image (under ϕ) of this representation has degree

$$(\lambda_1^{(0)})^2 + \ell^2(\lambda_1^{(1)})^2 + \ell^4(\lambda_1^{(2)})^2 \ldots,$$

However, in $\mathcal{S}(V_{\lambda_1})$ the total degree of the polynomial corresponding to the sign representation is $\lambda_1^2 = (\lambda_1^{(0)} + \ell\lambda_1^{(1)} + \ell^2\lambda_1^{(2)} + \ldots)^2 = \sum_i (\ell^i \lambda_1^{(i)})^2 + \sum_{i \neq j} 2\ell^{i+j}\lambda_1^{(i)}\lambda_1^{(j)}$. We see that we have not accounted for the cross terms for which the roots $e_i \pm e_j$ with e_i and e_j belong to different blocks.

We rectify this by introducing a correction factor, call it F_{λ_1}, of degree

$$\sum_{i\neq j} 2\ell^{i+j}\lambda_1^{(i)}\lambda_1^{(j)} \tag{2.4}$$

to raise the degree of the polynomial for $\epsilon_{B_{\lambda_1^{(0)}}\times B_{\lambda_1^{(1)}}\times B_{\lambda_1^{(2)}}\times\cdots}$.

Before we give the correction factor, it is worth describing how we intend to use it. By construction, $F_{\lambda_1}\cdot\Phi(\epsilon_{B_{\lambda_1^{(0)}}\times B_{\lambda_1^{(1)}}\times B_{\lambda_1^{(2)}}\times\cdots}) = \epsilon_{B_{\lambda_1}}$. Similarly, $F_{\lambda_i}\cdot\Phi(\epsilon_{B_{\lambda_i^{(0)}}\times B_{\lambda_i^{(1)}}\times B_{\lambda_i^{(2)}}\times\cdots}) = \epsilon_{B_{\lambda_i}}$.

We do the same for those factors of type D_{μ_i}, again noting that the image of $\epsilon_{D_{\mu_1^{(0)}}\times D_{\mu_1^{(1)}}\times D_{\mu_1^{(2)}}\times\cdots}$ under Φ in $\mathcal{S}(V_{\mu_1})$ has degree $\mu_1^{(0)}(\mu_1^{(0)}-1)+\sum_{i>0}\ell^i(\ell^i-1)\mu_1^{(i)}(\mu_1^{(i)}-1)$, where as the sign representation is given by a polynomial in $\mathcal{S}(V_{\mu_1})$ of total degree $\mu_1(\mu_1-1)$. We therefore seek to make up the difference in the terms:

$$\begin{aligned}\mu_1(\mu_1-1) &= (\mu_1^{(0)}+\ell\mu_1^{(1)}+\ldots)(\mu_1^{(0)}+\ell\mu_1^{(1)}+\ldots-1)\\ &= \sum \ell^{2i}(\mu_1^{(i)})^2 + 2\sum_{i<j}\ell^{i+j}\mu_1^{(i)}\mu_1^{(j)} - \sum \ell^i\mu_1^{(i)}\end{aligned}$$

while

$$\mu_1^{(0)}(\mu_1^{(0)}-1)+\sum_{i>0}\ell^i(\ell^i-1)\mu_1^{(i)}(\mu_1^{(i)}-1) = \sum \ell^{2i}(\mu_1^{(i)})^2 - \ell^i(\mu_1^{(i)})^2 - \ell^{2i}\mu_1^{(i)} + \ell^i\mu_1^{(i)}$$

Then, the difference between the terms is,

$$2\sum_{i<j}\ell^{i+j}\mu_1^{(i)}\mu_1^{(j)} - \sum 2\ell^i\mu_1^{(i)} + \sum \ell^{2i}\mu_1^{(i)} + \ell^i(\mu_1^{(i)})^2$$

which we rearrange as

$$2\sum_{i<j}\ell^{i+j}\mu_1^{(i)}\mu_1^{(j)} + \sum \ell^i(\mu_1^{(i)}(\mu_1^{(i)}-1)) + \sum \mu_1^{(i)}\ell^i(\ell^i-1). \tag{2.5}$$

We notice that the first term of 2.5 agrees with that of the type B degree correction term. Hence, we construct it analogously. We achieve the remaining two terms using polynomials in the Δ's. Namely, by applying the $\prod$ and Δ constructions in various ways to the element

$$\tilde{\Delta}_i = \prod_{1 \leq a, b \leq \mu_1^{(i)}} \left(\Delta_{[a,(i)]} - \Delta_{[b,(i)]}\right)\left(\Delta_{[a,(i)]} + \Delta_{[b,(i)]}\right).$$

Note that the total degree of $\tilde{\Delta}_i$ is $\ell^i(\ell^i - 1)\mu_1^{(i)}(\mu_1^{(i)} - 1)$.

The third term of 2.5 is achieved by the element

$$\prod_{1 \leq a \leq \mu_1^{(i)}} \Delta_{[a,(i)]}, \tag{2.6}$$

coming from the $\prod_{[\cdot,(i)]}$ construction applied to all the $\Delta_{[\cdot,(i)]}$'s . That is, 2.6 has degree $\sum \mu_1^{(i)} \ell^i(\ell^i - 1)$, as desired.

The second term of 2.5 is achieved by,

$$\prod_{1 \leq a \leq b \leq \mu_1^i} \left(\prod_{[a,(i)]} - \prod_{[b,(i)]}\right)\left(\prod_{[a,(i)]} + \prod_{[b,(i)]}\right), \tag{2.7}$$

which has the desired total degree $\sum \ell^i(\mu_1^{(i)}(\mu_1^{(i)} - 1))$.

The first term, compensates for the lack of cross-terms on which the group acts. We now outline the process. Recall that a part μ_1 of the larger partition μ has a modular reduction as shown below. We take progressively larger portions τ_i as follows:

$$\mu_1 = \underbrace{\underbrace{\underbrace{\mu_1^{(0)}}_{\tau_0} + \ell\mu_1^{(1)}}_{\tau_1} + \ell^2\mu_1^{(2)}}_{\tau_2} + \cdots \tag{2.8}$$

Then, taking the product of all positive roots which are constituents of different blocks, yields

$$\prod_{\substack{k<r \\ \tau_{k-1}+1\leq i\leq \tau_k \\ \tau_{r-1}+1\leq j\leq \tau_r}} (e_i - e_j)(e_i + e_j). \tag{2.9}$$

For fixed j, there are precisely $\ell^k \mu_1^{(k)}$ such terms of the product, and for fixed i there are precisely $\ell^r \mu_1^{(r)}$ such terms in the product. Then, the total degree of the product is $\sum_{k<r} \ell^{k+r} \mu_1^{(k)} \mu_1^{(r)}$. Then, squaring this term yields the desired degree. For partitions λ_1 arising from components of type B_{λ_1} we simply swap labels μ for λ. Denote by F_{μ_i} the endomorphism which is multiplication by these three terms for a given μ_i. And similarly F_{λ_i} the endomorphism which is multiplication by the degree correction term 2.9 where λ takes the place of μ. Combining the two, define $F_{\lambda_i,\mu_i} := F_{\lambda_i} \otimes F_{\mu_i}$

Then, the composition of Φ and F_{λ_i,μ_i} produces the map

$$\bigotimes_{i\geq 0} \left(\bigotimes_{j\geq 1} \mathcal{S}(V_{\lambda_j^{(i)}}) \otimes \bigotimes_{j\geq 1} \mathcal{S}(V_{\mu_j^{(i)}}) \right) \xrightarrow{F\cdot\Phi} \bigotimes_{j\geq 1} \mathcal{S}(V_{\lambda_j}) \otimes \bigotimes_{j\geq 1} \mathcal{S}(V_{\mu_j}) = \mathcal{S}(V_n) \tag{2.10}$$

2.4. Proof of Lemmas 4 and 5

The strategy is to answer the equivalent question on the level of the symmetric algebras. By our careful constructions in 2.2.1,2.2.2, and 2.3, we now have the tools necessary. We reformulate the lemmas as:

Lemma 6. *F_{λ_j} is B_{λ_j} stable for each j, and $F_{\lambda_j} \cdot \Phi$ sends the sign representation for $\bigotimes_{i,j\geq 1} \mathcal{S}(V_{\lambda_j^{(i)}})$ to an isomorphic copy of the sign representation for $\mathcal{S}(V_{\lambda_j})$.*

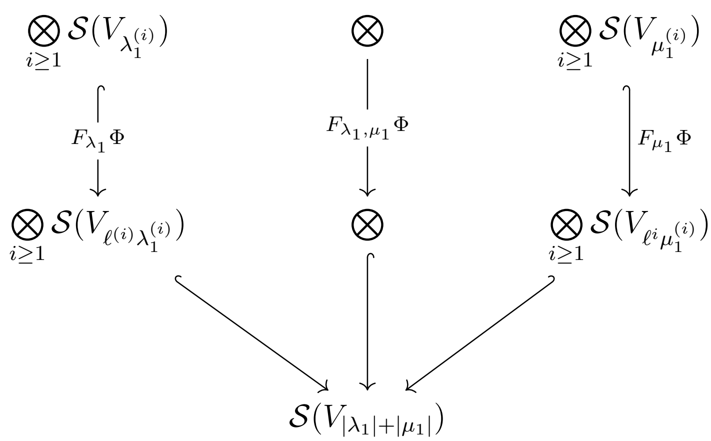
Lemma 7. *F_{μ_j} is D_{μ_j} stable for each j, and $F_{\mu_j} \cdot \Phi$ sends the sign representation for*

$\bigotimes_{i,j\geq 1} \mathcal{S}(V_{\mu_j^{(i)}})$ *to an isomorphic copy of the sign representation for* $\mathcal{S}(V_{\mu_j})$*.*

Theorem 2. *Combining lemmas 4,5,6,7 above and tensoring over* j*, the map*

$$F\cdot\Phi : \bigotimes_{i\geq 0}\left(\bigotimes_{j\geq 1}\mathcal{S}(V_{\lambda_j^{(i)}})\otimes\bigotimes_{j\geq 1}\mathcal{S}(V_{\mu_j^{(i)}})\right)\to\bigotimes_{j\geq 1}\mathcal{S}(V_{\lambda_j})\otimes\bigotimes_{j\geq 1}\mathcal{S}(V_{\mu_j})=\mathcal{S}(V_n)$$

is B_n*- equivariant and corresponds to the j-induction of the sign representation.*



Proof:(Lemma 6) By contstruction $B_{\lambda_j^{(i)}}$ acts on $\mathcal{S}(V_{\lambda_j^{(i)}})$, therefore, $B_{\ell^i\lambda_j^{(i)}}$ acts on $\bigotimes_{i\geq 0}\mathcal{S}(V_{\ell^i\lambda_j^{(i)}})$ by blocks of size ℓ^i. Hence, the sign representation acts by changing the signs of all elements in their respective blocks. What remains to be shown is that the map on symmetric algebras corresponding to the full composition for a given part λ_1

$$\bigoplus_i B_{\lambda_1^{(i)}} \xhookrightarrow{\phi|_{\lambda_1}} \bigoplus_i B_{\ell^i\lambda_1^{(i)}} \hookrightarrow B_{\lambda_1}$$

is fixed by B_{λ_1}. Namely, with respect to the map $e_i\mapsto -e_i$ in the target symmetric algebra $\mathcal{S}(V_{|\lambda_1|})$. We need only show that the terms involving roots from different blocks are fixed. Namely, the element

$$\prod_{\substack{k<r\\ \tau_{k-1}+1\leq i\leq\tau_k\\ \tau_{r-1}+1\leq j\leq\tau_r}}(e_i-e_j)(e_i+e_j). \tag{2.11}$$

is stable under $e_i \mapsto -e_i$. But this is clear since the change in sign merely permutes the terms of the product $(e_i - e_j) \mapsto -(e_i + e_j)$ and $(e_i + e_j) \mapsto -(e_i - e_j)$, hence to $(-(e_i - e_j))(-(e_i + e_j)) = (e_i - e_j)(e_i + e_j)$.

Proof:(Lemma 7) For components of type D, recall that we had three terms not arising from allowing the factors to act diagonally on the relevant blocks. The first, agreed with the degree correction term for type B components, and thus is likewise stable as previously demonstrated. What remains to show is that the other two terms are stable. We show that each of the degree correcting terms are in themselves stable.

$$\prod_{1\leq a\leq b\leq \mu_1^i}\left(\prod_{[a,(i)]}-\prod_{[b,(i)]}\right)\left(\prod_{[a,(i)]}+\prod_{[b,(i)]}\right),$$

We begin by showing that termwise, the product acts by sign, i.e. $\prod_{[a,(i)]} \mapsto -(\prod_{[a,(i)]})$. Recall the definition:

$$\prod_{[j,(i)]} := \prod_{\tau_{(i-1)}+(j-1)\ell^i < a \leq \tau_{(i-1)}+j\ell^i} e_a.$$

Note that this is a product of ℓ^i roots, on which the component of type D acts by a sign change, hence in total by $(-1)^{\ell^i} = -1$ since ℓ is odd, proving the claim.

Since this holds for each term of the product, we have

$$\prod_{1\leq a\leq b\leq \mu_1^i}\left(-\prod_{[a,(i)]}+\prod_{[b,(i)]}\right)\left(-\prod_{[a,(i)]}-\prod_{[b,(i)]}\right),$$

which is equivalent to our original product.

We now show that the last remaining term is also stable, namely

$$\prod_{1 \le a \le \mu_1^{(i)}} \Delta'_{[a,(i)]}.$$

Again we recall the definition

$$\Delta'_{[j,(i)]} = \prod_{\tau_{(i-1)}+(j-1)\ell^i \le a \le b \le \tau_{(i-1)}+j\ell^i} (e_a - e_b)(e_a + e_b)$$

And we note that the change in sign permutes the terms of the product $(e_i - e_j) \mapsto -(e_i + e_j)$ and $(e_i + e_j) \mapsto -(e_i - e_j)$, hence to $(-(e_i - e_j))(-(e_i + e_j)) = (e_i - e_j)(e_i + e_j)$. Since termwise, the product is fixed, the overall product is stabilized, hence $\Delta'_{[j,(i)]}$ is stabilized. This proves that the larger product of $\Delta'_{[j,(i)]}$'s is also fixed.

Chapter 3. Correspondences

We now set about describing two versions of the correspondence: one which is purely combinatorial (Lemma 10), and another which follows the precise statement of the modular generalized Springer correspondence as stated in 1.3.

3.1. The Combinatorial Correspondence

The unipotent classes in $G = \mathrm{Sp}_{2n}(k)$, are in 1-1 correspondence with the set $\mathrm{Part}_{\mathrm{Sp}}(2n) = \{\lambda \in \mathrm{Part}(2n) | \mathbf{m}_{2j+1}(\lambda) = \text{even}\}$; where $\lambda^{(i)}$ is the number of Jordan cells of size i of a unipotent element.

Let $\mathrm{F}_2[\Delta_\lambda]$ be the F_2-vector space with basis indexed by the set $\Delta_\lambda = \{i \ |\mathbf{m}_{2i}(\lambda) \neq 0\}$ where $\lambda \in \mathrm{Part}_{\mathrm{Sp}}$. The next lemma follows from Lusztig's characteristic zero correspondence for Sp_{2n} and takes considerable work to establish.

Lemma 8. *There is a bijection*

$$\bigsqcup_{\substack{n-m=\binom{k+1}{2} \\ 0\leq m\leq n}} Bipart(m) \leftrightarrow \bigsqcup_{\lambda\in Part_{Sp}(2n)} F_2[\Delta_\lambda]$$

Proof: The following diagram commutes.

$$\begin{array}{ccccc}
\displaystyle\bigsqcup_{\substack{n-m=\binom{k+1}{2}\\ 0\le m\le n}} \mathrm{Bipart}(m) & \longleftrightarrow & \displaystyle\bigsqcup_{\lambda\in \mathrm{Part}_{\mathrm{Sp}}(2n)} \mathrm{F}_2[\Delta_\lambda] & \longleftrightarrow & \Psi_{2n} \\
\updownarrow & & & & \Vert \\
 & & & & \displaystyle\bigsqcup_{d \text{ odd}} \Psi_{2n,d} \\
 & & & & \updownarrow \\
\displaystyle\bigsqcup_{\substack{j\ge 0\\ \frac{1}{2}j(j+1)\le n}} \mathrm{Irr}(\Bbbk[N_G(L_{n-\frac{1}{2}j(j+1)})/L_{n-\frac{1}{2}j(j+1)}]) & & \longleftrightarrow & & \displaystyle\bigsqcup_{d \text{ odd}} \Psi_{2n-d(d-1),1}
\end{array}$$

This follows from the following series of correspondences attributed to Lusztig [14]. Define $\tilde{\Psi}_N$, (N even), to be the set of ordered pairs $\binom{A}{B}$ where A is a finite subset of $\{0, 1, 2, \ldots\}$, B is a finite subset of $\{1, 2, \ldots\}$, with the following properties

1. For $i \in \mathbb{Z}$, $\{i, i+1\} \not\subset A, B$.
2. $|A| + |B|$ is odd, (Here $|\cdot|$ denotes cardinality).
3. $\sum_{a\in A} a + \sum_{b\in B} b = \frac{1}{2}[N + (|A| + |B|)(|A| + |B| - 1)]$.

Let Ψ_N be the set of equivalence classes on $\tilde{\Psi}_N$ with equivalence relations generated by

$$\begin{pmatrix} A \\ B \end{pmatrix} \sim \begin{pmatrix} \{0\} \bigcup (A+2) \\ \{1\} \bigcup (B+2) \end{pmatrix}. \tag{3.1}$$

Where context is clear, we shall again refer to equivalence classes for the pair (A, B) using the same notation, but will now call them *symbols*.

We shall say that two elements $\binom{A}{B)}$ and $\binom{A'}{B'}$ of Ψ_N are *similar* if they can be represented as a symbol with $A \bigcup B = A' \bigcup B'$, and $A \bigcap B = A' \bigcap B'$. To each similarity class there belongs a unique symbol $\binom{A}{B}$ we call *distinguished* with $A = \{a_1 < a_2 < \ldots < a_{m'}\}$, $B = \{b_1 < b_2 < \ldots < b_m$ having the following properties: $m' = m + 1$, $a_1 \le b_1 \le a_2 \le b_2 \le$

$\ldots \leq a_m \leq b_m\}$, and $b_m \leq a_{m+1}$.

Now we wish to associate a symbol to a partition $\lambda \in \mathrm{Sp}(2n)$. Begin with a partition $\lambda = (1\lambda^{(1)} + 2\lambda^{(2)} + 3\lambda^{(3)} + \ldots) \in \mathrm{Sp}(2n)$. Let $2m \geq \lambda^{(1)} + \lambda^{(2)} + \ldots$, and let $z_1 \leq z_2 \leq \ldots \leq z_{2m}$ be the sequence containing the number j precisely $\lambda^{(j)}$ times for each j, and the number 0 exactly $2m - (\lambda^{(1)} + \lambda^{(2)} + \ldots)$ times. We convert this sequence to a sequence of distinct integers $z'_1 < z'_2 < \ldots < z'_{2m}$ defined by $z'_i = z_i + (i-1)$. This sequence now contains precisely m even numbers $2y_1 < 2y_2 < \ldots < 2y_m$ and m odd numbers $2y'_1 + 1 < 2y'_2 + 1 < \ldots < 2y'_m + 1$.

Note that we now have a sequence

$$0 \leq y_1 + 1 \leq y'_1 + 2 \leq y_2 + 2 \leq y'_2 + 3 \leq \ldots \leq y_m + m \leq y'_m + (m+1).$$

We then collect an ordered pair,

$$A = \{0, y'_1 + 2, y'_2 + 3, \ldots, y'_m + (m+1)\}, \quad B = \{y_1 + 1, y_2 + 2, \ldots, y_m + m\}.$$

The resulting symbol is distinguished and its similarity class is stable when m is increased and, as such, depends only on the partition λ. The process can be reversed, hence we have a bijection

$$\Psi_N \leftrightarrow \mathrm{Part}(\mathrm{Sp}(2n)) \tag{3.2}$$

?? records the bijection for Sp(6).

We may now arrange the elements of a symbol's similarity class into an F_2-vectorspace. Begin with $\binom{A}{B}$ the distinguished symbol of the class. We may assume that $A \neq B$ so that

$C = (A \bigcup B) - (A \bigcap B)$ will be non-empty.

Define a non-empty subset I of C to be an *interval* if it is of the form $\{i, i+1, i+2, \ldots, i+k\}$ with $i-1, i+k+1 \notin C$ and $i \neq 0$. Let $\mathcal{I}$ denote the set of intervals of C. Note that $\mathcal{I}$ is non-empty. Define $\overline{I}$ to be the set of elements of C not belonging to any interval. Then we see that $\overline{I}$ is empty or $\{0, 1, 2, \ldots, h\}$ for some h.

For each subset $\alpha \subset \mathcal{I}$ let $\alpha' = \mathcal{I} - \alpha$ and set

$$A_\alpha = (\bigcup_{I \ni \alpha} (I \bigcap A)) \bigcup (\bigcup_{I \ni \alpha'} (I \bigcap B)) \bigcup (\overline{I} \bigcap A) \bigcup (A \bigcap B)$$

$$B_\alpha = (\bigcup_{I \ni \alpha} I \bigcap B)) \bigcap (\bigcup_{I \ni \alpha'} I \bigcap A) \bigcup (\overline{I} \bigcap B) \bigcup (A \bigcap B).$$

The resulting symbol (A_α, B_α) is in the similarity class of (A, B).

Lemma 9. *The map $\alpha \to \binom{A_\alpha}{B_\alpha}$ defines a bijection between $\mathcal{P}(\mathcal{I})$ and the set of elements in the similarity class of $\binom{A}{B}$. And with respect to the symmetric difference $\mathcal{P}(\mathcal{I})$ is an F_2-vector space with canonical basis the one-element subsets of $\mathcal{I}$.*

To wit, there is a bijection between the elements in the similarity class of $\binom{A}{B}$ and the F_2-vector space $F_2[\mathcal{I}]$. Our desired bijection, however, is $\bigsqcup_{\lambda \in \mathrm{Part}_{\mathrm{Sp}}(2n)} F_2[\Delta_\lambda] \leftrightarrow \Psi_n$ between symbols and the F_2 vector spaces with bases Δ_λ given by the distinct even parts of a given partition λ. We now give a bijection between the set Δ_λ and the set $\mathcal{I}_\lambda$ of intervals of the $C = (A \cup B) - (A \cap B)$ associated to the symbol $\binom{A}{B}$ obtained from 3.2.

Begin by arranging the intervals in $\mathcal{I}_\lambda$ in increasing order $I_1, I_2, \ldots, I_q$ wherein any element of I_i is smaller than any element of I_j provided $i < j$. We also arrange the elements of Δ_λ (recall these are distinct even integers) in increasing order, say $a_1 < a_2 < \ldots < a_{q'}$. Then we have that $q = q'$ and draw the correspondence I_h to a_h noting that I_h has length $\lambda^{(a_h)}$.

Thus, we have an isomorphism of vector spaces $F_2[\mathcal{I}_\lambda] \cong F_2[\Delta_\lambda]$. To recap, we have the first bijection of diagram 3.1

For the next bijection, define $d = |A| - |B|$ to be the integer associated to symbol $\binom{A}{B}$ called the *defect*. The defect d is independent of representative $\binom{A}{B}$ of the symbol, hence is well-defined and is always odd. In this way, we partition Ψ_{2n} according to symbols of a specific defect d we denote by $\Psi_{2n,\, d}$,

$$\Psi_{2n} = \bigsqcup_{d \text{ odd}} \Psi_{2n,\, d}.$$

We then convert symbols of defect d to symbols of defect 1; however, these new symbols will belong to $\Psi_{2n-d(d-1)}$. This will give us a bijection

$$\bigsqcup_{d \text{ odd}} \Psi_{2n,\, d} \leftrightarrow \bigsqcup_{d \text{ odd}} \Psi_{2n-d(d-1),1}$$

The conversion is as follows

$$\binom{A}{B} \mapsto \begin{cases} \binom{\{0,2,4,\ldots,2d-4\} \cup (A+2d-2)}{B} & \text{if } d \geq 1 \\ \\ \binom{A}{\{1,3,5,\ldots,1-2d\} \cup (B+2-2d)} & \text{if } d \leq -1. \end{cases}$$

Let $L_i = \mathrm{Gl}_1 \times \cdots \times \mathrm{Gl}_1 \times \mathrm{Sp}_{2n-2i}$, the Levi subgroups of $\mathrm{Sp}(2n)$. And let W_n denote the group of permutations of the set $\{1, 2, \ldots, n, n', \ldots, 2, 1\}$ commuting with the involution $j \leftrightarrow j'$. Then we have isomorphisms of groups $N(L_i)/L_i \cong (\mathbb{Z}/2\mathbb{Z})^i \wr S_i \cong W_i$ which we dub relative Weyl groups. The isomorphism is as follows.

Let V be a $2n$-dimensional vector space with non-singular symplectic form $(\cdot, \cdot)$ over $\Bbbk$ of odd-characteristic. Denote by $e_1, \ldots, e_n, e'_n, \ldots, e'_1$ a basis of V with $(e_i, e'_i) = 1 = -(e'_i, e_i)$ and zero otherwise. Assume without loss of generality that $L_i = \{g \in \mathrm{Sp}(2n) | g\cdot < e_i >=< e_i >, g\cdot < e'_i >=< e'_i > \ \ 1 \leq i \leq k, k \leq n\}$. Then elements of $N(L_i)/L_i$ correspond to permutations of the set of lines $< e_1 >, \ldots, < e_i >, < e'_i >, \ldots, < e'_1 >$ giving the desired isomorphism.

Now, letting $W_i^\vee = \mathrm{Irr}(\Bbbk[N_(L_i)/L_i]]$ denote the collection of isomorphism classes of irreducible representations of the group W_i, we describe the bijections $\Psi_{2n,1} \leftrightarrow W_n^\vee$. It is known that elements of $W_n^\vee$ can be parametrized by bipartitions of n. . That is, by partitions $\alpha = 0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{m'}$ and $\beta = 0 \leq \beta_1 \leq \beta_2 \leq \ldots \leq \beta_{m''}$ such that $\sum_i^{m'} \alpha_i + \sum_j^{m''} \beta_j = n$. Choose m' and m'' arbitrarily large, then by prepending zeros to the partitions as necessary, we can ensure $m' = m'' + 1$. Define $\binom{A}{B} \in \Psi_{2n,1}$ by

$$A = \{\alpha_1, \alpha_2 + 2, \alpha_3 + 4, \ldots, \alpha_{m'} + 2m' - 2\}$$

$$B = \{\beta_1 + 1, \beta_2 + 3, \beta_3 + 5, \ldots, \beta_{m''} + 2m'' - 1\}.$$

Our first deviation from the characteristic 0 case comes in the form of the following lemma which accounts for the new Levi subgroups in characteristic ℓ. By definition the set of bi-partitions of $p \in \mathbb{N}_{\geq 0}$ is simply the set of ordered pairs of partitions $\{(\rho_1, \rho_2) \mid \rho_2 \in \text{Part}(p - \sum \rho_1^{(i)})\}$, or equivalently the subset $\{(\rho_1, \rho_2) \in \text{Part}(p) \times \text{Part}(p) \mid \sum \rho_1^{(i)} + \sum \rho_2^{(i)} = p\}$. Furthermore, the condition that a bipartition is ℓ-regular, means each component is ℓ-regular.

We can therefore restrict our attention to componentwise cosideration of ordered pairs $(\text{Part}_\ell(r), \text{Part}_\ell(m-r))$. Denote by $(\mathbf{m}'(\nu), \mathbf{m}''(\nu))_\ell$ the image of $\text{Bipart}_\ell(\mathbf{m}(\nu))$ under this identification. Hence, $\mathbf{m}'(\nu) + \mathbf{m}''(\nu) = \mathbf{m}(\nu)$. Then:

$$\theta_\nu^{\text{co}} : (\mathbf{m}'(\nu), \mathbf{m}''(\nu))_\ell \to \text{Bipart}(m)$$

$$(\lambda_{\mathbf{1}}, \lambda_2) \mapsto (\sum_{i \geq 0} \ell^i (\lambda_1^{(\ell^i)})^t, \sum_{i \geq 0} \ell^i (\lambda_2^{(\ell^i)})^t).$$

Lemma 10. *The following map is a bijection*

$$\bigsqcup_{\nu \in Part(m,\ell)} \theta_\nu^{co} : \bigsqcup_{\nu \in Part(m,\ell)} Bipart_\ell(\boldsymbol{m}(\nu)) \to Bipart(m)$$

Then Lemma 10 is a consequence of the following lemma applied to each of the component partitions.

Lemma 11 ([3],3.9)**.** *The following map is a bijection:*

$$\Psi^{co} = \bigsqcup_{\nu \in Part(n,\ell)} \psi_\nu^{co} : \bigsqcup_{\nu \in Part(n,\ell)} \underline{Part}_\ell(\boldsymbol{m}(\nu)) \to Part(n),$$

where $\psi_v^{co} : \underline{Part}_\ell(\boldsymbol{m}(\nu)) \to Part(n)$ *is such that* $\lambda \mapsto \sum_{i \leq 0} \ell^i (\lambda^{(\ell^i)}) t$.

Combining lemmas 10 and 8, we establish the following combinatorial correspondence.

Theorem 3. *There is a bijection*

$$\Theta^{co} = \bigsqcup_{n-m=\binom{k+1}{2}} \bigsqcup_{\nu \in Part(m,\ell)} \theta^{co}_{\nu} \leftrightarrow \bigsqcup_{\lambda \in Part_{Sp}(2n)} F_2[\Delta_\lambda].$$

3.2. The Sheaf Theoretic Correspondence

3.2.1. Fourier-Sato Transform

Recall that we are considering sheaves whose coefficients come from $\mathbb{k}$, while varieties are defined over $\mathbb{C}$ which is endowed with the strong topology (i.e. that of a subspace of complex projective space). When letting a complex algebraic group H act on a variety X, we denote by $D^b_H(X, \mathbb{k})$ the constructible H-equivariant derived category which the reader may find in [6]. By $\mathrm{Perv}_H(X, \mathbb{k})$ we refer to the subcategory of $D^b_H(X, \mathbb{k})$ consisting of H-equivariant *perverse* $\mathbb{k}$-sheaves on X. We primarily concern ourselves with $X = \mathfrak{h}$ the Lie algebra of H or its nilpotent cone denoted $\mathcal{N}_H$. These are differentiated by letting $\mathcal{F} \in \mathrm{Perv}_H(\mathcal{N}_H, \mathbb{k})$ and denoting by $(a_H)_! \mathcal{F} \in \mathrm{Perv}_H(\mathfrak{h}, \mathbb{k})$ with $a_H : \mathcal{N}_H \hookrightarrow \mathfrak{h}$ the natural inclusion map.

We refer to [5] with adaptations to the task at hand. Let Y a topological space, and $p : E \to Y$ a (complex) vector bundle with a $\mathbb{C}^\times$-action defined on the fibers of p via homothety. Let E^* denote the dual vector bundle to E again viewed as a complex vector bundle; equivalently, let $\check{p} : E^* \to Y$ be the dual vector bundle to p. Then define $Q = E \times_Y E^*$ with $q : Q \to E$ and $\check{q} : Q \to E^*$ the projections onto each component. We consider specifically the case when $E, E^* = \mathfrak{g}$ where the dual $\mathfrak{g}^*$ is identified with $\mathfrak{g}$ as follows. Choose a fixed non-degenerate G-invariant symmetric bilinear form on the Lie algebra $\mathfrak{g}$ of G. Identify $\mathfrak{g}$ with its dual via $\langle \cdot, \mathfrak{g} \rangle : \mathfrak{g} \to \mathbb{C}$. Then, we define the Fourier-Sato Transform $\mathbb{T}_{\mathfrak{g}} = \check{q}_! \circ q^*[\dim(\mathfrak{g})]$, which is an equivalence of categories $\mathbb{T}_{\mathfrak{g}} : D^b_{\text{con}}(\mathfrak{g}, \Bbbk) \to D^b_{\text{con}}(\mathfrak{g}, \Bbbk)$. That is, $\mathbb{T}_{\mathfrak{g}}$ is an auto-equivalence of categories.

3.2.2. Induction and Restriction Functors

Let $L \subset P \subset G$ be a Levi factor and parabolic respectively. Denote by $\mathfrak{l}$ and $\mathfrak{p}$ their respective Lie algebras. It is possible to identify L with P/U_P via the composition $L \hookrightarrow P \twoheadrightarrow P/U_P$. Thus, with respect to the following diagram, we define three functors as in [[1],§2.4].

$$\mathfrak{l} \xleftarrow{q_{L\subset P}} \mathfrak{p} \xrightarrow{j_{L\subset P}} \mathfrak{g}$$

Define two restriction functors and an induction functor by

$$\mathbf{R}^G_{L\subset P} := (q_{L\subset P})_* \circ (j_{L\subset P})^! : D^B_G(\mathfrak{g}, \Bbbk) \to D^b_L(\mathfrak{l}, \Bbbk),$$

$$'\underline{\mathbf{R}}^G_{L\subset P} := (q_{L\subset P})_! \circ (j_{L\subset P})^* : D^b_G(\mathfrak{g}.\Bbbk) \to D^b_L(\mathfrak{l}, \Bbbk),$$

$$\underline{\mathbf{I}}^G_{L\subset P} := \gamma^G_P \circ (j_{L\subset P})_! \circ (q_{L\subset P})^* : D^b_L(\mathfrak{l}, \Bbbk) \to: D^b_G(\mathfrak{g}.\Bbbk)$$

The two restriction functors are then exchanged by Verdier duality, all are exact, and adjunctions are given by $'\mathbf{R}^G_{L\subset P} \dashv \mathbf{I}^G_{\text{L}\subset P} \dashv \mathbf{R}^G_{L\subset P}$.

3.2.3. Nilpotent Orbits

Note that nilpotent orbits $\mathcal{O}_\lambda \in \mathcal{N}_{\text{Sp}(n)}$ are labelled by $\lambda \in \text{Part}_{\text{Sp}}(n)$ such that the partition $(2n)$ corresponds to the principal nilpotent orbit and $(1, \ldots, 1)$ corresponds to the trivial orbit. For the Lie algebra $\mathfrak{sp}_n$ and $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_s$ its nilpotent orbits, we define a partial order $\preceq$ on these orbits such that $\mathcal{O}_i \preceq \mathcal{O}_j$ if and only if the closure of $\mathcal{O}_i$ is contained in the closure of $\mathcal{O}_j$.

For a Levi L let $\mathcal{O} \subset \mathcal{N}_L$ be a nilpotent orbit and let $\mathcal{E}$ be an L-equivariant local system on $\mathcal{O}$. Further, let $\mathfrak{z}_L$ denote the center of the lie algebra $\mathfrak{l}$ of L. Then, define $\mathfrak{z}_L^\circ := \{x \in \mathfrak{z}_L \mid G_x^\circ = L\}$. Recall that for $\mathcal{O}_L \subset \mathcal{N}_L$ an L-orbit, the G-orbit in $\mathcal{N}_{\mathrm{Sp}}(2n)$ induced by $\mathcal{O}$ is the unique dense G-orbit in $G \cdot (\overline{\mathcal{O}} + \mathfrak{u}_P)$, where $\mathfrak{u}_P$ denotes the nilpotent radical of $\mathfrak{p}$.

Let $Y_{(L,\mathcal{O})} := G \cdot (\mathcal{O} + \mathfrak{z}_L)$, these subsets are the strata in the Lusztig stratification for $\mathfrak{g}$. These are all locally closed smooth subvarieties of $\mathfrak{g}$. Let $X_{(L,\mathcal{O})} := G \cdot (\overline{\mathcal{O}} + \mathfrak{z}_L + \mathfrak{u}_p)$, then $X_{(L,\mathcal{O})} = \overline{Y_{(L,\mathcal{O})}}$ is the union of the strata.

We also define

$$\tilde{Y}_{(L.\mathcal{O})} := G \times^L (\mathcal{O} + \mathfrak{z}_L^\circ), \quad \tilde{X}_{(L,\mathcal{O})} := G \times^P (\overline{\mathcal{O}} + \mathfrak{z}_L + \mathfrak{u}_P).$$

and let $\varpi : \tilde{Y}_{L,\mathcal{O})} \to Y_{(L,\mathcal{O})}$ be the morphism induced by the adjoint G-action and $\pi_{(L,\mathcal{O})}$: $\tilde{X}_{(L,\mathcal{O})} \to X_{(L,\mathcal{O})}$ the restriction of $\pi_{L\subset P}$. We have a Cartesian square,

$$\begin{array}{ccc} \tilde{Y}_{(L,\mathcal{O})} & \hookrightarrow & \tilde{X}_{(L,\mathcal{O})} \\ {\scriptstyle \varpi_{(L,\mathcal{O})}}\downarrow & & \downarrow{\scriptstyle \pi_{(L,\mathcal{O})}} \\ Y_{(L,\mathcal{O})} & \hookrightarrow & X_{(L,\mathcal{O})} \end{array}$$

where the top-most horizontal map is induced from the natural map $G \times^L \mathfrak{l} \to G \times^P \mathfrak{p}$

Denote by $\tilde{\mathcal{E}}$ the unique G-equivariant local system on $\tilde{Y}_{(L,\mathcal{O})}$ whose pull-back to $G \times (\mathcal{O} + \mathfrak{z}_L^\circ)$ is precisely $\underline{\Bbbk}_G \boxtimes (\mathcal{E} \boxtimes \underline{\Bbbk}_{\mathfrak{z}_L^\circ})$. And let $N_G(L, \mathcal{O}) := \{n \in N_G(L) \mid n \cdot \mathcal{O} = \mathcal{O}\}$. A result by Letellier [12] asserts that the morphism $\varpi_{(L,\mathcal{O})}$ is a Galois covering with Galois group $N_G(L, \mathcal{O})/L$.

For any irreducible $\Bbbk[N_G(L,\mathcal{O})/L]-$module E there is a corresponding irreducible G-equivariant local system on $Y_{(L,\mathcal{O})}$ given by

$$\mathcal{L}_E := ((\overline{\omega}_{(L,\mathcal{O}*}\underline{\Bbbk} \otimes E)^{N_G(L,\mathcal{O})/L}$$

3.2.4. Cuspidal Pairs

We say that a simple object $\mathcal{F}$ in the abelian category $\mathrm{Perv}_G(\mathcal{N}, \Bbbk_\ell)$ is called **cuspidal** if for any proper parabolic $P \subsetneq G$ and Levi factor $L \subset P$ we have $\mathrm{Res}^G_{L\subset P}(\mathcal{F}) = 0$. A pair $(\mathcal{O}, \mathcal{E})$ labelling an object in $\mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk))$is called a **cuspidal pair** if $\mathcal{IC}(\mathcal{O}, \mathcal{E})$ is cuspidal. By abuse of notation, we will often write $(\mathcal{O}, \mathcal{E}) \in \mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk))^{\mathrm{cusp}}$ when referring to the cuspidal pair or the intersection cohomology complex with that label.

By [[1], Cor 2.7], every simple object of $\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk)$ appears as a quotient of $\mathbf{I}^G_{L\subset P}(\mathcal{IC}(\mathcal{O}_L, \mathcal{E})L))$ for some $L \subset P \subset G$ and $(\mathcal{O}_L, \mathcal{E}_L) \in \mathrm{Irr}(\mathrm{Perv}_L(\mathcal{N}_L, \Bbbk))^{\mathrm{cusp}}$. By [[1], Cor. 2.12], given any pair $(\mathcal{O}_L, \mathcal{E}_L) \in \mathrm{Irr}(\mathrm{Perv}_L(\mathcal{N}_L, \Bbbk))^{\mathrm{cusp}}$, there is a unique pair $(\mathcal{O}'_L, \mathcal{E}'_L) \in \mathrm{Irr}(\mathrm{Perv}_L(\mathcal{N}_L, \Bbbk))^{\mathrm{cusp}}$ with

$$\mathbb{T}_{\mathfrak{l}}(\mathcal{IC}(\mathcal{O}_L, \mathcal{E}_L)) \cong \mathcal{IC}(\mathcal{O}'_L + \mathfrak{z}_L, \mathcal{E}'_L \boxtimes \underline{\Bbbk}_{\mathfrak{z}_L}).$$

Then [[1], Cor. 2.18] states that the following canonical map is an isomorphism

$$\mathbb{T}_{\mathfrak{g}}(\mathbf{I}^G_{L\subset P}(\mathcal{IC}(\mathcal{O}_L, \mathcal{E}_L))) \cong \mathcal{IC}(Y_{(L,\mathcal{O}'_L)}, (\overline{\omega}_{(L,\mathcal{O}'_L)*}\tilde{\mathcal{E}}'_L).$$

Of note is that the argument on the left is independent of P up to canonical isomorphism. We term the collection of isomorphism classes of simple quotients of the perverse sheaf $\mathbf{I}^G_{L\subset P}(\mathcal{IC}(\mathcal{O}_L, \mathcal{E}_L))$ as the *induction series* for the triple $(L, \mathcal{O}_L, \mathcal{E}_L)$ which is preserved under conjugation by elements of G.

We say that a G-orbit $\mathcal{O} \subset \mathcal{N}_G$ is *distinguished* if it does not meet $\mathcal{N}_L$ for any proper Levi subgroup $L \leq G$. In particular, by [14], if $(\mathcal{O}, \mathcal{E}) \in \mathrm{Irr}(\mathrm{Perv}_G(\mathcal{N}_G, \Bbbk))^{\mathrm{cusp}}$, then $\mathcal{O}$ is distinguished. For $G = \mathrm{GL}(n)$, the only distinguished orbit is the regular orbit $\mathcal{O}_{(n)}$.

We recall some results of Lusztig presented with the characteristic zero correspondence

Lemma 12 ([14]. Thm 9.2(b),(d))**.** *Let $\Bbbk$ be a field of characteristic zero, and let $(\mathcal{O}, \mathcal{E})$ be a cuspidal pair such that $\mathcal{E}$ is absolutely irreducible. Then:*

1. *$N_G(L, \mathcal{O}) = N_G(L)$. Furthermore, the isomorphism class of $\mathcal{E}$ is preserved under the action of $N_G(L)/L$.*
2. *There is an isomorphism of $\Bbbk$-algebras $End((\overline{\omega}_{(L,\mathcal{O})*}\tilde{E}) \cong \Bbbk[N_G(L)/L]$ where we associate to $E \in Irr(\Bbbk[N_G(L)/L])$ the local system $\mathcal{L}_E \otimes \overline{\mathcal{E}}$.*

$$\left\{\begin{matrix} \textit{isomorphism classes of irreducible} \\ \textit{summands of } (\overline{\omega}_{(L,\mathcal{O})})_*\tilde{E} \end{matrix}\right\} \leftrightarrow Irr(\Bbbk[N_G(L)/L]).$$

3.2.5. Modular Reduction

Denote by $\mathbb{K}$ a finite extension of $\mathbb{Q}_\ell$, by $\mathbb{O}$ its ring of integers, and by $\mathbb{F}$ its residue field. Henceforth, the triple $(\mathbb{K}, \mathbb{O}, \mathbb{F})$ will be referred to a an *ℓ-modular system.* We make the necessary assumption that for any $x \in \mathcal{N}_G$, the irreducible representations of the (finite) component group $A_G(x)$ are defined over $\mathbb{K}$. Let $K_G(\mathcal{N}_G, \mathbb{E})$ denote the Grothendieck group for the category $D^b_G(\mathcal{N}_G, \mathbb{E})$ where $\mathbb{E}$ may be $\mathbb{K}$ or $\mathbb{F}$. This group forms a free $\mathbb{Z}$-module whose basis is given by $\{[\mathcal{F}]\}$ where the objects are isomorphism classes of simple objects $\mathcal{F} \in \mathrm{Perv}_G(\mathcal{N}_G, \mathbb{E})$.

In [[10],§2.9], Juteau describes a $\mathbb{Z}$-linear modular reduction

$$d : K_G(\mathcal{N}_G, \mathbb{K}) \to K_G(\mathcal{N}_G, \mathbb{F})$$

such that $d([\mathcal{F}]) = [\mathbb{F} \otimes^L_{\mathbb{O}} \mathcal{F}_{\mathbb{O}}]$ where $\mathcal{F}_{\mathbb{O}} \in \mathrm{Perv}_G(\mathcal{N}_G, \mathbb{O})$ is a torsion-free object with $\mathcal{F} \cong \mathbb{K} \otimes_{\mathbb{O}} \mathcal{F}_{\mathbb{O}}$. Given a simple object $\mathcal{G} \in \mathrm{Perv}_G(\mathcal{N}_G, \mathbb{F})$, when we say "$\mathcal{G}$ occurs in the modular reduction of $\mathcal{F}$ " if $[\mathcal{G}]$ appears with non-zero multiplicity in $d([\mathcal{F}])$.

Notably, when $\mathcal{F} = \mathcal{IC}(\mathcal{O}, \mathcal{E})$ for a given G-orbit $\mathcal{O} \subset N_G$ and irreducible G-equivariant local system $\mathcal{E}$ defined on $\mathcal{O}$, our assumptions on $\mathbb{K}$ are such that there exists a G-equivariant $\mathbb{O}$-free local system $\mathcal{E}_{\mathbb{O}}$ defined on $\mathcal{O}$ with $\mathcal{E} \cong \mathbb{K} \otimes_{\mathbb{O}} \mathcal{E}_{\mathbb{O}}$. Then, we can set $\mathcal{F}_{\mathbb{O}} = \mathcal{IC}(\mathcal{O}, \mathcal{E}_{\mathbb{O}})$ satisfying $d([\mathcal{F}]) = [\mathbb{F} \otimes^L_{\mathbb{O}} \mathcal{IC}(\mathcal{O}, \mathcal{E}_{\mathbb{O}})]$. To wit, if $\mathcal{E}'$ is any composition factor of the G-equivariant local system $\mathbb{F} \otimes^L_{\mathbb{O}} \mathcal{E}_{\mathbb{O}}$, then we are guaranteed that $\mathcal{IC}(\mathcal{O}, \mathcal{E}')$ occurs in the modular reduction of $\mathcal{F}$. We employ the following statement to classify those modular cuspidal pairs under consideration in this paper.

Lemma 13 ([1], Prop. 2.22)**.** *Let $\mathcal{G}$ be a simple object in $Perv_G(\mathcal{N}_G, \mathbb{F})$ occuring in the modular reduction of a cuspidal simple object $\mathcal{F}$ of $Perv_G(\mathcal{N}_G, \mathbb{K})$. Then $\mathcal{G}$ is cuspidal.*

3.2.6. Classification for GL(n)

AJHR provide the classification of the modular cuspidal pairs in GL(n) [1].

Theorem 4. *[1, Theorem 3.1] The group $GL(n)$ admits a cuspidal pair if and only if n is a power of ℓ, and it is unique, namely $(\mathcal{O}_{(n)}, \underline{\Bbbk})$*

To wit, $\mathfrak{L} := \{\mathbf{L}_\nu \mid \nu \in \mathrm{Part}(n, \ell)\}$ is exactly the set of conjugacy classes of Levi subgroups which admit a cuspidal pair and the unique cuspidal perverse sheaf for a choice of representative Levi $L \in \mathbf{L}_\nu$ is $\mathcal{IC}_{[\nu]}$.

Theorem 5. *[1, Theorem 3.3, 3.4] The following map is a bijection for $G = GL(n)$*

$$\Psi = \bigsqcup_{\mathbf{L}_\nu \in \mathfrak{L}} \psi_\nu : \bigsqcup_{\mathbf{L}_\nu \in \mathfrak{L}} Irr(\Bbbk[W_\nu]) \to Irr(Perv_G(\mathcal{N}_G, \Bbbk)), \tag{3.3}$$

where the map ψ_ν is given explicitly by

$$\psi_\nu(D^\lambda) = \mathcal{IC}_{\sum_{i\geq 0} \ell^i (\lambda^{(\ell^i)})^t} \tag{3.4}$$

for $\lambda = (\lambda^(1), \emptyset, \ldots, \emptyset, \lambda^{(\ell)}, \emptyset, \ldots, \emptyset, \lambda^{(\ell^2)}, \emptyset, \ldots) \in \underline{Part}_\ell(\mathbf{m}(\nu))$.

3.2.7. Classification for Sp$(2n)$

It is known that the set of G-conjugacy classes of Levi subgroups of G is in bijection with $\bigsqcup_{0\leq m\leq n}$ Part(m). For a given partition $\nu = (\nu_1, \nu_2, \cdots, \nu_s) \in$ Part(m) where $s = \text{length}(\nu)$ a corresponding Levi subgroup is of the form

$$L_\nu \cong \mathrm{GL}(\nu_1) \times \mathrm{GL}(\nu_2) \times \cdots \times \mathrm{GL}(\nu_s) \times \mathrm{Sp}(2(n-m)),$$

with the convention that Sp(0) is omitted when $n = m$.

In characteristic 2, the only irreducible L_ν-equivariant local system on any orbit in $\mathcal{N}_G$ is the trivial one. However, for characteristic $\ell \neq 2$, there are nontrivial local systems on the nilpotent orbits. Namely, for $x \in \mathcal{O}_\lambda$ with $\lambda \in \mathrm{Part}_{\mathrm{Sp}}(2n)$, the number of isomorphism classes of simple G-equivariant local systems on $\mathcal{O}_\lambda$ is $2^{|\{i|\mathbf{m}_{2i}(\lambda)\neq 0\}|}$, all of which are rank 1.

Define a partial (pre)-order on the set of pairs $(\mathcal{O}, \mathcal{L})$ by $(\mathcal{O}, \mathcal{L}) \leq^{\text{pre}}_{\text{geom}} (\mathcal{O}', \mathcal{L}')$ if $\mathcal{IC}(\mathcal{O}', \mathcal{L}'^{\mathbb{F}_\ell})$ occurs as a composition factor in the mod ℓ reduction of $\mathcal{IC}(\mathcal{O}, \mathcal{L}^{\mathbb{C}})$. Note that this is as yet not necessarily transitive, hence not a partial order, thus we define the partial order $\leq_{\text{geom}}$ to be the transitive closure of $\leq^{\text{pre}}_{\text{geom}}$.

3.3. The Characteristic 2 Correspondence for the Symplectic Group

For this section only, we will be considering the special case where $\ell = 2$ as detailed in [1]. We then are considering the correspondence for $G = \mathrm{Sp}_{2n}$ where $n \geq 2$. In the previous section, we noted that there were no non-trivial $L-$equivariant irreducible local systems on nilpotent orbits for L where L was a Levi subgroup of G. We saw in 3.2.4 that the Levis which supported cuspidal pairs were

$$L_\nu = \mathrm{GL}(\nu_1) \times \cdots \times \mathrm{GL}(\nu_m) \times \mathrm{Sp}_{2(n-k)}, \;\; 0 \leq k \leq n, \; \nu \in \mathrm{Part}(k, 2),$$

In fact, the following theorem shows this can be regarded as a full list of cuspidal pairs.

Theorem 6. *[4, Theorem 7.1] Every pair $(\mathcal{O}_\lambda, \underline{\Bbbk})$ for $\lambda \in Part_{2,Sp}(2n)$ is cuspidal, so the number of cuspidal pairs is $|Part_{2,Sp}(2n)| = |Part_2(n)|$.*

In summary, L_ν admits a unique cuspidal pair for each $\nu \in \mathrm{Part}(k, 2)$ and these all are of the form $(\mathcal{O}_{(\nu_1)} \times \cdots \times \mathcal{O}_{(\nu_m)} \times \mathcal{O}_{(\mu)}, \underline{\Bbbk})$ with μ running over all elements of $\mathrm{Part}_{2,\mathrm{Sp}}(2(n-k))$.

Let $P_\nu \subset \mathrm{Sp}_{2n}$ be a parabolic containing L_ν as its Levi factor, and let $W_\nu = N_{\mathrm{Sp}_{2n}}(L_\nu)/(L_\nu)$ be the relative Weyl group for L_ν. This relative Weyl group, as noted before, is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \wr \mathrm{Sp}_{\mathbf{m}(\nu)}$ and its irreducible representations (in the special case $\ell = 2$) are labelled by elements of $\underline{\mathrm{Part}}_2(\mathbf{m}(\nu))$.

Theorem 7. *[4, §7] The combinatorial bijection is then given by*

$$f : \bigsqcup_{0 \leq k \leq n} Part_{2,Sp}(2(n-k)) \times Part(k) \to Part_{Sp}(2n)$$

where $f(\mu, \lambda') = \mu \bigcup \lambda' \bigcup \lambda'$

3.4. The $\ell \neq 2$ Correspondence

By [[1], Theorem 7.2] for $\Bbbk$ a field of characteristic other than 2 there exists a unique cuspidal pair if $n = \binom{k+1}{2}$ and it is $(\mathcal{O}_{(2k,2(k-1),\dots,4,2)}, \Bbbk \otimes_{\mathbb{Z}_\ell} \mathcal{D}_k^{\mathbb{Z}_\ell})$, otherwise there is no cuspidal pair. Here, $\mathcal{D}_k^{\mathbb{Z}_\ell}$ is the $\mathbb{Z}_\ell$ form of the unique rank-one G-equivariant $\mathbb{Q}_\ell$-local system $\mathbb{Q}_k^{\mathbb{Q}_\ell}$ defined on the orbit $\mathcal{O}_{(2k,2(k-1),\dots,4,2)}$ as defined in [[14], Corollary 12.4(b)].

Thus those Levis occurring in a cuspidal pair are those whose GL factors form a cuspidal Levi for $\mathrm{GL}(\nu)$ and whose Sp factors are cuspidal for $\mathrm{Sp}(2n-2m)$.

Theorem 8. *The modular generalized Springer correspondence for* $G = Sp_{2n}(\Bbbk)$ *is given by*

$$\Theta^{co} = \bigsqcup_{\boldsymbol{L}_\nu \in \mathcal{L}} \theta^{co} : \bigsqcup_{\boldsymbol{L}_\nu \in \mathcal{L}} Irr(\Bbbk[W_\nu]) \to Irr((Perv_G(\mathcal{N}_G, \Bbbk)).$$

We now wish to show that this combinatorial correspondence of theorem 3 aligns with the statement of The Generalized Modular Springer Correspondence restated in theorem 8. Note that we can combinatorially identify the domains and codomains of the two correspondences Θ and Θ^{co}, thus we interpret Θ^{co} as a bijection

$$\Theta^{\mathrm{co}} = \bigsqcup_{n-m \in \binom{k+1}{2}} \bigsqcup_{\nu \in \mathrm{Part}(m,\ell)} \mathrm{Bipart}_\ell(\mathbf{m}(\nu)) \to \bigsqcup_{n-m \in \binom{k+1}{2}} \mathrm{Bipart}(m).$$

Proof. Consider the following maps

$$(L_{\mathbb{C}}, \mathcal{E}^{\mathbb{C}}) \xleftarrow{\Theta_0} (\mathcal{O}_L, \mathcal{L}) \xrightarrow{\Theta_\ell} (L_\ell, \mathcal{E}^{\mathbb{F}_\ell})$$

Recall that the objects under the characteristic zero generalized springer correspondence are labelled by Levi's of the form $L_{\mathbb{C}} = \mathrm{GL}(1)^m \times \mathrm{Sp}(2(n-m))$ and local systems $\mathcal{E}^{\mathbb{C}} \in \mathrm{Irr}(N_G(L)/L) \cong W_{n-r}$ which are enumerated by bipartitions of m. Yet the pairs under the mod ℓ generalized springer correspondence are labelled by Levi's $L_\ell = \mathrm{GL}(\ell^?) \times \mathrm{GL}(\ell^{??}) \times \cdots \times \mathrm{Sp}(m)$ and local system $\mathcal{E}^{\mathbb{F}_\ell} \in \mathrm{Irr}(N_G(L)/L)$ where representations are of groups $W_{r_0} \times W_{r_1} \times \ldots$ and labelled by a tuple of bipartitions.

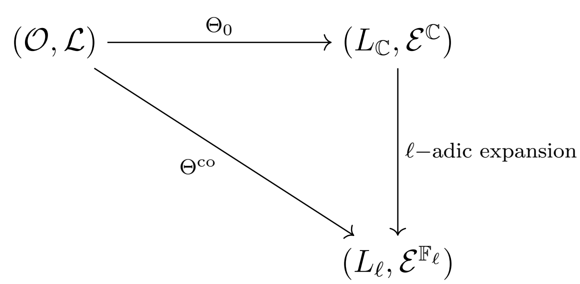
On the level of bipartitions, the correspondence is given by the commutativity of ℓ-adic expansion and recovery of these bipartitions, as described by Θ^{co}.

We have seen so far that Θ_ℓ, Θ_0, and Θ^{co} are all bijections with the same domains and codomains under this new interpretation. It therefore suffices to prove the following claim.

Claim: $\forall(\mathcal{O}, \mathcal{L})$,

$$\Theta_\ell^{-1}(\Theta^{\mathrm{co}}(\mathcal{O}, \mathcal{L})) \leq_{\mathrm{geom}} (\mathcal{O}, \mathcal{L})$$

Proof of claim: By construction, the following diagram commutes



When performing the modular reduction of a pair $(\mathcal{O}, \mathcal{L})$, we obtain several pairs $(\mathcal{O}', \mathcal{L}')$ as composition factors. We then consider the following not necessarily commutative diagram.

$$
\begin{array}{ccccc}
(\mathcal{O}_\lambda, \mathcal{L}^{\mathbb{C}}) & \longmapsto & & & \mathcal{E}^{\mathbb{C}} \in \operatorname{Irr}(W_{n-r}) \\
\downarrow & \mathcal{IC}(\mathcal{O}, \mathcal{L}^{\mathbb{C}}) & \xrightarrow{\Theta_0} & \operatorname{Irr}(W_{L_{\mathbb{C}}}) & \downarrow \\
\downarrow & \downarrow{\scriptstyle \text{mod } \ell \text{ redux}} & & \downarrow{\scriptstyle \text{mod } \ell \text{ redux}} & \downarrow \\
\downarrow & \mathcal{IC}(\mathcal{O}', \mathcal{L}'^{\mathbb{F}_\ell}) & \xrightarrow[\Theta_\ell]{} & \operatorname{Irr}(W_{L_{\mathbb{F}_\ell}}) & \downarrow \\
(\mathcal{O}_{\overline{\lambda}}, \mathcal{L}^{\mathbb{F}_\ell}) & \longmapsto & \mathcal{E}^{\mathbb{F}_\ell} \in \operatorname{Irr}(N_G(L_\nu)/L_\nu) = \operatorname{Irr}(W_{\binom{\lambda_1}{\mu_1}} \times W_{\binom{\lambda_2}{\mu_2}} \times \cdots \times W_{\binom{\lambda_s}{\mu_s}}) & & \ni \tilde{\mathcal{E}}^{\mathbb{F}_\ell} = \text{mod } \ell \text{ redux of } \mathcal{E}^{\mathbb{C}}|_{W_{\binom{\lambda_1}{\mu_1}} \times \cdots \times W_{\binom{\lambda_s}{\mu_s}}}
\end{array}
$$

Under Θ_ℓ, the pair $(\mathcal{O}', \mathcal{L}'^{\mathbb{F}_\ell})$ will correspond to a representation of a product of relative Weyls labelled by a tuple of bipartitions. Using the embedding of this product of Weyls into W_{n-r} coming from the characteristic zero correspondence, it is therefore enough to show that $\mathcal{E}^{\mathbb{F}_\ell} \in \bigotimes \mathcal{S}(V_i)$ occurs in the mod ℓ reduction of $\mathcal{E}^{\mathbb{C}} \in \mathcal{S}(V_n)$ when restricted. The content of chapter 2, particularly theorem 2 demonstrated that $\tilde{\mathcal{E}^{\mathbb{F}_\ell}} \subset \mathcal{E}^{\mathbb{C}}$, and so $\mathcal{E}^{\mathbb{F}_\ell}$ occurs in the modular reduction of $\mathcal{E}^{\mathbb{C}}$. Thus, while the diagram may not always commute, the pair $(L_\ell, \mathcal{E}^{\mathbb{F}_\ell})$ will occur as a constituent in the resultant compositions along both sides. We conclude that $\Theta_\ell^{-1}(\Theta^{\mathrm{co}}(\mathcal{O}, \mathcal{L})) \leq_{\mathrm{geom}} (\mathcal{O}, \mathcal{L})$ as desired. $\square$

Chapter 4. Explicit Calculations

4.1. Sp(6)

In this section we work out parts of the correspondence in excruciating detail. The reader is encouraged to refill their drink and get comfortable. We begin with the map which associates to a partition, a distinguished element of Ψ_6.

Figure 4.1. Sequences used in the construction of a distinguished pair

$\lambda \in \mathrm{Part}_{\mathrm{Sp}}(\mathbf{6})$	$(\mathbf{z_i})_{\mathbf{i=0}}^{\mathbf{6}}$	$(\mathbf{z'_i})_{\mathbf{i=0}}^{\mathbf{6}}$	$(\mathbf{y_i})$	$(\mathbf{y'_i})$
6	$(0,0,0,0,0,6)$	$(0,1,2,3,4,11)$	$(0,1,2)$	$(0,1,5)$
$2,4$	$(0,0,0,1,2,4)$	$(0,1,2,3,6,9)$	$(0,1,3)$	$(0,1,4)$
$1^2,4$	$(0,0,0,1,1,4)$	$(0,1,2,4,5,9)$	$(0,1,2)$	$(0,2,4)$
3^2	$(0,0,0,0,3,3)$	$(0,1,2,3,7,8)$	$(0,1,4)$	$(0,1,3)$
2^3	$(0,0,0,2,2,2)$	$(0,1,2,5,6,7)$	$(0,1,3)$	$(0,2,3)$
$1^2,2^2$	$(0,0,1,1,2,2)$	$(0,1,3,4,6,7)$	$(0,2,3)$	$(0,1,3)$
$1^4,2$	$(0,1,1,1,1,2)$	$(0,2,3,4,5,7)$	$(0,1,2)$	$(0,1,2,3)$
1^6	$(1,1,1,1,1,1)$	$(1,2,3,4,5,6)$	$(1,2,3)$	$(0,1,2)$

A'	**B'**	$(A,B) \in \Psi_6$	**C**	$\mathcal{I}_\lambda$
$(0,2,4,9)$	$(1,3,5)$	$((3),\emptyset)$	(3)	$\{3\}$
$(0,2,4,8)$	$(1,3,6)$	$((0,4),(2))$	$(0,2,4)$	$\{2\},\{4\}$
$(0,2,5,8)$	$(1,3,5)$	$((1,4),(1))$	(4)	$\{4\}$
$(0,2,4,7)$	$(1,3,7)$	$((0,3),(3))$	(0)	$\emptyset$
$(0,2,5,7)$	$(1,3,7)$	$((1,3),(2))$	$(1,2,3)$	$\{1,2,3\}$
$(0,2,4,7)$	$(1,4,6)$	$((0,2,5),(2,4))$	$(0,4,5)$	$\{4,5,\}$
$(0,2,4,7)$	$(1,3,5)$	$((1,3,5),(1,3))$	(5)	$\{5\}$
$(0,2,4,6)$	$(2,4,6)$	$((0,2,4,6),(2,4,6))$	(0)	$\emptyset$

Figure 4.2. Generating (A,B) for classes of pairs

Recall that each element of Ψ_n is a similarity class of a pair with similarity relation $(A,B) \sim (\{0\} \bigcup (A'+2), \{1\} \bigcup (B'+2))$ with a unique distinguished pair representing its class. For example, we reduce the pair $(A',B') = ((0,2,4,9),(1,3,5)) \sim ((0,2,7),(1,3)) \sim ((0,5),(1)) \sim ((3),(\emptyset))$. Then, to generate the remaining elements of Ψ_6, we permute the intervals between A and B. This is precisely the correspondence given by $\Psi_6 \leftrightarrow F_2[\mathcal{I}_\lambda]$.

For example, $\dim(F_2[\mathcal{I}_{(6)}]) = 1$, so consists of two elements, however we only listed the pair $(A, B) = ((3), \emptyset)$. We obtain the other element by sending the interval from A to B. Hence, when permuting the solitary interval $\{3\}$ in $((3), \emptyset)$, we obtain $(\emptyset, (3))$. Note that if an interval is contained in both A and B as is the case for $(A, B) = ((1, 3), (2))$ then we swap their containment between the two, e.g. $((2), (1, 3))$. Thus the bijection $\Psi_6 \leftrightarrow \bigsqcup_{\lambda \in \mathrm{Part}_{\mathrm{Sp}(6)}} F_2[\mathcal{I}_\lambda]$ is recorded in the following table

$\lambda \in \mathrm{Part}_{\mathrm{Sp}}(6)$	$(A, B) \in \Psi_6$	$\vec{v} \in F_2[\mathcal{I}_\lambda]$	$\vec{v} \in F_2[\boldsymbol{\Delta}_\lambda]$
6	$(\{3\}, \emptyset)$	$0\vec{v}_{\{3\}}$	$0\vec{v}_{[6]}$
	$(\emptyset, \{3\})$	$1\vec{v}_{\{3\}}$	$1\vec{v}_{[6]}$
$4+2$	$(\{0, 4\}, \{2\})$	$0\vec{v}_{\{2\}} + 0\vec{v}_{\{4\}}$	$0\vec{v}_{[2]} + 0\vec{v}_{[4]}$
	$(\{0, 2, 4\}, \emptyset)$	$1\vec{v}_{\{2\}} + 0\vec{v}_{\{4\}}$	$1\vec{v}_{[2]} + 0\vec{v}_{[4]}$
	$(\{0\}, \{2, 4\})$	$0\vec{v}_{\{2\}} + 1\vec{v}_{\{4\}}$	$0\vec{v}_{[2]} + 1\vec{v}_{[4]}$
	$(\{0, 2\}, \{4\})$	$1\vec{v}_{\{2\}} + 1\vec{v}_{\{4\}}$	$1\vec{v}_{[2]} + 1\vec{v}_{[4]}$
$4+1^2$	$(\{1, 4\}, \{1\})$	$0\vec{v'}_{\{4\}}$	$0\vec{v'}_{[4]}$
	$(\{1\}, \{1, 4\})$	$1\vec{v'}_{\{4\}}$	$1\vec{v'}_{[4]}$
3^2	$(\{0, 3\}, \{3\})$	$\vec{0}$	$\vec{0}$
2^3	$(\{1, 3\}, \{2\})$	$0\vec{v}_{\{1,2,3\}}$	$0\vec{v''}_{[2]}$
	$(\{2\}, \{1, 3\})$	$\vec{v}_{\{1,2,3\}}$	$1\vec{v''}_{[2]}$
2^2+1^2	$(\{0, 2, 5\}, \{2, 4\})$	$0\vec{v}_{\{4,5\}}$	$0\vec{v'''}_{[2]}$
	$(\{0, 2, 4\}, \{2, 5\})$	$1\vec{v}_{\{4,5\}}$	$1\vec{v'''}_{[2]}$
$2+1^4$	$(\{1, 3, 5\}, \{1, 3\})$	$0\vec{v}_{\{5\}}$	$0\vec{v^{(4)}}_{[2]}$
	$(\{1, 3\}, \{1, 3, 5\})$	$1\vec{v}_{\{5\}}$	$1\vec{v^{(4)}}_{[2]}$
1^6	$(\{0, 2, 4, 6\}, \{2, 4, 6\})$	$\vec{0}$	$\vec{0}$

Figure 4.3. The correspondence between symbols and elements of the F_2 vector spaces

Note that despite two objects from distinct partitions sharing the label $\vec{0}$, these vectors belong to disjoint vector spaces, and as such represent distinct objects.

Then using the following formula, we convert symbols of defect d into symbols of defect 1

$\Psi_{6,-1}$	$\Psi_{6,1}$	$\Psi_{6,3}$
	$\binom{3}{-}$	
$\binom{-}{3}$		
	$\binom{0,4}{2}$	
		$\binom{0,2,4}{-}$
$\binom{0}{2,4}$		
	$\binom{0,2}{4}$	
	$\binom{1,4}{1}$	
$\binom{1}{1,4}$		
	$\binom{0,3}{3}$	
	$\binom{1,3}{2}$	
$\binom{2}{1,3}$		
	$\binom{0,2,5}{2,4}$	
	$\binom{0,2,4}{2,5}$	
	$\binom{1,3,5}{1,3}$	
$\binom{1,3}{1,3,5}$		
	$\binom{0,2,4,6}{2,4,6}$	

Figure 4.4. Symbols and their defects

but for a different n, as illustrated in 4.1.

$$\binom{A}{B} \mapsto \begin{cases} \binom{\{0,2,4,\ldots,2d-4\}\cup(A+2d-2)}{B} & \text{if } d \geq 1 \\ \binom{A}{\{1,3,5,\ldots,1-2d\}\cup(B+2-2d)} & \text{if } d \leq -1. \end{cases}$$

Columns 2, 3, and 5 of 4.1 record the bijection given in 8. Columns 1 and 3 can be considered the characteristic 0 correspondence.

	$\Psi_{n',1}$		
$\Psi_{6,d}$	$\Psi_{6,1}$	$\Psi_{4,1}$	$\Psi_{0,1}$
$\binom{3}{-}$	$\binom{3}{-}$		
$\binom{-}{3}$		$\binom{0,2,4}{1,5}$	
$\binom{0,4}{2}$	$\binom{0,4}{2}$		
$\binom{0,2,4}{-}$			
$\binom{0}{2,4}$		$\binom{0,2,4}{2,4}$	
$\binom{0,2}{4}$	$\binom{0,2}{4}$		
$\binom{1,4}{1}$			$\binom{0}{-}$
$\binom{1}{1,4}$		$\binom{0,2,5}{1,4}$	
$\binom{0,3}{3}$	$\binom{0,3}{3}$		
$\binom{1,3}{2}$	$\binom{1,3}{2}$		
$\binom{2}{1,3}$		$\binom{0,2,6}{1,3}$	
$\binom{0,2,5}{2,4}$	$\binom{0,2,5}{2,4}$		
$\binom{0,2,4}{2,5}$	$\binom{0,2,4}{2,5}$		
$\binom{1,3,5}{1,3}$	$\binom{1,3,5}{1,3}$		
$\binom{1,3}{1,3,5}$		$\binom{0,3,5}{1,3}$	
$\binom{0,2,4,6}{2,4,6}$	$\binom{0,2,4,6}{2,4,6}$		

Figure 4.5. Converting symbols of defect d to symbols of defect 1

m	$\nu \in \mathrm{Part}(m,\ell)$ L_ν	$(\lambda^{(1)}, \lambda^{(3)} \in \mathbf{m}(\nu)$	$\mathrm{Bipart}_\ell(\mathbf{m}(\nu))$	$\mathrm{Bipart}(m)$
3	3 GL(3)	(0,1)	$\binom{1}{-}$	$\binom{3}{-}$
			$\binom{-}{1}$	$\binom{-}{3}$
	1^3 $\mathrm{GL}(1)^3$	(3,0)	$\binom{3}{-}$	$\binom{1,1,1}{-}$
			$\binom{-}{3}$	$\binom{-}{1,1,1}$
			$\binom{2}{1}$	$\binom{1,1}{1}$
			$\binom{1}{2}$	$\binom{1}{1,1}$
			$\binom{1,2}{-}$	$\binom{1,2}{-}$
			$\binom{-}{1,2}$	$\binom{-}{1,2}$
			$\binom{1,1}{1}$	$\binom{2}{1}$
			$\binom{1}{1,1}$	$\binom{1}{2}$
2	1^2 $\mathrm{GL}(2) \times \mathrm{Sp}(2)$	(2,0)	$\binom{2}{-}$	$\binom{1,1}{-}$
			$\binom{-}{2}$	$\binom{-}{1,1}$
			$\binom{1,1}{-}$	$\binom{2}{-}$
			$\binom{-}{1,1}$	$\binom{-}{2}$
			$\binom{1}{1}$	$\binom{1}{1}$
0	$\emptyset$ Sp(6)	(0,0)	$\emptyset$	$\emptyset$

Figure 4.6. The map Θ_ν

Table 4.1. Sp(6)

Orbit	π_1	Rep. π_1	Symbol	Bipart
6	$\mathbb{Z}_2$	1	$\binom{3}{-}$	$\binom{3}{-}$
		sgn	$\binom{-}{3}$	$\binom{-}{2}$
$4+2$	$\mathbb{Z}_2^2$	$1,1$	$\binom{0,4}{2}$	$\binom{2}{1}$
		1,sgn	$\binom{0}{2,4}$	$\binom{-}{1,1}$
		sgn,1	$\binom{0,2,4}{-}$	$\binom{-}{1,2}$
		sgn.sgn	$\binom{0,2}{4}$	$\binom{-}{3}$
$4+1^2$	$\mathbb{Z}_2$	1	$\binom{1,4}{1}$	$\binom{1.2}{-}$
		sgn	$\binom{1}{1,4}$	$\binom{1}{1}$
3^2	1	1	$\binom{0,3}{3}$	$\binom{1}{2}$
2^3	$\mathbb{Z}_2$	1	$\binom{1,3}{2}$	$\binom{1,1}{1}$
		sgn	$\binom{2}{1,3}$	$\binom{2}{-}$
2^2+1^2	$\mathbb{Z}_2$	1	$\binom{0,2,5}{2,4}$	$\binom{1}{1,1}$
		sgn	$\binom{0,2,4}{2,5}$	$\binom{-}{1,2}$
$2+1^4$	$\mathbb{Z}_2$	1	$\binom{1,3,5}{1,3}$	$\binom{1,1,1}{-}$
		sgn	$\binom{1,3}{1,3,5}$	$\binom{1,1}{-}$
1^2	1	1	$\binom{0,2,4,6}{2,4,6}$	$\binom{-}{1,1,1}$

$\binom{\lambda}{\mu} \in \text{Bipart}(6)$	$\binom{1,1,1}{-}$	$\binom{1,1}{1}$	$\binom{1}{1,1}$	$\binom{-}{1,1,1}$	$\binom{1,2}{-}$	$\binom{1}{2}$	$\binom{2}{1}$	$\binom{-}{1,2}$	$\binom{3}{-}$	$\binom{-}{3}$
$\binom{1,1,1}{-}$	1	1	1	1	−1	−1	−1	−1	1	1
$\binom{1,1}{1}$	3	1	−1	−3	−1	−1	1	1	–	–
$\binom{1}{1,1}$	3	−1	−1	3	−1	1	−1	1	–	–
$\binom{-}{1,1,1}$	1	−1	1	−1	−1	1	1	−1	1	−1
$\binom{1,2}{-}$	2	2	2	2	–	–	–	–	−1	−1
$\binom{1}{2}$	3	−1	−1	3	1	−1	1	−1	–	–
$\binom{2}{1}$	3	1	−1	−3	1	1	−1	−1	–	–
$\binom{-}{1,2}$	2	−2	2	−2	–	–	–	–	−1	1
$\binom{3}{-}$	1	1	1	1	1	1	1	1	1	1
$\binom{-}{3}$	1	−1	1	−1	1	−1	−1	1	1	−1

Figure 4.7. The Character Table of W_6

Table 4.2. The Characteristic 0 MGSC for Sp(8)

Orbit	π_1	Rep. π_1	Symbol
8	$\mathbb{Z}_2$	1	$\binom{-}{4}$
		sgn	$\binom{4}{-}$
$6+2$	$(\mathbb{Z}_2)^2$	1, 1	$\binom{0,5}{2}$
		1, sgn	$\binom{0,2}{5}$
		sgn,1	$\binom{0,2,5}{-}$
		sgn,sgn	$\binom{0,2}{5}$
$6+1^2$	$\mathbb{Z}_2$	1	$\binom{1,5}{1}$
		sgn	$\binom{1}{1,5}$
4^2	$\mathbb{Z}_2$	1	$\binom{0,4}{3}$
		sgn	$\binom{0,3}{4}$
$4+2^2$	$(\mathbb{Z}_2)^2$	1,1	$\binom{1,4}{2}$
		1,sgn	$\binom{1}{2,4}$
		sgn,1	$\binom{2,4}{1}$
		sgn,sgn	$\binom{2}{1,4}$
$4+2+1^2$	$(\mathbb{Z}_2)^2$	1,1	$\binom{0,2,6}{2,4}$
		1,sgn	$\binom{0,2}{2,4,6}$
		sgn,1	$\binom{0,2,4,6}{2}$
		sgn,sgn	$\binom{0,2,4}{2,6}$
$4+1^4$	$\mathbb{Z}_2$	1	$\binom{1,3,6}{1,3}$
		sgn	$\binom{1,3}{1,3,6}$
3^2+2	$\mathbb{Z}_2$	1	$\binom{1,3}{3}$
		sgn	$\binom{3}{1,3}$
3^2+1^2	1	1	$\binom{0,2,5}{2,5}$
2^4	$\mathbb{Z}_2$	1	$\binom{0,3,5}{2,4}$
		sgn	$\binom{0,2,4}{3,5}$
2^3+1^2	$\mathbb{Z}_2$	1	$\binom{1,3,5}{1,4}$
		sgn	$\binom{1,4}{1,3,5}$
2^2+1^4	$\mathbb{Z}_2$	1	$\binom{2,4,7}{2,4,6}$
		sgn	$\binom{0,2,4,6}{2,4,7}$
$2+1^6$	$\mathbb{Z}_2$	1	$\binom{1,3,5,7}{1,3,5}$
		sgn	$\binom{1,3,5}{1,3,5,7}$
1^8	1	1	$\binom{0,2,4,6,8}{2,4,6,8}$

λ	$\{z_i\}$	$\{z'_i\}$	$\{y'_i\}$	$\{y_i\}$	A	B
12	$0,0,0,0,0,0,0,0,0,0,0,12$	$0,1,2,3,4,5,6,7,8,9,10,23$	$0,1,2,3,4,11$	$0,1,2,3,4,5$	$\{0,2,4,6,8,10,18\}$	$\{1,3,5,7,9,11\}$
$10+2$	$0,0,0,0,0,0,0,0,0,0,2,10$	$0,1,2,3,4,5,6,7,8,9,12,21$	$0,1,2,3,4,10$	$0,1,2,3,4,6$	$\{0,2,4,6,8,10,17\}$	$\{1,3,5,7,10\}$
$10+1^2$	$0,0,0,0,0,0,0,0,0,1,1,10$	$0,1,2,3,4,5,6,7,8,10,11,21$	$0,1,2,3,5,10$	$0,1,2,3,4,5$	$\{0,2,4,6,8,11,17\}$	$\{1,3,5,7,9,11\}$
$8+2^2$	$0,0,0,0,0,0,0,0,0,2,2,8$	$0,1,2,3,4,5,6,7,8,11,12,19$	$0,1,2,3,5,9$	$0,1,2,3,4,6$	$\{0,2,4,6,8,11,16\}$	$\{1,3,5,7.9.12\}$
$8+2+1^2$	$0,0,0,0,0,0,0,0,1,1,2,8$	$0,1,2,3,4,5,6,7,9,10,12,19$	$0,1,2,3,4,9$	$0,1,2,3,5,6$	$\{0,2,4,6,8,10,16\}$	$\{1,3,5,7,10,12\}$
$8+1^4$	$0,0,0,0,0,0,0,1,1,1,1,8$	$0,1,2,3,4,5,6,8,9,10,11,19$	$0,1,2,4,5,8$	$0,1,2,3,4,5$	$\{0,2,4,6,9,11,15\}$	$\{1,3,5,7,9,11\}$
6^2	$0,0,0,0,0,0,0,0,0,0,6,6$	$0,1,2,3,4,5,6,7,8,9,16,17$	$0,1,2,3,4,8$	$0,1,2,3,4,8$	$\{0,2,4,6,8,10,15\}$	$\{1,3,5,7,9,14\}$
$6+4+2$	$0,0,0,0,0,0,0,0,0,2,4,6$	$0,1,2,3,4,5,6,7,8,11,14,17$	$0,1,2,3,5,8$	$0,1,2,3,4,7$	$\{0,2,4,6,8,11,15\}$	$\{1,3,5,7,9,13\}$
$6+4+1^2$	$0,0,0,0,0,0,0,0,1,1,4,6$	$0,1,2,3,4,5,6,7,9,10,14,17$	$0,1,2,3,4,8$	$0,1,2,3,5,7$	$\{0,2,4,6,8,10,15\}$	$\{1,3,5,7,10,13\}$
$6+3^2$	$0,0,0,0,0,0,0,0,0,3,3,6$	$0,1,2,3,4,5,6,7,8,12,13,17$	$0,1,2,3,6,8$	$0,1,2,3,4,6$	$\{0,2,4,6,8,12,15\}$	$\{1,3,5,7,9,12\}$
$6+2^3$	$0,0,0,0,0,0,0,0,2,2,2,6$	$0,1,2,3,4,5,6,7,10,11,12,17$	$0,1,2,3,5,8$	$0,1,2,3,5,6$	$\{0,2,4,6,8,11,15\}$	$\{1,3,5,7,10,12\}$
$6+2^2+1^2$	$0,0,0,0,0,0,0,1,1,2,2,6$	$0,1,2,3,4,5,6,8,9,11,12,17$	$0,1,2,4,5,8$	$0,1,2,3,4,6$	$\{0,2,4,6,9,11,15\}$	$\{1,3,5,7,9,12\}$
$6+2+1^4$	$0,0,0,0,0,0,1,1,1,1,2,6$	$0,1,2,3,4,5,7,8,9,10,12,17$	$0,1,2,3,4,8$	$0,1,2,4,5,6$	$\{0,2,4,6,8,10,15\}$	$\{1,3,5,8,10,12\}$
$6+1^6$	$0,0,0,0,0,1,1,1,1,1,1,6$	$0,1,2,3,4,6,7,8,9,10,11,17$	$0,1,3,4,5,8$	$0,1,2,3,4,5$	$\{0,2,4,7,9,11,15\}$	$\{1,3,5,7,9,11\}$
5^2+2	$0,0,0,0,0,0,0,0,0,2,5,5$	$0,1,2,3,4,5,6,7,8,11,15,16$	$0,1,2,3,5,7$	$0,1,2,3,4,8$	$\{0,2,4,6,8,11,14\}$	$\{1,3,5,7,9,14\}$
5^2+1^2	$0,0,0,0,0,0,0,0,1,1,5,5$	$0,1,2,3,4,5,6,7,9,10,15,16$	$0,1,2,3,4,7$	$0,1,2,3,5,8$	$\{0,2,4,6,8,10,14\}$	$\{1,3,5,7,9,13\}$
4^3	$0,0,0,0,0,0,0,0,0,4,4,4$	$0,1,2,3,4,5,6,7,8,13,14,15$	$0,1,2,3,6,7$	$0,1,2,3,4,7$	$\{0,2,4,6,8,12,14\}$	$\{1,3,5,7,9,13\}$
$4+3^2+2$	$0,0,0,0,0,0,0,0,2,3,3,4$	$0,1,2,3,4,5,6,7,10,12,13,15$	$0,1,2,3,6,7$	$0,1,2,3,5,6$	$\{0,2,4,6,8,12,14\}$	$\{1,3,5,7,11,13\}$
$4+3^2+1^2$	$0,0,0,0,0,0,0,1,1,3,3,4$	$0,1,2,3,4,5,6,8,9,12,13,15$	$0,1,2,4,6,7$	$0,1,2,3,4,6$	$\{0,2,4,6,9,12,14\}$	$\{1,3,5,7,9,12\}$
4^2+2^2	$0,0,0,0,0,0,0,0,2,2,4,4$	$0,1,2,3,4,5,6,7,10,11,14,15$	$0,1,2,3,5,7$	$0,1,2,3,5,7$	$\{0,2,4,6,8,11,14\}$	$\{1,3,5,7,10,13\}$
4^2+2+1^2	$0,0,0,0,0,0,0,1,1,2,4,4$	$0,1,2,3,4,5,6,8,9,11,14,15$	$0,1,2,4,5,7$	$0,1,2,3,4,7$	$0,2,4,6,9,11,14\}$	$\{1,3,5,7,9,13\}$
4^2+1^4	$0,0,0,0,0,0,1,1,1,1,4,4$	$0,1,2,3,4,5,7,8,9,10,14,15$	$0,1,2,3,4,7$	$0,1,2,4,5,7$	$\{0,2,4,6,8,10,17\}$	$\{1,3,5,8,10,13\}$
3^4	$0,0,0,0,0,0,0,0,3,3,3,3$	$0,1,2,3,4,5,6,7,11,12,13,14$	$0,1,2,3,5,6$	$0,1,2,3,6,7$	$\{0,2,4,6,8,11,13\}$	$\{1,3,5,7,11,13\}$
3^2+2^3	$0,0,0,0,0,0,0,2,2,2,3,3$	$0,1,2,3,4,5,6,9,10,11,13,14$	$0,1,2,4,5,6$	$0,1,2,3,5,7$	$\{0,2,4,6,9,11,13\}$	$\{1,3,5,7,10,16\}$
$3^2+2^2+1^2$	$0,0,0,0,0,0,1,1,2,2,3,3$	$0,1,2,3,4,5,7,8,10,11,13,14$	$0,1,2,3,5,6$	$0,1,2,4,5,7$	$\{0,2,4,6,8,11,13\}$	$\{1,3,5,8,10,13\}$
3^2+2+1^4	$0,0,0,0,0,1,1,1,1,2,3,3$	$0,1,2,3,4,6,7,8,9,11,13,14$	$0,1,3,4,5,6$	$0,1,2,3,4,7$	$\{0,2,4,7,9,11,13\}$	$\{1,3,5,7,9,13\}$
3^2+1^6	$0,0,0,0,1,1,1,1,1,1,3,3$	$0,1,2,3,5,6,7,8,9,10,13,14$	$0,1,2,3,4,6$	$0,1,3,4,5,7$	$\{0,2,4,6,8,10,13\}$	$\{1,3,6,8,10,13\}$
2^6	$0,0,0,0,0,0,2,2,2,2,2,2$	$0,1,2,3,4,5,8,9,10,11,12,13$	$0,1,2,4,5,6$	$0,1,2,4,5,6$	$\{0,2,4,6,9,11,13\}$	$\{1,3,5,8,10,12\}$
2^5+1^2	$0,0,0,0,0,1,1,2,2,2,2,2$	$0,1,2,3,4,6,7,9,10,11,12,13$	$0,1,3,4,5,6$	$0,1,2,3,5,6$	$\{0,2,4,7,9,11,13\}$	$\{1,3,5,7,10,12\}$
2^4+1^4	$0,0,0,0,1,1,1,1,2,2,2,2$	$0,1,2,3,5,6,7,8,10,11,12,13$	$0,1,2,3,5,6$	$0,1,3,4,5,6$	$\{0,2,4,6,8,11,13\}$	$\{1,3,6,8,10,12\}$
2^3+1^6	$0,0,0,1,1,1,1,1,1,2,2,2$	$0,1,2,4,5,6,7,8,9,11,12,13$	$0,2,3,4,5,6$	$0,1,2,3,4,6$	$\{0,2,5,7,9,11,13\}$	$\{1,3,5,7,9,12\}$
2^2+1^8	$0,0,1,1,1,1,1,1,1,1,2,2$	$0,1,3,4,5,6,7,8,9,10,12,13$	$0,1,2,3,4,6$	$0,2,3,4,5,6$	$\{0,2,4,6,8,10,13\}$	$\{1,4,6,8,10,12\}$
$2+1^{10}$	$0,1,1,1,1,1,1,1,1,1,1,2$	$0,2,3,4,5,6,7,8,9,10,11,13$	$1,2,3,4,5,6$	$0,1,2,3,4,5$	$\{0,3,5,7,9,11,13\}$	$\{1,3,5,7,9,11\}$
1^{12}	$1,1,1,1,1,1,1,1,1,1,1,1$	$1,2,3,4,5,6,7,8,9,10,11,12$	$0,1,2,3,4,5$	$1,2,3,4,5,6$	$\{0,2,4,6,8,10,12\}$	$\{2,4,6,8,10,12\}$

Table 4.3. Pairs (A,B) for Sp(12)

C	(A, B)
$\{0,1,2,3,4,5,6,7,8,9,10,11,18\}$	$(\{6\},\emptyset)$,$(\emptyset,\{6\})$
$\{0,1,2,3,4,5,6,7,10,12,17\}$	$(\{0,7\},2),(\{0,2\},\{7\}),(\{0,2,7\},\emptyset),(\{0\},\{2,7\})$
$\{0,1,2,3,4,5,6,7,8,9,17\}$	$((\{1,7\},\{1\}),(\{1\},\{1,7\})$
$\{0,1,2,3,4,5,6,7,8,9,11,12,16\}$	$(\{0,3,8\},\{2,4\}),(\{0,3\},(\{2,4,8\}),(\{0,2,4\},\{3,8\})\ (\{0,2,4,8\},\{3\})$
$\{0,1,2,3,4,5,6,7,8,12,16\}$	$(\{0,2,8\},\{2,4\}),(\{0,2,\},\{2,4,8\}),(\{0,2,4\},\{2,8\})\ (\{0,2,4,8\},\{2\})$
$\{0,1,2,3,4,5,6,7,15\}$	$(\{1,3,7\},\{1,3\}),(\{1,3\},\{1,3,7\})$
$\{0,1,2,3,4,5,6,7,8,9,10,14,15\}$	$(\{0,5\},\{4\}),(\{0,4\},\{5\})$
$\{0,1,2,3,4,5,6,7,8,9,11,13,15\}$	$(\{1,5\},\{3\}),(\{1,3,5\},\emptyset),(\emptyset,\{1,3,5\})$,$(\{1\},\{3,5\}),(\{3\},\{1,5\}),(\{5\},\{1,3\}),(\{1,3\},\{5\})$,$(\{3,5\},\{1\})$
$\{0,1,2,3,4,5,6,7,8,13,15\}$	$(\{0,2,7\},\{2,5\}),(\{0,2,5,7\},\{2\}),(\{0,2,5\},\{2,7\})$,$(\{0,2\},\{2,5,7\})$
$\{0,1,2,3,4,5,6,7,8,9,15\}$	$(\{2,5\},\{2\}),(\{2\},\{2,5\})$
$\{0,1,2,3,4,5,6,7,8,10,11,12,15\}$	$(\{0,3,7\},\{2,4\}),(\{0,3\},\{2,4,7\}),(\{0,2,4\},\{3,7\})$,$(\{0,2,4,7\},\{3\})$
$\{0,1,2,3,4,5,6,7,11,12,15\}$	$(\{1,3,7\},\{1,4\}),(\{1,4,7\},\{1,3\}),(\{1,3\},\{1,4,7\}),\ (\{1,4\},\{1,3,7\})$
$\{0,1,2,3,4,5,6,12,15\}$	$(\{1,3,5,9\},\{1,3,5\}),(\{1,3,5\},\{1,3,5,9\})$
$\{0,1,2,3,4,5,15\}$	$(\{1,4\},\{4\}),(\{4\},\{1,4\})$
$\{0,1,2,3,4,5,6,7,8,9,11\}$	$(\{0,4\},\{3\}),(\{0,3\},\{4\})$
$\{0,1,2,3,4,5,6,7,8,9,10,13,14\}$	$(\{2,4\},\{3\}),(\{3\},\{2,4\})$
$\{0,1,2,3,4,5,6,7,8,9,12,13,14\}$	$(\{0,2\},\{1\}),(\{0,1\},\{2\})$
$\{0,1,2,3,4,5,6,7,8,11,12,13,14\}$	$(\{1,4,6\},\{1,4\}),(\{1,4\},\{1,4,6\})$
$\{0,1,2,3,4,5,6,7,14\}$	$(\{0,3,6\},\{2,5\}),(\{0,3,5\},\{2,6\}),(\{0,2,5\},\{3,6\}),(\{0,2,6\},\{3,5\})$
$\{0,1,2,3,4,5,6,7,8,10,11,13,14\}$	$(\{1,3,6\},\{1,5\}),(\{1,3,5\},\{1,6\}),(\{1,5\},\{1,3,6\}),(\{1,6\},\{1,3,5\})$
$\{0,1,2,3,4,5,6,711,13,14\}$	$(\{0,2,4,11\},\{2,4,7\}),(\{0,2,4\},\{2,4,7,11\}),((\{0,2,4,7\},\{2,4,11\}),(\{0,2,4,7,11\},\{2,4\})$
$\{0,1,2,3,4,5,6,13,17\}$	$(\{1,3\},\{1,3\})$
$\{0,1,2,3,4,5,6,7,8\}$	$(\{1,3,5\},\{2,8\}),(\{2,8\},\{1,3,5\}),(\{1,2,5\},\{3,8\}),(\{3,8\},\{1,2,5\}),(\{1,3\},\{2,5,8\}),(\{2,5,8\},\{1,3\})$
$\{0,1,2,3,4,5,6,7,9,10,11,13,16\}$	$(\{0,2,5,7\},\{2,4,7\}),(\{0,2,4,7\},\{0,2,4,7\},\{2,5,7\})$
$\{0,1,2,3,4,5,10,11\}$	$(\{1,3,5,7\},\{1,3,7\}),(\{1,3,7\},\{1,3,5,7\})$
$\{0,1,2,3,4,5,11\}$	$(\{0,2,4,6,9\},\{2,4,6,9\})$
$\{0,1,2,3,4\}$	$(\{0,3,5,7\},\{2,4,6\}),(\{0,2,4,6\},\{3,5,7\})$
$\{0,1,2,3,4,5,6,8,910,11,12,13\}$	$(\{1,3,5,7\},\{1,4,6\}),(\{1,4,6\},\{1,3,5,7\})$
$\{0,1,2,3,4,5,9,10,11,12,13\}$	$(\{0,2,4,7,9\},\{2,4,6,8\}),(\{0,2,4,6,8\},\{2,4,7,9\})$
$\{0,1,2,3,4,11,12,13\}$	$(\{1,3,5,7,9\},\{1,3,5,8\}),(\{1,3,5,8\},\{1,3,5,7,9\})$
$\{0,1,2,3,11,12,13\}$	$(\{0,2,4,7,9\},\{2,4,6,8\}),(\{0,2,4,6,8\},\{2,4,7,9\})$
$\{0,1,2,12,13\}$	$(\{0,3,5,7,9\},\{1,3,5,8\}),(\{1,3,5,8\},\{1,3,5,7,9\})$
$\{0,1,13\}$	$(\{1,3,5,7,9,11\},\{1,3,5,7,9\}),(\{1,3,5,7,9\},\{1,3,5,7,9,11\})$
$\{0\}$	$(\{0,2,4,6,8,10,12\},\{2,4,6,8,10,12\})$

List of References

[1] Pramod Achar, Anthony Henderson, Daniel Juteau, and Simon Riche. "Modular generalized Springer correspondence II: classical groups". In: *Journal of the European Mathematical Society* 19.4 (2017), pp. 1013–1070. DOI: 10.4171/jems/687. URL: https://doi.org/10.4171%2Fjems%2F687 (cit. on pp. 2, 33, 35, 37–40).

[2] Pramod N. Achar. *Perverse sheaves and applications to representation theory*. American Mathematical Society, 2021.

[3] Pramod N. Achar, Anthony Henderson, Daniel Juteau, and Simon Riche. *Modular generalized Springer correspondence I: the general linear group*. 2014. arXiv: 1307.2702 [math.RT] (cit. on pp. 2, 31).

[4] Pramod N. Achar, Anthony Henderson, Daniel Juteau, and Simon Riche. "Modular generalized Springer correspondence III: exceptional groups". In: *Mathematische Annalen* 369.1-2 (Feb. 2017), pp. 247–300. DOI: 10.1007/s00208-017-1524-4. URL: https://doi.org/10.1007%2Fs00208-017-1524-4 (cit. on pp. vii, 2, 39).

[5] Pramod N. Achar, Anthony Henderson, Daniel Juteau, and Simon Riche. "Weyl group actions on the Springer Sheaf". In: *Proceedings of the London Mathematical Society* 108.6 (Dec. 2013), pp. 1501–1528. DOI: 10.1112/plms/pdt055 (cit. on p. 32).

[6] Joseph Bernstein and Valery Lunts. *Equivariant sheaves and functors*. Springer Berlin Heidelberg, 1994 (cit. on p. 32).

[7] David H. Collingwood and William M. McGovern. *Nilpotent orbits in semisimple lie algebras*. Van Nostrand Reinhold, 1993 (cit. on p. 1).

[8] Joseph Dorta, Samantha Jarvis, and Nelson Niu. "Monoidal structures on generalized polynomial categories". In: *Electronic Proceedings in Theoretical Computer Science* 397 (Dec. 2023), pp. 84–97. DOI: 10.4204/eptcs.397.6 (cit. on p. 55).

[9] I. Martin Isaacs and I. M. Isaacs. "Normal Subgroups". In: *Character theory of finite groups*. Academic Press, 1976, p. 80 (cit. on p. 8).

[10] Daniel Juteau. "Decomposition numbers for perverse sheaves". In: *Annales de l'institut Fourier* 59.3 (2009), pp. 1177–1229. DOI: 10.5802/aif.2461 (cit. on p. 36).

[11] Daniel Juteau. *Modular Springer correspondence, decomposition matrices and basic sets*. 2014. arXiv: 1410.1471 [math.RT] (cit. on p. 2).

[12] Emmanuel Letellier. “Fourier transforms of invariant functions on finite reductive lie algebras”. In: *Lecture Notes in Mathematics* (2005). DOI: 10.1007/b104209 (cit. on p. 34).

[13] G. Lusztig. “A class of irreducible representations of a Weyl group”. In: *Indagationes Mathematicae (Proceedings)* 82.1 (1979), pp. 323–335. DOI: 10.1016/s1385-7258(79)80026-8 (cit. on p. 10).

[14] G. Lusztig. “Intersection cohomology complexes on a reductive group”. In: *Inventiones Mathematicae* 75.2 (June 1984), pp. 205–272. DOI: 10.1007/bf01388564 (cit. on pp. 26, 36, 40).

[15] G. Lusztig and N. Spaltenstein. “On the generalized Springer correspondence for classical groups”. In: *Advanced Studies in Pure Mathematics, Algebraic Groups and Related Topics* 6 (1985), pp. 289–316. DOI: 10.2969/aspm/00610289 (cit. on p. 2).

[16] G. Lusztig and N. Spaltenstein. “On the generalized Springer correspondence for classical groups”. In: *Advanced Studies in Pure Mathematics* (). DOI: 10.2969/aspm/00610289.

[17] Gerhard J.A. Schneider. “Dixon’s character table algorithm revisited”. In: *Journal of Symbolic Computation* 9.5–6 (1990), pp. 601–606. DOI: 10.1016/s0747-7171(08)80077-6.

[18] N. Spaltenstein. “On the generalized springer correspondence for exceptional groups”. In: *Advanced Studies in Pure Mathematics, Algebraic Groups and Related Topics* 6 (1985), pp. 317–338. DOI: 10.2969/aspm/00610317 (cit. on p. 2).

[19] T. A. Springer. “A construction of representations of Weyl groups”. In: *Inventiones Mathematicae* 44.3 (1978), pp. 279–293. DOI: 10.1007/bf01403165 (cit. on p. 1).

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