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Analytic Wavefront Sets of Spherical Distributions on the De Sitter Space

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Abstract

In this work, we determine the wavefront set of certain eigendistributions of the Laplace-Beltrami operator on the de Sitter space. Let $G' = \mathrm{O}_{1,n}(\mathbb{R})$ be the Lorentz group, and let $H' = \mathrm{O}_{1,n-1}(\mathbb{R}) \subset G'$ be its subset. The de Sitter space dS^n is a one-sheeted hyperboloid in $\mathbb{R}^{1,n}$ isomorphic to G'/H'. A *spherical distribution* is an H'-invariant eigendistribution of the Laplace-Beltrami operator on dS^n. The space of spherical distributions with eigenvalue λ, denoted by $\mathcal{D}'_\lambda(\mathrm{dS}^n)$, has dimension 2. We construct a basis for the space of positive-definite spherical distributions as boundary value of sesquiholomorphic kernels on the crown domains, which are open complex domains in $\mathrm{dS}^n_{\mathbb{C}}$ containing dS^n on the boundary. We characterize the analytic wavefront set for such distributions.

Notations

- $[z, w] = -z_0w_0 + z_1w_1 + ... + z_nw_n$ for $z, w \in \mathbb{C}^{1+n}$,
- $\mathbb{R}^{1,n} = (\mathbb{R}^{1+n}, [\ ,\ ])$,
- $\mathrm{G}' = \mathrm{O}_{1,n}(\mathbb{R})$,
- $\mathrm{H}' = \mathrm{O}_{1,n-1}(\mathbb{R})$,
- $\mathrm{G} = \mathrm{SO}_{1,n}(\mathbb{R})_e$,
- $\mathrm{H} = \mathrm{SO}_{1,n-1}(\mathbb{R})_e \subset G$,
- $\mathrm{K} = \mathrm{SO}_n(\mathbb{R}) \subset G$,
- $G_{\mathbb{C}} = SO_{1+n}(\mathbb{C})$,
- $K_{\mathbb{C}} = SO_n(\mathbb{C})$,
- $\mathrm{dS}^n = \{x \in \mathbb{R}^{1+n} \mid [x, x] = 1\} \simeq G/H \simeq G'/H'$,
- $\mathrm{dS}^n_{\mathbb{C}} = \{z \in \mathbb{C}^{1+n} \mid [z, z] = 1\} = G_{\mathbb{C}}/K_{\mathbb{C}}$,
- $\mathbb{H}^n = \{ix \in i\mathbb{R}^{1+n} \mid x_0 > 0, -x_0^2 + \mathbf{x}^2 = -1\} \simeq G/K \subset \mathrm{dS}^n_{\mathbb{C}}$,
- $\overline{\mathbb{H}}^n = \{ix \in i\mathbb{R}^{1+n} \mid x_0 < 0, -x_0^2 + \mathbf{x}^2 = -1\} \simeq G/K \subset \mathrm{dS}^n_{\mathbb{C}}$,
- $\mathbb{S}^n = \{(ix_0, \mathbf{x}) \mid x_0^2 + \mathbf{x}^2 = 1\} \subset i\mathbb{R}e_0 + \mathbb{R}^n$,
- $\mathbb{S}^n_{\pm} = \{(ix_0, \mathbf{x}) \in \mathbb{S}^n \mid \pm x_0 > 0\}$,
- $\Xi = G.\mathbb{S}^n_+$,
- $\overline{\Xi} = G.\mathbb{S}^n_-$,
- $\Gamma^{\pm}(x) = \{y \in \mathrm{dS}^n \mid \text{for}\, x \in \mathrm{dS}^n, [y - x, y - x] < 0, \pm y_0 > x_0\}$,
- $\Gamma(x) = \Gamma^+ \cup \Gamma^-$,
- $\mathbb{L}_{n-1} = \{v \in \mathbb{R}^{1,n-1} \mid [v, v] = 0\}$,
- $\Omega = \{v \in \mathbb{R}^{1,n} \mid [v, v] < 0, v_0 > 0\}$,
- $T_\Omega = \mathbb{R}^{1+n} + i\Omega$,

- $\sigma(v) = \overline{v}$
- $\rho = (n-1)/2$ for $n \geq 2$.

Chapter 1. Introduction

For $x \in \mathbb{R}^{1+n}$, the Lorentzian bilinear form is given by $[x,x] = -x_0^2 + x_1^2 + ... + x_n^2$. Let $G' = O_{1,n}(\mathbb{R})$ be the group of isometries preserving the Lorentzian bilinear form and $H' = O_{1,n}(\mathbb{R})$ be its subgroup that fixes the point $e_n = (0,...,0,1)$. The n-dimensional de Sitter space dS^n is a one-sheeted hyperboloid (see Fig. 1.1) and a Lorentzian manifold. Mathematically, the de Sitter space is given by

$$\mathrm{dS}^n = \{x \in \mathbb{R}^{1+n} \mid [x,x] = 1\} \cong G' \cdot e_n.$$

A distribution Θ on dS^n is said to be *spherical* if it is an H'-invariant eigendistribution of the Laplace-Beltrami operator on dS^n. The space of such distributions with eigenvalue $\lambda \in \mathbb{C}$ is denoted by $\mathcal{D}'^{H'}_{\lambda}(\mathrm{dS}^n)$. In [D08], Van Djik has proven that the dimension of this space is 2. In this thesis (see also [ÓS23]) we will construct the basis of $\mathcal{D}'^{H'}_{\lambda}(\mathrm{dS}^n)$ for specific λ as boundary values of some sesquiholomorphic kernels defined in some open complex domains. We will study the singularities of the basis distributions and characterize them based on their singularities.

The de Sitter space is simple model of universe in special relativity. There have been several works done to understand the quantum field theories on dS^n including the papers [BM96, BM04, BV96, BV97] where the authors studied free fields and the related two

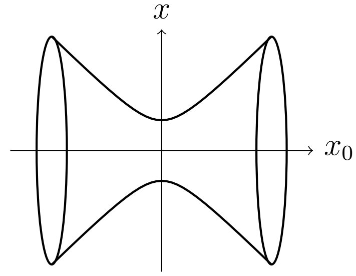

Figure 1.1. dS^2

point functions $\mathcal{W}_\lambda(x_1, x_2)$. The theory of interacting quantum fields on the de Sitter space is discussed in the paper [BJM13]. On the other hand, the authors in [NÓ22] study some aspects of algebraic quantum field theory on casual symmetric spaces of which the de Sitter space is an example. One of the tool used in [NÓ22] is an open complex domain called the crown domains Ξ .

In [NÓ18], the authors showed that the de Sitter space lies on the boundary of the open complex domains Ξ and its complex conjugate $\overline{\Xi}$ which are subsets of open complex unit sphere. The crown Ξ is holomorphically equivalent to the Lorentzian tuboid $\mathcal{T}^+$ in the complexified de Sitter space $\mathrm{dS}^n_{\mathbb{C}}$ defined in [BM96, BM04, BV96, BV97]. Similarly, $\overline{\Xi}$ is equivalent to the tuboid $\mathcal{T}^-$ in $\mathrm{dS}^n_{\mathbb{C}}$.

For $\rho = (n-1)/2$ and $\lambda \in i[0,\infty) \cup (0,\rho)$, the distribution $\mathcal{W}_\lambda(x_1, x_2)$ given in [BM96] satisfies the Klein-Gordon equation $\Delta + (\rho^2 - \lambda^2) = 0$ in both the variable. Moreover, this distribution is the boundary value of some analytic kernels in $\mathcal{T}^+$ and $\mathcal{T}^-$ called "perikernels". In parallel to perikernels, the authors in [NÓ18, NÓ20] have introduced the kernel Ψ_λ upto a constant and showed that Ψ_λ was represented as a hypergeometric function. Analogous to $\mathcal{W}_\lambda$, the boundary values of the kernels Ψ_λ and $\widetilde{\Psi}_\lambda$ defined below are studied in [ÓS23] where

$$\Psi_\lambda(z, w) = {}_2F_1\Big(\rho + \lambda, \rho - \lambda; \frac{n}{2}; \tfrac{1+[z,\bar{w}]}{2}\Big) \quad z, w \in \Xi,$$

and

$$\widetilde{\Psi}_\lambda(z, w) = {}_2F_1\Big(\rho + \lambda, \rho - \lambda; \frac{n}{2}; \tfrac{1+[z,\bar{w}]}{2}\Big) \quad z, w \in \overline{\Xi}.$$

These kernels were obtained by reflection positivity on sphere. Reflection positivity is one of the Osterwalder-Schrader axioms of constructive quantum field theory. It is a necessary and sufficient condition for a Euclidean field theory that has Euclidean symmetries to analytically continue to a relativistic field theory with Lorentzian symmetries. The initial attempt was done by E. Nelson in the paper [N73]. The breakthrough was done in the paper [OS73, OS75]. Reflection positivity implies that the kernels Ψ_λ and $\widetilde{\Psi}_\lambda$ are well-defined sesquiholomorphic, positive-definite, G-invariant kernels.

The "perikernels" are holomorphic in the cut domain of $\mathrm{dS}^n \times \mathrm{dS}^n$ of the form $\mathrm{dS}^n \times \mathrm{dS}^n \setminus \Sigma$ where, Σ is the set of tuples (x, y) with $[x - y, x - y] \leq 0$. In the paper [ÓS23], we showed that the boundary values which are defined as $\Psi^\lambda_x = \lim_{z \to x} \Psi_\lambda(z, \cdot)$ in Ξ and $\widetilde{\Psi}^\lambda_x = \lim_{z \to x} \widetilde{\Psi}_\lambda(z, \cdot)$ in $\overline{\Xi}$, are real analytic on the cut domain $\mathrm{dS}^n \times \mathrm{dS}^n \setminus \{(x, y) : [x-y, x-y] = 0\}$ and have jump discontinuities along the cut. The cut is where the distributions have singularities and studied these singularities in terms of analytic wavefront sets.

The wavefront set of a distribution was introduced by L. Hörmander in [H70] to study the propagation of singularities of pseudo-differential operators. For a distribution Θ on a real analytic manifold, the *analytic wavefront set* $WF_A(\Theta)$ describes the set of points where Θ is not given by a real-analytic function and the direction in which the singularity occurs (see [H90]).

The wavefront set is a crucial concept in quantum field theory(QFT). One of the initial papers using wavefront sets in QFT was [Di79]. Later the wavefront set was brought into the context of Hadarmard distributions in [RM96]. It was shown that the Hadamard condition of a two point distribution of a quasi-free quantum field is equivalent to a condition its wavefront set. In algebraic quantum field theory, the condition on the wavefront set of the states of quantum fields is related to Reeh-Schlieder property (see [SVW02, V99]). It was also conceptualized in the context of unitary representations of Lie groups in [Ho81] and studied for induced representations in [HHO16].

In this work we will show that for each $x \in \mathrm{dS}^n$, the boundary values Ψ_x^λ and $\widetilde{\Psi}_x^\lambda$ define distributions on dS^n. These distributions are eigendistributions of the Laplace Beltrami operator Δ. Moreover, $\Psi_{e_n}^\lambda + \widetilde{\Psi}_{e_n}^\lambda$ and $\Psi_{-e_n}^\lambda + \widetilde{\Psi}_{-e_n}^\lambda$ are $H'-$invariant distributions and span $\mathcal{D}_{m^2}^{H'}(\mathrm{dS}^n)$, the space of H'-invariant spherical distributions with eigenvalue $m^2 = \rho^2 - \lambda^2$. We will now state the main theorem.

Theorem 5.3.6. Let $n \geq 2$, then

1. The distributions $\Psi_{e_n}^\lambda + \widetilde{\Psi}_{e_n}^\lambda$ and $\Psi_{-e_n}^\lambda + \widetilde{\Psi}_{-e_n}^\lambda$ are H'- invariant spherical distributions and span $\mathcal{D}_{m^2}^{H'}(\mathrm{dS}^n)$, where $m^2 = \rho^2 - \lambda^2$ and, $\lambda \in \mathbb{C} \setminus (\{\rho + \mathbb{N}\} \cup \{-\rho - \mathbb{N}\})$.

2. The distributions $\Psi_{e_n}^\lambda + \widetilde{\Psi}_{e_n}^\lambda$ and $\Psi_{-e_n}^\lambda + \widetilde{\Psi}_{-e_n}^\lambda$ are positive definite for $\lambda \in i[0, \infty) \cup (0, \rho)$.

3. Moreover, the following holds for a non-zero spherical distribution $\Theta \in \mathcal{D}_{m^2}^{H'}(\mathrm{dS}^n)$:

   (a) $WF_A(\Theta) \subset WF_A(\Psi_{e_n}^\lambda + \widetilde{\Psi}_{e_n}^\lambda) \sqcup WF_A(\Psi_{-e_n}^\lambda + \widetilde{\Psi}_{-e_n}^\lambda)$.

   (b) If $WF_A(\Theta) = WF_A(\Psi_{e_n}^\lambda + \widetilde{\Psi}_{e_n}^\lambda)$ then there exists a nonzero constant c such that $\Theta = c(\Psi_{e_n}^\lambda + \widetilde{\Psi}_{e_n}^\lambda)$.

   (c) If $WF_A(\Theta) = WF_A(\Psi_{-e_n}^\lambda + \widetilde{\Psi}_{-e_n}^\lambda)$ then there is a non zero constant c such that $\Theta = c(\Psi_{-e_n}^\lambda + \widetilde{\Psi}_{-e_n}^\lambda)$.

The flow of this thesis is as follows: we will give a geometric description in Chapter 2 of the spaces: the hyperboloid, the crown and the de Sitter space. In Chapter 3, we will briefly recall reflection positivity and introduce the kernels Ψ_λ and $\widetilde{\Psi}_\lambda$. We will also discuss the relation to representation theory. In Chapter 4 we discuss the boundary values. The main theorem will be proven in Chapter 5 along with discussing the singularities of the boundary values.

Chapter 2. The De Sitter Space, The Hyperboloid and The Crown

In this chapter we recall some basic geometric facts about the hyperboloids $\mathbb{H}^n$, $\overline{\mathbb{H}}^n$ and the de Sitter space dS^n, the two main spaces that we will discuss in this article. The material is well known. We are going to follow the geometrical setup which is also described in the paper [ÓS23]. However, the setup in [NÓ20, BM96, ÓS23] are all equivalent.

We write elements in $\mathbb{C}^{n+1}$ as $z = (z_0, \mathbf{z})$ with $z_0 \in \mathbb{C}$ and $\mathbf{z} \in \mathbb{C}^n$. We write $\mathbf{z} \cdot \mathbf{w} = \sum_{j=1}^n z_j w_j$ and $\mathbf{z}^2 = \mathbf{z} \cdot \mathbf{z}$. The bilinear form $[\cdot,\cdot]$ on $\mathbb{C}^{1+n}$ is given by

$$[z, w] = -z_0 w_0 + \sum_{j=1}^n z_j w_j = -z_0 w_0 + \mathbf{z} \cdot \mathbf{w}.$$

We denote by $\mathbb{R}^{1,n}$ the space $\mathbb{R}^{1+n}$ viewed as a Lorentzian space with Lorentzian form $[\cdot,\cdot]$. We say that a vector $v \in \mathbb{R}^{1,n}$ is time-like if $[v, v] < 0$ and space-like if $[v, v] > 0$.

2.1. The hyperboloids and the de Sitter space

The *hyperbolic spaces* $\mathbb{H}^n$ and $\overline{\mathbb{H}}^n$ are given as follows:

$$\mathbb{H}^n = \{x \in i\mathbb{R}^{1+n} \mid [x, x] = 1, x_0 > 0\} \quad \text{and} \quad \overline{\mathbb{H}}^n = \{x \in i\mathbb{R}^{1+n} \mid [x, x] = 1, x_0 < 0\}.$$

The de Sitter space dS^n is described as

$$\mathrm{dS}^n = \{x \in \mathbb{R}^{1+n} \mid [x, x] = 1\}.$$

All the spaces defined above are closed submanifolds of the complex manifold

$$\mathrm{dS}^n_{\mathbb{C}} = \{z \in \mathbb{C}^{1+n} \mid [z, z] = 1\}.$$

Let σ be the complex conjugation $\sigma(z) = \overline{z}$. We also write $V = i\mathbb{R}^{1+n}$ and $\sigma_V = -\sigma$, the conjugation w.r.t. V. Then $\overline{\mathbb{H}}^n = \sigma(\mathbb{H}^n)$.

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We are mostly interested in the de Sitter space so we restrict our discussion to that case. Let $x \in \mathrm{dS}^n$, then we denote the future (past) cone of x as $\Gamma^+(x)(\Gamma^-(x))$ where

$$\Gamma^{\pm}(x) := \{y \in \mathrm{dS}^n \mid [y-x, y-x] < 0, \pm(y-x)_0 > 0\}.$$

For $x \in \mathrm{dS}^n$, the set $\{y \in \mathrm{dS}^n \mid [y-x, y-x] = 0\}$ is called the light cone of x in dS^n.

Definition 2.1.1. Let G be a semisimple Lie group and τ be an involution on G, such that $\tau^2 = id$. Let G^σ be the fixed point group and H an open subgroup of G^σ such that

$$(G^\sigma)_e \leq H \leq G^\sigma.$$

Then the space G/H is called a symmetric space.

Let $G' = O_{1,n}(\mathbb{R})$ be the isometry group of $[\cdot, \cdot]$ and $G = \mathrm{SO}(1,n)_e$ be the connected component of identity of G'. Let $\tau : G' \to G'$ be the involution given by $\tau(g) = JgJ$, where J is the orthogonal reflection in the hyperplane $x_n = 0$. The restriction $\tau|_G$ is also an involution on G which will be denoted by τ as well. Let

$$\begin{aligned} H' = \{h \in G' \mid h \cdot e_n = e_n\} &= \left\{ \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \middle| \; h \in \mathrm{O}(1, n-1) \right\} \\ &= \mathrm{O}(1, n-1), \end{aligned}$$

and

$$\begin{aligned} H = \{h \in G \mid h \cdot e_n = e_n\} &= \left\{ \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \middle| \; h \in \mathrm{SO}(1, n-1)_e \right\} \\ &= \mathrm{SO}(1, n-1)_e. \end{aligned}$$

Then $H = G^\tau$ is the fixed point group of τ in G and $H' = G'^\tau$ is the fixed point group in

G' of tau. The de Sitter space $\mathrm{dS}^n \simeq G/H \simeq G'/H'$ is a symmetric space with Lorentzian metric on the the tangent space.

Furthermore, we have

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q},$$

where $\mathfrak{g}$ is the Lie algebra of G and G' with $\mathfrak{h} = \ker(\tau - 1)$ and $\mathfrak{q} = \ker(\tau + 1)$. It is easy to see that $\mathfrak{h}$ is the Lie algebra of H and H'.

We now describe the following groups:

$$K = \{k \in G \mid g \cdot e_0 = e_0\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \middle| \; a \in \mathrm{SO}(n) \right\},$$

$$K' = \{k \in G' \mid g \cdot e_0 = e_0\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \middle| \; a \in \mathrm{O}(n) \right\},$$

$$A = \left\{ a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} : t \in \mathbb{R} \right\}$$

and by

$$N = \left\{ n_z = \begin{pmatrix} 1 + \frac{1}{2}||v||^2 & v^t & -\frac{1}{2}||v||^2 \\ v & I_{n-1} & -v \\ \frac{1}{2}||v||^2 & v^t & 1 - \frac{1}{2}||v||^2 \end{pmatrix} : v \in \mathbb{R}^{n-1} \right\}.$$

The following lemma has been proved in [NÓ20, Lemma 6.3] and in [D08].

Lemma 2.1.2. *We have the decomposition* $G = HAK = KAH$ *and* $G' = K'AH' =$

$H'AK'$. *Moreover,*

$$G/H = KA.e_n = \mathrm{dS}^n$$

$$G'/H' = K'A.e_n = \mathrm{dS}^n.$$

There exists unique upto a constant a G-invariant and a G'- invariant measure on dS^n (for more discussions see [D08, p. 159]).

Further, the Iwasawa decomposition (see [D08, Theorem 7.5.3 and Sec 9.2]) is given by

Lemma 2.1.3. $G = KAN$ *and* $G' = K'AN$.

Remark 2.1.4. Let

$$\Lambda_1 = \begin{pmatrix} -1 & 0 \\ 0 & Id_n \end{pmatrix}, \qquad \Lambda_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & Id_{n-1} \end{pmatrix}.$$

Then $O_{1,n}(\mathbb{R}) = SO_{1,n}(\mathbb{R})_e \sqcup \Lambda_1 SO_{1,n}(\mathbb{R})_e \sqcup \Lambda_2 SO_{1,n}(\mathbb{R})_e \sqcup \Lambda_1\Lambda_2 SO_{1,n}(\mathbb{R})_e$.

The groups G' acts transitively on dS^n and G acts transitively on $\mathbb{H}^n$, $\overline{\mathbb{H}}^n$ and dS^n. We have that

$$\mathbb{H}^n = G \cdot ie_0 \simeq G/K \simeq \overline{\mathbb{H}}^n = G \cdot (-ie_0).$$

From now on we will consider the connected group G until Section 5.3. Since K is a maximal compact subgroup of G, the hyperboloids $\mathbb{H}^n$ and $\overline{\mathbb{H}}^n$ are Riemannian symmetric space. The metric at a point $p \in \mathbb{H}^n, \overline{\mathbb{H}}^n$ is given as $g_p(v,v) = [v,v]$. The tangent space at the point p is given as

$$T_p(\mathbb{H}^n) = \{v \in \mathbb{R}^{1,n} | \, [p,v] = 0\}.$$

That is, $p_0 v_0 = p_1 v_1 + ... + p_n v_n$. By definition $p_0 \neq 0$ and we obtain $v_0^2 = 1/p_0^2(p_1 v_1 + ... +$

$p_nv_n)^2$. By Cauchy- Schwarz inequality

$$\begin{aligned} v_0^2 &\leq \frac{1}{p_0^2}(p_1^2 + ...p_n^2)(v_1^2 + ... + v_n^2) \\ &\leq \frac{p_0^2 - 1}{p_0^2}(v_1^2 + ... + v_n^2) \\ &\leq (v_1^2 + ... + v_n^2). \end{aligned}$$

Thus, g_p is positive definite. The same holds for $\overline{\mathbb{H}}^n$.

The tangent space at $x \in \mathrm{dS}^n$ is

$$\mathrm{T}_x(\mathrm{dS}^n) = \{y \in \mathbb{R}^{1+n} \mid [x, y] = 0\} \cong \mathbb{R}^{1,n-1}.$$

In particular, we have

$$\mathrm{T}_{e_n}(\mathrm{dS}^n) = \{y \in \mathbb{R}^{1+n} \mid y_n = 0\} \cong \mathfrak{q} \cong \mathbb{R}^{1,n-1}.$$

The tangent bundle is then given by

$$\mathrm{T}(\mathrm{dS}^n) = \{(x, v) \in \mathbb{R}^{1+n} \times \mathbb{R}^{1+n} \mid x \in \mathrm{dS}^n \text{ and } [x, v] = 0\}.$$

We will write ℓ_g for the diffeomorphism $\ell_g x = gx$. The group G acts on the tangent bundle by

$$g \cdot (x, v) = (d\ell_g)_x(v) = (gx, gv),$$

where the action on the right is the natural linear action. It is well know that if $(x, v), (y, w) \in T(\mathrm{dS}^n)$ with $[v, v] = [w, w]$ then there exists a $g \in G$ such that $g \cdot (x, v) = (y, w)$.

The exponential function can be written using analytic functions $\mathrm{C}, \mathrm{S} : \mathbb{C} \to \mathbb{C}$ defined by

$$\mathrm{C}(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^k \quad \text{and} \quad \mathrm{S}(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^k.$$

Thus, $\mathrm{C}(z) = \cos\sqrt{z}$ and $\mathrm{S}(z) = \frac{\sin\sqrt{z}}{\sqrt{z}}$. Note that this is well defined as the functions $y \mapsto \cos(y), \sin(y)/y$ are both even. With this notation we have [NÓ20, p. 15]

Lemma 2.1.5. *The exponential function* $\mathrm{Exp}_x : \mathrm{T}_x(\mathrm{dS}^n) \to \mathrm{dS}^n$ *is given by*

$$\mathrm{Exp}_x(v) = C([v,v])x + S([v,v])v, \quad v \in \mathrm{T}_x(\mathrm{dS}^n)$$

and satisfies

$$\ell_g \circ \mathrm{Exp}_x = \mathrm{Exp}_{g\cdot x} \circ (d\ell_g)_x.$$

Let $U_{e_n} = \{v \in \mathrm{T}_{e_n}(\mathrm{dS}^n) \mid [v,v] < \pi/2\}$ and note that if $x = g \cdot e_n$ then $U_x = d\ell_g \cdot U_{e_n}$. Let $V_x = \mathrm{Exp}_x U_x \subset \mathrm{dS}^n$. Then the following holds

Lemma 2.1.6. V_x *is open and* $\mathrm{Exp}_x : U_x \to V_x$ *is an analytic diffeomorphism.*

Proof. Clearly, the map is analytic. It is enough to prove this for $x = e_n$. Let the map α be given by $u = (u_0, \mathbf{u}) \in \mathrm{T}_{e_n}\mathrm{dS}^n \to X_u \in \mathfrak{q}$ where,

$$\alpha(u) := X_u := \left(\begin{array}{c|c|c} 0 & 0 & u_0 \\ \hline 0 & \mathbf{0}_{n-1} & \mathbf{u} \\ \hline u_0 & -\mathbf{u}^t & 0 \end{array}\right)$$

The map α is an isomorphism. Consider the map from $\mathrm{T}_{e_n}\mathrm{dS}^n$ into G given by

$$u \mapsto \exp(\alpha(u)) = \left(\begin{array}{cc|c} C[u,u] & 0 & \multirow{2}{*}{$S[u,u]u$} \\ 0 & \mathrm{Id}_{n-1} & \\ \hline \multicolumn{2}{c|}{-S[u,u]u^T} & C[u,u] \end{array}\right).$$

We claim that the restriction of this map to the set U_{e_n} is injective. Suppose, $\exp(X_u) = \mathrm{Id}_{1+n}$. It follows that $C[u,u] = 1$ and $S[u,u]u = 0$. This is true only if either $u = 0$ or

$[u,u] = 4m^2\pi^2$, for $m \in \mathbb{Z} \setminus 0$. Thus, the claim follows for the restriction to U_{e_n}. Observe that

$$\exp(X_u)e_n = \mathrm{Exp}_{e_n}(u). \tag{2.1.1}$$

Since $u \to \exp(X_u)$ is injective, the lemma is proved. $\square$

We will also use the following co-ordinates as some of the computations will be easier. Let $t \in \mathbb{R}$ and $u \in \mathbb{S}^n$, then

$$x = \sinh(t)e_0 + \cosh(t)u \tag{2.1.2}$$

are real analytic co-ordinates from $\mathbb{S}^n \times \mathbb{R}$ to dS^n.

The metric g at the point $x(t,u)$ on dS^n is given by

$$g = -dt^2 + \cosh^2(t)\left(\sum_{i=1}^{n} du_i^2\right). \tag{2.1.3}$$

2.1.1. Invariant differential operator

Let L be a Lie group and assume that L acts on the manifold X by $g \cdot x = \ell_g(x)$. Then a differential operator $D : C_c^\infty(X) \to C_c^\infty(X)$ is *invariant* if for all $f \in C_c^\infty(X)$ and all $g \in L$, we have $D(f \circ \ell_g) = (Df) \circ \ell_g$. We denote by $\mathbb{D}(X)$ the algebra of invariant differential operators. It is known [F79] that $\mathbb{D}(\mathrm{dS}^n) = \mathbb{C}[\Delta]$, the algebra of polynomials in the Laplacian which we define in two equivalent ways.

First let

$$\square_{n+1} = -\frac{\partial^2}{\partial x_0^2} + \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

in $\mathbb{R}^{1,n}$. Let $\varphi \in C_c^\infty(\mathbb{R})$, $\varphi = 1$ in a neighborhood of 1 and, $\varphi(t) = 0$ for $|t-1| > 1/2$. For $f \in C_c^\infty(\mathrm{dS}^n)$ define

$$\tilde{f}(x) = \varphi([x,x])f(x/|[x,x]|^{1/2}), \quad x \in \mathbb{R}^{1,n}.$$

Then $\widetilde{f} \in C_c^\infty(\mathbb{R}^{1,n})$ and we define

$$\widetilde{\Delta} f := (\Box_{n+1} \widetilde{f})|_{\mathrm{dS}^n}.$$

It is a well defined G-invariant differential operator on dS^n, see [D08, p. 110,160].

We can also define Δ using the tangent space and the exponential map. Note that $\Box_n$ is a well defined $H = \mathrm{SO}(1, n-1)_e$-invariant differential operator on $T_{e_n}(\mathrm{dS}^n) \simeq \mathbb{R}^{1,n-1}$. Define

$$(\Delta f) \circ \mathrm{Exp}_{e_n} := \Box_{e_n}(f \circ \mathrm{Exp}_{e_n}), \quad f \in C_c^\infty(\mathrm{dS}^n).$$

As Δ_{e_n} is H-invariant we have a well defined G-invariant differential operator Δ on dS^n given by

$$\Delta f(g \cdot e_n) = \Delta(f \circ \ell_g)(e_n).$$

As both $\widetilde{\Delta}$ and Δ are second order invariant differential operators annihilating the constants it follows that there exists a $c > 0$ such that $\Delta = c\widetilde{\Delta}$.

In the co-ordinates given by Eq. (2.1.2) the Laplace-Beltrami operator is then a positive constant multiple of the operator given by

$$\Delta' f = -\frac{\partial^2 f}{\partial t^2} - (n-1)\tanh(t)\frac{\partial f}{\partial t} + \frac{1}{\cosh^2(t)}\Box_{\mathbb{S}^n} f \tag{2.1.4}$$

where $\Box_{\mathbb{S}^n}$ is the Laplacian on the sphere $\mathbb{S}^n$ and $f \in C_c^\infty(\mathrm{dS}^n)$.

Hence, from now on with abuse of notations we will use Δ as the Laplace-Beltrami operator defined in all the three ways.

2.2. The Crown Ξ, $\widetilde{\widetilde{\Xi}}$

The complex crown Ξ of a Riemannian symmetric space G/K is a natural complex open domain in the complexification $G_{\mathbb{C}}/K_{\mathbb{C}}$ with the property that the eigenfunctions of the algebra of G-invariant differential operators on G/K extends to Ξ. This domain was introduced in [AG90]. It was studied by several authors but for us the articles [GK02a, GK02b, KSt04] are of most importance in particular, the articles [GK02a, KSt04] finished the description of the crown. The article [GK02b] showed that a non-compactly causal symmetric space [HÓ97], including the de Sitter space, can be realized as open orbit in the boundary of the crown. The crown showed up in a natural way in [NÓ20] in relation to reflection positivity and we will collect those results here.

Let $h = E_{0n} + E_{n0} \in \mathfrak{so}(1,n)$ be the operator

$$h(x_0, x_1, \dots, x_{n-1}, x_n) = (x_n, 0, \dots, 0, x_1).$$

Then ad$h$ has the eigenvalues $0, 1, -1$. Thus, we have the eigenspace decomposition

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$$

and the space $\mathfrak{g}_{\pm 1}$ are $\mathfrak{g}_0$-invariant. The crown of $\mathbb{H}^n$ is defined to be:

$$\Xi = G \exp(i(-\pi/2, \pi/2)h) \cdot ie_0 = G \cdot \{(i \cos t, 0, \dots, -\sin t) \mid |t| < \pi/2\}.$$

Similarly, for $\overline{\mathbb{H}}^n$:

$$\overline{\Xi} = G \exp i(-\pi/2, \pi/2)h \cdot -ie_0 = \sigma(\Xi).$$

The crown domain $\Xi \subset \mathrm{SO}_{d+1}(\mathbb{C})(ie_0) \simeq G_{\mathbb{C}}/K_{\mathbb{C}} \simeq \mathrm{dS}^n_{\mathbb{C}}$ and same follows for $\overline{\Xi}$. We now recall the description of Ξ and its properties, see [NÓ20]. The corresponding statements for $\overline{\Xi}$ follows by taking the complex conjugation σ.

Remark 2.2.1. Recall that an element $h \in \mathfrak{g}$, $h \neq 0$, is called an Euler element if ad$h$ has eigenvalues $0, 1, -1$. The crown domain depends on $(\mathfrak{g}, \mathfrak{k})$ where $\mathfrak{g}, \mathfrak{k}$ are the Lie algebra of G and K respectively. It does not depend the choice of Lie group G with Lie algebra $\mathfrak{g}$.

Consider the open future light cone Ω given by

$$\Omega = \{x \in \mathbb{R}^{1,n} : [x,x] < 0, x_0 > 0\}.$$

The corresponding future tube is given by

$$T_\Omega = \mathbb{R}^{1,n} + i\Omega.$$

Similarily, the past tube is

$$T_{-\Omega} = \mathbb{R}^{1,n} - i\Omega.$$

We realize the unite sphere in $i\mathbb{R}e_0 + \mathbb{R}^n$ by $\mathbb{S}^n = \{x \in i\mathbb{R}e_0 + \mathbb{R}^n \mid [x,x] = 1\}$. Set $\mathbb{S}^n_+ = \{x \in \mathbb{S}^n \mid x_0 > 0\}$ and $\mathbb{S}^n_- = \{x \in \mathbb{S}^n \mid x_0 < 0\} = \sigma(\mathbb{S}^n_+)$.

Lemma 2.2.2 (NÓ 2020)**.** *The crowns can be described as*

$$\begin{aligned}
\Xi &= G \cdot \mathbb{S}^n_+ = T_\Omega \cap \mathbb{S}^n_{+,\mathbb{C}} = T_\Omega \cap \mathrm{dS}^n_{\mathbb{C}} \\
&= \{u + iv : [u,u] - [v,v] = 1, [u,v] = 0, [v,v] < 0, v_0 > 0\}; \\
\overline{\Xi} &= G \cdot (\mathbb{S}^n_-) = T_{-\Omega} \cap \mathbb{S}^n_{-,\mathbb{C}} = T_{-\Omega} \cap \mathrm{dS}^n_{\mathbb{C}} \\
&= \{u - iv : [u,u] - [v,v] = 1, [u,v] = 0, [v,v] < 0, v_0 > 0\}.
\end{aligned}$$

Proof. The first part is [NÓ20, Lem. 3.1] and [NÓ20, Prop. 3.2]. The second part follows by applying σ to Ξ. □

The following proposition is the key for the kernels Ψ_λ and $\widetilde{\Psi}_\lambda$, which we will see in Chapter 3, to be well defined on the crown domains Ξ and $\overline{\Xi}$ respectively.

Proposition 2.2.3. *We have*

$$\{[z, \sigma(w)] \mid z, w \in \Xi\} = \mathbb{C} \setminus [1, \infty) = \{[z, \sigma(w)] \mid z, w \in \overline{\Xi}\}.$$

Proof. The crown is invariant under the conjugation $z \mapsto -\sigma(z)$. The claim therefore follows from [NÓ20, Lem. 3.5] using that the Lorentz form in [NÓ20] is the negative of the form considered here. The claim for $\overline{\Xi}$ follows from the first part using that $\overline{\Xi} = \sigma(\Xi)$. $\square$

For $U \subset \mathbb{C}^{n+1}$ denote by $\mathrm{cl}\,(U)$ the closure of U in $\mathbb{C}^n$. The boundary ∂U is then $\partial U = \mathrm{cl}\,(U) \setminus U$.

Lemma 2.2.4. *The boundary of* Ξ*, and respectively* $\overline{\Xi}$*, in* $\mathrm{dS}^n_{\mathbb{C}}$ *is given by*

$$\begin{aligned}\partial\,\Xi &= \{x+iy : x, y \in \mathbb{R}^{1,n},\ [x,x]=1, [y,y]=0, y_0 \geq 0, [x,y]=0\};\\ \partial\,\overline{\Xi} &= \{x-iy : x, y \in \mathbb{R}^{1,n},\ [x,x]=1, [y,y]=0, y_0 \geq 0, [x,y]=0\}\\ &= \sigma(\partial\Xi).\end{aligned}$$

Proof. The first part is [NÓ20, Lem. 3.7] and the second claim then follows from $\overline{\Xi} = \sigma(\Xi)$. $\square$

Corollary 2.2.5. $\mathrm{dS}^n = \partial\,\Xi \cap \partial\,\overline{\Xi}$.

Proof. The above description of the boundary implies that $\mathrm{dS}^n \subset \partial\,\Xi \cap \partial\,\overline{\Xi}$. If $z = x+iy \in \partial\,\Xi \cap \partial\,\overline{\Xi}$ then $y_0 \geq 0$. Hence $0 = [y,y] = \mathbf{y}^2$ which happens if and only if $\mathbf{y} = 0$. Hence $y = 0$ and, $[x,x]=1$ implies that $x \in \mathrm{dS}^n$. $\square$

The next proposition shows that around each point $x \in \mathrm{dS}^n$, the crown can be represented locally as a tuboid of the form $U + i\Omega'$ where U is an open set and Ω is a pointed cone in the tangent space of x.

Let U_x be the coordinate chart around $x = g \cdot e_n \in \mathrm{dS}^n$. Let $\Omega'_{e_n} = \{v \in \mathrm{T}_{e_n}(\mathrm{dS}^n) : [v,v] < 0, v_0 > 0\}$ be the open future H-invariant cone in $\mathbb{R}^{1,n-1}$. Write $\Omega'_x = d\ell_g \cdot \Omega'_{e_n} \subset \mathrm{T}_x(\mathrm{dS}^n)$.

Proposition 2.2.6. *Let* $g = ka_th$ *and,* $x = g \cdot e_n \in \mathrm{dS}^n$.

1. *The map* $\kappa_x : U_x + i\Omega'_x \to \Xi$ *where,*
$$\kappa_x : u + iv \mapsto (\sqrt{1+[v,v]})\mathrm{Exp}_x(u) + i\exp(X_{(ka_t)^{-1}\cdot u})\cdot v \tag{2.2.1}$$
is well-defined and biholomorphic onto its image.

2. *The map* $\widetilde{\kappa}_x : U_x - i\Omega'_x \to \overline{\Xi}$ *where,*
$$\widetilde{\kappa}_x : u - iv \mapsto (\sqrt{1+[v,v]})\mathrm{Exp}_x(u) - i\exp(X_{(ka_t)^{-1}\cdot u})\cdot v \tag{2.2.2}$$
is well-defined and bi-antiholomorphic onto its image.

Proof. For $u \in U_{e_n}$, $v \in \Omega'_{e_n}$, $h \in H$ and, using the fact that $\exp(X_u) = \exp(X_{h\cdot u})$, we obtain

$$\begin{aligned} \kappa_x \circ (d\ell_g)_{e_n}(u+iv) &= g\cdot \kappa_{e_n}(u+iv);\\ \widetilde{\kappa}_x \circ (d\ell_g)_{e_n}(u-iv) &= g\cdot \widetilde{\kappa}_{e_n}(u-iv). \end{aligned} \tag{2.2.3}$$

Thus, it is enough to prove for $x = e_n$. Note that if $[v,v] < -1$ for $v \in \Omega'$, then $k_{e_n}(u + iv) \in \mathbb{H}^n$. That is because $k_{e_n}(u+iv) = i\exp(X_u)\cdot(\sqrt{-[v,v]-1}e_n + v)$ and the group G preserves the direction of time-like vector with $v_0 \neq 0$. Following the proof of Lemma 2.1.6 and Lemma 2.1.2, the maps κ and $\widetilde{\kappa}$ are well defined and biholomorphic and bi-antiholomorphic, respectively. □

Corollary 2.2.7. *We have that*

$$\Xi = G\cdot\kappa_{e_n}(U_{e_n} + i\Omega'_{e_n}) = \bigcup_{x\in \mathrm{dS}^n} \kappa_x(U_x + i\Omega'_x)$$

and,

$$\overline{\Xi} = G\cdot\widetilde{\kappa}_{e_n}(U_{e_n} - i\Omega'_{e_n}) = \bigcup_{x\in \mathrm{dS}^n} \widetilde{\kappa}_x(U_x - i\Omega'_x).$$

Proof. Because of Eq. (2.2.3), it is enough to prove the first equality. From the above proposition we clearly have that $G\cdot\kappa_{e_n}(U_{e_n} + i\Omega') \subseteq \Xi$. Now, let $z \in \Xi$. As $\Xi = G\cdot\mathbb{S}^n_+$ it follows that $z = g\cdot(i\cos(t)e_0 + \sin(t)e_n)$ for $t \in (-\pi/2, \pi/2)$. Clearly, $i\cos(t)e_0 + \sin(t)e_n = \kappa_{e_n}(i\cos(t)e_0)$. Thus, the first equality holds. We follow the same arguments for $\overline{\Xi}$. □

Chapter 3. Reflection positivity, Kernels Ψ_λ and $\widetilde{\Psi}_\lambda$

We will now recall reflection positivity on the sphere [NÓ20], see also [NÓ22], which lead to a positive definite kernel Ψ_λ (with a different normalization in [NÓ20]) for $\lambda \in i[0,\infty) \cup [0, \frac{n-1}{2})$ and $\rho = (n-1)/2$ by

$$\Psi_\lambda(z,w) = {}_2F_1\left(\rho+\lambda, \rho-\lambda; \frac{n}{2}; \frac{1+[z,\overline{w}]}{2}\right), \quad z,w \in \Xi. \tag{3.0.1}$$

We will also consider the following kernel

$$\widetilde{\Psi}_\lambda(z,w) = {}_2F_1\left(\rho+\lambda, \rho-\lambda; \frac{n}{2}; \frac{1+[z,\overline{w}]}{2}\right), \quad z,w \in \overline{\Xi}. \tag{3.0.2}$$

As both n and λ are fixed most of the time we simplify our notation and write

$${}_2F_1(z) = {}_2F_1\left(\rho+\lambda, \rho-\lambda; \frac{n}{2}; z\right).$$

Here ${}_2F_1(a,b;c;z)$ denotes the Gauss hypergeometric function

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

where $(d)_n = d(d+1)\cdots(d+n-1)$, $c \notin -\mathbb{N}_0$ and $|z| < 1$. The hypergeometric function ${}_2F_1$ extends to a holomorphic function on $\mathbb{C} \setminus [1,\infty)$ (see [LS66]).

3.1. Reflection Positivity

Reflection positivity is a property which is used to construct relativistic quantum fields that satisfy Wightmann's axioms from Euclidean fields with Euclidean symmetries. We will now understand reflection positivity in the context of [N73, Di18, JR08].

Let $\mathcal{E}$ be a Hilbert space with $\mathcal{E}_+$ its closed subspace and a unitary involution θ.

This chapter has appeared in the article: G. Olafsson, I. Sitiraju. *Analytic wavefront sets of spherical distributions on the de Sitter space.* arXiv:2309.10685

Definition 3.1.1. The triple $(\mathcal{E}, \mathcal{E}_+, \theta)$ is called reflection positive Hilbert space if

$$\langle v, \theta v\rangle \geq 0 \quad \text{for all} \quad v \in \mathcal{E}_+.$$

Consider the unit sphere $\mathbb{S}^n$ realized in $i\mathbb{R}e_0 + \mathbb{R}^n$. Let $\sigma : \mathbb{S}^n \to \mathbb{S}^n$ be the reflection $\sigma(ix_0, \mathbf{x}) = (-ix_0, \mathbf{x})$. Let us denote $\square$ as the laplacian on $\mathbb{S}^n$. Then, $(-\square + (\rho^2 - \lambda^2))^{-1}$ is a bounded positive operator on $L^2(\mathbb{S}^n)$ for $\lambda \in i[0, \infty) \cup (0, \rho)$.

Let $\mathcal{H}^{-1}$ be the completion of $C_c^\infty(\mathbb{S}^n)$ with respect to the inner product

$$\langle \phi, (-\square + (\rho^2 - \lambda^2))^{-1}\psi\rangle.$$

This space does not depend on λ. Let $\mathcal{H}_+^{-1} = \{\phi \in \mathcal{H}^{-1} : \text{supp}\phi \subset \overline{\mathbb{S}_+^n}\}$ and $\theta(\phi) = \phi \circ \tau$. Then we obtain the following which can be found in [NÓ18, Di18].

Theorem 3.1.2. *The triple $(\mathcal{H}^{-1}, \mathcal{H}_+^{-1}, \theta)$ is a reflection positive Hilbert space.*

We will now consider a distribution Φ_λ on $\mathbb{S}^n \times \mathbb{S}^n$ as follows:

$$\Phi_\lambda(\phi \otimes \overline{\psi}) = \int_{\mathbb{S}^n} \overline{\phi(x)}(-\square + (\rho^2 - \lambda^2))^{-1}\psi(x)d\mu(x) \quad \text{for} \quad \phi, \psi \in C_c^\infty(\mathbb{S}^n).$$

The following corollary follows from Theorem 3.1.2(proven in [NÓ20, Sec 2]).

Corollary 3.1.3. *The distribution Φ_λ is reflection positive with respect to $(\mathbb{S}^n, \mathbb{S}_+^n, \sigma)$. That is, the distribution $\Phi_\lambda^\sigma = \Phi_\lambda \circ (id, \sigma)$ is positive definite on $\mathbb{S}_+^n \times \mathbb{S}_+^n$.*

It was proven in [NÓ20] that the distribution Φ_λ^σ is given by the kernel $\Psi_\lambda(x, y)$ for $x, y \in \mathbb{S}_+^n$. This kernel has a natural extension on $\Xi \times \Xi$. Part (1) of the following is in [NÓ20, NÓ22] and part (3) follows by part (2) :

Theorem 3.1.4. *Let $\rho = \frac{n-1}{2}$. For $\lambda \in i[0, \infty) \cup (0, \rho)$*

(1) *The kernel $\Psi_\lambda(z, w)$ is a positive definite G invariant kernel on $\Xi \times \Xi$ which is holomorphic in first variable and anti-holomorphic in the second variable. It is given by*

$$\Psi_\lambda(z, w) = {}_2F_1\left(\rho + \lambda, \rho - \lambda; \frac{n}{2}; \tfrac{1 + [z, \sigma(w)]}{2}\right), \quad z, w \in \Xi.$$

(2) *Let $z, w \in \Xi$. Then*

$$\overline{\Psi_\lambda(z,w)} = \Psi_\lambda(\sigma(z), \sigma(w)).$$

(3) *The kernel $\widetilde{\Psi}_\lambda(z,w)$ is a G-invariant positive definite kernel on $\Xi \times \Xi$ holomorphic in the first variable and anti-holomorphic in the second variable given by:*

$$\widetilde{\Psi}_\lambda(z,w) = {}_2F_1\Big(\rho+\lambda, \rho-\lambda; \frac{n}{2}; \tfrac{1+[z,\sigma(w)]}{2}\Big) \quad z, w \in \Xi.$$

We also have that $\Psi_\lambda(z,w) = \overline{\widetilde{\Psi}_\lambda(\bar{z},\bar{w})} = \widetilde{\Psi}_\lambda(\bar{w},\bar{z})$.

Proof. We use the simplified notation ${}_2F_1(u) = {}_2F_1(\rho+\lambda, \rho-\lambda; n/2; u)$. (1) is [NÓ20, Thm. 4.12] and (2) follows from (1) and the fact that for $\lambda \in i\mathbb{R} \cup \mathbb{R}$ we have for $z \in \mathbb{C} \setminus [1,\infty)$:

$$\overline{{}_2F_1(z)} = {}_2F_1(\bar{z})$$

as ${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z)$.

For (3) let $\widetilde{\phi}_\lambda(z) = \widetilde{\Psi}_\lambda(z, -ie_0)$, $z \in \overline{\mathbb{H}}^n$, be the spherical function on $\overline{\mathbb{H}}^n$. Then, using that $\exp(th)\overline{x}_0 = -i(\cosh(t)e_0 + \sinh(t)e_n)$, we get

$$\begin{aligned}\widetilde{\phi}(\exp th\overline{x}_0) &= {}_2F_1\left(\frac{1+\cosh t}{2}\right)\\ &= {}_2F_1\left(\frac{1+[\exp(th)\overline{x}_0, \sigma(\overline{x}_0)]}{2}\right).\end{aligned}$$

From this it follows that for all $z \in \overline{\mathbb{H}}^n$ we have

$$\widetilde{\phi}(z) = {}_2F_1\left(\frac{1+[z,\sigma(\overline{x}_0)]}{2}\right)$$

because $\overline{\mathbb{H}}^n = K \exp \mathbb{R}h \cdot (-ie_0)$ and K fixes $\pm ie_0$. As $\widetilde{\Psi}(\cdot, \overline{x}_0)$ is holomorphic on Ξ it follows that

$$\widetilde{\Psi}(z, \overline{x}_0) = {}_2F_1\left(\frac{1+[z,\sigma(\overline{x}_0)]}{2}\right), \quad z \in \Xi.$$

Using that $\widetilde{\Psi}$ is G-invariant it follows that

$$\widetilde{\Psi}(\cdot, w) = {}_2F_1\left(\frac{1+[z,\sigma(w)]}{2}\right), \qquad \text{for all } z \in \overline{\Xi} \text{ and } w \in \overline{\mathbb{H}}^n.$$

The claim now follows using that $w \mapsto \widetilde{\Psi}_\lambda(z,w)$ is antiholomorphic and hence determined by the restriction to $\overline{\mathbb{H}}^n$. The last claim follows by $\overline{\Xi} = \sigma(\Xi)$ and $\Psi_\lambda(z,w) = \widetilde{\Psi}(\overline{w}, \overline{z})$. □

The following lemma has been proved in [NÓ20, Lem. 6.4].

Lemma 3.1.5. *We have*

$$[\mathrm{dS}^n, \Xi] \cap \mathbb{R} = [\mathrm{dS}^n, \overline{\Xi}] \cap \mathbb{R} = (-1, 1).$$

From this and the properties of the hypergeometric function we get:

Proposition 3.1.6. *The kernel Ψ_λ can be extended continuously to $\Xi \times (\mathrm{dS}^n \cup \Xi)$ and the kernel $\widetilde{\Psi}_\lambda$ can be extended continuously to $\overline{\Xi} \times (\mathrm{dS}^n \cup \overline{\Xi})$.*

For $y \in \mathrm{dS}^n$ and $z = e_n$, we have that $(1 + [z,y])/2 \notin [1,\infty)$ iff $y_n < 1$. In particular, for $y \in \mathrm{dS}^n$ we have that $y \mapsto \Psi_\lambda(y, e_n), \Psi_\lambda(e_n, y), \widetilde{\Psi}_\lambda(y, e_n), \widetilde{\Psi}(e_n, y)$ is analytic on $\{y \in \mathrm{dS}^n : y_n < 1\}$. We will discuss these singularities in Section 4.2 and Section 5.2.

3.2. Representation Theory Perspective

We will now discuss how the kernel Ψ_λ obtained by reflection positivity is related to representations.

Definition 3.2.1. An irreducible unitary representation $(\pi, \mathcal{H})$ is said to *spherical* if the K-fixed vectors $\mathcal{H}^K$ is a non-empty set. Then the dimension of $\mathcal{H}^K$ turns out to be 1. Let e_π be a unit vector in $\mathcal{H}^K$ then the function $\phi_\pi(g) = \langle \pi(g)e_\pi, e_\pi\rangle$ is called a *spherical function.*

It is K- biinvariant and can be defined on G/K. This definition of spherical function is equivalent to the definition that a smooth function ϕ on G/K with $\phi(eK) = 1$ is called *spherical* if ϕ is K-biinvariant and is an eigenfunction for the algebra of G-invariant differential operators $\mathbb{D}(G/K)$ on G/K. For reference see [He62, Chap. X].

We will now review the spherical representations on $\mathbb{H}^n = G/K$. The material is well known (see [D08]). We have the Iwasawa decomposition $G = KAN$. We write $g = k(g)a(g)n(g)$ and G acts on S^{n-1} by $g \cdot v = k(g)v$. The principal series representation π_λ with spectral parameter $\lambda \in i[0, \infty)$ acting on the Hilbert space $\mathrm{H}_\lambda = L^2(\mathrm{S}^{n-1})$ is given by

$$\pi_\lambda(g)f(v) = a(g^{-1}k)^{-\lambda-\rho}f(g^{-1} \cdot v)$$

where $v \in \mathrm{S}^{n-1}$, $g \in G$ and $f \in L^2(\mathrm{S}^{n-1})$. That is, for $g = ka_tn_z$

$$\pi_\lambda(g)f(v) = e^{(\lambda-\rho)t}f(g^{-1} \cdot v).$$

These representations are unitary and irreducible for $\lambda \in i\mathbb{R}$ and are called *principal series representations* ([D08]).

The constant function $e_\lambda(v) = 1$ is K-invariant with norm 1 and the associated spherical function is

$$\phi_\lambda(g) = \langle \pi_\lambda(g)e_\lambda, e_\lambda \rangle = \int_{\mathrm{S}^n} a(g^{-1}v)^{-\lambda-\rho}dv.$$

We note that $g \mapsto \pi_\lambda(g)e_\lambda$ is right K-invariant, hence $\pi_\lambda(z)e_\lambda$ is well defined for z and w in $\mathbb{H}^n$ and the kernel Ψ_λ is given by

$$\Psi_\lambda(z, w) = \langle \pi_\lambda(z)e_\lambda, \pi_\lambda(w)e_\lambda \rangle.$$

Therefore, it follows from [NÓ20, Theorem 5.10] that

$$\phi_\lambda(x) = \Psi_\lambda(x, ie_0) = {}_2F_1\left(\rho + \lambda, \rho - \lambda; \frac{n}{2}; \frac{1 + ix_0}{2}\right), \quad ix \in \mathbb{H}^n,$$

is an eigenfunction of algebra of G-invariant differential operators on $\mathbb{H}^n$.

Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be defined as above. Denote by $\mathcal{H}_\lambda^\infty$, the space of smooth vectors and by $\mathcal{H}^{-\infty}$ the space of continuous conjugate linear maps $\mathcal{H}_\lambda^\infty \to \mathbb{C}$, the space of distribution vectors. The group G leaves $\mathcal{H}^\infty$ invariant and then defines a representation $\pi^{-\infty}$ by duality. For $\eta \in \mathcal{H}^{-\infty}$ and $\phi \in C_c^\infty(G)$ it is well known that $\pi^{-\infty}(\phi)\eta = \int_G \phi(x)\pi^{-\infty}(x)\eta dx$ is in $\mathcal{H}^\infty$. Hence,

$$\Phi_\eta(\phi) = \eta(\pi^{-\infty}(\phi)\eta) = \langle \eta, \pi^{-\infty}(\phi)\eta \rangle \tag{3.2.1}$$

is well defined positive-definite distribution on G. If η is H-invariant then Φ_η defines a distribution on G/H (see [NÓ18] for more discussion). If π is irreducible then, Φ_η is an eigendistribution for the algebra of differential operators coming from the center of $U(\mathfrak{g})$.

The above setup leads to the element $s = \exp(\frac{i\pi}{2}h) \in G_\mathbb{C}$, where h is the Euler element(Remark 2.2.1), such that $sK_\mathbb{C}s^{-1} = H_\mathbb{C}$ and $G/H = G \cdot sK_\mathbb{C}$ is on the boundary of the crown and $\exp(ith)K_\mathbb{C}$ belongs to Ξ for $|t| < \pi/2$. Hence, $\pi(\exp(itX))v$ is well defined for $|t| < \pi/2$.

For $g \in G$ we have $g\exp(-ith)e_n \in \Xi$ and, the orbit map

$$(-\pi/2, \pi/2) \mapsto \pi_\lambda(g\exp(-ith))e_\lambda$$

is analytic and

$$e_\lambda^H = \lim_{t\to\pi/2} \pi_\lambda(\exp(-ith))e_\lambda \tag{3.2.2}$$

exists in $\mathrm{H}_\lambda^{-\infty}$ and defines a H-invariant distribution vector [FNÓ23, Sec. 5]. Furthermore $\pi_\lambda^{-\infty}(\varphi)e_\lambda^H \in \mathrm{H}_\lambda^\infty$ for $\varphi \in C_c^\infty(G/H)$, see [NÓ18, Chap. 7]. Hence,

$$\Theta^\lambda(\varphi) = \langle e_\lambda, \pi_\lambda^{-\infty}(\varphi)e_\lambda^H \rangle$$

defines an H-invariant distribution. Furthermore,

$$\Delta\Theta^\lambda = (\rho^2 - \lambda^2)\Theta^\lambda.$$

This can be reformulated in terms of the kernel Ψ_λ. For that let $z \in \Xi$ and $g \in G$. Then

$$t \mapsto \Psi_\lambda(z, g\exp(-ith)e_n) = \Psi_\lambda(z, g(i\cos te_0 + \sin te_n))$$

is analytic on an open interval containing $(-\pi/2, \pi/2)$ with limit

$${}_2F_1\left(\rho+\lambda, \rho-\lambda; \frac{n}{2}; \frac{1+[z, ge_n]}{2}\right) = \Psi_\lambda(z, ge_n).$$

is analytic and extend to a continuous map to an open interval containing $\pi/2$, see more detailed discussion in a moment. We then get a distribution on dS^n by

$$\Theta^\lambda(z;\varphi) = \int_{\mathrm{dS}^n} \overline{\varphi(y)}\Psi_\lambda(z,y)d\mu_{\mathrm{dS}^n}(y) = \langle \pi(z)e_\lambda, \pi_\lambda^{-\infty}(\varphi)e_\lambda^H\rangle$$

where μ_{dS^n} is a G-invariant measure on dS^n. Taking the limit $z \to e_n$ leads then to the eigendistribution Θ^λ:

$$\Theta^\lambda(\varphi) = \lim_{t\to\pi/2^-} \int_{\mathrm{dS}^n} \overline{\varphi(y)}\Psi_\lambda(\exp(-ith)e_n, y)d\mu_{\mathrm{dS}^n}(y) \tag{3.2.3}$$

or

$$\Theta^\lambda = \lim_{t\to\pi/2^-} \Psi_\lambda(\exp(-ith)e_n, \cdot).$$

Furthermore, we also obtain that

$$\Delta\Theta^\lambda = (\rho^2 - \lambda^2)\Theta^\lambda. \tag{3.2.4}$$

Similar discussion holds for $\overline{\Xi}$, $\overline{\mathbb{H}}^n$ and $\widetilde{\Psi}_\lambda$.

It was proved in [GKÓ04] that $\lim_{\pi/2>t\to\pi/2} \Psi_\lambda(\exp(-ith)\cdot e_n, y)$ exists as a distribution on dS^n. It was proved for all ncc symmetric spaces using the *Automatic Continuation Theorem* of van den Ban, Brylinski and Delorme, see [vdBD88, Thm. 2.1] and [BD92, Thm. 1] and Hardy space approximation and restated in [NÓ18] for the specific case of dS^n. A different and less abstract proof was given in [FNÓ23]. A third proof of this fact was proved in [ÓS23] independent of representation theory and without using the existence of the H-invariant distribution vector e_λ^H which we will see in next chapter.

Chapter 4. Distributions as boundary values of holomorphic functions

In Proposition 3.1.6 we saw that the kernels $\Psi_\lambda(z,y)$ and $\widetilde{\Psi}_\lambda(z,y)$ are analytic for z in their respective crown domains and $y \in \mathrm{dS}^n$. In this chapter we prove that the boundary value of the kernels $\Psi_\lambda(z,.)$ and $\widetilde{\Psi}_\lambda(\bar{z},.)$ are distributions as z and $\bar{z}$ tends to an element in dS^n. As a motivation we start with simpler kernel Φ_λ and $\widetilde{\Phi}_\lambda$. We use the usual notation $\mathcal{D}(\mathrm{dS}^n) = C_c^\infty(\mathrm{dS}^n)$, $\mathcal{E}(\mathrm{dS}^n) = C^\infty(\mathrm{dS}^n)$ with the standard topology, $\mathcal{E}'(\mathrm{dS}^n)$ the space of distributions with compact support and, $\mathcal{D}'(\mathrm{dS}^n)$ the space of distributions on the de Sitter space. From this section onwards we will denote the elements in $\overline{\Xi}$ as $\bar{z}$, since $\sigma(\bar{z}) = z$ lies in Ξ.

Let us define the limit in Ξ, where it is understood following Proposition 2.2.6 and Corollary 2.2.7. Let $\Omega'_{e_n} = \{v \in \mathrm{T}_{e_n}(\mathrm{dS}^n) : [v,v] < 0, v_0 > 0\}$ and $\kappa = \kappa_{e_n}$ be the map defined in Eq. (2.2.1). Then define the limit $z = \sqrt{1-[v,v]}e_n + iv \to e_n$ as $v \to 0$ in Ω'_{e_n} as follows

$$\begin{aligned}
\Psi^\lambda_{e_n}(y) &= \lim_{z\to e_n} \Psi^\lambda_z(y) = \lim_{z\to e_n} \Psi_\lambda(z,y) \\
&= \lim_{v\to 0} \Psi_\lambda(\kappa(e_n + iv), y) \\
&= \lim_{v\to 0} \Psi_\lambda(\sqrt{1-[v,v]}e_n + iv, y).
\end{aligned}$$

Similarly for $\bar{z} \in \overline{\Xi}$, using the definition of $\widetilde{\kappa} = \widetilde{\kappa}_{e_n}$ as in Eq. (2.2.2) then define

$$\widetilde{\Psi}^\lambda_{e_n}(y) = \lim_{\bar{z}\to e_n} \widetilde{\Psi}^\lambda_{\bar{z}}(y) = \lim_{\bar{z}\to e_n} \widetilde{\Psi}_\lambda(\bar{z},y) = \lim_{v\to 0} \widetilde{\Psi}_\lambda(\widetilde{\kappa}(e_n - iv), y).$$

However, we will consider the limit in Eq. (3.2.3), which is weaker than the above limit as $\exp(-ith)e_n = i\cos(t)e_0 + \sin(t)e_n \in \Omega'_{e_n}$.

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By the discussions so far we will calculate the boundary value for $z = z_t = i\cos(t)e_0 + \sin(t)e_n = \exp(-ith)$ and $\bar{z} = \bar{z}_t = -i\cos(t)e_0 + \sin(t)e_n = \exp(ith)$ as $t \to \frac{\pi}{2}^-$.

4.1. The kernels Φ_λ and $\widetilde{\Phi}_\lambda$

For $\lambda \in \mathbb{C}$ and $z, w \in \mathbb{C}^{n+1}$ with $[z, w] \notin [1, \infty)$ let

$$\Phi(z, w) = \frac{1 - [z, \bar{w}]}{2}$$

and for $\lambda \in \mathbb{C}$

$$\Phi_\lambda(z, w) = \left(\frac{1 - [z, \bar{w}]}{2}\right)^\lambda$$

where ever defined. Note that Φ_λ is well defined for $z, w \in \Xi$ and $z, w \in \overline{\Xi}$ or if one of the points z or w is in Ξ, respectively $\overline{\Xi}$ and the other is from dS^n. In the case of $\overline{\Xi}$ we sometimes write $\widetilde{\Phi}$ to indicate the domain that we are looking at. We note that the kernels Ψ_λ and $\widetilde{\Psi}_\lambda$ behave approximately as a constant multiple $\Phi_{\frac{2-n}{2}}$ and $\widetilde{\Phi}_{\frac{2-n}{2}}$ respectively near $[z, \bar{w}] = 1 = [\bar{z}, w]$, where the constant depends on λ and n. Fix $z \in \Xi$ and $\bar{z} \in \overline{\Xi}$. Then the the functions $\Phi_\lambda(z, \cdot)$ and $\widetilde{\Phi}_\lambda(\bar{z}, \cdot)$ extends to analytic functions on dS^n and hence defines distributions Φ_z^λ and $\widetilde{\Phi}_{\bar{z}}^\lambda$ on dS^n.

For $y \in \mathrm{dS}^n$, we want to prove that $\lim\limits_{z \to x} \Phi_\lambda(z, y)$ is a distribution for any $x \in \mathrm{dS}^n$. For simplicity we start by taking $x = e_n$.

If $z \in \Xi$ then, as mentioned earlier, there exists $t \in (-\pi/2, \pi/2)$ and $g \in G$ such that $z(g, t) = g.z_t = g\exp(-ith)x_0$ and we have $z(g, t) \to ge_n \in \mathrm{dS}^n$ as $t \to \pi/2$. Similary for $\bar{z} \in \overline{\Xi}$, there exists a $t \in (-\pi/2, \pi/2)$ such that $\bar{z}(g, t) = g.\bar{z}_t$. In particular, if we take $g = Id$ then $z(Id, t) = z_t \to e_n$.

For $\mathrm{Re}(\lambda) > 0$, the limit is well defined in distributions. We will use analytic continuation to extend the definition to $\mathrm{Re}(\lambda) < 0$. Using the local co-ordinates and after some calculations we arrive at the following:

Lemma 4.1.1. *Let $L_\lambda = \Delta + \lambda(\lambda - 1 + n)$, where Δ is the Laplace-Beltrami operator on* dS^n. *Then the distributions Φ_z^λ and $\widetilde{\Phi}_{\bar z}^\lambda$ satisfy*

$$L_{\lambda+1}\Phi_z^{\lambda+1} = (\lambda+1)\left(\lambda + \frac{n}{2}\right)\Phi_z^\lambda$$

and

$$L_{\lambda+1}\widetilde{\Phi}_{\bar z}^{\lambda+1} = (\lambda+1)\left(\lambda + \frac{n}{2}\right)\widetilde{\Phi}_{\bar z}^\lambda.$$

Proof. As Δ is a G-invariant operator on dS^n and G acts transitively on Ξ and $\overline{\Xi}$, it is enough to calculate it for $z_t = i\cos(t)e_0 + \sin(t)e_n$. Then for $y = \sinh(s)e_0 + \cosh(s)u$, $s \in \mathbb{R}$ and $u = \sin(\theta)\widetilde{u} + \cos(\theta)e_n$ where $\widetilde{u} \in \mathbb{S}^{n-1}$ we obtain that

$$\Phi_{z_t}^\lambda(y) = \left(\frac{1 + i\cos(t)\sinh(s) - \sin(t)\cosh(s)\cos(\theta)}{2}\right)^\lambda.$$

As the function is $K = SO(n)$ invariant then $\Delta\Phi_{z_t}^\lambda(y)$ reduces to the following

$$\Delta\Phi_{z_t}^\lambda(s,\theta) = \left[-\frac{\partial^2}{\partial s^2} - (n-1)\tanh(s)\frac{\partial}{\partial s} + \frac{1}{\cosh^2(s)}\left(\frac{\partial^2}{\partial\theta^2} + (n-2)\cot(\theta)\frac{\partial}{\partial\theta}\right)\right]\Phi_{z_t}^\lambda(s,\theta).$$

Calculating each term we obtain the lemma. We follow the same steps for $\widetilde{\Phi}_{\bar z}^\lambda$. □

It follows from the above lemma that

$$\begin{aligned}\Phi_z^\lambda &= L_{\lambda+1}\ldots L_{\lambda+k}\,\Phi_z^{\lambda+k}\\ \widetilde{\Phi}_{\bar z}^\lambda &= L_{\lambda+1}\ldots L_{\lambda+k}\,\widetilde{\Phi}_{\bar z}^{\lambda+k}.\end{aligned} \tag{4.1.1}$$

For $\lambda \neq -1, -2, ..., -n/2, -n/2-1, ...$, we can thus define the analytic continuation of Φ_z^λ and $\widetilde{\Phi}_{\bar z}^\lambda$. For the residue at the singular points we refer to [GS64, Sec III.2]. Therefore, we obtain that the limits

$$\lim_{z\to x}\Phi_z^\lambda, \quad \lim_{\bar z\to x}\widetilde{\Phi}_{\bar z}^\lambda$$

are distributions on dS^n for $\lambda \in \mathbb{C}$.

Corollary 4.1.2. *For $\lambda = (2-n)/2$ the distributions Φ_z^λ and $\widetilde{\Phi}_{\bar{z}}^\lambda$ and their limits Φ_x^λ, $\widetilde{\Phi}_x^\lambda$ are eigendistributions of the Laplace-Beltrami operator Δ with eigenvalue $\frac{n}{2}(\frac{n-2}{2})$.*

Proof. For $\lambda = 1 - n/2$, it follows from Lemma 4.1.1 that,

$$\Delta \Phi_z^{\frac{2-n}{2}} = \frac{n}{2}\Big(\frac{n-2}{2}\Big)\Phi_z^{\frac{2-n}{2}}.$$

and same for $\widetilde{\Phi}_z^{\frac{2-n}{2}}$. Since differentiation is continuous on the space of distributions and Δ is invariant under the group G we have that

$$\Delta \Phi_x^{\frac{2-n}{2}} = \frac{n}{2}\Big(\frac{n-2}{2}\Big)\Phi_x^{\frac{2-n}{2}}, \qquad \Delta \widetilde{\Phi}_x^{\frac{2-n}{2}} = \frac{n}{2}\Big(\frac{n-2}{2}\Big)\widetilde{\Phi}_x^{\frac{2-n}{2}}. \tag{4.1.2}$$

□

We start with the special case $g = \mathrm{Id}$ and write $y' = (1-y_n)/2$. From now on we denote $\Phi_x = \Phi_x^{\frac{2-n}{2}}$ and $\widetilde{\Phi}_x = \widetilde{\Phi}_x^{\frac{2-n}{2}}$. From Appendix B we obtain that

$$\Phi_{e_n}(y) = \lim_{t \to \frac{\pi}{2}^-} \Phi_{z_t} = (y' \pm i0)^{\frac{2-n}{2}} \quad \text{for } \pm y_0 > 0,$$

and

$$\widetilde{\Phi}_{e_n}(y) = \lim_{t \to \frac{\pi}{2}^-} \widetilde{\Phi}_{\bar{z}_t} = (y' \mp i0)^{\frac{2-n}{2}} \quad \text{for } \pm y_0 > 0,$$

because

$$\frac{1-[z_t, y]}{2} = \frac{1-\sin(t)y_n}{2} + i\frac{\cos(t)y_0}{2}.$$

For n even we have by (B.0.8):

$$\Phi_{e_n}(y) = (y')_+^{\frac{2-n}{2}} + (-1)^{\frac{n-2}{2}}(y')_-^{\frac{2-n}{2}} - (-1)^{\frac{n-2}{2}}\mathrm{sgn}(y_0)\frac{i\pi}{(n/2-2)!}\delta^{\frac{n-2}{2}}(y'),$$

and if n is odd then (B.0.6) leads to

$$\Phi_{e_n}(y) = (y')_+^{\frac{2-n}{2}} + (-i\,\mathrm{sgn}(y_0))^{n-2}(y')_-^{\frac{2-n}{2}}.$$

Correspondingly, when n is even we have

$$\begin{aligned}\widetilde{\Phi}_{e_n}(y) &= (y' \mp \mathrm{sgn}(y_0)\, i0)^{\frac{2-n}{2}} \\ &= (y')_+^{\frac{2-n}{2}} + (-1)^{\frac{n-2}{2}}(y')_-^{\frac{2-n}{2}} + (-1)^{\frac{n-2}{2}}\mathrm{sgn}(y_0)\frac{i\pi}{(n/2-2)!}\delta^{\frac{n-2}{2}}(y'),\end{aligned}$$

and when n is odd

$$\widetilde{\Phi}_{e_n}(y) = (y' \mp \mathrm{sgn}(y_0)\, i0)^{\frac{2-n}{2}} = (y')_+^{\frac{2-n}{2}} + (i\,\mathrm{sgn}(y_0))^{n-2}(y')_-^{\frac{2-n}{2}},$$

where sgn is the signature function. Observe that $\widetilde{\Phi}_x = \overline{\Phi_x}$.

Thus, we obtain the following theorem:

Theorem 4.1.3. *The limits*

$$\lim_{t\to\pi/2^-} \Phi(g.z_t, \cdot) = \Phi_x \quad \textit{and} \quad \lim_{t\to\pi/2^-} \widetilde{\Phi}(g.\bar{z}_t, \cdot) = \widetilde{\Phi}_x, \quad x = ge_n$$

exist in $\mathcal{D}'(\mathrm{dS}^n)$. *The distributions* Φ_x *and* $\widetilde{\Phi}_x$ *satisfy Eq.* (4.1.2). *Finally we have*

$$\Phi_x(y) = \begin{cases} \left(\frac{1-[x,y]}{2}\right)_+^{\frac{2-n}{2}} + (-1)^{\frac{n-2}{2}}\left(\frac{1-[x,y]}{2}\right)_-^{\frac{2-n}{2}} - \\ (-1)^{\frac{n-2}{2}}\mathrm{sgn}((y-x)_0)\frac{i\pi}{(n/2-2)!}\delta^{\frac{n-2}{2}}\left(\frac{1-[x,y]}{2}\right) & \textit{if } n \textit{ even;} \\ \left(\frac{1-[x,y]}{2}\right)_+^{\frac{2-n}{2}} + (-i\,\mathrm{sgn}((y-x)_0))^{n-2}\left(\frac{1-[x,y]}{2}\right)_-^{\frac{2-n}{2}} & \textit{if } n \textit{ is odd.}\end{cases} \tag{4.1.3}$$

and,

$$\widetilde{\Phi}_x(y) = \begin{cases} \left(\frac{1-[x,y]}{2}\right)_+^{\frac{2-n}{2}} + (-1)^{\frac{n-2}{2}}\left(\frac{1-[x,y]}{2}\right)_-^{\frac{2-n}{2}} + \\ (-1)^{\frac{n-2}{2}}\mathrm{sgn}((y-x)_0)\frac{i\pi}{(n/2-2)!}\delta^{\frac{n-2}{2}}\left(\frac{1-[x,y]}{2}\right) & \textit{if } n \textit{ even;} \\ \left(\frac{1-[x,y]}{2}\right)_+^{\frac{2-n}{2}} + (i\,\mathrm{sgn}((y-x)_0))^{n-2}\left(\frac{1-[x,y]}{2}\right)_-^{\frac{2-n}{2}} & \textit{if } n \textit{ is odd.}\end{cases} \tag{4.1.4}$$

Proof. Clearly, the limits are well-defined . The rest follows from the above discussion and the fact that $\Phi_{\frac{2-n}{2}}(g.z_t, y) = \Phi_{\frac{2-n}{2}}(z_t, g^{-1}.y)$ and, $\widetilde{\Phi}_{\frac{2-n}{2}}(g.\bar{z}_t, y) = \widetilde{\Phi}_{\frac{2-n}{2}}(\bar{z}_t, g^{-1}.y)$. $\square$

Immediately, we obtain the following corollary:

Corollary 4.1.4. *The distributions Φ_{e_n} and, $\widetilde{\Phi}_{e_n}$ are H-invariant distributions.*

4.2. The kernels Ψ_λ and $\widetilde{\Psi}_\lambda$

In this section we will consider the kernels Ψ_λ and $\widetilde{\Psi}_\lambda$ for $\lambda \in \mathbb{C} \setminus (\{\rho + \mathbb{N}\} \cup \{-\rho - \mathbb{N}\})$. As usual $a = \rho + \lambda$, $b = \rho - \lambda$ and, $c = n/2$ we will denote ${}_2F_1(z) = {}_2F_1(a, b; c; z)$.

Remark 4.2.1. If $\lambda \in \{\rho + \mathbb{N}\} \cup \{-\rho - \mathbb{N}\}$, we have that either a or b is a negative integers for which ${}_2F_1(z)$ reduces to a polynomial. In this case we obtain that

$$\lim_{z \to x} \Psi_\lambda(z, y) = \lim_{\bar{z} \to x} \widetilde{\Psi}_\lambda(\bar{z}, y), \quad \text{if} \quad \lambda \in \{\rho + \mathbb{N}\} \cup \{-\rho - \mathbb{N}\}.$$

$\square$

From the Eq. (2.2.1), Eq. (2.2.2) and following previous section it is enough to prove that the limits

$$\lim_{t \to \pi/2^-} \Psi_\lambda(g.z_t, y) \quad \text{and} \quad \lim_{t \to \pi/2^-} \widetilde{\Psi}_\lambda(g.\bar{z}_t, y)$$

are distributions.

We drop the dependence on λ for the limit distributions as it will be clear from the context.

From Theorem A.0.1 the point-wise limit is the following:

$$
\begin{aligned}
\Psi_{e_n}^{\lambda}(y) &= \lim_{t\to\pi/2^-} {}_2F_1\left(\frac{1+[z_t,y]}{2}\right) \\
&= \lim_{t\to\pi/2^-} {}_2F_1\Big(\frac{1+\sin(t)y_n - i\cos(t)y_0}{2}\Big) \\
&= \begin{cases}
{}_2F_1\left(\frac{1+y_n}{2}\right) & \text{if } y_n < 1, \\[2ex]
{}_2F_1\left(\frac{1+y_n}{2} - i0\right) & \text{if } y_n > 1, y_0 > 0, \\[2ex]
{}_2F_1\left(\frac{1+y_n}{2} + i0\right) & \text{if } y_n > 1, y_0 < 0;
\end{cases}
\end{aligned}
$$

where ${}_2F_1(x \pm i0)$ has been calculated for $x > 1$ in Appendix A.

For the other kernel we get

$$
\begin{aligned}
\widetilde{\Psi}_{e_n}^{\lambda}(y) &= \lim_{t\to\pi/2^-} {}_2F_1\Big(\frac{1+\sin(t)y_n + i\cos(t)y_0}{2}\Big) \\
&= \begin{cases}
{}_2F_1\left(\frac{1+y_n}{2}\right) & \text{if } y_n < 1, \\[2ex]
{}_2F_1\left(\frac{1+y_n}{2} + i0\right) & \text{if } y_n > 1, y_0 > 0, \\[2ex]
{}_2F_1\left(\frac{1+y_n}{2} - i0\right) & \text{if } y_n > 1, y_0 < 0.
\end{cases}
\end{aligned}
$$

From the Theorem A.0.1 we have that in each of the disjoint region the limit is uniform on compact sets. Next step is to prove that the limit actually converges to a distribution.

Let $n \geq 2$ and φ be such that $\mathrm{supp}(\varphi) \cap \{y_n = 1\} = \emptyset$. Since $\Psi_\lambda(z_t, y)$ and $\widetilde{\Psi}_\lambda(\bar{z}_t, y)$ converges to $\Psi_{e_n}^{\lambda}(y)$ and $\widetilde{\Psi}_{e_n}^{\lambda}(y)$ uniformly on compact sets in the region $\mathrm{dS}^n \setminus \{y_n = 1\}$, we have that

$$
\lim_{t\to\pi/2^-} \int_{\mathrm{dS}^n} \Psi_\lambda(z_t, y)\varphi(y)dy \longrightarrow \int_{\mathrm{dS}^n} \Psi_{e_n}^{\lambda}(y)\varphi(y)dy
$$

and

$$
\lim_{t\to\pi/2^-} \int_{\mathrm{dS}^n} \widetilde{\Psi}_\lambda(\bar{z}_t, y)\varphi(y)dy \longrightarrow \int_{\mathrm{dS}^n} \widetilde{\Psi}_{e_n}^{\lambda}(y)\varphi(y)dy.
$$

Case: dimension 2

On the other hand if $\mathrm{supp}(\varphi) \cap \{y_2 = 1\} \neq \{\emptyset\}$ for $\varphi \in \mathcal{D}(X)$, without loss of generality we can take φ such that in local co-ordinates, $\max\left[\mathrm{d}(y, \{y_2 = 1\})\right] < \epsilon$, for $y \in \mathrm{supp}(\varphi)$ and very small $\epsilon > 0$. We know that close to the set $\{y_2 = 1\}$,

$$\Psi_\lambda(z_t, y) \approx -\frac{\Gamma(1)}{\Gamma(\frac{1}{2}+\lambda)\Gamma(\frac{1}{2}-\lambda)} \ln\left(\frac{1-[z_t, y]}{2}\right)$$

and

$$\widetilde{\Psi}_\lambda(\bar{z}_t, y) \approx -\frac{\Gamma(1)}{\Gamma(\frac{1}{2}+\lambda)\Gamma(\frac{1}{2}-\lambda)} \ln\left(\frac{1-[\bar{z}_t, y]}{2}\right).$$

Since logarithm is locally integrable function and by appendix B and [GS64, Sec 2.4, Example 4] we see that the limit convergences in distribution.

Case: $n \geq 3$

For $n \geq 3$, we have that $\mathrm{Re}(c - a - b) = (2 - n)/2 < 0$. Without loss of generality, we choose φ as we did in the 2-dimensional case. Close to $y_n = 1$, the kernels behave as:

$$\Psi_\lambda(z_t, y) \approx \frac{\Gamma(n/2)\Gamma((n-2)/2)}{\Gamma(\rho+\lambda)\Gamma(\rho-\lambda)} \left(\frac{1-[z_t,y]}{2}\right)^{\frac{2-n}{2}};$$

$$\widetilde{\Psi}_\lambda(\bar{z}_t, y) \approx \frac{\Gamma(n/2)\Gamma((n-2)/2)}{\Gamma(\rho+\lambda)\Gamma(\rho-\lambda)} \left(\frac{1-[\bar{z}_t,y]}{2}\right)^{\frac{2-n}{2}}.$$

Therefore by Theorem 4.1.3, appendix B and [GS64, Sec 3.6], as we take $t \to \pi/2^-$ the kernels $\Psi_\lambda(z_t, y)$ and $\widetilde{\Psi}_\lambda(\bar{z}_t, y)$ converge to corresponding distributions $\Psi^\lambda_{e_n}$ and $\widetilde{\Psi}^\lambda_{e_n}$. These limits are well defined as $\Psi^\lambda_{e_n}$ and $\widetilde{\Psi}^\lambda_{e_n}$ are H-invariant. To see that let $h \in H$ for which $h \cdot e_n = e_n$. Let $\varphi \in \mathcal{D}(\mathrm{dS}^n)$. If $\mathrm{supp}(\varphi) \cap \{y_n = 1\} = \emptyset$ then clearly

$$\lim_{t\to\pi/2^-} \int_{\mathrm{dS}^n} \Psi_\lambda(h \cdot z_t, y)\varphi(y)dy = \lim_{t\to\pi/2^-} \int_{\mathrm{dS}^n} \Psi_\lambda(z_t, y)\varphi(y)dy$$

and

$$\lim_{t\to\pi/2^-}\int_{\mathrm{dS}^n}\widetilde{\Psi}_\lambda(h\cdot\bar{z}_t,y)\varphi(y)dy=\lim_{t\to\pi/2^-}\int_{\mathrm{dS}^n}\widetilde{\Psi}_\lambda(\bar{z}_t,y)\varphi(y)dy.$$

If $\operatorname{supp}(\varphi)\cap\{y_n=1\}\neq\emptyset$ as in previous steps. For $n\geq 3$ and, some constant c, we obtain that

$$|\langle\lim_{t\to\pi/2^-}(\Psi_\lambda(h.z_t,.)-\Psi_\lambda(z_t,.)),\varphi\rangle|\leq const.\lim_{t\to\pi/2^-}|\langle\Phi_{\frac{2-n}{2}}(h.z_t,.)-\Phi_{\frac{2-n}{2}}(z_t,.),\varphi\rangle|$$

$$=0.$$

The last equality is due to Corollary 4.1.4. The same steps can be followed for $n=2$ and also for $\widetilde{\Psi}^\lambda_{e_n}$. Thus proving that the limits are well-defined.

For $g\in G$ and $x=g\cdot e_n$

$$\Psi^\lambda_x(y)=\lim_{t\to\pi/2^-}\Psi_\lambda(g.z_t,y)=\lim_{t\to\pi/2^-}\Psi_\lambda(z_t,g^{-1}.y)$$

and

$$\widetilde{\Psi}^\lambda_x(y)=\lim_{t\to\pi/2^-}\widetilde{\Psi}_\lambda(g.\bar{z}_t,y)=\lim_{t\to\pi/2^-}\widetilde{\Psi}_\lambda(\bar{z}_t,g^{-1}.y)$$

are also distributions.

Now, we claim that $(\Delta-m^2)\Psi^\lambda_{e_n}=0$ where $m^2=\rho^2-\lambda^2$. Using fact that differentiation is a continuous linear map on space of distributions, we obtain

$$\lim_{t\to\pi/2^-}(\Delta-m^2)\Psi_\lambda(z_t,y)=(\Delta-m^2)\Psi^\lambda_{e_n}.$$

Now, let $a=\rho+\lambda$, $b=\rho-\lambda$, $c=n/2$ and $w_t=\frac{1+[z_t,y]}{2}$.

Following the same steps as in proof of Lemma 4.1.1, we arrive at

$$(\Delta - m^2)\Psi_\lambda(z_t, y) = \frac{ab}{c}\Big[w_t(1-w_t)\frac{(a+1)(b+1)}{(c+1)}{}_2F_1(a+2, b+2, c+2, w_t)$$

$$+\left(\frac{n}{2} - nw_t\right){}_2F_1(a+1, b+1, c+1, w_t) - c\,{}_2F_1(a, b, c, w_t)\Big]$$

$$= 0.$$

using the properties of hypergeometric function. We obtain that as distributions $(\Delta - m^2)\Psi^\lambda_{e_n} = 0$. Following the same steps we obtain $(\Delta - m^2)\widetilde{\Psi}^\lambda_{e_n} = 0$. As $(\Delta - m^2)$ is a G invariant operator, we have that $(\Delta - m^2)\Psi^\lambda_x = 0 = (\Delta - m^2)\widetilde{\Psi}^\lambda_x$.

Therefore, we have proved that :

Theorem 4.2.2. *For $n \geq 2$ and $\lambda \notin \{\rho + \mathbb{N}\} \cup \{-\rho - \mathbb{N}\}$ we have:*

1. *The limits $\lim\limits_{t\to\pi/2^-} \Psi_\lambda(g.z_t, y)$ and $\lim\limits_{t\to\pi/2^-} \widetilde{\Psi}_\lambda(g.\bar{z}_t, y)$ converge to distributions Ψ^λ_x and $\widetilde{\Psi}^\lambda_x$ respectively, on dS^n with $x = g \cdot e_n$.*
2. *The limits satisfy $(\Delta - m^2)\Psi^\lambda_x = 0 = (\Delta - m^2)\widetilde{\Psi}^\lambda_x$.*
3. *Moreover, Also, Ψ^λ_x and $\widetilde{\Psi}^\lambda_x$ can be represented as analytic functions in the following regions:*

$$\Psi^\lambda_x(y) = \begin{cases} {}_2F_1\left(\frac{1+[x,y]}{2}\right) & \text{if } y \notin \overline{\Gamma(x)}, \\ {}_2F_1\left(\frac{1+[x,y]}{2} - i0\right) & \text{if } y \in \Gamma^+(x), \\ {}_2F_1\left(\frac{1+[x,y]}{2} + i0\right) & \text{if } y \in \Gamma^-(x); \end{cases}$$

$$\widetilde{\Psi}^\lambda_x(y) = \begin{cases} {}_2F_1\left(\frac{1+[x,y]}{2}\right) & \text{if } y \notin \overline{\Gamma(x)}, \\ {}_2F_1\left(\frac{1+[x,y]}{2} + i0\right) & \text{if } y \in \Gamma^+(x), \\ {}_2F_1\left(\frac{1+[x,y]}{2} - i0\right) & \text{if } y \in \Gamma^-(x). \end{cases}$$

As a conclusion it implies that $\Psi^\lambda_x = \overline{\widetilde{\Psi}^\lambda_x}$.

In particular, the singular support of Ψ^λ_x and $\widetilde{\Psi}^\lambda_x$ are exactly the points $\{y \in \mathrm{dS}^n \mid [y - x, y - x] = 0\}$. Fig. 4.1 shows the singular support on dS^2 of these distributions when

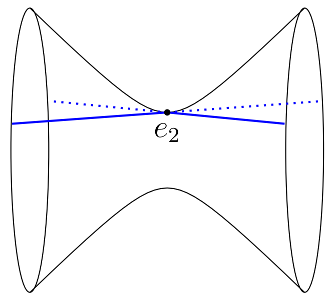


Figure 4.1. Singularities on dS^2

$x = e_2$ in blue lines.

Chapter 5. Wavefront Sets of Spherical Distributions

The wavefront set of a distribution was introduced by L. Hörmander in 1970. It gives more information about singularities. In particular, it gives the singular support of a distribution and the direction where the distribution is not smooth or analytic. We apply this notion to the distributions Ψ_x^λ and $\widetilde{\Psi}_x^\lambda$. As we have seen in previous section, the distributions Ψ_x^λ and $\widetilde{\Psi}_x^\lambda$ can be written as analytic functions everywhere on the de Sitter except at the boundary of the light cone of x. That is where the distributions are singular. We will now recall the wavefront set of distributions.

5.1. Wavefront Sets

Let $X \subset \mathbb{R}^{1,n}$ be an open subset. Suppose, $\Theta \in \mathcal{E}'(X)$ is a distribution with compact support then we can define Fourier transform of Θ at $\xi \in (\mathbb{R}^{1,n} \setminus 0)$ as follows:

$$\widehat{\Theta}(\xi) = \Theta(e^{-2\pi i[x,\xi]}).$$

where $[x,\xi] = -x_0\xi_0 + x_1\xi_1 ... + x_n\xi_n$.

Definition 5.1.1. Let Θ in $\mathcal{D}'(X)$ be a distribution. We say $(x_0, \xi_0) \in \mathrm{T}^*(X) \setminus \{0\}$ is a regular directed point if there exist an open neighbourhood U of x_0, a conical neighbourhood V of ξ_0 and $\varphi \in C_c^\infty(U)$ with $\varphi(x_0) \neq 0$ such that for all $N \in \mathbb{N}$:

$$|\widehat{\varphi\Theta}(\tau\xi)| \leq C_{N,\varphi}(1 + |\tau|)^{-N}, \quad \forall \xi \in V. \tag{5.1.1}$$

The wavefront set $WF(\Theta) \in \mathrm{T}^*(X) \setminus \{0\}$ is the complement of the regular directed set.

Definition 5.1.2. Let $\Theta \in \mathcal{D}'(X)$. The *singular support* of Θ is set of all points x such that there is no neighbourhood of x to which the restriction of Θ is a C^∞ function .

This chapter has appeared in the article: G. Olafsson, I. Sitiraju. *Analytic wavefront sets of spherical distributions on the de Sitter space.* arXiv:2309.10685

Lemma 5.1.3. *If $\Theta \in \mathcal{D}'(X)$, then the projection of $WF(\Theta)$ onto X is the singular support of Θ.*

Remark 5.1.4. The $WF(\Theta)$ is a conic set, that is if $(x,\xi) \in WF(\Theta)$, then for $\tau > 0$, $(x, \tau\xi) \in WF(\Theta)$.

Here are some examples.

Example 5.1.5. We will consider the Dirac-delta distribution in $\mathbb{R}^{1,n}$. Then the $\operatorname{supp}(\delta_0) = \{0\}$. Let $\varphi \in C_c^\infty(\mathbb{R}^{1,n})$ with $\varphi(0) = c \neq 0$. Now, choose any $\xi \in (\mathbb{R}^{1,n} \setminus 0)$, we see that because

$$\widehat{\varphi\delta_0}(\xi) = \delta_0(\varphi(x)e^{-2\pi i[x,\xi]}) = \varphi(0) \neq 0,$$

the Fourier transform is not rapidly decreasing in ξ for any $\xi \in (\mathbb{R}^{1,n} \setminus 0)$. Hence

$$\mathrm{WF}(\delta_0) = \{(0,\xi) : \xi \in \mathbb{R}^{1,n} \setminus 0\}. \qquad \square$$

Example 5.1.6. Consider the Heaviside function as distribution. That is,

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Clearly it is smooth function away from zero. Let $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi(0) \neq 0$. Then using integration by parts we obtain:

$$\begin{aligned} \widehat{\varphi H}(\xi) &= \int_0^\infty \varphi(x)e^{-2\pi i x\xi}dx. \\ &= \frac{\varphi(0)}{2\pi i\xi} + \int_0^\infty \varphi'(x)e^{-2\pi i x\xi}dx \\ &= \frac{\varphi(0)}{2\pi i\xi} + \frac{\varphi'(0)}{(2\pi i\xi)^2} + \frac{1}{(2\pi i\xi)^2}\int_0^\infty \varphi''(x)e^{-2\pi i x\xi}dx. \end{aligned}$$

The first term is of order 1 and the rest are atleast of order 2. Thus, the Fourier transform does not decay rapidly enough for any $\xi \neq 0$ in $\mathbb{R}$. Hence,

$$WF(H) = \{0\} \times (\mathbb{R} \setminus 0). \qquad \square$$

We will now introduce analytic wavefront sets. We follow the definition from [H90, def. 8.4.3]. Since, multiplying the distribution with smooth function will only increase the analytic wavefront set and there is no non-zero real analytic function with compact support. To circumvent this problem the following proposition (see [H90, Proposition 8.4.2]) is the basis for the definition of analytic wavefront set.

Proposition 5.1.7. *Let X be an open subset of $\mathbb{R}^{1,n}$ and $\Theta \in \mathcal{D}'(X)$. Then Θ is real analytic in a neighbourhood U of x_0 if and only if there is a bounded sequence Θ_N of distributions with compact support which is equal to Θ in U satisfying,*

$$|\widehat{\Theta}_N(\xi)| \leq C^{N+1}(N/|\xi|)^N, \quad N = 1, 2, ...$$

for $C > 0$.

Definition 5.1.8. If X is an open subset of $\mathbb{R}^{1,n}$ and $\Theta \in \mathcal{D}'(X)$, we denote $WF_A(\Theta)$ to be the complement in $X \times (\mathbb{R}^{1,n} \setminus 0)$ of the set (x_0, ξ_0) such that there is an open neighbourhood $U \subset X$ of x_0, a conic neighbourhood Γ of ξ_0 and a bounded sequence of $\Theta_N \in \mathcal{E}'(X)$ which is equal to Θ in U and satisfies

$$|\widehat{\Theta_N}(\xi)| \leq C^{N+1}(N/|\xi|)^N \quad N = 1, 2, ...$$

when $\xi \in \Gamma$ and for some $C > 0$.

The following lemma shows that Θ_N can always be chosen as a product of Θ with some suitable functions.

Lemma 5.1.9. *Let $\Theta \in \mathcal{D}'(X)$. Let Γ and U be as in the definition above. We have that $(x_0, \xi_0) \notin WF_A(\Theta)$ if and only if for K a compact neighbourhood of x_0 in U, Θ a closed conic neighbourhood of ξ_0 in Γ, there exists functions $\chi_N \in C_c^\infty(U)$ such that $\chi_N = 1$ on K with*

$$|D^{t+\beta}\chi_N| \leq C_t^{N+1} N^{|\beta|}, \quad |\beta| \leq N,$$

then, it follows that the sequence $\chi_N \Theta$ is bounded in $\mathcal{E}'$ and satisfies the following:

$$|\widehat{\chi_N \Theta}(\xi)| \leq C(C(N+1)/|\xi|)^N. \tag{5.1.2}$$

The proof of the above lemma can be found in [H90, Chap 8].

Example 5.1.10. Let $u = \delta_0$ in $\mathbb{R}^{1,n}$. We can see that $WF_A(\delta_0) \subset \{0\} \times (\mathbb{R}^{1,n} \setminus 0)$. Let χ_N be a sequence of functions as in the above lemma. Then for $\xi \neq 0$

$$\widehat{\chi_N \delta_0}(\xi) = \chi_N(0) = 1,$$

which does not decay at infinity. Therefore, $WF_A(\delta_0) = \{0\} \times (\mathbb{R}^{1,n} \setminus 0)$. □

The following lemma tells us the relation between wavefront sets and analytic wavefront set.

Lemma 5.1.11. *Let $\Theta \in \mathcal{D}'(X)$, we have that $WF(\Theta) \subset WF_A(\Theta)$.*

Proof. Suppose $(x_0, \xi_0) \notin WF_A(\Theta)$ then there exist an open neighbourhood $U \ni x_0$, an open cone $\Gamma \ni \xi_0$ and a bounded sequence of Θ_N with compact support such that $\Theta_N = \Theta$ in U and

$$|\widehat{\Theta}_N(\xi)| \leq C^{N+1}(N/|\xi|)^N, \quad \xi \in \Gamma.$$

Then for $x \in U$,

$$D^\alpha \Theta(x) = D^\alpha \Theta_N(x) = \int \xi^\alpha \widehat{\Theta}_N(\xi) e^{2\pi i [x,\xi]} d\xi.$$

It follows since $\xi^\alpha \widehat{\Theta}_N(\xi)$ is integrable for $N = |\alpha| + n + 1$, as $1/|\xi|^{1+n}$ is integrable outside unit ball and $|\widehat{\Theta}_N(\xi)| \le C(1+|\xi|)^M$. Hence Θ is smooth in U. $\square$

We will now show an examples of a distribution whose analytic wavefront set is strictly bigger than the smooth wavefront set. Before that, let us look at a characterization of real analytic function. A smooth function Θ is real analytic if and only if for every compact set $K \subset \mathbb{R}$ there is a constant C_K with

$$|D^N \Theta(x)| \le C_K^{N+1} (N)^N, \quad x \in K,$$

for all $N \ge 0$. Indeed, by Taylor's theorem

$$\Theta(x) = \sum_{i=0}^{n} \Theta^{(i)}(x_0) \tfrac{(x-x_0)^i}{i!} + \frac{1}{n!} \int_{x_0}^{x} \Theta^{(n+1)}(t)(x-t)^n dt.$$

We have that, for $|x - x_0| < \delta < 1/(3C_K)$ and $N^N \le 3^N N!$,

$$\begin{aligned} \Big| \frac{1}{n!} \int_{x_0}^{x} \Theta^{(n+1)}(t)(x-t)^n dt \Big| &\le \frac{C_K^{N+1}(N)^N}{N!} \Big| \int_{x_0}^{x} (x-t)^N dt \Big| \\ &= \frac{C_K^{N+1}(N)^N}{(N+1)!} |x - x_0|^{N+1} \\ &\le (3C_K\delta)^{N+1} \to 0, \text{ as } N \to \infty. \end{aligned}$$

Hence Θ is real analytic function. On the other hand, we get that Θ satisfies the above conditions if it is real analytic by Cauchy's inequalities.

Example 5.1.12. We know that the function

$$\Theta(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ \\ 0 & \text{if } x \le 0 \end{cases}$$

is smooth everywhere but not real analytic at origin. It is obvious that Θ is a distribution. Let φ be a smooth function with compact support in a small neighbourhood of 0 with $\varphi(0) = 1$. Then,

$$\begin{aligned}|\xi^N \widehat{\varphi\Theta}(\xi)| &= |\int_0^\infty D^N(\varphi e^{-1/x})e^{-2\pi i x\xi}dx| \\ &\leq \int_0^\infty |D^N(\varphi e^{-1/x})|dx \\ &\leq C_N,\end{aligned}$$

where the last inequality is because all the derivatives of $e^{-1/x}$ are bounded and φ is smooth with compact support. Therefore, $D^N(\varphi\Theta)$ is integrable for all N. Hence,

$$WF(\Theta) = \emptyset.$$

Now, let $K = [-\epsilon, \epsilon]$, for ϵ very small. We have that

$$D^N(e^{-1/x}) = \frac{e^{-1/x}p_N(x)}{x^{2N}},$$

where $p_N(x)$ is a polynomial of degree N with constant coefficient 1. Thus for x in K, $D^N(e^{-1/x}) \approx \frac{e^{-1/x}}{x^{2N}}$. Hence the maximum is approximately at $x = 1/2N$ and the maximum value is $e^{-2N}(2N)^{2N}$. For sufficiently large N,

$$\max_{x\in K}|D^N(e^{-1/x})| \approx e^{-2N}(2N)^{2N} > N^N.$$

We see that the derivatives of Θ do not have the desired growth near zero. Hence Θ is not real analytic at 0 and $\emptyset \neq WF_A(\Theta) \subset \{0\} \times (\mathbb{R} \setminus 0)$. From Theorem 5.2.2, we obtain that if $WF_A(\Theta) \cap -WF_A(\Theta) = \emptyset$, then Θ can not vanish on any open set of $\mathbb{R}$. This implies that

$$WF_A(\Theta) = \{0\} \times (\mathbb{R} \setminus 0). \qquad \square$$

We will also be needing the following theorem which again can be found in [H90].

Theorem 5.1.13. *1. Let a be a real analytic function. Then $WF_A(a\Theta) \subset WF_A(\Theta)$.*

2. $WF_A(\Theta_1 \pm \Theta_2) \subset WF_A(\Theta_1) \cup WF_A(\Theta_2)$

Generally, the pull back of a distribution under a map is not continuous. For example, consider the map $\iota : \mathbb{R} \to \mathbb{R}^2$ by $\iota(x) = (x, 0)$. Then the pull back must be defined such that $\iota^*(\Theta) = \Theta \circ \iota$ for Θ a smooth map.

For $\Theta \geq 0$, smooth with $\text{supp}(\Theta) \subseteq \overline{\text{B}(0,1)}$, $f_k = k^2 f(kx)$, we have $f_k \to \delta_{(0,0)}$. Let $\varphi \in C_c^\infty(\mathbb{R})$ and $\varphi \geq 0$,

$$\begin{aligned} < \iota^*(f_k), \varphi > &= \int_{\mathbb{R}} (f_k \circ \iota)(x)\varphi(x)dx \\ &= k\int_{-1}^{1} \Theta(x,0)\varphi(x)dx \to \infty \quad \text{as } k \to \infty. \end{aligned}$$

Therefore the pull back is not continuous. Define the normal set of the map ι by

$$\begin{aligned} N_\iota &= \{(\iota(x), \xi) \in \mathbb{R}^2 \times \mathbb{R}^2 : {}^t d\iota_x(\xi) = 0\}; \\ &= \{((x,0); (0, \xi_2)) : x, \xi_2 \in \mathbb{R}\}. \end{aligned}$$

where ${}^t d\iota_x = [1, 0]$.

We have that $WF_A(\delta_{(0,0)}) = \{((0,0); (\xi_1, \xi_2)\}$.

Observe that $WF_A(\delta_{(0,0)}) \cap N_\iota \neq \emptyset$. We will now see the relation between the set of normals, wavefront set and pullback of distribution.

The following theorem says under what condition we can define a pull back of a distribution. The proof can be found in [H90, Theorem 8.2.4, Theorem 8.5.1].

Theorem 5.1.14. *Let X and Y be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and let ι : $X \to Y$ be a real analytic map. Denote the normal set of the map by*

$$N_\iota = \{(\iota(x), \xi) \in Y \times \mathbb{R}^n : {}^t d\iota_x(\xi) = 0\}.$$

Then the pull back $\iota^*\Theta$ can be defined in one and only one way for all $\Theta \in \mathcal{D}'(Y)$ with

$$N_\iota \cap WF_A(\Theta) = \emptyset$$

so that $\iota^*(\Theta) = \Theta \circ \iota$ when $\Theta \in C^\infty$ and for any closed conic subset Γ of $Y \times (\mathbb{R}^n \setminus 0)$ with $\Gamma \cap N_\iota = \emptyset$ we have

$$\iota^*(\Gamma) = \{(x, {}^t d\iota_x(\xi)) : (\iota(x), \xi) \in \Gamma\}.$$

In particular, if $\Theta \in \mathcal{D}'(Y)$ with $N_\iota \cap WF_A(\Theta) = \emptyset$ then

$$WF_A(\iota^*\Theta) \subset \iota^* WF_A(\Theta).$$

The above theorem lets us define the analytic wavefront set if X is a real analytic manifold.

Definition 5.1.15. If X is a real analytic manifold, and (U_k, k) be the analytic local co-ordinates on X. We define $WF_A(\Theta) \subset T^*(X) \setminus 0$ to be the set

$$k^* WF((k^{-1})^*\Theta) := \{(x, {}^t dk_x^{-1}(\eta)); (k^{-1}(x), \eta) \in WF((k^{-1})^*\Theta),$$

where $(k^{-1})^*\Theta(\varphi) = \Theta(\varphi \circ k^{-1})$ for $\varphi \in C_c^\infty(U_k)$.

The Theorem 5.1.14 tells us that the above definition is invariant under co-ordinate change.

The next theorem describes the analytic wavefront sets of distributions which are boundary value of analytic functions. Let Γ be an open convex cone, then the *dual cone* Γ° is defined as

$$\Gamma^\circ = \{\eta \in \mathbb{R}^{1+n} : \eta_0\xi_0 + ... + \eta_n\xi_n \geq 0, \ \forall \xi \in \Gamma\}.$$

Theorem 5.1.16. *Let $X \subset \mathbb{R}^{1,n}$ be an open set and Γ an open convex cone in $\mathbb{R}^{1,n}$ and for some $\gamma > 0$,*

$$Z = \{z \in \mathbb{C}^{1+n} : \mathrm{Re}\,\mathrm{z} \in \mathrm{X}, \mathrm{Im}\,\mathrm{z} \in \Gamma, |\mathrm{Imz}| < \gamma\}.$$

If Θ is an analytic function in Z such that

$$|\Theta(z)| \leq C|\mathrm{Im}\,\mathrm{z}|^{-\mathrm{N}}$$

for some N and some constant $C > 0$, the $\lim\limits_{y \searrow 0} \Theta(.+iy) = \Theta_0$ exists in terms of distribution and is of order N. We also have that

$$WF_A(\Theta_0) \subset X \times (\Gamma^\circ \setminus 0).$$

Proof. See theorem 3.1.15 and theorem 8.4.8 in [H90]. □

Example 5.1.17. Consider the distribution $\Theta = (x + i0)^{\frac{2-n}{2}}$ on $\mathbb{R}$. It is the limit of the analytic function $(x + iy)^{\frac{2-n}{2}}$ for $x \in \mathbb{R}$ and $y \in \Gamma = \mathbb{R}_+$. Then its dual cone is $\Gamma^\circ = \mathbb{R}_{\geq 0}$. By Theorem 5.1.16, $WF_A(\Theta) \subset \mathbb{R} \times \mathbb{R}_+$. It is obvious that the distribution has singularities only at $x = 0$. Therefore,

$$WF_A((x + i0)^{\frac{2-n}{2}}) = \{(0, \tau) : \tau > 0\}.$$

Similarily,

$$WF_A((x - i0)^{\frac{2-n}{2}}) = \{(0, \tau) : \tau < 0\}.$$ □

Example 5.1.18. Let $\Theta = \ln(x + i0)$ which is a boundary value of holomorphic function $\ln(x + iy)$ for $y > 0$. Since logarithm grows slower than any negative power of $|y|$, the limit $y \to 0$ is a distribution on $\mathbb{R}$. It follows from Theorem 5.1.16 that

$$WF_A(\ln(x + i0)) = \{(0, \tau) : \tau > 0\}.$$

Likewise we have that,

$$WF_A(\ln(x - i0)) = \{(0, \tau) : \tau < 0\}. \qquad \square$$

Example 5.1.19. Let $\Theta = {}_2F_1(x + i0)$, the boundary value of the holomorphic function ${}_2F_1(x + iy)$ for $y > 0$. We have proved in Appendix A that it is a distribution which has analytic singularity at $x = 1$. As a result of Theorem 5.1.16, the analytic wavefront set is

$$WF_A({}_2F_1(x + i0)) = \{(1, \tau) : \tau > 0\},$$

and it also follows that

$$WF_A({}_2F_1(x - i0)) = \{(1, \tau) : \tau < 0\}. \qquad \square$$

Let $P(x, D) = \sum\limits_{|t|\leq m} a_t(x)D^\alpha$ be a differential operator on X with analytic coefficients. Then we have that

$$WF_A(P(x, D)\Theta) \subset WF_A(\Theta).$$

The following theorem is a converse to the above statement which can be found in [H90].

Theorem 5.1.20. *If P(x,D) is a differential operator of order m with real analytic coefficients in X, then*

$$WF_A(\Theta) \subset WF_A(Pf) \cup \mathrm{Char}(P),$$

where the characteristic set of P is defined by

$$\mathrm{Char}P = \{(x, \xi) \in T^*(X) \setminus 0 : P_s(x, \xi) := \sum_{|\alpha|=s} a_\alpha \xi^\alpha = 0\}.$$

Consider the differential operator $P(x, D)$ in a manifold X with real analytic co-eficients,. In local coordinates, the principle symbol is $P_s = \sum\limits_{|\alpha|=s} a_\alpha \xi^t$. We say that the curve

$(x(t), \xi(t))$ in $T^*(\mathrm{dS}^n)$ is a bicharacteristic strip if $P_s(x(t), \xi(t)) = 0$ for all with initial data $(x_0, \xi_0) \in \mathrm{CharP_s}$ and satisfies Hamiltonian equations defined as:

$$\frac{dx}{dt} = \frac{\partial P_s(x, \xi)}{\partial \xi}, \qquad \frac{d\xi}{dt} = -\frac{\partial P_s(x, \xi)}{\partial x}.$$

Let S be a closed conic set in $T^*(X)$. We say that it is *invariant under the Hamiltonian vector field of P_s* if $S \subseteq \mathrm{CharP_s}$ and for a bicharacteristic strip $(x(t), \xi(t))$ passing through $(x_0, \xi_0) \in S$, then $(x(t), \xi(t))$ must lie in S for all t.

The following result can be found in [H71].

Theorem 5.1.21 (Propagation of Singularities)**.** *Let P be a differential operator with analytic coefficients and P_s be its real principle symbol. If $\Theta \in \mathcal{D}'(X)$ and $P\Theta = f$, it follows that $WF_A(\Theta) \setminus WF_A(f)$ is invariant under the Hamiltonian vector field of P_s when $\partial P_s(x, \xi)/\partial \xi \neq 0$.*

We say that a curve $(x(t), \xi(t))$ is a *null geodesic strip* if $[\dot{x(t)}, \dot{x(t)}] = 0$ and $\xi(t)$ is the dual of $\dot{x}(t)$. The following proposition is a well known fact. The projection on dS^n of the curve $(x(t), \xi(t))$ is a *bicharacteristics curve for P* if it is given by $P_s = 0$ and satisfies the Hamiltonian equations.

Proposition 5.1.22. *On dS^n, the bicharacteristics curve for $\Delta - m^2$ are exactly the null geodesic strip for dS^n.*

Proof. In the coordinates given by Eq. (2.1.2) the principle symbol for $\Delta - m^2$ is given as

$$P(x, \xi) = -\xi_0^2 + \frac{1}{\cosh^2(x_0)}(\xi_1^2 + ... + \xi_n^2).$$

Let $(x(\tau), \xi(t))$ be a bicharacteristic curve. Then

$$\frac{\partial x_0(\tau)}{\partial \tau} = -2\xi_0, \qquad \frac{\partial \xi_0(\tau)}{\partial \tau} = 2\frac{\sinh(x_0)}{\cosh^3(x_0)}(\xi_1^2 + ... + \xi_n^2),$$

$$\frac{\partial x_i(\tau)}{\partial \tau} = \frac{1}{\cosh^2(x_0(\tau))}2\xi_i, \qquad \frac{\partial \xi_i(\tau)}{\partial \tau} = 0.$$

The condition that $P(x(\tau), \xi(\tau)) = 0$ implies that $\xi_1^2 + ... + \xi_n^2 = \cosh^2(x_0)\xi_0^2$. Using this we obtain that

$$g_{x(\tau)}(\dot{x}(t), \dot{x}(t)) = -4\xi_0^2 + 4(\xi_1^2 + ... + \xi_n^2) = 0.$$

Hence, $(x(\tau), \xi(\tau))$ is a null strip. Again from the Hamiltonian equations we obtain that the curve $(x(\tau), \xi(\tau))$ satisfies the geodesic equation given in these coordinates as follows:

$$\ddot{x}_0 + \sum_{i=1}^{n} \ddot{x}_i + \cosh(x_0)\sinh(x_0)\dot{x}_i\dot{x}_i + \frac{4\sinh(x_0)}{\cosh^3(x_0)}\dot{x}_0\dot{x}_i = 0.$$

Thus, $x(t)$ is a null geodesic proving the proposition.

□

5.2. Wavefront set of Ψ_x^λ and $\widetilde{\Psi}_x^\lambda$

In this section we will state one of the main theorem and its implications.

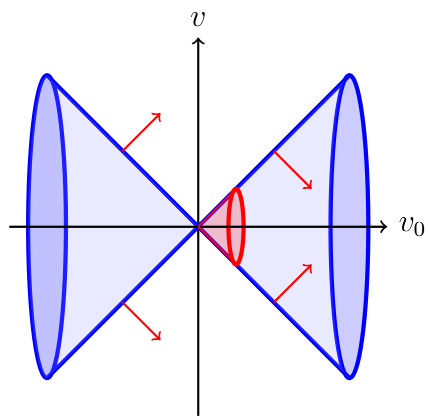
Theorem 5.2.1. *Let* $\Psi_x^\lambda = \lim_{t\to\pi/2^-} \Psi_\lambda(g.z_t, y)$ *and* $\widetilde{\Psi}_x^\lambda = \lim_{t\to\pi/2^-} \Psi_\lambda(g.\bar{z}_t, y)$, *where* $x = g \cdot e_n$ *and* $\lambda \in \mathbb{C} \setminus (\{\rho + \mathbb{N}\} \cup \{-\rho - \mathbb{N}\})$. *Then the analytic wavefront sets of these distributions are given by*

$$WF_A(\Psi_x^\lambda) = \{(x + v, \tau(-v_0, v_1, ..., v_{n-1}), v_0 > 0\} \cup$$
$$\{(x + v, \tau(v_0, -v_1, ..., -v_{n-1})), v_0 < 0\} \cup \{(x, v) : v_0 < 0\};$$

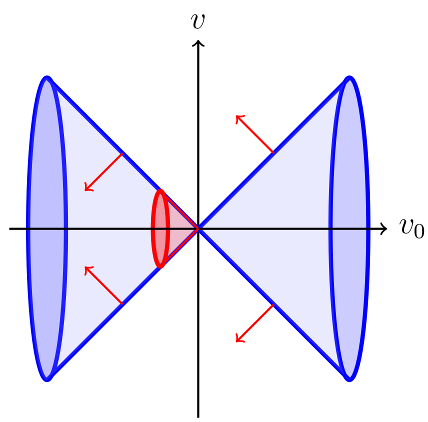
$$WF_A(\widetilde{\Psi}_x^\lambda) = \{(x + v, \tau(v_0, -v_1, ..., -v_{n-1}), v_0 > 0\} \cup$$
$$\{(x + v, \tau(-v_0, v_1, ..., v_{n-1})), v_0 < 0\} \cup \{(x, v) : v_0 > 0\},$$

for $v \in \{T_x(\mathrm{dS}^n) \mid [v-x, v-x] = 0\}$.

In Fig. 5.1 we can see the analytic wavefront set in $T_x\mathrm{dS}^n$.



(a) Analytic wavefront set of $\widetilde{\Psi}^{\lambda}_{x}$.



(b) Analytic wavefront set of Ψ^{λ}_{x} .

Figure 5.1. In this figure, the tangent space at a point x has been identified with its cotangent space at x. The blue region is the light cone of 0 and, the red arrows and red region are in the cotangent space at that point. The light cone together with red arrows and red region is the analytic wavefront set.

Proof. First step is to find the analytic wavefront sets of $\Psi^{\lambda}_{e_n}$ and $\widetilde{\Psi}^{\lambda}_{e_n}$. Since the distributions are solutions of $P = \Delta_y - m^2$, therefore, we have that

$$WF_A(\Psi^{\lambda}_{x}), WF_A(\widetilde{\Psi}^{\lambda}_{x}) \subset \mathrm{Char}P$$

where

$$\mathrm{Char}P = \{(x, \xi) \in T^*(X) \setminus 0, P_n(x, \xi) = 0\},$$

and P_n is the principle symbol of the differential operator P.

(1) Let $U = U_{e_n}$ be the local chart around e_n and the co-ordinate map be the exponential map:

$$y = \mathrm{Exp}_{e_n}(v) = C([v,v])e_n + S([v,v])v.$$

As we know that the singularities of $\Psi^{\lambda}_{e_n}$ and $\widetilde{\Psi}^{\lambda}_{e_n}$ lie on the boundary of the light cone of e_n, it is enough to calculate the wavefront set in U.

Now consider the map $f : U \to \mathbb{R}$ defined by

$$f(v) = \tfrac{1+C[v,v]}{2}.$$

The distribution $\Psi^{\lambda}_{e_n}$ is the distribution ${}_2F_1(f(v) - i0)$ in the open set $U|_{v_0>0}$ and is ${}_2F_1(f(v)+i0)$ in $U|_{v_0<0}$.

The differential operator P in U is given by

$$\Delta - m^2 = -\frac{\partial^2}{\partial v_0^2} + \sum_{i=1}^{n-1} \frac{\partial^2}{\partial v_i^2} - m^2.$$

Moreover, the principle symbol of the differential operator P is $P_n = \xi_0^2 - \sum_{i=1}^{n-1} \xi_i^2$. Therefore the analytic wavefront set lies in the set $\{(v,\xi) : v \in \mathbb{L}_{n-1}, \xi \in \mathbb{L}^*_{n-1}\}$, where $\mathbb{L}_{n-1}$ is the light cone in $U \subset \mathbb{R}^{1,n-1}$ and $\mathbb{L}^*_{n-1}$ is the dual of the light cone.

(2) The tangent map of f is

$$df_v = \frac{-S[v,v]}{4}\begin{bmatrix} -2v_0 & 2v_1 & ... & 2v_n \end{bmatrix}, \quad v \in U.$$

For $\eta \in \mathbb{R}$,$v \in U$, if ${}^t df_v(\eta) = 0$ then either $\eta = 0$ or $S[v,v] = 0$. Now, using the relation that $S(z^2) = \sin z/z$ when $z \neq 0$, we obtain that $S[v,v] = 0$ when $[v,v] = m^2\pi^2$ for $m \in \mathbb{Z} \setminus 0$. Such a v does not belong to the set U. Thus the set of normals of f is $N_f = \{(f(v), 0) : v \in U\}$.

(3) In this step we will calculate the singularities when $v \neq 0$. We will write $v = (v_0, \mathbf{v})$. Consider the distributions ${}_2F_1(x+i0)$ and ${}_2F_1(x-i0)$. Then from Example 5.1.19, it follows $N_f \cap WF_A({}_2F_1(x-i0)) = \emptyset$ and $N_f \cap WF_A({}_2F_1(x+i0)) = \emptyset$. As a result of Theorem 5.1.14, the distribution $\Psi_{e_n}^{\lambda}$ is the pullback of the distributions ${}_2F_1(x-i0)$ and ${}_2F_1(x+i0)$ under the restriction of f at $U|_{v_0>0}$ and $U|_{v_0<0}$, respectively. Consequently, in $U_{v_0>0}$

$$WF_A({}_2F_1(f(v)-i0)) \subseteq \{(v, {}^t df_v(\Phi)) : v_0 > 0, (f(v), \Phi) \in WF_A({}_2F_1(x-i0))\};$$

and in $U|v_0 < 0$,

$$WF_A({}_2F_1(f(v)+i0)) \subseteq \{(v, {}^t df_v(\Phi)) : v_0 < 0, (f(v), \Phi) \in WF_A({}_2F_1(x+i0))\}.$$

That is ,

$$WF_A({}_2F_1(f(v)-i0)) = \{(v,\xi) : [v,v] = 0, \xi = \tau(-v_0, \mathbf{v}), \tau > 0, v_0 > 0\},$$

and

$$WF_A({}_2F_1(f(v)+i0)) = \{(v,\xi) : [v,v] = 0, \xi = \tau(v_0, -\mathbf{v}), \tau > 0, v_0 < 0\}.$$

We get the equality since the analytic wavefront set cannot be empty as the points $[v,v] = 0$ lies in analytic singular support of ${}_2F_1(f(v)-i0)$ and ${}_2F_1(f(v)+i0)$ in their respective domains.

(4) Now that we have calculated wavefront set of $\Psi_{e_n}^{\lambda}$ when $v \neq 0$. The next step is to calculate at $v = 0$. For that we will use Propagation of Singularity theorem, which says that analytic wavefront set is invariant under Hamiltonian P_n when $\frac{\partial P_n}{\partial \xi} \neq 0$. We have that $\frac{\partial P_n}{\partial \xi} = 0$ only if $\xi = \mathbf{0}$. Hence, we can apply Theorem 5.1.21. Now the Hamiltonian equations in the local coordinates are

$$\frac{\partial v}{\partial t} = \frac{\partial P_n}{\partial \xi}, \qquad \frac{\partial \xi}{\partial t} = -\frac{\partial P_n}{\partial v}.$$

That is, for $\xi \in \mathbb{L}^*_{n-1}$

$$\dot{v_0} = 2\xi_0, \qquad \dot{\xi_0} = 0$$

$$\dot{v_i} = -2\xi_i, \qquad \dot{\xi_i} = 0, \quad \text{for } i = 1, ..., n.$$

which gives us,

$$v_0(t) = 2\xi_0 t,\ v_i(t) = -2\xi_i t; \quad \xi(t) = const.$$

with $v(0) = 0$. That is, $v(t)$ lies on the light cone of 0. From what we have calculated in step (3), choose $\tau = 1$ then $\xi_0 = -v_0 < 0$, $\xi_i = v_i$ when $v_0 > 0$ and $\xi_0 = v_0 < 0$, $\xi_i = -v_i$ when $v_0 < 0$. This says that $\xi_0 < 0$ and thus the null geodesic $v(t)$ is the past directed curve. At $t = 0$, $(0, (\dot{v_0}, -\dot{\mathbf{v}}))$ must be in the wavefront set for all null geodesics $v(t)$ satisfying the Hamiltonian equations and fitting in what we have calculated in step (3). Thus so far what we have calculated is

$$WF_A((\mathrm{Exp}_{e_n}^{-1})^* \Psi^\lambda_{e_n}) = WF_A\left(\lim_{t\to\pi/2} {}_2F_1\left(\frac{1+[z_t, \mathrm{Exp}_{e_n}(v)]}{2}\right)\right)$$

$$= \{(0, \tau v) : v_0 < 0\} \cup \{(v, \tau(-v_0, \mathbf{v}), v_0 > 0\}) \cup \{(v, \tau(v_0, -\mathbf{v}), v_0 < 0\};$$

for $v \in \mathbb{L}_{n-1}$ and $\tau > 0$.

(5) Now that the wavefront set has been calculated in local coordinates, we pull the wavefront set back to the de Sitter space. If $v \in \mathbb{L}_{n-1}$, then $y = C[v,v]e_n + S[v,v]v = e_n + v$ and $[y - e_n, y - e_n] = 0$ which implies that y lies on the light cone of e_n. We now conclude that the wavefront of $\Psi^\lambda_{e_n}$ is given by

$$WF_A(\Psi^\lambda_{e_n}) = (\mathrm{Exp_{e_n}})^* WF_A((\mathrm{Exp_{e_n}}^{-1})^* \Psi^\lambda_{e_n}).$$

That is, for all $v \in \mathbb{L}_{n-1}$ and $\tau > 0$,

$$WF_A(\Psi^\lambda_{e_n}) = \{(e_n, v) : v_0 > 0\} \cup \{(e_n + v, \tau(-v_0, v_1, ..., v_{n-1}), v_0 > 0\} \cup$$

$$\{(e_n + v, \tau(v_0, -v_1, ..., -v_{n-1})), v_0 < 0\}.$$

(6) Lastly, consider $\Psi^\lambda_x = \lim_{t \to \pi/2^-} \Psi_\lambda(g \cdot z_t, y)$. For $x = g \cdot e_n$, define a map $l_g : \mathrm{dS}^n \to \mathrm{dS}^n$ by $l_g(y) = g^{-1} \cdot y$. Since l_g is an analytic diffeomorphism, we have that dl_g is an isomorphism of tangent spaces. Therefore, $N_{l_g} = \{(y, 0) : y \in \mathrm{dS}^n\}$ and the pull back of the distribution $\Psi^\lambda_{e_n}$ under the map l_g is Ψ^λ_x. Thus, $WF_A(\Psi^\lambda_x) = l_g^* WF_A(\Psi^\lambda_{e_n})$. That is, $(y, \xi) \in WF_A(\Psi^\lambda_x)$ if $(l_g(y), {}^t dl_{g^{-1}} \xi) \in WF_A(\Psi^\lambda_{e_n})$. This implies that $y = x + v$ for $v \in \mathbb{L}_{n-1}$ as the G acts transitively on light cone and $({}^t dl_{g^{-1}} \xi) = g \cdot \xi$ is also on the dual light cone, as a result we obtain:

$$WF_A(\Psi^\lambda_x) = \{(x, v) : v_0 < 0\} \cup \{(x + v, \tau(-v_0, v_1, ..., v_{n-1}), v_0 > 0\} \cup$$

$$\{(x + v, \tau(v_0, -v_1, ..., -v_{n-1})), v_0 < 0\}.$$

for $v \in \mathbb{L}_{n-1}$.

(7) Finally, in local coordinates $\widetilde{\Psi}^\lambda_{e_n}$ is the distribution ${}_2F_1(f(v)+i0)$ in the open set $U|_{v_0>0}$ and as ${}_2F_1(f(v) - i0)$ in $U|_{v_0<0}$. Following all the steps above we obtain for $v \in \mathbb{L}_{n-1}$,

$$WF_A(\widetilde{\Psi}^\lambda_{e_n}) = \{(e_n, v) : v_0 > 0\} \cup \{(e_n + v, \tau(v_0, -v_1, ..., -v_{n-1}), v_0 > 0\} \cup$$

$$\{(e_n + v, \tau(-v_0, v_1, ..., v_{n-1})), v_0 < 0\},$$

and

$$WF_A(\widetilde{\Psi}^\lambda_x) = \{(x, v) : v_0 > 0\} \cup \{(x + v, \tau(v_0, -v_1, ..., -v_{n-1}), v_0 > 0\} \cup$$

$$\{(x + v, \tau(-v_0, v_1, ..., v_{n-1})), v_0 < 0\}.$$

Thus, we have proved the theorem. $\square$

Using the analytic wavefront sets, we can prove that the distributions can not vanish on any non-empty open set O of dS^n.

The following theorem is due to Strohmaier, Verch and, Wollenberg, see [SVW02, Proposition 5.3].

Theorem 5.2.2. *Let X be a real analytic manifold and $\Theta \in \mathcal{D}'(X)$. If $WF_A(\Theta) \cap -WF_A(\Theta) = \emptyset$ then for an open region O in X*

$$\Theta|_O \Rightarrow \Theta = 0,$$

where $-WF_A(\Theta) = \{(x, \xi) : (x, -\xi) \in WF_A(\Theta)\}$.

This theorem is not true in the case of smooth wavefront set. Consider the distribution Θ from Example 5.1.12. The wavefront set of Θ satisfies the condition that $WF(\Theta) \cap -WF(\Theta) = \emptyset$. Obviously, Θ is not a zero distribution however, it is zero in the open region $(-\infty, 0)$.

Since the wavefront sets of the distributions Ψ_x^λ is such that $WF_A(\Psi_x^\lambda) \cap -WF_A(\Psi_x^\lambda) = \emptyset$ for all x, which is same for $\widetilde{\Psi}_x^\lambda$, immediately as a corollary we obtain that,

Corollary 5.2.3. *The distributions Ψ_x^λ, $\widetilde{\Psi}_x^\lambda$ can not vanish on any open regions of* dS^n*.*

The following theorem will be key when studying the spherical distributions which distinguishes them.

Theorem 5.2.4. *The wavefront set of the sum $\Psi_x^\lambda + \widetilde{\Psi}_x^\lambda$ (see Fig. 5.2) is given as follows:*

$$WF_A(\Psi_x^\lambda + \widetilde{\Psi}_x^\lambda) = WF_A(\Psi_x^\lambda) \cup WF_A(\widetilde{\Psi}_x^\lambda).$$

Proof. As we have $\overline{\Psi_x^\lambda + \widetilde{\Psi}_x^\lambda} = \Psi_x^\lambda + \widetilde{\Psi}_x^\lambda$, the proof of the theorem follows immediately from the below lemma. $\square$

Lemma 5.2.5. *Let Θ be a distribution. Then*

$$(x,\xi) \in WF_A(\Theta) \iff (x,-\xi) \in WF_A(\overline{\Theta}).$$

Proof. Let $(x_0,\xi_0) \notin WF_A(\Theta)$. Then there exists a neighbourhood U around x_0 and a conic neighbourhood around ξ_0 and a sequence $\chi_N \in C_C^\infty(U)$ from Lemma 5.1.9 such that

$$|\widehat{(\chi_N\Theta)}(\xi)| \leq C^{N+1}((N+1)/|\xi|)^N, \quad \xi \in \Gamma.$$

But $\widehat{(\chi_N\Theta)}(\xi) = \Theta(\chi_N e^{2\pi i[x,\xi]}) = \overline{\overline{\Theta}(\overline{\chi}_N e^{-2\pi i[x,\xi]})}$. Hence, we have the decay

$$|\widehat{(\overline{\chi}_N\overline{\Theta})}(-\xi)| \leq C^{N+1}((N+1)/|\xi|)^N \quad -\xi \in -\Gamma,$$

and vice-versa proving the lemma. $\square$

From Theorem 4.2.2 we obtain the following:

Corollary 5.2.6. *The wavefront set of the distribution $\Psi_x^\lambda - \widetilde{\Psi}_x^\lambda$ (see Fig. 5.2) is*

$$WF_A(\Psi_x^\lambda - \widetilde{\Psi}_x^\lambda) = WF_A(\Psi_x^\lambda) \cup WF_A(\widetilde{\Psi}_x^\lambda).$$

Moreover,

1. *For odd n and $c_1 = (-1)^{\frac{n+1}{2}} \frac{2i\Gamma(n/2)\Gamma((n-2)/2)}{\Gamma(\rho+\lambda)\Gamma(\rho-\lambda)}$, the distribution is given by: $(\Psi_x^\lambda - \widetilde{\Psi}_x^\lambda)(y) =$*

$$c_1 \begin{cases} 0 & \text{if } y \notin \overline{\Gamma(x)} \\ \left(\frac{[x,y]-1}{2}\right)^{\frac{2-n}{2}} {}_2F_1(1/2-\lambda, 1/2+\lambda; \frac{4-n}{2}; \frac{1-[x,y]}{2}) & \text{if } y \in \Gamma^+(x) \\ -\left(\frac{[x,y]-1}{2}\right)^{\frac{2-n}{2}} {}_2F_1(1/2-\lambda, 1/2+\lambda; \frac{4-n}{2}; \frac{1-[x,y]}{2}) & \text{if } y \in \Gamma^-(x). \end{cases}$$

2. *For even n and $c_2 = (-1)^{\frac{n}{2}} \frac{2\pi i}{\Gamma(1/2+\lambda)\Gamma(1/2-\lambda)}$, the distribution is given by: $(\Psi_x^\lambda - \widetilde{\Psi}_x^\lambda)(y) =$*

$$c_2 \begin{cases} 0 & \text{if } y \notin \overline{\Gamma(x)} \\ {}_2F_1(\rho+\lambda, \rho-\lambda; n/2; \frac{1-[x,y]}{2}) & \text{if } y \in \Gamma^+(x) \\ -{}_2F_1(\rho+\lambda, \rho-\lambda; n/2; \frac{1-[x,y]}{2}) & \text{if } y \in \Gamma^-(x). \end{cases}$$

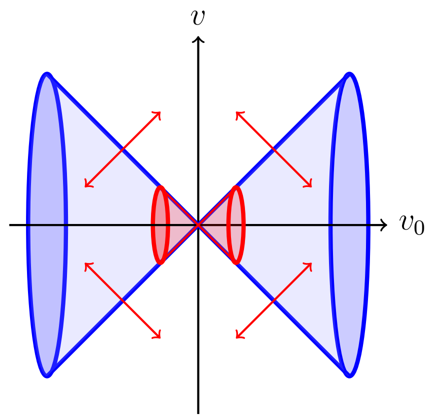


Figure 5.2. Analytic wavefront set of $\Psi_x^\lambda + \widetilde{\Psi}_x^\lambda$ and $\Psi_x^\lambda - \widetilde{\Psi}_x^\lambda$ as they coincide on $T_x(\mathrm{dS}^n)$.

Proof. We have that $WF_A(\Psi_x^\lambda - \widetilde{\Psi}_x^\lambda) \subset WF_A(\Psi_x^\lambda) \cup WF_A(\widetilde{\Psi}_x^\lambda)$ and is a not empty set as $\Psi_x^\lambda - \widetilde{\Psi}_x^\lambda$ has singularities on the boundary of $\Gamma(x)$. Observe that $\overline{\Psi_x^\lambda - \widetilde{\Psi}_x^\lambda} = -(\Psi_x^\lambda - \widetilde{\Psi}_x^\lambda)$. Therefore, it is enough to prove that if $(y, \xi) \in WF_A(\Theta) \iff (y, \xi) \in WF_A(-\Theta)$. Let $(y_0, \xi_0) \notin WF_A(\Theta)$. Let $U, \Gamma, \{\chi_N\}$ be as in Lemma 5.1.9. Then

$$|\widehat{\chi_N(-\Theta)}(\xi)| = |\widehat{(\chi_N\Theta)}(\xi)| \leq C^{N+1}((N+1)/|\xi|)^N, \quad \xi \in \Gamma.$$

Therefore, combining this with Lemma 5.2.6 we obtain the corollary. ☐

5.3. Spherical Distributions

Let (G_0, H_0) be a symmetric space and Θ be a distribution on G_0/H_0. Then G_0 acts on Θ by

$$\pi_{-\infty}(g)\Theta(\varphi) = \Theta(\pi_\infty(g^{-1})\varphi), \quad \varphi \in \mathcal{D}(G_0/H_0),$$

where $\pi_\infty(g)\varphi(x) = \varphi(g^{-1} \cdot x)$.

Definition 5.3.1. We say that a distribution Θ is H_0-invariant if $\pi_{-\infty}(h)\Theta = \Theta$ for all $h \in H_0$.

Definition 5.3.2. A distribution Θ is said to be a spherical distribution if it is H_0-invariant eigendistribution of the Laplace-Beltrami operator Δ on G_0/H_0. This space is denoted by $\mathcal{D}_\lambda^{H_0}(\mathrm{dS}^n)$.

Let $G' = \mathrm{O}_{1,n}(\mathbb{R})$ and $H' = \mathrm{O}_{1,n-1}(\mathbb{R})$ is the closed subgroup of G'. Let $\mathcal{D}_\lambda^{H'}(\mathrm{dS}^n)$ be the space of spherical distributions on the de Sitter space with $\Delta(\Theta) = \lambda\Theta$. Then according to [D08, Theorem 9.2.5]

Theorem 5.3.3. *The dimension of $\mathcal{D}_\lambda^{H'}(\mathrm{dS}^n)$ is 2.*

Remark 5.3.4. From Remark 2.1.4 we obtain that $\pi_{-\infty}(\Lambda_1)(\Psi_{e_n}^\lambda) = \widetilde{\Psi}_{e_n}^\lambda$, $(\Lambda_1)(\Psi_{-e_n}^\lambda) = \widetilde{\Psi}_{-e_n}^\lambda$ and vice-versa. Also, $\pi_{-\infty}(\Lambda_1\Lambda_2)(\Psi_{e_n}^\lambda) = \widetilde{\Psi}_{e_n}^\lambda$, $\pi_{-\infty}(\Lambda_1\Lambda_2)(\Psi_{-e_n}^\lambda) = \widetilde{\Psi}_{-e_n}^\lambda$ and vice-versa. So, they are not H' invariant. However, the sums $\Psi_{e_n}^\lambda + \widetilde{\Psi}_{e_n}^\lambda$ and $\Psi_{-e_n}^\lambda + \widetilde{\Psi}_{-e_n}^\lambda$ are H'-invariant.

We will now restate the main theorem.

Theorem 5.3.5. *Let $n \geq 2$, then*

1. *The distributions $\Psi_{e_n}^\lambda + \widetilde{\Psi}_{e_n}^\lambda$ and $\Psi_{-e_n}^\lambda + \widetilde{\Psi}_{-e_n}^\lambda$ are H'- invariant spherical distributions and span $\mathcal{D}_{m^2}^{H'}(\mathrm{dS}^n)$, where $m^2 = \rho^2 - \lambda^2$ and, $\lambda \in \mathbb{C} \setminus (\{\rho + \mathbb{N}\} \cup \{-\rho - \mathbb{N}\})$.*

2. *The distributions $\Psi_{e_n}^\lambda + \widetilde{\Psi}_{e_n}^\lambda$ and $\Psi_{-e_n}^\lambda + \widetilde{\Psi}_{-e_n}^\lambda$ are positive definite for $\lambda \in i[0,\infty) \cup (0,\rho)$.*

3. *Moreover, the following holds for a non-zero spherical distribution $\Theta \in \mathcal{D}_{m^2}^{H'}(\mathrm{dS}^n)$:*

   (a) *$WF_A(\Theta) \subset WF_A(\Psi_{e_n}^\lambda + \widetilde{\Psi}_{e_n}^\lambda) \sqcup WF_A(\Psi_{-e_n}^\lambda + \widetilde{\Psi}_{-e_n}^\lambda)$.*

   (b) *If $WF_A(\Theta) = WF_A(\Psi_{e_n}^\lambda + \widetilde{\Psi}_{e_n}^\lambda)$ then there exists a nonzero constant c such that $\Theta = c(\Psi_{e_n}^\lambda + \widetilde{\Psi}_{e_n}^\lambda)$.*

   (c) *If $WF_A(\Theta) = WF_A(\Psi_{-e_n}^\lambda + \widetilde{\Psi}_{-e_n}^\lambda)$ then there is a non zero constant c such that $\Theta = c(\Psi_{-e_n}^\lambda + \widetilde{\Psi}_{-e_n}^\lambda)$.*

Proof.

(1) Clearly, it follows from Theorem 4.2.2 that $\Psi_{e_n}^\lambda + \widetilde{\Psi}_{e_n}^\lambda$ and $\Psi_{-e_n}^\lambda + \widetilde{\Psi}_{-e_n}^\lambda$ are linearly independent distribution with eigenvalue $\rho^2 - \lambda^2$ for the operator Δ. It follows from the

remark above that they are H'-invariant. Hence, they form a basis for $\mathcal{D}^{H'}_{m^2}(\mathrm{dS}^n)$

(2) When $\lambda \in i[0,\infty) \cup (0,\rho)$ the kernels Ψ_λ and $\widetilde{\Psi}_\lambda$ are positive definite from Theorem 3.1.4 and thus the limits are also positive definite and the sums are also positive definite.

(3) The analytic wavefront sets of $\Psi^\lambda_{e_n} + \widetilde{\Psi}^\lambda_{e_n}$ and $\Psi^\lambda_{-e_n} + \widetilde{\Psi}^\lambda_{-e_n}$ are disjoint as $\Psi^\lambda_{e_n} + \widetilde{\Psi}^\lambda_{e_n}$ has singularities at $y_n = 1$ and $\Psi^\lambda_{-e_n} + \widetilde{\Psi}^\lambda_{-e_n}$ has singularities at $y_n = -1$. The rest of the proof follows similar to that of part (3) of Theorem 5.3.3. □

We will now study the case when $G = \mathrm{SO}_{1,n}(\mathbb{R})_e$ and $H = \mathrm{SO}_{1,n-1}(\mathbb{R})$ is its closed subgroup. Then according to Oshima and Sekiguchi [OSe80], the dimension of the H-invariant eigenspaces of Laplace-Beltrami operator Δ with eigenvalue $\lambda \in \mathbb{C}$ (which is denoted by $\mathcal{D}^H_\lambda(\mathrm{dS}^n)$) turns out to be 4. Therefore, we obtain the following theorem:

Theorem 5.3.6. *Let $n \geq 2$, then*

1. *The distributions $\Psi^\lambda_{e_n}$, $\widetilde{\Psi}^\lambda_{e_n}$, $\Psi^\lambda_{-e_n}$ and $\widetilde{\Psi}^\lambda_{-e_n}$ are spherical distributions. They span $\mathcal{D}^H_{m^2}(\mathrm{dS}^n)$ where, $m^2 = \rho^2 - \lambda^2$ and $\lambda \in \mathbb{C} \setminus \{(\rho + \mathbb{N}) \cup (-\rho - \mathbb{N})\}$.*
2. *The distributions $\Psi^\lambda_{e_n}$, $\widetilde{\Psi}^\lambda_{e_n}$, $\Psi^\lambda_{-e_n}$ and $\widetilde{\Psi}^\lambda_{-e_n}$ are positive definite for $\lambda \in i\mathbb{R} \cup (-\rho, \rho)$.*
3. *Moreover, the following holds for a non-zero spherical distribution $\Theta \in D^H_{m^2}(\mathrm{dS}^n)$:*
   (a) *$WF_A(\Theta) \subset WF_A(\Psi^\lambda_{e_n}) \sqcup WF_A(\widetilde{\Psi}^\lambda_{e_n}) \sqcup WF_A(\Psi^\lambda_{-e_n}) \sqcup WF_A(\widetilde{\Psi}^\lambda_{-e_n})$.*
   (b) *If $WF_A(\Theta)$ coincides with any of the analytic wavefront set of the distribution $\Psi^\lambda_{e_n}$, $\widetilde{\Psi}^\lambda_{e_n}$, $\Psi^\lambda_{-e_n}$ and $\widetilde{\Psi}^\lambda_{-e_n}$, then it has to be a constant multiple of that distribution.*

Proof.

(1) We obtain from Theorem 4.2.2 that all of them are eigen-distribution of Δ. It was proven in the discussion before Theorem 4.2.2 that $\Psi^\lambda_{e_n}$ and $\widetilde{\Psi}^\lambda_{e_n}$ are H-invariant. As H

also fixes $-e_n$, we can follow the same arguments by allowing $z_t = i\cos(t)e_0 - \sin(t)e_n$ and then taking limit as t goes to $\pi/2$. Clearly, all the four distributions are linearly independent. Thus, they span $\mathcal{D}^H_{m^2}(\mathrm{dS}^n)$.

(2) Notice that $\Psi_\lambda = \Psi_{-\lambda}$ and $\widetilde{\Psi}_\lambda = \widetilde{\Psi}_{-\lambda}$.So, when $\lambda \in i[0, \infty \cup (0, \rho)$ the kernels Ψ_λ and $\widetilde{\Psi}_\lambda$ are positive definite from Theorem 3.1.4 and thus the limits are also positive definite.

(3) The first part follows from Theorem 5.1.13. From Theorem 5.2.1 we obtain that all the four distributions have disjoint wavefront sets. Suppose that $WF_A(\Theta) = WF_A(\Psi^\lambda_{e_n})$. Let $\Theta = a\Psi^\lambda_{e_n} + b\widetilde{\Psi}^\lambda_{e_n} + c\Psi^\lambda_{-e_n} + d\widetilde{\Psi}^\lambda_{e_n}$ and without loss of generality let $b \neq 0$. Then $\widetilde{\Psi}^\lambda_{e_n} = \frac{1}{b}(\Theta - a\Psi^\lambda_{e_n} + c\Psi^\lambda_{-e_n} + d\widetilde{\Psi}^\lambda_{e_n})$ and from Theorem 5.1.13 we obtain that $WF_A(\widetilde{\Psi}^\lambda_{e_n}) \subset WF_A(\Psi^\lambda_{e_n}) \sqcup WF_A(\Psi^\lambda_{-e_n}) \sqcup WF_A(\widetilde{\Psi}^\lambda_{e_n})$. Then we arrive at a contradiction and $b = 0$. Similarly, we obtain that $c = d = 0$. We can then repeat the same argument for rest of the distributions and have established the last claim. $\square$

Thus, in the case of H'-invariant spherical distributions, we were able to distinguish between the basis elements by looking at their singular support. However, $\Psi^\lambda_{e_n}$ and $\widetilde{\Psi}^\lambda_{e_n}$ have the same singular support and the same is true for $\Psi^\lambda_{-e_n}$ and $\widetilde{\Psi}^\lambda_{-e_n}$. In this case, we look at their wavefront sets to distinguish between them.

Chapter 6. Future Work

Let G be a Lie group and H a closed subgroup. Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of G. Denote by $\mathcal{H}_\pi^\infty$, the space of smooth vectors and by $\mathcal{H}^{-\infty}$ the space of continuous conjugate linear maps $\mathcal{H}^\infty \to \mathbb{C}$, the space of distribution vectors. The group G leaves $\mathcal{H}^\infty$ invariant and then defines a representation $\pi^{-\infty}$ by duality. For $\eta \in \mathcal{H}^{-\infty}$ and $\phi \in C_c^\infty(G)$ it is well known that $\pi^{-\infty}(\phi)\eta = \int_G \phi(x)\pi^{-\infty}(x)\eta dx$ is in $\mathcal{H}^\infty$. Hence,

$$\Phi_\eta(\phi) = \eta(\pi^{-\infty}(\phi)\eta) = \langle \eta, \pi^{-\infty}(\phi)\eta \rangle \tag{6.0.1}$$

is well defined positive-definite distribution on G. If η is H-invariant then Φ_η defines a distribution on G/H (see [NÓ18] for more discussion). If π is irreducible then, Φ_η is an eigen-distribution for the algebra of differential operators coming from the center of $U(\mathfrak{g})$. It is a natural question to study the wavefront set of those distributions.

We will now assume that G is connected, semisimple and linear. Let K be the maximal compact subgroup and assume H is symmetric. Let θ be the corresponding Cartan involution and τ the involution corresponding to H. We assume that θ and τ commute. Our assumption is that G/H is casual (see [HÓ97]). This is the setup in the research project by Neeb-Ólafsson and their collaborators. We assume that $v \in \mathcal{H}_\pi$ is $K-$finite. Then $g \to \pi(g)v$ extends as a holomorphic function to the crown Ξ (see [FNÓ23]), where Ξ is an open complex domain in the complexification $G_\mathbb{C}/K_\mathbb{C}$.

The above setup leads to an element $s = \exp(\frac{i\pi}{2}X) \in G_\mathbb{C}$ such that $sK_\mathbb{C}s^{-1} = H_\mathbb{C}$ and $G/H = G.sK_\mathbb{C}$ is on the boundary of the crown and $\exp(itX)K_\mathbb{C}$ belongs to Ξ for $|t| < \pi/2$. Hence, $\pi(\exp(itX))v$ is well defined for $|t| < \pi/2$. The following has been answered for several cases [FNÓ23].

Question 1. Does

$$\lim_{t \to \pi/2} \pi(\exp(itX))v = \eta$$

exists in $\mathcal{H}^{-\infty}$.

So far it has been proven for principle series representations of G when v is K-fixed vector (see [GKÓ04]). In this case η is H-invariant distribution vector. Hence, the next question.

Question 2. What can be said about the wavefront set of Φ_η defined in Eq. (A.0.5), if the limit exists?

If $\mathcal{H}_\pi \subset L^2(G/H)$ (discrete series for G/H) and $\mathrm{pr}_\pi : L^2 \to \mathcal{H}_\pi$ is the orthogonal projection then $\mathrm{pr}_\pi(C_c^\infty(G/H)) \subset \mathcal{H}_\pi^\infty$ and $f \to \mathrm{pr}_\pi(f)(eH)$ is a H-invariant distribution η and hence Φ_η is well defined.

Note that in the case $G \times G/\mathrm{diag}(G) \cong G$ then Φ_η is up to a constant the character of π, $f \to \mathrm{Tr}\pi(f)$. So, the above questions reduces to the wave front set of π as defined by Howe [Ho81].

Question 3. What can be said if G/H has holomorphic discrete series representations?

In the case of dS^2, the authors in [BM04] constructed complex domains in $\mathrm{dS}^n_{\mathbb{C}}$ where the kernels related to discrete series live. We can consider the domains $G \cdot h_t$ and $G \cdot \bar{h_t}$ where $h_t = i\sinh(t)e_1 + \cosh(t)e_n$ and $t > 0$. We obtain that $h_t \to e_n$ as $t \to 0$. So, we can proceed with the same kind of questions as we did in case of principle series spherical distributions.

Question 4. What is the wavefront set of the distribution Ψ_η?

Question 5. In the case of anti-de Sitter space $A\mathrm{dS}^n$, where

$$A\mathrm{dS}^n = SO(2,n)/SO(1,n),$$

is a compactly casual symmetric space (see [FNÓ23]), are the above questions valid?

Let us define the operator T for $f \in C_c^\infty(\mathrm{dS}^n)$ as follows:

$$Tf(x) = \lim_{z \to x} \int_{\mathrm{dS}^n} \Psi_\lambda(z, y) f(y) dy,$$

where Ψ_λ is given Chapter 1. We have that $|Tf(x)| < \infty$ because the limit exists in distribution.

Question 6. What can be said about the L^p estimates? What can be its implications?

Appendix A. Boundary value of Hypergeometric function

For simplicity we will write ${}_2F_1(a,b;c,z) = {}_2F_1(z)$. In this section we will show that ${}_2F_1(x+i0) := \lim_{y\to 0} {}_2F_1(x+iy)$ for $y > 0$ and ${}_2F_1(x-i0) := \lim_{y\to 0} {}_2F_1(x-iy)$ for $y > 0$, are distributions for $a = \rho+\lambda$, $b = \rho-\lambda$ and $c = n/2$. It is a fact that ${}_2F_1(z)$ has a branch cut on $[1,\infty)$. Hence, the convergence for $x < 1$ is uniform on compact sets. The case when $x > 1$ and the growth near $z = 1$ will determine whether it will be a distribution or not.

Theorem A.0.1. *The limit* $\lim_{y\to 0} {}_2F_1(\rho+\lambda, \rho-\lambda, n/2, x \pm iy)$ *for* $y > 0$ *exists in the sense of distributions where for* $\mathrm{Re(z)} > 1$ *the limit converges uniformly on compact sets. For* $1 < x < 2$, *if* n *is odd*

$$\begin{aligned} {}_2F_1(x\pm i0) &= \frac{\Gamma(n/2)\Gamma((2-n)/2)}{\Gamma(1/2+\lambda)\Gamma(1/2-\lambda)} {}_2F_1(\rho+\lambda, \rho-\lambda; \frac{n}{2}; 1-x) \\ &+ e^{\mp i\pi(\frac{2-n}{2})}(x-1)^{\frac{2-n}{2}} \frac{\Gamma(n/2)\Gamma((n-2)/2)}{\Gamma(\rho+\lambda)\Gamma(\rho-\lambda)} {}_2F_1(1/2-\lambda, 1/2+\lambda; \frac{4-n}{2}; 1-x). \end{aligned} \tag{A.0.1}$$

and if n *is even*

$$\begin{aligned} {}_2F_1(x\pm i0) =& \frac{\Gamma(n/2)}{\Gamma(\rho+\lambda)\Gamma(\rho-\lambda)} \sum_{k=0}^{n/2-2} \frac{(-1)^k (n/2-k-2)!(1/2+\lambda)_k(1/2-\lambda)_k}{k!}(1-x)^{k+1-\frac{n}{2}} \\ &+ \frac{(-1)^{\frac{n-2}{2}}\Gamma(n/2)}{\Gamma(1/2+\lambda)\Gamma(1/2-\lambda)} \sum_{k=0}^{\infty} \frac{(\rho+\lambda)_k(\rho-\lambda)_k}{k!(n/2-1+k)!}[\psi(k+1)+\psi(n/2+k) \\ &-\psi(\rho+\lambda+k)-\psi(\rho-\lambda+k)-\ln(x-1)\pm i\pi](1-x)^k, \end{aligned} \tag{A.0.2}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ *Furthermore, the behaviour of the hypergeometric function near* $z = 1$ *as distributions is given as follows: for* $n = 2$,

$${}_2F_1(z) \approx \frac{1}{\Gamma(\rho+\lambda)\Gamma(\rho-\lambda)}(-\ln(1-z)) \tag{A.0.3}$$

This appendix has appeared in the article: G. Olafsson, I. Sitiraju. *Analytic wavefront sets of spherical distributions on the de Sitter space.* arXiv:2309.10685 *and for $n \geq 3$,*

$$ {}_2F_1(z) \approx \frac{\Gamma(n/2)\Gamma((n-2)/2)}{\Gamma(\rho+\lambda)\Gamma(\rho-\lambda)}(1-z)^{\frac{2-n}{2}}. \tag{A.0.4} $$

Proof. Let $n \geq 2$. Suppose that n is odd. Then $c-a-b = \frac{2-n}{2}$ is not an integer. Therefore, for $|z-1| < 1$ and $|\arg(1-z)| < \pi$ we can use the following transformation

$$ \begin{aligned} {}_2F_1(z) &= \tfrac{\Gamma(n/2)\Gamma((2-n)/2)}{\Gamma(1/2+\lambda)\Gamma(1/2-\lambda)}{}_2F_1(\rho+\lambda, \rho-\lambda; \frac{n}{2}; 1-z) \\ &+ (1-z)^{\frac{2-n}{2}} \tfrac{\Gamma(n/2)\Gamma((n-2)/2)}{\Gamma(\rho+\lambda)\Gamma(\rho-\lambda)}{}_2F_1(1/2-\lambda, 1/2+\lambda; \frac{4-n}{2}; 1-z). \end{aligned} \tag{A.0.5} $$

Suppose that $1 < x < 2$ then,

$$ \begin{aligned} {}_2F_1(x \pm i0) &= \tfrac{\Gamma(n/2)\Gamma((2-n)/2)}{\Gamma(1/2+\lambda)\Gamma(1/2-\lambda)}{}_2F_1(\rho+\lambda, \rho-\lambda; \frac{n}{2}; 1-x) \\ &+ e^{\mp i\pi(\frac{2-n}{2})}(x-1)^{\frac{2-n}{2}} \tfrac{\Gamma(n/2)\Gamma((n-2)/2)}{\Gamma(\rho+\lambda)\Gamma(\rho-\lambda)}{}_2F_1(1/2-\lambda, 1/2+\lambda; \frac{4-n}{2}; 1-x). \end{aligned} \tag{A.0.6} $$

For $x \geq 2$ we can use linear transformations of hypergeometric functions to extend ${}_2F_1(x \pm i0)$ analytically.

If n is even, we obtain Eq. (A.0.2) for $1 < x < 2$ from [GS64, Eq 9.7.5, 9.7.6].

Similarly, we can extend ${}_2F_1(x \pm i0)$ for $x > 2$ using the formulae from [GS64, Sec 9.7]

Now let us calculate the behaviour of the hypergeometric function near $x = 1$. Let $n \geq 3$.

We have that for $\mathrm{Re}(c-a-b) = 1-n/2 < 0$ and $x < 1$,

$$ \lim_{x \to 1^-} \frac{{}_2F_1(x)}{(1-x)^{\frac{2-n}{2}}} = \frac{\Gamma(n/2)\Gamma((n-2)/2)}{\Gamma(\rho+\lambda)\Gamma(\rho-\lambda)}. \tag{A.0.7} $$

From Eq. (A.0.6) and Eq. (A.0.2) we obtain that

$$ \lim_{x \to 1^+} \frac{{}_2F_1(x \pm i0)}{(x-1)^{c-a-b}} = e^{\mp i\pi(\frac{2-n}{2})} \frac{\Gamma(n/2)\Gamma((n-2)/2)}{\Gamma(\rho+\lambda)\Gamma(\rho-\lambda)}. \tag{A.0.8} $$

From this we can say that around 1 (see Appendix B),

$$
{}_2F_1(z) \approx \frac{\Gamma(n/2)\Gamma((n-2)/2)}{\Gamma(\rho+\lambda)\Gamma(\rho-\lambda)}(1-z)^{\frac{2-n}{2}}. \tag{A.0.9}
$$

The growth of ${}_2F_1(z)$ near $z=1$ is

$$
|{}_2F_1(z)| \approx \left|\tfrac{\Gamma(n/2)\Gamma((n-2)/2)}{\Gamma(\rho+\lambda)\Gamma(\rho-\lambda)}\right|(|1-z|^{\frac{2-n}{2}}) \leq \text{const. } |\mathrm{y}|^{\frac{2-\mathrm{n}}{2}}.
$$

Hence, it follows from [H90, Theorem 3.1.11] that the limit converges to a distribution.

If $n=2$, then $c=a+b$ and

$$
\lim_{x\to 1^-} \frac{{}_2F_1(x)}{-\ln(1-x)} = \frac{1}{\Gamma(\rho+\lambda)\Gamma(\rho-\lambda)}. \tag{A.0.10}
$$

For $n=2$, the first summation in Eq. (A.0.2) does not appear. Thus, we obtain that

$$
\lim_{x\to 1^-} \frac{{}_2F_1(x\pm i0)}{-\ln(x-1)\pm i\pi} = \frac{1}{\Gamma(\rho+\lambda)\Gamma(\rho-\lambda)}. \tag{A.0.11}
$$

Therefore, around $z=1$

$$
{}_2F_1(z) \approx \frac{1}{\Gamma(\rho+\lambda)\Gamma(\rho-\lambda)}(-\ln(1-z)). \tag{A.0.12}
$$

Since logarithm is an integrable function on compact sets we have that for $n=2$, ${}_2F_1(z)$ is a distribution. $\square$

Appendix B. Distributions: $(x+i0)^{\frac{2-n}{2}}$ and $\log(x+i0)$

Here, we will recall the distributions $(x \pm i0)^{\frac{2-n}{2}}$ and $\ln(x \pm i0)$ (see [GS64]). First for n an odd number, we look at the distributions $x_+^{\frac{2-n}{2}}$ and $x_-^{\frac{2-n}{2}}$. Let $\varphi \in C_c^\infty(\mathbb{R})$. We will look at the case when n is odd dimension. For $n = 3$

$$(x_+^{-\frac{1}{2}}, \varphi) = \int_0^\infty x^{-\frac{1}{2}} \varphi(x) dx, \tag{B.0.1}$$

is the regular distribution. However, for $n \geq 5$, $m = (n-5)/2$ we have

$$(x_+^{\frac{2-n}{2}}, \varphi) = \int_0^\infty x^{\frac{2-n}{2}} \Big[\varphi(x) - \varphi(0) - x\varphi'(0) - ... - \frac{x^m}{(m)!}\varphi^m(0)\Big] dx. \tag{B.0.2}$$

The distribution $x_-^{\frac{2-n}{2}}$ is defined as follows:

$$(x_-^{\frac{2-n}{2}}, \varphi(x)) = (x_+^{\frac{2-n}{2}}, \varphi(-x)). \tag{B.0.3}$$

Now we will look at the case when n is even dimension:

For $k = (n-2)/2$ and k is even,

$$\begin{aligned}(x^{-k}, \varphi) = \int_0^\infty x^{-k} \Big(\varphi(x) + \varphi(-x) \\ -2\Big[\varphi(0) + \frac{x^2}{2!}\varphi''(0) + ... + \frac{x^{k-2}}{(k-2)!}\varphi^{k-2}(0)\Big]\Big) dx.\end{aligned} \tag{B.0.4}$$

For $k = (n-2)/2$ and k an odd number:

$$\begin{aligned}(x^{-k}, \varphi) = \int_0^\infty x^{-k} \Big(\varphi(x) - \varphi(-x) \\ -2\Big[x\varphi'(0) + \frac{x^3}{3!}\varphi'''(0) + ... + \frac{x^{k-2}}{(k-2)!}\varphi^{k-2}(0)\Big]\Big) dx.\end{aligned} \tag{B.0.5}$$

Let us consider the distributions given as follows:

This appendix has appeared in the article: G. Olafsson, I. Sitiraju. *Analytic wavefront sets of spherical distributions on the de Sitter space.* arXiv:2309.10685

$$(x \pm i0)^{\frac{2-n}{2}} = \lim_{y \to 0^+} (x \pm iy)^{\frac{2-n}{2}}.$$

When n is odd,

$$(x + i0)^{\frac{2-n}{2}} = x_+^{\frac{2-n}{2}} + e^{i\pi\frac{2-n}{2}} x_-^{\frac{2-n}{2}}, \tag{B.0.6}$$

$$(x - i0)^{\frac{2-n}{2}} = x_+^{\frac{2-n}{2}} + e^{-i\pi\frac{2-n}{2}} x_-^{\frac{2-n}{2}}. \tag{B.0.7}$$

When n is even, $k = (n-2)/2$ we have

$$(x + i0)^{-k} = x^{-k} - \frac{i\pi(-1)^{k-1}}{(k-1)!}\delta^{k-1}(x), \tag{B.0.8}$$

$$(x - i0)^{-k} = x^{-k} + \frac{i\pi(-1)^{k-1}}{(k-1)!}\delta^{k-1}(x). \tag{B.0.9}$$

Finally, we have the distribution

$$\ln(x \pm i0) = \lim_{y \to 0} \ln(x \pm iy), \tag{B.0.10}$$

where

$$\ln(x \pm i0) = \begin{cases} \ln|x| \pm i\pi & \text{for } x < 0, \\ \ln x & \text{for } x > 0. \end{cases} \tag{B.0.11}$$

Appendix C. Copyright information

Copyright information for Gestur Ólafsson and Iswarya Sitiraju. "Analytic Wavefront Sets of Spherical Distributions on the De Sitter Space". arXiv:2309.10685 used in Chapters 1-5.

ANALYTIC WAVEFRONT SETS OF SPHERICAL DISTRIBUTIONS ON DE SITTER SPACE

GESTUR ÓLAFSSON AND ISWARYA SITIRAJU

Abstract. In this work we determine the wavefront set of certain eigendistributions of the Laplace-Beltrami operator on the de Sitter space. Let $G = \mathrm{SO}_{1,n}(\mathbb{R})_e$ be the connected component of identity of Lorentz group and let $H = \mathrm{SO}_{1,n-1}(\mathbb{R})_e \subset G$. The de Sitter space dS^n, is the one-sheeted hyperboloid in $\mathbb{R}^{1,n}$ isomorphic to G/H. A *spherical distribution*, is an H-invariant, eigendistribution of the Laplace-Beltrami operator on dS^n. The space of spherical distributions with eigenvalue λ, denoted by $\mathcal{D}'_\lambda(\mathrm{dS}^n)$, has dimension 2. In this article we construct a basis for the space of positive-definite spherical distributions as boundary value of sesquiholomorphic kernels on the crown domains, an open G-invariant domain in $\mathrm{dS}^n_{\mathbb{C}}$. It contains dS^n as a G-orbit on the boundary. We characterize the analytic wavefront set for such distributions. Moreover, if a spherical distribution $\Theta \in \mathcal{D}'_\lambda(\mathrm{dS}^n)$ has the wavefront set same as one of the basis element, then it must be a constant multiple of that basis element. Using the analytic wavefront sets we show that the basis elements of $\mathcal{D}'_\lambda(\mathrm{dS}^n)$ can not vanish in any open region.

Contents

1

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Vita

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