The Lowest Discriminant Ideal of Cayley-Hamilton Hopf Algebras

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THE LOWEST DISCRIMINANT IDEAL OF CAYLEY-HAMILTON HOPF ALGEBRAS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
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Abstract

Discriminant ideals are defined for an algebra $R$ with central subalgebra $C$ and trace $\text{tr} : R \to C$. They are indexed by positive integers and more general than discriminants. Usually $R$ is required to be a finite module over $C$. Unlike the abundance of work on discriminants, there is hardly any literature on discriminant ideals. The levels of discriminant ideals relate to the sums of squares of dimensions of irreducible modules over maximal ideals of $C$ containing these discriminant ideals. We study the lowest level when $R$ is a Cayley-Hamilton Hopf algebra, i.e. $C$ is also a Hopf subalgebra, and all irreducible representations over the kernel of the counit of $C$ are one-dimensional. The level is determined by considering the actions of these one-dimensional modules through tensor product and then we look into the orbits of elements in the zero set of the lowest discriminant ideal under winding automorphisms, which fix $C$. We then apply the results to the examples of the group algebra of certain central extensions of finitely generated Abelian groups by finite Abelian groups and the quantum Borel subalgebra at roots of unity.
Chapter 1. Introduction

The discriminant and discriminant ideals (Definition 2.3.6) have been studies in noncommutative algebra for an algebra $R$ that is a finite module over a central subalgebra $C$ with a trace $\text{tr} : R \to C$ (Definition 2.3.1). Discriminants have been intensively studied lately. One direction is the computation of discriminants and using different techniques such as smash products [19], Poisson geometry [24][32], cluster algebra [33] and reflexive hulls [10]. Another direction is using discriminants to study algebraic structures such as the automorphism groups of noncommutative algebras [8][9][10] and the Zariski cancellation problem[2][10]. The Zariski cancellation problem is about whether $A$ and $B$ are isomorphic as algebras when $A[X]$ and $B[X]$ are. However, very few results are dedicated to discriminant ideals, which are more general and do not require some ideal to be principal. When $(R, C, \text{tr})$ satisfies

$$(CH_n) \ (R, C, \text{tr}) \text{ is a finitely generated Cayley-Hamilton algebra (Definition 2.5.1) of degree } n, \mathbb{k} \text{ is algebraically closed and } \text{char } \mathbb{k} \notin [1, n],$$

there is a description of the zero sets of discriminant ideals by dimensions of irreducible representations [7]:

$$\nu_k = \mathcal{V}(D_k(R/C, \text{tr})) = \mathcal{V}(MD_k(R/C, \text{tr})) = \left\{ m \in \text{MaxSpec}(C) \bigg| \sum_{V \in \text{Irr}(R/mR)} \dim(V)^2 < k \right\}.$$  \hspace{1cm} (1.1)

where $\text{Irr}(R/mR)$ denotes isomorphism classes of irreducible representations of $R$ over $m \in \text{MaxSpec}(C)$. If $R$ is furthermore prime, $C$ is its center and its center is integrally closed

The zero set of the highest discriminant ideal $D_h(R/C, \text{tr})$ (Definition 2.5.6) correspond

---

This section is mostly a shorter version of the introduction in [30].
to the complement of the Azumaya locus (Definition 2.2.32). The purpose of this work is
to explore the lowest discriminant ideal $D_l(R/C, \text{tr})$ (Definition 2.5.7. Unlike its highest
counterpart, there is no clean description of representations over it even under our strong
conditions nor is the concept of Azumaya locus helpful. In this work we consider Cayley-
Hamilton Hopf algebras with basic identity fiber (Definition 2.5.1) over some field $k$ and
try to give an account of

1. the level $l$ for the lowest discriminant ideal $D_l(H/C, \text{tr})$;

2. representations over $D_l(H/C, \text{tr})$ and

3. the orbit of the identity $m_\epsilon$, i.e. the kernel of cokernel of $C$ under some $k$-algebra
   automorphisms of $H$ that fix $C$.

There is also some relation to projective representations and homological algebra. This
dissertation is large based on [30] and [29] with possibly more details.

First, we consider a Hopf algebra $H$ over an algebraically closed field $k$ that is a
finite module over a central Hopf subalgebra $C$ and study some action of the group $G_0$ on
the isomorphism classes of irreducible representations over a given maximal ideal $m$ of $C$
to get [30, Theorem A]:

**Theorem A.** Assume that $H$ is a finitely generated Hopf algebra over an algebraically
closed field $k$ and $C$ is a central Hopf subalgebra such that $H$ is a finitely generated $C$-
module. Then the following hold:

(a) For all $V, W \in \text{Irr}(H/mH)$, there exist $M', M'' \in \text{Irr}(H/m_\epsilon H)$ such that $W$ is a
   quotient of $M' \otimes V$ and a submodule of $M'' \otimes V$.

(b) The group

$$G_0 := G((H/m_\epsilon H)^o) \subseteq \text{Irr}(H/m_\epsilon H)$$

(1.2)
acts on \( \text{Irr}(H/mH) \) by \( V \in \text{Irr}(H/mH) \mapsto \chi \otimes V \) for \( \chi \in G_0 \).

(c) Every module \( V \in \text{Irr}(H/mH) \) for \( m \in \text{MaxSpec} C \) satisfies

\[
|\text{Stab}_{G_0}(V)| \leq \dim(V)^2. \tag{1.3}
\]

(d) An equality in (1.3) holds (we call such a module \textbf{maximally stable}) if and only if one of the following two equivalent conditions holds:

(ii) \( V \otimes V^* \) is a direct sum of nonisomorphic one-dimensional \( H \)-modules.

(iii) \( V \otimes V^* \cong \bigoplus_{\chi \in \text{Stab}_{G_0}(V)} \chi \).

(e) If \( V \in \text{Irr}(H/mH) \) is a maximally stable irreducible module, then the primitive quotient

\[
H/\text{Ann}_H(V)
\]

is isomorphic to a twisted group algebra

\[
\mathbb{k}\gamma_V \text{Stab}_{G_0}(V)
\]

for a canonically defined 2-cocycle \( \gamma_V : \text{Stab}_{G_0}(V) \times \text{Stab}_{G_0}(V) \to \mathbb{k}^* \), see Theorem 3.5.1 (a). Both algebras are isomorphic to \( \text{End}_k(V) \).

If, in addition, the identity fiber algebra \( H/m_H \) is a basic algebra, then we have:

(f) [4, Proposition III.4.11] The inclusion in (1.2) is an equality and the action in part (b) is transitive.

The results are then used to answer 1. and 2. mentioned before [30, Theorem B]

Theorem B. Assume that \((H,C,\text{tr})\) is a Cayley–Hamilton Hopf algebra of degree \( n \) over an algebraically closed field \( \mathbb{k} \) of characteristic \( \text{char}\mathbb{k} \notin [1,n] \), such that \( H \) is a finitely generated \( \mathbb{k} \)-algebra and the identity fiber \( H/m_H \) is a basic algebra. The following hold:
(a) For all \( m \in \text{MaxSpec} C \),

\[
\min \left\{ k \in \mathbb{Z}_+ \mid D_k(H/C, \text{tr})(m) = 0 \right\} = \frac{|G_0| \dim(V)^2}{|\text{Stab}_{G_0}(V)|} + 1
\]

for any \( V \in \text{Irr}(H/mH) \), recall the definition (1.2) of the group \( G_0 \).

(b) The lowest discriminant ideal of \((H, C, \text{tr})\) is of level

\[ l = |G_0| + 1. \]

(c) The following are equivalent for \( m \in \text{MaxSpec} C \):

(i) \( m \) belongs to the zero set \( V_{|G_0|+1} \) of the lowest discriminant ideal of \((H, C, \text{tr})\);

(ii) There exists \( V \in \text{Irr}(H/mH) \) that is maximally stable;

(iii) All modules \( V \in \text{Irr}(H/mH) \) are maximally stable.

(d) Under the assumptions of the theorem, but without the assumption that the identity fiber \( H/m_\epsilon H \) is a basic algebra, we have that the group \( G_0 \) contains a copy of the cyclic group of order equal to the integral order \( \text{io}(H) \) of \( H \), so \( \text{io}(H) \) divides \( |G_0| \).

And if \( l \) is the order of the lowest discriminant ideal of \( H \),

\[ l > \text{io}(H). \]

Next, we look into the orbit of \( m_\epsilon \) under winding automorphisms and conclude [30, Theorem C]:

**Theorem C.** Assume that \((H, C, \text{tr})\) is a Cayley–Hamilton Hopf algebra of degree \( n \) satisfying the assumptions of Theorem B.

(a) For \( m \in \text{MaxSpec} C \) the following are equivalent:

(i) The algebra \( H/mH \) has a one-dimensional module;
(ii) The algebra $H/mH$ is basic;

(iii) $H/mH$ and $H/m_\tau H$ are isomorphic as $k$-algebras;

(iv) $m$ belongs to the orbit of $m_\tau$ under the group $\text{Aut}_{k-\text{alg}}(H, C)$ of all $k$-algebra automorphisms of $H$ that preserve $C$;

(v) $m$ belongs to the orbit of $m_\tau$ under the left winding automorphism group $W_1(G(H^\circ))$;

(vi) $m$ belongs to the orbit of $m_\tau$ under the right winding automorphism group $W_r(G(H^\circ))$.

(b) If $m \in \text{MaxSpec } C$ satisfies any of the six equivalent conditions in part (a), then it belongs to the zero set $\mathcal{V}_{G_0}+1$ of the lowest discriminant ideal of $(H, C, \text{tr})$.

(c) If every maximally stable irreducible $H$-module is one-dimensional, then the maximal ideal $m \in \text{MaxSpec } C$ belongs to the zero set $\mathcal{V}_{G_0}+1$ of the lowest discriminant ideal of $(H, C, \text{tr})$ if and only if it satisfies any of the six equivalents conditions in part (a).

Last, we look into examples of group algebras of certain central extensions of products of two finitely generated Abelian groups and big quantum Borel subgroups at roots of unity. The second example satisfies the condition in Theorem C(c) but the first example does not. In fact, it can happen that the orbit of $m_\tau$ is trivial. In the first example all maximal ideals of $C$ are in the zero set of the lowest discriminant ideal.
Chapter 2. Background

In this work, we consider noncommutative rings with multiplicative identity $1 \neq 0$. Noncommutative means the ring is not necessarily commutative under multiplication.

2.1. Rings and modules

In this section, we quickly review the definition of Noetherian and Artinian ring, prime and primitive ring, socle and top for modules. For completeness, a basic definition of right quotient ring is also given but we are going to considering a multiplicatively closed subset in the center of prime rings, which are nonzero divisors (not zero divisors), so this part can be skipped and the reader may formally treat it as localization in the commutative setting.

2.1.1. Artinian and Noetherian conditions

Notation 2.1.1. We denote an ideal $A$ of the ring $R$ by $A \triangleleft R$. Let $L$ be a left $R$-module (sometimes denoted $_RL$), then we denote the left annihilators of $L$ in $R$ as $\text{lann}(R_L)$ and omit the subscript or simply write $\text{ann}(L)$ if the meaning is clear from the context. Similarly $\text{rann}(M_R)$ is the set of right annihilators of a right $R$-module $M$ (sometimes denoted $M_R$).

Definition 2.1.2. Let $C$ be a central subring of a ring $H$. The central character map

$$\kappa : \text{Irr}(R) \mapsto \text{MaxSpec}(C)$$

is defined by

$$\kappa(V) := \text{ann}_C(V), \quad \forall V \in \text{Irr}(R).$$

(2.1)

Definition 2.1.3. A ring $R$ is called left Noetherian if $_RR$ is Noetherian and similarly one can define right Noetherian, left Artinian and right Artinian. A ring is
Noetherian if it is both left and right Noetherian and similarly for Artinian.

Example 2.1.4. Finite dimensional algebras are always Artinian since the dimension can only strictly decrease finitely many times.

For rings, Artinian is a stronger condition [27, Corollary 1.1.13]

Lemma 2.1.5 (Hopkins lemma). If a ring $R$ is right (left) Artinian, then it is right (left) Noetherian.

Remark 2.1.6. There is no analog of Lemma 2.1.5 for modules. Consider for example $\mathbb{Z}[\frac{1}{p}]$ as a $\mathbb{Z}$-module for a prime number $p$.

Definition 2.1.7. A ring $R$ is called simple if its only ideals are zero and itself; similarly for modules.

Example 2.1.8. In an Artinian module, a smallest (nonzero) module is simple.

2.1.2. Prime and primitive ideals

Definition 2.1.9. A ring $R$ is called prime if $AB \neq 0$ whenever $0 \neq A, B \triangleleft R$. And an ideal $P$ is called prime if the quotient $R/P$ is prime.

Remark 2.1.10. There is still the notion of integral domain in noncommutative ring theory, this means there are no nonzero divisors. But simple rings, necessarily prime, may not be domains, e.g. $M_n(\mathbb{k})$.

Proposition 2.1.11. The following are equivalent for a ring $R$ [38, Proposition 2.1.13]:

(i) $R$ is prime;

(ii) For very left ideal $L$ of $R$, $\text{lann}(R_L) = 0$;

(iii) If $0 \neq r_1, r_2 \in R$, then $r_1 R r_2 \neq 0$.

Remark 2.1.12. It is easy to show using Zorn’s lemma that every prime ideal in a ring $R$
contains a minimal prime ideal. In fact if $R$ is left or right Noetherian, 0 is a finite product of minimal prime ideals \([27, \text{Proposition 2.2.17}]\); the proof is the same as Noether’s proof of the commutative case \([17, \text{Ex 1.11}]\).

**Remark 2.1.13.** The equivalent condition Proposition 2.1.11(iii) implies that central elements of a prime ring are nonzero divisors.

**Definition 2.1.14.** A ring $R$ is semiprime if the intersection of all its prime ideals is zero or equivalently $R$ is the semidirect product of $\mathcal{P} = \{P \lhd R : P \text{ is prime}\}$, i.e. $\phi : R \to \prod_{P \in \mathcal{P}} R/P$ is an injection.

**Proposition 2.1.15.** If $R$ is a semiprime ring and $A \lhd R$, then

(i) $\text{lann}(RA) = \text{rann}(AR) := \text{ann}(A)$;

(ii) $\text{ann}(A)$ is the intersection of minimal prime ideals of $R$ not contained in $A$.

Another interesting result is \([27, \text{Proposition 2.2.14}]\)

**Definition 2.1.16.** An $A \lhd R$ is an annihilator ideal if $A = \text{ann}(B)$ for some $B \lhd R$.

**Theorem 2.1.17.** For a ring $R$ the conditions below are equivalent \([38, \text{Theorem 2.6.17}]/[27, \text{Theorem 2.2.15}]\):

(i) $R$ is semiprime;

(ii) every nilpotent left (right) ideal of $R$ is zero;

(iii) every nilpotent ideal of $R$ is zero;

(iv) if $A \lhd R$, $A^2 = 0$, then $A = 0$;

(v) $R$ has finitely many minimal prime ideals;

(vi) $R$ has finitely many annihilator ideals;

(vii) the annihilator ideals of $R$ satisfy ascending chain conditions (ACC).

**Definition 2.1.18.** An $R$-module $M$ is called faithful if $\text{ann}_R(M) = 0$. A ring $R$ is prim-
itive if it has a faithful simple module. If $P \triangleleft R$ and $R/P$ is primitive, then $P$ is a primitive ideal.

**Remark 2.1.19.** Every primitive ideal is prime [38, Remark 2.1.4].

**Remark 2.1.20.** By Artin-Wedderburn Theorem, every primitive ideal in an Artinian ring is maximal.

**Definition 2.1.21.** The Jacobson radical $\text{Jac}(R)$ of a ring $R$ is the intersection of all primitive ideals of $R$. An element $r \in R$ is quasi-invertible if $1 - a$ is a unit in $R$.

There are alternative characterizations of the Jacobson radical [27, Theorem 0.3.8]

**Theorem 2.1.22.** Let $A$ be an ideal of a ring $R$ the following are equivalent:

(i) $A = \text{Jac}(R)$;

(ii) $A$ is the intersection of all maximal left (right) ideals of $R$;

(iii) $A$ is the largest quasi-invertible ideal of $R$.

The next result concerns with Jacobson radicals of homomorphism image [38, Proposition 2.5.6].

**Proposition 2.1.23.** (i) Let $\phi : R_1 \to R_2$ be a surjective ring homomorphism, then

$\phi(\text{Jac}(R_1)) \subseteq \text{Jac}(R_2)$.

(ii) For $A \triangleleft R$, $(\text{Jac}(R) + A) \subseteq \text{Jac}(R/A)$. Equality holds when $A \subseteq \text{Jac}(R)$.

There is also an analog of Nakayama’s lemma [17, Corollary 4.8] in commutative algebra [38, Proposition 2.5.24][27, Lemma 0.3.10]

**Proposition 2.1.24.** Let $M$ be a finitely generated $R$-module and abbreviate $\text{Jac}(R)$ as $J$.

(i) Suppose $N$ is a submodule of $M$ and $M = N + JM$, then $N = M$.

(ii) Let $x_1, \ldots, x_n \in M$ and $x_1 + JM, \ldots, x_n + JM$ span $M/(JM)$, then $x_1, \ldots, x_n$ also span $M$. 

9
2.1.3. Socle and top

**Definition 2.1.25.** Similar to [1], Let $R$ be a ring and $M$ a $R$-module. Define socle as the semisimple $R$-module generated by all simple submodules of $M$.

We can generalize part(a)(b) in [1, Exercise 1.17] a bit using Schur’s lemma to get the following.

**Lemma 2.1.26.** (i) If $M \neq 0$ is an $R$-module and has a nonzero Artinian submodule, then socle$(M) \neq 0$. This hold in particularly true when $R$ or $M$ is finite dimensional.

(ii) Let $f : M \rightarrow N$ be an $R$-module homomorphism, then $f(\text{socle}(M)) \subseteq \text{socle}(N)$ and $\text{socle}(f(M)) \neq 0$ if $f \neq 0$ and $\text{socle}(M) \neq 0$.

**Remark 2.1.27.** Example 2.1.8 shows that socle$(M) \neq 0$ for any $M$ Artinian.

**Definition 2.1.28.** Let $M$ be an $R$-module, define rad of $M$ as the intersection of all maximal submodules of $M$ and top of $M$ by top$(M) = M/\text{rad}(M)$.

The following observations are immediate from the definition and, Schur’s and Nakayama’s lemma [1, Proposition 3.7, Corollary 3.9]:

**Lemma 2.1.29.** Let $M$ and $N$ be $R$-modules.

(i) An $m \in M$ is in rad$(M)$ $\iff$ $f(m) = 0 \ \forall f \in \text{Hom}_R(M, S)$, $S$ simple.

(ii) rad$(M) = \text{Jac}(R)M$.

(iii) $M$ is semisimple $\iff$ rad$(M) = 0$.

(iv) rad$(M \oplus N) = \text{rad}(M) \oplus \text{rad}(N)$.

(v) If $f \in \text{Hom}_R(M, N)$, then $f(\text{rad}(M)) \subseteq \text{rad}(N)$. In particular, any $g \in \text{Hom}_R(M, L)$, $L$ semisimple factors through top$(M)$.
(vi) \( f \in \text{Hom}_R(M,N) \) is surjective \iff the induced map \( \text{top}(M) \mapsto \text{top}(N) \) is surjective.

**Remark 2.1.30.** By Nakayama’s remark, \( \text{top}(M) \) is nonzero for any nonzero finite generated module over a ring \( R \).

### 2.1.4. Quotient rings

Similarly to commutative rings, localization with respect to a multiplicatively closed set \( S \) of a ring \( R \) can be defined. Set

\[
\text{ass}S = \{ r \in R : rs = 0 \text{ for some } s \in S \}. \quad (2.2)
\]

A reasonable set of conditions on the localization \( Q \) is for it to be a **right quotient ring** \( R_S \), i.e. there is a ring homomorphism \( \phi : R \rightarrow Q \) such that

(i) \( \phi(s) \) is a unit in \( Q \) \quad \forall s \in S; \n
(ii) if \( q \in Q \), then \( q = \phi(r)\phi(s)^{-1} \) for some \( r \in R, s \in S \); \n
(iii) \( \ker \phi = \text{ass}S \).

**Remark 2.1.31.** If such a right quotient ring exists, then it can be easily seen to be universal in the following sense: for every ring homomorphism \( \phi : R \mapsto T \) sending \( S \) to units, there is a unique ring homomorphism \( p : Q \mapsto T \) making the diagram below commutative.

\[
\begin{array}{ccc}
R & \xrightarrow{\theta} & Q \\
\phi \downarrow & & \downarrow p \\
T & & \\
\end{array}
\]

Thus \( R_S \) is unique up to isomorphism and a right quotient ring is isomorphic to a left quotient ring [27, Corollary 2.1.4].

**Definition 2.1.32.** An \( r \in R \) is **left regular** if \( rt \neq 0 \) for any \( 0 \neq t \in R \). **Right regular** can be defined similarly.
A necessary condition for the existence of a right quotient ring is the right Ore condition, i.e. for any \( r \in R, s \in S \), there exist \( r' \in R, s' \in S \) that satisfy \( rs' - sr' = 0 \) [27, Proposition 2.1.6]. In this case \( \text{ass}S \) is indeed an ideal of \( R \) [27, Lemma 2.1.9] and all elements in \( \tilde{S} \): the image of \( S \) under the natural projection \( R \twoheadrightarrow R/(\text{ass}S)R \) is left regular. However, this condition is not sufficient: some elements in \( \tilde{S} \) may not be right regular [27, Example 2.1.11], this is a contraction with (i). The question of existence is settled below [27, Theorem 2.1.12]

**Theorem 2.1.33.** Let \( S \) be a multiplicatively closed subset of a ring \( R \). The right quotient ring \( R_S \) exists \( \iff \) \( S \) satisfies the right Ore condition and \( \tilde{S} \) are nonzero divisors in \( R/(\text{ass}S)R \).

Such a multiplicatively closed subset of \( R \) is called a right denominator set. If \( S \) is a right denominator set, a right quotient ring can be constructed from some equivalence class of right \( R \)-module homomorphism [27, Section 2.1.12]. Set

\[
\mathcal{F} = \{ A \in R_R : A \cap S \neq \emptyset \}. \quad (2.3)
\]

then \( \forall A_1, A_2 \in \mathcal{F}, \alpha \in \text{Hom}(A_1, R)_R \)

(i) \( A_1 \cap A_2 \in \mathcal{F} \) and

(ii) \( \alpha^{-1}A_2 := \{ \alpha \in A_1 : \alpha(a) \in A_2 \} \in \mathcal{F} \).

We can give the set \( \bigcup_{A \in \mathcal{F}} \text{Hom}(A, R)_R \) an equivalence relation \( \sim \) as follows: for \( \alpha_1 \in \text{Hom}(A_1, R)_R, \alpha_2 \in \text{Hom}(A_2, R)_R, \alpha_1 \sim \alpha_2 \) if they coincide on some \( \mathcal{F} \ni A \subseteq A_1 \cap A_2 \). Now

\[
\bigcup_{A \in \mathcal{F}} \text{Hom}(A, R)_R / \sim \quad (2.4)
\]

can be identified with \( R_S \) as below
(i) for $r \in R$, $\phi(r) \sim \alpha \in \text{Hom}(R, R)$, $\alpha : x \mapsto rx$, $\forall x \in R$;

(ii) for $s \in S$, $\phi(s^{-1}) \sim \alpha \in \text{Hom}(sR, R)$, $\alpha : sx \mapsto x$, $\forall x \in R$;

(iii) let $\alpha \in \text{Hom}(A, R)$, then $\alpha \sim \phi(as^{-1})$, where $s \in S \cap A$, $a = \alpha(s)$.

Addition is defined on $A_1 \cap A_2$ and multiplication by composition on $\alpha^{-1}A_2$. This construction turns out to be useful for later discussion in [27].

**Remark 2.1.34.** Any multiplicatively closed set of nonzero divisors of the center is a right denominator set. In particular, by Remark 2.1.13 any multiplicatively closed subset $S$ of the center of a prime ring $R$ is a right denominator set.

### 2.2. Polynomial identity algebras

This section is a quick review of polynomial identity rings, Azumaya locus and some results related to representation theory.

A ring $R$ satisfies a a polynomial $f \in \mathbb{Z}\langle x_1, \cdots, x_m \rangle$ and $f$ is a polynomial identity for $R$ if for any $r_1, \cdots, r_m \in R$, $f(r_1, \cdots, r_n) = 0$. Such a polynomial is called monic if at least one highest degree term has coefficient one.

**Definition 2.2.1.** If a ring $R$ satisfies a monic polynomial $f \in \mathbb{Z}\langle x_1, \cdots, x_n \rangle$ for some integer $n$, it is called a polynomial identity (PI) ring. The least degree of such a polynomial is called the **minimal degree** of a PI ring $R$.

**Example 2.2.2.** The simplest example is commutative rings with the polynomial identity $f = x_1x_2 - x_2x_1$, whose minimal degree is two since we assume $1 \neq 0$.

We note that all multilinear polynomials in $\mathbb{Z}\langle x_1, \cdots, x_n \rangle$ are in the form

$$
\sum_{\sigma \in S_n} a_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(m)}, \quad a_{\sigma} \in \mathbb{Z} \forall \sigma
$$

for some $1 \leq m \leq n$. Each polynomial identity can be reduced to a multilinear one, in fact
there is a stronger result in [4, Theorem 13.1.9]

**Proposition 2.2.3.** If a ring $R$ has a polynomial identity $f$ of degree $d$ then it has a multilinear polynomial identity $g$ of degree at most $d$ and the coefficients of each term in $g$ is the coefficient of some term in $f$. In particular, if $f$ is monic, so is $g$.

The benefit of working with multilinear polynomial identities is that if the ring is a module over a central subalgebra such an identity only needs to be checked on the generators [27, Lemma 13.1.10] and thus central extensions of PI rings, i.e. rings generated by a PI ring $R$ and elements commutating with $R$ are PI rings [27, Corollary 13.1.11].

A very important polynomial identity is the standard identity $s_m$ of degree $m$ obtained by setting $a_\sigma = \text{sgn}(\sigma)$ in (2.5). We note that

**Proposition 2.2.4.** A ring satisfying $s_d$ satisfies $s_t$ for all $t \geq d$ [27, Theorem 13.3.2].

The standard identities are alternating, i.e. swapping two indeterminants changes the sign and wedge products can be used to prove the following [11, Theorem 9.5.8]

**Theorem 2.2.5** (Amitsur-Levitzki 1950). Let $C$ be a commutative ring, then $M_n(C)$ satisfies $s_{2n}$.

Indeed, $2n$ is the minimal degree of $M_n(C)$:

**Proposition 2.2.6.** Suppose $C$ is commutative ring, then $M_n(C)$ satisfies no monic polynomial of degree smaller than $2n$ [27, Proposition 13.3.2].

There are PI rings that not module finite over the center [39, Example 5.1.18].

**Definition 2.2.7.** A finite dimensional algebra over a field $k$ is called a **central simple algebra over** $k$ if its center is just $k$. The field is the center if not explicitly given.

A primitive PI-ring $R$ is a central simple algebra by the following theorem [27, Theorem 13.3.9]
Theorem 2.2.8 (Kaplansky). If a primitive PI ring $R$ is of minimal degree $d$, then $d$ is even and $R$ is a central simple algebra over its center of dimension $(d/2)^2$.

Definition 2.2.9. A finitely generated algebra $R$ over a commutative ring $C$ is called an affine algebra over $C$ or affine $C$-algebra.

Lemma 2.2.10 (Artin-Tate lemma). Let $A \subseteq B$ be central subrings of an affine $A$-algebra $R$ and $R$ is a finite module over $B$. Then

(i) There exists an affine $A$-algebra $B'$ of $B$ such that $R$ is a finitely generated $B'$ module.

(ii) If $A$ is Noetherian or $B$ is a direct summand of $R$ a right $B$-module, then $B$ is finitely generated $B'$-module and an affine $A$-algebra.

Remark 2.2.11. A very important case of Lemma 2.2.10(ii) is when $A = \mathbb{k}$. This is sometimes also called Artin-Tate lemma [4, I.13.4].

Remark 2.2.12. If the primitive PI ring $R$ is also an affine algebra over an algebraically close field $\mathbb{k}$, then $R \cong M_{d/2}(\mathbb{k})$ [4, I.13.5]: the center of $R$ is a finitely generated algebra over $\mathbb{k}$ by Artin-Tate lemma and it is a field by Kaplansky’s theorem and equal to $\mathbb{k}$ by Hilbert’s Nullstellensatz. The rest follows from Artin-Wedderburn theorem.

For each prime PI ring $R$, there is some multilinear $g_m \in \mathbb{Z}\langle X \rangle$ for some $m$ such that $g_m(r)$ is nonzero and in the center of $R$ for any $r \in R$ Theorem 2.2.8[27, Proposition 13.2.7][27, Corollary 13.6.3]. This can be used to derive [27, Theorem 6.4]

Theorem 2.2.13. Let $R$ be a semiprime PI ring with center $Z$ and $I \neq 0$ an ideal of $R$, then for some positive integer $m$, $0 \neq g_m(I) \subseteq I \cap Z$.

Consequently stronger result can be derived [27, Theorem 13.6.5]

Theorem 2.2.14 (Posner). Let $R$ be a prime PI ring with minimal degree $d$ and $Z$ be its
center, $S = Z \setminus \{0\}$, $Q = R_S$ and $F = Z_S$ (quotient field of $Z$). Then $Q$ is a central simple algebra over $F$ with center $F$ and $\dim_F Q = (d/2)^2$.

**Definition 2.2.15.** The number $d/2$ above is called the **PI-degree** of a prime PI ring.

Another way to say Theorem 2.2.14 is that under those conditions, $S = R \otimes_Z F$ is primitive. Let $\bar{k}$ be the algebraic closure of $k$, then by Remark 2.2.12 $R \otimes_Z \bar{k} \cong M_k(d/2)$.

**Definition 2.2.16.** Any field $H \supseteq Z$ such that $R \otimes_Z H \cong M_t(H)$ for some integer $t$ is called a **splitting field** of $R$.

The splitting field may be smaller than $\bar{k}$ if $k \neq \bar{k}$. As a consequence of Jacobson’s density theorem [38, Proposition 2.1.7][27, Corollary 3.7] and Proposition 2.2.6, $S \cong M_t(D)$ for the division ring $D = \text{End}(V_S)$, $V$ an irreducible right module of $S$. Then a maximal field $Z \subseteq H \subseteq S$ is a splitting field of $S$ by the following lemma [27, Lemma 13.3.4]:

**Lemma 2.2.17.** Let $R = M_t(D)$ for some division ring $D$, $H$ be a maximal field in $D$, $C$ be the center of $D$ and $V_R$ be simple. Note by Schur’s lemma $D \cong \text{Hom}(V_S, V_S)$.

(i) $S = R \otimes_C H$ is simple and $V_S$ is simple with $\text{End}(V_S) \cong H$.

(ii) If $\dim V_H = m < \infty$, then $S \cong M_m(H)$, $\dim D_H = \dim H_C = m/t$ and $\dim R_C = m^2$.

**Definition 2.2.18.** A prime ideal $P$ is **regular** in a prime PI ring $R$ if PI-deg$(R/P) = \text{PI-deg}(P)$.

There is a way to remove non-regular primes through localization [27, Proposition 13.7.4]:

**Proposition 2.2.19.** If $R$ be a prime PI ring with PI-degree $n$ and $0 \neq c \in \mathfrak{g}_n(R)$, $S = \{c^n\};$ then each prime ideal in $R_S$ is regular.
Denote

(FinFin): \( R \) is an affine algebra over a field \( k \) and \( R \) is a finite module (module finite) over a central subalgebra \( C \).

Under this condition, there is relationship between regular prime and regular maximal ideal [4, Lemma III.2]

**Lemma 2.2.20.** Suppose \( R \) satisfies (FinFin), then

(i) a prime ideal of \( R \) is regular (resp. non regular) \( \iff \) it is an intersection of regular (resp. non-regular) maximal ideals of \( R \).

(ii) The set of non-regular prime (resp. non-regular maximal) ideals of \( R \) is a proper closed subset of \( \text{Spec} R \) (reps. \( \text{MaxSpec} R \)). In particular, regular maximal ideals exist and their intersection is zero.

**Definition 2.2.21.** A ring \( R \) is an **Azumaya algebra** over its center \( Z \) if

(i) \( R_Z \) (equivalently \( Z R \)) is finitely generated and projective and

(ii) \( \theta : R \otimes_Z R^{\text{op}} \to \text{End}_Z(R) \) defined by

\[
    r_1 \otimes r_2 \mapsto (x \mapsto r_1 x r_2) \quad (2.6)
\]

is a ring isomorphism.

**Example 2.2.22.** Two simple examples are [27, Proposition 13.7.7]

(i) every central simple algebra,

(ii) the matrix \( M_n(C) \) for any commutative ring \( C \).

Ideals of \( R \) and \( Z \) are related by [27, Proposition 13.7.9]:

**Proposition 2.2.23.** In an Azumaya algebra \( R \) over its center \( Z \), there is a bijection be-
tween ideals $I$ of $R$ and $J$ of $Z$ given by

$$I \mapsto I \cap Z, \quad J \mapsto JR.$$  \hspace{1cm} (2.7)

Combined with the definition of Azumaya algebras, this shows [27, Corollary 13.7.10]

**Corollary 2.2.24.** Let $R$ be an Azumaya algebra over its center $Z$, the following are equivalent

(i) $R$ satisfies ACC on ideals;

(ii) $R$ is left and right Noetherian;

(iii) $Z$ is Noetherian.

Next result concerns with quotient of an Azumaya algebra by a maximal ideal of its center.

**Proposition 2.2.25.** Suppose $R$ is an Azumaya algebra over its center $Z$ and $m \in \text{MaxSpec}(Z)$. Then $R/mR$ is a central simple algebra with center $Z/m$.

**Remark 2.2.26.** The set $Z \setminus m$ is an Ore set, hence localizations $R_m$ and $Z_m$ exist. Furthermore $R_m$ is a localization of the central simple algebra $R/mR$, so $R_m/mR_m$ is a finitely generated free module over $Z/m$ by Nakayama’s lemma and $R/mR \cong R_m/mR_m$ [27, 13.7.12].

**Definition 2.2.27.** Let $R$ be an Azumaya algebra over its center $Z$. Then $R$ is an Azumaya algebra of rank $n^2$ if for every $m \in \text{MaxSpec}(Z)$, $R/mR$ has rank $n^2$ over $Z/m$.

**Example 2.2.28.** If $R$ is a prime Azumaya algebra over its center $Z$, then $\dim_{Z/m} R/mR = \dim_{\text{Fract}(Z)} R \otimes_Z \text{Fract}(Z) = n^2$ is a constant independent of $m \in \text{MaxSpec}(Z)$ [4, III.1.4].

A key result about Azumaya algebra is the following [27, 13.7.14][4, III.1.4]
Theorem 2.2.29 (Artin-Procesi theorem). For a prime ring $R$, the following are equivalent:

(i) $R$ is an Azumaya algebra of rank $n^2$;

(ii) $R$ is a prime PI ring of PI-degree $n$ and all its prime ideals are regular;

(iii) $R$ is a prime PI ring of PI-degree $n$ and all its maximal ideals are regular;

(iv) $g_n(R)R = R$.

There is an analog of finite morphisms being closed in algebraic geometry under some conditions:

Lemma 2.2.30. Suppose $R$ is a Noetherian subring of the center of a ring $S$ that is module finite over $R$. Then the contraction maps

$$
\pi : \text{Spec}(S) \mapsto \text{Spec}(R), \quad \tilde{\pi} : \text{MaxSpec}(S) \mapsto \text{MaxSpec}(R)
$$

defined in (2.9) are closed in Zariski topology.

$$
\pi : P \mapsto P \cap R, \forall P \in \text{Spec}(R); \quad \pi : M \mapsto M \cap R, \forall M \in \text{MaxSpec}(R).
$$

There is a characterization of regular maximal ideals under some conditions [4, Theorem III.1.6]:

Theorem 2.2.31. Let $R$ be a prime algebra over an algebraically closed field $k$ and satisfy (FinFin), $n$ be its PI-degree and $Z$ its center. If $M \in \text{MaxSpec}(R)$, $m = M \cap Z \in \text{MaxSpec}(Z)$, then the following are equivalent:

(i) $M$ is a regular maximal ideal of $R$;

(ii) $R_m$ is Azumaya over $Z_m$;
(iii) $M = mR$;

(iv) All irreducible $R/M$-modules are isomorphic and have the maximal $k$-dimension possible among irreducible $R$-modules, i.e. $n = PI\text{-deg}(R)$;

(v) $R/M \cong M_n(k)$.

**Definition 2.2.32.** Let $R$ be a prime $k$-algebra that satisfies (FinFin) and $Z$ be its center. The **Azumaya locus** of $R$ over $Z$ is defined as

$$A_R := \{ m \in \text{MaxSpec}(Z) : R_m \text{ is Azumaya over } Z_m \}.$$  \hspace{1cm} (2.10)

The theorem below can be deduced using Lemma 2.2.20, Lemma 2.2.30 and Theorem 2.2.29 [4, Theorem III.1.7]

**Theorem 2.2.33.** Let $R$ be a prime algebra over an algebraically closed field $k$ satisfying (FinFin) and $Z$ be its center. Then $A_R$ is a nonempty open subset of $\text{MaxSpec}(Z)$.

**Definition 2.2.34.** Let $C$ be a commutative Noetherian ring, its **singular locus** is defined as

$$S_C := \{ m \in \text{MaxSpec}(C) : \text{gl\text{-}dim}(C_m) = \infty \}.$$  \hspace{1cm} (2.11)

In the previous setting, we have [4, Lemma III.1.8]

**Lemma 2.2.35.** Let $R$ be a prime $k$-algebra that satisfies (FinFin), $Z$ be its center and $\text{gl\text{-}dim}(R) < \infty$. Then $A_R \cap S_Z = \emptyset$.

2.3. Discriminants and discriminant ideals

Next part is the definition of discriminants and discriminant ideals and examples of algebra with trace.

**Definition 2.3.1.** Let $R$ be an algebra over a field $k$ with central subalgebra $C$. Following
the definition in [7], we call a nonzero map \( tr : R \rightarrow C \) a trace if \( tr \) is

(i) \( C \)-linear:

\[
tr(c_1 r_1 + c_2 r_2) = c_1 tr(r_1) + c_2 tr(r_2), \quad \forall c_1, c_2 \in C, \ r_1, r_2 \in R
\]  \hspace{1cm} (2.12)

and

(ii) cyclic:

\[
tr(r_1 r_2) = tr(r_2 r_1) \quad \forall r_1, r_2 \in R.
\]  \hspace{1cm} (2.13)

In this case, \((R, C, tr)\) is an algebra with trace over \( k \).

If \((R, C, tr)\) is an algebra with trace and \( tr(1) \) has no \( \mathbb{Z} \)-torsion, then \( R \) has invariant basis number (IBN), i.e. \( R^n \cong R^m \) implies \( n = m \) [38, Proposition 13.29].

**Example 2.3.2** (Matrix over a field). Let \( R = M_n(k) \) and \( C \) be the scalar matrices, i.e. the center of \( R \) and \( tr \) be the usual trace obtained by summing up the diagonal entries.

Then \((R, C, tr)\) is an algebra with trace. The field can actually be replaced by any commutative algebra.

**Example 2.3.3** (Reduce trace into the center). Suppose \( R \) is prime and a finite module over its center \( Z = Z(R) \), which is integrally closed and denote \( Q = \text{Frac}Z \). Then for some field \( F \supseteq Q \)

\[
R \hookrightarrow R \otimes_Z Q \hookrightarrow R \otimes_Z F \cong M_n(F) \xrightarrow{\text{tr}} F.
\]  \hspace{1cm} (2.14)

By [36, Theorem 10.1], the image of the composition is in \( Z \), and the compositions in (2.14) defines the reduced trace.

**Example 2.3.4** (Reduce trace into a central subalgebra). Let \( R \) be prime, and a finite module over a central subalgebra \( C \), \( F = \text{Frac}C \), \( S = R \otimes_C F \) and; \( C \) be integrally closed.
Denote the center of $S$ by $Z$, then since $S$ is simple, $Z$ is a finite field extension of $F$, following the notation in [16] denote

$$p = [Z : F] = \dim_F Z.$$  

If the extension is separable, denote the algebraic closure of $Z$ by $\bar{Z}$, then

$$S \otimes_Z \bar{Z} \cong M_n(\bar{Z}).$$

for some integer $n$. Define

$$[R : C] = p \cdot n.$$ 

The composition

$$R \hookrightarrow S = R \otimes_C F \hookrightarrow S \otimes_Z \bar{Z} \cong M_n(\bar{Z}) \xrightarrow{\text{tr}} \bar{Z}$$

(2.15)

defines a $C$-linear cyclic map $\text{tr} : R \to F$ by Galois theory. The image of $\text{tr}$ is actually in $C$ when $C$ is algebraically closed [16, Theorem 4.1]; this defines a reduce trace $\text{tr}_{[R:A]} : R \mapsto C$ with $\text{tr}_{[R:C]}(1) = [R : C]$.

**Example 2.3.5** (Regular trace). Let $R$ be an algebra over $\mathbb{k}$ and $C$ a central subalgebra. Suppose $R$ is a free module over $C$ of rank $n$ with basis $\{v_1, \cdots, v_n\}$. Then each $\phi \in \text{End}_C(R)$ can be represented by $\Theta(\phi) \in M_n(C)$ and there is an inclusion $\eta : R \hookrightarrow \text{End}_C(R)$ defined by left multiplication. Let $\text{tr}$ be the trace on $M_n(C)$ defined by summing up the diagonal elements. Then the regular trace denoted $\text{tr}_{\text{reg}}$ is defined by the composition

$$R \xrightarrow{\eta} \text{End}_C(R) \xleftarrow{\Theta} M_n(C) \xrightarrow{\text{tr}} C.$$ 

Note that in the case of $C = \mathbb{k}$, $\text{tr}_{\text{reg}}(x)$ is $m$-times the usual trace for $x \in M_m(\mathbb{k})$ because $x$ has $m$ columns [7, 2.2-(4)]. So $\Theta$ is proper inclusion for $m > 1$ ($n = m^2$).
Definition 2.3.6.  (i) The \textbf{n-th discriminant ideal} $D_n(R/C, \text{tr})$ is defined as the ideal of $C$ generated by elements of the form

$$\det(\text{tr}(y_iy_j))_{1 \leq i,j \leq n}, \quad (y_1, \cdots, y_n) \in R^n$$

(2.16)

(ii) and the \textbf{n-th modified discriminant ideal} $MD_n(R/C, \text{tr})$ is defined as the ideal of $C$ generated by the elements of the form

$$\det(\text{tr}(y_iy'_j))_{1 \leq i,j \leq n}, \quad (y_1, \cdots, y_n), (y'_1, \cdots, y'_n) \in R^n.$$  

(2.17)

(iii) When $R$ is furthermore a free (left) $C$-module of rank $N$, the discriminant

$$D(R/C, \text{tr})$$

is given by

$$D(R/C, \text{tr}) = \det(\text{tr}(a_ia_j))_{1 \leq i,j \leq n}$$

(2.18)

for a basis $\{a_1, \cdots, a_n\}$ of $R$ as a $C$-module.

The discriminant is determined up to the square of a unit computed from the determinant of the change of basis matrix and $D_{n^2}(R/C, \text{tr}) = MD_{n^2}(R/C, \text{tr}) = \langle D(R/C, \text{tr}) \rangle$. In literature, $R$ is usually a finite module over $C$, so $R$ is a PI ring. If $R$ is furthermore prime and $n$ is its PI-degree, then $D_k(R/C, \text{tr}) = MD_k(R/C, \text{tr}) = 0$ for $k > n^2$ [7, Corollary 2.4]. Since the determinant of a matrix can be computed using $C$-linear combination of smaller matricies

$$C = MD_1(R/C, \text{tr}) \supseteq \cdots \supseteq MD_k(R/C, \text{tr}) \supseteq MD_{k+1}(R/C, \text{tr}) \cdots,$$

(2.19)

thus

$$\emptyset = \mathcal{V}(MD_1(R/C, \text{tr})) \subseteq \cdots \subseteq \mathcal{V}(MD_k(R/C, \text{tr})) \subseteq \mathcal{V}(MD_{k+1}(R/C, \text{tr})) \cdots.$$  

(2.20)
2.4. Adjoints

Before moving on to Cayley-Hamilton algebras, we recall adjoints in category theory [26].

**Definition 2.4.1.** Let \( F : \mathcal{D} \to \mathcal{C} \) be a functor and \( c \in \mathcal{C} \). A **universal arrow** from \( c \) to \( F \) is a pair \( \langle r, u \rangle \) with \( r \in \mathcal{D} \), \( u : c \to Fr \) such that for every pair \( \langle d, f \rangle \) with \( f : c \to Fd \), there is a unique \( f' : r \to d \) making the diagram below commutative

\[
\begin{array}{ccc}
  c & \xrightarrow{u} & Fr \\
  f \downarrow & & \downarrow f' \\
  Fd & & \\
\end{array}
\]

We also say \( u : c \to Fr \) is **universal to** \( F \) **from** \( c \).

There is also a dual version of universal by reversing arrows.

**Definition 2.4.2.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories. An **adjunction from** \( \mathcal{C} \) **to** \( \mathcal{D} \) is a triple \( \langle F, G, \phi \rangle : \mathcal{C} \to \mathcal{D} \) where \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{C} \) are functors and for any \( c \in \mathcal{C} \), \( d \in \mathcal{D} \),

\[
\phi_{c,d} : \text{Hom}_\mathcal{C}(Fc, d) \to \text{Hom}_\mathcal{D}(c, Gd) \quad (2.21)
\]

is a bijection of sets. In this case \( F \) is called the **left adjoint** of \( G \) and \( G \) the right adjoint of \( F \).

There are a few equivalent formulations of an adjunction triple: [26, Theorem 1.2]

**Theorem 2.4.3.** The following are equivalent for \( F, G \) and \( \phi \) defined above.

(i) \( \langle F, G, \phi \rangle \) is an adjunction from \( \mathcal{C} \) to \( \mathcal{D} \).

(ii) There is a natural transformation \( \eta : I_\mathcal{C} \to GF \) such that each \( \eta_c : c \to GFc \) is universal to \( G \) from \( c \). And \( \phi : c \to Gd \) is given for each \( f : Fc \to d \) by

\[
\phi f = Gf \circ \eta_c. \quad (2.22)
\]
(iii) For each $c \in C$, there is a universal arrow $\eta_c : c \to GFc$ from $c$ to $G$. Furthermore, on morphisms $h : c_1 \to c_2$, $F$ is defined by $GFh \circ \eta_{c_1} = \eta_{c_2} \circ h$ and $\phi$ is given by (2.22).

(iv) There is a natural transformation $\varepsilon : FG \to ID$ such that each $\varepsilon_d : FGd \to d$ is universal to $F$ from $d$. And $\phi^{-1} : c \to Gd$ is given for each $g : c \to Gd$ by

$$\phi^{-1}g = \varepsilon_c \circ Fg.$$  (2.23)

(v) For each $d \in D$, there is a universal arrow $\varepsilon_c : FGd \to d$ from $F$ to $d$. Furthermore, on morphisms $g : d_1 \to d_2$, $G$ is defined by $\varepsilon_{d_1} \circ F Gh = g \circ \varepsilon_{c_2} \circ h$ and $\phi^{-1}$ is given by (2.23).

(vi) There are natural transformations $\eta : IC \to GF$ and $\eta : FG \to ID$ such that the compositions

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\eta} G, F \xrightarrow{F\eta} FGF \xrightarrow{\eta F} F$$  (2.24)

are identities. Then maps $\phi$ and $\phi^{-1}$ are given by (2.22) and (2.23), respectively.

2.5. Cayley-Hamilton algebras

This section is a very short introduction to Cayley-Hamilton algebras, in particular its representations. The field $k$ is always algebraically closed.

2.5.1. Definitions

The $k$-th power sum in $k[x_1, \cdots, x_n]$ is defined by

$$\psi_k(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i^k$$  (2.25)

and for $1 \leq j \leq n$ define

$$e_j(x_1, \cdots, x_n) = \sum_{\{i_1, \cdots, i_j\} \subseteq \{1, \cdots, n\}} \prod_{l=1}^{j} x_{i_l}.$$  (2.26)
Then by Newton identities there are $p_i \in \mathbb{Z}[(il)^{-1}][x_1, \ldots, x_n]$ for $1 \leq i \leq n$ such that

$$p_i(\psi_1, \ldots, \psi_i) = e_i$$

(2.27) as formal polynomials in $\mathbb{Z}[x_1, \ldots, x_n]$.

**Definition 2.5.1.** Let $(R, C, \text{tr})$ be an algebra with trace over $\mathbb{k}$, then the \textbf{n-characteristic polynomial} of $r \in R \in C[x]$ is defined by

$$\chi_{n,r}(x) := x^n + \sum_{i=1}^{n} (-1)^{n-i} p_i(\text{tr}(r), \ldots, \text{tr}(r^{n-i})) x^i.$$  

(2.28)

(i) An algebra with trace $(R, C, \text{tr})$ is a \textbf{Cayley-Hamilton algebra of degree d} if it satisfies

(i) $\text{tr}(1) = d,$

(ii) $\chi_{d,r}(r) = 0, \forall r \in R.$

(ii) It is called a \textbf{Cayley-Hamilton Hopf algebra of degree d} if furthermore $R$ is a Hopf algebra and $C$ is a Hopf subalgebra.

(iii) The \textbf{category of Cayley-Hamilton algebra of d} $C_d$ is defined as follows: the objects are Cayley-Hamilton algebras of degree $d$ and the morphisms are algebra homomorphism that are compatible with trace, i.e. algebra homomorphisms of the form $\phi : R_1 \rightarrow R_2$ s.t. $\text{tr}_{R_2} \circ \phi = \phi \circ \text{tr}_{R_1}.$

We use the convention of calling the kernel of the cokernel of $C$ denoted by $m_\varepsilon$ the \textbf{identity} and $R/m_\varepsilon R$ the \textbf{identity fiber algebra}. An algebra is called \textbf{basic} if all its irreducible representations are one-dimensional.

**Remark 2.5.2.** If $\text{char} \mathbb{k} = 0$, the trace of a Cayley-Hamilton algebra factors through its Jacobson radical.
Example 2.5.3. Let $R = M_n(k)$ and $C$ be the scalar matrices, i.e. the center of $R$ and $\text{tr}$ be the usual trace obtained by summing up the diagonal entries. Then (2.28) becomes the characteristic polynomial, hence $(R, C, \text{tr})$ is a Cayley-Hamilton algebra by Cayley-Hamilton theorem.

Example 2.5.4. Let $R = M_n(C)$ for some commutative algebra $C$ and $\text{tr} : R \to C$ be defined by summing up the diagonal elements, then $(R, C, \text{tr})$ is a Cayley-Hamilton algebra of degree $n^2$. If $C$ is a domain, and $F$ its field of fractions, then $R$ is a subring of $M_n(F)$ with $\text{tr}$ equal to the restriction; hence, it follows from the example above. In general $C$ is a quotient of an integral domain and $\text{tr}$ is a polynomial of entries of $M_n(C)$; thus, the claim is also true.

Example 2.5.5. Subalgebras of Cayley-Hamilton algebras of degree $d$ with trace defined by restriction are also Cayley-Hamilton algebras of degree $d$. So a prime PI ring of PI-degree $n$ with reduced trace is a Cayley-Hamilton algebra of degree $n$ and a free-module of rank $m$ of a central subalgebra is a Cayley-Hamilton algebra of degree $m^2$ with the regular trace.

2.5.2. Representations over commutative rings

For simplicity let $\text{char } k = 0$. Denote by $C_d$ the category of Cayley-Hamilton algebras of degree defined earlier and $C_{\text{Alg}}$ the category of commutative algebras over $k$ with morphisms being $k$-algebra homomorphism. In [15] and [16], a functor $F : C_d \to C_{\text{Alg}}$ is defined along with $i_R : R \to M_d(F(R))$ such that for any $A \in C_{\text{Alg}}$ and $j \in$
Hom$_{C_d}(R, M_d(A))$ there is $\bar{j} : F(R) \to A$ and $\bar{j}_d$ determined from $\bar{j}$ such that

$$
\begin{array}{ccc}
R & \xrightarrow{i} & M_d(F(R)) \\
\downarrow{j} & & \downarrow{\bar{j}_d} \\
M_d(A) & \xrightarrow{j} & A
\end{array}
$$

(2.29)

is commutative. A functor $G : C_{Alg} \to C_d$ can be defined using the vertical arrow in Diagram (2.29) such that $G(A) = M_d(A)$ and $G(\bar{j}) = \bar{j}_d$. Consequently, Diagram (2.29) can be relabeled as

$$
\begin{array}{ccc}
R & \xrightarrow{i} & GF(R) \\
\downarrow{j} & & \downarrow{G(\bar{j})} \\
G(A) & \xrightarrow{j} & A
\end{array}
$$

and $i$ is a universal arrow from $R$ to $G$. By Theorem 2.4.3 $F$ is a left adjoint of $G$ or for any $R \in C_d$, $A \in C_{Alg}$ there is a bijection of sets

$$
\text{Hom}_{C_d}(R, M_d(A)) \cong \text{Hom}_{C_{Alg}}(F(R), A).
$$

In fact, the map $i$ in Diagram (2.29) is injective. Thus an algebra with trace has a faithful representation $\phi : R \hookrightarrow M_d(B)$ compatible with trace for some commutative algebra $B$ if and only if $R$ is a Cayley-Hamilton algebra of degree $d$ [34, Theorem 0.3].

### 2.5.3. Representations over a field

Next we are going to focus on finitely generated Cayley-Hamilton algebras. A Cayley-Hamilton algebra $(R, C, \text{tr})$ is a finite module over $\text{tr}(R)$, so in particular it is a finite module over $C$ [16, Theorem 2.6]. If $(R, C, \text{tr})$ satisfies ($\text{CH}_d$), and denote its Jacobson radical by $J$, then [16, Proposition 3.4]

$$
R/(JR) \cong \bigoplus_{i=1}^k M_{n_i}(k)
$$
for some positive integers $k$, $n_i$ and there are positive integer $s_i$ such that

$$
\sum_{i=1}^{k} s_i n_i = d. \tag{2.30}
$$

If $(R, C, \text{tr})$ satisfies $(\text{CH}_d)$ for some $d$, define $V_k$ as the zero set of $D_k(R/C, \text{tr})$ (or equivalently of $MD_k(R/C, \text{tr})$) as in (1.1) then

$$
\emptyset = V_1 \subseteq \cdots \subseteq V_k \subseteq V_{k+1} \cdots. \tag{2.31}
$$

Recall (1.1) and $R$ is module finite over $C$, so there is a smallest positive integer $h$ such that

$$
\emptyset = V_1 \subseteq \cdots \subsetneq V_h \subsetneq V_{h+1} = \text{MaxSpec}(C). \tag{2.32}
$$

**Definition 2.5.6.** Then $D_h(R/C, \text{tr})$ is called the **highest discriminant ideal**.

Similarly there is a smallest $l$ such that

$$
\emptyset = V_1 = \cdots \subsetneq V_l \subseteq \cdots. \tag{2.33}
$$

**Definition 2.5.7.** Then $D_l(R/C, \text{tr})$ is called the **lowest discriminant ideal**.

### 2.6. The second cohomology group and group algebras of central extensions of Abelian groups

A short introduction to the second cohomology group can be found in [28] and we refer to reader to reference material e.g. [23] for more background. The second cohomology group classifies central extensions of groups. Our goal is to relate some of those extension to matrices with integer entries and then use theory of modules over PID.

**Definition 2.6.1.** Let $G$ be a group, and $M$ be an Abelian group in additive notation. A functions $f : G \times G \rightarrow M$ is called a **2-cocycle** if

$$
\beta(h, k) - \beta(gh, k) + \beta(g, hk) - \beta(g, h) = 0 \quad \forall g, h, k \in G \tag{2.34}
$$

---

This section is Section 3 of [29].
and a **2-coboundary** if there a function $f : G \to M$ such that

$$\beta(g, h) = f(h) - f(gh) + f(g) \quad \forall g, h \in G.$$  

Denote the set of 2-cocyles and 2-coboundaries as $Z^2(G, M)$ and $B^2(G, M)$. Then these have structures of Abelian groups from $M$. And the **2nd-cohomology group** is defined as the quotient group

$$H^2(G, M) = \frac{Z^2(G, M)}{B^2(G, M)}.$$  

Two 2-cocyles are called **cohomologous** if they are in the same cohomology class, i.e. the same in $H^2(G, M)$. Every 2-cocycle satisfies [28, Lemma 1.2.1]

$$\beta(1, g) = \beta(1, 1) = \beta(g, 1) \quad \text{and}$$

$$\beta(g, g^{-1}) = \beta(g^{-1}, g). \quad \forall g \in G.$$  

In fact, for every $\gamma \in Z^2(G, M)$, there is $\beta \sim_{H^2(G, M)} \gamma$ such that

$$\beta(1, g) = \beta(1, 1) = \beta(g, 1) = 1 \quad \text{and}$$

$$\beta(g, g^{-1}) = \beta(g^{-1}, g). \quad \forall g \in G.$$  

We now switch to additive notations for Abelian groups in the rest of the section.

**Definition 2.6.2.** Let $G$ be a group, $M$ be an Abelian group and $\beta \in Z^2(G, M)$, denote by $G_\beta$ the central extension of $G$ by $M$ with

$$G_\beta := \{(m, g) : m \in M, g \in G\},$$

$$(m_1, g_1)(m_2, g_2) := (m_1 + m_2 + \beta(g_1, g_2), g_1g_2), \quad m_i \in M, g_i \in G.$$
Remark 2.6.3. Any central extension of $G$ by $M$ is equivalent to $G_{\beta}$ for some $\beta \in Z^2(G, M)$ and two such extensions are equivalent if and only if the cocyles are cohomologous [28, Theorem 3.2.3].

Notation 2.6.4. We will use the notation that $\Lambda = A \times B$ where $A$ and $B$ are finitely generated abelian groups, $\Delta$ is a finite abelian group and $\Sigma = \Delta \rtimes_\gamma \Lambda = \Lambda_{\gamma}$ defined in (2.39) and (2.40) for some $\beta \in Z^2(\Lambda, \Delta)$. In $\Delta \rtimes_\beta \Lambda$, use the shorthand $(A, 0) = \{(0, (a, 0)) : a \in A\}$ and $(0, B) = \{(0, (0, b)) : b \in B\}$.

Definition 2.6.5. Denote the centralizers of a set $S$ in a group $G$ by $C_G(S)$. Define a set

$$\mathcal{B} = \{\beta \in Z^2(\Lambda, \Delta) : (A, 0) \subseteq C_\Sigma((A, 0)) \text{ and } (0, B) \subseteq C_\Sigma((0, B))\}. \quad (2.41)$$

Then $\mathcal{B}$ is a group. Set $\Sigma = \Delta \rtimes_\beta \Lambda$ and denote by $N^2(\Lambda, \Delta)$ the set of $\beta \in Z^2(\Lambda, \Delta)$ that satisfy

$$\beta((a_1, 0), (a_2, 0)) = \beta((0, b_1), (0, b_2)) = \beta((a_1, 0), (0, b_1)) = 0$$

$$\forall a_1, a_2 \in A \text{ and } b_1, b_2 \in B.$$  

Lemma 2.6.6. Using the notation above, then there is a split short exact sequence of Abelian groups where $i$ and the splitting homomorphism are inclusions.

$$0 \longrightarrow B^2(\Lambda, \Delta) \cap \mathcal{B} \overset{i}{\longrightarrow} \mathcal{B} \overset{\Phi}{\longrightarrow} N^2(\Lambda, \Delta) \longrightarrow 0. \quad (2.42)$$

Thus if $(A, 0)$ and $(0, B)$ are Abelian in $\Sigma$, then $\Sigma \cong \Delta \rtimes_\gamma \Lambda$ for some $\gamma \in N^2(\Lambda, \Delta)$.

Proof. $(A, 0)$ and $(0, B)$ are Abelian in $\Sigma$ is equivalent to

$$\beta((a_1, 0), (a_2, 0)) = \beta((a_2, 0), (a_1, 0)) \quad \forall a_1, a_2 \in A \quad \text{and} \quad (2.43)$$

$$\beta((0, b_1), (0, b_2)) = \beta((0, b_2), (0, b_1)) \quad \forall b_1, b_2 \in B. \quad (2.44)$$
Define $f : \Sigma \to \Delta$ by
\[
 f((a, b)) = \beta((a, 0), (0, b)) - \beta((0, 0), (0, 0)) \tag{2.45}\n\]
and $\Phi$ by
\[
 \Phi(\beta)(e_1, e_2) = \beta(e_1, e_2) - \beta((0, 0), (0, 0)) + f(e_1) + f(e_2) - f(e_1 + e_2). \tag{2.46}\n\]

In this work the field $k$ is assumed to be algebraically closed and we use $kG$ to denote the group algebra of $G$ over $k$.

**Notation 2.6.7.** In $A \times B$, for brevity denote $a = (a, 0)$ and $b = (0, b)$ for any $a \in A$ and $b \in B$. In a general group $G$, let $|g|_G$ denote the order of the element $g$. If no finite positive integer power of $g$ equals the identity, define $|g|_G = \infty$ and use the convention that any positive integer divides $\infty$ and $\gcd(l, \infty) = l$ for any positive integer $l$.

**Lemma 2.6.8 (compatibility).** Let $A, B$ be finitely generated Abelian groups, $\Lambda = A \times B$ and $\Delta$ be a finite Abelian group; $\beta \in N^2(\Lambda, \Delta)$.

(i) Then $\beta(a_i + a_j, b_k) = \beta(a_i, b_k) + \beta(a_j, b_k)$ and $\beta(a_i, b_k + b_l) = \beta(a_i, b_k) + \beta(a_i, b_l)$ for any $a_i, a_j \in A$ and $b_k, b_l \in B$.

(ii) If $a \in A$, $|a|_A = l$ and $b \in B$, $|b|_B = k$ then $|\beta(a, b)|_\Delta$ divides $\gcd(l, k)$.

(iii) There is a group isomorphism $\Phi : N^2(\Lambda, \Delta) \to \text{Hom}_\mathbb{Z}(A \otimes \mathbb{Z} B, \Delta)$. 

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Proof. (i) Since \( \Lambda \) is Abelian and \( \Delta \) is central, in the extension in the group algebra of \( k\Delta \)

\[
a_i a_j b_k b_l = a_i \beta(a_j, b_k) b_k a_j b_l = a_i (\beta(a_j, b_k) + \beta(a_j, b_l)) b_k b_l a_j
\]

\[
= (\beta(a_i, b_k) + \beta(a_i, b_l) + \beta(a_j, b_k) + \beta(a_j, b_l)) b_k b_l a_i a_j
\]

\[
= \beta(a_i + a_j, b_k + b_l) (b_k b_l) (a_i a_j)
\]

\[
= \beta(a_i + a_j, b_k + b_l) b_k b_l a_i a_j.
\]

Hence

\[
\beta(a_i + a_j, b_k + b_l) = \beta(a_i, b_k) + \beta(a_i, b_l) + \beta(a_j, b_k) + \beta(a_j, b_l).
\]

(ii) Recall \( \gcd(l, k) = x l + y k \) for some integers \( x \) and \( y \). From (i) in \( k\Delta \)

\[
1 \cdot b = a_1 = a_1 a_1 a_0 1 a_0 1 b = |a|_A \beta(a, b) b \cdot a_1 a_1 a_0 1 a_0 1 b = |a|_A \beta(a, b) b
\]

So

\[
|a| \beta(a, b) = 0 \quad \text{in } \Delta.
\]

This shows \( |\beta(a, b)|_\Delta \) divides \( |a|_A \).

(iii) Use the standard identification of Abelian groups with \( \mathbb{Z} \)-modules. Set

\[
\Phi(\beta)(a, b) = \beta((b, 0), (0, a)) \quad \forall \beta \in N^2(\Lambda, \Delta), a \in A, b \in B,
\]

\[
\Theta(\beta)((b, 0), (0, a)) = f(a, b) \quad \forall f \in \text{Hom}_\mathbb{Z}(A \otimes \mathbb{Z} B, \Delta), a \in A, b \in B.
\]

Well-definedness and injectivity of \( \Phi \) follow from (i)(ii) and universal property of tensor products. It is easy to verify that \( \Theta \) is well-defined and \( \Phi \circ \Theta \) is identity, so \( \Phi \) is surjective as well.

\( \square \)
Remark 2.6.9. Let \( \{e_1, \ldots, e_m\}, \{f_1, \ldots, f_n\} \) be the generators of \( A \) and \( B \), respectively. Then \( \beta \in N^2(\Lambda, \Delta) \) is uniquely determined by \( T \in M_{m,n}(\Delta) \) with entries \( t_{ij} = \beta(e_i, f_j) \) for \( 1 \leq i \leq m, 1 \leq j \leq n \) such that \( |t_{ij}|_\Delta \) divides \( \gcd(|e_i|_A, \gcd(|f_j|_B)) \).

Let \( n = |\Delta|, C = k\Delta, m \in \text{MaxSpec}(C) \) and \( \xi \) be a primitive \( n \)-th root of unity in \( k \). Then the images of \( t_{ij} \in \Delta \) above under the natural projection \( C \to C/m \) are \( \phi(t_{ij}) = \xi^{s_{ij}} \) for some integers \( 0 \leq s_{ij} < n \).

Remark 2.6.10. By the fundamental theorem of finitely generated Abelian groups

\[
A \cong \mathbb{Z}/\tilde{l}_1\mathbb{Z} \times \cdots \times \mathbb{Z}/\tilde{l}_m\mathbb{Z} \times \mathbb{Z}^{r_1},
\]

\[
B \cong \mathbb{Z}/\hat{l}_1\mathbb{Z} \times \cdots \times \mathbb{Z}/\hat{l}_n\mathbb{Z} \times \mathbb{Z}^{r_2}
\]

for some nonnegative integers \( \tilde{l}_i, \hat{l}_i, r_1 \) and \( r_2 \). Let \( \{e_1, \ldots, e_{m+r_1}\}, \{f_1, \ldots, f_{n+r_2}\} \) be the associated standard generators. Define

\[
x_i = \gcd_{1 \leq j \leq n+r_2} |t_{ij}|_\Delta,
\]

\[
y_j = \gcd_{1 \leq i \leq m+r_1} |t_{ij}|_\Delta.
\]

Then \( \Delta \) and \( \{e_1^{x_1}, \ldots, e_{m+r_1}^{x_{m+r_1}}\}, \{f_1^{y_1}, \ldots, f_{n+r_2}^{y_{n+r_2}}\} \) generate a central subgroup \( \Omega \) of \( \Sigma = \Delta \rtimes_\beta \Lambda \), denote \( H = k\Sigma \) and \( C = k\Omega \). Set

\[
l = \prod_{i=1}^{m+r_1} x_i \prod_{j=1}^{n+r_2} y_j.
\]

Then \( H \) is a free \( C \)-module of rank \( l \) and \( \langle \text{Im}(\beta) \rangle \subseteq \Delta \subseteq C \). Hence if \( \text{char} \ k \notin [1, l] \), \( (H, C, \text{tr}_{\text{reg}}) \) is a Cayley-Hamilton Hopf algebra of degree \( l \).

Corollary 2.6.11. Let \( \Sigma = \Delta \rtimes_\beta \Lambda \) with \( \beta \in N^2(\Lambda, \Delta) \) satisfying (2.38) and (2.44), and \( A \cong B \cong A_1 \times \cdots \times A_l \) be the elementary divisor form of the finite Abelian group \( A \), i.e.
\[ |A_j| = p_j^{\alpha_j} \text{ for distinct primes } p_j \text{ and } \Lambda_i = A_i \times A_i \text{ then:} \]

\[
\Sigma \cong (\Delta_1 \times_{\beta_1} \Lambda_1) \times \cdots \times (\Delta_l \times_{\beta_l} \Lambda_l)
\]

\[
k\Sigma \cong k(\Delta_1 \times_{\beta_1} \Lambda_1) \otimes_k \cdots \otimes_k k(\Delta_l \times_{\beta_l} \Lambda_l)
\]

for suitable Abelian groups \( \Delta \) with \( |\Delta_i| = p_i^{n_i} \) for some integers \( n_i \) and \( \beta_i \in N^2(\Lambda_i, \Delta_i) \) defined by restrictions.

2.7. Hopf algebras

A collection of results and definitions about Hopf algebras are given in this part, including group-like elements in the Hopf algebra of finite dual of a Hopf algebra, conditions for the antipode being bijective, smash product and winding automorphisms.

In a Hopf algebra \( H, \text{Alg}_k(H, k) \) is the same as the group-like elements in the finite dual of \( H \) denoted by \( G(H^\circ) \) [31, 1.3.5]. In fact \( G(H^\circ) \) has a group structure defined by convolution:

\[
(\chi_1 \cdot \chi_2)(h) = \sum \chi_1(h_1)\chi_2(h_2)
\]

in Sweedler notation and the inverse of \( \chi \in G(H^\circ) \) is the composition.

\[
\chi^{-1} = \chi \circ S.
\]

The following proposition gives two conditions for the antipode \( S \) itself being bijective [40]

Proposition 2.7.1 (Skryabin). The antipode \( S \) of a Noetherian Hopf algebra \( H \) is bijective if one of the two conditions below holds:

(i) \( H \) is semiprime;

(ii) \( H \) is affine PI.
Next part is the definitions of module, comodule algebra and smash product.

**Definition 2.7.2.** Let $H$ be a Hopf algebra over a field $k$. A $k$-algebra $A$ is a **left $H$-module algebra** if it satisfies the following conditions

(i) $A$ is a (left) $H$-module via $h \otimes a \mapsto h \cdot a$;

(ii) $h \cdot (ab) = \sum(h_1 \cdot a)(h_2 \cdot b)$;

(iii) $h \cdot 1_A = \varepsilon(h)1_A$

for any $h \in H, a, b \in A$.

**Definition 2.7.3.** An algebra $A$ is a **right $H$-comodule algebra** if

(i) $A$ is a right $H$-comodule via $\rho : A \mapsto A \otimes H$;

(ii) the multiplication $m_A : A \otimes A \mapsto A$ and injection $u_A : 1 \mapsto 1_A$ in the definition of the algebra $(A, m_A, u_A)$ are morphisms of right $H$-comodule.

**Definition 2.7.4.** Suppose $A$ is a left $H$-module algebra. The **smash product** $A \# H$ is defined for $a, a' \in A, h, h' \in H$ as below:

(i) $A \# H = A \otimes$ as vector spaces over $k$, the elements are formally written as $a \# h$.

(ii) Multiplication is defined by

$$(a \# h)(a' \# h') = \sum a(h_1 \cdot a') \# h_2 h'.$$  \hspace{1cm} (2.47)

In this thesis, winding $k$-algebra automorphisms of Hopf algebras play an important role.

**Definition 2.7.5.** Let $\phi \in G(H^\circ)$, then the **left and right winding** $k$-algebra automorphisms $W_l(\phi)$ and $W_r(\phi)$ are defined by

$$W_l(\phi)(h) = \sum \phi(h_1)h_2, \quad W_r(\phi)(h) = \sum \phi(h_2)h_1, \quad h \in H$$  \hspace{1cm} (2.48)

in Sweedler notation. It can be shown that they are indeed $k$-algebra automor-
morphisms of $H$ with inverse

$$W_l(\phi)^{-1} = W_l(\phi) \circ S, \quad W_r(\phi)^{-1} = W_r(\phi) \circ S. \quad (2.49)$$

And if $C$ is a Hopf subalgebra of $H$; $W_l(\phi)(C) = C$ and $W_r(\phi)(C) = C$ for any $\phi \in G(H^\circ)$. An immediate but useful result is

**Lemma 2.7.6.** Let $V$ be an $H$-module, $\phi \in G(H^\circ)$, then $\phi \otimes V$ is an $H$-module defined by the coproduct of $H$, i.e. $h \cdot (\phi \otimes v) = \sum \phi(h_1) h_2 \cdot v$ for $v \in V$ and

$$W_l(\phi)(\text{ann}(\phi \otimes V)) = \text{ann}(V). \quad (2.50)$$

Thus $\phi \otimes -$ is a bijection of latticies of left $H$-modules. In particular, if $V$ is a simple left $H$-module, $\phi \otimes V$ is simple.

**Proof.** This follows directly from the definition of the left winding automorphism and the fact that it is an algebra automorphism. \hfill \square

Lemma 2.7.6 has a right analog with similar proof. An important feature of winding automorphisms is that they action transitively on the set of codimension one ideals of $H$.

**Proposition 2.7.7.** If $H$ is a Hopf algebra over the field $k$. Let $M_1$ and $M_2$ be ideals of $H$ such that their codimension is one, then there are $\Phi_l \in W_l(G(H^\circ)) \subseteq \text{Aut}_{k-\text{Alg}} H$ and $\Phi_r \in W_r(G(H^\circ)) \subseteq \text{Aut}_{k-\text{Alg}} H$ such that $M_2 = \Phi_r(M_1)$.

**Proof.** It suffices to shows this for $M_2 = \ker \varepsilon$.

Let $\phi = H/M_1$ and define $\Phi_l = W_l(\phi)$ and $\Phi_r = W_r(\phi)$.

$$\varepsilon \circ \Phi_l(h) = \varepsilon(\sum \phi(h_1)h_2) = \sum \phi(h_1)\varepsilon(h_2) = \sum \phi(h_1\varepsilon(h_2)) = \phi(h).$$

The proof for $W_r$ is similar. \hfill \square
Corollary 2.7.8. If $H$ is a Hopf algebra over the field $\mathbb{k}$ and $V_1$ and $V_2$ are two one-dimensional left $H$-modules. Denote their left annihilators as $M_1 = \text{ann}(V_1)$ and $\text{ann}(V_2) = M_2$. Then there is a $\mathbb{k}$-algebra automorphism sending $M_1$ to $M_2$.

Corollary 2.7.9. Let $H$ be a Hopf algebra over the field $\mathbb{k}$ and $C$ its center or a central sub-Hopf algebra. If $m_1, m_2 \in \text{MaxSpec}(C)$ such that $H/m_1 H$ and $H/m_2 H$ have at least one one-dimensional representation, then there is a $\mathbb{k}$-algebra automorphism of $H$ sending $m_1$ to $m_2$.

Proof. Let $M_1 \supset m_1$, $M_2 \supset m_2$ be codimension-one ideals of $H$, then there is a $\mathbb{k}$-algebra automorphism of $H$ sending $M_1$ to $M_2$. Now $\Phi \in \text{Aut}_{\mathbb{k}-\text{Alg}}H$ defined in the proof of Proposition 2.7.7 sends $C$ to $C$ and $m_1 = M_1 \cap C$ and $m_2 = M_2 \cap C$. The proof is complete. \qed
Chapter 3. Fiberwise Actions on Irreducible Modules of Hopf Algebras

This chapter is concerned with an action of the identity fiber algebra defined in Definition 2.5.1 on the isomorphism classes of irreducible representations (equiv. irreducible modules) over an arbitrary \( m \in \text{MaxSpec}(C) \) when \( H \) is a finite module over \( C \).

We also defined and study maximally stable irreducible representations. Key results are summarized in Theorem A and will be used later for the investigation of the lowest discriminant ideal.

3.1. Adjunction from left and right duals

For a \( \mathbb{k} \)-algebra \( R \), denote the category of finite dimensional left \( R \)-modules by \( \text{mod}(R) \). left \( H \) be a Hopf algebra over a field \( \mathbb{k} \). For \( U, V \) and \( W \) in \( \text{mod}(H) \), denote by \( a_{U,V,W} \) the canonical left \( H \)-module isomorphism

\[
(U \otimes V) \otimes W \xrightarrow{a_{U,V,W}} U \otimes (V \otimes W)
\]

and identify \( U \) with \( \mathbb{k} \otimes U \).

**Definition 3.1.1.** Let \( W \in \text{mod}(H) \). Then \( W^* \in \text{mod}(H) \) is a left dual of \( W \) if there exist morphisms \( \text{ev}_W : W^* \otimes W \to \mathbb{k} \) and \( \text{coev} : \mathbb{k} \to W \otimes W^* \) in \( \text{mod}(H) \), called **evaluation** and **coevaluation**, respectively; such that the compositions of morphisms

\[
W \xrightarrow{\text{coev}_W \otimes \text{id}_W} (W \otimes W^*) \otimes W \xrightarrow{a_{W,W^*,W}} W \otimes (W^* \otimes W) \xrightarrow{\text{id}_W \otimes \text{ev}_W} W; \tag{3.1}
\]

\[
W^* \xrightarrow{\text{id}_{W^*} \otimes \text{coev}_W} W^* \otimes (W \otimes W^*) \xrightarrow{a_{W^*,W,W^*}^{-1}} (W^* \otimes W) \otimes W^* \xrightarrow{\text{ev}_W \otimes \text{id}_{W^*}} W^* \tag{3.2}
\]

are identities.

**Definition 3.1.2.** Let \( W \in \text{mod}(H) \). Then \( ^*W \in \text{mod}(H) \) is a right dual of \( W \) if there...
exist morphisms $\text{ev}_W' : W \otimes^* W \to k$ and $\text{coev}_W' : k \to *W \otimes W$ in $\text{mod}(H)$, respectively; such that the compositions of morphisms

$$
W \xrightarrow{\text{id}_W \otimes \text{coev}_W'} W \otimes (*W \otimes W) \xrightarrow{a_{W,*W,W}^{-1}} (W \otimes^* W) \otimes W \xrightarrow{\text{ev}_W' \otimes \text{id}_W} W, \tag{3.3}
$$

$$
*W \xrightarrow{\text{coev}_W' \otimes \text{id}_{*W}} (*W \otimes W) \otimes^* W \xrightarrow{a_{*W,*W,W}^*} *W \otimes (W \otimes^* W) \xrightarrow{\text{id}_{*W} \otimes \text{ev}_W'} *W, \tag{3.4}
$$

are identities.

Left and right dual can be used to make adjunctions [18, Proposition 2.10.8]

**Proposition 3.1.3.** If $V \in \text{mod} H$ has a left dual $V^*$ and a right dual $^*V$ and $M, W \in \text{mod} H$, then there are natural adjunction isomorphisms

$$
\Phi' : \text{Hom}_H(M, W \otimes V^*) \xrightarrow{\cong} \text{Hom}_H(M \otimes V, W) \quad \text{and} \quad \Phi'' : \text{Hom}_H(W \otimes ^*V, M) \xrightarrow{\cong} \text{Hom}_H(W, M \otimes V), \tag{3.5}
$$

given by

$$
\Phi'(\eta') := (\text{id}_W \otimes \text{ev}_V) \circ (\eta' \otimes \text{id}_V) \quad \text{and} \quad (\Phi'')^{-1}(\eta'') := (\text{id}_M \otimes \text{ev}_V^*) \circ (\eta'' \otimes \text{id}_{^*V}). \tag{3.6}
$$

for all $\eta' \in \text{Hom}_H(M, W \otimes V^*), \eta'' \in \text{Hom}_H(W, M \otimes V)$ and $m \in M, v \in V, w \in W, \xi \in ^*V$.

**Proof.** By definitions of left and right duals and Theorem 2.4.3, $- \otimes V^*$ is a left adjoint of $- \otimes V^*$ and a right adjoint of $^*V$. The rest follows from Theorem 2.4.3. \hfill \square

**Remark 3.1.4.** If the antipode $S$ is invertible then left and right duals exist. For $V \in \text{mod} H$, let $V^*$ and $^*V$ be vector spaces of $k$-linear maps $V \mapsto k$. Define the evaluation maps directly by

$$
\text{ev}_V : f \otimes v = f(v) \quad \text{and} \quad \text{ev}_V^* : v \otimes f = f(v).
$$
Let \( \{v_1, \cdots, v_n\} \) be a basis of \( V \) and \( \{f_1, \cdots, f_n\} \) be a corresponding dual basis, i.e. \( f_i(v_j) = \delta_{ij} \), then the coevaluations are defined by

\[
\text{coev}_V(1) = \sum_{i=1}^n v_i \otimes f_i \quad \text{and} \quad \text{coev}'_V(1) = \sum_{i=1}^n f_i \otimes v_i.
\]

Then equations (3.1)(3.2)(3.3)(3.4) can be immediately verified at the level of \( \mathbb{k} \)-vector space. The difficulty lie in finding the correct left \( H \)-module structure. As the antipode is bijective, we can define

\[
\langle h.f, v \rangle := \langle f, S(h).v \rangle \quad \text{and} \quad \langle h.f, v \rangle := \langle f, S^{-1}(h).v \rangle.
\]

Then \( \text{ev}_V \) and \( \text{ev}'_V \) are easily seen to be morphisms in \( \text{mod } H \). The map

\[
\text{ev}'_V|_{\text{coev}(\mathbb{k})} \circ \text{coev}_V = n \cdot \text{id}_\mathbb{k}
\]

is an isomorphism in \( \text{mod } H \) and so is \( \text{ev}_V|_{\text{coev}(\mathbb{k})} \), thus \( \text{coev} \) is a morphism in \( \text{mod } H \). The proof for \( \text{coev}' \) is similar. In this set-up the expressions for the adjunction isomorphisms (3.7) and (3.8) become

\[
\Phi'(\eta')(m \otimes v) := (\text{id}_W \otimes v)\eta'(m) \quad \text{and} \quad (\Phi''^{-1}(\eta'')(w \otimes f) := (\text{id}_M \otimes f)\eta''(w).
\]

### 3.2. A fiberwise action on irreducible modules

In this section, we restrict to the following setting:

(FinHopf) \( H \) is a finitely generated Hopf algebra over an algebraically closed field \( \mathbb{k} \) and \( C \) is a central Hopf subalgebra such that \( H \) is module finite over \( C \).

By Artin–Tate Lemma, \( C \) is a finitely generated commutative Hopf algebra, and thus is the coordinate ring of an affine algebraic group with identity element \( \mathfrak{m}_\mathbb{F} \) [20, Section 7.6].
The antipode of $H$ is bijective by Proposition 2.7.1 and we can consider both the left and right duals of a finite dimensional $H$-module $V$, which will be denoted by $V^*$ and $^*V$.

For a character $\varphi \in G(C^o)$, denote the maximal ideal

$$m_\varphi := \text{Ker } \varphi \in \text{MaxSpec } C.$$  

For $m \in \text{MaxSpec } C$, $C/m \cong \mathbb{k}$ by Hilbert’s Nullstellensatz; that is, $m = m_\varphi$ for some $\varphi \in G(C^o)$. For any two characters $\varphi, \psi \in G(C^o)$ of $C$, we have

$$(\varphi \otimes \psi)\Delta(m_{\varphi\psi}) = 0, \quad \text{and thus,} \quad \Delta(m_{\varphi\psi}) \subseteq m_\varphi \otimes C + C \otimes m_\psi.$$ 

Therefore, the coproduct $\Delta : H \to H \otimes H$ induces the algebra homomorphisms

$$\Delta : H/m_{\varphi\psi}H \to H/m_\varphi H \otimes H/m_\psi H.$$  

(3.9)

They turn $H/m_\varepsilon H$ into a finite dimensional Hopf algebra and $H/mH$ into finite dimensional left and right comodule algebras over $H/m_\varepsilon H$:

$$\Delta : H/mH \to H/m_\varepsilon H \otimes H/mH, \quad \Delta : H/mH \to H/mH \otimes H/m_\varepsilon H.$$  

(3.10)

It follows from (3.10) that $\text{mod}(H/m_\varepsilon H)$ is a tensor category and $\text{mod}(H/mH)$ are left and right module categories over it; i.e., we have the bifunctors

$$\text{mod}(H/m_\varepsilon H) \times \text{mod}(H/mH) \to \text{mod}(H/mH) \quad \text{and}$$  

(3.11)

$$\text{mod}(H/mH) \times \text{mod}(H/m_\varepsilon H) \to \text{mod}(H/mH),$$  

(3.12)

induced by (3.10). Both bifunctors will be denoted by $\otimes$ as they are tensor products of $H$-modules that filter through the quotients $H \to H/mH$ for $m \in \text{MaxSpec } C$. Furthermore,
the homomorphisms (3.9) give rise to the bifunctors

\[ \otimes : \text{mod}(H/m_\varphi H) \times \text{mod}(H/m_\psi H) \to \text{mod}(H/m_{\varphi \psi} H) \]  

(3.13)

for all characters \( \varphi, \psi \in G(C^o) \).

The next theorem addresses the action of the identity fiber \( \text{Irr}(H/m_\varepsilon H) \) of the central character map \( \kappa : \text{Irr}(H) \to \text{MaxSpec } C \) from (2.1) on the other fibers \( \text{Irr}(H/mH) \) of \( \kappa \).

**Theorem 3.2.1.** Assume that \( (H, C) \) is a pair of a Hopf algebra and a central subalgebra satisfying \( \text{(FinHopf)} \). Let \( m \in \text{MaxSpec } C \).

(a) For all \( V, W \in \text{Irr}(H/mH) \), there exist \( M', M'' \in \text{Irr}(H/m_\varepsilon H) \), such that \( W \) is a quotient of \( M' \otimes V \) and a submodule of \( M'' \otimes V \).

(b) The group

\[ G_0 := G((H/m_\varepsilon H)^o) \subseteq \text{Irr}(H/m_\varepsilon H) \]  

(3.14)

acts on \( \text{Irr}(H/mH) \) by \( V \in \text{Irr}(H/mH) \mapsto \chi \otimes V \) for \( \chi \in G_0 \).

(c) [4, Proposition III.4.11] If, in addition the algebra \( H/m_\varepsilon H \) is a basic, then the inclusion in (3.14) is an equality

\[ G_0 = G((H/m_\varepsilon H)^o) = \text{Irr}(H/m_\varepsilon H) \]

and the action in part (b) is transitive.

**Proof of Theorem 3.2.1.** (a) Since \( V, W \in \text{mod}(H/mH) \), \( V^* \in \text{mod}(H/S^{-1}(m)H) \), and by (3.13), \( W \otimes V^* \in \text{mod}(H/m_\varepsilon) \). Choose an irreducible module \( M' \in \text{mod}(H/m_\varepsilon) \) in the socle of \( W \otimes V^* \) (Remark 2.1.27). It gives rise to a nontrivial homomorphism

\[ \eta' \in \text{Hom}_H(M', W \otimes V^*) \].
The first adjunction formula (3.5) produces a nonzero homomorphism

$$\Phi'(\eta') \in \text{Hom}_H(M' \otimes V, W).$$

Since $W$ is irreducible, this homomorphism is surjective. Therefore, $W$ is a quotient of $M' \otimes V$. This proves the first statement of part (a) of the theorem.

Similarly, we have that $W \otimes \ast V \in \text{mod}(H/m)$. Let $M'' \in \text{mod}(H/m)$ be an irreducible $H$-module in the top of $W \otimes \ast V$, (Remark 2.1.30). We obtain a nontrivial homomorphism

$$\eta'' \in \text{Hom}_H(W \otimes \ast V, M'').$$

The second adjunction formula (3.6) gives a nonzero homomorphism

$$\Phi''(\eta'') \in \text{Hom}_H(W, M'' \otimes V).$$

This homomorphism is injective because $W$ is irreducible. Therefore, $W$ is a submodule of $M \otimes V$. This proves the second statement of part (a) of the theorem.

(b) For all $\chi \in G_0$ and $V \in \text{Irr}(H/mH)$, $\chi \otimes V$ is an irreducible $H$-module since $\chi \otimes -$ is a bijection of lattice of $H$-modules by Lemma 2.7.6. More directly, for every $H$-submodule $W \subseteq \chi \otimes V$, $\chi^{-1} \otimes W$ is a submodule of $V$. And if $W \neq 0$, $\chi^{-1} \otimes W \neq 0$ as $\chi \otimes \chi^{-1} \otimes W \cong W$. Alternatively, one can use Lemma 2.7.6. By (3.11), $\chi \otimes V \in \text{mod}(H/mH)$, and thus, $\chi \otimes V \in \text{Irr}(H/mH)$.

(c) A proof of this part was given in [4, Proposition III.4.11]. We include a proof based on part (a) for completeness. If the algebra $H/mH$ is a basic, then all of its irreducible modules are one-dimensional. So,

$$G((H/mH)^\circ) = \text{Irr}(H/mH).$$
Fix $V \in \text{Irr}(H/\mathfrak{m}H)$. By part (a), every $W \in \text{Irr}(H/\mathfrak{m}H)$ is a submodule of $\chi \otimes V$ for some $\chi \in G_0$. Since $\chi \otimes V$ is itself irreducible, this forces $W = \chi \otimes V$. Therefore, the $G_0$-action on $\text{Irr}(H/\mathfrak{m}H)$ is transitive. \hfill \Box

**Remark 3.2.2.** One can prove analogous statements to those in Theorem 3.2.1(a) for the tensor products $V \otimes M$ for $V \in \text{Irr}(H/\mathfrak{m}H)$ and $M \in \text{Irr}(H/\mathfrak{m}_\varepsilon H)$. This gives rise to a right action of $G_0$ on $\text{Irr}(H/\mathfrak{m}H)$ with the properties in Theorem 3.2.1(b)–(c). We will not need those facts in this paper and leave the details to the reader.

### 3.3. Stabilizers

For the rest of this section we continue to work under the assumption that $(H, C)$ is a pair of a Hopf algebra and a central subalgebra satisfying (FinHopf).

For $V \in \text{Irr}(H)$, denote the stabilizer of $V$ under the action in Theorem 3.2.1(b):$$\text{Stab}_{G_0}(V) = \{ \chi \in G_0 \mid \chi \otimes V \cong V \}.$$\[\text{Lemma 3.3.1.}\]

(a) For $V, W \in \text{Irr}(H)$ and $\chi \in G_0$, the following are equivalent:

(i) $\chi \otimes V \cong W$;

(ii) $\chi$ is a submodule of $W \otimes V^*$;

(iii) $\chi$ is a quotient of $W \otimes V$.

(b) For $V, W \in \text{Irr}(H)$ and $\chi \in G_0$, the following are equivalent:

(i) $\chi \in \text{Stab}_{G_0}(V)$;

(ii) $\chi$ is a submodule of $V \otimes V^*$;

(iii) $\chi$ is a quotient of $V \otimes V$.

The proof of part (a) is similar to the proofs of Theorem 3.2.1(a)–(b), using the
adjunction formulas (3.5)—(3.6), and is left to the reader. The second part of the lemma follows from the first by letting $W = V$.

**Proposition 3.3.2.** For all pairs $(H, C)$ of a Hopf algebra and a central subalgebra satisfying $(\text{FinHopf})$, and a module $V \in \text{Irr}(H)$, the following hold:

(a) $\bigoplus_{\chi \in \text{Stab}_{G_0}(V)} \chi$ is a submodule of $V \otimes V^*$ and a quotient of $V \otimes \ast V$.

(b) $|\text{Stab}_{G_0}(V)| \leq \dim(V)^2$.

**Proof.** (a) By Lemma 3.3.1(b), $\chi$ is a submodule of $V \otimes V^*$ for all $\chi \in \text{Stab}_{G_0}(V)$. Therefore, $\bigoplus_{\chi \in \text{Stab}_{G_0}(V)} \chi$ is a submodule of $V \otimes V^*$. The second statement is proved in a similar way by using Lemma 3.3.1(b) or by dualizing the first statement and using that $\chi^* = \chi S = \chi^{-1}$ for $\chi \in G_0$, so

$$
\left( \bigoplus_{\chi \in \text{Stab}_{G_0}(V)} \chi \right)^* \cong \bigoplus_{\chi \in \text{Stab}_{G_0}(V)} \chi.
$$

Part (b) follows from part (a) since

$$
\dim \left( \bigoplus_{\chi \in \text{Stab}_{G_0}(V)} \chi \right) = |\text{Stab}_{G_0}(V)|.
$$

\[\square\]

### 3.4. Maximally stable irreducible modules

By Proposition 3.3.2(b),

$$
|\text{Stab}_{G_0}(V)| \leq \dim(V)^2.
$$

**Definition 3.4.1.** An irreducible $H$-module $V$ will be called *maximally stable* if

$$
|\text{Stab}_{G_0}(V)| = \dim(V)^2.
$$
For example, all one-dimensional $H$-modules (i.e., all characters) are maximally stable.

**Lemma 3.4.2.** For each $\chi \in G_0$, $V \in \text{Irr}(H/mH)$ is maximally stable if and only if $\chi \otimes V$ is maximally stable.

**Proof.** This follows from the facts that

$$\text{Stab}_{G_0}(\chi \otimes V) = \chi \text{Stab}_{G_0}(V) \chi^{-1}$$

and

$$\dim(\chi \otimes V) = \dim(V).$$

\[\square\]

**Proposition 3.4.3.** For all pairs $(H, C)$ of a Hopf algebra and a central subalgebra satisfying (FinHopf), and $V \in \text{Irr}(H)$, the following are equivalent:

(i) $V$ is a maximally stable module;

(ii) $V \otimes V^*$ is a direct sum of nonisomorphic one-dimensional $H$-modules;

(iii) $V \otimes V^* \cong \bigoplus_{\chi \in \text{Stab}_{G_0}(V)} \chi$.

**Proof.** (i) $\Rightarrow$ (iii) Proposition 3.3.2(a) implies that $\bigoplus_{\chi \in \text{Stab}_{G_0}(V)} \chi$ is a submodule of $V \otimes V^*$. If $V$ is maximally stable, then the two modules have the same dimension, and thus, are equal.

(iii) $\Rightarrow$ (ii) This is obvious.

(ii) $\Rightarrow$ (i) Assume that

$$V \otimes V^* \cong \bigoplus_{\chi \in \mathcal{T}} \chi$$
for some subset $T \subseteq G_0$. Lemma 3.3.1(b) implies that $\text{Stab}_{G_0}(V) = T$. Therefore,

$$|\text{Stab}_{G_0}(V)| = |T| = \dim(V \otimes V^*) = \dim(V)^2,$$

and hence, $V$ is maximally stable.

3.5. Actions of twisted group algebras

For a group $G$ and a 2-cocycle

$$\gamma : G \times G \to k^*, \quad \gamma : G \times G \to k^*,$$

one defines the twisted group algebra [28, Definition 2.2.3]

$$k\gamma G := \bigoplus_{g \in G} kg$$

with the product

$$g.h = \gamma(g, h)gh,$$

expended by $k$-bilinearity. Associativity of multiplication is guaranteed by the 2-coycle condition (2.34). For 2-cocycles $\beta$ and $\gamma$ that are cohomologous, the corresponding $k$-algebras $k\beta G$ and $k\gamma G$ are isomorphic. A 2-cocycle $\gamma$ is normalized, if

$$\gamma(1, g) = \gamma(g, 1) = 1, \quad \forall g \in G.$$

For such a cocycle, $1 \in G$ is the identity element of $k\gamma G$. Each 2-cocycle is cohomologous to a normalized one.

Given a $G$-graded algebra

$$A = \bigoplus_{g \in G} A_g$$

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and a 2-cocycle $\gamma : G \times G \to k^*$, one defines the 2-cocycle twist $A_\gamma$, which is a $k$-algebra isomorphic to $A$ as a vector space, equipped with the new product

\[ a \bullet b := \gamma(g, h)ab, \quad \forall g, h \in G, \ a \in A_G, b \in A_h. \]

The group algebra $kG$ is naturally $G$-graded by putting $kg$ in degree $g$ for all $g \in G$, and the twist of it by a 2-cocycle $\gamma$ is canonically isomorphic to the corresponding twisted group algebra

\[ k_\gamma G \cong (kG)_\gamma. \]

Returning to our setting (FinHopf), for a subgroup $G \subseteq G(H^\circ)$, define

\[ \chi h = W_1(\chi)^{-1}(h)\chi. \]

Then this is a left $kG$-module algebra. Definition 2.7.2(i) follows from the fact that $W_1 : G(H^\circ) \to \text{Aut}_{k-\text{alg}}(H, C)$ is a group antihomomorphism. Definition 2.7.2(iii) is easily verified. Recall that for a group algebra $kG$, the coproduct is defined by $\Delta g = g \otimes g$. To verify Definition 2.7.2(ii), pick $\chi \in G$, $h, k \in H$.

\[ (\chi \cdot h)(\chi \cdot k) = W_1(\chi)^{-1}(hk) \]

since $W_1(\chi)$ is an algebra homomorphism. Thus the smash product $H\#kG$ can be defined following (2.47)

The algebra $H\#kG$ is $G$-graded by putting $H\chi$ in degree $\chi$ for $\chi \in G$. For a 2-cocycle $\gamma : G \times G \to k^*$, we denote the $k$-algebra

\[ H\#k_\gamma G := (H\#kG)_\gamma. \]
It contains $H$ and $k, G$ as subalgebras and is isomorphic to $H \otimes k, G$ as an $(H, k, G)$-bimodule with the standard structure on the tensor algebra. Furthermore,

$$\chi h = W_{i}(\chi)^{-1}(h)\chi, \quad \forall \chi \in G(H^{\circ}), \quad h \in H.$$ 

Consider an irreducible $H$-module $V$ and denote by

$$\pi : H \to \text{End}_{k}(V)$$

the $H$-action on $V$. For each $\chi \in \text{Stab}_{G_{0}}(V)$, fix a linear operator

$$L_{\chi} \in \text{End}_{k}(V),$$

giving the $H$-module isomorphism $V \xrightarrow{\pi} \chi^{-1} \otimes V$, where in the right hand side we use the canonical identification of $k$-vector spaces $k \otimes_{k} V \cong V$ by $k \otimes v = kv$. Since the module $V$ is irreducible, $L_{\chi}$ is uniquely defined up to rescaling.

We view $G_{0} = G((H/m_{\varepsilon}H)^{\circ})$ as a subgroup of $G(H^{\circ})$ by composing the projection $H \to H/m_{\varepsilon}H$ with the characters $\chi : H/m_{\varepsilon}H \to k$.

**Theorem 3.5.1.** Assume that $(H, C)$ is a pair of a Hopf algebra and a central Hopf subalgebra satisfying (FinHopf).

(a) For each irreducible $H$-module $V$, there exists a uniquely defined 2-cocycle

$$\gamma_{V} : \text{Stab}_{G_{0}}(V) \times \text{Stab}_{G_{0}}(V) \to k^{*}$$

such that the map

$$\chi \in \text{Stab}_{G_{0}}(V) \mapsto L_{\chi} \in \text{End}_{k}(V)$$

defines an injective algebra homomorphism

$$L : k_{\gamma_{V}} \text{Stab}_{G_{0}}(V) \to \text{End}_{k}(V).$$
(b) The homomorphisms $L$ and $\pi$ (from (3.15)) turn $V$ into an $H \# k_\gamma \operatorname{Stab}_{G_0}(V)$-module.

In other words, part (a) states that $L : \operatorname{Stab}_{G_0} \to \operatorname{GL}(V)$ is a section of a projective representation of the group $\operatorname{Stab}_{G_0}(V)$ with Schur multiplier $\gamma : G \times G \to k^*$, see [28, Section 2.1] or [23] for background on projective group representations.

Proof. (a) For each two characters $\chi, \theta \in G_0$, $L_\chi L_\theta \in \operatorname{End}_k(V)$ gives an isomorphism

$$V \xrightarrow{\cong} \theta^{-1} \otimes \chi^{-1} \otimes V.$$  

Therefore,

$$L_\chi L_\theta = \gamma(\chi, \theta)L_\chi \theta$$  \hspace{1cm} (3.16)

for a (unique) scalar $\gamma(\chi, \theta) \in k^*$.

Consider the canonical identification $V \otimes V^* \cong \operatorname{End}_k(V)$ as $k$-vector spaces and the embedding

$$\bigoplus_{\chi \in \operatorname{Stab}_{G_0}(V)} \chi \hookrightarrow V \otimes V^*$$

from Proposition 3.3.2. Under this identification, $L_\chi \in \operatorname{End}_k(V)$ corresponds to picking up a nonzero vector in the image of $\chi \hookrightarrow V \otimes V^*$. We note that this can be used as an alternative definition of the operators $L_\chi$. The identification and Proposition 3.3.2(a) imply that $L_\chi \in \operatorname{End}_k(V)$ are linearly independent for $\chi \in \operatorname{Stab}_{G_0}(V)$. The associativity property of the algebra $\operatorname{End}_k(V)$ now implies that $\gamma : G \times G \to k^*$ is a 2-cocycle and the map

$$\chi \mapsto L_\chi$$ defines a $k$-algebra embedding $k_\gamma \operatorname{Stab}_{G_0}(V) \hookrightarrow \operatorname{End}_k(V)$.

(b) This homomorphism and the homomorphism $\pi : H \to \operatorname{End}_k(V)$ from (3.15)
combine to give an $H\#k_\gamma\text{Stab}_{G_0}(V)$-module structure on $V$ because

$$L_\chi \pi(h) = \pi(W_i(\chi)^{-1}(h))L_\chi, \quad \forall h \in H, \chi \in \text{Stab}_{G_0}(V),$$

which follows from the fact that $L_\chi \in \text{End}_k(V)$ gives an $H$-module isomorphism $V \xrightarrow{\cong} \chi^{-1} \otimes V$.

\[\text{(3.16)}\]

**Remark 3.5.2.** It follows from (3.16) that rescaling $L_\chi$ for $\chi \in \text{Stab}_{G_0}(V)$ changes the cocycle $\gamma_V$ by a coboundary, so the cohomology class

$$[\gamma_V] \in H^2(\text{Stab}_{G_0}(V), k^*)$$

is an invariant of $V$.

**Theorem 3.5.3.** Let $(H, C)$ be pair of a Hopf algebra and a central subalgebra satisfying $(\text{FinHopf})$, and $V$ be a maximally stable irreducible module of $H$. Then the primitive quotient

$$H/\text{Ann}_H(V)$$

is isomorphic to a twisted group algebra

$$k_{\gamma_v}\text{Stab}_{G_0}(V)$$

for the 2-cocycle $\gamma_V : \text{Stab}_{G_0}(V) \times \text{Stab}_{G_0}(V) \to k^*$ from Theorem 3.5.1. Both algebras are isomorphic to $\text{End}_k(V)$.

**Proof of Theorem 3.5.3.** Denote for simplicity $\gamma := \gamma_V$. The homomorphism $L : k_{\gamma}\text{Stab}_{G_0}(V) \to \text{End}_k(V)$ is injective. By the assumption that $V$ is maximally stable,

$$\dim k_{\gamma}\text{Stab}_{G_0}(V) = \dim(V)^2 = \dim \text{End}_k(V).$$
Hence, the homomorphism $L$ is an isomorphism:

$$L : k_\gamma Stab_{G_0}(V) \xrightarrow{\simeq} \text{End}_k(V).$$

(3.17)

Since $V$ is an irreducible $H$-module (3.15)

$$H/\text{Ann}_H(V) \cong \text{End}_k(V).$$

(3.18)

is an immediate consequence of Remark 2.2.12. The theorem is proved by combining (3.17) with (3.18).

Remark 3.5.4. This theorem places a strong constraint on what Hopf algebras admit maximally stable irreducible modules of dimension $> 1$.

If $G$ is a finite group and $\text{char} \, k \nmid |G|$, then the group algebra $kG$ is semisimple by Maschke’s Theorem. Such a group algebra is simple, if and only if $G$ has order 1 because the blocks of $kG$ correspond to the irreducible $G$-modules, which in turn correspond to the conjugacy classes of $G$ (and the identity element $1 \in G$ forms a single conjugacy class).

On the other hand, there are nontrivial examples of twisted group algebras that are simple as described by the next example.

Example 3.5.5. For a positive integer $l$ consider the abelian group

$$\Lambda := \mathbb{Z}/l \times \mathbb{Z}/l\mathbb{Z}$$

and denote its standard generators by $a$ and $b$ in multiplicative notation. Let $k$ be an algebraically closed field of characteristic that does not divide $l$. Consider the 2-cocycle

$$\gamma : \Lambda \times \Lambda \rightarrow k^*, \quad \gamma(a^i b^j, a^k b^m) = \varepsilon^{jk}, \quad \forall i, j, k, m \in \mathbb{Z}/l,$$

(3.19)
where $\varepsilon \in \mathbb{k}$ is a primitive $l$-th root of unity. The corresponding twisted group algebra is isomorphic to a truncated quantum torus:

$$\mathbb{k}_\gamma \Lambda \cong \frac{\mathbb{k}\langle a, b \rangle}{(a^l - 1, b^l - 1, ba - \varepsilon ab)}.$$ 

We have the well known isomorphism $\mathbb{k}_\gamma \Lambda \cong M_l(\mathbb{k})$, given by

$$b \mapsto \text{diag}(1, \varepsilon, \ldots, \varepsilon^{l-1}), \quad a \mapsto (g_{st})^l_{s,t=1}, \quad g_{st} := \begin{cases} 
1, & \text{if } t \equiv s + 1 \pmod{l} \\
0, & \text{otherwise}.
\end{cases}$$

A more general example is given in Theorem 5.1.6(ii).

Theorem A in the introduction is the combination of Theorem 3.2.1, Propositions 3.3.2 and 3.4.3, and Theorem 3.5.3.
Chapter 4. The Lowest Discriminant Ideal and Winding Automorphisms

In this section we prove two results that characterize the lowest level set of the square dimension function of a Hopf algebra $H$ which is module finite over a central Hopf subalgebra $C$. They are in turn used to prove Theorems B and C from the introduction.

4.1. The square dimension function in the setting (FinFin)

We start with (FinFin), i.e. when $R$ is an affine $k$-algebra that is module finite over a central subalgebra $C$.

Definition 4.1.1. Define the square dimension function of the pair $(A, C)$,

$$Sd : \text{MaxSpec } C \to \mathbb{Z} \text{ by } \quad Sd(m) := \sum_{V \in \text{Irr}(A/mA)} \dim(V)^2.$$ (4.1)

In terms of the central character map $\kappa : \text{Irr}(A) \to \text{MaxSpec } C$ from (2.1), the right hand side of the defining equation for $Sd(m)$ ranges over the preimages $\kappa^{-1}(m)$.

We will be concerned with describing the lowest level set of the square dimension function of $(A, C)$,

$$Sd^{-1}(l) \subseteq \text{MaxSpec } C, \quad \text{where } \quad l := \min Sd(\text{MaxSpec } C)$$

in the Hopf case.

4.2. First theorem for the lowest level set of the square dimension function of a Hopf algebra

Through the rest of the section we will work in the following setting:

(FinHopfBasic) $H$ is a finitely generated Hopf algebra over an algebraically closed field $k$ and $C$ is a central Hopf subalgebra such that $H$ is module finite over $C$ and the algebra $H/m\pi H$ is basic.

This chapter is published in Section 4 of [30].
Lemma 4.2.1. For any \( m \in \text{MaxSpec } C, V, W \in \text{Irr}(H/mH), \)

(i) \( \dim(V) = \dim(W). \)

(ii) If furthermore \((H, C, \text{tr})\) is a prime Cayley-Hamilton Hopf algebra of degree \( d, C \) is integrally closed, then

\[
\frac{|G_0| \dim(V)}{|\text{Stab}_{G_0}(V)|} \bigg| d.
\]

In particular, \( |G_0| \bigg| d. \)

Proof. (i) By Theorem 3.2.1(c) the action of \( G_0 \) on \( \text{Irr}(H/mH) \) is transitive, so there is \( \chi \in G_0 \) such that \( \chi \otimes V \cong W \) and for any \( \chi \in G_0, \dim(\chi \otimes V) = \dim V; \) therefore, \( \dim V = \dim W. \)

(ii) Under these assumptions, in Example 2.3.4 \( \text{char } k = 0 \) or

\[
\text{char } k > \text{tr}(1) \geq [Z : F],
\]

so \( Z \) is a separable extension of \( F, \) there is an intrinsically defined trace \( \text{tr}_{[H:C]} \) such that \((H, C, \text{tr}_{[H:C]})\) is a Cayley-Hamilton Hopf algebra and \( \text{tr} = r\text{tr}_{[H:C]} \) for some integer \( r \) [16, Theorem 4.2]. Let \( \chi \in G_0 \) such that \( \chi \otimes V \cong W, \) then by Lemma 2.7.6

\[
H/\text{Ann}(W) = W_1(\chi)^{-1}(V).
\]

Hence for \( \text{tr}_{[H:C]} \) in (2.30) all \( s_i's \) are equal, i.e.

\[
\sum_{U \in \text{Irr}(R/mR)} s \cdot \dim(U) = s \cdot \frac{|G_0| \dim(V)}{|\text{Stab}_{G_0}(V)|} = \text{tr}_{[H:C]}(1)
\]

for some integer \( s. \) Denote \([H : C] := \text{tr}_{[H:C]}(1), \) then

\[
\frac{|G_0| \dim(V)}{|\text{Stab}_{G_0}(V)|} \bigg| [H : C] \bigg| d.
\]
Theorem 4.2.2. Let \((H, C)\) be a pair of a Hopf algebra and a central Hopf subalgebra satisfying (FinHopfBasic). The following hold:

(a) For any \(m \in \text{MaxSpec } C\) and \(V \in \text{Irr}(H/mH)\),

\[
\text{Sd}(m) = \frac{|G_0| \dim(V)^2}{|\text{Stab}_{G_0}(V)|},
\]

(b) The lowest level set of the square dimension function of \((H, C)\) is

\[
\tilde{l} = |G_0|.
\]

(c) The following are equivalent for \(m \in \text{MaxSpec}(C)\):

(i) \(m\) belongs to the lowest level set \(\text{Sd}^{-1}(|G_0|)\) of the square dimension function of \((H, C)\);

(ii) There exists \(V \in \text{Irr}(H/mH)\) that is maximally stable;

(iii) All modules \(V \in \text{Irr}(H/mH)\) are maximally stable.

Proof. (a) Thus,

\[
\text{Sd}(m) = \frac{|G_0| \dim(V)^2}{|\text{Stab}_{G_0}(V)|},
\]

which proves (a).

(b) We have

\[
\text{Sd}(m) = |\text{Irr}(H/m_H)| = |G_0|.
\]

Part (a) of the theorem and Proposition 3.3.2(b) imply that for all \(m \in \text{MaxSpec } C\),

\[
\text{Sd}(m) = \frac{|G_0| \dim(V)^2}{|\text{Stab}_{G_0}(V)|} \geq |G_0|,
\]

which proves (b).
(c) Lemma 3.4.2 and the transitivity of the $G_0$-action on $\text{Irr}(H/\mathfrak{m}H)$ from Theorem 3.2.1(c) imply that (ii) $\iff$ (iii). By parts (a) and (b), $\mathfrak{m} \in \text{MaxSpec } C$ belongs to the lowest level set $\text{Sd}^{-1}(|G_0|)$ if and only if

$$|G_0| = \text{Sd}(\mathfrak{m}) = \frac{|G_0|}{|\text{Stab}_{G_0}(V)|} \dim(V)^2,$$

i.e., if and only if $|\text{Stab}_{G_0}(V)| = \dim(V)^2$ for one $V \in \text{Irr}(H/\mathfrak{m}H)$, and thus, for any $V \in \text{Irr}(H/\mathfrak{m}H)$. This proves that (i) $\iff$ (ii).

4.3. Second theorem for the lowest level set of the square dimension function of a Hopf algebra

**Theorem 4.3.1.** Let $(H, C)$ be a pair of a Hopf algebra and a central Hopf subalgebra satisfying (FinHopfBasic).

(a) For $\mathfrak{m} \in \text{MaxSpec } C$ the following are equivalent:

(i) The algebra $H/\mathfrak{m}H$ has a one-dimensional module;

(ii) The algebra $H/\mathfrak{m}H$ is basic;

(iii) $H/\mathfrak{m}H$ and $H/\mathfrak{m}_eH$ are isomorphic as $\mathbb{k}$-algebras;

(iv) $\mathfrak{m}$ belongs to the orbit of $\mathfrak{m}_e$ under the group $\text{Aut}_{\mathbb{k}\text{-alg}}(H, C)$ of all $\mathbb{k}$-algebra automorphisms that preserve $C$;

(v) $\mathfrak{m}$ belongs to the orbit of $\mathfrak{m}_e$ under the left winding automorphism group $W_l(G(H^0))$;

(vi) $\mathfrak{m}$ belongs to the orbit of $\mathfrak{m}_e$ under the right winding automorphism group $W_r(G(H^0))$.

(b) If $\mathfrak{m} \in \text{MaxSpec } C$ satisfies any of the six equivalent conditions in part (a), then it belongs to the lowest level set $\text{Sd}^{-1}(|G_0|)$ of the square dimension function of
(c) If every maximally stable irreducible $H$-module is one-dimensional, then a maximal ideal $m \in \text{MaxSpec} C$ belongs to the lowest level set $Sd^{-1}(|G_0|)$ of the square dimension function of $(H, C)$ if and only if it satisfies any of the six equivalents conditions in part (a).

**Proof.** (a) We prove the implications

$$(i) \Rightarrow (v) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i).$$

Analogously to the first two implications, one shows that $(i) \Rightarrow (vi) \Rightarrow (iv)$.

$[(i) \Rightarrow (v)]$ Corollary 2.7.9.

$[(v) \Rightarrow (iv)]$ This implication is obvious since $W_1(G(H^\circ))$ is a subgroup of the automorphism group $\text{Aut}_{k-\text{alg}}(H, C)$.

$[(iv) \Rightarrow (iii)]$ This implication follows from the fact that, if $m$ and $m'$ are in the same orbit of $\text{Aut}_{k-\text{alg}}(H, C)$, then

$$H/mH \cong H/m'H$$

as $k$-algebras.

$[(iii) \Rightarrow (ii)]$ Since $H/mH \cong H/m_\tau H$ and $H/m_\tau H$ is a basic algebra, the $k$-algebra $H/mH$ is also basic.

$[(ii) \Rightarrow (i)]$ This implication is obvious.

(b) If $m \in \text{MaxSpec} C$ satisfies any of the six equivalent conditions in part (a) of the theorem, then $H/mH \cong H/m_\tau H$, and thus,

$$\text{Sd}(m) = \text{Sd}(m_\tau) = |G_0|.$$
By Theorem 4.2.2(b), $m$ belongs to the lowest level set $S_d^{-1}(|G_0|)$ of the square dimension function of $(H,C)$

(c) This part follows from Theorem 4.2.2(c) by comparing condition (ii) in Theorem 4.2.2(c) and condition (i) in Theorem 4.3.1(a).

4.4. Proofs of Theorems B(a)-(c) and C

Assume that $(H,C,\text{tr})$ is a Cayley–Hamilton Hopf algebras such that $H$ is a finitely generated $k$-algebra and the identity fiber $H/mH$ is a basic algebra. Theorem 4.5 in [15] implies that $H$ is module finite over $C$ and $C$ is a finitely generated $k$-algebra. Therefore the assumptions in Theorems 4.2.2 and 4.3.1 are satisfied.

Theorem 4.1(b) in [7] implies that

$$V(D_k(H/C, \text{tr})) = V(MD_k(H/C, \text{tr}))$$

$$= \{ m \in \text{MaxSpec } C \mid S_d(m) < k \},$$

recall (4.1). Theorems B(a)-(c) and C now follow from this relation and Theorems 4.2.2 and 4.3.1.

4.5. Integral orders and proof of Theorem B(d)

For the convenience of the reader, we recall some facts and the definition of AS-regular algebras. Suppose $R$ is a Noetherian ring, if the injective dimensions $\text{inj. dim}(R_R)$ and $\text{inj. dim}(R_R)$ are both finite then they are equal [37, Theorem 8.27] and it is a known fact that they are equal to the respective projective dimensions. There are slight differences in the definitions in [41] [25] [6] and we follow [6] to use the results in it. The convention is that in a Hopf algebra over a field $k$, the field is identified with the bimodule given by the counit $\varepsilon : H \mapsto k$.  

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Definition 4.5.1.

(i) A Noetherian Hopf algebra $H$ over a field $\mathbb{k}$ is **AS-Gorenstein** if

- (AS1) $\text{inj.dim}(H) = d < \infty$;
- (AS2) $\dim_k \text{Ext}^d_H(\mathbb{k}_H H) = 1$ and $\dim_k \text{Ext}^i_H(\mathbb{k}_H H) = 0$, $\forall i \neq d$,
- (AS3) the right $H$-module version of (AS1) and (AS2) hold.

(ii) $H$ is **AS-regular** if the global dimension of $H$ is also $d$.

The Hopf algebras we consider, i.e. those satisfying (FinHopf) or (FinHopfBasic) are Noetherian PI Hopf algebras where all irreducible modules are finite dimensional, so AS-Gorenstein [41, Theorem 3.5].

**Remark 4.5.2.** Condition (AS2) is equivalent to [3, Lemma 1.11]

(AS2)' for each finite dimensional simple left $H$-module $M$, $\dim_k \text{Ext}^d_H(H_M, H) = \dim M$

and $\dim_k \text{Ext}^i_H(H_M, H) = 0$, $\forall i \neq d$.

In [41] the condition formulated for a Noetherian algebra $A$ is

(AS2)'' for each simple $A$-module $M$, $\text{Ext}^d_A(A_M, A)$ is a simple $A^{\text{op}}$-module and

$\text{Ext}^i_A(A_M, A) = 0$, $\forall i \neq d$.

In order to get the $A^{\text{op}}$-module, we apply $\text{Ext}^d_A(A_M, -)$ to the injective resolution of $A$

as a left $A \otimes_k A^{\text{op}}$-module. (AS2)'' is a stronger condition than (AS2)' and is satisfied when

$A$ is a Noetherian PI Hopf algebra with all irreducible modules finite dimensional. When

$A$ is only a Noetherian PI algebra, the left and right versions of (AS1)+(AS2)'' are both

equivalent to the following [41, Proposition 3.2]:

There is $\mathcal{I} \subset \mathbb{Z}$ such that

(i) If $i \notin \mathcal{I}$, then $\text{Ext}^i_A(A_M, A) = 0$ for all $A_M$ simple.

(ii) If $i \in \mathcal{I}$, then $\text{Ext}^i_A(A_M, A) \neq 0$ for all $A_M$ simple and $\text{Ext}^i_A(-, A)$ is an exact
functor on left $A$-modules of finite length.

**Definition 4.5.3.** Let $H$ be an AS-Gorenstein Hopf algebra of injective dimension $d$.

(a) The left and right homological integrals of $H$ are the bimodules

$$
\int^l = \text{Ext}^d_H(\mathbb{k}, H H) \quad \text{and} \quad \int^r = \text{Ext}^d_{H^{\text{op}}}(\mathbb{k}, H H),
$$

(b) The integral order of $H$, denoted $io(H)$, is the minimal positive integer $n$ such that $(\int^r)^{\otimes n}$ is isomorphic to the trivial $H$-bimodule $\varepsilon$. If such a number does not exits, $io(H) = \infty$.

Here $\int^l$ and $\int^r$ are both $H$-bimodules; $\int^l$ has a trivial left $H$-module structure coming from the right $H$-module structure of $\mathbb{k}$ and $\int^r$ has a trivial right $H$-module structure coming from the left $H$-module structure of $\mathbb{k}$.

**Proof of Theorem B(d).** Proposition 4.5(c) in [6] implies that

$$
\int^l \cong 1_{\mathbb{k}^{\text{gr}}}(\chi)
$$

for some character $\chi$ of $H$, where the right hand side denotes the trivial $H$-bimodule given by the counit $\varepsilon$, twisted on the right side by the winding automorphism $W_l(\chi)$. By [6, Theorem 0.3], the Nakayama automorphism $\nu$ of $H$ equals

$$
\nu = S^2 W_l(\chi).
$$

We now apply [6, Proposition 4.4(b)], stating that $\nu$ acts trivially on $Z(H)$, to deduce that $W_l(\chi)$ acts trivially on $C$. This is equivalent to saying that

$$
\chi|_C = \varepsilon|_C.
$$

Therefore, $\chi \in G_0 = G((H/m_\varepsilon)^\circ)$. 

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If $H$ has a bijective antipode, then

$$S(f^l) = f^r \quad \text{and} \quad S(f^r) = f^l$$

by [25, Lemma 2.1]. In the cases of (FinHopf) and (FinHopfBasic), the antipode is indeed bijective by Proposition 2.7.1; thus, $io(H)$ equals the minimal positive integer $n$ such that $(f^l)^{\otimes n}$ is isomorphic to the trivial $H$-bimodule $\varepsilon$.

Hence, $io(H)$ equals the order of the element $\chi \in G_0$, which implies the first statement in Theorem B(d).

For the second statement in Theorem B(d), note that $l = |G_0| + 1$. By (2.30), $|G_0|$ is finite, so Lagrange’s theorem completes this statement. 

$\blacksquare$
Chapter 5. Applications

In this section we present applications of Theorems A-C from the introduction to group algebras of central extensions of abelian groups and (big) quantum groups at roots of unity. This illustrates the theorems and answers questions naturally arising from them.

5.1. Group algebras of some central extensions

The goal of this section is to consider group algebras of central extensions of the product of two finitely generated Abelian groups \( \Lambda = A \times B \) by a finitely Abelian group \( \Delta \). We will start the basic cases of \( \Delta = \mathbb{Z}/l\mathbb{Z} \) and \( A = B = \mathbb{Z}/l\mathbb{Z} \) or \( A = B = \mathbb{Z} \). We denote the central of an algebra \( R \) by \( Z(R) \).

5.1.1. Finite-dimensional basic case

Fix a positive integer \( l \). Consider the abelian group

\[ \Lambda := \mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z} \]

as in Example 3.5.5, written in multiplicative notation

\[ \Lambda := \langle a, b \mid a^l = b^l = 1, ab = ba \rangle, \]

and the cyclic group

\[ \Delta := \mathbb{Z}/l\mathbb{Z}, \]

also written in multiplicative notation

\[ \Delta := \langle c \mid c^l = 1 \rangle. \]

It is easy to verify that the map

\[ \beta : \Lambda \times \Lambda \to \Delta, \quad \text{given by} \quad \beta(a^i b^j, a^k b^m) = c^{jk}, \quad \forall i, j, k, m \in \mathbb{Z}/l \]  \hspace{1cm} (5.1)

Subsection 5.3 is section 4 in [29] and the rest of this chapter is published in Subsections 5.1-5.3 of [30].
is a 2-cocycle. We are going to consider this one. Denote the corresponding central extension
\[ \Sigma := \Delta \rtimes_\beta \Lambda, \]
i.e.,
\[ \Sigma := \{ c^s a^i b^j \mid a^i b^j \in \Lambda, c^s \in \Delta \} \]
with the product
\[ (c^s a^i b^j)(c^t a^k b^m) = c^{s+t+jk} a^{i+k} b^{j+m}, \quad \forall i, j, k, m, s, t \in \mathbb{Z}/l\mathbb{Z}. \]

Let \( k \) be an algebraically closed field of characteristic 0 or \( > l^2 \). Consider the group algebras
\[ H := k\Sigma \cong \frac{k\langle a, b, c \rangle}{(ac - ca, bc - cb, ba - cab, a^l - 1, b^l - 1, c^l - 1)} \]
\[ C := k\Delta \cong \frac{k\langle c \rangle}{(c^l - 1)}. \]

Since \( \Sigma \) is a central extension of \( \Lambda \) by \( \Delta \), \( C \subseteq Z(H) \). It is easy to verify from the presentation of \( H \) that
\[ C = Z(H). \]

The Hopf algebra \( H \) is a free module over the Hopf subalgebra \( C \) of rank \( l^2 \) with basis
\[ \{ a^i b^j \mid i, j \in \mathbb{Z}/l\mathbb{Z} \}. \]

By Example 2.5.5, the regular trace function \( \text{tr}_{\text{reg}} : H \to C \) makes the triple \( (H, C, \text{tr}_{\text{reg}}) \) a Cayley–Hamilton algebra of degree \( l^2 \).

Denote by \( \varepsilon \in k \) a primitive \( l \)-th root of 1. We have the isomorphism
\[ C \cong k[c]/(c^l - 1), \]
and thus,

\[ \text{MaxSpec } C = \{ m_s \mid s \in \mathbb{Z}/l\mathbb{Z} \}, \quad \text{where } m_s := (c - \varepsilon^s). \]

The kernel of the counit \( \varepsilon : C \to k \) is \( m_0 = m_0 \). For \( s \in \mathbb{Z}/l\mathbb{Z} \) consider the 2-cocycles

\[ \beta_s : \Lambda \times \Lambda \to k^*, \quad \beta_s(a^ib^j, a^kb^m) = \varepsilon^{jks}, \quad \forall i, j, k, m \in \mathbb{Z}/l\mathbb{Z}. \]

For \( s = 1 \) we recover the cocycle from (3.19), \( \beta_1 = \gamma \), and for \( s = 0 \) the trivial one.

The fiber algebras of \( H \) with respect to all maximal ideals \( m_s \in \text{MaxSpec } C \) are truncated quantum tori, which can be viewed as twisted group algebras of \( \Lambda \),

\[ H/m_sH = \frac{k\langle a, b \rangle}{(a^i - 1, b^j - 1, ba - \varepsilon^sab)} \cong k_{\beta_s}\Lambda, \quad \forall s \in \mathbb{Z}/l\mathbb{Z}. \]

In particular, the identity fiber algebra is isomorphic to the group algebra of the abelian group \( \Lambda \),

\[ H/m_0H = H/m_0H = k[a, b]/(a^i - 1, b^j - 1) \cong k\Lambda, \]

and so is a basic algebra. The corresponding group \( G_0 \) from Theorem A(a) is the dual group \( \hat{\Lambda} \):

\[ G_0 = G((H/m_0H)^\circ) = \hat{\Lambda} = \{ \chi_{i,j} \mid i, j \in \mathbb{Z}/l\mathbb{Z} \}, \quad \chi_{i,j}(a) = \varepsilon^i, \chi_{i,j}(b) = \varepsilon^j. \]

For \( s \in \mathbb{Z}/l \), denote

\[ d := \gcd(l, s). \]

It is easy to verify that the map

\[ b \mapsto \text{diag}(1, \varepsilon^s, \ldots, \varepsilon^{(l/d - 1)s}), \quad a \mapsto (g_{rt})_{r,t=1}^{l/d}, \quad g_{rt} := \begin{cases} 1, & \text{if } t \equiv r + 1 \pmod{l/d} \\ 0, & \text{otherwise} \end{cases} \]
is an irreducible $H/\mathfrak{m}_sH$-module of dimension $l/d$. Denote it by $V_s$ with standard basis \{\(e_1, \cdots, e_{l/d}\}\}. Next we want to show

\[
\text{Stab}_{G_0}(V_s) = \Lambda^d = \{\chi_{i,j} \mid i, j \in d(\mathbb{Z}/l\mathbb{Z})\}.
\]  

(5.2)

To this end, we may assume $s = d$; this corresponds to another choice of the primitive $l$-th root of unity $\varepsilon$. First step is to prove

\[
\chi_{i,j} \otimes V_s \cong V_s, \quad \forall i, j \in \mathbb{Z}/l\mathbb{Z} \text{ such that } d|i, j.
\]  

(5.3)

Indeed, the representation $\chi_{i,j} \otimes V_s$ can be identified with the action of $H/\mathfrak{m}_sH$ on $V_s$ given by

\[
b \mapsto \text{diag}(\varepsilon^j, \varepsilon^{j+s}, \ldots, \varepsilon^{j+(l/d-1)s}), \quad a \mapsto (\varepsilon^t g_{rt})^{l/d}.
\]

Denote this module $W_s$ with standard basis \{\(f_1, \cdots, f_{l/d}\}\}. The goal is to find a $\mathbb{C}$-linear map $L : V_s \to W_s$ given by

\[
L(e_\alpha) = \sum_{1 \leq \beta \leq (l/d)} k_{\alpha,\beta} f_\beta \quad \text{for } k_{\alpha,\beta} \in \mathbb{C},
\]  

(5.4)

such that

\[
b_{W_s} \circ L = L \circ b_{V_s},
\]  

(5.5)

\[
a_{W_s} \circ L = L \circ a_{V_s}.
\]  

(5.6)

In the language of (5.4), (5.5) means

\[
\varepsilon^{j+(\beta-1)s} \sum_{\beta} k_{\alpha,\beta} f_\beta = \varepsilon^{(\alpha-1)s} \sum_{\beta} k_{\alpha,\beta} f_\beta.
\]

Thus $k_{\alpha,\beta} = 0$ unless $\beta = \alpha - (j/s)$, (5.6) can be written as

\[
\varepsilon^t k_{\alpha,\alpha-(j/s)} e_{\alpha-(j/s)+1} = k_{\alpha+1,\alpha+1-(j/s)} e_{\alpha-(j/s)+1}
\]
and the \( \mathbb{k} \)-linear map
\[
L(e_\alpha) = \varepsilon^{i\alpha} e_{\alpha-(j/s)}
\]
is the desired module isomorphism.

The number of the elements \( \chi_{i,j} \) of \( G_0 \) in (5.3) equals \( l^2/d^2 \). Proposition 3.3.2(b) implies that \( |\text{Stab}_{G_0}(V_s)| \leq \dim(V_s)^2 = l^2/d^2 \), and (5.2) is proved. (Here and below, we use \( d \)-th powers because, by convention, all of our abelian groups are written in the multiplicative notation.) The corresponding 2-cocycle from Theorem 3.5.1 is
\[
\gamma_{V_s} = \beta_k|_{\Lambda^d}
\]
in terms of the isomorphism \( \Lambda \cong (\mathbb{Z}/l\mathbb{Z}) \times (\mathbb{Z}/l\mathbb{Z}) \cong \Lambda \). Theorem 3.2.1(c) implies
\[
\text{Irr}(H/mH) = \{\chi_{i,j} \otimes V_s \mid i, j \text{ running over a set of representatives of } \Lambda/\Lambda^d\}. \tag{5.7}
\]
Hence, the square dimension function (4.1) is given by
\[
\text{Sd}(m_s) = d^2 \frac{l^2}{d^2} = l^2, \quad \forall s \in \mathbb{Z}/l\mathbb{Z}.
\]

Applying (4.2) yields
\[
\mathcal{V}(D_k(A/C, \text{tr})) = \mathcal{V}(MD_k(A/C', \text{tr})) = \begin{cases} 
\varnothing, & k \leq l^2 \\
\text{MaxSpec } C, & k > l^2.
\end{cases}
\]
Another consequence of (5.7) is that the group of characters of \( H \) is
\[
G(H^0) = G((H/m_0)^0) = G_0.
\]
Theorem 4.3.1(a) implies that the orbits of \( m_0 \) under the left winding automorphisms of \( H \), the right winding automorphisms of \( H \) and \( \text{Aut}(H, C) \) consist of the maximal ideal \( m_0 \) alone. Thus, we obtain the following:
Theorem 5.1.1. Let $l$ be a positive integer and $k$ be an algebraically closed field of characteristic 0 or $> l^2$.

(a) The Hopf algebra

$$H := k \left( \left( \mathbb{Z}/l\mathbb{Z} \right) \rtimes \beta \left( \mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z} \right) \right)$$

is a free module over its center

$$Z(H) = k(\mathbb{Z}/l\mathbb{Z})$$

of rank $l^2$. This gives rise to the Cayley–Hamilton Hopf algebra structure

$$(H, Z(H), \text{tr}_{\text{reg}})$$

doing of degree $l^2$.

(b) $H$ has maximally stable representations of dimension $g$ for all positive divisors $g$ of $l$. All irreducible modules of $H$ are maximally stable.

(c) The zero sets of the discriminant/modified discriminant ideals of $(H, Z(H), \text{tr})$ are

\[ \mathcal{V}(D_k(H/Z(H), \text{tr})) = \mathcal{V}(MD_k(H/Z(H), \text{tr})) = \begin{cases} \emptyset, & k \leq l^2 \\ \Omega_l, & k > l^2 \end{cases} \]

where

$$\Omega_l := \{ \omega \in k \mid \omega^l = 1 \}.$$ 

(d) The left and right winding automorphism groups of $H$, as well as the group

$\text{Aut}(H, Z(H))$, fix the identity element of the group $\text{MaxSpec} Z(H) \cong \Omega_l$.

Part (b) of the proposition shows that, in general, the maximally stable representations $V$ of a Hopf algebra $H$ can have arbitrary dimension $\dim(V)$ such that

$$\dim(V)^2 \mid |G_0|.$$
Part (d) of the proposition shows that the inclusion in Theorem C(b) can be proper. Even worse, it can happen that

$$\text{Aut}(H,C).m_r = \{m_r\},$$

while the zero set of the lowest discriminant ideal of $(H,C,\text{tr})$ is the whole MaxSpec $C$.

5.1.2. Infinite dimensional basic case

The phenomena from the previous subsection are not restricted to the case of finite dimensional Cayley–Hamilton Hopf algebras. Here we describe how they appear in the infinite dimensional case.

Fix again a positive integer $l$ and an algebraically closed field $k$ of characteristic $\text{char} k \notin [1, l^2]$, and retain the notation from the previous subsection. Consider the free abelian group

$$\tilde{\Lambda} := \langle x, y \mid xy = yx \rangle \cong \mathbb{Z} \times \mathbb{Z}$$

and the projection

$$\pi : \tilde{\Lambda} \to \Lambda, \quad \pi(x) = a, \pi(y) = b.$$ 

Denote the 2-cocycle

$$\tilde{\beta} := \beta \circ \pi : \tilde{\Lambda} \times \tilde{\Lambda} \to \Delta$$

and consider the central extension

$$\tilde{\Sigma} := \Delta \rtimes_{\beta} \tilde{\Lambda}.$$ 

Its group algebra is

$$\tilde{H} := \mathbb{k}\tilde{\Sigma} \cong \frac{\mathbb{k}\langle x^\pm 1, y^\pm 1, c \rangle}{(xc - cx, yc - cy, yx - cxy, c^l - 1)}.$$
Denote by $\tilde{\Delta}$ the subgroup of $\tilde{\Sigma}$ generated by $x^l, y^l$ and $c$. Clearly, $\tilde{\Delta} \cong \mathbb{Z} \times \mathbb{Z} \times (\mathbb{Z}/l\mathbb{Z})$.

One verifies that

$$Z(\tilde{H}) = k\tilde{\Delta} \cong \frac{k[X^{\pm 1}, Y^{\pm 1}, c]}{(c^l - 1)}$$

(5.8)

where the isomorphism is given by $x^l \mapsto X$, $y^l \mapsto Y$ and $c \mapsto c$. This implies in particular that $Z(\tilde{H})$ is a Hopf subalgebra of $\tilde{H}$. The group $\Sigma$ is isomorphic to the factor group of $\tilde{\Sigma}$ by the free abelian subgroup generated by $x^l, y^l$ by sending $x \mapsto a$, $y \mapsto b$, $c \mapsto c$, so

$$H \cong \tilde{H}/(x^l - 1, y^l - 1).$$

The Hopf algebra $\tilde{H}$ is a free module over $Z(\tilde{H})$ of rank $l^2$ with basis

$$\{x^iy^j \mid 0 \leq i, j \leq l - 1\},$$

which gives rise to the regular trace function $\text{tr}_{\text{reg}} : H \to C$, making $(H, C, \text{tr})$ a Cayley–Hamilton algebra of degree $l^2$ by Example 2.5.5.

It follows from (5.8) that

$$\text{MaxSpec } Z(\tilde{H}) = \{m_{u,v,s} \mid u, v \in k^*, s \in \mathbb{Z}/l\mathbb{Z}\}, \text{ where } m_{u,v,s} := (x^l - u, y^l - v, c - \varepsilon^s).$$

The kernel of the counit $\varepsilon : Z(\tilde{H}) \to k$ is

$$m_\varepsilon = m_{1,1,0}.$$

The fiber algebras of $\tilde{H}$ are

$$\tilde{H}/m_{u,v,s}\tilde{H} = \frac{k\langle x, y \rangle}{(x^l - u, y^l - v, yx - \varepsilon^s xy)} \cong \frac{k\langle a, b \rangle}{(a^l - 1, b^l - 1, ba - \varepsilon^s ab)} \cong k_{\beta_s} \Lambda, \quad \forall u, v \in k^*, s \in \mathbb{Z}/l\mathbb{Z},$$

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where the first isomorphism is given by \( x \mapsto \sqrt[7]{u}a, \) \( y \mapsto \sqrt[7]{v}b. \) From here on, \( \sqrt[7]{u} \) and \( \sqrt[7]{v} \) will denote two fixed choices of \( 7 \)-th roots of \( u \) and \( v \) in \( \mathbb{k}. \) The identity fiber algebra is basic:

\[
\tilde{H}/\mathfrak{m}_z \tilde{H} = \tilde{H}/\mathfrak{m}_{1,1,0} \tilde{H} = \mathbb{k}[x, y]/(x^l - 1, y^l - 1) \cong \mathbb{k} \Lambda.
\]

The group \( G_0 \) from Theorem A(a) is the dual group \( \hat{\Lambda}: \)

\[
G_0 = G((\tilde{H}/\mathfrak{m}_{1,1,0} \tilde{H})^o) = \hat{\Lambda} = \{ \chi_{i,j} \mid i, j \in \mathbb{Z}/l\mathbb{Z} \}, \quad \chi_{i,j}(x) = \varepsilon^i, \chi_{i,j}(y) = \varepsilon^j.
\]

In the notation \( d := \gcd(l, s) \), the map

\[
y \mapsto \sqrt[7]{v} \text{diag}(1, \varepsilon^s, \ldots, \varepsilon^{(l/d-1)s}), \quad x \mapsto (g_{rt})_{r,t=1}^{l/d}, \quad g_{rt} := \begin{cases} 
\sqrt[7]{u}, & \text{if } t \equiv r + 1 \pmod{l/d} \\
0, & \text{otherwise}
\end{cases}
\]

defines an irreducible \( \tilde{H}/\mathfrak{m}_{u,v,s} \tilde{H} \)-module of dimension \( l/d \), to be denoted by \( V_{u,v,s} \), and

\[
\text{Stab}_{G_0}(V_{u,v,s}) = \hat{\Lambda}^d = \{ \chi_{i,j} \mid i, j \in d(\mathbb{Z}/l\mathbb{Z}) \}.
\]

In particular, \( V_{u,v,s} \) is maximally stable. The corresponding 2-cocycle from Theorem 3.5.1 is again computed to be \( \gamma_V = \beta_s|_{\hat{\Lambda}^d} \), assuming the identification \( \hat{\Lambda} \cong \mathbb{Z} \times \mathbb{Z} \cong \Lambda. \) Theorem 3.2.1(c) implies

\[
\text{Irr}(\tilde{H}/\mathfrak{m}_{u,v,s} \tilde{H}) = \{ \chi_{i,j} \otimes V_s \mid i, j \text{ running over a set of representatives of } \hat{\Lambda}/\hat{\Lambda}^d \},
\]

and thus,

\[
\text{Sd}(\mathfrak{m}_{u,v,s}) = l^2, \quad \forall u, v \in \mathbb{k}^*, s \in \mathbb{Z}/l\mathbb{Z}.
\]

Equation (4.2) and Theorem 4.3.1(a) imply the following:

**Theorem 5.1.2.** Let \( l \) be a positive integer, \( \mathbb{k} \) be an algebraically closed field of characteristic \( \text{char} \mathbb{k} \notin [1, l^2] \) and

\[
\tilde{H} := \mathbb{k} \left( (\mathbb{Z}/l\mathbb{Z}) \times \beta (\mathbb{Z} \times \mathbb{Z}) \right).
\]
(a) The Hopf algebra $\tilde{H}$ is free over its center $Z(\tilde{H})$, which is isomorphic to the coordinate ring of the affine algebraic group $\mathbb{G}_m^2 \times \Omega_l$ as a Hopf algebra. This makes the triple $(H, Z(H), \text{tr}_{\text{reg}})$ a Cayley–Hamilton Hopf algebra of degree $l^2$.

(b) All irreducible modules of $\tilde{H}$ are maximally stable and those of the fiber algebras $H/\mathfrak{m}_{u,v,s}H$ have dimension $l^2/\gcd(s, l)^2$ for all $u, v \in k^*$, $s \in \mathbb{Z}/l\mathbb{Z}$.

(c) We have

$$V(D_k(\tilde{H}/Z(\tilde{H}), \text{tr})) = V(MD_k(\tilde{H}/Z(\tilde{H}), \text{tr})) = \begin{cases} \emptyset, & k \leq l^2 \\ \mathbb{G}_m^2 \times \Omega_l, & k > l^2. \end{cases}$$

(d) The winding automorphism groups $W_l(\tilde{H})$ and the automorphism group $W_r(\tilde{H})$ and the group $\text{Aut}(\tilde{H}, Z(\tilde{H}))$ fix the identity element of $\text{MaxSpec} Z(\tilde{H}) \cong \mathbb{G}_m^2 \times \Omega_l$.

Remark 5.1.3. Similar calculation to [25, Example 2.7] yields $io(H) = l$ in this and the previous example, so the integral order equals the order the largest cyclic subgroup of $G_0$ in these two examples and is strictly smaller than $G_0$ if $l \neq 1$.

5.1.3. General cases

We now return to the general case of central extension of $\Lambda = A \times B$ by $\Delta$ for $A$, $B$ finitely generated Abelian groups and $\Delta$ finite Abelian such that $(A, 0) \subseteq C((0, B))$ and $(0, B) \subseteq C((A, 0))$ in the extension.

Lemma 5.1.4. Let

$$H = \frac{k < x, y >}{(x^l - 1, y^m - 1, xy - \xi yx)}, \quad C = k$$

where $\xi$ is a primitive root of unity of order $n$ and $n|\gcd(l, m)$, then

(i) $R$ has an irreducible representation $V$ of dimension $n$;
(ii) The stabilizer of $V$ is given by

$$\text{Stab}_{G_0}(V) = \{ \chi \in G(H^\circ) : \chi(x) = \xi^i, \chi(y) = \xi^j, i, j \in \mathbb{Z} \}$$

$$\cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$ 

Proof. (i) Let $V$ be a vector space with basis $\{v_0, \cdots, v_{n-1}\}$. Define

$$x \cdot v_i = \xi^i v_i, \quad y \cdot v_i = v_j, \quad j = i + 1 \mod n.$$ 

Then $V$ is an $n$-dimensional representation of $R$.

(ii) Similar to the finite dimensional basic case. $\square$

Lemma 5.1.5. Let $A$ and $B$ be finitely generated Abelian groups, $\Delta$ be a finite cyclic group of order $d$, $\Lambda = A \times B$ and $\Sigma = \Delta \rtimes_\beta \Lambda$ for some $\beta \in B$. Then there are minimal sets of generators of $A$ and $B$ such that the matrix $T$ defined in Remark 2.6.9 is in the form

$$\begin{bmatrix}
t_{11} & \cdots & 0 \\
t_{22} & 0 & \cdots \\
\vdots & & \ddots \\
0 & \cdots & t_{kk}
\end{bmatrix}$$

for some $t_{11} | \cdots | t_{kk} | d$. Minimal means there is no proper subset of the generators that generated the group.

Proof. This is an application of theory of finitely generated modules over a PID [22, Theorem 3.8]. $\square$
Theorem 5.1.6. Let $A$ and $B$ be finitely generated Abelian groups, $\Delta$ be a finite Abelian group, $\Lambda = A \times B$ and

$$1 \to \Delta \to \Sigma \overset{f}{\to} \Lambda \to 1$$

be a central extension of $\Lambda$ by $\Delta$ such that $f^{-1}((A,0)) \subseteq C_{\Sigma}(f^{-1}((A,0)))$ as well as $f^{-1}((0,B)) \subseteq C_{\Sigma}(f^{-1}((0,B)))$, $\beta$ be a 2-cocycle associated with $\Sigma$. There is a central subgroup $\Omega \triangleleft \Sigma$ containing $\langle \text{Im}(\beta) \rangle$ of finite index, i.e. $m = [\Sigma : \Omega] < \infty$. Choose an algebraically closed field $k$ with $\text{char } k \not\in [1,m]$ and, define $H = k\Sigma$ and $C = k\Omega$. Let $m \in \text{MaxSpec}(C)$ and $\text{tr}_{\text{reg}}$ denote the regular trace then

(i) $(H, C, \text{tr}_{\text{reg}})$ is a Cayley-Hamilton Hopf algebra of rank $m$ with basic identity fiber,

$$\text{GKdim}(H) = \text{GKdim}(C) \text{ equals free rank of } \Omega.$$  

(ii) The algebra $R/mR$ is simple. Let $n = |\Delta|$ and $\xi$ be a primitive $n$-th root of unity.

Then there are positive integers $k$ and $l_i$, $1 \leq i \leq k$ such that

$$l_k \mid l_{k-1} \mid \cdots \mid l_1 \mid n.$$

$$R/mR \cong R_1 \otimes_k \cdots \otimes_k R_k \quad \text{where} \tag{5.9}$$

$$R_i \cong \frac{k \langle x_i, y_i \rangle}{(x_i^{l_i} - 1, y_i^{l_i} - 1, x_iy_i - \xi^n y_i x_i)}.$$

(iii) $R/mR$ has an irreducible representation $V$ with

$$\dim(V) = \prod_{i=1}^{k} l_i \quad \text{and} \tag{5.10}$$

$$\text{Stab}_{G_0}(V) \cong (\mathbb{Z}/l_1\mathbb{Z} \times \cdots \mathbb{Z}/l_k\mathbb{Z})^2.$$

Hence all irreducible representations of $H$ are maximally stable.
(iv) \[
\mathcal{V}(D_k(H/C, tr_{reg})) = \begin{cases} 
\emptyset, & k \leq m, \\
\text{MaxSpec}(C), & k > m.
\end{cases}
\] (5.11)

(v) Define the orbit of \( m_{\xi} \) under \( \text{Aut}_{k-\text{Alg}}(H, C) \) as \( \mathcal{O} \)
\[
\mathcal{O} = \text{Aut}_{k-\text{Alg}}(H, C) \cdot m_{\xi} = W_1(G(H^\circ)) \cdot m_{\xi} = W_i(G(H^\circ)) \cdot m_{\xi} \quad (5.12)
\]
\[
\cong \Omega / \langle \text{Im} \beta \rangle.
\]

As a group \( \mathcal{O} \) is isomorphic to the subgroup of \( G(C^\circ) \) with basic fibers.

Proof of Theorem 5.1.6. By Lemma 2.6.6, we may assume \( \Sigma \cong \Delta \times_\beta \Lambda \) for some \( \beta \in N^2(\Lambda, \Delta) \). The existence of \( \Omega \) is shown in Remark 2.6.10.

(i) The GK-dimension is routine. \( H = k\Sigma \) is a free module over \( C = k\Omega \) of rank \( [\Sigma : \Omega] \).

(ii) Similar to the proof of Lemma 5.1.5, the matrix \( S \) defined in Remark 2.6.9 can also be written in some \( \mathbb{Z} \)-linearly independent basis of \( A \) and \( B \) as
\[
\begin{bmatrix}
s_{11} & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & s_{kk}
\end{bmatrix}
\] (5.13)

for some \( s_{11} | \cdots | s_{kk} | n = |\Delta| \) thus (5.9) holds. To shown \( R/mR \) is simple, define \( V_i \) as irreducible representations of \( R_i \) with \( \text{dim } V_i = l_i \) and denote the left annihilators of \( V_i \) in \( R_i \) by \( \text{lann}(R_i V_i) \), then
\[
R/mR \cong R_1/\text{lann}(R_1 V_1) \otimes_k \cdots \otimes_k R_k/\text{lann}(R_k V_k) \cong R/\text{lann}(R V_1 \otimes_k \cdots \otimes_k V_k)
\]
is simple as a tensor product of simple \( \mathbb{k} \)-algebras whose centers are \( \mathbb{k} \) [38, Theorem 1.7.27]. The rest is similar to the proof for (5.2) in the basic finite case using Lemma 5.1.4.

(iii) This is an immediate consequence of (ii).

(iv) It is easy to see from the invariant factor form of modules over PID that \( G_0 \cong \Sigma/\Omega \) and \( |G_0| = m \).

(v) (5.12) is Theorem 4.3.1(a). From (5.13) \( m \in \mathcal{O} \iff \) all \( s_{ij} \) defined in Remark 2.6.9 are zero \( \iff \) \( \text{Im}(\Phi(\beta)) = \langle \text{Im}(\beta) \rangle \subset m\mathcal{C} \) for \( \Phi \) defined in Lemma 2.6.8(iii).

Remark 5.1.7. (i) For studying representations, one can take \( \Delta = \langle \text{Im}(\beta) \rangle \) w.l.o.g. and then \( \mathcal{O} \cong \Omega/\Delta \). Other choices of \( \mathcal{C} \) may also be possible.

In general \( G(H^\circ) \) does not act transitively on irreducible representations of higher dimensions by

\[ \chi \cdot V = \chi \otimes V, \quad \chi \in G(H^\circ), V \in \text{Irr}(H), \]

e.g. when any of \( l_i \) in (5.10) is greater than 2. This can be seen as follows: Denote the corresponding irreducible representation as \( V \) and let \( c = \text{ann}_C(V) \), then \( c \in \text{MaxSpec}(C) \) and any irreducible representation over \( c^{-1} \) has the same dimension. However any irreducible module over \( c^2 \) has dimension greater than one since by Lemma 5.1.4 \( c^2 \) does not have basic fiber by Theorem C(c).

Example 5.1.8. Let \( A = B = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}, \Delta = (\mathbb{Z}/2\mathbb{Z})^2 \) and \( \Lambda = A \times B \). Denote the standard generators as \( a_1, a_2 \in A, b_1, b_2 \in B, c_1, c_2 \in \Delta \). Let \( \beta \in N^2(\Lambda, \Delta) \) such that the
associated matrix defined in Remark 2.6.9 is
\[
\begin{bmatrix}
c_1 & c_2 \\
0 & 0
\end{bmatrix}.
\]
Choose \( \Omega = \langle c_1, c_2, a_2, b_2^2 \rangle \) and \( k = \mathbb{C} \) then MaxSpec(\( C \)) \( \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathcal{G}_m^2 \) where \( \mathcal{G}_m = (k, \ast) \), more concretely define \( b := b_2^2 \) then
\[
\text{MaxSpec}(C) = \{ m_{u,v,w,x} = \langle c_1 - u, c_2 - v, a_2 - w, b - x \rangle : u, v \in \{ \pm 1 \}; w, x \in k \setminus \{ 0 \} \}.
\]
Characters over \( m_\bar{\varepsilon} \) are
\[
G_0 = \{ \chi \in G(H^\circ) : \chi(c_1) = \chi(c_2) = \chi(a_2) = 1; \chi(a_1), \chi(b_1), \chi(b_2) \in \pm 1 \}
\cong \Sigma/\Omega \cong (\mathbb{Z}/2\mathbb{Z})^3.
\]
And (5.11) becomes
\[
\mathcal{V}(D_k(H/C, \text{tr}_{\text{reg}})) = \begin{cases} 
\varnothing, & k \leq 8, \\
\text{MaxSpec}(C) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathcal{G}_m^2, & k > 8.
\end{cases}
\]
\( H \) is a free \( C \)-module of rank 8 with basis
\[
\{ a_1^i b_1^j b_2^k : i, j, k \in \{ 0, 1 \} \}.
\]
Pick \( m_{u,v,w,x} \in \text{MaxSpec}(C) \), then
\[
H/(m_{u,v,w,x} H) \cong \frac{k\langle X, Y, Z^\pm 1 \rangle}{(XY - uYX, XZ - vZX, Z^2 - 1)}.
\]
There are four cases depending on the choice of \( u \) and \( v \). The bases for matrices \( S \) in invariant factor form are given in (Table 5.1). Thus
Choose $m = m_{-1,-1,w,x}$ in case (iv), the irreducible representations are two dimensional. In some basis $\{v_1, v_2\}$ the matrices

$$
c_1 = c_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix}, \quad b_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
$$

$$
b_2 = \begin{bmatrix} 0 & \sqrt{x} \\ \sqrt{x} & 0 \end{bmatrix};
$$

where $\sqrt{x}$ is some square root of $x$, is a representation $V$ of $H$ over $m$. The stabilizers are
given by

\[ \text{Stab}_{G_0}(V) = \{ \chi \in G(H^\circ) : \chi(c_1) = \chi(c_2) = \chi(a_2) = 1; \]
\[ \chi(a_1), \chi(b_1) = -\chi(b_2) \in \{ \pm 1 \} \}. \]

5.2. Big quantum Borel subgroups at roots of unity

Let \( g \) be a finite dimensional, complex, simple Lie algebra of rank \( r \) and \( G \) be the corresponding connected, simply connected, complex algebraic group. Denote by \( B^\pm \) a pair of opposite Borel subgroups and by \( T := B^+ \cap B^- \) the corresponding maximal torus of \( G \). The coordinate rings of \( G \) and \( B^- \) will be denoted by \( \mathcal{O}(G) \) and \( \mathcal{O}(B^-) \). The Cartan matrix of \( g \) will be denoted by \( (a_{ij}) \). Let \( d_1, \ldots, d_r \) be the relatively prime, positive integers that symmetrize the Cartan matrix \( (a_{ij}) \). Denote by \( \Phi_+ \) and \( P_+ \) the sets of positive roots and dominant integral weights of \( g \), respectively. Let \( \{ \alpha_1^\vee, \ldots, \alpha_r^\vee \} \) be the simple coroots of \( g \).

Let \( W \) be the Weyl group of \( g \) and \( l : W \to \{0, 1, \ldots\} \) be its length function. The reflection length function \( s : W \to \{0, 1, \ldots\} \) is defined by setting \( s(w) \) to be equal to the minimal length of a presentation of \( w \in W \) as a product of reflections. It also equals

\[ s(w) = \dim \text{coker}(id - w) \]

for the action of \( w \) on \( \text{Lie} T \). The double Bruhat cells of \( G \) are defined by

\[ G^{w_1, w_2} := B^+ w_1 B^+ \cap B^- w_2 B^- \quad \text{for} \quad w_1, w_2 \in W. \]

Let \( \epsilon \in \mathbb{C} \) be a root of unity of odd order \( \text{ord}(\epsilon) \), which is coprime to 3 if \( g \) is of type \( G_2 \). In other words, \( \text{ord}(\epsilon) \) is odd and coprime to \( d_i \) for \( 1 \leq i \leq r \). Denote by \( \mathcal{U}_\epsilon(g) \)
the De Concini–Kac [12] (big) quantized universal enveloping algebra of $\mathfrak{g}$ at the root of unity $\epsilon$. It is a non-cocommutative Hopf algebra with Chevalley generators

$$\{K_i^{\pm 1}, E_i, F_i \mid 1 \leq i \leq r\}. $$

Denote by $\mathcal{U}_\epsilon(\mathfrak{g})^\geq$ the quantum Borel subalgebra of $\mathcal{U}_\epsilon(\mathfrak{g})$, [13, 15], generated by the nonnegative Chevalley generators $E_i, K_i^{\pm 1}, 1 \leq i \leq r$, subject to the relations

$$K_i K_j = K_j K_i, \quad K_i E_j K_i^{-1} = \epsilon^{d_{aij}} E_j, \quad \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} \epsilon^{d_{aij}} E_j E_i^k = 0,$$

where in the last identity $i \neq j$ and

$$[n]_{\epsilon^d} := \frac{\epsilon^{nd} - \epsilon^{-nd}}{\epsilon^d - \epsilon^{-d}}, \quad [n]_{\epsilon^d}! := [1]_{\epsilon^d} \cdots [n]_{\epsilon^d}$$

for the positive integer values of $k \leq n$ for which the denominators involved do not vanish.

$\mathcal{U}_\epsilon(\mathfrak{g})^\geq$ is a Hopf subalgebra of $\mathcal{U}_\epsilon(\mathfrak{g})$ with coproduct, antipode and counit given by

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,$$

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -K_i^{-1} E_i,$$

$$\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = 0.$$

De Concini and Kac [12] introduced the central Hopf subalgebra

$$C_\epsilon(\mathfrak{g})^\geq := \mathbb{C}[K_i^{\pm \text{ord}(\epsilon)}, E_\alpha^{\text{ord}(\epsilon)} \mid 1 \leq i \leq r, \alpha \in \Phi_+] \subset Z(\mathcal{U}_\epsilon(\mathfrak{g})^\geq),$$

where $\{E_\alpha \mid \alpha \in \Phi_+\}$ are the positive root vectors of $\mathcal{U}_\epsilon(\mathfrak{g})$, [12, Corollaries 3.1 and 3.3], and interpreted it as the image of a quantum Frobenius map. There is a canonical Hopf algebra isomorphism

$$C_\epsilon(\mathfrak{g})^\geq \cong \mathcal{O}(B^-). \quad (5.14)$$
Theorem 14.1] deals with the image of the quantum Frobenius map of $\mathcal{U}_\epsilon(g)$ and the above is its restriction to $C_\epsilon(g)^\geq$. We will identify the maximal ideals of the two algebras in (5.14) via this isomorphism. The maximal ideal of the counit $m_\epsilon \in \text{MaxSpec } C_\epsilon(g)^\geq$ corresponds to the identity element $1 \in B^-$. The algebra $\mathcal{U}_\epsilon(g)^\geq$ is a free module over $C_\epsilon(g)^\geq$ of rank $\text{ord}(\epsilon)^{\dim B^-}$ by using PBW bases. This gives rise to the regular trace function

$$\text{tr}_{\text{reg}} : \mathcal{U}_\epsilon(g)^\geq \to C_\epsilon(g)^\geq,$$

turning $\langle \mathcal{U}_\epsilon(g)^\geq, C_\epsilon(g)^\geq, \text{tr}_{\text{reg}} \rangle$ into a Cayley–Hamilton Hopf algebra of degree equal to $\text{ord}(\epsilon)^{\dim B^-}$, cf. Example 2.5.5.

**Lemma 5.2.1.**

(a) The characters of $\mathcal{U}_\epsilon(g)^\geq$ are given by

$$\chi_t(K_i) = t_i, \quad \chi_t(E_i) = 0, \quad \forall \ 1 \leq i \leq r \quad (5.15)$$

for $t := (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$.

(b) We have

$$\left(\mathcal{U}_\epsilon(g)^\geq/m_\epsilon \mathcal{U}_\epsilon(g)^\geq\right)/\text{rad}\left(\mathcal{U}_\epsilon(g)^\geq/m_\epsilon \mathcal{U}_\epsilon(g)^\geq\right)$$

$$\cong \mathbb{C}[K_1, \ldots, K_r]/(K_i^{\text{ord}(\epsilon)} - 1 \mid 1 \leq i \leq r).$$

In particular, the identity fiber algebra $\mathcal{U}_\epsilon(g)^\geq/m_\epsilon \mathcal{U}_\epsilon(g)^\geq$ is basic and

$$G_0 = G((\mathcal{U}_\epsilon(g)^\geq/m_\epsilon \mathcal{U}_\epsilon(g)^\geq)^\circ) \cong (\mathbb{Z}/\text{ord}(\epsilon))^r, \quad (5.16)$$

recall (1.2).
Proof. (a) If $\chi : U_\epsilon(g) \geq \to \mathbb{C}^*$ is a character, then

$$\chi(K_i) = t_i, \quad \forall 1 \leq i \leq r$$

for some $t_i \in \mathbb{C}^*$ because the elements $K_i$ are units. The assumption that the order of the root of unity $\epsilon$ is odd and coprime to the symmetrizing integers $d_i$ of the Cartan matrix $a_{ij}$ implies that $\epsilon^{d_ia_{ij}} \neq 1$. The identity $K_iE_iK_i^{-1} = \epsilon^{2d_i}E_i$ implies $\chi(E_i) = 0$.

It is obvious from the presentation of $U_\epsilon(g) \geq$ that (5.15) defines a character of $U_\epsilon(g) \geq$ for all $t_1, \ldots, t_n \in \mathbb{C}^*$.

(b) By the definition of the counit of $U_\epsilon(g) \geq$,

$$m_\epsilon = (K_i^{\text{ord}(\epsilon)} - 1, E_\alpha^{\text{ord}(\epsilon)} | 1 \leq i \leq r, \alpha \in \Phi_+),$$

and thus,

$$E_\alpha \in \text{rad} \left( U_\epsilon(g) \geq / m_\epsilon U_\epsilon(g) \geq \right).$$

Therefore, we have a surjective homomorphism

$$\mathbb{C}[K_1, \ldots, K_r]/(K_i^{\text{ord}(\epsilon)} - 1 | 1 \leq i \leq r) \rightarrow \left( U_\epsilon(g) \geq / m_\epsilon U_\epsilon(g) \geq \right) / \text{rad} \left( U_\epsilon(g) \geq / m_\epsilon U_\epsilon(g) \geq \right).$$

This homomorphism is an isomorphism since $\mathbb{C}[K_1, \ldots, K_r]/(K_i^{\text{ord}(\epsilon)} - 1 | 1 \leq i \leq r)$ is a semisimple sublagebra of $U_\epsilon(g) \geq / m_\epsilon U_\epsilon(g) \geq$. This proves the first statement in part (b) of the lemma. The last two statements of part (b) follow at once from it. \hfill \Box

We will need the following result.

**Theorem 5.2.2.** [5, Theorem 2.3(a)] Let

$$m \in G^{w.1} = B^+ w B^+ \cap B^- \subseteq B^- \cong \text{MaxSpec} \ C_\epsilon(g) \geq$$

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for \( w \in W \). There are precisely \( \text{ord}(\epsilon)^{r-s(w)} \) nonisomorphic irreducible modules of the fiber algebra \( \mathcal{U}_c(\mathfrak{g})^\geq / \mathfrak{m} \mathcal{U}_c(\mathfrak{g})^\geq \) and they all have dimension \( \text{ord}(\epsilon)^{(l(w)+s(w))/2} \).

The next theorem shows that the big quantum Borel subgroups at roots of unity \( \mathcal{U}_c(\mathfrak{g})^\geq \) satisfy the assumptions of Theorem C(c) of the introduction and derives consequences of this fact for the zero sets of their discriminant ideals.

**Theorem 5.2.3.** Let \( \mathfrak{g} \) be a finite dimensional complex simple Lie algebra of rank \( n \) and \( \epsilon \in \mathbb{C} \) be a root of unity of odd order, coprime to 3 if \( \mathfrak{g} \) is of type \( G_2 \). The following hold:

(a) \( (\mathcal{U}_c(\mathfrak{g})^\geq, \mathcal{C}_c(\mathfrak{g})^\geq, \text{tr}_{\text{reg}}) \) is a finitely generated Cayley–Hamilton Hopf algebra of degree \( \text{ord}(\epsilon)^{\dim B^-} \) with the properties that

(i) the identity fiber \( \mathcal{U}_c(\mathfrak{g})^\geq / \mathfrak{m}_- \mathcal{U}_c(\mathfrak{g})^\geq \) is a basic algebra and

(ii) all maximally stable irreducible modules of \( \mathcal{U}_c(\mathfrak{g})^\geq \) have dimension 1.

(b) The lowest discriminant ideal of \( (\mathcal{U}_c(\mathfrak{g})^\geq, \mathcal{C}_c(\mathfrak{g})^\geq, \text{tr}_{\text{reg}}) \) is of level

\[ l := \text{ord}(\epsilon)^{r} + 1 \]

and

\[ \mathcal{V}(D_l(\mathcal{U}_c(\mathfrak{g})^\geq / \mathcal{C}_c(\mathfrak{g})^\geq, \text{tr}_{\text{reg}})) = \mathcal{V}(MD_l(\mathcal{U}_c(\mathfrak{g})^\geq / \mathcal{C}_c(\mathfrak{g})^\geq, \text{tr}_{\text{reg}})) \]

\[ = W_l(G((\mathcal{U}_c(\mathfrak{g})^\geq)^\circ)) \cdot \mathfrak{m}_- = W_l(G((\mathcal{U}_c(\mathfrak{g})^\geq)^\circ)) \cdot \mathfrak{m}_- \]

\[ = \text{Aut}(\mathcal{U}_c(\mathfrak{g})^\geq; \mathcal{C}_c(\mathfrak{g})^\geq) \cdot \mathfrak{m}_- = T \subset B^- \cong \text{MaxSpec} \mathcal{C}_c(\mathfrak{g})^\geq. \]

(c) For \( k > \text{ord}(\epsilon)^{r} + 1 \),

\[ \mathcal{V}(D_k(\mathcal{U}_c(\mathfrak{g})^\geq / \mathcal{C}_c(\mathfrak{g})^\geq, \text{tr}_{\text{reg}})) = \mathcal{V}(MD_k(\mathcal{U}_c(\mathfrak{g})^\geq / \mathcal{C}_c(\mathfrak{g})^\geq, \text{tr}_{\text{reg}})) \]

\[ = \bigcup_{w \in W, \text{ord}(\epsilon)^{r+l(w)} < k} \mathcal{G}^{w,1}. \]
Proof. (a) The algebra $\mathcal{U}_\epsilon(\mathfrak{g})^\geq$ is clearly finitely generated. We proved that the triple $(\mathcal{U}_\epsilon(\mathfrak{g})^\geq, \mathcal{C}_\epsilon(\mathfrak{g})^\geq, \text{tr}_{\text{reg}})$ is a Cayley–Hamilton Hopf algebra of degree $\text{ord}(\epsilon)^{\dim B^-}$. Its identity fiber $\mathcal{U}_\epsilon(\mathfrak{g})^\geq / \mathfrak{m}_\epsilon \mathcal{U}_\epsilon(\mathfrak{g})^\geq$ is a basic algebra by Lemma 5.2.1(b).

Assume that $V$ is an irreducible module of $\mathcal{U}_\epsilon(\mathfrak{g})^\geq$ of dimension $> 1$. Theorem 5.2.2 implies that

$$V \in \text{Irr} \left( \mathcal{U}_\epsilon(\mathfrak{g})^\geq / \mathfrak{m} \mathcal{U}_\epsilon(\mathfrak{g})^\geq \right)$$

for some $\mathfrak{m} \in G^w_1$ with $w \in W$, $w \neq 1$. Theorems A(a) and 5.2.2 and the formula (5.16) for the group $G_0$ imply that

$$\dim(V)^2 = \text{ord}(\epsilon)^{l(w)+s(w)} \quad \text{and} \quad |\text{Stab}_{G_0}(V)| = \text{ord}(\epsilon)^r / \text{ord}(\epsilon)^{r-s(w)} = \text{ord}(\epsilon)^{s(w)}.$$ 

Hence, $|\text{Stab}_{G_0}(V)| < \dim(V)^2$ because $l(w) \geq 1$, and thus, $V$ is not maximally stable. This proves property (ii).

(b) The fact that the level of the lowest discriminant ideal of $\mathcal{U}_\epsilon(\mathfrak{g})^\geq$ equals

$$\text{ord}(\epsilon)^r + 1$$

follows from Theorem B(b) and Equation (5.16). The coproduct formulas for the generators of $\mathcal{U}_\epsilon(\mathfrak{g})^\geq$ imply that for all $c = (c_1, \ldots, c_n) \in (\mathbb{C}^*)^n$,

$$W_l(\chi_c) \cdot \mathfrak{m}_\epsilon = W_r(\chi_c) \cdot \mathfrak{m}_\epsilon = \left( K_i^{\text{ord}(\epsilon)} - c_i^{-1}, E_\alpha^{\text{ord}(\epsilon)} \mid 1 \leq i \leq r, \alpha \in \Phi_+ \right),$$

recall (5.15). The description of the character group $G((\mathcal{U}_\epsilon(\mathfrak{g})^\leq)^\circ)$ in Lemma 5.2.1(a) implies that

$$W_l(G((\mathcal{U}_\epsilon(\mathfrak{g})^\leq)^\circ)) \cdot \mathfrak{m}_\epsilon = W_r(G((\mathcal{U}_\epsilon(\mathfrak{g})^\leq)^\circ)) \cdot \mathfrak{m}_\epsilon = T.$$ 

The other equalities in part (b) now follow from Theorem C(c).

Part (c) follows by combining Equation (4.2) and Theorem 5.2.2. \qed
Bibliography


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Vita

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