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# Subroups of Coxeter Groups and Stallings Foldings

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# SUBGROUPS OF COXETER GROUPS AND STALLINGS FOLDINGS

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## Abstract

For each finitely generated subgroup of a Coxeter group, we define a cell complex called a completion. We show that these completions characterizes the index and normality of the subgroup. We construct a completion corresponding to the intersection of two subgroups and use this construction to characterize malnormality of subgroups of right-angled Coxeter groups. Finally, we show that if a completion of a subgroup is finite, then the subgroup is quasiconvex. Using this, we show that certain reflection subgroups of a Coxeter are quasiconvex.

# Chapter 1. Introduction

Coxeter groups are well studied in geometric group theory and have pleasant geometric representations. For example, any Coxeter group can be generated by a set of combinatorial reflections of a certain cell complex called the Davis complex. Our goal is to investigate finitely generated subgroups of these Coxeter groups and provide geometric characterizations of algebraic and geometric properties of these subgroups. The technique we will be using for this was first used in the study of free groups.

In [23], Stallings introduced a technique now known a Stallings fold, which is an operation applied to graphs with labeled edges where two adjacent edges with the same label and orientation are identified. Given a finite generating set of a subgroup of a free group, Stallings used this operation to associate a directed labeled graph to the subgroup. One key property that these labeled graphs have is that any reduced word $w$ representing an element of the corresponding subgroup is the label of a loop at some fixed base point $b$. By applying Stallings' fold operation, this loop is the unique loop labeled $w$ based at $b$. Stallings used these graphs to provide alternative proofs of various results about subgroups of free groups, such as if $H$ and $K$ are finitely generated subgroups of a free group, then $H \cap K$ is also finitely generated. In [15], Kapovich–Myasnikov used Stallings folds in a more combinatorial way to provide algorithmic solutions to questions of the structure of subgroups of free groups. These questions include finding a generating set for intersections of subgroups, determining the rank and index of a subgroup, and determining whether a given subgroup is normal. Arzhantseva and Ol'shanskii were the first to use Stallings folds for non-free groups in [2]. Beeker and Lazarovich created an analogue of Stallings folds for groups acting on CAT(0) cube complexes in [3]. A survey of applications of Stallings folds can be found in [9].

Dani–Levcovitz adapted Stallings folds to the setting of right-angled Coxeter groups in [6]. Right-angled Coxeter groups are generated by involutions and the only other relations are commuting relations between some generators. To each finitely generated subgroup of a right angled Coxeter group, they associated a cube complex called a *completion*. They showed that these completions characterize several properties of these subgroups, including the index, normality, and whether the subgroup has torsion. The major difference between the right-angled Coxeter group case and the free group case is the presence of relators. To deal with the generators being involutions, the 1-skeleton of the cube complex is not directed. To take care of commuting relations, they attach squares to the complex with boundary labeled by the relator. The result is a cube complex with labeled edges which again has the property that for any reduced word $w$ representing an element in the corresponding subgroup, there exists a loop in the complex labeled by $w$.

Here, we extend the construction of completions in [6] to all Coxeter groups. Since general Coxeter groups have relations other than the commuting relations of the right-angled case, instead of only attaching squares to our complex, we will need to attach larger polygons representing these relations. We will show that these complexes once again satisfy the property that any reduced word representing an element of the corresponding subgroup is the label of some loop at the base point. We show that some of the characterizations of subgroups of right-angled Coxeter groups carry over to subgroups of general Coxeter groups.

**Theorem 1.1.** *For a subgroup $H$ of a Coxeter group $W$, there exists characterizations of the index of $H$ and whether $H$ is normal in terms of the properties of a completion of $H$.*

We use a construction of Kapovich–Myasnikov to produce completions for intersections of subgroups of Coxeter groups. We refer a completion obtained by this method as a pullback. Once we have a pullback, we can use it to create a generating set for the intersection of the subgroups. In particular, if the original subgroups have finite completions, then their pullback is also finite. This gives us the following result for quasiconvex subgroups of Coxeter groups.

**Theorem 1.2.** *The pullback of completions for subgroups $H$ and $K$ of a Coxeter group is a completion for $H \cap K$. In particular, if $H$ and $K$ are quasiconvex there exists an algorithm which computes a generating set for $H \cap K$.*

We say a subgroup $H$ of a group $G$ is malnormal if for any $g \in G \setminus H$ we have that $H \cap gHg^{-1} = \{1\}$. Using our construction of a pullback, we are able to create completions for $H \cap gHg^{-1}$. We are then able to characterize whether a subgroup is malnormal in right-angled Coxeter groups.

**Theorem 1.3.** *There exists a characterization of malnormality in terms of completions for subgroups of right-angled Coxeter groups. In particular, for a quasiconvex subgroup $H$ of a right-angled Coxeter group, there exists an algorithm to determine whether $H$ is malnormal.*

More results on malnormal subgroups of right-angled Coxeter groups can be found in [24], where Tran showed that any malnormal subgroup of a one-ended right-angled Coxeter group is quasiconvex.

Similar to the completions constructed for right-angled Coxeter groups, a notion of a completion is defined for subgroup of fundamental groups of non-positively curved cube complexes in [4]. They show that these completions give a solution to the membership problem for these subgroups, as well as characterize whether a subgroup is normal or finite index. Applying the same construction of a pullback to this setting to create a completion for intersections of subgroups, we obtain the following theorem.

**Theorem 1.4.** *For quasiconvex subgroups $H$ and $K$ of a fundamental group of a non-positively curved cube complex, there exists an algorithm which computes a generating set for $H \cap K$.*

In [6], a completion of a subgroup of a right-angled Coxeter group $W$ is shown to be finite if and only if the subgroup is quasiconvex with respect to the standard generating set of $W$. They proved that any subgroup generated by reflections of a right-angled Coxeter group has a finite completion, and is therefore quasiconvex. In general, a subgroup of a Coxeter group generated by finitely many reflections was shown to be a Coxeter group by Deodha in [10] and by Dyer in [11].

**Theorem 1.5.** *If a completion of a subgroup $H$ of a Coxeter group is finite, then $H$ is quasiconvex.*

We use this to demonstrate that the following family of subgroups generated by reflections of a Coxeter group is quasiconvex.

**Theorem 1.6.** *Suppose $H$ is a subgroup of a Coxeter group $W$ generated by reflections $\langle w_1 r_1 w_1^{-1}, \ldots, w_n r_n w_n^{-1} \rangle$ where each $r_i$ is a standard generator of $W$, such that for any $i, j$ no reduced expression for $w_j^{-1} w_i$ ends in a letter $s$ such that $sr_i$ has finite order. Then $H$ is quasiconvex in $W$.*

Coxeter groups have been an object of study in geometric group theory for a while. In [22], Schupp used a technique similar to a completion, which he called a 2-completion, to show that a large family of Coxeter groups are locally quasiconvex, meaning every finitely generated subgroup is quasiconvex. Niblo and Reeves showed that Coxeter groups act properly discontinuously on CAT(0) cube complexes in [21] and Haglund and Wise showed in [12] that Coxeter groups are virtually special. Davis and Shapiro showed that all Coxeter groups are automatic in [8]. Many algorithmic problems regarding quasiconvex subgroups of automatic groups were shown to have solutions in [17].

In [22], Schupp used a technique similar to a completion, which he called a 2-completion, to show that a large family of Coxeter groups are locally quasiconvex, meaning every finitely generated subgroup is quasiconvex. In this paper, Schupp asked whether every Coxeter group of extra-large type was locally quasiconvex. While this question was already shown to be false, the author would like to thank Kevin Schreve for suggesting this counter example.

A stronger condition than local quasiconvexity is coherence. A group is coherent if every finitely generated subgroup is also finitely presented. Any locally quasiconvex subgroup is also coherent. It is known that for a group $G$ of cohomological dimension 2, if there exists a map $G \to \mathbb{Z}$ with finitely presented kernel, then $\chi(G) = 0$. If $\chi(G) > 0$ and the map $G \to \mathbb{Z}$ has finitely generated kernel, then the kernel is not finitely presented. The following theorem is a result of Kielak.

**Theorem 1.7** ([18])**.** *Let $G$ be an infinite finitely generated group which is residually finite and virtually solvable. Then $G$ admits a finite-index subgroup mapping onto $Z$ with a finitely generated kernel, if and only if $\beta_1^{(2)} = 0$.*

It is known that the extra-large type Coxeter group $W$ with defining graph $K_{n,n}$ with edges labeled by 4 and $n \geq 7$ satisfies the conditions of Kielak's theorem and has $\chi(W) > 0$. This means $W$ is not coherent and therefore is not locally quasiconvex.

Jankiewicz and Wise showed that many Coxeter groups are incoherent in [14]. Some more results for coherence and quasiconvexity for Coxeter groups and related groups can be found in [19], [13], [1], and [16].

Chapter 2 of this dissertation will cover background material on Coxeter groups and relevant terms from geometric group theory. Chapter 3 will start by defining a completion for finitely generated subgroups of Coxeter groups and prove some basic properties of these completions. Then we will cover the algebraic properties of these subgroups characterized by the completion.

# Chapter 2. Background

We will first cover some basic tools used in geometric group theory. Let $G$ be a finitely generated group with generating set $S$. We denote by $S^{-1}$ the set of inverses of elements of $S$. A *word* $w$ in $S$ is a string $s_1 \dots s_n$, where each $s_i$ is an element of $S \cup S^{-1}$. We define the *length* of $w = s_1 \dots s_n$ to be $n$ and denote it by $l(w)$. For an element $g$ in $G$, we define its length $l(g)$ to be the shortest length of a word in $S \cup S^{-1}$ representing $g$ in $G$. If $w$ is a word representing $g$ in $G$ such that $l(w) = l(g)$, we say that $w$ is *reduced*. We define a metric $d_S$ called the *word metric* on $G$ by $d_S(g,h) = l(gh^{-1})$. Note that the word metric depends on the choice of $S$.

**Example 2.1.** First, consider the group $\mathbb{Z}$ with generating set $S = \{a\}$. Then $d_S$ is the standard metric on $\mathbb{Z}$, as $d_S(a^n, a^m) = a^{n-m} = |n-m|$. Now choose another generating set $S' = \mathbb{Z}$. Then $d_{S'}(g,h) = 1$ for any choice of $g$ and $h$. In particular, we have that $d_S \neq d_{S'}$.

One way to visualize this metric is through the Cayley graph.

**Definition 2.2.** (Cayley Graph) The *Cayley graph* $\mathcal{C}(G,S)$ is the graph with vertex set $G$ and an edge $(g,h)$ labeled $s$ if $g = hs$ for some $s \in S$.

Note that this graph also depends on the choice of generating set of $G$.

**Example 2.3.** We again consider the group $\mathbb{Z}$. Let $S = \{a\}$ and $S' = \{a^2, a^3\}$ be two generating sets of $\mathbb{Z}$. Figure 2.1 shows the Cayley graph of $\mathbb{Z}$ with respect to each of these generating sets. Note that each the black edges correspond to the generator $a$, the blue edges are labeled by $a^2$, and the red edges are labeled by $a^3$.

If we consider each edge of the Cayley graph to have length 1, then the distance between two vertices $h$ and $g$ is precisely $d_S(h,g)$. Also, note that there is a natural group action of $G$ on $\mathcal{C}(G,S)$ where $g \in G$ sends $h \in V(\mathcal{C}(G,S))$ to the vertex $gh$. In Exam-

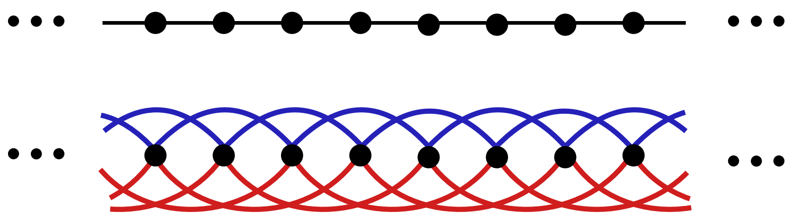

Figure 2.1. The top graph is $\mathcal{C}(\mathbb{Z}, S)$ and the bottom graph is $\mathcal{C}(\mathbb{Z}, S')$.

ple 2.3, $a^n$ acts on $\mathcal{C}(\mathbb{Z}, S)$ by translating $n$ units to the right.

**Definition 2.4.** (Quasiconvex) Let $U$ be a subspace of a geodesic metric space $(M, d)$. If for any two points $x, y \in U$ and any geodesic segment $\gamma$ between $x$ and $y$, we have that $\gamma$ is contained in a $k$-neighborhood of $U$, we say that $U$ is $k$-*quasiconvex* in $M$. We say that $U$ is *quasiconvex* in $M$ if it is $k$-quasiconvex for some $k \geq 0$. A subgroup $H$ of a finitely generated group $(G, S)$ is *quasiconvex* with respect to $S$ if $H$ is quasiconvex in $\mathcal{C}(G, S)$.

Note that whether a subgroup is quasiconvex depends on the choice of generating set. For example, consider the group $\mathbb{Z}^2$ with a standard generating set $\{a, b\}$. With this generating set, the subgroup $H = \langle ab \rangle$ is not quasiconvex. However, $H$ is quasiconvex with respect to the generating set $\{a, ab\}$.

**Definition 2.5.** (Hyperbolic Group) Let $(M, d)$ be a metric space. For $\delta > 0$, we say that $(M, d)$ is $\delta$-*hyperbolic* if for any geodesic triangle $(\gamma_1, \gamma_2, \gamma_3)$ we have that $\gamma_i$ is contained in a $\delta$ neighborhood of $\gamma_j \cup \gamma_k$ for $i \neq j \neq k$. We say that $(M, d)$ is *hyperbolic* if it is $\delta$-hyperbolic for some $\delta$. A group $G$ is *hyperbolic* if its Cayley graph is hyperbolic for some finite generating set of $G$.

It is known that if a subgroup $H$ of a hyperbolic group $G$ is quasiconvex with respect to a generating set $S$, then $H$ is quasiconvex in $G$ with respect to any choice of generating set of $G$.

## 2.1. Cube complexes and non-positive curvature

In this section we will define cube complexes and give a combinatorial description of when a cube complex has non-positive curvature.

**Definition 2.6.** (Cube complex) We say that $[0,1]^n$ is an *$n$-cube.* A *face* of an $n$-cube $C$ is a subset of $C$ obtained by setting at least one coordinate of $C$ to 0 or 1. A *cube complex* $X$ is a collection of $n$-cubes $S$ together with a collection of isometries called gluings between faces of cubes in $S$. We give a cube complex $X$ the metric induced by the Euclidean metric on each $n$-cube.

We can determine whether a cube complex $X$ is non-positively curved by looking at links of vertices of $X$.

**Definition 2.7.** (Link of a vertex) For a vertex $v$ of a cube complex $X$, let $S_\epsilon(v)$ be an $\epsilon$ sphere around $v$ in $X$ for some $0 < \epsilon < 1$. The *link* of $v$, denoted $lk(v)$, is given by $X \cap S_\epsilon(v)$.

Note that these links can be thought of as simplicial complexes.

**Definition 2.8.** (Non-positive curvature) A cube complex $X$ is *non-positively curved* if $lk(v)$ is a flag complex for any vertex $v$ of $X$. We say $X$ is CAT(0) if $X$ is non-positively curved and simply connected.

Next we will look at maps between cube complexes.

**Definition 2.9.** (Local isometry) For cube complexes $X$ and $Y$ with basepoints $b_X$ and $B_Y$, respectively. We say that $f : (X, b_X) \to (Y, b_Y)$ is a *cubical map* if $f$ sends $n$-cubes isometrically to $n - cubes$. We say that $f$ is a *local isometry* if for any vertex $v$ of $X$, the image of the restriction of $f$ to $lk(v)$ is a full subcomplex of $lk(f(v))$.

An important fact from [5], if $f$ is a cubical map and a local isometry, then the induced map $f_* : \pi_1(X, b_X) \to \pi_1(Y, b_Y)$ is injective.

### 2.2. Coxeter groups

Now we will introduce Coxeter groups and cover some of their useful properties. For a more comprehensive introduction to Coxeter groups, we refer the reader to [7].

**Definition 2.10.** (Coxeter group) A *Coxeter group* $W$ is defined by the group presentation $W = \langle s_1, \ldots s_n \mid (s_i s_j)^{m_{i,j}} \rangle$ where $m_{i,i} = 1$ and $m_{i,j} \in \{2, 3, \ldots\} \cup \{\infty\}$ if $i \neq j$. If $m_{i,j} = \infty$, then there are no relations between $s_i$ and $s_j$.

We often represent Coxeter groups with labeled simplicial graphs. For a simplicial graph $\Gamma$ with finitely many vertices $V(\Gamma)$ and with edges $E(\Gamma)$ labeled by elements of $\{2, 3, \ldots\}$, we define the Coxeter group $W_\Gamma$ by

$$W_\Gamma = \langle V(\Gamma) \mid s^2 \text{ for } s \in V(\Gamma),\ (s_i s_j)^{m_{i,j}} \text{ if } (s_i, s_j) \in E(\Gamma) \text{ is labeled by } m_{i,j} \rangle.$$

Note that if there is no edge between vertices $s_i$ and $s_j$, then there is no relation between $s_i$ and $s_j$. One type of Coxeter group we consider are *right-angled Coxeter groups*. These are obtained from simplicial graphs whose edges are all labeled by 2. Note that if $m_{i,j} = 2$, then $(s_i s_j)^2 = 1$ implies that $s_i s_j = s_j s_i$, since $s_i = s_i^{-1}$.

Several properties of a Coxeter group $W_\Gamma$ can be detected from its defining graph $\Gamma$. For example, all finite Coxeter groups have been classified by their defining graph and Moussong in [20] classified all hyperbolic Coxeter groups.

All Coxeter groups satisfy the *exchange condition* and the *deletion condition*. These conditions are extremely helpful when reducing a word of a Coxeter group.

1. (Exchange condition) For a reduced expression $s_1 \ldots s_n$ of an element $w$ of a Cox-

eter group $W$ and a generator $s$ of $W$, either $l(sw) = n+1$ or $sw = s_1 \ldots \widehat{s_i} \ldots s_n$ for some $i$, where $\widehat{s_i}$ denotes the absence of $s_i$.

2. (Deletion condition) If $s_1 \ldots s_n$ is an expression for a word $w$ in $W$ but is not reduced in $W$, then $w = s_1, \ldots, \widehat{s_i}, \ldots, \widehat{s_j} \ldots s_n$ for some $i$ and $j$.

Note that the deletion and exchange conditions are equivalent conditions [7].

Another helpful tool for studying how words in a Coxeter group reduce comes from Tits' solution to the word problem. His solution states there are only two operations needed to reduce a word $w$ in a Coxeter group, namely deletions and slides.

1. (Deletion) A *deletion* removes a subword of the form $ss$ from $w$.

2. (Slide) A *slide* replaces a subword of the form $s_i s_j s_i \ldots$ of length $m_{i,j}$ with the subword $s_j s_i s_j \ldots$ of length $m_{i,j}$.

We refer to this second operation as a slide because if we consider a word $w$ as a path $\gamma$ in the Cayley graph of $W$, applying a slide to a subword $s_i s_j s_i \ldots$ corresponds to "sliding" the corresponding subpath of $\gamma$ across a $2m_{i,j}$-gon.

**Theorem 2.11.** *(Tits) If $w$ is a reduced word in a Coxeter group $W$ and $w'$ is a word which represents the same element as $w$ in $W$, then there exists a sequence of deletions and slides which turns $w'$ into $w$. In particular, if both $w$ and $w'$ are reduced, then there exists a sequence of slides which turns $w$ into $w'$.*

## Chapter 3. Completions

In this section we define completions associated to subgroups of Coxeter groups. These completions are cell complexes with edges labeled by generators of a Coxeter group. We will start by defining the operations we will use to construct these completions.

**Definition 3.1.** (Labeled complex) For a finitely generated group $(G, S)$, we define a $(G, S)$-*complex* $X$ to be a cell complex whose edges are each labeled by an element of $S$. If $S$ is clear, we will refer to $X$ as a $G$-complex. For an edge $e$ of $X$, we denote the label of $e$ by $\mu(e)$. A $(G, S)$-complex is said to be *folded* if any two adjacent edges have distinct labels.

Let $W$ be a Coxeter group and $X$ be a $W$-complex. We use three operations on $X$ to construct a completion of $X$. The first of these operations is *folding*.

1. (Folding) If two distinct adjacent edges $e_1$ and $e_2$ of $X$ share the same label, we create a new $W$-complex by identifying $e_1$ and $e_2$.

**Definition 3.2.** (Reduced path) A *path* in a $W$-complex $X$ is a sequence of vertices and edges $v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n$ where $e_i$ and $e_{i+1}$ are both adjacent to $v_i$ for all $i$. We say a path is *reduced* if $e_i$ and $e_{i+1}$ are distinct edges for all $i$.

Note that the label of a reduced path in a folded complex will have no subwords of the form $ss$.

Our goal to construct a complex such that if $w$ is the label of a path $\alpha$ in the complex and $w'$ is a word obtained from $w$ by applying a slide, then $w'$ is the label of a path $\beta$ which is fixed point homotopy equivalent to $\alpha$. To do this, we will need to attach 2-cells corresponding to relators of our Coxeter group.

**Definition 3.3.** (Relator disks and deficient paths) For a Coxeter group $W$, we say that a reduced path in an $W$-complex is an $(a,b)$-*path* if it is labeled by $abab\ldots$. Similarly, if an $(a,b)$-path forms a cycle, we say that it is an $(a,b)$-*cycle*.

An $(a,b)$-*relator disk* is a cycle of length $2m_{a,b}$ which bounds a 2-cell and whose edge labels alternate between $a$ and $b$. If $\gamma$ is an $(a,b)$-path of length at least $m_{a,b}$ which is not part of an $(a,b)$-relator disk or if $\gamma$ is an $(a,b)$-cycle which does not bound a 2-cell, we say that $\gamma$ is *deficient*.

Our last two completion operations involve attaching and identifying these relator disks.

2. (Relator Attachment) If there is a deficient $(a,b)$-path $\gamma$ in $X$ then we attach an $(a,b)$-relator disk $C$ to $X$ by identifying a vertex of $C$ to a vertex of $\gamma$.
3. (Relator Identification) If two relator disks $C_1$ and $C_2$ of $X$ have the same 1-skeleton, then we identify $C_1$ and $C_2$.

Similar to folded graphs, we give a name to complexes which contain all necessary 2-cells.

**Definition 3.4.** (Cell-full complex) We say that a $W$-complex is *cell-full* if there are no deficient paths and there are no possible relator attachments or relator identifications.

We consider a complex that is both folded and cell-full to be complete. This gives us our next definition.

**Definition 3.5.** (Completion) If $\Omega$ is the direct limit of a possibly infinite sequence of these three operations applied to $X$ and $\Omega$ is folded and cell-full, then we say that $\Omega$ is a *completion* of $X$.

Note that there is a natural image of $X$ in $\Omega$. We now show that such a completion always exists.

**Lemma 3.6.** *For any finite $W$-complex $X$, there exists a completion of $X$.*

*Proof.* Let $X$ be a finite $W$-complex. We will describe a process which will result in a completion $\Omega$ of $X$. Since any fold operation reduces the number of edges in our complex, only a finite number of folds are required to turn $X$ into a folded complex. Similarly, relator identifications decrease the number of 2-cells of $X$. Let $X = \Omega_0 \to \Omega_1 \to \cdots \to \Omega_i$ be a sequence of folds and relator identifications where $\Omega_i$ is folded and finite and no more relator identifications are possible. Now for each deficient path in $\Omega_i$, we apply relator attachment operations until each deficient path is incident to a corresponding relator disk. This gives a sequence $\Omega_i \to \Omega_{i+1} \to \cdots \to \Omega_j$. Then we repeat this procedure by applying folds and relator identifications until we get a folded complex, and then applying relator attachments until there are no more deficient paths.

Let $\Omega$ be the direct limit of this process. Suppose $\Omega$ is not folded. Suppose edges $e$ and $e'$ are adjacent and share the same label. Then there exists some $i$ such that the preimage of $e$ and $e'$ in $\Omega_i$ are adjacent and share the same label. However, by our construction, there exists an $n > i$ such that for all $j > n$ we have that the image of $e$ and $e'$ are identified in $\Omega_j$, which is a contradiction. Therefore $\Omega$ is folded.

Now suppose $\Omega$ is not cell-full. Let $\gamma$ be a deficient path in $\Omega$. Then there exists an $i$ such that the preimage of $\gamma$ in $\Omega_i$ is deficient. Again, by construction there exists an $n > i$ such that for every $j > n$ we have that the preimage of $\gamma$ in $\Omega_j$ is incident to a corresponding relator disk. Thus $\Omega$ is cell-full and therefore is a completion of $X$. $\square$

We refer to the completion created by this process as a *standard completion* of $X$. In [6], completions were defined for right-angled Coxeter groups. They attach a square to paths $e_1e_2$ where the labels of $e_1$ and $e_2$ commute, meaning wherever there is a path labeled by half of a relator. We generalize this to all Coxeter groups by attaching larger polygons to any path which contains at least half of a relator. In fact, if we restrict our definition of a completion to right-angled Coxeter case, we obtain the 2-skeleton of the completions from [6].

Now we define a complete complex for a subgroup of a Coxeter group.

**Definition 3.7.** (Completion of a subgroup) For a subgroup $H$ of a Coxeter group $W$, we say that a connected $W$-complex $\Omega$ with basepoint $b$ is a completion of $H$ if:

1. $\Omega$ is folded and cell-full.
2. The label of any loop in $\Omega$ based at $b$ is an element of $H$.
3. For any element $g$ of $H$, there exists an expression $w$ of $g$ such that $w$ is the label of a loop in $\Omega$ based at $b$.

We can construct a completion for each finitely generated subgroup $H$ of a Coxeter group $W$ as follows. Let $\langle h_1, \ldots, h_n \rangle$ be a generating set for $H$. Let $X$ be a wedge of $n$ loops based at a vertex $b$. For each generator $h_i$, we subdivide the $i$th loop into $l(h_i)$ edges and label the edges by the letters of $h_i$ in order starting from an edge adjacent to $b$. We say that the standard completion of $X$ is a standard completion of $H$.

**Example 3.8.** Figure 3.1 shows the process of creating a standard completion for the subgroup $H = \langle abca, bacab \rangle$ of the Coxeter group $W$ with presentation

$$W = \langle a, b, c \mid a^2, b^2, c^2, (ab)^3 \rangle.$$

The leftmost complex is $X$ and the rightmost complex is the standard completion of $H$. The first step is given by three folds. The second step attaches an $(a, b)$-relator disk to the

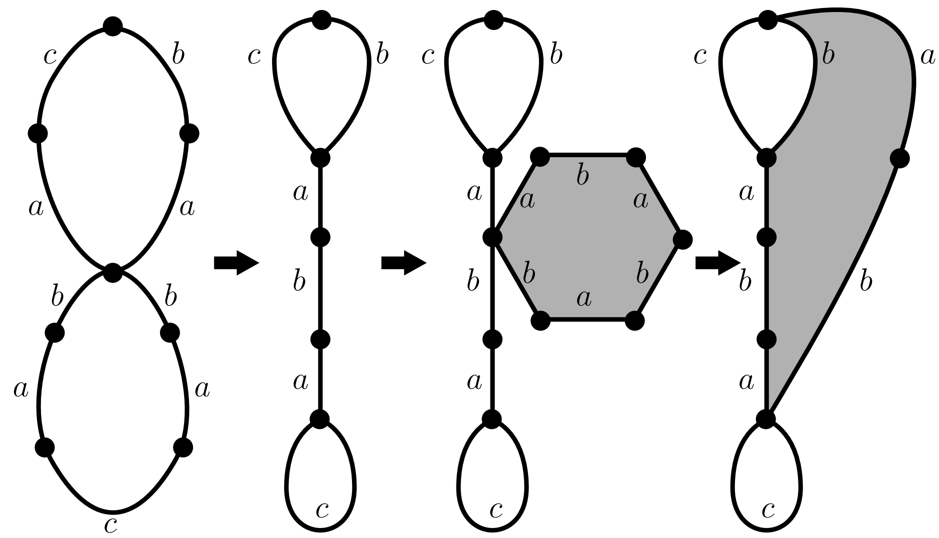


Figure 3.1. The process of creating a completion of $H$ for Example 3.8.

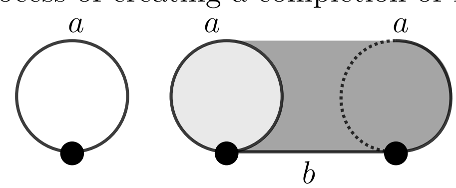


Figure 3.2. Two completions of the subgroup $K$ from Example 3.8.

deficient $(a, b)$-path. The final step folds the $(a, b)$-relator disk onto the deficient path.

Note that a completion of a subgroup is not unique. In fact, different choices of generating sets of a subgroup result in different standard completions. For the Coxeter group $W$ given by

$$W = \langle a, b \mid a^2, b^2, (ab)^2 \rangle,$$

Figure 3.2 shows two completions of the subgroup $K = \langle a \rangle$.

Now we will define hyperplanes of a complex, which will be a useful tool for some of our proofs.

**Definition 3.9.** (Parallel edges) For a relator disk of length $n$ consisting of edges $e_1, \ldots, e_n$ in order, we say that the edges $e_i$ and $e_{i+n}$ are *parallel* for $1 \leq i \leq n$. Let $v$ be a vertex of an $(a, b)$-cycle $C'$ of a $W$-complex $X$ which bounds a 2-cell and let $C$ be an

$(a, b)$-relator disk. We attach a copy of $C$ to $v$, fold the edges of $C$ onto the edges of $C'$, and identify their 2-cells. We use this process define a map $f : C \to C'$ by sending $C$ to its image folded onto $C'$. If $e$ and $e'$ are parallel in $C$, then we say that $f(e)$ and $f(e')$ are parallel in $C'$.

We say that two edges $e$ and $e'$ in $X$ are parallel if there exists a sequence of edges $e = e_1, e_2, \ldots, e_n = e'$ such that $e_i$ and $e_{i+1}$ are parallel in some relator disk in $X$ for all $i$.

Note that the set of edges parallel to some edge $e$ forms an equivalence class. We will associate a hyperplane to each of these equivalence classes.

**Definition 3.10.** (Hyperplanes) For a relator disk $C$ of a $W$-complex $X$, *mid-line* of $C$ is a line contained in $C$ which bisects two parallel edges in $C$. A *hyperplane* $H$ of $X$ is a maximal collection of mid-lines such that for any two mid-lines $h$ and $h'$ there exists a sequence $h = h_1, h_2, \ldots, h_n = h'$ in $H$ such that $h_i \cap h_{i+1}$ is nonempty for all $i$.

The *support* of a hyperplane $H$ of $X$, denoted *supp*$(H)$, is the set of all cells containing $H$.

One useful property of hyperplanes is that the number of hyperplanes of a complex never increases when completing a complex.

**Lemma 3.11.** *If $\Omega$ is a completion of a $W$-complex $X$, then any hyperplane of $\Omega$ must intersect the image of $X$ in $\Omega$.*

*Proof.* Let $X = \Omega_0 \to \Omega_1 \to \ldots \to \Omega$ be the sequence of completion operations resulting in $\Omega$. Let $\Omega_{n_1}, \Omega_{n_2}, \ldots$ be the possibly infinite subsequence of folded complexes. We will prove the lemma by showing that for any $n$, there exists $n_i > n$ such that any hyperplane of $\Omega_{n_i}$ intersects the image of $X$. If $\Omega_{n+1}$ is obtained from $\Omega_n$ via a fold, then the image of

$\Omega_n$ is surjective on $\Omega_{n+1}$, so any hyperplane in $\Omega_{n+1}$ must intersect the image of $\Omega_n$. Similarly, the image of $\Omega_n$ is surjective on $\Omega_{n+1}$ if $\Omega_{n+1}$ is the result of a relator identification, so any hyperplane of $\Omega_{n+1}$ must intersect the image of $\Omega_n$.

Suppose $\Omega_{n+1}$ is the result of a relator attachment and $C$ is the cycle attached to a deficient $\Omega_n$. If $C$ is attached to a deficient cycle $C'$, then it is possible to fold every edge of $C$ onto an edge of $C'$. Since $\Omega$ is folded, we can choose $i$ such that $n_i > n$. Since $\Omega_{n_i}$ is folded, the image of $C$ must be contained in the image of $C'$. Therefore any hyperplane intersecting the image of $C$ in $\Omega_{n_i}$ must also intersect the image of $C'$, and therefore intersects the image of $\Omega_n$.

If $C$ is attached to a deficient path $\gamma$, then the length of $\gamma$ must be at least half the length of $C$. Again, we choose $i$ such that $n_i > n$. Since $\Omega_{n_i}$ is folded, at least half of the edges of $C$ in $\Omega_{n_i}$ are contained in the image of $\gamma$. Then any pair of parallel edges in the image of $C$ in $\Omega_{n_i}$ must have an edge in the image of $\gamma$. Therefore, any hyperplane intersecting the image of $C$ in $\Omega_{n_i}$ must also intersect the image of $\gamma$, and therefore must intersect the image of $\Omega_n$.

Suppose $H$ is a hyperplane of $\Omega$ which does not intersect the image of $X$. Then there exists an $n$ such that $H$ contains the image of a hyperplane of $\Omega_n$ and for any $m > n$ the image of $H$ in $\Omega_m$ does not intersect the image of $X$, which is a contradiction. $\square$

The reason we require completions to be folded and cell-full is for completions to have the following property.

**Lemma 3.12.** *Let $\Omega$ be a folded and cell-full $W$-complex for some Coxeter group $W$ and $w$ be the label of a path $\gamma$ in $\Omega$. If $w'$ is a reduced word equivalent to $w$ in $W$, then there is* *a fixed end point homotopy from $\gamma$ to a path $\gamma'$ with label $w'$.*

*Proof.* By Tits' solution to the word problem for Coxeter groups in Theorem 2.11, a word $w$ is equal to a reduced word $w'$ in $W$ if and only $w'$ can be obtained from $w$ by a sequence of deletions and slides. First, suppose a deletion is applied to $w$. This means that $w$ has an expression $s_1s_2\cdots ss\cdots s_k$. Since $\Omega$ is folded, this means that $\gamma$ traverses the same edge $e$ twice in a row. By removing this edge from $\gamma$ we obtain a new path with label $s_1s_2\cdots\hat{s}\hat{s}\cdots s_k$.

Next, suppose a slide is applied to $w$. This means that $w$ contains a sub-word of the form $(abab\ldots)$ of length $m_{a,b}$. Therefore $\gamma$ contains an $(a,b)$-path of length $m_{a,b}$. Since $\Omega$ is cell-full, we can replace this sub-path with a $(b,a)$-path of equal length. Proceeding this way, we obtain a path $\gamma'$ with label $w'$. □

The key property of completions which follows directly from Lemma 3.12 is that any reduced word in $H$ is the label of a loop in the completion of $H$.

**Corollary 3.13.** *Let $(\Omega,b)$ be a completion of a subgroup $H$ of a Coxeter group $W$. If $w$ is a reduced word in $H$, then there exists a loop in $\Omega$ based at $b$ with label $w$.*

*Proof.* By the definition of a completion, for any element $g$ of $W$ there exists a loop in $\Omega$ based at $b$ whose label is equal to $g$. By Lemma 3.12, if $w$ is a reduced word equal to $g$ in $W$, there must be a loop based at $b$ with label $w$. □

We now show that changing the basepoint of a completion of a subgroup $H$ gives us a completion for a conjugate of $H$.

**Lemma 3.14.** *Let $(\Omega,b)$ be a completion of a subgroup $H$ of a Coxeter group $W$. If $v$ is a vertex of $\Omega$ and $g$ is the label of a path $\alpha$ from $v$ to $b$, then $(\Omega,v)$ is a completion of*

$gHg^{-1}$.

*Proof.* First, note that part 1 and 2 of Definition 3.7 are satisfied as $\Omega$ is folded and cell-full and the label of any loop based at $v$ is equal to the label of a loop at $b$ conjugated by $g$. To prove that part 3 is satisfied, we will show that for any element $w$ of $gHg^{-1}$ there is a loop based at $v$ whose label is equal to $w$ in $W$. First, we know that $w = ghg^{-1}$ for some reduced word $h \in H$. Since $h$ is a reduced word in $H$ and $(\Omega, b)$ is a completion of $H$, there must be a loop $\gamma$ based at $b$ with label $H$. Then $\alpha\gamma\alpha^{-1}$ is a loop based at $v$ with label $ghg^{-1}$, where $\alpha$ is labeled by $g$. Therefore $(\Omega, v)$ is a completion of $gHg^{-1}$. □

Similarly, if there is a loop in $\Omega$ whose label is contained in a finite special subgroup, we show that a conjugate of that element lives in $H$.

**Lemma 3.15.** *Let $H$ be a subgroup of a Coxeter group $W$ and let $(\Omega, b)$ be a completion of $H$. If there exists a loop in $\Omega$ whose label is a nontrivial reduced word in a finite special subgroup of $W$, then $H$ has torsion.*

*Proof.* Suppose there exists a loop labeled $u$ based at some vertex $v$ in $\Omega$ where $u$ is an element of a finite special subgroup. Let $g$ be the label of a path from $v$ to $u$. Then $gug^{-1}$ is a nontrivial element of $H$ with finite order. □

### 3.1. Index of a subgroup

In this section, we will show that completions characterize the index of a subgroup. To do so, we will look at the valence of the vertices in our completion.

**Definition 3.16.** (Full valence completion) A completion $\Omega$ of a subgroup of a Coxeter group $W$ is *full valence* if for every vertex $v$ of $\Omega$ and every generator $s$ of $W$, there exists an edge labeled $s$ incident to $v$.

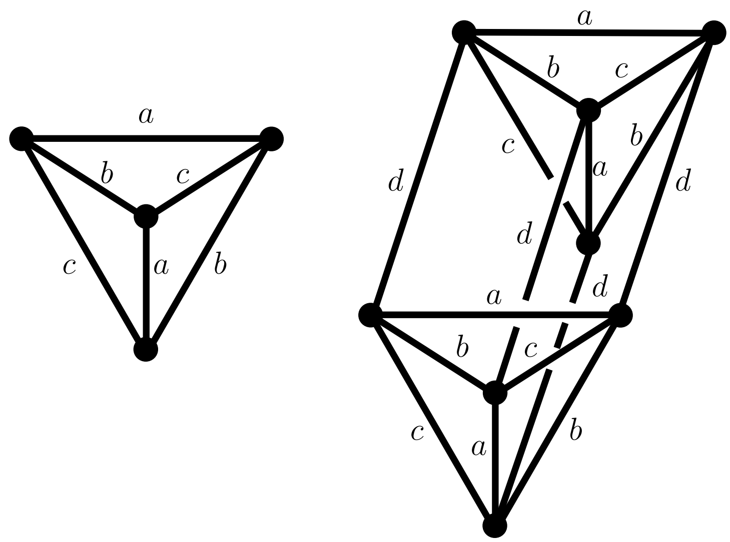


Figure 3.3. Two completions of $H$. Only the completion on the right is full-valence.

In the case of a free group, it sufficed to say that a subgroup had finite index if and only if its Stallings graph was full-valence [15]. However, for Coxeter groups it is possible to have a finite index subgroup with a completion that is not full-valence.

**Example 3.17.** Let $W$ be the Coxeter group with presentation

$$W = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, (ab)^4, (ac)^4, (ad)^2, (bd)^2, (cd)^2 \rangle.$$

The subgroup $H = \langle abc, bca, cab \rangle$ has index 8 in $W$. Figure 3.3 contains a full-valence completion of $H$ and a completion of $H$ which is not full valence.

In general, we can take any Coxeter group $W$ and consider a new Coxeter group given by $W' = \mathbb{Z}_2 \times W$. In this case, a completion of the subgroup $W$ will not necessarily contain an edge labeled by the generator of $\mathbb{Z}_2$.

Now we provide a condition for when a completion of a finite index subgroup is full valence.

**Theorem 3.18.** *Suppose $H$ is a subgroup of an infinite Coxeter group $W$ and $(\Omega, b)$ is a finite completion of $H$ such that if $s \in Z(W)$ is a generator of $W$, then $s$ is the label of* *some edge in $\Omega$, where $Z(W)$ is the center of $W$. Then $H$ has finite index if and only if $\Omega$ is full valence.*

*Proof.* First suppose $\Omega$ is full valence. We will show that $[W, H] = |V(\Omega)|$. For each vertex $v$ of $\Omega$, choose a path $\alpha_v$ from $b$ to $v$ and let $g_v$ be its label. For any $g \in W$, since $\Omega$ is full valence, there exists a path from $b$ labeled $g$. Let $u$ be the endpoint of this path so $g_u^{-1}$ labels a path based at $b$. Then we have that $g_u^{-1}g$ is the label of a loop in $\Omega$ based at $b$ and therefore is an element of $H$. Therefore, $g \in g_u H$ and each vertex of $\Omega$ corresponds to a coset of $H$.

Now suppose $\Omega$ does not have full valence. We can assume that there exists a generator $s \notin Z(W)$ such that there is no edge incident to $b$ with label $s$, as otherwise, we can take a conjugate of $H$ by Theorem 3.14. This means that no reduced word in $H$ begins with $s$. Since $W$ is not finite, there exists generators $s_1, \dots, s_k$ such that $(ss_1 \cdots s_k)^n$ is reduced and is therefore not an element of $H$ for any $n > 0$. This means that each coset $(ss_1 \cdots s_k)^n H$ is distinct for each $n$. Otherwise, if $(ss_1 \cdots s_k)^n H = (ss_1 \cdots s_k)^m H$ for $m \neq n$, then $(ss_1 \cdots s_k)^{n-m} \in H$, which is a contradiction. Therefore $H$ has infinite index in $W$. □

**Remark 3.19.** Note that we can easily obtain a completion which satisfies the assumptions of Theorem 3.18 taking a completion $\Omega$ and attaching an edge to $b$ labeled $s$ for each $s \in Z(W)$ and taking the completion of this new complex.

### 3.2. Normal Subgroups

In this section, we will define a subgraph, called the core, of a completion which characterizes the normality of a subgroup. While a completion of a subgroup is not unique to the subgroup, the core of the subgroup is unique, as we will see in Lemma 3.22.

**Definition 3.20.** (Core Graph) For a $W$-complex $(\Omega, b)$, we define the *core graph* $C(\Omega, b)$ at $b$ to be the subset of the 1-skeleton consisting of all loops based at $b$ whose labels are reduced words in $W$. If $(\Omega, b)$ is a completion of a subgroup $H$, we say that $C(\Omega, b)$ is the core of $H$.

The idea of a core graph is to remove nonessential edges and 2-cells from a completion. For example, aside from possibly the basepoint, the core will not contain any vertex of degree 1, since any loop containing such a vertex can be reduced via a deletion. Example 3.21 shows an example where taking the core of a complex removes half of a Coxeter polytope.

**Example 3.21.** Figure 3.4 shows a completion $\Omega$ and the core graph of the subgroup $H = \langle bcda, cbea \rangle$ for the Coxeter group $W$ given by the presentation

$$W = \langle a, b, c, d, e \mid a^2, b^2, c^2, d^2, e^2, (ab^2), (ac^2), (bc)^4 \rangle.$$

Note any loop based at $B$ in $\Omega$ which contains a blue edge has a homotopy to a shorter loop containing only black edges.

We show that the core graph of a given subgroup is unique.

**Lemma 3.22.** *For a subgroup $H$ of $W$, if $(\Omega_1, b_1)$ and $(\Omega_2, b_2)$ are two completions of $H$, then there is an isomorphism $f : C(\Omega_1, b_1) \to C(\Omega_2, b_2)$ such that $f(b_1) = b_2$.*

*Proof.* Let $w$ be a reduced word in $H$. By Corollary 3.13 there exists a loop $\gamma_i$ in $\Omega_i$ based at $b_i$ with label $w$ for $i = 1, 2$. Then, by definition of a core graph, $\gamma_i$ is also contained in $C(\Omega_i, b_i)$. We define $f$ by setting $f(\gamma_1) = \gamma_2$. This map is well-defined since $C(\Omega_1, b_1)$ and $C(\Omega_2, b_2)$ are both folded as they are subcomplexes of folded complexes. We define a map

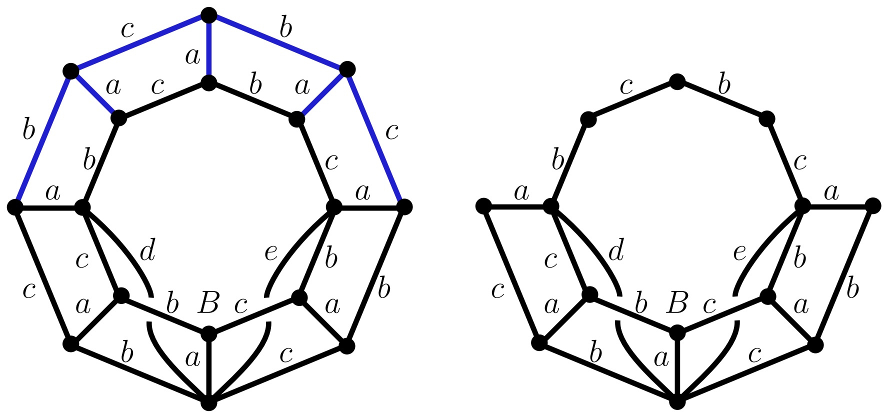


Figure 3.4. The left figure is a completion $\Omega$ of $H$ as defined in Example 3.21 and the right figure is the core graph $C(\Omega, B)$. Note that the 2-cells of $\Omega$ have been omitted for simplicity. However, each $(a, b)$-square, $(a, c)$-square, and $(b, c)$-octogon bounds a disk, so $\Omega$ is homeomorphic to $S^2$ with two edges attached as handles.

$g : C(\Omega_2, b_2) \to C(\Omega_1, b_1)$ the same way we defined $f$.

We will show that $g = f^{-1}$. Let $e$ be an edge of $C(\Omega_1, b_1)$. By the definition of a core graph, there exists a reduced word $w$ in $H$ such that a loop $\gamma$ based at $b_1$ contains $e$ and has label $w$. Then $f(\gamma)$ is a loop in $C(\Omega_2, b_2)$ based at $b_2$, so $g(f(\gamma))$ is a loop based at $b_1$ labeled $w$. However, such a loop is unique since $C(\Omega_1, b_1)$ is folded, so $g(f(\gamma)) = \gamma$. In particular, we have that $g(f(e)) = e$. Similarly, for any edge $e'$ in $C(\Omega_2, b_2)$, we have that $f(g(e')) = e'$. Therefore $g = f^{-1}$ and $f$ is an isomorphism. $\square$

Intuitively, a subgroup should be normal if and only if its core is vertex transitive. However, this is not always the case. For example, if $W$ is a Coxeter group, we can consider $W$ as a subgroup of a new Coxeter group $W * \mathbb{Z}_2$. Then $W$ is not normal in $W * \mathbb{Z}_2$, but the completion of $W$ consisting of a single vertex and a graph loop for each generator of $W$ is a core graph and is clearly vertex transitive. Therefore, we will need some more conditions on our core graph to accurately detect normality.

**Theorem 3.23.** *Let $(\Omega, b)$ be a completion of a subgroup $H$ of a Coxeter group $W$ with* *standard generating set $S$. We define*

$$\Delta = \{s \in S \mid s \textit{ commutes with every element of } H\}$$

*Then $H$ is normal in $W$ if and only if there is an edge incident to $b$ in $\Omega$ labeled by $s$ for every $s \in S \setminus \Delta$ and for every $v \in \Omega$ there exists a label preserving isomorphism from $C(\Omega, b)$ to $C(\Omega, v)$ which takes $b$ to $v$.*

*Proof.* First, suppose $H$ is normal. Let $v$ be a vertex of $\Omega$ and $g$ be the label of a path from $v$ to $b$. Then $(\Omega, v)$ is a completion of the subgroup $gHg^{-1} = H$. Therefore $C(\Omega, b)$ is isomorphic to $C(\Omega, v)$ by Lemma 3.22. Now we will show that for every $s \in S \setminus \Delta$ there exists an edge incident to $b$ labeled by $s$. Let $w = s_1 \ldots s_k$ be a reduced word representing an element of $H$ which does not commute with $s$, which exists since $s \notin \Delta$. If $w$ has a reduced expression beginning or ending with $s$, then we are done, as there must be a loop in $\Omega$ based at $b$ with this expression as its label by Corollary 3.13. Suppose no reduced expression of $w$ begins or ends with $s$. Then $sws$ is an element of $H$, since $H$ is normal. If $sws$ is not reduced, then by the deletion condition, at least one of $sw$ and $ws$ is not reduced, as otherwise $sws = w$, which is a contradiction. Without loss of generality, suppose $sw$ is not reduced. By the exchange condition, we have $sw = s_1 \ldots \hat{s}_i \ldots s_n$ for some $i$. Therefore $ss_1 \ldots \hat{s}_i \ldots s_n$ is a reduced expression for $w$ which starts with $s$, which is a contradiction. Therefore $sws$ is reduced and there must be a path based at $b$ with label $sws$.

Now suppose there is an edge incident to $b$ in $\Omega$ labeled by $s$ for every $s \in S \setminus \Delta$ and for every $v \in \Omega$ there exists an isomorphism from $C(\Omega, b)$ to $C(\Omega, v)$ which takes $b$ to $v$. Clearly if $s \in \Delta$ we have $sHs = H$. If $s \in S \setminus \Delta$, then there is an edge $e$ incident to $b$ in with label $s$. Let $u$ be the other endpoint of $e$. Then $(\Omega, u)$ is a completion for $sHs$ and

$C(\Omega, b)$ and $C(\Omega, u)$ are isomorphic by assumption. Let $w$ be a reduced word representing an element of $H$. Then $w$ is the label of a loop in $C(\Omega, b)$ based at $b$, and therefore is also the label of a loop in $C(\Omega, u)$ based at $u$. Therefore $w$ represents an element of $sHs$ and $H \subset sHs$. Similarly, if $w$ is a reduced word representing an element of $sHs$, then $w$ is the label of a loop in $C(\Omega, u)$ based at $u$, and therefore is the label of a loop in $C(\Omega, b)$ based at $b$. Thus $w$ represents an element of $H$, so $H = sHs$ and $H$ is normal in $W$. $\square$

### 3.3. Intersections of subgroups

In this section we will use completions to find a generating set of intersections of finitely generated subgroups of Coxeter groups. To accomplish this, for completions $\Omega_H$ and $\Omega_K$ of subgroups $H$ and $K$, respectively, we will construct a completion $\Omega_H * \Omega_K$ for $H \cap K$. We call this completion the pullback of $\Omega_H$ and $\Omega_K$.

**Definition 3.24.** (Pullback) For completions $\Omega_1$ and $\Omega_2$, let $M$ be the graph with vertex set $V(\Omega_1) \times V(\Omega_2)$ and an edge from $(v_1, v_2)$ to $(u_1, u_2)$ labeled $s$ if and only if there exists edges labeled $s$ from $v_1$ to $u_1$ and from $v_2$ to $u_2$ in $\Omega_1$ and $\Omega_2$, respectively. To add the necessary 2-cells to $M$, we define the *pullback* $\Omega_1 * \Omega_2$ to be the completion of $M$.

Note that completing $M$ does not add any additional edges, as if there exists a deficient $(a, b)$-path $\gamma$ in $M$, then the corresponding $(a, b)$-paths in $\Omega_1$ and $\Omega_2$ must be contained in relator disks, as otherwise $\Omega_1$ and $\Omega_2$ would not be cell-full. Therefore $\gamma$ must be contained in an $(a, b)$-cycle $C$ in $M$. Let $D$ be an $(a, b)$-relator disk attached to $\gamma$. Then the image of the boundary of $D$ in $\Omega_1 * \Omega_2$ is contained in the image of $C$. Therefore, the image of $M$ is surjective onto $\Omega_1 * \Omega_2$.

**Example 3.25.** The pullback is not necessarily connected. For example the Coxeter

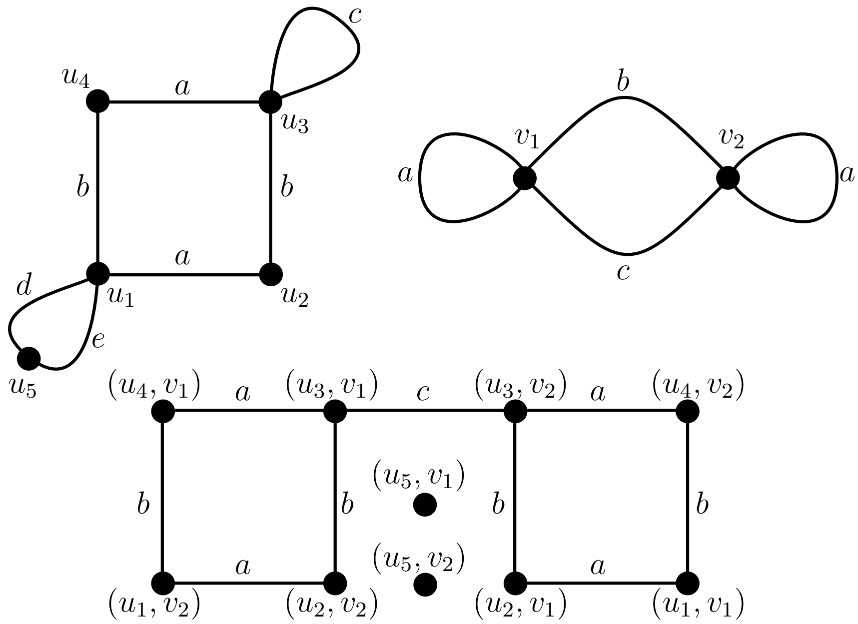


Figure 3.5. The top figures are the completions $\Omega_1$ and $\Omega_2$ from Example 3.25. The bottom figure is the pullback $\Omega_1 * \Omega_2$. We omit the 2-cells for clarity.

group $W$ defined by

$$W = \langle a, b, c, d, e \mid a^2, b^2, c^2, d^2, e^2, (ab)^2 \rangle$$

and subgroups $H = \langle abcab, de \rangle$ and $K = \langle a, bac \rangle$, Figure 3.5 shows a completion $(\Omega_1, u_1)$ for $H$, a completion $(\Omega_2, b_1)$ for $K$, and the pullback $\Omega_1 * \Omega_2$.

Now we show that a pullback of completions of two subgroups gives a completion for the intersection of the subgroups.

**Theorem 3.26.** *Let $H_1$ and $H_2$ be finitely generated subgroups of a Coxeter group $W$, and let $(\Omega_1, b_2)$ and $(\Omega_2, b_2)$ be their respective completions. Then $(\Omega_1 * \Omega_2, (b_1, b_2))$ is a completion for $H_1 \cap H_2$.*

*Proof.* To show that $(\Omega_1 * \Omega_2, (b_1, b_2))$ satisfies the definition of a completion, first note that $\Omega_1 * \Omega_2$ is folded and cell-full and therefore satisfies part 1 of Definition 3.7. To prove part 2 of the definition is satisfied, we prove that any loop based at $(b_1, b_2)$ in $\Omega_1 * \Omega_2$ is labeled by a word $w$ representing an element of $H_1 \cap H_2$. If $w$ is the label of a loop based at $(b_1, b_2)$, then by construction there must exist loops based at $b_1$ in $\Omega_1$ and at $b_2$ in $\Omega_2$ both labeled by $w$. Therefore, $w$ is an expression of an element of $H_1 \cap H_2$.

Finally, we show that part 3 of the definition is satisfied by proving that any reduced word $w$ in $H_1 \cap H_2$ is the label of a loop in $(b_1, b_2)$ in $\Omega_1 * \Omega_2$. By Corollary 3.13, for any such $w$, there exists a loop based at $b_1$ in $\Omega_1$ and a loop based at $b_2$ in $\Omega_2$, both with label $w$. By construction, there must be a loop based at $(b_1, b_2)$ with label $w$ in $M$ and therefore also in $\Omega_1 * \Omega_2$. $\square$

Since changing basepoints corresponds to conjugating our subgroup, we can use a pullback to produce completions for intersections of conjugates of subgroups.

**Corollary 3.27.** *Let $H_1$ and $H_2$ be subgroups of a Coxeter group $W$. Let $(\Omega_1, b_1)$ be a completion of $H_2$ and $(\Omega_2, b_2)$ be a completion of $H_2$. If there is path in $(\Omega_1, b_1)$ from some vertex $u$ to $b_1$ labeled $g_1$ and a path in $(\Omega_2, b_2)$ from some vertex $v$ to $b_2$ labeled $g_2$, then $(\Omega_1 * \Omega_2, (v, u))$ is a completion for $g_1 H_1 g_1^{-1} \cap g_2 H_2 g_2^{-1}$.*

*Proof.* By Lemma 3.14, we know that $(\Omega_1, u)$ is a completion for $g_1 H_1 g_1^{-1}$ and $(\Omega_2, v)$ is a completion for $g_2 H_2 g_2^{-1}$. Therefore, by Theorem 3.26, we have that $(\Omega_1 * \Omega_2, (u, v))$ is a completion for $g_1 H_1 g_1^{-1} \cap g_2 H_2 g_2^{-1}$. $\square$

Note that if we want to find a completion for $g_1 H_1 g_1^{-1} \cap g_2 H_2 g_2^{-1}$ for arbitrary $g_1, g_2 \in W$ which do not necessarily label paths in $(\Omega_i, b_i)$, we can take a path labeled $g_i^{-1}$ and attach its origin to $b_i$ in $\Omega_i$ and take the completion of this new complex to get a new completion $\Omega_i'$ of $H_i$ for $i = 1, 2$. Then $g_1$ and $g_2$ will be labels of paths in $\Omega_1'$ ending at $b_1$ and in $\Omega_2$ ending at $b_2$, respectively, which allows us to apply Corollary 3.27.

### 3.3.1. Nonpositively curved cube complexes

We can use this construction of a pullback to find completions for intersections of subgroups of fundamental groups of non-positively curved cube complexes, which are not Coxeter groups. This adds to the results in [4]. We will first cover the definition of a completion for subgroups of fundamental groups of nonpositively curved cube complexes. This construction comes from [4].

Let $Y$ be a nonpositively curved cube complex and let $G = \pi_1(Y, b_Y)$. Let $H = \langle h_1, \ldots, h_k$ be a finitely generated subgroup of $G$. We can consider the generator $h_i$ as loop a $\gamma_i$ in the 1-skeleton of $Y$ based at $b_Y$. Let $X$ be the wedge of $k$ circle and let $b_X$ be the basepoint. We can define a map $f : (X, b_X) \to (Y, b_Y)$ and subdivide each loop in $X$ so that $f$ is a cubical map. The goal is to turn $X$ into a complex such that $f$ becomes a local isometry, as then $f_*$ is injective and $\pi_1(X, b_X)$ can be viewed as a subgroup of $\pi_1(Y, b_Y)$. To do this, we use the following operations.

1. (Folding) If $e$ and $e'$ are adjacent edges in $X$ and $f(e) = f(e')$, then we identify $e$ and $e'$ to get a new complex $X'$. Note that $f$ factors through the quotient map $X \to X'$ to give us a new cubical map $f' : X' \to Y$.
2. (Cube Identification) If $n$-cubes $C$ and $C'$ of $X$ share the same 1-skeleton, then we identify $C$ and $C'$ to obtain a new cube complex $X'$. Note that since $Y$ is non-positively curved and $f(C)$ and $f(C')$ share the same 1-skeleton, we know that $f(C) = f(C')$. Therefore $f$ again factors through the quotient map to give a new cubical map $f' : X' \to Y$.
3. (Cube Attachment) For a vertex $v$ of $X$, if $f(lk(v))$ is not a full subcomplex of

$lk(f(v))$, then there exists vertices $v_1, \ldots, v_n$ of $lk(v)$ that do not span a simplex in $lk(v)$, but $f(v_1), \ldots, f(v_n)$ does span a vertex in $lk(f(v))$. Let $e_1, \ldots, e_n$ be the edges adjacent to $v$ such that $e_1$ is the edge containing $v_i$. We attach a vertex of $n$ cube to $v$ and identify the $n$ edges of the $n$-cube adjacent to $v$ with the edges $e_1, \ldots, e_n$ to obtain $X'$. Note $f$ naturally extends to a map $f' : X' \to Y$.

The direct limit of applying these operations to $X$ is a cube complex $(\Omega, b'_X)$ with a local isometry $f' : \Omega \to Y$ and $\pi_1(\Omega, b'_X) = H$. We say that $(\Omega, b'_X)$ is a *cubical completion* of $H$. It was shown in [4] that $\Omega$ is finite if and only if $H$ is quasiconvex.

**Definition 3.28.** (Cubical word [4]) Let $T$ be a spanning tree of a cube complex $y$ and let $S$ be the set of squares in $Y$. Then

$$\pi_1(Y, b) = \langle E(Y) \mid \{e \mid e \in T\} \cup \{\partial s \mid s \in S\} \rangle.$$

A word $w$ in this presentation is a *cubical word* if it lifts to a path in the universal cover $\widetilde{Y}$ of $Y$. We say $w$ is *reduced* if it lifts to a geodesic in $\widetilde{Y}$.

Now we will define a pullback for cube complexes.

**Definition 3.29.** (Pullback of cube complexes) Let $Y$ be a non-positively curved cube complex with oriented edges. Give each edge a distinct label. For cube complexes $M_1$ and $M_2$ with locally isometric cubical maps $f_1 : M_1 \to Y$ and $f_2 : M_2 \to Y$, label each edge $e \in M_i$ by the label of $f_i(e)$. We define a new cube complex $M$ with vertices given by $V(M_1) \times V(M_2)$ and an edge $((v_1, u_1), (v_2, u_2))$ labeled $a$ in $M$ if there are edges labeled $a$ connecting $v_1$ and $v_2$ in $M_1$ and connecting $u_1$ and $u_2$ in $M_2$. We define $f : M \mapsto Y$ to be the map which sends each edge labeled $x$ to the edge labeled $x$ in $Y$ and the *pullback* $M_1 * M_2$ to be the completion of $M$.

Now we show that a pullback of cubical completions behaves like the pullback of completions of subgroups of Coxeter groups.

**Theorem 3.30.** *For a non-positively curved cube complex $Y$, $G = \pi_1(Y, b_Y)$, and subgroups $H, K \leqslant G$, let $(M_1, p_1)$ and $(M_2, p_2)$ be cubical completions associated with $H$ and $K$, respectively. Then $M_1 * M_2$ is a completion of $H \cap K$.*

*Proof.* We claim that $\pi_1(M_1 * M_2, (p_1, p_2)) \cong H \cap K$. For $g \in H \cap K$, we can express $g$ as a reduced cubical word $w$ based at $q$. Since $M_1$ and $M_2$ are completions, there exists a loop in $M_1$ based at $p_1$ and a loop in $M_2$ based at $p_2$, both having the label $w$. By construction, there must exist a loop in $M_1 * M_2$ with the label $w$.

Conversely, given a loop in $M_1 * M_2$ labeled $w$ based at $p_1 \times p_2$, there must be a loop in $M_1$ based at $p_1$ and a loop $M_1$ based at $p_1$ both with the label $w$. Therefore, $w \in H \cap K$. So $f_*(\pi_1[M_1 * M_2, p_1 \times p_2]) = H \cap K$. Since $M_1 * M_2$ is a cubical completion, $f_*$ is injective, thus proving our claim. □

Since cubical completions are finite for quasiconvex subgroups, we get the following algorithm.

**Corollary 3.31.** *Let $H$ and $K$ be quasiconvex subgroups of the fundamental group of a nonpositively curved cube complex $Y$. There exists an algorithm which computes the generating set of $H \cap K$.*

*Proof.* By Theorem 5.1 of [4], the cubical completions $M_1$ and $M_2$ of $H$ and $K$, respectively are both finite since $H$ and $K$ are quasiconvex, so $M_1 * M_2$ is finite. Therefore $M_1 * M_2$ can be constructed in a finite number of steps. □

### 3.3.2. Malnormal subgroups

Given a completion $(\Omega, b)$ of a subgroup $H$ of a right-angled Coxeter group $W$, we can use the pullback $\Omega * \Omega$ to characterize whether $H$ is malnormal. Note that the diagonal of $\Omega * \Omega$ is precisely $\Omega$ and is a connected component of the pullback. Furthermore, a path labeled $w$ in $\Omega * \Omega$ not contained in the diagonal exists if and only if there are two distinct paths in $\Omega$ with label $w$.

**Definition 3.32.** A nontrivial subgroup $H$ of a group $G$ is malnormal if for any element $g \in G \setminus H$ we have $H \cap gHg^{-1} = \emptyset$.

Now we will start finding conditions on a completion for when a subgroup is malnormal.

**Lemma 3.33.** *For a subgroup $H$ of a right-angled Coxeter group $W$, let $(\Omega, b)$ be a completion of $H$. For an element $g \in W \setminus H$ such that $H \cap gHg^{-1} \neq 1$, if $g$ is the label a path in $\Omega$ based at $b$, then there exists a component of $\Omega * \Omega$ which has a nontrivial fundamental group and does not contain $b \times b$.*

*Proof.* Suppose $w$ is a reduced nontrivial word representing an element of $H \cap gHg^{-1}$ and $g$ is the label of a path in $\Omega$ with endpoints $b$ and $v$. Then there exists a loop in $\Omega$ based at $b$ and a loop based at $v$ both with label $w$. Therefore, there exists a loop $\gamma$ labeled $w$ based at $b \times v$ in $\Omega * \Omega$. Note that $\gamma$ is not null-homotopic, as otherwise $w$ would represent the identity in $W$, which is a contradiction. Therefore, the component of $\Omega * \Omega$ containing $b \times v$ is not simply connected. $\square$

We now need to show that such a path $g$ must exist in our completion. To do this, we will need the following lemma.

**Lemma 3.34.** *For a cyclically reduced word $w \in H \cap gHg^{-1}$, if there exists reduced words $w'$, $g_1$, $g_2$, $g_3$, and $g_4$ such that $g_1g_2$ and $g_3g_4$ are both reduced expressions for $g$, $g_2^{-1}w'g_4$ is a reduced word in $g^{-1}Hg$, and $g_1w'g_3^{-1}$ is a reduced expression for $w$, then $g$ is the label of a path in $\Omega$ based at $b$.*

*Proof.* First note that $g_1w'g_3^{-1}$ is the label of a loop in $\Omega$ based at $b$, since it is a reduced word in $H$. Let $v$ be the vertex at the end of the path labeled $g_1$. Then $(\Omega, v)$ is a completion for $g_1^{-1}Hg_1$. Since $w$ is cyclically reduced, $w'g_3^{-1}g_1$ is a reduced word in $g_1^{-1}Hg_1$. Since $w'g_3^{-1}g_1 = w'g_4g_2^{-1}$, there exists a loop labeled $w'g_4g_2^{-1}$ based at $v$. Therefore, there is a path labeled $g_2$ from $v$, so there is a path labeled $g$ from $b$. □

We use these lemmas to show that malnormality can be characterized by a completion.

**Theorem 3.35.** *Suppose $(\Omega, b)$ is a completion of a subgroup $H$ of a right-angled Coxeter group $W$. For each generator $s$ of $W$, attach an edge labeled $s$ to each vertex of $\Omega$. Let $\Omega'$ be a completion of this new complex. Then $H$ is malnormal if and only if $\Omega' * \Omega'$ has exactly one component with trivial fundamental group.*

*Proof.* First, suppose $\Omega' * \Omega'$ has multiple components with nontrivial fundamental group. Let $v \times u$ be a vertex in one such component which does not contain $b \times b$. Let $\gamma$ be a loop based at $u \times v$ with label $w$. Then there exists a loop in $\Omega'$ based at $u$ with label $w$ and a loop based at $v$ with label $w$ which is not null-homotopic. Let $g_u$ be the label of a path from $b$ to $u$ and $g_v$ be the label of a path from $b$ to $v$. Then $w$ is a non-trivial element of $g_u^{-1}Hg_u$ and $g_v^{-1}Hg_v$. Therefore $g_vg_u^{-1}Hg_ug_v^{-1} \cap H \neq 1$, so $H$ is not malnormal.

Now suppose $H$ is not malnormal. Let $g \notin H$ such that $H \cap gHg^{-1} \neq 1$ and $w$

be a reduced word representing an element of $H \cap gHg^{-1}$. We can assume without loss of generality that $w$ is also cyclically reduced, as otherwise we can pass to a conjugate of $H \cap gHg^{-1}$. There is $w' \in H$ such that $gw'g^{-1} = w$. Suppose $s$ is a generator which commutes with $w'$ and $s \notin H$. By construction, $b$ is incident to an edge labeled $s$ in $\Omega'$. Let $v$ be the other endpoint of this edge. If $v = b$, then $s$ is an element of $H$, which is a contradiction. Therefore, assume $v \neq b$.

Since $w'$ is reduced, there exists a loop based at $b$ and a loop based at $v$ both with label $w'$. Therefore there exists a nontrivial loop labeled $w'$ based at $b \times v$ in $\Omega' \times \Omega'$. If $b \times v$ is in the same component as $b \times b$, then there exists a path $\alpha_1$ from $b$ to $b$ and a path $\alpha_2$ from $b$ to $v$ with the same label. Since $\Omega'$ is folded, this implies that $\alpha_1 = \alpha_2$ and $v = b$, which is a contradiction. Therefore, $b \times v$ is not in the component containing $b \times b$, so there are multiple components of $\Omega' \times \Omega'$ with nontrivial fundamental group.

Now suppose no such $s$ exists. Since $W$ satisfies the deletion and exchange conditions, we can write $gw'$ as the reduced word $g_1w_1$ where $g = g_1g_2$ and $w' = g_2^{-1}w_1$. Similarly, $w_1g^{-1}$ can be written as the reduced word $w_2g_3^{-1}$ with $g = g_3g_4$ and $w_1 = w_2g_4$. If $g_1w_2g_4^{-1}$ is not reduced, then a letter of $g_1$ must cancel with some letter of $g_4^{-1}$, which is a contradiction. Therefore $g$ and $w$ satisfy the conditions of Lemma 3.34, so $\Omega' * \Omega'$ has multiple components with nontrivial fundamental group by Lemma 3.33. □

This allows us to construct an algorithm for determining when a quasiconvex subgroup of a right-angled Coxeter group is malnormal.

**Corollary 3.36.** *For a quasiconvex subgroup $H$ of a right-angled Coxeter group $W$, there exists an algorithm which determines whether $H$ is malnormal.*

*Proof.* By Theorem A of [6], we know that there exists a finite completion $\Omega$ of $H$, since $H$ is quasiconvex. Therefore $\Omega'$ is a standard completion of $H$, so $\Omega'$ is also finite by [6]. Therefore $\Omega' * \Omega'$ is finite and its construction requires a finite number of operations. □

**Example 3.37.** Let $W$ be the Coxeter group given by the presentation

$$W = \langle a, b, c \mid a^2, b^2, c^2, (ab)^2 \rangle.$$

Then the subgroup $H = \langle abcab \rangle$ is malnormal in $W$. Figure 3.6 shows a completion $\Omega_H$ of $H$ and the pullback $\Omega_H * \Omega_H$ where $\Omega_H = \Omega'_H$, where $\Omega'$ is constructed as in Theorem 3.35. In the figure, edges labeled $a$ are colored red, edges labeled $b$ are colored green, and edges labeled $c$ are colored blue. The bottom of the figure shows the pullback "untangled." Note that all components aside from the diagonal are simply connected.

Now, let $K = \langle a, bac \rangle$ be a subgroup of $W$. Figure 3.7 shows a completion $\Omega_K$ of $K$, where $\Omega_K = \Omega'_K$, and the pullback $\Omega_K * \Omega_K$. Note that $K$ is not malnormal as $\Omega_K = \Omega'_K$ has two components which are not simply connected.

### 3.4. Quasiconvexity

In [6] and [4], completions are shown to be finite if and only if the corresponding subgroup is quasiconvex in the cases of subgroups of right-angled Coxeter groups and of subgroups of fundamental groups of non-positively curved cube complexes. We will prove that if a completion of a subgroup of a general Coxeter group is finite, then the corresponding subgroup is quasiconvex. Then we will apply this to show that a family of reflection subgroups are quasiconvex.

**Definition 3.38.** For a geodesic metric space $(M, d)$, a subset $U$ is said to be $k$-*quasiconvex* for $k > 0$ if for any two points $x, y \in U$ and any geodesic $\gamma$ from $x$ to $y$,

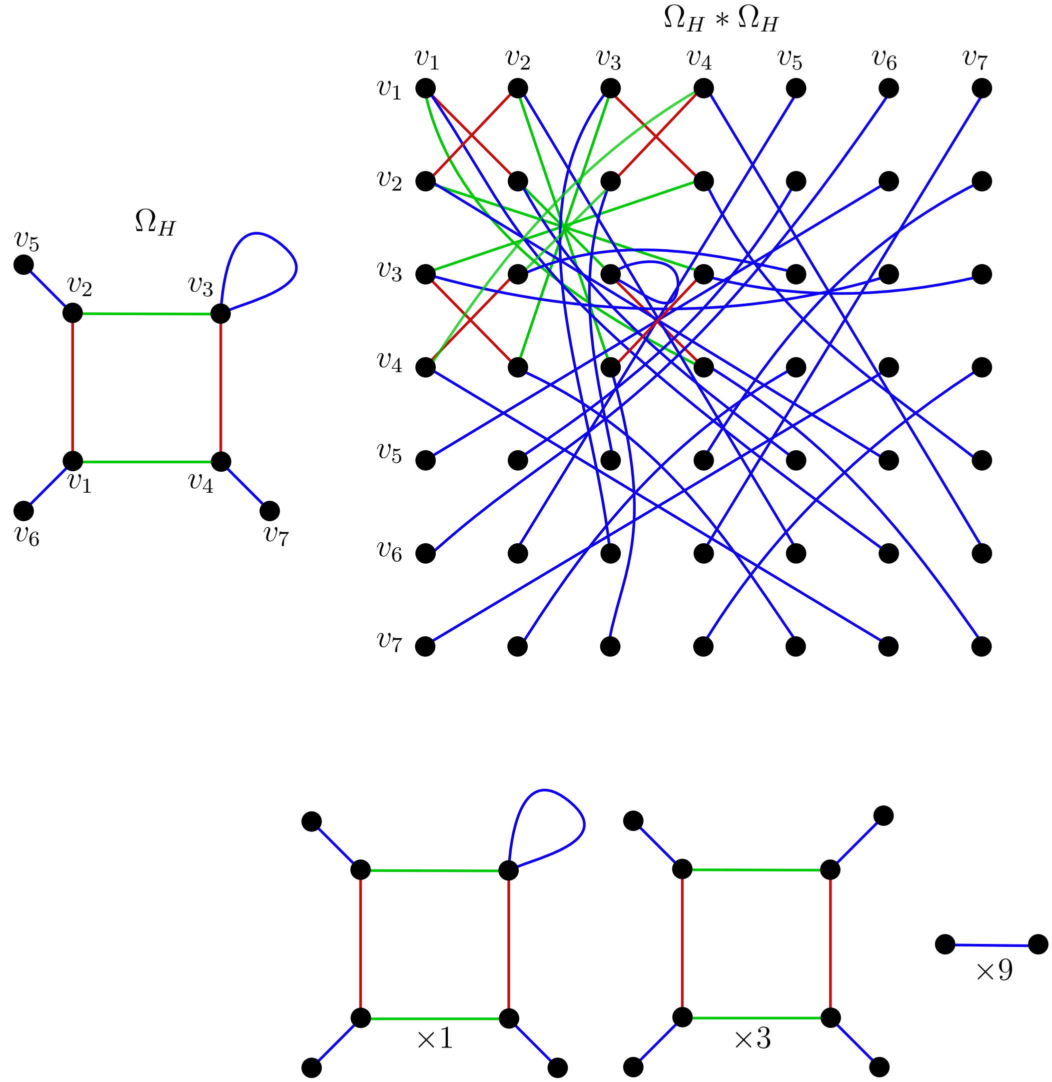


Figure 3.6. The top left figure is $\Omega_H$ from Example 3.37. The top right figure is $\Omega_H * \Omega_H$. The bottom three figures are a rearrangement of $\Omega_H * \Omega_H$. Note that each square bounds a 2-cell, so the pullback has exactly one component with nontrivial fundamental group, so $H$ is malnormal.

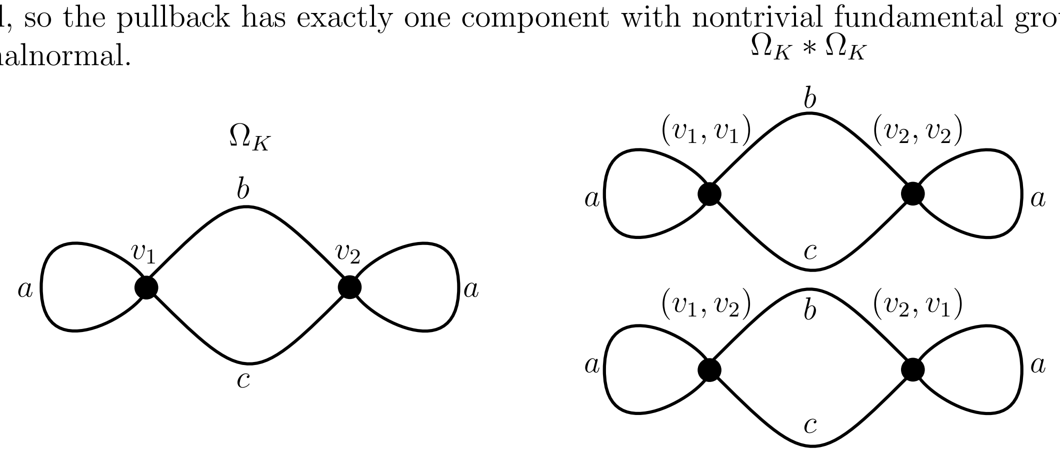


Figure 3.7. The left figure is the completion $\Omega_K$ from Example 3.37. The right figure is the pullback $\Omega_K * \Omega_K$. Both components of the pullback are not simply connected, so $K$ is not malnormal.

we have that $\gamma$ is contained in a $k$-neighborhood of $U$. A subset $U$ is said to be *quasiconvex* if it is $k$-quasiconvex for some $k$.

We say that a subgroup $H$ of a group $G$ with generating set $S$ is quasiconvex in $G$ if $H$ is quasiconvex in $Cay(G, S)$.

To relate quasiconvexity with completions of a subgroup, we first need to relate distances in a completion with the word metric of our group.

**Theorem 3.39.** *Let $(\Omega, b)$ be a completion of a subgroup $H$ of a Coxeter group $W$. We will denote the metric on $\Omega$ by $d_\Omega$. Let $Cay(W)$ be the Cayley graph of $W$ with respect to its standard generating set with metric $d_C$. Then for some path from $b$ to some vertex $v$ in $\Omega$ labeled $g$, we have that $d_\Omega(v, b) = d_C(g, H)$.*

*Proof.* Let $\alpha$ be a geodesic from $v$ to $b$ in $\Omega$ labeled $w$. Then $w$ is a reduced word in $W$ with length $d_\Omega(v, b)$. Since $\gamma\alpha$ is a loop in $\Omega$ based at $b$, we have that $ga$ is a word in $H$. Therefore, the path in $Cay(W)$ from $g$ labeled by $w$ is a geodesic to a vertex of $H$, so $d_\Omega(v, b) \geq d_C(g, H)$.

Now let $w$ be the label of a geodesic from $g$ to $H$ in $Cay(W)$ which realizes $d_C(g, H)$. Then $gw$ is an element of $H$, so there exists a loop in $\Omega$ whose label is equal to $gw$ in $\Omega$. Therefore, there is a path from $v$ to $b$ in $\Omega$ whose label is equal to $w$ in $W$. By Lemma 3.12, since $w$ is reduced, there exists a path from $v$ to $b$ labeled by $w$. Therefore, $d_\Omega(v, b) \leq d_C(g, H)$. □

Using this equivalence between distances in the completion and in the Cayley graph, we can relate quasiconvexity with the completion of a subgroup.

**Corollary 3.40.** *If $H$ is a subgroup of a Coxeter group $W$ and $(\Omega, b)$ is a finite comple-*

*tion of $H$, then $H$ is quasiconvex in $W$.*

*Proof.* Let $D = max\{d(b, v) \mid v \in \Omega\}$ and let $\gamma$ be a geodesic in $Cay(W)$ between vertices in $H$. By Lemma 3.39, we know $\gamma$ is contained within a $D$-neighborhood of $H$. Therefore, $H$ is quasiconvex in $W$. □

Another application of Theorem 3.39 is that the core of a quasiconvex subgroup must be finite.

**Corollary 3.41.** *Let $H$ be an $M$-quasiconvex subgroup of a Coxeter group $W$ with completion $(\Omega, b)$. Then $C(\Omega, b)$ has diameter at most $2M + 1$.*

*Proof.* Suppose $C(\Omega, b)$ has diameter greater than $2M + 1$. Then there exists an edge $e$ such that $d(e, b) > M$. Since $e$ is an edge in $C(\Omega, b)$, there exists a loop $\gamma$ based at $b$ containing $e$ whose label is a reduced word $w$. Then the path from 1 in $Cay(W)$ contains an edge which is a distance more than $m$ away from $b$. This is a contradiction, as $H$ is $M$-quasiconvex. □

We will use Corollary 3.41 to construct an algorithm to find a generating set for the intersection of quasiconvex subgroups of a Coxeter group.

**Theorem 3.42.** *For two quasiconvex subgroups, $H$ and $K$, of a Coxeter group There exists an algorithm which computes the generating set $H \cap K$.*

*Proof.* Let $M$ be a positive number such that both $H$ and $K$ are $M$-quasiconvex. Let $S = \{s_1, \ldots, s_n\}$ be a standard generating set for $W$. Let $T_0 = Cay(F_n, S)$ and $b$ be the vertex corresponding to 1 in $T_0$. We now focus only on edges of $T_0$ that are distance at most $M$ away from $b$, so let $T = T_0 \cap B_M(b)$. Our goal is to use $T$ to construct finite complexes for

$H$ and $K$ which contain their respective core graphs.

We will go through this construction for $H$. Let $\{h_1, \ldots, h_m\}$ a generating set of $H$ such that each $h_i$ is reduced. Let $X$ be the standard wedge of $m$ loops, with the $i$th loop subdivided and labeled by $h_i$. We define $\Omega_0$ to be the graph obtained by identifying the wedge vertex of $X$ with $b$ in $T$. We call the image of $b$ in $\Omega_0$ to be $b_H$. We now attach the appropriate relator disks to every deficient path in $\Omega_0$ to obtain $\Omega_1$. Then we fold $\Omega_1$ until we obtain a folded complex $\Omega_H$. Note that $C(\Omega_H, b_H)$ is the core graph for $H$. We repeat this same process to obtain $C(\Omega_K, b_K)$ for $K$.

The pullback $(C(\Omega_H, b_H) * C(\Omega_K, b_K), (b_H, b_K))$ contains the core graph for $H \cap K$, as any reduced word in $H \cap K$ must be the label of a loop in $(C(\Omega_H, b_H) * C(\Omega_K, b_K)$ based at $(b_H, b_K)$ by construction. Now we can find a generating set for $H \cap K$ by taking a spanning tree of $(C(\Omega_H, b_H) * C(\Omega_K, b_K)$ and looking at the edges in its complement. $\square$

### 3.4.1. Reflection subgroups

We will show that a class of reflection subgroups of Coxeter groups are quasiconvex subgroups.

**Definition 3.43.** (Reflection Subgroup) A conjugate of a standard generator of a Coxeter group $W$ is called a *reflection*. A subgroup of $W$ generated by reflections is a *reflection subgroup.*

We will use Corollary 3.40 to show that the following reflection subgroups are quasiconvex.

**Theorem 3.44.** *Suppose $H$ is a subgroup of a Coxeter group $W$ generated by reflections $\langle w_1 r_1 w_1^{-1}, \ldots, w_n r_n w_n^{-1} \rangle$ such that for any $i, j$ there is no reduced expression for $w_j^{-1} w_i$*

*ending in a letter $s$ such that $m_{s,r_i} < \infty$, where $r_i$ is a generator of $W$. Then $H$ is quasi-convex in $W$.*

*Proof.* We will prove this by constructing a finite completion for $H$. First, let $X$ be a wedge of $n$ loops where the $i$th loop is subdivided and labeled by the letters of $w_i r_i w_i^{-1}$ for $1 \leq i \leq n$. Let $\Omega_0$ be the result of applying all possible folds to $X$. Then $\Omega_0$ is a tree $T$ with graph-loops attached.

Let $T = T_0 \to T_1 \to \ldots \to T'$ be a sequence of completion operations which results in a completion $T'$. We will prove that any hyperplane of $T'$ must separate $T'$ into two components. This is clearly true for $T$, as it is a tree. Suppose any hyperplane of $T_i$ separates $T_i$ into two components. First we consider the case where $T_{i+1}$ is obtained by folding edges $e_1$ and $e_2$. Any hyperplane of $T_{i+1}$ is the image of a hyperplane of $T_i$. Let $h$ be a hyperplane of $T_i$ and let $\hat{h}$ be its image in $T_{i+1}$. If $h$ does not intersect $e_1$ or $e_2$, then $T_{i+1} \setminus \hat{h}$ is the same complex as $T_i \setminus h$ with $e_1$ and $e_2$ identified. Since $e_1$ and $e_2$ are contained in the same component of $T_i \setminus h$, we have that $T_{i+1} \setminus \hat{h}$ is also two components.

Now suppose $h_1$ is the hyperplane dual to $e_1$ and $h_2$ be the hyperplane dual to $e_2$. Let $v$ be the vertex incident to $e_1$ and $e_2$. Let $e_{i,1}$ be the component of $e_i \setminus h_i$ containing $v$ and let $e_{i,2}$ be the other component of $e_i \setminus h_i$ for $i = 1, 2$. Then $T_{i+1} \setminus \hat{h_1}$ is the same complex as $T_i \setminus h_1 \cup h_2$ with $e_{1,1}$ identified with $e_{2,1}$ and $e_{1,2}$ identified with $e_{2,2}$. Since $T_i \setminus h_1 \cup h_2$ has three components, the component containing $v$, the component containing $e_{1,2}$, and the component containing $e_{2,2}$, we have that $T_{i+1} \setminus \hat{h_1}$ has exactly two components.

Suppose that $T_{i+1}$ is obtained from $T_i$ by attaching a relator disk $C$ to a vertex $v$ of a deficient path $\gamma$. A hyperplane $h$ of $t_{i+1}$ is either contained in $C$ or contained in the image of $T_i$. If a hyperplane is contained in $C$, then let $C_1$ and $C_2$ be the two components of $C \setminus h$. Assume that $v$ is contained in $C_1$. Then $T_{i+1}$ has two components. The first component is the image of $T_i$ with $C_1$ attached to $v$ and the second is $C_2$. If $h$ is contained in the image of $T_i$, then $T_i \setminus h$ has two components, $U_1$ and $U_2$. Assume that $v$ is contained in $U_1$. Then $T_{i+1} \setminus h$ has two components, namely $U_1$ with $C$ attached to $v$ and $U_2$.

Now we will show that $T'$ is finite by showing that any geodesic intersects a given hyperplane of $T'$ at most once. Suppose that $\gamma$ is a geodesic which intersects the hyperplane $h$ twice. Let the starting vertex of $\gamma$ be $u$ and the ending vertex be $u$. Let $e$ and $e'$ be the edges of $\gamma$ dual to $h$. Without loss of generality, suppose that $e$ is the edge closest to the start of $\gamma$. Let $\gamma_1$ be the subpath of $\gamma$ from $u$ to the start of $e$ and let $\gamma_2$ be the subpath from the end of $e'$ to $v$. Let $\alpha$ be the geodesic from the start of $e$ to the end of $e'$ contained in $supp(h) \setminus h$. This path exists since $h$ separates $T'$ into two components. The path $\gamma_1\alpha\gamma_2$ is a path from $u$ to $v$ which is shorter than $\gamma$, which contradicts the fact that $\gamma$ is a geodesic. By Lemma 3.11, $T'$ contains at most $|E(T)|$ hyperplanes. Since any geodesic of $T'$ intersects a given hyperplane at most once, the diameter of $T'$ is at most $|E(T)|$, and therefore $T'$ is finite.

Now suppose that $l$ is a graph loop in $\Omega_0$ labeled $s$ incident to a vertex $v$. Let $\hat{v}$ be the image of $v$ in $T'$. We will show that no edge incident to $\hat{v}$ in $T'$ is labeled by a generator with a relation to $s$. For the sake of contradiction, suppose this is not the case and there is an edge $e$ incident to $\hat{v}$ labeled by $x$ with $m_{x,s} < \infty$. Let $h$ be the hyperplane dual to $e$ and $e'$ be the closest edge to $\hat{v}$ in $T$ which is dual to $h$. Let $u$ be the vertex of $e'$ which is farthest from $v$, $\alpha$ be geodesic in $T$ from $u$ to $\hat{v}$, and $\beta$ be the geodesic in $supp(h) \setminus h$ from $u$ to $e$. Let $a$ be the label of $\alpha$ and $b$ be the label of $\beta$. Then $\beta e$ is a geodesic in $T'$. Since $T'$ is simply connected, we know that $bx$ is a reduced expression for $a$. This contradicts the assumption of our theorem, as $a$ is a suffix of some $w_j^{-1}w_i$.

Therefore, attaching $l$ to $T'$ is still folded and does not create any deficient paths, so attaching all such graph loops from $\Omega_0$ to $T'$ creates a finite completion of $H$. Therefore $H$ is quasiconvex by Corollary 3.40. $\square$

## Bibliography

[1] GN Arzhantseva. Generic properties of finitely presented groups and howson's theorem. *Communications in Algebra*, 26(11):3783–3792, 1998.

[2] Goulnara N Arzhantseva and Alexander Yu Ol'shanskii. Generality of the class of groups in which subgroups with a lesser number of generators are free. *Mat. Zametki*, 59(4):489–496, 1996.

[3] Benjamin Beeker and Nir Lazarovich. Stallings' folds for cube complexes. *Israel Journal of Mathematics*, 227(1):331–363, 2018.

[4] Michael Ben-Zvi, Robert Kropholler, and Rylee Alanza Lyman. Folding-like techniques for cat (0) cube complexes. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 173, pages 227–238. Cambridge University Press, 2022.

[5] Martin R Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319. Springer Science & Business Media, 2013.

[6] Pallavi Dani and Ivan Levcovitz. Subgroups of right-angled coxeter groups via stallings-like techniques. *Journal of Combinatorial Algebra*, 5(3):237–295, 2021.

[7] Michael W Davis. *The Geometry and Topology of Coxeter Groups.(LMS-32)*. Princeton University Press, 2012.

[8] Michael W Davis and Michael D Shapiro. Coxeter groups are automatic. *Ohio State University*, 1991.

[9] Jordi Delgado and Enric Ventura. A list of applications of stallings automata. *arXiv preprint arXiv:2109.01268*, 2021.

[10] Vinay V Deodhar. A note on subgroups generated by reflections in coxeter groups. *Archiv der Mathematik*, 53(6):543–546, 1989.

[11] Matthew Dyer. Reflection subgroups of coxeter systems. *Journal of Algebra*, 135(1):57–73, 1990.

[12] Frédéric Haglund and Daniel T Wise. Coxeter groups are virtually special. *Advances in Mathematics*, 224(5):1890–1903, 2010.

[13] Kasia Jankiewicz, Sergey Norin, and Daniel T Wise. Virtually fibering right-angled coxeter groups. *Journal of the Institute of Mathematics of Jussieu*, 20(3):957–987, 2021.

[14] Kasia Jankiewicz and Daniel Wise. Incoherent coxeter groups. *Proceedings of the*

*American Mathematical Society*, 144(5):1857–1866, 2016.

[15] Ilya Kapovich and Alexei Myasnikov. Stallings foldings and the subgroup structure of free groups. *arXiv preprint math/0202285*, 2002.

[16] Ilya Kapovich and Paul Schupp. Bounded rank subgroups of coxeter groups, artin groups and one-relator groups with torsion. *Proceedings of the London Mathematical Society*, 88(1):89–113, 2004.

[17] Olga Kharlampovich, Alexei Miasnikov, and Pascal Weil. Stallings graphs for quasi-convex subgroups. *Journal of Algebra*, 488:442–483, 2017.

[18] Dawid Kielak. Residually finite rationally solvable groups and virtual fibring. *Journal of the American Mathematical Society*, 33(2):451–486, 2020.

[19] Jonathan P McCammond and Daniel T Wise. Coherence, local quasiconvexity, and the perimeter of 2-complexes. *Geometric & Functional Analysis GAFA*, 15:859–927, 2005.

[20] Gabor Moussong. *Hyperbolic coxeter groups*. The Ohio State University, 1988.

[21] Graham A Niblo and Lawrence D Reeves. Coxeter groups act on cat (0) cube complexes. 2003.

[22] Paul E Schupp. Coxeter groups, 2-completion, perimeter reduction and subgroup separability. *Geometriae Dedicata*, 96:179–198, 2003.

[23] John R Stallings. Topology of finite graphs. *Inventiones mathematicae*, 71(3):551–565, 1983.

[24] Hung Cong Tran. Malnormality and join-free subgroups in right-angled coxeter groups. *arXiv preprint arXiv:1703.09032*, 2017.

## Vita

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