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Unavoidable Topological Minors of Large or Infinite 3-connected Rooted Graphs

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UNAVOIDABLE TOPOLOGICAL MINORS OF LARGE OR INFINITE 3-CONNECTED ROOTED GRAPHS

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Abstract

The goal of this thesis is to determine the unavoidable topological minors of large and infinite 3-connected rooted graphs, where a rooted graph is a graph G together with a specified subset X of $V(G)$ or $E(G)$. We have two results for finite graphs. First, every 3-connected finite graph G with a sufficiently large $X \subseteq E(G)$ must contain a topological minor $K_{3,n}, W_n$, or V_n, using many edges of X, where W_n is a wheel with n spokes and V_n is obtained from a ladder with n rungs by adding two grips and a handle. Second, every 3-connected finite graph G with a sufficiently large $X \subseteq V(G)$ must contain a topological minor $K_{3,n}, K^1_{3,n}, K^2_{3,n}, K^3_{3,n}, W_n$, or V_n, using many vertices of X, where $K^i_{3,n}$ $(i = 1, 2, 3)$ is obtained by gluing the leaves of i combs and $3 - i$ stars in the natural way.

We also have two results for infinite graphs. First, every 3-connected graph G with an infinite $X \subseteq E(G)$ must contain a topological minor $K_{3,\infty}, FF, FL$, or LL, using infinitely many edges of X, where FF, FL, and LL are obtained, respectively, by gluing i $(i = 0, 1, 2)$ infinite ladders and $2 - i$ infinite fans along their rails. Second, every 3-connected graph G with an infinite $X \subseteq V(G)$ must contain as a subgraph a subdivision of $K_{3,\infty}, FF, FL$, or LL, containing infinitely many vertices of X. We also discuss similar results for lower connectivities, which in fact are corollaries of results listed above.

Chapter 1. Introduction

This thesis is about the structure of unavoidable topological minors of large and infinite 3-connected rooted graphs. In this chapter, we provide some relevant background and outline our main results. We first list some relevant research that has been done in this area. Next, we present the statements of our results, whose proofs will be detailed in the next three chapters. Finally, we define basic terminology and state standard theorems in graph theory that are used in later chapters.

1.1. Background Survey

All graphs in this thesis are simple. In this section, we present some related results to our main topic. For undefined terms used here, we refer the readers to the last section of this chapter. There are two questions that we are interested in.

1. Given a large or infinite k-connected graph G, what are the unavoidable large or infinite k-connected structures in G? This topic has been studied extensively, and many results are now known for graphs of small connectivity.

2. Given a k-connected graph G together with a large or infinite subset X of $V(G)$ or $E(G)$, what are the unavoidable large or infinite k-connected structures in G that contain many elements of X? Not much has been known about this topic, which is the focus of this thesis.

In the next few sections, we discuss the research that has been established for each question and then we describe our main results.

1.1.1. Connected Graphs

Let G be a complete graph. If G is a finite graph with n vertices, then we denote G as K^n. Otherwise, G is an infinite graph, and we denote G as K^∞. The complement of K^n and K^∞ are $\overline{K^n}$ and $\overline{K^\infty}$ respectively.

We first state Ramsey Theorem for finite graphs.

Theorem 1.1.1 (Ramsey Theorem for Finite Graphs, Theorem 9.1.1 in [4])**.** *For every $r \geq 1$, there exists a positive integer n such that every finite graph with at least n vertices contains K^r or $\overline{K^r}$ as an induced subgraph.*

A more general version of this result is Ramsey Theorem for infinite graphs. The formulation we provided below is obtained from a more general version of Theorem 9.1.2 in [4] by setting $k = c = 2$.

Theorem 1.1.2 (Ramsey Theorem for Infinite Graphs)**.** *Every infinite graph G contains K^∞ or $\overline{K^\infty}$ as an induced subgraph.*

If in addition, we know that G is also connected, then we can say a little more. The following theorem gives us the unavoidable induced subgraphs for large connected graphs.

Theorem 1.1.3 (Theorem 9.4.1 in [4])**.** *For every $r \geq 3$, there exists a positive integer n such that every finite connected graph with at least n vertices contains K^r, $K_{1,r}$, or a path of length r as an induced subgraph.*

The unavoidable induced subgraphs of infinite connected graphs are determined in the following theorem. Even though it is a well-known result, we could not find its original proof (see Theorem 1.6 in [1] for a reference). We want to remark that the proof is very similar to that of Theorem 9.4.1 in [4] and can be obtained by applying Theorem 1.1.2 above and Lemma 8.1.2 in [4].

Theorem 1.1.4. *Every infinite connected graph G contains K^∞, $K_{1,\infty}$, or a one-way infinite path as an induced subgraph.*

1.1.2. 2-connected Graphs

In their papers, Allred, Ding, and Oporowski proved two results about the unavoidable induced subgraphs of large and infinite 2-connected graphs (see [2] and [1]). We will not describe all the graphs involved since they are not needed for our main results. For 2-connectivity, we will instead consider the much weaker result on unavoidable minors and topological minors.

A graph G' is a **subdivision** of a graph G if G' is obtained from G by replacing every edge e of G with a path P_e between the two endpoints of e such that the internal vertices of P_e do not contain any vertex of G and two distinct $P_e, P_{e'}$ are internally disjoint. We call P_e a **component path** of G'. In G', the original vertices of G are called **branching vertices** and the new vertices are called **subdividing vertices**.

Let H be a subgraph of G such that H is a subdivision of a graph J. We say G contains a **subdivided** J or J is a **topological minor** of G.

The following result is known for finite graphs.

Theorem 1.1.5 (Theorem 9.4.2 in [4]). *For every $r \geq 3$, there exists a positive integer n such that every finite 2-connected graph with at least n vertices contains $K_{2,r}$ or a cycle of length r as a topological minor.*

We want to point out that the result of Allred, Ding, and Oporowski in [2] implies Theorem 1.1.5. However, their result is a lot stronger and the proof is a lot more complicated. For infinite 2-connected graphs, we have the following result due to Ding and Chun.

Theorem 1.1.6 (Theorem 1.3 in [3]). *Every infinite 2-connected graph contains a graph in $\{K_{2,\infty}, F_\infty, L_\infty\}$ as a topological minor.*

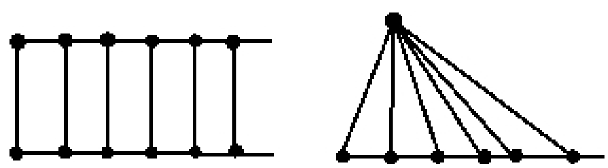

Figure 1.1. Left: L_∞, right: F_∞

This implies that every infinite 2-connected graph contains $K_{2,\infty}$ or F_∞ as a minor.

1.1.3. 3-connected Graphs

Currently, no theorem about the unavoidable induced subgraphs has been established for graphs of connectivity 3 and higher. Thus, we will consider unavoidable minors and topological minors. For finite 3-connected graphs, we have the following result, proven by Oporowski, Oxley, and Thomas in 1993.

Theorem 1.1.7 (Theorem 1.3 in [6]). *For every $r \geq 3$, there exists a positive integer n such* *that every finite 3-connected graph with at least n vertices contains a graph in $\{W_r, V_r, K_{3,r}\}$ as a topological minor.*

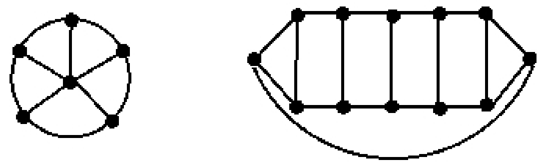

Figure 1.2. Left: W_r, right: V_r

This implies that every sufficiently large 3-connected graph contains a large wheel or a large $K_{3,r}$ as a minor. For infinite graphs, we have the following result due to Ding and Chun.

Theorem 1.1.8 (Theorem 1.3 in [3]). *Every infinite 3-connected graph contains a graph in $\{K_{3,\infty}, FF, FL, LL\}$ as a topological minor.*

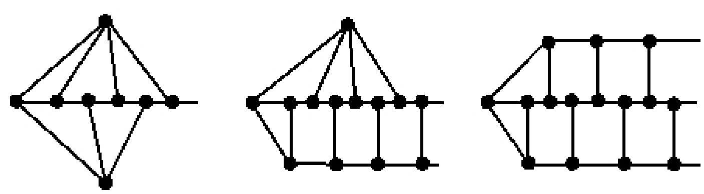

Figure 1.3. Left: FF, middle: FL, right: LL

This implies that every infinite 3-connected graph contains $K_{3,\infty}$ or FF as a minor.

1.1.4. Graphs with Connectivity 4 and Higher

Currently, no result about unavoidable topological minors for k-connected ($k \geq 4$) finite graphs exists. In their paper, Oxley, Oporowski, and Thomas determined the unavoidable topological minors of sufficiently large quasi 4-connected graphs. A 3-connected graph $G = (V, E)$ with $|V| \geq 7$ is **quasi 4-connected** if for every subset X of V where $|X| = 3$, either $G - X$ is connected or $G - X$ has two components, one of which is a single vertex.

Theorem 1.1.9 (Theorem 1.4 in [6])**.** *For every $r \geq 4$, there exists a positive integer n such that every finite quasi 4-connected graph with at least n vertices contains a graph in $\{A_r, O_r, M_r, K_{4,r}, K'_{4,r}\}$ as a topological minor.*

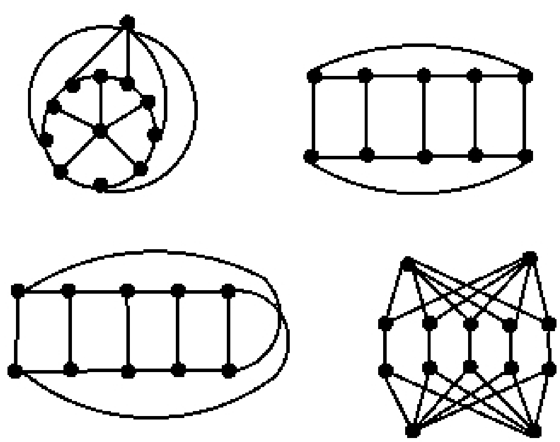

Figure 1.4. Top left: A_r, top right: O_r, bottom left: M_r, bottom right: $K'_{4,r}$

We remark that every graph in $\{A_r, O_r, M_r, K_{4,r}, K'_{4,r}\}$ is quasi 4-connected.

For $k = 5$, we have a result about the unavoidable minors of sufficiently large 5-connected graphs due to Shantanam in [7]. We will not describe all the unavoidable minors here since the list contains 30 graphs.

For $k \geq 6$, there is currently no known result for finite graphs. For infinite graphs, Ding and Chun determined the unavoidable topological minors of infinite loosely k-connected graphs, for all $k \geq 4$, in [3]. An infinite graph G is **loosely k-connected** if there exists a number d depending on G such that deleting fewer than k vertices from G leaves precisely one infinite component and a graph containing at most d vertices. We will not go into details their construction since the graphs involved are not needed in our main results.

1.1.5. Rooted Graphs

By a **rooted graph** we mean a graph G together with a subset $X \subseteq V(G)$ or $X \subseteq E(G)$. Rooted graphs play a central role in this thesis. We first consider finite rooted graphs.

Let $n \geq 3$. Let $P = x_1x_2 \ldots x_n$ and $Q = y_1y_2 \ldots y_n$ be disjoint paths. A **ladder** L_n is obtained by adding edges x_iy_i for $i = 1, 2, \ldots, n$. We call P, Q the **rails** and each edge x_iy_i a **rung**. For a subdivided L_n, we use the terms rail and rung to mean its subdivided rail and subdivided rung respectively.

Let $n \geq 3$ and let $P = x_1x_2 \ldots x_n$ be a path. Let u be a vertex not on P. A **fan** F_n is obtained by adding edges ux_i for $i = 1, 2, \ldots, n$. We call P the **rail** and each edge ux_i a **spoke**. For a subdivided F_n, we use the terms rail and spoke to mean its subdivided rail and subdivided spoke respectively.

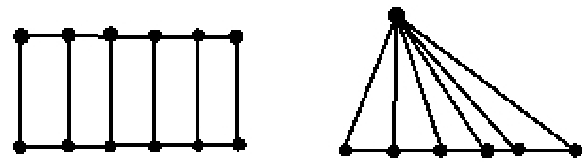

Figure 1.5. Left: L_n, right: F_n

The following results, due to Wang, determine the unavoidable topological minors of large 2-connected rooted graphs.

Theorem 1.1.10 (Vertex Version, Theorem 3.1.5 in [8]). *There exists a function $f_{1.1.10}(t)$ where $t \geq 3$ with the following property. Let G be a finite 2-connected graph and let $X \subseteq V(G)$ with $|X| \geq f_{1.1.10}(t)$. Then G contains one of the following subgraphs*

1. *a cycle containing at least t vertices of X,*
2. *a subdivided $K_{2,t}$ containing vertices of X in at least t component paths,*
3. *a subdivided F_t where each spoke contains at least one vertex of X in its interior,*
4. *a subdivided L_t where each rung contains at least one vertex of X in its interior.*

Theorem 1.1.11 (Edge Version, Theorem 3.1.1 in [8]). *There exists a function $f_{1.1.11}(t)$ where $t \geq 3$ with the following property. Let G be a finite 2-connected graph and let $X \subseteq E(G)$ with $|X| \geq f_{1.1.11}(t)$. Then G contains one of the following subgraphs*

1. *a cycle containing at least t edges of X,*
2. *a subdivided $K_{2,t}$ containing edges of X in at least t component paths,*
3. *a subdivided F_t where each spoke contains at least one edge of X,*

4. a subdivided L_t where each rung contains at least one edge of X.

The previous two theorems imply the corresponding results for large connected rooted graphs.

Let $n \geq 3$ and let $u, x_1, x_2, \ldots, x_n$ be distinct vertices. A **star** $K_{1,n}$ is obtained by adding an edge between u and x_i for $i = 1, 2, \ldots, n$. We call u the **center** of the star. For a subdivided $K_{1,n}$, we also use the term center to denote its degree-n vertex.

Let $n \geq 3$ and let $P = x_1 x_2 \ldots x_n$ be a path. A **comb** $\mathcal{C}_n$ is obtained from P by joining each x_i with a pendent edge $x_i v_i$. We call P the **spine** and each $x_i v_i$ an $x_i v_i$-**tooth** of $\mathcal{C}_n$. By a leaf sequence of $\mathcal{C}_n$ we mean the sequence of its leaves, listed in the order as they appear, that is $v_1, v_2 \ldots, v_n$. For a subdivided comb, we use the terms spine and tooth to mean its subdivided spine and subdivided tooth respectively.

For connected graphs, we have the following results. The first one is explicitly stated in [8], whereas the second one is not, but it has been implicitly obtained in [8].

Theorem 1.1.12 (Vertex Version, Theorem 2.1.4 in [8])**.** *There exists a function $f_{1.1.12}(t)$ where $t \geq 3$ with the following property. Let G be a finite connected graph and let $X \subseteq V(G)$ with $|X| \geq f_{1.1.12}(t)$. Then G contains one of the following subgraphs*

1. a path containing at least t vertices of X,

2. a subdivided $K_{1,t}$ whose leaves belong to X,

3. a subdivided $\mathcal{C}_t$ whose leaves belong to X.

Theorem 1.1.13 (Edge Version)**.** *There exists a function $f_{1.1.13}(t)$ where $t \geq 3$ with the following property. Let G be a finite connected graph and let $X \subseteq E(G)$ with $|X| \geq f_{1.1.13}(t)$.*

Then G contains one of the following subgraphs

1. *a path containing at least t edges of X,*
2. *a subdivided $K_{1,t}$ where each component path contains at least one edge of X,*
3. *a subdivided $\mathcal{C}_t$ where each tooth contains at least one edge of X.*

We want to point out that Theorem 1.1.13 can be obtained easily from Theorem 1.1.11. Let G be a finite connected graph and let X be a sufficiently large subset of $E(G)$. Let v be a vertex not in G and let G' be obtained from G by adding edges from v to every vertex in G. Then G' is 2-connected and $X \subseteq E(G')$. Thus, G' contains one of the subgraphs listed in Theorem 1.1.11, call it H. Now $H - v$ contains a desired subgraph in G.

Two main results of ours settle the $k = 3$ case. For $k \geq 4$, there is no known result at this point.

We now consider infinite rooted graphs.

A **ray** is an infinite graph R whose vertex set is $\{x_1, x_2, \ldots\}$ and whose edge set is $\{x_i x_{i+1} \mid i = 1, 2, \ldots\}$. We call x_1 the **endpoint** and $x_2, x_3, \ldots$ the **internal vertices**. We denote R by listing its vertices, in the order as they appear on R, so we will write $R = x_1 x_2 \ldots$. A **double ray** is a graph obtained by identifying the two endpoints of two disjoint rays.

Let $\{x_1, x_2, \ldots\}$ be an infinite set of vertices. A **star** $K_{1,\infty}$ is obtained by adding an edge between x_1 and x_i for all $i \geq 2$. For a $K_{1,\infty}$ or its subdivision, we use the term **center** to denote its infinite degree vertex.

Let $R = x_1 x_2 \ldots$ be a ray. A **comb** $\mathcal{C}_\infty$ is obtained from R by joining each x_i with a pendent edge x_iy_i. We call R the **spine** and each x_iy_i an x_iy_i-**tooth**. For a subdivided $\mathcal{C}_\infty$, we use the terms spine and tooth to mean subdivided spine and subdivided tooth, respectively. There are some differences between our definition of a comb and the one used in [4]. A comb in [4] may have only one leaf, in which case it is a ray, or finitely many leaves, or infinitely many leaves. A comb under our definition always has infinitely many leaves. We prefer to use our definition of a comb instead of the one in [4] since we want to distinguish between a ray and a comb for the case analysis in later theorems.

The following theorem is a reformulation of Lemma 8.2.2 in [4].

Theorem 1.1.14 (Vertex Version)**.** *Let G be an infinite connected graph and let X be an infinite subset of $V(G)$. Then G contains one of the following subgraphs*

1. *a ray containing infinitely many vertices of X,*
2. *a subdivided $K_{1,\infty}$ whose leaves belong to X,*
3. *a subdivided $\mathcal{C}_\infty$ whose leaves belong to X.*

Four other main results of ours settle the $k = 2, 3$ cases. For $k \geq 4$, there is no known result at this point. In addition, as we will justify later on, our Theorem 1.2.4 implies the following theorem.

Theorem 1.1.15 (Edge Version)**.** *Let G be an infinite connected graph and let X be an infinite subset of $E(G)$. Then G contains one of the following subgraphs*

1. *a ray containing infinitely many edges of X,*
2. *a subdivided $K_{1,\infty}$ where each component path contains at least one edge of X,*

3. a subdivided $\mathcal{C}_\infty$ where each tooth contains at least one edge of X.

1.2. Main Results

We now state all of our main results, whose proofs are deferred to the next three chapters.

1.2.1. Finite Graphs

Let G be a finite graph and let H' be a subgraph of G where H' is a subdivision of a graph H. Suppose $X \subseteq E(G)$. Then a component path of H' is **heavy** if it contains at least one edge of X and is **light** otherwise. The **edge-weight** of H' is the number of heavy component paths. On the other hand, suppose $X \subseteq V(G)$. Let U be the set of branching vertices of H'. Then the **vertex-weight** of H' is the number of elements in $U \cap X$.

We want to emphasize that for a finite graph G together with $X \subseteq V(G)$ or $X \subseteq E(G)$, we are interested in unavoidable structures of G containing many elements of X in many components paths. This is because a subgraph of G may contain many elements of X, but those elements are in very few component paths. In this case, it is not good since we want the elements of X to be spread out to fully capture the k-connectivity property. For infinite graphs, this does not matter because if a subgraph contains infinitely many elements of X, then infinitely many different component paths contain elements of X.

Theorem 1.2.1 (Edge Version)**.** *There exists a function $f_{1.2.1}(t)$ where $t \geq 3$ with the following property. Let G be a finite 3-connected graph and let X be a subset of $E(G)$ with*

$|X| \geq f_{1.2.1}(t)$. *Then* G *contains a subdivided* H *with edge-weight at least* t *for some* H *in* $\{K_{3,n}, W_n, V_n \mid \textit{for some } n \geq t\}$.

Theorem 1.2.2 (Vertex Version). *There exists a function* $f_{1.2.2}(t)$ *where* $t \geq 3$ *with the following property. Let* G *be a finite* 3*-connected graph and let* X *be a subset of* $V(G)$ *with* $|X| \geq f_{1.2.2}(t)$. *Then* G *contains a subdivided* H *with vertex-weight at least* t *for some* H *in* $\{K_{3,n}, K^1_{3,n}, K^2_{3,n}, K^3_{3,n}, W_n, V_n \mid \textit{for some } n \geq t\}$.

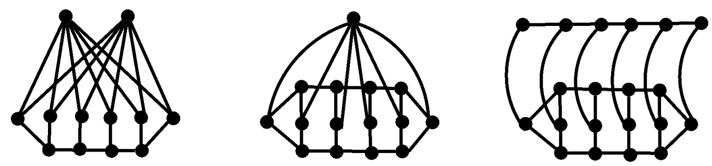

Figure 1.6. Left: $K^1_{3,n}$, middle: $K^2_{3,n}$, right: $K^3_{3,n}$

We want to point out that Theorem 1.2.1 and Theorem 1.2.2 extend the results of Theorem 1.1.11 and Theorem 1.1.10 to 3-connectivity. For 2-connectivity, the list of unavoidable graphs in Theorem 1.1.11 and Theorem 1.1.10 contains $K_{2,n}, F_n$, and L_n. For 3-connectivity, $K_{2,n}$ becomes $K_{3,n}$, F_n becomes W_n, and L_n becomes V_n.

1.2.2. Infinite Graphs

Let G be an infinite graph and let X be an infinite subset of $V(G)$. Assume a subgraph G' of G is a subdivision of a graph H such that $V(G') \cap X$ is infinite. Then we call G' an X**-rich** H. Note that the elements of X in G' might not be branching vertices. In the definition of vertex-weight, we are counting the number of branching vertices that are in X whereas in the definition of X-rich, we are counting the number of vertices, branching or subdividing, that are in X.

For 2-connectivity, we have the following results.

Theorem 1.2.3 (Vertex Version). *Let G be an infinite 2-connected graph and let X be an infinite subset of $V(G)$. Then G contains an X-rich H for some H in $\{K_{2,\infty}, F_\infty, L_\infty\}$.*

Theorem 1.2.4 (Edge Version). *Let G be an infinite 2-connected graph and let X be an infinite subset of $E(G)$. Then G contains a subdivided H containing infinitely many edges of X for some H in $\{K_{2,\infty}, F_\infty, L_\infty\}$.*

As mentioned before, Theorem 1.2.4 implies Theorem 1.1.15. To see this, let G be an infinite connected graph and let X be an infinite subset of $E(G)$. Let v be a vertex not in G and let G' be obtained from G by adding edges from v to every vertex in G. Then G' is 2-connected and $X \subseteq E(G')$. Thus, G' contains one of the subgraphs listed in Theorem 1.2.4, call it H. Now $H - v$ contains a desired subgraph in G.

For 3-connectivity, we have the following results.

Theorem 1.2.5 (Vertex Version). *Let G be an infinite 3-connected graph and let X be an infinite subset of $V(G)$. Then G contains an X-rich H for some H in $\{K_{3,\infty}, FF, FL, LL\}$.*

Theorem 1.2.6 (Edge Version). *Let G be an infinite 3-connected graph and let X be an infinite subset of $E(G)$. Then G contains a subdivided H containing infinitely many edges of X for some H in $\{K_{3,\infty}, FF, FL, LL\}$.*

As we shall see in the next few chapters, we will prove a stronger result, which implies Theorem 1.2.5 and Theorem 1.2.6 immediately. Note that by setting $X = V(G)$, Theorem

1.2.3 and Theorem 1.2.5 imply Theorem 1.1.6 and Theorem 1.1.8 respectively.

1.3. Basic Definitions and Theorems

All definitions and theorems in this section are standard in graph theory and are taken from [4]. All undefined terms will also follow [4].

1.3.1. Graphs

For a set X, we use $|X|$ to denote the number of elements in X, which can be finite or infinite. By convention, elements in a set are distinct. Let G be a graph. We write $V(G)$ to mean its **vertex set** and $E(G)$ to mean its **edge set**. The **order** of G is the number of vertices and is denoted as $|G|$, so $|G| = |V(G)|$. We say G is a **finite graph** if $V(G)$ is finite and is an **infinite graph** if $V(G)$ is infinite. Graphs in this section can be either finite or infinite. Two graphs are **disjoint** if their vertex sets are disjoint and are **edge-disjoint** if their edge sets are disjoint.

Let $e = uv$ be an edge. We call u, v the **endpoints** of e. Let $v \in V(G)$. We denote $N_G(v)$ (or simply $N(v)$ when G is clear) to be the set of neighbors of v. We denote $\deg_G v$ to be the **degree** of v in G, which can be finite or infinite. When the underlying graph G is clear, we will simply write $\deg v$. We define the **minimum degree** of G as $\delta(G) = \min\{\deg v \mid v \in V(G)\}$ and the **maximum degree** of G as $\Delta(G) = \max\{\deg v \mid v \in V(G)\}$. Note that both $\delta(G)$ and $\Delta(G)$ can be finite or infinite. We say G is **locally finite** if all of its vertices have finite degree.

When we say a graph G contains another graph H, we mean H is a subgraph of G and we write $H \subseteq G$. We denote $G - H$ to be the graph obtained from G by deleting all vertices of H. In addition, if $e \in E(G)$, then we write $G \backslash e$ to mean deleting e from G.

A set of vertices is called a **stable** if its elements are pairwise non-adjacent and is called a **clique** if its elements are pairwise adjacent.

Let $n \geq 1$. A **path** is a graph whose vertex set is $\{x_1, \ldots, x_n\}$ and whose edge set is $\{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$. We call x_1, x_n the **endpoints** and $x_2, x_3, \ldots, x_{n-1}$ the **internal vertices**. We denote P by listing its vertices, in the order as they appear on P, so we will write $P = x_1x_2 \ldots x_n$. The **length** of a path is the number of its edges. If P has length at least one, then $\overset{\circ}{P}$ is obtained from P by removing its two endpoints and we call it the **interior** of P. (If P has length one, then $\overset{\circ}{P}$ is an empty graph.)

Let P be a path and let x, y be two vertices of P. We define the following terms

- $P[xy]$ is the xy-subpath of P,
- $P(xy] = P[xy] - x$,
- $P[xy) = P[xy] - y$,
- $P(xy) = P[xy] - \{x, y\}$.

Let H be a subgraph of G with at least two vertices. A path P in G is called an H**-path** if $E(P \cap H) = \emptyset$ and $V(P \cap H)$ consists of the two endpoints of P.

Let $A, B \subseteq V(G)$. We say a path $P = x_0x_1 \ldots x_k$ is an AB**-path** (and AB**-edge** if P is an edge) if $V(P) \cap A = x_0$ and $V(P) \cap B = x_k$. When $A = \{a\}$, we use the notation aB-path (and aB-edge) to mean an $\{a\}B$-path (and $\{a\}B$-edge). Sometimes, it is more convenient to talk about an AB-path in the context of graphs. Let A, B be subgraphs of G. By an **AB-path** (and **AB-edge**), we mean a $V(A)V(B)$-path (and $V(A)V(B)$-edge). When A is a single vertex graph a, we again adopt the notation aB-path (and aB-edge) to mean an $\{a\}V(B)$-path (and $\{a\}V(B)$-edge).

Two paths are **internally disjoint** if they do not share any common internal vertices. Let a be a vertex and $B \subseteq V(G) - a$. Two aB-paths are **weakly disjoint** if they only have a in common.

This paragraph defines the concept of a separator. We will make a distinction between different types of separators, which we will clarify below. First, we define a separator of two sets of vertices. Let $A, B \subseteq V(G)$ and $X \subseteq V(G)$. We say X **separates** A, B if every AB-path in G contains a vertex of X. We call X a **separator** of A, B in this case. Next, we define a separator of a vertex and a set of vertices. Let a be a vertex of G and $B \subseteq V(G)$. We say X **separates** a, B if it separates $\{a\}, B$ and $a \notin X$. We call X a **separator** of a, B in this case. Finally, we define a separator of two vertices. Let a, b be two vertices. We say X **separates** a, b if it separates $\{a\}, \{b\}$ and $a, b \notin X$. We call X a **separator** of a, b. From the previous three definitions, we make a distinction between different types of separators. For example, a separator of a, B is conceptually different from a separator of $\{a\}, B$. Consider a $K_{1,3}$ where each edge is subdivided exactly once. Let u be the cubic vertex and let b_i $(i = 1, 2, 3)$ be the three leaves. Let a_i be the internal vertex of the ub_i-path. Then $X = \{u\}$ is a separator of $\{u\}$ and $B = \{b_1, b_2, b_3\}$, but X is not a separator of u and B because $u \in X$. Now $X = \{a_1, a_2, a_3\}$ is a separator of u and B.

1.3.2. Minors

Graphs in this section can be either finite or infinite.

Let G' be a connected subgraph of G and let N be the set of vertices of $G - G'$ with a neighbor in G'. The graph G/G' is obtained from $G - G'$ by adding a vertex v not in G and then adding edges from v to all vertices in N. We call G/G' the graph obtained by **contracting** G'. A **minor** of G is a graph obtained from a subgraph H of G by contracting disjoint connected subgraphs of H.

Sometimes, it is more convenient to talk about minors without mentioning the contraction operation. We now introduce an alternative definition of minors. We say H is a **minor** of G if there is a function π, called an **embedding**, with domain $V(H) \cup E(H)$ satisfying the following

1. $\pi(v)$ is a nonempty, connected subgraph of G for every $v \in V(H)$,
2. $\pi(u)$ and $\pi(v)$ are disjoint for every distinct $u, v \in V(H)$,
3. if $e = uv \in E(H)$, then $\pi(e)$ is an edge of G between $\pi(u)$ and $\pi(v)$.

The union of $\pi(v)$ and $\pi(e)$ for all $v \in V(H)$ and all $e \in E(H)$ is called an **expansion** of H in G and is denoted as $G|H$. If H is a minor of G, then we also say G contains an H-**minor**.

It is easy to see that the two definitions of minors are equivalent. The difference is that using the language of an embedding, we can refer directly the disjoint connected subgraphs that are contracted. We want to remark that if H is a minor of G and G is a minor of G', then H is also a minor of G'. We will not justify this fact here since our proofs do not rely on it.

Let H be a minor of G and let $v \in V(H)$. We say v is **firm** if there exists an embedding π such that $\pi(v)$ has only one vertex (that is, $\pi(v)$ is a vertex of G).

1.3.3. Bridges

Let H be a subgraph of G. An H-**bridge** is a connected subgraph B of $G \backslash E(H)$ satisfying one of the following

1. B has one edge and $V(B) \subseteq V(H)$, which we call a **trivial bridge**,

2. there exists a connected component C of $G - H$ such that $E(B)$ consists of all edges incident with at least one vertex of C.

For a bridge B, vertices that belong to $B \cap H$ are called its **feet**. The following properties of bridges are easy to verify. First, if x, y are two distinct feet of a bridge B, then B contains an xy-path. Next, every edge $e \notin E(H)$ belongs to a unqiue bridge. Finally, if $x \in V(B_1) \cap V(B_2)$ where B_1, B_2 are distinct bridges, then $x \in V(H)$.

1.3.4. Crossing and Positions

Let S be a path, finite or infinite and let a, b, c, d be distinct vertices on S. We say $\{a, b\}$ **crosses** $\{c, d\}$ with respect to S if one vertex in $\{c, d\}$ belongs to $S(ab)$ and the other vertex in $\{c, d\}$ does not belong to $S[ab]$. Let P, Q be disjoint S-paths where P has endpoints $\{a, b\}$ and Q has endpoints $\{c, d\}$. We say P **crosses** Q with respect to S if $\{a, b\}$ crosses $\{c, d\}$ with respect to S. Let B_1, B_2 be distinct S-bridges of a graph G. We say B_1 **crosses**

B_2 with respect to S if there exist two feet a, b of B_1 and two feet c, d of B_2 such that $\{a, b\}$ crosses $\{c, d\}$ with respect to S. It is easy to see that crossing is a symmetric relation.

Let S be a path, finite or infinite. We label all the vertices as they appear on S as a sequence

$$\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots.$$

If S has a finite end, then the sequence terminates at that end. Otherwise, the sequence continues indefinitely on that end. When S is a ray, we assume its endpoint x_i has smallest index i. Let x_a, x_b be two distinct vertices of S. We say with respect to S, x_a is **on the left** of x_b, or x_b is **on the right** of x_a, if $a < b$. Let P, Q be disjoint S-paths where P has endpoints x_a, x_b with $a < b$ and Q has endpoints x_c, x_d with $c < d$. We say with respect to S, Q is **on the right** of P, or P is **on the left** of Q, if x_b is on the left of x_c with respect to S.

1.3.5. Connectivity

Graphs in this section can be either finite or infinite.

Let $k \geq 1$. A graph G is k**-connected** if $|G| > k$ and $G - X$ is connected for every $X \subseteq V(G)$ with $|X| < k$. Note that for graphs with at least two vertices, being 1-connected is equivalent to being connected.

The best known theorem related to connectivity is Menger Theorem.

Theorem 1.3.1 (Menger Theorem, Theorem 8.4.1 in [4])**.** *Let G be a graph and let k be an integer. Let $A, B \subseteq V(G)$ that cannot be separated by fewer than k vertices. Then G*

contains k disjoint AB-paths.

The following corollary to Menger Theorem is also very useful.

Corollary 1.3.2. *Let a be a vertex of G and let $B \subseteq V(G) - a$. If a, B cannot be separated by fewer than k vertices, then G contains k weakly disjoint aB-paths.*

Proof. Let A be the set of neighbors of a in G.

Claim 1.3.2.1. *In $G - a$, A and B cannot be separated by fewer than k vertices.*

Suppose there exists a separator X of size less than k separating A and B in $G - a$. We show that X is a separator of a and B in G. Let P be an aB-path in G. Then $P - a$ is a path in $G - a$ with one endpoint in A and the other endpoint in B. Thus, $P - a$ contains a subpath P' that is an AB-path. This means that P' contains a vertex of X and so does P. We have shown that every aB-path in G contains a vertex of X. In addition, $a \notin X$ since $X \subseteq V(G) - a$. Hence, X is a separator of a and B in G. Since $|X| < k$, we get a contradiction. This proves the claim.

By Theorem 1.3.1, $G - a$ contains k disjoint AB-paths. Therefore, G contains k weakly disjoint aB-paths. □

Menger Theorem guarantees the existence of many disjoint paths between two set of vertices if they cannot be separated by a small set. The next theorem shows that if we are given a set of k weakly disjoint aB-paths in G and we know G contains a set of $k+1$ weakly disjoint aB-paths, then we can obtain those $k+1$ paths so that they contain the same set of endpoints as the given k paths.

Theorem 1.3.3. *Let a be a vertex of G and $B \subseteq V(G) - a$. Let $\mathcal{P}$ be a set of k weakly disjoint aB-paths. If G has more than k weakly disjoint aB-paths, then G has a set $\mathcal{Q}$ of $k+1$ weakly disjoint aB-paths such that every end of a path of $\mathcal{P}$ in B is an end of a path of $\mathcal{Q}$.*

Proof. For every set $\mathcal{Q}$ of $k+1$ weakly disjoint aB-paths, let H be the union of all paths in $\mathcal{P}, \mathcal{Q}$. Since H is a finite graph, we can choose $\mathcal{Q}$ so that $|E(H)|$ is minimal. We prove that this set $\mathcal{Q}$ satisfies the conclusion of the theorem. Suppose for contradiction that there exists a path $P \in \mathcal{P}$ having an end $x \in B$ that is not an end of any path in $\mathcal{Q}$. Since a belongs to every path in $\mathcal{P} \cup \mathcal{Q}$, there exists a $z \in P$ such that $z \in Q$ for some $Q \in \mathcal{Q}$, but no other vertex of $P[zx]$ belongs to any path in $\mathcal{Q}$. First, suppose $z = a$. This means that P only intersects every path in $\mathcal{Q}$ at a. Since $|\mathcal{P}| = k$ and $|\mathcal{Q}| = k+1$, H has an edge e incident with a such that e does not belong to any path in $\mathcal{P}$. Let Q be the path on $\mathcal{Q}$ containing e. By replacing Q with P, we obtain a set $\mathcal{Q}'$ of $k+1$ weakly disjoint aB-paths and the union of all paths in $\mathcal{P}, \mathcal{Q}'$ yields a graph H with smaller $|E(H)|$ value, contradicting the minimality of $E(H)$. Hence, $z \neq a$. Now $z \in Q$ for some $Q \in \mathcal{Q}$, but no other vertex of $P[zx]$ belongs to any path in $\mathcal{Q}$. Let y be the endpoint of Q in B. Note that $Q[zy]$ contains an edge e where $e \notin E(P)$. Let Q' be the path obtained from Q by replacing $Q[zy]$ with $P[zx]$ and let $\mathcal{Q}'$ be obtained from $\mathcal{Q}$ by replacing Q with Q'. Now $\mathcal{Q}'$ is a set of $k+1$ weakly disjoint aB-paths whose resulting graph H has smaller $|E(H)|$ value, contradicting the minimality of $E(H)$. Therefore, $\mathcal{Q}$ satisfies the conclusion of the theorem. $\square$

Chapter 2. Unavoidable Topological Minors of Large 3-connected Rooted Graphs

Graphs in this chapter are finite.

2.1. Definitions and Lemmas

This section defines more terminology and states some theorems that are needed for the proof of our main result. First, we examine the properties of 3-connected graphs. We discuss how local operations affect 3-connectivity. The following theorem was proven by Tutte.

Theorem 2.1.1 (Chapter 3, Exercise 10 in [4])**.** *Let $G \neq K_4$ be a 3-connected graph and let $e \in E(G)$. Then G/e is 3-connected or $G \backslash e$ is a subdivision of a 3-connected graph.*

Given a 3-connected graph G together with a subset X of $E(G)$, if we know that every proper minor of G no longer contains X, then we can say something about how X interacts with edges not in X. This is the main idea of the next theorem.

Theorem 2.1.2. *Let $G \neq K_4$ be a 3-connected graph and let X be a subset of $E(G)$. Assume that for every proper 3-connected minor H of G, we have $X \not\subseteq E(H)$. Then for every $e \in E(G) - X$, one of the following must be true*

1. *one endpoint of e is cubic in G and is incident with two edges of X,*
2. *e and two edges of X form a triangle.*

Proof. By Theorem 2.1.1, either $H = G/e$ is 3-connected or $G \backslash e$ is a subdivision of a 3-connected graph H. In both cases, H is a proper minor of G, so by the minimality assumption, $X \not\subseteq E(H)$. If $H = G/e$, then after identifying the two ends of e, there exist two parallel edges that are both in X. Thus, e and two edges of X form a triangle in G, so statement 2 is satisfied. Otherwise, $G\backslash e$ is a subdivision of H. Since H is simple and 3-connected and $X \not\subseteq E(H)$, it follows that $G\backslash e$ has a vertex v of degree 2 that is incident with two edges of X and v is incident with e in G. Hence, in G, one endpoint of e is cubic and is incident with two edges of X, so statement 1 is satisfied. □

The next theorem asserts that a sufficiently large connected graph contains a vertex of high degree or a long path starting from any vertex.

Theorem 2.1.3. *Let $d, t \geq 3$ and let $f_{2.1.3}(t) = 1+(d-1)+(d-1)(d-2)+(d-1)(d-2)^2+\cdots+(d-1)(d-2)^{t-1}$. Let G be a connected graph with $|G| \geq f_{2.1.3}(d,t)$. Then $\Delta(G) \geq d$ or G contains a path of length t starting from any vertex.*

Proof. Assume $\Delta(G) \leq d-1$, for otherwise we are done. Let $v \in V(G)$ be chosen arbitrarily and let n_k be the number of vertices in G of distance k from v. Then $n_0 = 1$, $n_1 = \deg_G v \leq d-1$, and $n_k \leq n_{k-1}(d-2)$ for all $k \geq 2$. Hence,

$$n_0 + n_1 + \cdots + n_{t-1} \leq 1 + (d-1) + (d-1)(d-2) + (d-1)(d-2)^2 + \cdots + (d-1)(d-2)^{t-2}.$$

In addition, when $d, t \geq 3$,

$$1 + (d-1) + (d-1)(d-2) + (d-1)(d-2)^2 + \cdots + (d-1)(d-2)^{t-2} < |G|.$$

This implies that $n_t \neq 0$. Hence, G contains a path of length t starting from v. □

The next theorem is a special case of Theorem 1.1.12. Given a connected graph G and a large subset X of $V(G)$, the unavoidable topological minors containing many elements of X are a path, a subdivided star, or a subdivided comb. If in addition, we know that vertices of X have degree 1 in G, then we can eliminate the path possibility.

Theorem 2.1.4. *There exists a function $f_{2.1.4}(d,t)$ where $d,t \geq 3$ with the following property. Let T be a tree with at least $f_{2.1.4}(d,t)$ leaves. Then T contains a subdivided $K_{1,d}$ or a subdivided $\mathcal{C}_t$ whose leaves are the leaves of T.*

Proof. Let $k = \max(d,t)$ and let X be the set of leaves of T. Let $f_{2.1.4}(d,t) = f_{1.1.12}(k)$. Note that every element of X has degree one in T. By Theorem 1.1.12, T contains one of the following subgraphs

1. a path containing at least k vertices of X,
2. a subdivided $K_{1,k}$ whose leaves belong to X,
3. a subdivided $\mathcal{C}_k$ whose leaves belong to X.

Note that statement 1 is not possible because vertices of X have degree 1 in T. Therefore, T contains a subdivided $K_{1,d}$ or a subdivided $\mathcal{C}_t$ whose leaves are the leaves of T. □

We will need a stronger version of Theorem 2.1.4. We want to insist that in case T contains a subdivided a comb, the leaves of the comb are arranged in a nice way with respect to T.

Definition 2.1.5. Let T be a tree whose leaves are labeled $u_1, u_2, \ldots, u_k$ where $k \geq 3$. Suppose T contains K, a subdivided comb $\mathcal{C}_n$ whose leaves are the leaves of T. If a leaf sequence $u_{i_1}, u_{i_2}, \ldots, u_{i_n}$ of K satisfies $i_1 < i_2 < \cdots < i_n$ or $i_n < i_{n-1} < \cdots < i_1$, then we say K is **straight with respect to** T (or simply **straight** when the tree T is clear).

Theorem 2.1.6. *There exists a function $f_{2.1.6}(m,n)$ where $m,n \geq 3$ with the following property. Let T be a tree with at least $f_{2.1.6}(m,n)$ leaves. Then T contains a subdivided $K_{1,m}$ or a subdivided straight $\mathcal{C}_n$ whose leaves are the leaves of T.*

Proof. Let $k = R(n,n)$ and let $f_{2.1.6}(m,n) = f_{2.1.4}(m,k)$. We label the leaves of T as $u_1, u_2, \ldots, u_l$ where $l \geq f_{2.1.4}(m,k)$. By Theorem 2.1.4, T contains a subdivided $K_{1,m}$ or a subdivided $\mathcal{C}_k$ whose leaves are the leaves of T. If T contains a subdivided $K_{1,m}$, then the theorem holds. Otherwise, T contains a subdivided $\mathcal{C}_k$ whose leaf sequence is labeled $v_1, v_2, \ldots, v_k$, so that each v_i corresponds to a leaf u_{i_j} of T. Let $H = K^k$ be a complete graph on $\{v_1, v_2, \ldots, v_k\}$. We color an edge $v_a v_b$ of H red if $a < b$ and $i_a < i_b$ and blue if $a < b$ and $i_a > i_b$. By the definition of k, the graph H contains a monochromatic subgraph K^n. This yields a subdivided straight $\mathcal{C}_n$. □

We now turn back to discuss unavoidable structures of large graphs. Recall that at the beginning, we have a theorem about the unavoidable topological minors of 3-connected graphs with many vertices. Since all graphs in this thesis are simple, we can also determine the unavoidable topological minors of 3-connected graphs with many edges as well. The following is a reformulation of Theorem 1.1.7. We want to use this theorem because it is essential later on in our proof.

Theorem 2.1.7. *There exists a function $f_{2.1.7}(t)$ where $t \geq 3$ with the following property. Let G be a 3-connected graph with at least $f_{2.1.7}(t)$ edges. Then G contains a subdivided H for some H in $\{K_{3,t}, W_t, V_t\}$.*

Proof. Let n be determined as in Theorem 1.1.7 and let $f_{2.1.7}(t) = \binom{n}{2}$. Then $|G| \geq n$

because G is simple. The theorem then follows from Theorem 1.1.7. $\square$

Finally, we discuss the concepts of cycles and chords and examine chord arrangements.

Definition 2.1.8. Let $n \geq 3$. A **cycle** C is a graph whose vertex set is $\{x_1, x_2, \ldots, x_n\}$ and whose edge set is $\{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$. We denote C by listing its vertices, in the order as they appear on C, so we will write $C = x_1x_2 \ldots x_n$. The **length** of a cycle is the number of its edges, which is also the same as the number of its vertices. A cycle of length n is denoted as C_n. We call a C-path a C**-chord** (or simply **chord** when the cycle C is clear).

Definition 2.1.9. Let $\{M_1, M_2, \ldots, M_k\}$ be a set of k pairwise internally disjoint chords of a cycle C. For each i, let x_i, y_i be the endpoints of M_i on C. The set $\{M_1, M_2, \ldots, M_k\}$ is of

- arrangement 1 if $x_1 = x_2 = \cdots = x_k$ and $y_1, y_2, \ldots, y_k$ are distinct,
- arrangement 2 if the chords are pairwise disjoint and their endpoints appear in the order $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$,
- arrangement 3 if the chords are pairwise disjoint and their endpoints appear in the order $x_1, x_2, \ldots, x_k, y_k, \ldots, y_2, y_1$,
- arrangement 4 if the chords are pairwise disjoint and their endpoints appear in the order $x_1, y_1, x_2, y_2, \ldots, x_k, y_k$.

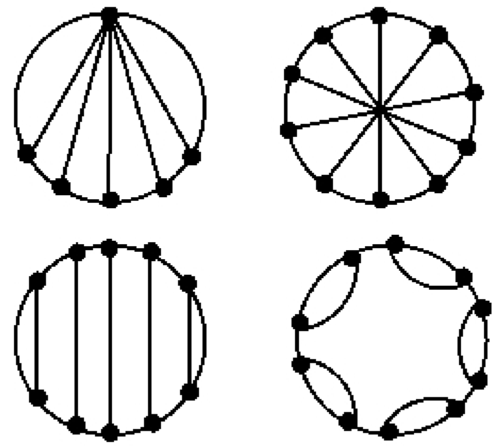

Figure 2.1. Top left: arrangement 1, top right: arrangment 2, bottom left: arrangement 3, bottom right: arrangement 4

To examine chord arrangements, we need a stronger version of Ramsey Theorem stated at the beginning of this thesis. In fact, Ramsey Theorem can be formulated using the language of coloring, which gives us the following theorem. We will refer to this result as Ramsey Theorem from now on.

Theorem 2.1.10 (Ramsey Theorem)**.** *For any positive integers $t_1, t_2, \dots, t_n$, there exists an integer N satisfying the following. For any function $\pi : E(K_N) \to \{1, 2, \dots, n\}$, there exists an $i \in \{1, 2, \dots, n\}$ such that the subgraph formed by edges e with $\pi(e) = i$ contains a clique of size t_i. The smallest such N is denoted as $R(t_1, t_2, \dots, t_n)$.*

Using Ramsey Theorem, we prove that if a cycle has many chords, then many of them will be of the same arrangement.

Theorem 2.1.11. *Let $t_1, t_2, t_3, t_4 \geq 3$ be integers. Then there exists a function $f_{2.1.11}(t_1, t_2, t_3, t_4)$ with the following property. Let C be a cycle with at least $f_{2.1.11}(t_1, t_2, t_3, t_4)$*

chords. Then in C, we can find a set of t_i chords of arrangement i for some $i \in \{1, 2, 3, 4\}$.

Proof. Let $f_{2.1.11}(t_1, t_2, t_3, t_4) = R(t_1, t_2, t_3, t_4) = N$ and let C be a cycle with at least $f_{2.1.11}(t_1, t_2, t_3, t_4)$ chords. Let $\pi : E(K_N) \to \{1, 2, 3, 4\}$ be a function. Then there exists an $i \in \{1, 2, 3, 4\}$ such that the subgraph formed by edges e with $\pi(e) = i$ contains a clique of size t_i. This yields a set of t_i chords of arrangement i in C. □

2.2. Edge Version

In this section, we prove Theorem 1.2.1. To do so, we prove the minor version of Theorem 1.2.1 and then open up the contracted vertices to obtain the topological minor result. We need the following definitions.

Definition 2.2.1. Let $n \geq 3$ and let $\{x_1, x_2, \ldots, x_n\}$ be a set of vertices. Let u, v be vertices not in $\{x_1, x_2, \ldots, x_n\}$. A $K_{2,n}$ is obtained by adding edges ux_i and vx_i for $i = 1, 2, \ldots, n$.

Definition 2.2.2. Let $n \geq 3$ and let G_1, G_2, G_3 be disjoint graphs such that each G_i is either a star or a comb with n leaves. For $i = 1, 2, 3$, we label the leaves of G_i as $x_1^i, x_2^i, \ldots, x_n^i$ (if G_i is a comb, then we label the leaves according to one of its leaf sequences). Let G be the graph obtained by identifying x_i^1, x_i^2, x_i^3, for $i = 1, 2, \ldots, n$, and then unsubdividing all vertices of degree two.

- If all of the G_i are stars, then we call G a $K_{3,n}$. For a $K_{3,n}$ or its subdivision, we use the term **cores** to denote its degree-n vertices and the term **children** to denote its cubic vertices.

- If exactly two of the G_i are stars, then we call G a $K^1_{3,n}$.
- If exactly one of the G_i is a star, then we call G a $K^2_{3,n}$.
- If none of the G_i is a star, then we call G a $K^3_{3,n}$.

Definition 2.2.3. Let $n \geq 3$ and let $C = x_1x_2 \ldots x_n$ be a cycle. A **wheel** W_n is obtained by adding a vertex u, called the **center**, and edges $ux_1, ux_2, \ldots, ux_n$. For $i = 1, 2, \ldots, n$, an edge ux_i is called a **spoke** and an edge x_ix_{i+1} (with $x_{n+1} = x_1$) is called a **rim**. For a subdivided wheel, we will use the terms spoke and rim to denote its subdivided spoke and subdivided rim respectively.

Definition 2.2.4. Let $n \geq 3$. Let $P = x_1x_2 \ldots x_n$ and $Q = y_1y_2 \ldots y_n$ be two disjoint paths. For every $i \in \{1, 2, \ldots, n\}$, we add an edge between x_i and y_i. A **ladder with a handle** V_n is the graph obtained by adding two vertices u, v, called the **grips**, and edges $uv, ux_1, uy_1, vx_n, vy_n$. We call x_iy_i $(i = 1, 2, \ldots, n)$ a **rung** and x_ix_{i+1} or y_iy_{i+1} $(i = 1, 2, \ldots, n-1)$ a **rail edge** of V_n. We call P, Q the **rails** and uv the **handle** of V_n. For a subdivided ladder, we will use the terms rung, rail, and handle to denote its subdivided rung, subdivided rail, and subdivided handle respectively.

Lemma 2.2.5. *There exists a function $f_{2.2.5}(t)$ where $t \geq 3$ with the following property. Let G be a 3-connected graph and let X be a subset of $E(G)$ with $|X| \geq f_{2.2.5}(t)$. Then G contains a $K_{3,m}$- or a W_m-minor, for some m, each containing at least t edges of X.*

Remark. *We can actually insist that $m = t$ in the statement of the lemma. However, to facilitate the case analysis in Theorem 1.2.1, we are not concerned with how big m is, as long as the minor contains at least t edges of X.*

Proof. Let $f_{2.2.5}(t) = \min\{f_{2.1.7}(t^2+3t), 7\}$. By proving the lemma on the largest 3-connected minor of G containing X, we may assume that no 3-connected proper minor of G contains X. Note that $G \neq K_4$ because K_4 has only six edges whereas $|X| \geq 7$. By Theorem 2.1.2, for every $e \in E(G) - X$, one of the following must be true

1. one endpoint of e is cubic in G and is incident with two edges of X,

2. e and two edges of X form a triangle.

Let E_1 be the set of edges in $E(G) - X$ satisfying statement 1 and let $E_2 = E(G) - (X \cup E_1)$. Since $|X| \geq f_{2.1.7}(t^2+3t)$, G contains a subdivision H of J for some J in $\{K_{3,n}, W_n, V_n \mid n \geq t^2+3t\}$. We choose H with the largest number of heavy component paths. The following observation is immediate.

Claim 2.2.5.1. *Let P be a light component path of H. Then G does not contain a P-path P' such that P' is also an H-path and P' contains an edge of X.*

Assume for a contradiction that there exists such a path P'. Let x, y be the two endpoints of P'. Since P is light, we can replace $P[xy]$ by P' and obtain a subdivision of J with more heavy component paths than H, contradicting the choice of H. Consequently, no such P' exists. This proves the claim.

We divide the remain of this proof into three cases.

Case 1: H is a subdivided $K_{3,n}$ where $n \geq t^2+3t$. Let u_1, u_2, u_3 be the cores and let $v_1, v_2, \ldots, v_n$ be the children of H. Let $P_{i,j}$ be the component path between u_i and v_j. A component path is called good if it contains at least one edge of $X \cup E_1$ and is called bad otherwise. A child vertex is called good if it belongs to at least one good component path and is called bad otherwise.

Claim 2.2.5.2. *If H contains at least t good children, then the lemma holds.*

Let $v_1, v_2, \ldots, v_t$ be the good children of H and let $H' \subseteq H$ be the subdivided $K_{3,t}$ with u_1, u_2, u_3 as the cores and $v_1, v_2, \ldots, v_t$ as the children. Then for every $k \in \{1, 2, \ldots, t\}$, there exists a component path $P_{i,k}$, for some $i \in \{1, 2, 3\}$, containing at least one edge of $X \cup E_1$ because v_k is good. If a component path $P_{i,k}$ contains an edge $e_k \in X$, then we associate v_k with this e_k. Otherwise, $P_{i,k}$ does not contain any edge belonging to X. Thus, in $P_{i,k}$, there exists an edge $e'_k \in E_1$. By definition, one endpoint of e'_k is cubic in G, call it x, and is incident with two edges of X. Observe that $x \neq u_i$ because $\deg_G(u_i) \geq n > 3$. Hence, $x = v_k$ or it is an internal vertex of $P_{i,k}$. If $x = v_k$, then v_k is cubic in G and is incident with two edges of X. Since $\deg_{H'}(v_k) = 3$, there exists an $i' \neq i$ such that the component path $P_{i',k}$ contains an edge $e_k \in X$. In this case, we associate v_k with this e_k. Otherwise, x is an internal vertex of $P_{i,k}$. But then this implies that $P_{i,k}$ contains an edge $e_k \in X$, which is not possible. We have shown that for every $k \in \{1, 2, \ldots, t\}$, we can associate a good child v_k with an edge $e_k \in X$. In addition, $e_1, e_2, \ldots, e_t$ are distinct because every e_k belongs to $E(P_{1,k} \cup P_{2,k} \cup P_{3,k})$. We now perform contraction in H' to obtain the desired $K_{3,t}$-minor according to the following procedure. For a component path containing an $e_k \in X$ that has been associated with a good v_k, we contract all edges except e_k in that component path. For every other component path, we contract it into an edge. Doing so yields a $K_{3,t}$-minor containing at least t edges of X. This proves the claim.

From the previous claim, we may assume that H has fewer than t good children and so it has at least t^2 bad children because $n \geq t^2 + 3t$. We choose t^2 bad children and label them $v_1, v_2, \ldots, v_{t^2}$. By the definition of being bad, $E(P_{i,k}) \subseteq E_2$ for every $i \in \{1, 2, 3\}$ and every $k \in \{1, 2, \ldots, t^2\}$. Let w_k be the neighbor of u_1 on $P_{1,k}$ for $k = 1, 2, \ldots, t^2$. Then for every k, there exists a z_k such that $u_1 w_k z_k$ is a triangle with $z_k u_1, z_k w_k \in X$. We call z_k the tip of v_k.

Claim 2.2.5.3. *For every $k = 1, 2, \ldots, t^2$, $z_k \in H$.*

If $z_k \notin H$ for some k, then we get a contradiction of Claim 2.2.5.1 by setting $P = P_{1,k}$ and $P' = z_k u_1 \cup z_k w_k$. This proves the claim.

Claim 2.2.5.4. *For every $k = 1, 2, \ldots, t^2$, $z_k \notin \{u_1, u_2, u_3\}$.*

Clearly, $z_k \neq u_1$ because G is simple. Assume for a contradiction that $z_k = u_2$ for some k. By replacing $P_{2,k}$ with $u_2 w_k$, we obtain a subdivided $K_{3,n}$ with more heavy component paths than H and this contradicts the choice of H. This proves the claim.

Claim 2.2.5.5. *For every $k = 1, 2, \ldots, t^2$, z_k does not belong to a component path for which one of its endpoint is a bad child.*

Assume for a contradiction that some z_k belongs to a $P_{i,l}$, for a bad child v_l, where $l = k$ is possible. If $i = 1$, then we get a contradiction of Claim 2.2.5.1 by setting $P = P_{1,l}$ and $P' = z_k u_1$. Hence, $i \neq 1$. By replacing $P_{1,l}$ with $u_1 z_k$, we obtain a subdivided $K_{3,n}$ with more heavy component paths than H and this contradicts the choice of H. This proves the claim.

From the previous three claims, we deduce that for every bad child v_k, its tip z_k is not a core vertex and belongs to a component path for which one of its endpoint is a good child. Since H has at least t^2 bad children and fewer than t good ones, there exists a good child v_l, for some l, such that $(P_{1,l} \cup P_{2,l} \cup P_{3,l}) - \{u_1, u_2, u_3\}$ contains at least t tips z_k of at least t bad children. We choose t bad children and label them $v_1, v_2, \ldots, v_t$. We now describe the process to obtain the desired $K_{3,m}$-minor. We first contract $(P_{1,l} \cup P_{2,l} \cup P_{3,l}) - \{u_2, u_3\}$ into a vertex u'. Next, for $k = 1, 2, \ldots, t$, we have the paths $P_{1,k}[w_k v_k] \cup w_k u'$ between u' and v_k, each of which contains at least one edge of X. Finally, for $k = 1, 2, \ldots, t$, we have the paths $P_{2,k}$ between u_2 and v_k and the paths $P_{3,k}$ between u_3 and v_k. This yields a subdivided $K_{3,t}$-minor with cores u', u_2, u_3 and children $v_1, v_2, \ldots, v_t$ where every $u'v_i$-path (for $i = 1, 2, \ldots, t$) contains at least one edge of X. This yields a $K_{3,t}$-minor in G containing at least t edges of X.

Case 2: H is a subdivided W_n where $n \geq t^2 + 3t$. We orient the rim cycle of H clockwise and call it C. Let u be the center of H. A spoke is called good if it contains at least one edge of $X \cup E_1$ and is called bad otherwise.

Claim 2.2.5.6. *If H contains at least t good spokes, then the lemma holds.*

Let H' consists of C and t good spokes of H. In H', let $v_1, v_2, \ldots, v_t$ be the cubic vertices on C, listed in the order as they appear on C. For $i = 1, 2, \ldots, t$, let S_i be the uv_i-spoke of H' and let Q_i be the directed $v_i v_{i+1}$-rim on C (with $v_{n+1} = v_1$). Observe that each S_i contains an edge of $X \cup E_1$ by the definition of being good. If an S_i contains an edge $e_i \in X$, then we associate S_i with this e_i. Otherwise, S_i does not contain any edge belonging to X. Thus, it contains an edge $e_i' \in E_1$. By definition, one endpoint of e_i' is cubic in G, call it x, and is incident with two edges of X. Observe that $x \neq u$ because $\deg_G(u) \geq n > 3$. Hence, $x = v_i$ or x is an internal vertex of S_i. If x is an internal vertex of S_i, then S_i contains an edge $e_i \in X$, which is not possible. Otherwise, $x = v_i$. This means that v_i is cubic and is incident with two edges of X. Since $\deg_{H'}(v_i) = 3$, Q_i contains an edge $e_i \in X$. We associate S_i with this e_i. We have shown that every good S_i can be associated with an edge $e_i \in X$. In addition, $e_1, e_2, \ldots, e_t$ are distinct because every e_i belongs to $E(S_i \cup Q_i)$. To obtain the desire W_t-minor, we contract H' as following. For a component path containing an $e_i \in X$ that has been associated with a good S_i, we contract all edges except e_i in that component path. For every other component path, we contract it into an edge. Since there are t good spokes, we obtain a W_t-minor containing at least t edges of X. This proves the claim.

From the previous claim, we may assume that H has fewer than t good spokes and so it has at least $t^2 + 2t$ bad spokes because $n \geq t^2 + 3t$. Let S be a bad spoke and let v be the endpoint of S on C. By definition, $E(S) \subseteq E_2$. Let w be the neighbor of u on S. Then there exists a vertex z such that uwz forms a triangle with $zu, zw \in X$. We call z the tip of S.

Claim 2.2.5.7. *$z \in H - u$.*

If $z \notin H$, then we get a contradiction of Claim 2.2.5.1 by setting $P = S$ and $P' = zu \cup zw$. In addition, $z \neq u$ because G is simple. This proves the claim.

Claim 2.2.5.8. *z belongs to a good spoke.*

Since $z \in H$ and $z \neq u$, either z is an internal vertex of a rim or z belongs to a spoke. If z is an internal vertex of a rim, then $H \cup uz$ is a subdivided W_{n+1} with more heavy component paths than H, contradicting the choice of H. This means that z belongs to a spoke. If z belongs to a bad spoke S', where $S' = S$ is possible, then we get a contradiction of Claim 2.2.5.1 by setting $P = S'$ and $P' = zu$. Therefore, if z belongs to a spoke, then it belongs to a good spoke. This proves the claim.

We have shown that for every bad spoke S, its tip z is not the center and belongs to a good spoke. Since H has at least $t^2 + 2t$ bad spokes and fewer than t good spokes, there exists a good spoke S_g such that $S_g - u$ contains at least $t + 2$ such tips z. We choose $t + 2$ of those bad spokes and label them as $S_1, S_2, \dots, S_{t+2}$, so that all of their corresponding tips $z_1, z_2, \dots, z_{t+2}$ belong to $S_g - u$. Let v_i be the endpoint of S_i on C and let w_i be the neighbor of u on S_i for $i = 1, 2, \dots, t+2$. Let w be the neighbor of u on S_g and let v be the endpoint of S_g on C. Then on C, we may assume, without loss of generality, that v is between v_1 and v_2. We construct the desire W_t-minor as following. Let Q be the v_1v_2-subpath of C that is disjoint from S_g and let $D = S_1 \cup S_2 \cup Q$. Then D is a cycle. Let $M = D \cup (S_g - u)$ and let $R_i = (S_i \cup z_iw_i) - u$ for $i = 3, 4, \dots, t+2$. The subgraph $(\bigcup_{i=3}^{t+2} R_i) \cup M$ contains a W_t-minor containing at least t edges of X.

Case 3: H is a subdivided V_n where $n \geq t^2 + 3t$. Let u, v be the grips and let P, Q be the rails of H. A rung is called good if it contains at least one edge of $X \cup E_1$ and is called bad otherwise.

Claim 2.2.5.9. *If H contains at least $3t + 1$ good rungs, then the lemma holds.*

Let $R_1, R_2, \dots, R_{3t+1}$ be $3t+1$ good rungs of H, listed in the order they appear along the ladder, where each R_i has endpoints $x_i \in P, y_i \in Q$. Let H' be the subgraph of H that is obtained from H by deleting edges and internal vertices of other rungs. Then H' is a subdivided V_{3t+1} whose rungs are $R_1, R_2, \ldots, R_{3t+1}$. In H', let P_i be the subdivided x_ix_{i+1}-rail edge and Q_i be the subdivided y_iy_{i+1}-rail edge for $i = 1, 2, \ldots, 3t$. Observe that each R_i contains an edge of $X \cup E_1$ by the definition of being good. If R_i contains an edge $e_i \in X$, then we associate R_i with this e_i. Otherwise, R_i does not contain any edge belonging to X. Thus, it contains an edge $e_i' \in E_1$. By definition, one endpoint of e_i' is cubic in G, call it x, and is incident with two edges of X. If x is an internal vertex of R_i, then R_i contains an edge $e_i \in X$, which is not possible. Otherwise, $x = x_i$ or $x = y_i$. This means that x_i or y_i is cubic and is incident with two edges of X. Since $\deg_{H'}(x_i) = \deg_{H'}(y_i) = 3$, P_i or Q_i contains an edge $e_i \in X$. We associate R_i with this e_i. We have shown that every good R_i can be associated with an edge $e_i \in X$. In addition, $e_1, e_2, \ldots, e_{3t}$ are distinct because every e_i belongs to $E(R_i \cup P_i \cup Q_i)$. Note that each chosen $e_i \in X$ is on a rung or a rail. Since there are $3t$ such chosen e_i, at least t of them are on the rungs or at least t of them are on the same rail. To obtain the desire W_m-minor, we do the following to H'. For a component path contains an $e_i \in X$ that has been associated with a good R_i, we contract all edges except e_i in that component path. For every other component path, we contract it into an edge. First, suppose at least t of those e_i are on the rungs. By contracting one of the rails into a single vertex, we obtain a W_m-minor with at least t edges of X. Now suppose at least t of them are on a rail, say P. By contracting Q into a single vertex, we obtain a W_m-minor with at least t edges of X. This proves the claim.

From the previous claim, we may assume that H has fewer than $3t+1$ good rungs. This implies that H has at least t^2 bad rungs because $n \geq t^2+3t$. We choose t^2 of them and label them as $R_1, R_2, \ldots, R_{t^2}$, in the order as they appear along the ladder. For each R_i, let $x_i \in P$ and $y_i \in Q$ be its two endpoints. By the definition of being bad, $E(R_i) \subseteq E_2$ for every $i \in \{1, 2, \ldots, t^2\}$. Let w_i be the neighbor of x_i on R_i. Then there exists a vertex z_i such that $z_i x_i w_i$ forms a triangle with $z_i x_i, z_i w_i \in X$.

Claim 2.2.5.10. *$z_i \in H$ for $i = 1, 2, \ldots, t^2$.*

If $z_i \notin H$, then we get a contradiction of Claim 2.2.5.1 by setting $P = R_i$ and $P' = z_i x_i \cup z_i w_i$. This proves the claim.

Let A be the handle of H and let a, b be the endpoints of P such that a is adjacent to u and b is adjacent to v. We define B to be the union of P and the ua, bv-component paths of H and let $D = H - (A \cup B)$. From the previous claim, we deduce that each z_i belongs to one of the subgraphs A, B, or D because $V(H) = V(A) \cup V(B) \cup V(D)$. Since there are t^2 such z_i, at least t of them belong to one of the following

- $V(B)$. In this case, we contract $\mathring{B}$ into a vertex.
- $V(A - B)$. In this case, we contract $\mathring{A}$ into a vertex.
- $V(D)$. In this case, we contract D into a vertex.

This yields a minor of G, which contains a W_m-minor with at least t edges of X. □

We will now prove the edge version.

Proof of Theorem 1.2.1. Let $a = f_{2.1.6}(t, t)$, $b = f_{2.1.6}(a, a)$, and $c = f_{2.1.6}(b, b)$. Let

$f_{1.2.1}(t) = f_{2.2.5}(3c)$. We divide the proof into two cases.

Case 1: G contains a W_m-minor, for some m, containing at least $3c$ edges of X. Let H be this W_m-minor. Then in G, there exist a cycle C, a tree T disjoint from C, and m edges $\{e_1, e_2, \ldots, e_m\}$ between T and C where the endpoints of e_i on C are disjoint. Since H contains at least $3c$ edges of X, either C contains at least c heavy component paths or at least c edges e_i belong to X. If C contains at least c heavy component paths, then let $\mathcal{S} = \{e_i \mid e_i \text{ is incident with a heavy component path}\}$. Otherwise, at least c edges e_i belong to X, and we let $\mathcal{S}$ be the set of those e_i. Observe that $|\mathcal{S}| \geq c$. Let S be the union of all edges in $\mathcal{S}$ and let $Y = V(C) \cap V(S)$. Then $|Y| \geq c$. Let T' be the minimal subtree of $T \cup S$ such that the leaves of T' are elements of Y. Then T' contains a subdivided $K_{1,b}$ or a subdivided straight $\mathcal{C}_b$ whose leaves are the leaves of T'. This yields a subdivided W_b or a subdivided V_b with edge-weight at least t in G.

Case 2: G does not contain a W_m-minor, for any m, containing at least $3c$ edges of X. Then G contains a $K_{3,n}$, for some n, containing at least $3c$ edges of X. Let H be this $K_{3,n}$-minor and let u_1, u_2, u_3 be the cores of H. A child v of H is called type i, for some $i \in \{1, 2, 3\}$, if $vu_i \in X$. Note that a child may belong to more than one types. Since H has at least $3c$ edges of X, we may assume, without loss of generality, that it has at least c children of type 1. Let $v_1, v_2, \ldots, v_l$ be all the children of type 1 in H for some $l \geq c$. Let H' be the $K_{3,l}$ whose cores are u_1, u_2, u_3, whose children are $v_1, v_2, \ldots, v_l$, and whose edges are edges of H between u_i, v_j for $i \in \{1, 2, 3\}$ and $j \in \{1, 2, \ldots, l\}$. Note that by the construction of H', $u_1v_i \in X$ for all $i \in \{1, 2, \ldots, l\}$.

Since H' is also a minor of G, there exists an embedding π. In $G|H'$, let

$$M_1 = \left(\bigcup_{j=1}^{l} \pi(u_1 v_j)\right) \cup \pi(u_1)$$

and let $N_1 = V(M_1) \cap \left(\bigcup_{j=1}^{l} \pi(v_j)\right)$. Then M_1 is connected and every vertex in N_1 has degree 1 in M_1. Let T_1 be the minimal subtree of M_1 whose leaves are elements of N_1. Since $|N_1| = l \geq c$, T_1 contains Z_1 that is subdivided $K_{1,b}$ or a subdivided straight $\mathcal{C}_b$ whose leaves are the leaves of T_1.

Let $x_1, x_2, \ldots, x_b$ be the leaves of Z_1 where each x_j belongs to some $\pi(v_{i_j})$. Let

$$M_2 = \left(\bigcup_{j=1}^{b} \pi(u_2 v_{i_j})\right) \cup \pi(u_2)$$

and let $N_2 = V(M_2) \cap \left(\bigcup_{j=1}^{b} \pi(v_{i_j})\right)$. Then M_2 is connected and every vertex in N_2 has degree 1 in M_2. Let T_2 be the minimal subtree of M_2 whose leaves are elements of N_2. Since $|N_2| = b$, T_2 contains Z_2 that is subdivided $K_{1,a}$ or a subdivided straight $\mathcal{C}_a$ whose leaves are the leaves of T_2.

Let $y_1, y_2, \ldots, y_a$ be the leaves of Z_2 where each y_j belongs to some $\pi(v_{k_j})$. Let

$$M_3 = \left(\bigcup_{j=1}^{a} \pi(u_3 v_{k_j})\right) \cup \pi(u_3)$$

and let $N_3 = V(M_3) \cap \left(\bigcup_{j=1}^{a} \pi(v_{k_j})\right)$. Then M_3 is connected and every vertex in N_3 has degree 1 in M_3. Let T_3 be the minimal subtree of M_3 whose leaves are elements of N_3. Since $|N_3| = a$, T_3 contains Z_3 that is subdivided $K_{1,t}$ or a subdivided straight $\mathcal{C}_t$ whose leaves are the leaves of T_3.

Recall that each of the Z_1, Z_2, or Z_3 has two possibilities, a subdivided star or a subdivided comb. To complete the proof, we divide the analysis into subcases, depending on the choice of Z_1, Z_2, and Z_3.

Case 2a: At least two of them are subdivided combs. Then G contains a subdivided V_t with edge-weight at least t.

Case 2b: Exactly one of them is a subdivided comb. Then G contains a subdivided W_t with edge-weight at least t.

Case 2c: All of them are subdivided stars. Then G contains a subdivided $K_{3,t}$ with edge-weight at least t. □

2.3. Vertex Version

In this section, we prove Theorem 1.2.2. To do so, we prove the minor version of Theorem 1.2.2 and then open up the contracted vertices to obtain the topological minor result.

First, it is helpful to mention the notion of suppressing a vertex of G. As we will see below, this operation produces a graph that is isomorphic to a minor of G while still preserving the vertices in X. We then discuss the idea of X-preserving minor that is central to the proof of the vertex version.

Definition 2.3.1. Let z be a vertex of degree 2 in G and let u, v be the neighbors of z. By **suppressing** z we mean deleting z and in addition, adding an edge between u, v if $uv \notin E(G)$.

Remark. *Let z be a vertex of degree 2 in G and let u, v be the neighbors of z. Note that* *suppressing z produces a graph that is isomorphic to a minor of G (the minor is G/uz or G/vz). However, we want to distinguish between suppressing z from contracting uz (or vz), even though both produce two isomorphic graphs. When we suppress z, the vertices u, v in the resulting graph are still vertices of G. This is not the case if we contract uz or vz.*

Definition 2.3.2. Let H be a subgraph of G and let $X \subseteq V(H)$ such that for every $v \in V(H) - X$, $\deg_H v \geq 2$. Assume a graph G' can be obtained from G by a sequence of the following operations, in any order

- deleting an edge uv where $u, v \notin H$,
- contracting an edge uv where $u \notin H$ and $v \notin X$,
- suppressing a vertex not in X.

In addition, G' has a subgraph H' where H' is obtained from H by suppressing vertices of $V(H) - X$ and $X \subseteq V(H')$. Then we say (G', H') is an X**-preserving minor** of (G, H).

Remark. *We want to point out that G' is not minor of G, but it is isomorphic to a minor of G.*

Lemma 2.3.3. *Let $G \neq K_4$ be a 3-connected graph and let H be a subgraph of G. Let $X \subseteq V(H)$ such that for every $v \in V(H) - X$, $\deg_H v \geq 2$. Then there exists an X-preserving minor (G', H') of (G, H) satisfying the following*

1. *G' is 3-connected,*
2. *for every $v \in V(G') - V(H')$, all neighbors of v belong to X.*

Proof. Let $e \in E(G)$ whose both endpoints are not in H. Then G/e is 3-connected or $G\backslash e$

is a subdivision of a 3-connected graph by Theorem 2.1.1. If G/e is 3-connected, then we contract e. Otherwise, $G\backslash e$ is a subdivision of a 3-connected graph, for which we delete e and suppress any resulting degree-2 vertices. By repeating this process for all edges of G whose both endpoints are not in H, we obtain an X-preserving minor (G', H) of (G, H) where G' is 3-connected. In addition, $V(G') - V(H)$ is stable.

Let $v \in V(G')-V(H)$ and suppose v has a neighbor $u \in V(H)-X$. Let $e = uv$. Then G'/e is 3-connected or $G'\backslash e$ is a subdivision of a 3-connected graph. If G'/e is 3-connected, then we contract e. Otherwise, $G'\backslash e$ is a subdivision of a 3-connected graph, for which we delete e and suppress any resulting degree-2 vertices. By repeating this process, we obtain the desired X-preserving minor. □

We now turn our attention to rooted trees, which in essence is a tree with a specified vertex as a root. Let T be a tree and let $u, v \in V(T)$. Then there exists a unique path between u and v in T. We denote this unique path as uTv and we adopt this notation for the next few definitions and lemmas.

Definition 2.3.4. Let r be a vertex in a tree T. We call (T, r) a **rooted tree** with r as its **root**. For two vertices $x, y \in T$, we say y is a **child** of x if $x \in rTy$ and x is adjacent to y in T. We say x, y are **comparable** if $x \in rTy$ or $y \in rTx$. Let G be a graph and let (T, r) be a rooted tree in G. We say (T, r) is a **normal tree** of G if the endpoints of every T-path in G are comparable.

We have the following two rephrases in [4].

Lemma 2.3.5 (Lemma 1.5.5 in [4])**.** *Every connected graph contains a normal spanning tree* *with any specified vertex as its root.*

Lemma 2.3.6 (Lemma 1.5.6 in [4])**.** *Let (T, r) be a normal tree of G and let $x, y \in V(T)$. Then x, y are separated in G by $V(rTx) \cap V(rTy)$.*

The following corollary is needed.

Corollary 2.3.7. *Let (T, r) be a normal tree of G and let $v \in V(T)$. Then in $G - rTv$, no two children of v belongs to the same component.*

Proof. Let x, y be two distinct children of v. Then $V(rTx) \cap V(rTy) = V(rTv)$. The corollary then follows from the previous lemma. □

We have seen that a large 3-connected graph contains a large wheel or a large $K_{3,n}$ as a minor. In their paper, Ding, Dziobiak, and Wu determines the requirement to have each of these two as an unavoidable minor. Informally, their result states that a large 3-connected graph containing a long path must contain a large wheel as a minor and conversely, a large 3-connected graph without a long path must contain a large $K_{3,n}$ as a minor. The following result is a reformulation of Theorem 3.8 in [5].

Lemma 2.3.8. *There exists a function $f_{2.3.8}(t)$ where $t \geq 3$ with the following property. Let G be a 3-connected graph that contains a path of length $f_{2.3.8}(t)$. Then G contains a W_t-minor.*

In the next two lemmas, we prove an equivalence of Theorem 2.1 and Theorem 3.8 in [5] for rooted graphs. Our results also establish that the existence of a long path (or a lack thereof) determines whether a large wheel (or a large $K_{3,n}$) exists as a minor.

Lemma 2.3.9. *Let $t \geq 3$, $n = f_{2.1.11}(t, t, t, t)$, and $a = f_{2.3.8}(tn + t)$. Let G be a 3-*

connected graph such that G has no path of length a. Let X be a subset of $V(G)$ such that $|X| \geq f_{2.1.3}\left(t\binom{a}{3}+a, t\binom{a}{3}+a\right)$. Then G contains a subdivided $K_{3,t}$ where all cubic vertices belong to X.

Proof. We define the height $h(T)$ of a tree T to be the length of its longest path. Let (T, r) be a normal spanning tree of G for some specified r, whose existence is guaranteed by Lemma 2.3.5. Now $h(T) < a$ because G has no path of length a. Let T' be the minimal subtree of T containing $X \cup \{r\}$. Then (T', r) is a rooted tree with $h(T') < a$. In addition, every leaf of T' belongs to X.

Claim 2.3.9.1. *T' has a vertex with at least $t\binom{a}{3}$ children.*

Since $|X| \geq f_{2.1.3}\left(t\binom{a}{3}+a, t\binom{a}{3}+a\right)$ and $X \subseteq V(T')$, either $\Delta(T') \geq t\binom{a}{3}+a$ or T' contains a path of length $t\binom{a}{3}+a$. The latter is not possible because G has no path of length a. Thus, $\Delta(T') \geq t\binom{a}{3}+a$, so T' has a vertex with at least $t\binom{a}{3}$ children. This proves the claim.

Let $v \in V(T')$ that has at least $t\binom{a}{3}$ children. We choose $t\binom{a}{3}$ of those children and we label them as $u_1, u_2, \ldots, u_{t\binom{a}{3}}$. By Corollary 2.3.7, in $G - rTv$, no two children of v belongs to the same component. Let G_i be the component containing u_i in $G - rTv$ for $i = 1, 2, \ldots, t\binom{a}{3}$. Observe that every G_i contains a leaf l_i of T', which belongs to X. By Menger Theorem, there exist three weakly disjoint $l_i(rTv)$-paths P_i, Q_i, R_i in G. Note that if $i \neq j$, then $P_i \cup Q_i \cup R_i$ only intersects $P_j \cup Q_j \cup R_j$ on $V(rTv)$. Let a_i, b_i, c_i be the endpoints of P_i, Q_i, R_i in $V(rTv)$ respectively. Since there are fewer than $\binom{a}{3}$ possible choices for a_i, b_i, c_i (because $|V(rTv)| < a$), whereas there are $t\binom{a}{3}$ possible l_i, at least t of those l_i all have the same a_i, b_i, c_i. The union of all such P_i, Q_i, R_i yields the desired subdivided $K_{3,t}$ in G. $\square$

Lemma 2.3.10. *Let $t \geq 3$, $n = f_{2.1.11}(t,t,t,t)$, and $a = f_{2.3.8}(tn+t)$. Let G be a 3-connected graph such that G has a path of length a. Let X be a subset of $V(G)$ such that $|X| \geq f_{2.1.3}\left(t\binom{a}{3}+a, t\binom{a}{3}+a\right)$ and $V(G)-X$ is a stable set. Then G contains a minor H where H is isomorphic to a graph obtained from W_t by subdividing its rims. In addition, all non-center cubic vertices of H are firm and belong to X.*

Proof. Since G has a path of length a, by Lemma 2.3.8, G has a W_{tn+t}-minor. This means that G has subgraph H, consisting of a cycle C, a tree T disjoint from C, and edges $\{e_i \mid i = 1,2,\ldots,tn+t\}$ between C and T where the endpoints of all e_i are disjoint on C. For each $i \in \{1,2,\ldots,tn+t\}$, let v_i be the endpoint of e_i on C. If at least t vertices, say $v_1, v_2, \ldots, v_t$, belong to X, then the union of those v_i and C and T contains the desired minor. Otherwise, fewer than t vertices v_i belong to X, so at least tn vertices v_i do not belong to X. We relabel those vertices as $v'_1, v'_2, \ldots, v'_{tn}$ and for each v'_i, let e'_i be the edge of H with v'_i as one of its endpoints and the other endpoint belongs to T. Let K be the union of C, T, and those e'_i (for $i = 1,2,\ldots,tn$).

Claim 2.3.10.1. *Every path of $C - \{v'_1, v'_2, \ldots, v'_{tn}\}$ has a vertex $w'_i \in X$ for $i = 1,2,\ldots,tn$.*

Since $V(G)-X$ is stable, for every $v \in V(G)-X$, all of its neighbors are in X. Thus, for every v'_i, both its neighbors in C are in X. This proves the claim.

Let $X' = \{w'_1, w'_2, \ldots, w'_{tn}\}$. Clearly, $X' \subseteq V(K)$. We now apply Lemma 2.3.3 on (G,K) to obtain an X-preserving minor (G',K'). Note that G' is 3-connected and for every vertex in $V(G') - V(K')$, all of its neighbors belong to X'. In G'/T, let u be the contracted T.

Claim 2.3.10.2. *G'/T is 3-connected.*

Clearly, G'/T is connected. Assume for a contradiction that G'/T has a separator Y of size 1 or 2 separating $A, B \subseteq V(G'/T)$ where A, B is a partition of $V(G'/T)$. Note that $u \in Y$ for otherwise, G' has a separator of size 1 or 2. This means that $V(C) \subseteq A - u$ or $V(C) \subseteq B - u$. Without loss of generality, we may assume $V(C) \subseteq A - u$. Let $b \in B - Y$. Then b has at least 3 neighbors because G' is 3-connected. But every neighbor of b must be in X' and $X' \subset V(C) \subseteq A - u$. Hence, b has at least one neighbor in $A - Y$ and this is not possible. Therefore, no such separator Y exists. This proves the claim.

For the remain of this proof, by bridge we mean a (K'/T)-bridge of G'/T and by chord we mean a C-chord. Note that for every bridge, its feet belong to X'.

Claim 2.3.10.3. *If there exists a bridge with at least t feet, then the lemma holds.*

Let B be a bridge with at least t feet and let Y be the set of feet of B. Then $Y \subseteq X'$ and $|Y| \geq t$. By contracting $B - Y$ into a single vertex, we obtained the desired minor. This proves the claim.

From the previous claim, we may assume that every bridge has fewer than t feet. Since every foot of a bridge belongs to X' and $|X'| = tn$, there are at least n bridges. Now every bridge B has two distinct feet $x, y \in X'$. Let Q be an xy-path in B. Then Q is a chord. We have shown that every bridge contains at least one chord, so there are at least n chords because there are at least n bridges. Additionally, two different chords are internally disjoint because two different chords are subpaths of two different bridges. By the definition of n, we can find a set $\mathcal{S}$ of t chords of arrangement i for some $i \in \{1,2,3,4\}$. Let S be the union of all chords in $\mathcal{S}$. To make the last part of the proof more convenient, in the set of t chords of arrangement i, we relabel each chords to have endpoints x_j, y_j for $j = 1, 2, \dots, t$. Note that $x_j, y_j \in X'$ for every $j \in \{1, 2, \dots, t\}$.

First, suppose $i = 1$. This means that $x_1 = x_2 = \cdots = x_t$ and $y_1, y_2, \dots, y_t$ are distinct. Without loss of generality, we may assume that the endpoints of the chords appear in the order $x_1, y_1, y_2, \dots, y_t$. Let P be the x_1y_1-subpath of C that does not contain y_t and let Q be the x_ty_t-subpath of C that does not contain y_1. By the construction of X', there exist a $v'_a \in \mathring{P}$ and a $v'_b \in \mathring{Q}$ such that both v'_a, v'_b are adjacent to u. Let R be the $v'_av'_b$-subpath of C that does not contain x_1. Then $R \cup v'_au \cup v'_bu$ is a cycle, call it C_1. The subgraph $C_1 \cup S$ yields the desired minor.

Next, suppose $i = 2$. This means that the chords are pairwise disjoint and their endpoints appear in the order $x_1, x_2, \dots, x_t, y_1, y_2, \dots, y_t$. Let P be the x_1y_t-subpath of C that does not contain x_t and let Q be the x_ty_1-subpath of C that does not contain x_1. By the construction of X', there exist a $v'_a \in \mathring{P}$ and a $v'_b \in \mathring{Q}$ such that both v'_a, v'_b are adjacent to u. Let R be the $v'_av'_b$-subpath of C that does not contain x_1. Then $R \cup v'_au \cup v'_bu$ is a cycle, call it C_2. Let R' be the x_1x_t-subpath of C that is disjoint from R. The subgraph $C_2 \cup S \cup R'$ yields the desired minor.

Next, suppose $i = 3$. This means that the chords are pairwise disjoint and their endpoints appear in the order $x_1, x_2, \dots, x_t, y_t, \dots, y_2, y_1$. Let P be the x_1y_1-subpath of C

that does not contain x_t and let Q be the x_ty_t-subpath of C that does not contain x_1. By the construction of X', there exist a $v'_a \in \mathring{P}$ and a $v'_b \in \mathring{Q}$ such that both v'_a, v'_b are adjacent to u. Let R be the $v'_av'_b$-subpath of C that does not contain x_1. Then $R \cup v'_au \cup v'_bu$ is a cycle, call it C_3. Let R' be the x_1x_t-subpath of C that is disjoint from R. The subgraph $C_3 \cup S \cup R'$ yields the desired minor.

Finally, suppose $i = 4$. This means that the chords are pairwise disjoint and their endpoints appear in the order $x_1, y_1, x_2, y_2, \dots, x_t, y_t$. For each i, let Q_i be the x_iy_i-subpath of C that does not contain any other x_j. By the construction of X', there exist a $v'_{a_i} \in \mathring{Q}_i$ such that v'_{a_i} is adjacent to u for every $i \in \{1, 2, \dots, t\}$. Let $Z = \bigcup_{i=1}^{t} Q_i$ and let $C_4 = (C \cup S)\backslash E(Z)$. Then C_4 is a cycle. The subgraph $(\bigcup_{i=1}^{t} uv'_{a_i} \cup Q_i[x_iv'_{a_i}]) \cup C_4$ yields the desired minor. □

We will now prove the minor version of Theorem 1.2.2.

Lemma 2.3.11. *There exists a function $f_{2.3.11}(t)$ where $t \geq 3$ with the following property. Let G be a 3-connected graph and let X be a subset of $V(G)$ such that $|X| \geq f_{2.3.11}(t)$. Then G contains one of the following*

1. *a minor H that is isomorphic to a $K_{3,t}$ where all cubic vertices are firm and belong to X,*

2. *a minor H that is isomorphic to a graph obtained from W_t by subdividing its rims. In addition, all non-center cubic vertices of H are firm and belong to X.*

Proof. Let $n = f_{2.1.11}(t, t, t, t)$ and let $a = f_{2.3.8}(tn+t)$. Let $f_{2.3.11}(t) = f_{2.1.3}\left(t\binom{a}{3} + a, t\binom{a}{3} + a\right)$. We first prove that there exists a 3-connected graph G' containing X such that G' is isomorphic to a minor of G and $V(G') - X$ is a stable set. Let $e = uv \in E(G)$ where $u, v \notin X$. Then G/e is 3-connected or $G \backslash e$ is a subdivision of a 3-connected graph. If G/e is 3-connected, then we contract e. Otherwise, $G \backslash e$ is a subdivision of a 3-connected graph, for which we delete e and suppress any resulting degree-2 vertices. By repeating this process for all edges of G whose both endpoints are not in X, we obtain the desired G'. Since G' is isomprhic to a minor of G, it suffices to show that G' contains a minor satisfying statement 1 or statement 2 in the lemma.

Now G' either has a path of length a or it does not. In both cases, by applying Lemma 2.3.9 and Lemma 2.3.10, we obtain the desired conclusion. (Note that if G' contains a minor that is isomorphic to a subdivided $K_{3,t}$ where all cubic vertices are firm and belong to X, then G contains a minor that is isomorphic to a $K_{3,t}$ with the same property.) □

We conclude this chapter with the proof of the vertex version.

Proof of Theorem 1.2.2. Let $a = f_{2.1.6}(t, t)$, $b = f_{2.1.6}(a, a)$, and $c = f_{2.1.6}(b, b)$. Let $f_{1.2.2}(t) = f_{2.3.11}(c)$. We apply the previous lemma and divide the proof into two cases.

Case 1: G contains a minor H that is isomorphic to a $K_{3,c}$ where all cubic vertices are firm and belong to X.

Let u_1, u_2, u_3 be the cores and let $v_1, v_2, \ldots, v_c$ be the children of H. Since H is a minor of G, there exists an embedding π. In $G|H$, let

$$M_1 = \left(\bigcup_{j=1}^{c} \pi(u_1 v_j) \right) \cup \pi(u_1).$$

Then M_1 is connected and every v_j has degree 1 in M_1. Let T_1 be the minimal subtree of M_1 whose leaves are $v_1, v_2, \ldots, v_c$. Then T_1 contains Z_1 that is subdivided $K_{1,b}$ or a subdivided straight $\mathcal{C}_b$ whose leaves are the leaves of T_1.

Let $v_{i_1}, v_{i_2}, \ldots, v_{i_b}$ be the leaves of Z_1. Let

$$M_2 = \left(\bigcup_{j=1}^{b} \pi(u_2 v_{i_j}) \right) \cup \pi(u_2).$$

Then M_2 is connected and every v_{i_j} has degree 1 in M_2. Let T_2 be the minimal subtree of M_2 whose leaves are $v_{i_1}, v_{i_2}, \ldots, v_{i_b}$. Then T_2 contains Z_2 that is subdivided $K_{1,a}$ or a subdivided straight $\mathcal{C}_a$ whose leaves are the leaves of T_2.

Let $z_{i_1}, z_{i_2}, \ldots, z_{i_a}$ be the leaves of Z_2. Let

$$M_3 = \left(\bigcup_{j=1}^{a} \pi(u_3 z_{i_j}) \right) \cup \pi(u_3).$$

Then M_3 is connected and every z_{i_j} has degree 1 in M_3. Let T_3 be the minimal subtree of M_3 whose leaves are $z_{i_1}, z_{i_2}, \ldots, z_{i_a}$. Then T_3 contains Z_3 that is subdivided $K_{1,t}$ or a subdivided straight $\mathcal{C}_t$ whose leaves are the leaves of T_3.

Recall that each of the Z_1, Z_2, or Z_3 has two possibilities, a subdivided star or a subdivided comb. To complete this case, we divide the analysis into subcases, depending on the choice of Z_1, Z_2, and Z_3.

Case 1a: All of them are subdivided stars. Then G contains a subdivided $K_{3,t}$ with vertex-weight at least t.

Case 1b: Exactly two of them are subdivided stars. Then G contains a subdivided $K^1_{3,t}$ with vertex-weight at least t.

Case 1c: Exactly one of them is a subdivided star. Then G contains a subdivided $K^2_{3,t}$ with vertex-weight at least t.

Case 1d: All of them are subdivided combs. Then G contains a subdivided $K^3_{3,t}$ with vertex-weight at least t.

Case 2: G contains a minor H where H is isomorphic to a graph obtained from W_c by subdividing its rims. In addition, all non-center cubic vertices of H are firm and belong to X. Then G contains a subgraph K consisting of a cycle C, a tree T disjoint from C, and edges $\{e_i \mid i = 1, 2, \ldots, c\}$ where the endpoints of all e_i are disjoint on C. For each e_i, let v_i be the endpoint of e_i on C. Now $v_i \in X$ for $i = 1, 2, \ldots, c$. Let S be the union of all e_i for $i = 1, 2, \ldots, c$. Let T' be the minimal subtree of $T \cup S$ such that the leaves of T' are $\{v_1, v_2, \ldots, v_c\}$. Then T' contains a subdivided $K_{1,b}$ or a subdivided straight $\mathcal{C}_b$ whose leaves are the leaves of T'. This yields a subdivided W_b or a subdivided V_b with vertex-weight at least t in G. □

Chapter 3. Unavoidable Topological Minors of Infinite 2-connected Rooted Graphs

Graphs in this chapter are infinite.

3.1. Definitions and Lemmas

This section defines more terminology and states some theorems that are needed for the proof of our main result. We first prove two standard results from real analysis and set theory.

Lemma 3.1.1. *Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of distinct positive integers. Then $\{x_i\}_{i=1}^{\infty}$ contains an increasing infinite subsequence.*

Proof. We call an index n good if $x_n < x_m$ for all $m > n$ and is bad otherwise.

Claim 3.1.1.1. *There are infinitely many good indices.*

Suppose there are only finitely many good indices $n_1, n_2, \ldots, n_k$ for some k. Then there exists an index a_1 that is greater than every n_i. Now a_1 is bad, so there exists an index $a_2 > a_1$ such that $x_{a_2} < x_{a_1}$. Next, a_2 is also bad, so there exists an index $a_3 > a_2$ such that $x_{a_3} < x_{a_2}$. Note that we can choose $a_1 < a_2 < a_3 < \ldots$ indefinitely whereas we cannot choose $x_{a_1} > x_{a_2} > x_{a_3} > \ldots$ indefnitely since $\{x_i\}_{i=1}^{\infty}$ is a sequence of positive integers. This proves the claim.

From the previous claim, we can choose infinitely many good indices $n_1 < n_2 < n_3 < \ldots$. Now $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ is an increasing infinite subsequence as wanted. □

Lemma 3.1.2. *Let A, B be infinite sets. Then A has an infinite subset A' and B has an infinite subset B' such that $A' \cap B' = \emptyset$.*

Proof. If $A \cap B$ is finite, then $A' = A - B$ and $B' = B - A$ are the desired subsets. Otherwise, $A \cap B$ is infinite and so it contains two disjoint infinite subsets A', B'. □

The following is an immediate application of Theorem 1.1.14.

Lemma 3.1.3. *Let H be a subgraph of G and let B be an H-bridge. Let X be the set of feet of B. If X is infinite, then B contains a subdivided $K_{1,\infty}$ or comb whose leaves belong to X.*

Proof. For every $x \in X$, we delete all but one edge of B that is incident with x. Let B' be the subgraph of B obtained after performing this operation. Then B' is connected and X is an infinite subset of $V(B')$. In addition, every $x \in X$ has degree 1 in B'. By Lemma 1.1.14, B' contains one of the following subgraphs

1. an X-rich ray,
2. an X-rich $K_{1,\infty}$ whose leaves belong to X,
3. an X-rich comb whose leaves belong to X.

Note that statement 1 is not possible because every $x \in X$ has degree 1 in B'. Therefore, B contains a subdivided $K_{1,\infty}$ or comb whose leaves belong to X. □

The following lemma is also very useful.

Lemma 3.1.4 (Lemma 3.1 in [3])**.** *Every locally finite, connected graph contains an induced ray starting from any vertex.*

We now describe the graphs $K_{2,\infty}, F_\infty, L_\infty$ that are important in our later discussion.

Definition 3.1.5. Let $\{x_1, x_2, \ldots\}$ be an infinite set of vertices. A $K_{2,\infty}$ is obtained by adding edges x_1x_i and x_2x_i for every $i \geq 3$.

Definition 3.1.6. Let $R = x_1x_2\ldots$ be a ray and let u be a vertex not on R. We then add an edge e_i between u and x_i for $i = 1, 2, \ldots$. The resulting graph is called a **fan** and is denoted as F_∞. We call R the **rail** and each edge e_i a **spoke**. For a subdivided F_∞, we use the terms rail and spoke to mean its subdivided rail and subdivided spoke respectively.

Definition 3.1.7. Let $P = x_1x_2\ldots$ and $Q = y_1y_2\ldots$ be disjoint rays. We then add an edge e_i between x_i and y_i for $i = 1, 2, \ldots$. The resulting graph is called a **ladder** and is denoted as L_∞. We call P, Q the **rails** and each edge e_i a **rung**. For a subdivided L_∞, we use the terms rail and rung to mean its subdivided rail and subdivided rung respectively.

The ladder L_∞ is an important unavoidable graph since it is 2-connected and serves as the basis where more complicated 3-connected graphs are built upon. However, in many case analyses, we obtain something that is very close to a true ladder (a locally finite graph consisting of two disjoint rays together with infinitely many internally disjoint rungs in between). In the next three lemmas, we will clean up those types of messy ladders to obtain an L_∞.

Lemma 3.1.8. *Let G be the union of a ray R and infinitely many internally disjoint R-paths $Q_1, Q_2, \ldots$ such that with respect to R, Q_{i+1} crosses Q_i but does not cross Q_j for any $j < i$. Then $G = H_1 \cup H_2$ where H_1 is a finite graph and H_2 is the union of two disjoint rays A, B and infinitely many internally disjoint AB-paths. In addition, H_1 and H_2 are edge-disjoint and G is locally finite.*

Proof. Let r be the endpoint of R. For each Q_i, we denote its two endpoints as a_i, b_i where a_i is on the left of b_i with respect to R. Since R is a ray, there exists an index i_0 such that for every $i \neq i_0$, neither a_i nor b_i is on the left of a_{i_0} with respect to R. Let $H_1 = (\bigcup_{i=1}^{i_0-1} Q_i) \cup R[ra_{i_0})$. Then H_1 is a finite graph. Let R' be the subray of R with a_{i_0} as the endpoint and let $H_2 = (\bigcup_{i=i_0}^{\infty} Q_i) \cup R'$. Clearly, $G = H_1 \cup H_2$ and H_1, H_2 are edge-disjoint.

We now show that H_2 is the union of two disjoint rays A, B and infinitely many internally disjoint AB-paths. For the remain of this proof, every crossing and left, right position is with respect to R'. For convenience, we relabel the Q_i in H_2. Let $Q_1 = Q_{i_0},\ Q_2 = Q_{i_0+1}, \ldots$, so that Q_{i+1} crosses Q_i but does not cross Q_j for any $j < i$. For each Q_i, we denote its two endpoints as a_i, b_i where a_i is on the left of b_i. Note that a_1 is the endpoint of R'.

Claim 3.1.8.1. *For any i, j with $j > i + 1$, if a_j or b_j belongs to $R'(a_i b_i)$, then $R'[a_j b_j] \subseteq R'[a_i b_i]$ and at least one of the a_{j+1} or b_{j+1} belongs to $R'(a_i b_i)$.*

Since Q_j does not cross Q_i and one of the a_j or b_j belongs to $R'(a_i b_i)$, it follows that both a_j and b_j belongs to $R'[a_i b_i]$. Hence, $R'[a_j b_j] \subseteq R'[a_i b_i]$. Additionally, since Q_{j+1} crosses Q_j, at least one of the a_{j+1} or b_{j+1} belongs to $R'(a_j b_j) \subseteq R'(a_i b_i)$. This proves the claim.

Claim 3.1.8.2. *For any i, j with $j > i + 1$, neither a_j nor b_j belongs to $R'(a_i b_i)$.*

Assume for a contradiction that such i, j exist. By induction on k using the previous claim, we deduce that $R'[a_k b_k] \subseteq R'[a_i b_i]$ for all $k \geq j$. Since $R'[a_i b_i]$ is finite, there exist

m, n such that $n > m \geq j$ and $R'[a_m b_m] = R'[a_n b_n]$. But this implies that Q_{n+1}, which crosses Q_n, also crosses Q_m, a contradiction. This proves the claim.

Claim 3.1.8.3. *If x is an endpoint of Q_j and x is not a_1, then a_i is on the left of x for every $i < j$.*

Assume for a contradiction that there exists an $i < j$ where a_i is on the right of x. Since x is not a_1, there exists an a_k with $k < i$, namely a_1, such that a_k is on the left of x. We choose the largest such k. Since $j > i \geq k + 1$, by Claim 3.1.8.2, $x \notin R'(a_k b_k)$. This implies that $b_k = x$ or b_k is on the left of x. Since Q_{k+1} crosses Q_k, either a_{k+1} or b_{k+1} belongs to $R'(a_k b_k)$. If $a_{k+1} \in R'(a_k b_k)$, then a_{k+1} is on the left of x and this contradicts the maximality of k. Hence, $b_{k+1} \in R'(a_k b_k)$. But then a_{k+1}, being on the left of b_{k+1}, is on the left of x and this again contradicts the maximality of k. Therefore, no such i exists. This proves the claim.

Claim 3.1.8.4. *We have $a_{n+1} \in R'(a_n b_n)$ for all $n \geq 1$.*

Assume there exists such an n where the statement is false. This means that $b_{n+1} \in R'(a_n b_n)$ and a_{n+1} is on the left of a_n since Q_{n+1} crosses Q_n. If a_{n+1} is not a_1, then this contradicts Claim 3.1.8.3 because a_{n+1} is on the left of a_n. Thus, $a_{n+1} = a_1$. Since Q_{n+2} crosses Q_{n+1}, it has an endpoint $x \in R'(a_{n+1} b_{n+1})$. By Claim 3.1.8.3, x is on the right of a_n. But since x is also on the left of b_{n+1}, which is on the left of b_n, it follows that $x \in R'(a_n b_n)$ and this contradicts Claim 3.1.8.2. Therefore, no such n exists. This proves the claim.

It follows from the previous claim that starting from the endpoint a_1 of R' and going from left to right, the endpoints of $Q_1, Q_2, \dots$ are $a_1, a_2, b_1, a_3, b_2, a_4, \dots, b_i, a_{i+2}, \dots$, where

$b_i = a_{i+2}$ is possible. This implies that G is locally finite because $a_i \neq b_i$ for every i. Let

$$A = \bigcup_{k=0}^{\infty} Q_{2k+1} \cup R'[b_{2k+1}a_{2k+3}] = Q_1 \cup R'[b_1a_3] \cup Q_3 \cup R'[b_3a_5] \cup \dots$$

and let

$$B = \bigcup_{k=0}^{\infty} Q_{2k+2} \cup R'[b_{2k+2}a_{2k+4}] = Q_2 \cup R'[b_2a_4] \cup Q_4 \cup R'[b_4a_6] \cup \dots .$$

Then A, B are disjoint rays. Let

$$M = \bigcup_{k=1}^{\infty} R'[a_{2k+1}b_{2k}] = R'[a_3b_2] \cup R'[a_5b_4] \cup \dots$$

and let

$$N = \bigcup_{k=0}^{\infty} R'[a_{2k+2}b_{2k+1}] = R'[a_2b_1] \cup R'[a_4b_3] \cup \dots .$$

Then $M \cup N \cup R'[a_1a_2]$ is the set of infinitely many internally disjoint AB-paths. Finally, $H_2 = A \cup B \cup M \cup N \cup R'[a_1a_2]$, which completes the proof. □

Lemma 3.1.9. *Let A, B be disjoint rays and let $\mathcal{P}$ be an infinite set of internally disjoint AB-paths. Let H be the union of A, B, and all paths in $\mathcal{P}$. Assume additionally that H is locally finite. Then H contains a subdivided L_∞ whose rails are contained in A, B and whose rungs belong to $\mathcal{P}$.*

Proof. Since H is locally finite, $\mathcal{P}$ has an infinite subset $\mathcal{P}'$ such that two paths in $\mathcal{P}'$ are disjoint. Starting at the endpoint of A, we label the vertices of A that are incident with a path in $\mathcal{P}'$ as a sequence $\{x_i\}_{i=1}^{\infty}$, in the order as they appear on A. Let y_i be the endpoint on B of the path in $\mathcal{P}'$ with x_i as one of its endpoints. Starting at the endpoint of B, we list the vertices y_i in the order as they appear on B. This yields a sequence $\{y_{i_j}\}_{j=1}^{\infty}$ where

$\{i_j\}_{j=1}^{\infty}$ is a sequence of distinct positive integers. By Lemma 3.1.1, the sequence $\{i_j\}_{j=1}^{\infty}$ contains an increasing infinite subsequence $\{i'_j\}_{j=1}^{\infty}$. Let P_j be the path in $\mathcal{P}'$ with endpoints $x_{i'_j}, y_{i'_j}$. The graph $\bigcup_{j=1}^{\infty} P_j \cup A \cup B$ contains a subdivided L_∞ that is the desired subgraph of H. □

Lemma 3.1.10. *Let A, B be disjoint rays and let $\mathcal{P}$ be an infinite set of internally disjoint AB-paths. Let H be the union of A, B, and all paths in $\mathcal{P}$ and let X be an infinite subset of $V(H)$. Assume additionally that H is locally finite. Then H contains a subdivided L_∞, whose rails are contained in A, B and whose rungs belong to $\mathcal{P}$, such that one of its rails contains infinitely many elements of X or every of its rungs contains at least one element of X.*

Proof. Since $X \subseteq V(H)$, one of the following is true

1. $A \cup B$ contains infinitely many elements of X,
2. there exists an infinite subset $\mathcal{P}'$ of $\mathcal{P}$ such that each path in $\mathcal{P}'$ contains at least one element of X.

If statement 1 is true, then let $\mathcal{P}' = \mathcal{P}$. Otherwise, let $\mathcal{P}'$ be an infinite subset of $\mathcal{P}$ such that each path in $\mathcal{P}'$ contains at least one element of X. Let H' be the union of A, B, and all paths in $\mathcal{P}'$. By Lemma 3.1.9, H' contains the desired subdivided L_∞. □

3.2. Vertex Version

For connected rooted graphs, their unavoidable rooted topological minors are a path, a subdivided star, or a subdivided comb. Thus, it is natural to consider the simplest case when a 2-connected rooted graphs contains a rich path. We begin with the following lemma.

Lemma 3.2.1. *Let G be a 2-connected graph and let X be an infinite subset of $V(G)$. Assume G contains an X-rich ray. Then G contains an X-rich F_∞ or an X-rich L_∞.*

Proof. Let R be the ray that contains infinitely many elements of X in G. For the remain of the proof, every bridge and crossing is with respect to R.

Claim 3.2.1.1. *If there exists a bridge with infinitely many feet, then the lemma holds.*

Suppose there exists a bridge B with infinitely many feet. Let Y be the set of feet of B. By Lemma 3.1.3, one of the following is true

1. B contains a subdivided $K_{1,\infty}$, call it K, whose leaves belong to Y. Then the subgraph $K \cup R$ contains an X-rich F_∞.

2. B contains a subdivided comb, call it K, whose leaves belong to Y. Starting from the endpoint of R, we label the leaves of K as $x_1, x_2, \ldots$, in the order as they appear on R. Let W be the spine of K and let $y_i \in W$ such that $x_i y_i$ is a tooth of K. Starting from the endpoint of W, we list the vertices y_i in the order as they appear on W. This yields a sequence $\{y_{i_j}\}_{j=1}^{\infty}$ where $\{i_j\}_{j=1}^{\infty}$ is a sequence of distinct positive integers. By Lemma 3.1.1, the sequence $\{i_j\}_{j=1}^{\infty}$ contains an increasing infinite subsequence $\{i'_j\}_{j=1}^{\infty}$. Let $P_{i'_j}$ be the tooth of K with endpoints $x_{i'_j}, y_{i'_j}$. Then the union of R, W, and all $P_{i'_j}$ contains an X-rich L_∞.

This proves the claim.

By the previous claim, we may assume that every bridge has finitely many feet. We now define the peak of a bridge and the reach of a vertex in R. Starting from the endpoint of R, we list all of its vertices from left to right as a sequence $x_1, x_2, \ldots$. The peak of a bridge B is the largest i such that x_i is a foot of B and is denoted as $p(B)$. Note that $p(B)$ is finite because B has finitely many feet. Let x_i be a vertex of R. If no bridge contains x_i as a foot, then the reach $r(x_i)$ of x_i is 0. Otherwise, we define its reach $r(x_i)$ to be the largest $p(B)$, among all bridges B that contain x_i, or $r(x_i) = \infty$ if no such $p(B)$ exists.

Claim 3.2.1.2. *If $r(x_i) = \infty$ for some i, then the lemma holds.*

Since $r(x_i) = \infty$ and every bridge has finitely many feet, there exists a sequence of bridges $B_1, B_2, \ldots$ each containing x_i such that $p(B_1) < p(B_2) < \ldots$. Let P_k be the $x_i x_{p(B_k)}$-path in B_k. Let R' be the subray of R with $x_{p(B_1)}$ as its endpoint. The subgraph $(\bigcup_{k=1}^{\infty} P_k) \cup R'$ is an X-rich F_∞. This proves the claim.

From the previous claim, we may assume additionally that every vertex in R has finite reach. We now construct a sequence $Q_1, Q_2, \ldots$ of internally disjoint R-paths such that Q_{i+1} crosses Q_i but does not cross Q_j for any $j < i$. Intuitively, we construct the sequence using a greedy process; at each step, we choose Q_i with its reach as large as possible and also crosses Q_{i-1}.

We first construct Q_1. Let $y_1 = x_{r(x_1)}$ and let B_1 be a bridge containing x_1, y_1. Let Q_1 be an $x_1 y_1$-path in B_1. Next, we construct Q_2. Since $G - y_1$ is connected, it has an R-path from $R[x_1 y_1)$ to $R - R[x_1 y_1]$. In addition, this aforementioned path cannot has x_1 as its endpoint by the choice of Q_1. Hence, $G - y_1$ has a vertex in $R(x_1 y_1)$ whose reach exceeds $r(x_1)$. Among all such vertices in $R(x_1 y_1)$, we choose one with the largest reach and call it

x_2. Let $y_2 = x_{r(x_2)}$ and let B_2 be a bridge containing x_2, y_2. Let Q_2 be an x_2y_2-path in B_2. Observe that Q_2 crosses Q_1 since $x_2 \in R(x_1y_1)$ and $y_2 \notin R[x_1y_1]$. In addition, Q_1 and Q_2 are internally disjoint because $B_1 \neq B_2$ as $p(B_1) < p(B_2)$.

Suppose $Q_1, Q_2, \ldots, Q_n$ are constructed such that Q_{i+1} crosses Q_i but does not cross Q_j for any $j < i$. In Q_{n-1}, let x_{n-1} be one of its endpoint with its corresponding $y_{n-1} = x_{r(x_{n-1})}$. In Q_n, let x_n be one of its endpoint with its corresponding $y_n = x_{r(x_n)}$. Since $G - y_n$ is connected, it has an R-path from $R[x_1y_n)$ to $R - R[x_1y_n]$. This aforementioned path must have an endpoint in $R[y_{n-1}y_n)$ by the construction of $Q_1, Q_2, \ldots, Q_n$. Hence, $G - y_n$ has a vertex in $R[y_{n-1}y_n)$ whose reach exceeds $r(x_n)$. Among all such vertices in $R[y_{n-1}y_n)$, we choose one with the largest reach and call it x_{n+1}. Let $y_{n+1} = x_{r(x_{n+1})}$ and let B_{n+1} be a bridge containing x_{n+1}, y_{n+1}. Let Q_{n+1} be an $x_{n+1}y_{n+1}$-path in B_{n+1}. Observe that Q_{n+1} crosses Q_n since $x_{n+1} \in R[y_{n-1}y_n) \subseteq R(x_ny_n)$ and $y_{n+1} \notin R[x_ny_n]$. In addition, Q_{n+1} does not cross Q_j for any $j < n+1$ because x_{n+1}, y_{n+1} are not in $R[x_1y_{n-1})$.

We have constructed a sequence $Q_1, Q_2, \ldots$ of internally disjoint R-paths such that Q_{i+1} crosses Q_i but does not cross Q_j for any $j < i$. Let $K = (\bigcup_{i=1}^{\infty} Q_i) \cup R$. By Lemma 3.1.8, $K = H_1 \cup H_2$ where H_1 is a finite graph and H_2 is the union of two disjoint rays A, B and infinitely many internally disjoint AB-paths. In addition, H_1 and H_2 are edge-disjoint and K is locally finite. Since R contains infinitely many elements of X, so does K. Since H_1 is finite, H_2 contains infinitely many elements of X. By Lemma 3.1.10, H_2 contains an X-rich L_∞. □

The next lemma asserts that given a rooted graph consisting of a ray and infinitely many paths in a nice configuration, we can obtain a rich ray, for which the analysis is reduced to the previous lemma.

Lemma 3.2.2. *Let H be the union of a ray R and infinitely many disjoint R-paths $Q_1, Q_2, \ldots$ such that with respect to R, Q_i is on the left of Q_{i+1} for every i. Let X be an infinite subset of $V(H)$. Then H contains an X-rich ray.*

Proof. For each Q_i, let x_i, y_i be its endpoints on R. Since H contains infinitely many elements of X, either R contains infinitely many elements of X or infinitely many Q_i each contains at least one element of X in its interior. If R contains infinitely many elements of X, then the lemma holds. Otherwise, infinitely many Q_i each contains at least one element of X in its interior. If a Q_i does not contain any element of X in its interior, then we delete $E(Q_i)$. Otherwise, it contains at least one element of X in its interior and we delete edges of $R[x_i y_i]$. By repeating this process, we obtain an X-rich ray in H. □

We will now prove the vertex version.

Proof of Theorem 1.2.3. For a subgraph H of G, an H-path is called an H-ear if its interior contains at least one element of X.

Claim 3.2.2.1. *Every finite subgraph H of G with at least two vertices has an H-ear.*

Since H is finite and X is infinite, there exists an $a \in X - V(H)$. Since G is 2-connected, a and $V(H)$ cannot be separated by fewer than two vertices. By Corollary 1.3.2, G contains two weakly disjoint $aV(H)$-paths. The union of these two paths yields an H-ear. This proves the claim.

Back to our proof, we first construct an infinite sequence of subgraphs $H_0, H_1, H_2, \ldots$ of G such that for every $n \geq 1$, $H_n = H_{n-1} \cup Q_n$ where Q_n is an H_{n-1}-ear and is chosen according to the rule which we will describe in the next paragraph. Let H_0 be a cycle of G containing at least one element of X and let $e_0 = x_0y_0$ be an edge of H_0. Let $P_0 = H_0 \backslash e_0$ and let $T_0 = P_0$. Note that T_0 is a spanning tree of H_0. To illustrate, we will construct H_1. Every H_0-ear has two distinct endpoints $x, z \in H_0$ and we denote x as the endpoint so that $||P_0[xx_0]|| < ||P_0[zx_0]||$. Among all H_0-ears, we choose one such that $||P_0[xx_0]||$ is the smallest and then $||P_0[zx_0]||$ is the smallest. Let Q_1 be such an H_0-ear and let $H_1 = H_0 \cup Q_1$. Let x_1, z_1 be the two endpoints of Q_1 where $||P_0[x_1x_0]|| < ||P_0[z_1x_0]||$ by construction. Let $e_1 = y_1z_1$ be the edge of Q_1 with z_1 as an endpoint and let $P_1 = Q_1 - z_1$. Let $T_1 = P_0 \cup P_1$. Note that T_1 is a spanning tree of H_1.

Suppose $H_0, H_1, H_2, \ldots, H_{n-1}$ are defined and let $T_{n-1} = P_0 \cup P_1 \cup \cdots \cup P_{n-1}$. Note that T_{n-1} is a spanning tree of H_{n-1}. For every vertex $v \in H_{n-1}$, we define $l(v) = (i, d)$ where i is the smallest index such that $v \in P_i$ and $d = ||P_i[vx_i]||$. For any two distinct vertices $u, v \in H_{n-1}$, by $l(u) < l(v)$ we mean $l(u)$ is lexicographically smaller than $l(v)$. Every H_{n-1}-ear has two distinct endpoints $x_n, z_n \in H_{n-1}$ and we denote x_n as the endpoint so that $l(x_n) < l(z_n)$. Among all H_{n-1}-ears, we choose one with endpoints x_n, z_n where $l(x_n) < l(z_n)$ such that $l(x_n)$ is the smallest and then $l(z_n)$ is the smallest. Let Q_n be such an H_{n-1}-ear and let $H_n = H_{n-1} \cup Q_n$. Let $e_n = y_nz_n$ be the edge of Q_n with z_n as an endpoint and let $P_n = Q_n - z_n$. Let $T_n = T_{n-1} \cup P_n$. Note that T_n is a spanning tree of H_n.

Let $H = H_0 \cup H_1 \cup H_2 \cup \ldots$ and let $T = T_0 \cup T_1 \cup T_2 \cup \ldots$. Observe that for every

i, H_i is 2-connected and $H_i \subseteq H_{i+1}$. Let a, b be two distinct vertices of H. Then we may assume $a \in H_i$ and $b \in H_j$ for some $i \leq j$. Thus, $a \in H_j$ because $H_i \subseteq H_j$. Since H_j is 2-connected, it contains a cycle containing a, b. Hence, H contains a cycle containing a, b. This proves that H is 2-connected. In addition, we can naturally extend the definition of $l(v)$ for every vertex $v \in H$ as $l(v) = (i, d)$ where i is the smallest index such that $v \in P_i$ and $d = ||P_i[vx_i]||$. Note that by definition, $u = v$ if and only if $l(u) = l(v)$.

Claim 3.2.2.2. *We have $l(x_i) \leq l(x_{i+1})$ for every i.*

Assume for a contradiction that $l(x_{i+1}) < l(x_i)$ for some i. Then $x_{i+1} \notin Q_i$. This means that $x_{i+1} \in V(H_{i-1}) - \{x_i, z_i\}$. If $z_{i+1} \in H_{i-1}$, then Q_{i+1} is an H_{i-1}-ear. But $l(x_{i+1}) < l(x_i)$ implies that Q_{i+1} must be chosen before Q_i and this is not possible. Thus, $z_{i+1} \in V(Q_i) - \{x_i, z_i\}$. Let Q' be the $x_i z_{i+1}$-subpath of Q_i. Then $Q_{i+1} \cup Q'$ is an H_{i-1}-ear. But $l(x_{i+1}) < l(x_i)$ again implies that $Q_{i+1} \cup Q'$ must be chosen before Q_i and this is not possible. This proves the claim.

Let $F = \{e_0, e_1, e_2, \ldots\}$. Then T is a spanning tree of H and $H = T \cup F$. We divide the proof into two cases.

Case 1: H contains a vertex v of infinite degree. We further divide this case into two subcases.

Case 1a: v is incident with infinitely many edges of F. This means that the set $I = \{i \mid v = z_i\}$ is infinite. Let $l(v) = (n, d)$. Since $l(x_i) < l(z_i)$ for every $i \in I$, it follows that $x_i \in H_n$ for all $i \in I$. Since H_n is finite, it contains a vertex u such that $u = x_j$ for infinitely many $j \in I$. The union of all such Q_j yields an X-rich $K_{2,\infty}$.

Case 1b: v is incident with only finitely many edges of F and no vertex in H is incident with infinitely many edges of F. This means that the set $I = \{i \mid v = x_i\}$ is infinite. Let j be the smallest index in I.

Claim 3.2.2.3. *Every $k \geq j$ is in I.*

Since I is infinite, there exists a $k' \in I$ such that $k' \geq k$. Hence, by Claim 3.2.2.2, $l(x_j) \leq l(x_k) \leq l(x_{k'})$. But $l(x_{k'}) = l(x_j)$ because $k' \in I$. Thus, $l(x_j) = l(x_k) = l(x_{k'})$. This implies $x_k = x_{k'}$, so $k \in I$. This proves the claim.

This means that we can write $I = \{k \mid k \geq j\}$. Since H is 2-connected, $H - v$ is connected and it can be obtained from $H_{j-1} - v$ by repeatedly adding paths $Q_k - v$ for all $k \in I$. Note that every Q_k has v, z_k as its two endpoints. Now $H - v$ is locally finite since none of its vertices is incident with infinitely many edges of F. Thus, $H - v$ contains a ray R by Theorem 3.1.4. Since $H_{j-1} - v$ is a finite subgraph of $H - v$, this ray R contains subpaths of infinitely many $Q_k - v$. Let

$$I' = \{k \in I \mid R \text{ contains at least one edge of } Q_k - v\}.$$

Then I' is infinite and for every k in I' that is not the smallest element, $z_k \in R$. Let $M = (\bigcup_{k \in I'} Q_k) \cup R$. Then M is a union of R and infinitely many weakly disjoint vR-paths. Hence, it contains a subdivided F_∞. Furthermore, since every Q_k contains at least one element of X, M contains an X-rich F_∞.

Case 2: H is locally finite. This means that T is also locally finite and contains a ray R starting from x_0. Let

$$I = \{i \mid P_i \text{ contains at least one edge of } R\}.$$

For an $i \in I$, let span(y_i) be the union of all P_j for all $j \in I$ with $j \leq i$. Let $S = \bigcup_{i \in I} P_i$. We can also label the elements of I in increasing order as $i_1 < i_2 < \dots$ where $x_{i_{n+1}} \in P_{i_n}$ for $n = 1, 2, \dots$. For the remain of this proof, by bridge we mean an S-bridge of H.

Claim 3.2.2.4. *If there exists a bridge B containing infinitely many y_i with $i \in I$, then the lemma holds.*

Let $Y = \{y_i \mid i \in I \text{ and } y_i \in B\}$. Then Y is infinite. By Corollary 3.1.3 and the assumption that H is locally finite, B contains a subdivided comb, call it K, whose leaves belong to Y. The subgraph $K \cup S$ contains an X-rich L_∞. This proves the claim.

From the previous claim, we may assume that every bridge contains finitely many y_i with $i \in I$.

Claim 3.2.2.5. *For every $i = i_n \in I$ with $n \geq 2$, H has an S-path L_{i_n} with y_{i_n} as an endpoint and the other endpoint belongs to span$(y_{i_{n-1}})$.*

Observe that $z_i \in T_{i-1}$ and span$(y_{i_{n-1}})$ is nonempty and is contained in T_{i-1}. Hence, T_{i-1} has an S-path P from z_i to span$(y_{i_{n-1}})$. The path $P \cup e_i$ is the desired S-path L_{i_n}. This proves the claim.

We now construct a sequence of disjoint R-paths $M_1, M_2, \dots$ such that with respect to R, M_i is on the left of M_{i+1} for every i. We first construct M_1 and we consider i_2. By the previous claim, H has an S-path L_{i_2} with y_{i_2} as an endpoint and the other endpoint belongs to span(y_{i_1}). Let B_{i_2} be the bridge containing L_{i_2}. Let $M_1 = L_{i_2} \cup P_{i_2}[x_{i_3} y_{i_2}]$. Then M_1 is an R-path. Next, we construct M_2. Let $j \in I$ be the largest index such that y_j is a foot of B_{i_2}. Since G is locally finite and span(y_{i_1}) is a finite graph, there exists an $i_k > \max(i_2, j)$

such that L_{i_k} has y_{i_k} as an endpoint and the other endpoint does not belong to $\mathrm{span}(y_{i_1})$. Let B_{i_k} be the bridge containing L_{i_k}. Note that L_{i_k} is disjoint from L_{i_2} because $B_{i_k} \neq B_{i_2}$. Let $M_2 = L_{i_k} \cup P_{i_k}[x_{i_{k+1}} y_{i_k}]$. Then M_2 is an R-path. Clearly, M_2 and M_1 are disjoint and M_1 is on the left of M_2 with respect to R. By repeating this process, we obtain the desired sequence $M_1, M_2, \ldots$. The subgraph $(\bigcup_{i=1}^{\infty} M_i) \cup R$ satisfies the hypotheses in Lemma 3.2.2, so H contains an X-rich ray. Therefore, it contains an X-rich F_∞ or an X-rich L_∞ by Lemma 3.2.1. Since H is also locally finite, it contains an X-rich L_∞. □

3.3. Edge Version

As described below, the edge version is a simple application of the vertex version. The following theorem asserts that the subdivision operation still preserves 2-connectivity.

Theorem 3.3.1. *Let G be a 2-connected graph and let G' be a subdivision of G. Then G' is 2-connected.*

Proof. Clearly, G' is connected and $|G'| > 2$ since G is 2-connected. Let v be a vertex in G'. Assume for contradiction that $G' - v$ is not connected. If v is a subdividing vertex, then there exists an edge $e \in E(G)$ such that $G \backslash e$ is not connected, which is not possible. Hence, v is a branching vertex. But this means that $G - v$ is not connected, a contradiction. Therefore, $G' - v$ is connected for every v, so G' is 2-connected. □

We conclude this chapter with the proof of the edge version.

Proof of Theorem 1.2.4. Let G' be obtained from G by subdividing each edge in X exactly once. Then G' is 2-connected by Theorem 3.3.1. Let Y be the set of subdividing vertices of G'. Then Y is infinite because X is infinite. In addition, every vertex of Y has degree 2 in G'. By Theorem 1.2.3, G' contains a Y-rich H' for some H' in $\{K_{2,\infty}, F_\infty, L_\infty\}$. Consequently, G contains a subdivided H containing infinitely many edges of X for some H in $\{K_{2,\infty}, F_\infty, L_\infty\}$. $\square$

Chapter 4. Unavoidable Topological Minors of Infinite 3-connected Rooted Graphs

Graphs in this chapter are infinite.

4.1. Definitions and Lemmas

This section defines more terminology and states some theorems that are needed for the proof of our main result. We will prove a stronger version of Theorem 1.2.5 by weakening the 3-connectivity assumption. In particular, we prove Theorem 1.2.5 under the assumption that G is weakly 3-connected.

Definition 4.1.1. A graph G' is **weakly 3-connected** if G' is obtained from a 3-connected graph G by subdividing every edge of G at most once. We call G the **underlying** 3-connected graph of G'.

In the next few lemmas, we establish some properties of weakly 3-connected graphs.

Lemma 4.1.2. *Every weakly 3-connected graph is 2-connected.*

Proof. By definition, every weakly 3-connected graph is a subdivision of a 2-connected graph, so the lemma follows from Theorem 3.3.1. □

Lemma 4.1.3. *Let G be a weakly 3-connected graph and let a, b be vertices of degree at least 3 in G. Then G does not contain a separator of size 2 separating a from b.*

Proof. Suppose for contradiction that such a separator X of size 2 exists. This means that there is no ab-path in $G - X$. Let G' be the underlying 3-connected graph of G. Then $a, b \in V(G')$ since a, b has degree at least 3 in G. Now deleting X in G is equivalent to deleting $\{m, n\}$ in G' where each m, n is either a vertex or an edge. Thus, since there is no ab-path in $G - X$, there is no ab-path in $G' - \{m, n\}$. But this is not possible since G' is 3-connected. Therefore, no such X exists. □

Lemma 4.1.4. *Let G be a weakly 3-connected graph and let a be a vertex of degree at least 3 in G. Let $B \subseteq V(G) - a$ contain at least three vertices of degree at least 3. Then G does not contain a separator of size 2 separating a from B.*

Proof. Suppose for contradiction that such a separator X of size 2 exists. By the definition of separating a vertex and a set, $a \notin X$, so $a \in G - X$. In $G - X$, let C_1 be the component containing a. If C_1 contains a vertex of $B - X$, then there exists an $a(B - X)$-path in $G - X$. Thus, there exists an aB-path in G that does not meet X, which is not possible. Hence, C_1 and $B - X$ are disjoint. Since B contains at least three vertices of degree at least 3 in G and $|X| = 2$, there exists a vertex $b \in B - X$ of degree at least 3 in G. Now X is an ab-separator of size 2 in G, contradicting Lemma 4.1.3. Therefore, no such X exists. □

Lemma 4.1.5. *Let G be a connected graph and $X = \{X_1, X_2, \dots\}$ be an infinite set of disjoint connected subgraphs of G. Then one of the following is true in G*

1. *There exists an infinite subset of $Y = \{Y_1, Y_2, \dots\}$ of X and internally disjoint $(Y_1 \cup Y_2 \cup \dots)$-paths $P_1, P_2, \dots$ of G where P_i is between Y_i and Y_{i+1} for $i = 1, 2, \dots$;*

2. *G contains K, a subdivided $K_{1,\infty}$ or a subdivided comb, such that each leaf of K belongs to an X_i and this X_i does not contain any other vertices of K.*

Proof. Let G' be the graph obtained from G by contracting each X_i into a vertex x'_i. Then

G' is a minor of G, so there exists an embedding π'. For every $v \in V(G')$ whose degree is at most three in G', we first define the process of truncating $\pi'(v)$ in $G|G'$ as following. In $G|G'$, let A be the set of vertices of $\pi'(v)$ that are adjacent to a vertex not in $\pi'(v)$. Since v has degree at most three in G', at most three vertices of $\pi'(v)$ are adjacent to a vertex not in $\pi'(v)$ in $G|G'$, so $|A| \leq 3$. First, suppose $A = \emptyset$. In this case, we delete all but one vertex in $\pi'(v)$ from $G|G'$. Next, suppose $|A| = 1$, so A contains a vertex a. In this case, we delete $\pi'(v) - a$ from $G|G'$. Next, suppose $|A| = 2$, so A contains distinct vertices a, b. Since $\pi'(v)$ is connected, there exists an ab-path P in $\pi'(v)$. In this case, we delete $\pi'(v) - P$ from $G|G'$. Finally, suppose $|A| = 3$, so A contains distinct vertices a, b, c. Since $\pi'(v)$ is connected, there exist an ab-path P and a cP-path Q in $\pi'(v)$. In this case, we delete $\pi'(v) - (P \cup Q)$ from $G|G'$.

Next, since G' is connected and $X' = \{x'_1, x'_2, \ldots\}$ is an infinite subset of $V(G')$, by Theorem 1.1.14, G' contains one of the following subgraphs

1. A ray R with infinitely many elements of X'. Now R is a minor of G, so there exists an embedding π. A vertex p on R is called good if $\pi(p) = X_i$ for some i. Starting from the endpoint of R, we label the good vertices of R as a sequence $p_1, p_2, \ldots$. In $G|R$, let $Y_i = \pi(p_i)$ and let P_i be the path between Y_i and Y_{i+1}. By the definition of being good, every Y_i is an X_j for some j. Furthermore, P_i and P_j are internally disjoint when $i \neq j$. Thus, statement 1 is satisfied.

2. A subdivided $K_{1,\infty}$, denoted by K, whose leaves belong to X'. Let u be the infinite

degree vertex of K. For every leaf v of K, if the uv-path Q of K contains a vertex w of degree 2 in K that belongs to X', then we delete the tv-subpath of Q from K where t is the neighbor of w in the wv-subpath of Q. By doing this to every uv-path where v is a leaf of K, we may assume that every degree-2 vertex of K does not belong to X'. Since K is also a minor of G, there exists an embedding π mapping each leaf of K to an X_i in G. Clearly, this X_i does not contain any other vertices of K. In $G|K$, let F be the set of edges with one end in $\pi(u)$ and the other end not in $\pi(u)$ and let $Y = V(F) - V(\pi(u))$. Now $\pi(u) \cup F$ is a connected graph and Y is an infinite subset of $V(\pi(u) \cup F)$. In addition, $(\pi(u) \cup F) - Y$ is connected. By Theorem 3.1.3, the graph $\pi(u) \cup F$ contains a subdivided $K_{1,\infty}$ whose leaves belong to Y or a subdivided comb whose leaves belong to Y. Suppose $\pi(u) \cup F$ contains a subdivided $K_{1,\infty}$ whose leaves belong to Y, call it K'. Let $y_1, y_2, \ldots$ be the leaves of K'. For every y_i, there exists a $y_i v_i$-path Q_i in $G|K$ that is disjoint from $\pi(u)$ where v_i belongs to an X_i. Now $\bigcup_{i=1}^{\infty} K' \cup Q_i$ is a subdivided $K_{1,\infty}$ satisfying statement 2. Otherwise, $\pi(u) \cup F$ contains a subdivided comb whose leaves belong to Y, call it K'. Let $y_1, y_2, \ldots$ be the leaves of K'. For every y_i, there exists a $y_i v_i$-path Q_i in $G|K$ that is disjoint from $\pi(u)$ where v_i belongs to an X_i. Now $\bigcup_{i=1}^{\infty} K' \cup Q_i$ is a subdivided comb satisfying statement 2.

3. A subdivided $\mathcal{C}$, denoted by K, whose leaves belong to X'. Let P be the spine of K. For every leaf v of K, if the Pv-path Q of K contains a vertex w of degree 2

in K that belongs to X', then we delete the tv-subpath of Q from K where t is the neighbor of w in the wv-subpath of Q. By doing this to every Pv-path where v is a leaf of K, we may assume that every degree-2 vertex not on P of K does not belong to X'. Since K is also a minor of G, there exists an embedding π mapping each leaf of K to an X_i in G. Clearly, this X_i does not contain any other vertices of K. Let $u \in V(K)$. Then u has degree at most 3 in K. Thus, we can perform truncation on $\pi(u)$ in $G|K$. By doing this truncation process for every vertex in K, statement 3 is satisfied.

This completes the proof. □

We now describe the graphs $K_{3,\infty}, FF, FL, LL$ that are important in our later discussion.

Definition 4.1.6. Let $\{x_1, x_2, \ldots\}$ be an infinite set of vertices. A $K_{3,\infty}$ is obtained by adding edges x_1x_i, x_2x_i, and x_3x_i for every $i \geq 4$.

Definition 4.1.7. We define the graph FF as following. Let $R = x_1y_1x_2y_2\ldots$ be a ray and let u, v be vertices not on R. We add edges ux_i and edges vy_i for $i = 1, 2, \ldots$. Finally, we add an edge bewteen v and x_1.

We define the graph FL as following. Let $P = x_1y_1x_2y_2\ldots$ and $Q = z_1z_2\ldots$ be disjoint rays. We add an edge between x_i and z_i for $i = 1, 2, \ldots$. Let u, v be vertices not on $P \cup Q$. We add an edge between u and y_i for $i = 1, 2, \ldots$. Finally, we add edges uv, vx_1, vz_1.

We define the graph LL as following. Let $P = x_1y_1x_2y_2\ldots$, $Q = z_1z_2\ldots$, and $R = r_1r_2\ldots$ be disjoint rays. We add an edge between x_i and z_i and an edge between y_i and

r_i for $i = 1, 2, \ldots$. Let u be a vertex not on $P \cup Q \cup R$. Finally, we add edges ux_1, uz_1, ur_1.

Next, we examine six classes of graphs $\{\mathcal{FF}_1, \mathcal{FF}_2, \mathcal{FL}_1, \mathcal{FL}_2, \mathcal{LL}_1, \mathcal{LL}_2\}$ that are essential in the analysis of Theorem 1.2.5.

Definition 4.1.8. Let $\mathcal{FF}_1$ be the set of graphs defined as follows. Let R be a ray, which we call the **rail**, and let u, v be vertices not on R. We then add infinitely many edges from u to R, which we call **spokes at** u, and infinitely many edges from v to R, which we call **spokes at** v.

Let $\mathcal{FF}_2$ be the set of graphs defined as follows. Let R be a ray, which we call the **rail**, and let u, v be vertices not on R. We then add infinitely many uR-edges $e_1, e_2, \ldots$, which we call **spokes at** u, and hook v to infinitely many e_i such that each edge e_i is hooked at most once. Note that in this process, some e_i become two-edge paths if they are hooked; we still consider those two-edge paths spokes at u. We call each edge incident with v a **spoke** at v.

Let $\mathcal{FL}_1$ be the set of graphs defined as follows. Let A, B be disjoint rays, which we call **rails**, and let u be a vertex not in $A \cup B$. We first add infinitely many AB-edges, which we call **rungs**, such that no vertex in $A \cup B$ is incident with infinitely many rungs. We then add infinitely many uA-edges, which we call **spokes**.

Let $\mathcal{FL}_2$ be the set of graphs defined as follows. Let A, B be disjoint rays, which we call **rails**, and let u be a vertex not in $A \cup B$. We first add infinitely many AB-edges, which we call **rungs**, such that no vertex in $A \cup B$ is incident with infinitely many rungs. We then hook u to infinitely many rungs such that each rung is hooked at most once. Note that in this process, some rungs become two-edge paths if they are hooked; we still consider those two-edge paths rungs. We call each edge incident with u a **spoke**.

Let $\mathcal{LL}_1$ be the set of graphs defined as follows. Let A, B, C be disjoint rays, which we call **rails**. We then add infinitely many AB-edges and infinitely many BC-edges, which we call **rungs**, such that no vertex in $A \cup B \cup C$ is incident with infinitely many rungs.

Let $\mathcal{LL}_2$ be the set of graphs defined as follows. Let A, B, C be disjoint rays, which we call **rails**. We then add infinitely many AB-edges, which we call **rungs**, such that no vertex in $A \cup B$ is incident with infinitely many rungs. We then choose an infinite subset $\{x_1, x_2, \ldots\}$ of $V(C)$ and hook each x_i to a rung such that each rung is hooked at most once. Note that in this process, some rungs become two-edge paths if they are hooked; we still consider those two-edge paths rungs. A **spoke** is an edge with an endpoint on C and the other endpoint on the interior of a rung.

For a subdivision of a graph in $\{\mathcal{FF}_1, \mathcal{FF}_2, \mathcal{FL}_1, \mathcal{FL}_2, \mathcal{LL}_1, \mathcal{LL}_2\}$, we use the terms rail, spoke, and rung where applicable to mean its subdivided rail, subdivided spoke, and subdivided rung, respectively.

The six classes of graphs $\{\mathcal{FF}_1, \mathcal{FF}_2, \mathcal{FL}_1, \mathcal{FL}_2, \mathcal{LL}_1, \mathcal{LL}_2\}$ are important because graphs in each class can be reduced to one of the graphs in $\{FF, FL, LL\}$, which we now justify in the next few lemmas.

Lemma 4.1.9. *Let R be a ray and let A, B be two infinite subsets of $V(R)$. Then R contains a sequence of vertices $\{x_i\}_{i=1}^{\infty}$, listed in the order as they appear on R, such that for every nonnegative integer k, $x_{2k+1} \in A$ and $x_{2k+2} \in B$.*

Proof. We label all vertices of R as a sequence $\{r_i\}_{i=1}^{\infty}$ in the order as they appear on R. We define $\{x_i\}_{i=1}^{\infty}$ inductively. Let x_1 be a vertex $r_i \in R$, for some i, that is in A. Since there are infinitely many vertices in R that are in B, there exists an $r_j \in R$ with $j > i$ that is in B. Let $x_2 = r_j$. Next, since there are infinitely many vertices in R that are in A, there exists an $r_k \in R$ with $k > j$ that is in A. Let $x_3 = r_k$. By repeating this process, we obtain the desired sequence $\{x_i\}_{i=1}^{\infty}$. □

Lemma 4.1.10. *Let H be a subdivision of a graph in $\mathcal{FF}_1$ and let X be an infinite subset of $V(H)$. Then H contains an X-rich FF.*

Proof. Let R be the rail and let u, v be the infinite-degree vertices of H. Since $X \subseteq V(H)$, one of the following is true

1. R contains infinitely many elements of X,
2. there exist infinitely many spokes at u, each contains at least one element of $X - u$,
3. there exist infinitely many spokes at v, each contains at least one element of $X - v$.

Thus, H contains a subgraph H', which is also a subdivision of a graph in $\mathcal{FF}_1$ and with the same u, v, R, satisfying one of the following

1. R contains infinitely many elements of X,
2. every spoke at u of H' contains at least one element of $X - u$,
3. every spoke at v of H' contains at least one element of $X - v$.

In H', let

$$A = \{x \in V(R) \mid x \text{ is an endpoint of a spoke at } u\}$$

and let

$$B = \{x \in V(R) \mid x \text{ is an endpoint of a spoke at } v\}.$$

Then A and B are infinite subsets of $V(R)$. By Lemma 4.1.9, R contains a sequence of vertices $\{x_i\}_{i=1}^{\infty}$, listed in the order as they appear on R, such that for every nonnegative integer k, $x_{2k+1} \in A$ and $x_{2k+2} \in B$. For a nonnegative integer i, let P_{2i+1} be the spoke with endpoints u, x_{2i+1} and let Q_{2i+2} be the spoke with endpoints v, x_{2i+2}. The subgraph $(\bigcup_{i=0}^{\infty} P_{2i+1} \cup Q_{2i+2}) \cup R$ contains an X-rich FF. □

Lemma 4.1.11. *Let H be a subdivision of a graph in $\mathcal{FF}_2$ and let X be an infinite subset of $V(H)$. Then H contains an X-rich FF.*

Proof. Let R be the rail and let u, v be the infinite-degree vertices of H. Let $\mathcal{P}$ be the set of spokes at u so that every spoke in $\mathcal{P}$ has an endpoint on R. Let $\mathcal{Q}$ be the set of spokes at v. Since $X \subseteq V(H)$, one of the following is true

1. R contains infinitely many elements of X,

2. there exists an infinite subset $\mathcal{P}'$ of $\mathcal{P}$ where each path in $\mathcal{P}'$ contains at least one element of $X - u$,

3. there exists an infinite subset $\mathcal{Q}'$ of $\mathcal{Q}$ where each path in $\mathcal{Q}'$ contains at least one element of $X - v$.

We divide the proof into two cases.

Case 1: Statement 1 or statement 3 is true.

If statement 1 is true, let $\mathcal{Q}' = \mathcal{Q}$. Otherwise, let $\mathcal{Q}'$ be determined as in statement 3. We label the paths in $\mathcal{P}$ that are hooked by a path in $\mathcal{Q}'$ as $P_1, P_2, \ldots,$ in the order as their endpoints appear on R. For a $P_i \in \mathcal{P}$, let $Q_i \in \mathcal{Q}'$ be the path that is hooked to P_i. Let t_i be the endpoint of P_i on R and let z_i be the endpoint of Q_i on $\mathring{P}_i$. Let R_i be the z_it_i-subpath of P_i. The subgraph $(\bigcup_{i=0}^{\infty} P_{2i+1} \cup Q_{2i+2} \cup R_{2i+2}) \cup R$ contains an X-rich FF.

Case 2: Statement 2 is true.

A path in $\mathcal{P}'$ is called good if it is hooked and is bad otherwise. First, suppose there are infinitely many good paths in $\mathcal{P}'$. We label the good paths in $\mathcal{P}'$ as $P_1, P_2, \ldots$, in the order as their endpoints appear on R. For each i, let $Q_i \in \mathcal{Q}$ be the path that is hooked to P_i. Let t_i be the endpoint of P_i on R and let z_i be the endpoint of Q_i on $\mathring{P}_i$. Let R_i be the z_it_i-subpath of P_i. The subgraph $(\bigcup_{i=0}^{\infty} P_{2i+1} \cup Q_{2i+2} \cup R_{2i+2}) \cup R$ contains an X-rich FF.

Now suppose there are only finitely many good paths in $\mathcal{P}'$, so there are infinitely many bad paths in $\mathcal{P}'$. We label the bad paths in $\mathcal{P}'$ as $P'_1, P'_2, \ldots$. In addition, we label the paths in $\mathcal{P}$ that are hooked as $P''_1, P''_2, \ldots$. For a $P''_i \in \mathcal{P}$, let $Q''_i \in \mathcal{Q}$ be the path that is hooked to P''_i. Observe that the two sets $\{P'_1, P'_2, \ldots\}$ and $\{P''_1, P''_2, \ldots\}$ are disjoint. Let t_i be the endpoint of P''_i on R and let z_i be the endpoint of Q''_i on $\mathring{P}''_i$. Let R''_i be the z_it_i-subpath of P''_i. The subgraph $(\bigcup_{i=1}^{\infty} P'_i \cup Q''_i \cup R''_i) \cup R$ is a subdivision of a graph in $\mathcal{FF}_1$ containing infinitely many elements of X. By Lemma 4.1.10, it contains an X-rich FF. □

Lemma 4.1.12. *Let H be a subdivision of a graph in $\mathcal{FL}_1$ and let X be an infinite subset of $V(H)$. Then H contains an X-rich FL.*

Proof. In H, let u be the infinite-degree vertex and let A, B be the rails. Without loss of generality, let A be the rail that contains the endpoints of the spokes of H. Let R be the union of all rungs. Since $X \subseteq V(H)$, one of the following is true

1. $A \cup B \cup R$ contains infinitely many elements of X,

2. there exist infinitely many spokes each contains at least one element of $X - u$.

First, suppose statement 1 is true. By Lemma 3.1.10, $A \cup B \cup R$ contains a subdivided L_∞, which we call L, whose rails are contained in A, B and whose rungs are rungs of H, such that one of its rails contains infinitely many elements of X or every of its rungs contains at least one element of X. Let

$$M = \{x \in V(A) \mid x \text{ is an endpoint of a spoke of } H\}$$

and let

$$N = \{x \in V(A) \mid x \text{ is an endpoint of a rung of } L\}.$$

Then both M and N are infinite subsets of $V(A)$. By Lemma 4.1.9, A contains a sequence of vertices $\{x_i\}_{i=1}^{\infty}$, listed in the order as they appear on A, such that for every nonnegative integer k, $x_{2k+1} \in M$ and $x_{2k+2} \in N$. For a nonnegative integer i, let S_{2i+1} be the spoke of H with endpoints u, x_{2i+1} and let R_{2i+2} be the rung of L with x_{2i+2} as its endpoint in A. The subgraph $(\bigcup_{i=0}^{\infty} S_{2i+1} \cup R_{2i+2}) \cup A \cup B$ contains an X-rich FL.

Now suppose statement 2 is true. Let $\mathcal{S}$ be an infinite set of the spokes of H such that every spoke in $\mathcal{S}$ contains at least one element of $X - u$. By Lemma 3.1.9, $A \cup B \cup R$ contains a subdivided L_∞, which we call L, whose rails are contained in A, B and whose rungs are rungs of H. Let

$$M = \{x \in V(A) \mid x \text{ is an endpoint of a spoke in } \mathcal{S}\}$$

and let

$$N = \{x \in V(A) \mid x \text{ is an endpoint of a rung of } L\}.$$

Then both M and N are infinite subsets of $V(A)$. By Lemma 4.1.9, A contains a sequence of vertices $\{x_i\}_{i=1}^{\infty}$, listed in the order as they appear on A, such that for every nonnegative integer k, $x_{2k+1} \in M$ and $x_{2k+2} \in N$. For a nonnegative integer i, let S_{2i+1} be the spoke in $\mathcal{S}$ with endpoints u, x_{2i+1} and let R_{2i+2} be the rung of L with x_{2i+2} as its endpoint in A. The subgraph $(\bigcup_{i=0}^{\infty} S_{2i+1} \cup R_{2i+2}) \cup A \cup B$ contains an X-rich FL. □

Lemma 4.1.13. *Let H be a subdivision of a graph in $\mathcal{FL}_2$ and let X be an infinite subset of $V(H)$. Then H contains an X-rich FL.*

Proof. In H, let A, B be the rails and let u be the infinite-dergree vertex. Let $\mathcal{S}$ be the set of spokes and let $\mathcal{R}$ be the set of rungs. Since $X \subseteq V(H)$, one of the following is true

1. $A \cup B$ contains infinitely many elements of X,
2. there exists an infinite subset $\mathcal{S}'$ of $\mathcal{S}$ where every spoke in $\mathcal{S}'$ contains at least one element of $X - u$,
3. there exists an infinite subset $\mathcal{R}'$ of $\mathcal{R}$ where every rung in $\mathcal{R}'$ contains at least one element of X.

We divide the proof into two cases.

Case 1: Statement 1 or statement 2 is true.

If statement 1 is true, let $\mathcal{S}' = \mathcal{S}$. Otherwise, let $\mathcal{S}'$ be determined as in statement 2. Let $\mathcal{R}''$ be the set of rungs of H that are hooked by a spoke in $\mathcal{S}'$ and let R'' be the union of all rungs in $\mathcal{R}''$. If statement 1 is true, then by Lemma 3.1.10, $A \cup B \cup R''$ contains a subdivided L_∞, whose rails are contained in A, B and whose rungs belong to $\mathcal{R}''$, such that one of its rails contains infinitely many elements of X. Otherwise, by Lemma 3.1.9, $A \cup B \cup R''$ contains a subdivided L_∞, whose rails are contained in A, B and whose rungs belong to $\mathcal{R}''$. In either case, we denote L to be this subdivided L_∞. We label the rungs of L as $R_1, R_2, \ldots,$ in the order as their endpoints appear on A. Note that every R_i is hooked by a spoke in $\mathcal{S}'$ by the choice of $\mathcal{R}''$. For an R_i, let $S_i \in \mathcal{S}'$ be the path that is hooked to R_i. Let t_i be the endpoint of R_i on A and let z_i be the endpoint of S_i on $\mathring{R}_i$. Let M_i be the $z_i t_i$-subpath of R_i. The subgraph $(\bigcup_{i=0}^{\infty} R_{2i+1} \cup S_{2i+2} \cup M_{2i+2}) \cup A \cup B$ contains an X-rich FL.

Case 2: Statement 3 is true.

A rung in $\mathcal{R}'$ is called good if it is hooked and is bad otherwise. First, suppose there are infinitely many good rungs in $\mathcal{R}'$. Let $\mathcal{R}'' \subseteq \mathcal{R}'$ be an infinite subset of good rungs and let R'' be the union of all rungs in $\mathcal{R}''$. By Lemma 3.1.10, $A \cup B \cup R''$ contains a subdivided L_∞, which we call L, whose rails are contained in A, B and whose rungs belong to $\mathcal{R}''$, such that one of its rails contains infinitely many elements of X or every of its rungs contains at least one element of X. We label the rungs of L as $R_1, R_2, \ldots,$ in the order as their endpoints appear on A. Note that every R_i is hooked by the definition of being good. For an R_i, let $S_i \in \mathcal{S}$ be the spoke that is hooked to R_i. Let t_i be the endpoint of R_i on A and let z_i be the endpoint of S_i on $\mathring{R}_i$. Let M_i be the $z_i t_i$-subpath of R_i. The subgraph $(\bigcup_{i=0}^{\infty} R_{2i+1} \cup S_{2i+2} \cup M_{2i+2}) \cup A \cup B$ contains an X-rich FL.

Now suppose there are only finitely many good rungs in $\mathcal{R}'$, so there are infinitely many bad rungs in $\mathcal{R}'$. Let $\mathcal{R}'' \subseteq \mathcal{R}'$ be an infinite subset of bad rungs and let R'' be the union of all rungs in $\mathcal{R}''$. By Lemma 3.1.10, $A \cup B \cup R''$ contains a subdivided L_∞, which we call L, whose rails are contained in A, B and whose rungs belong to $\mathcal{R}''$, such that L contains infinitely many elements of X. Let $\mathcal{R}''' \subseteq \mathcal{R}$ be an infinite subset of rungs that are hooked and let R''' be the union of all rungs in $\mathcal{R}'''$. By Lemma 3.1.9, $A \cup B \cup R'''$ contains a subdivided L_∞, which we call L', whose rails are contained in A, B and whose rungs belong to $\mathcal{R}'''$. We label the rungs of L as $R'_1, R'_2, \ldots$. We label the rungs of L' as $R''_1, R''_2, \ldots$. Note that every R''_i is hooked by the definition of being good. For an R''_i, let $S''_i \in \mathcal{S}$ be the spoke that is hooked to R''_i. Observe that the two sets $\{R'_1, R'_2, \ldots\}$ and $\{R''_1, R''_2, \ldots\}$ are disjoint. Let t_i be the endpoint of R''_i on A and let z_i be the endpoint of S''_i on $\mathring{R}''_i$. Let M''_i be the $z_i t_i$-subpath of R''_i. The subgraph $(\bigcup_{i=1}^{\infty} R'_i \cup R''_i \cup M''_i) \cup A \cup B$ is a subdivision of a graph in $\mathcal{FL}_1$ containing infinitely many elements of X. By Lemma 4.1.12, it contains an X-rich FL. □

Lemma 4.1.14. *Let H be a subdivision of a graph in $\mathcal{LL}_1$ and let X be an infinite subset of $V(H)$. Then H contains an X-rich LL.*

Proof. In H, let A, B, C be its rails, let $\mathcal{M}$ be the set of rungs between A, B, and let $\mathcal{N}$ be the set of rungs between B, C. Let M be the union of all rungs in $\mathcal{M}$ and let N be the union of all rungs in $\mathcal{N}$. Since $X \subseteq V(H)$, we may assume, without loss of generality, that $A \cup B \cup M$ contains infinitely many elements of X. By Lemma 3.1.10, $A \cup B \cup M$ contains a subdivided L_∞, which we call L, whose rails are contained in A, B and whose rungs belong to $\mathcal{M}$, such that L contains infinitely many elements of X. In addition, by Lemma 3.1.9,

$B \cup C \cup N$ contains a subdivided L_∞, which we call L', whose rails are contained in B, C and whose rungs belong to $\mathcal{N}$. Let

$$S = \{x \in V(B) \mid x \text{ is an endpoint of a rung of } L\}$$

and let

$$T = \{x \in V(B) \mid x \text{ is an endpoint of a rung of } L'\}.$$

Then both S and T are infinite subsets of $V(B)$. By Lemma 4.1.9, B contains a sequence of vertices $\{x_i\}_{i=1}^{\infty}$, listed in the order as they appear on B, such that for every nonnegative integer k, $x_{2k+1} \in M$ and $x_{2k+2} \in N$. For a nonnegative integer i, let S_{2i+1} be the rung of L with x_{2i+1} as its endpoint in B and let T_{2i+2} be the rung of L' with x_{2i+2} as its endpoint in B. The subgraph $(\bigcup_{i=0}^{\infty} S_{2i+1} \cup T_{2i+2}) \cup A \cup B \cup C$ contains an X-rich LL. □

Lemma 4.1.15. *Let H be a subdivision of a graph in $\mathcal{LL}_2$ and let X be an infinite subset of $V(H)$. Then H contains an X-rich LL.*

Proof. In H, let A, B, C be the rails, let $\mathcal{R}$ be the set of rungs between A, B, and let $\mathcal{S}$ be the set of spokes. Since $X \subseteq V(H)$, one of the following is true

1. $A \cup B$ contains infinitely many elements of X,
2. C contains infinitely many elements of X,
3. there exists an infinite subset $\mathcal{S}'$ of $\mathcal{S}$ where every spoke in $\mathcal{S}'$ contains at least one element of X,
4. there exists an infinite subset $\mathcal{R}'$ of $\mathcal{R}$ where every rung in $\mathcal{R}'$ contains at least one element of X.

We divide the proof into two cases.

Case 1: Statement 1, statement 2, or statement 3 is true.

If statement 1 or statement 2 is true, let $\mathcal{S}' = \mathcal{S}$. Otherwise, let $\mathcal{S}'$ be determined as in statement 3. Let $\mathcal{R}''$ be the set of rungs of H that are hooked by a spoke in $\mathcal{S}'$ and let R'' be the union of all rungs in $\mathcal{R}''$. If statement 1 is true, then by Lemma 3.1.10, $A \cup B \cup R''$ contains a subdivided L_∞, whose rails are contained in A, B and whose rungs belong to $\mathcal{R}''$, such that one of its rails contains infinitely many elements of X. Otherwise, by Lemma 3.1.9, $A \cup B \cup R''$ contains a subdivided L_∞, whose rails are contained in A, B and whose rungs belong to $\mathcal{R}''$. In either case, we denote L to be this subdivided L_∞. We label the rungs of L as $R_1, R_2, \ldots,$ in the order as their endpoints appear on A. Note that every R_i is hooked by a spoke in $\mathcal{S}'$ by the choice of $\mathcal{R}''$. For an R_i, let $S_i \in \mathcal{S}'$ be the path that is hooked to R_i. Let t_i be the endpoint of R_i on A and let z_i be the endpoint of S_i on $\mathring{R}_i$. Let M_i be the $z_i t_i$-subpath of R_i. The subgraph $(\bigcup_{i=0}^{\infty} R_{2i+1} \cup S_{2i+2} \cup M_{2i+2}) \cup A \cup B \cup C$ is a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely many elements of X. By Lemma 4.1.14, it contains an X-rich LL.

Case 2: Statement 4 is true.

A rung in $\mathcal{R}'$ is called good if it is hooked and is bad otherwise. First, suppose there are infinitely many good rungs in $\mathcal{R}'$. Let $\mathcal{R}'' \subseteq \mathcal{R}'$ be an infinite subset of good rungs and let R'' be the union of all rungs in $\mathcal{R}''$. By Lemma 3.1.10, $A \cup B \cup R''$ contains a subdivided L_∞, which we call L, whose rails are contained in A, B and whose rungs belong to $\mathcal{R}''$, such that one of its rails contains infinitely many elements of X or every of its rungs contains at least one element of X. We label the rungs of L as $R_1, R_2, \ldots$, in the order as their endpoints appear on A. Note that every R_i is hooked by the definition of being good. For an R_i, let $S_i \in \mathcal{S}$ be the spoke that is hooked to R_i. Let t_i be the endpoint of R_i on A and let z_i be the endpoint of S_i on $\mathring{R}_i$. Let M_i be the $z_i t_i$-subpath of R_i. The subgraph $(\bigcup_{i=0}^{\infty} R_{2i+1} \cup S_{2i+2} \cup M_{2i+2}) \cup A \cup B$ is a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely many elements of X. By Lemma 4.1.14, it contains an X-rich LL.

Now suppose there are only finitely many good rungs in $\mathcal{R}'$, so there are infinitely many bad rungs in $\mathcal{R}'$. Let $\mathcal{R}'' \subseteq \mathcal{R}'$ be an infinite subset of bad rungs and let R'' be the union of all rungs in $\mathcal{R}''$. By Lemma 3.1.10, $A \cup B \cup R''$ contains a subdivided L_∞, which we call L, whose rails are contained in A, B and whose rungs belong to $\mathcal{R}''$, such that one of its rails contains infinitely many elements of X or every of its rungs contains at least one element of X. Let $\mathcal{R}''' \subseteq \mathcal{R}$ be an infinite subset of rungs that are hooked and let R''' be the union of all rungs in $\mathcal{R}'''$. By Lemma 3.1.9, $A \cup B \cup R'''$ contains a subdivided L_∞, which we call L', whose rails are contained in A, B and whose rungs belong to $\mathcal{R}'''$. We label the rungs of L as $R'_1, R'_2, \ldots$. We label the rungs of L' as $R''_1, R''_2, \ldots$. Note that every R''_i is hooked by the definition of being good. For an R''_i, let $S''_i \in \mathcal{S}$ be the spoke that is hooked to R''_i. Observe that the two sets $\{R'_1, R'_2, \ldots\}$ and $\{R''_1, R''_2, \ldots\}$ are disjoint. Let t_i be the endpoint of R''_i on A and let z_i be the endpoint of S''_i on $\mathring{R}''_i$. Let M''_i be the $z_i t_i$-subpath of R''_i. The subgraph $(\bigcup_{i=1}^{\infty} R'_i \cup R''_i \cup M''_i) \cup A \cup B$ is a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely many elements of X. By Lemma 4.1.14, it contains an X-rich LL. □

Lemma 4.1.16. *Let A, B be disjoint rays and let $\mathcal{R}$ be an infinite set of internally disjoint*

AB-paths. Let R be the union of all paths in $\mathcal{R}$ and let $H = A \cup B \cup R$. Assume H is locally finite. Let u be a vertex not in H and let $\mathcal{S}$ be an infinite set of weakly disjoint uH-paths. Let S be the union of all paths in $\mathcal{S}$ and let $G = H \cup S$. Let X be an infinite subset of $V(G)$. Then G contains an X-rich FL.

Proof. Since $X \subseteq V(G)$, one of the following is true

1. H contains infinitely many elements of X,

2. $\mathcal{S}$ has an infinite subset $\mathcal{S}'$ such that every path in $\mathcal{S}'$ contains an element of $X - u$.

We divide the proof into two cases.

Case 1: Statement 1 is true.

If infinitely many paths in $\mathcal{S}$ has endpoints in $A \cup B$, then G contains a subdivision of a graph in $\mathcal{FL}_1$ containing infinitely many elements of X. By Lemma 4.1.12, it contains an X-rich FL. Otherwise, infinitely many paths in $\mathcal{S}$ has endpoints in $R - (A \cup B)$. If a path in $\mathcal{R}$ contains endpoints of more than one path in $\mathcal{S}$, then we delete edges of all but one path in $\mathcal{S}$. By repeating this process, we obtain a subdivision of a graph in $\mathcal{FL}_2$ containing infinitely many elements of X. By Lemma 4.1.13, G contains an X-rich FL.

Case 2: Statement 2 is true. If a path in $\mathcal{R}$ contains endpoints of more than one path in $\mathcal{S}'$, then we delete edges of all but one path in $\mathcal{S}'$. By repeating this process, we obtain a subdivision of a graph in $\mathcal{FL}_2$ containing infinitely many elements of X. By Lemma 4.1.13, G contains an X-rich FL. □

In addition to cleaning up the graphs in $\{\mathcal{FF}_1, \mathcal{FF}_2, \mathcal{FL}_1, \mathcal{FL}_2, \mathcal{LL}_1, \mathcal{LL}_2\}$, we also need to clean up graphs of an $\mathcal{LL}_1$ or $\mathcal{LL}_2$ nature, but with extra jumps inside. The next three lemmas make this idea more precise.

Lemma 4.1.17. *Let A, B be disjoint rays and let $\mathcal{R}$ be an infinite set of internally disjoint AB-paths. Let R be the union of all paths in $\mathcal{R}$ and let $H = A \cup B \cup R$. Assume H is locally finite. Let $\mathcal{J}$ be an infinite set of disjoint H-paths. Let G be the union of H and all paths in $\mathcal{J}$ and let X be an infinite subset of $V(G)$ such that every path in $\mathcal{J}$ contains at least one element of X. Then G contains an X-rich L_∞.*

Proof. First, observe that G is locally finite since H is locally finite and paths in $\mathcal{J}$ are disjoint. By definition, every path in $\mathcal{J}$ has its two endpoints on H. Up to symmetry, we may assume that each path in $\mathcal{J}$ is exactly one of the following types

- type 1: both endpoints belong to A,
- type 2: one endpoint belongs to A and the other endpoint belongs to B,
- type 3: one endpoint belongs to A and the other endpoint belongs to $R - (A \cup B)$.

Note that infinitely many paths in $\mathcal{J}$ are of one type. For convenience, in this proof, a path in $\mathcal{R}$ is called a rung.

Claim 4.1.17.1. *If there exist infinitely many paths of $\mathcal{J}$ of type 1, then the lemma holds.*

Since paths in $\mathcal{J}$ are disjoint, we can find infinitely many paths $J_1, J_2, \dots$ in $\mathcal{J}$ such that with respect to A, J_i is on the left of J_{i+1} for $i = 1, 2, \dots$. For each J_i, let a_i, b_i be its two endpoints on A. Observe that every $A[a_i b_i]$ contains endpoints of finitely many rungs because G is locally finite. First, if an $A[a_i b_i]$ contains endpoints of more than one rung, then we delete edges of all but one rung with an endpoint in $A[a_i b_i]$. Hence, we may assume every $A[a_ib_i]$ contains endpoint of at most one rung. Next, suppose a rung has an endpoint r in an $A[a_ib_i]$. If $r \in A(a_ib_i)$, then we delete edges of $A[a_ir]$. Otherwise, $r \in \{a_i, b_i\}$, and we delete edges of $A[a_ib_i]$. By repeating this process, we obtained a graph satisfying the conditions in Lemma 3.1.10. Thus, G contains an X-rich L_∞. This proves the claim.

Claim 4.1.17.2. *If there exist infinitely many paths of $\mathcal{J}$ of type 2, then the lemma holds.*

Let $\mathcal{J}'$ be an infinite subset of $\mathcal{J}$ such that every path in $\mathcal{J}'$ is of type 2. Let J' be the union of all paths in $\mathcal{J}'$. The subgraph $A \cup B \cup J'$ satisfies the conditions in Lemma 3.1.10, so it contains an X-rich L_∞. This proves the claim.

Claim 4.1.17.3. *If there exist infinitely many paths of $\mathcal{J}$ of type 3, then the lemma holds.*

Let $\mathcal{J}'$ be an infinite subset of $\mathcal{J}$ such that every path in $\mathcal{J}'$ is of type 3. Since G is locally finite, $\mathcal{J}'$ has an infinite subset $\mathcal{J}'' = \{P_1, P_2, \ldots\}$ such that every rung contains endpoints of at most one path in $\mathcal{J}''$. For each i, let Q_i be the unique rung such that $\mathring{Q}_i$ contains an endpoint of P_i. Let x_i be the endpoint of Q_i on B. and let y_i be the endpoint of P_i on $\mathring{Q}_i$. Let M_i be the x_iy_i-subpath of Q_i. Now $Q_i' = P_i \cup M_i$ is an AB-path. Let $R' = \bigcup_{i=1}^{\infty} Q_i'$. The subgraph $A \cup B \cup R'$ satisfies the conditions in Lemma 3.1.10, so it contains an X-rich L_∞. This proves the claim.

We have shown that if infinitely many paths of $\mathcal{J}$ are of type i, for any $i \in \{1, 2, 3\}$, then the lemma holds. This completes the proof. □

Lemma 4.1.18. *Let H be a subdivision of a graph in $\mathcal{LL}_1$ and let $\mathcal{J}$ be an infinite set of disjoint H-paths. Let G be the union of H and all paths in $\mathcal{J}$ and let X be an infinite subset* *of $V(G)$ such that every path in $\mathcal{J}$ contains at least one element of X. Then G contains an X-rich LL.*

Proof. First, observe that G is locally finite since H, being a subdivision of a graph in $\mathcal{LL}_1$, is locally finite and paths in $\mathcal{J}$ are disjoint. In H, let A, B, C be its rails and let $\mathcal{S}_1$ be the sets of rungs between A, B and let $\mathcal{S}_2$ be the sets of rungs between B, C. Let S_i be the union of all paths in $\mathcal{S}_i$ for $i = 1, 2$. By definition, every path in $\mathcal{J}$ has its two endpoints on H. Up to symmetry, we may assume that each path in $\mathcal{J}$ is exactly one of the following types

- type 1: both endpoints belong to A,
- type 2: one endpoint belongs to A and the other endpoint belongs to $B \cup C$,
- type 3: one endpoint belongs to A and the other endpoint belongs to $S_1 - (A \cup B)$,
- type 4: one endpoint belongs to A and the other endpoint belongs to $S_2 - (B \cup C)$,
- type 5: both endpoints belong to B,
- type 6: one endpoint belongs to B and the other endpoint belongs to $S_1 - (A \cup B)$,
- type 7: both endpoints belong to $S_1 - (A \cup B)$,
- type 8: one endpoint belongs to $S_1 - (A \cup B)$ and the other endpoint belongs to $S_2 - (B \cup C)$.

Note that infinitely many paths in $\mathcal{J}$ are of one type.

Claim 4.1.18.1. *If there exist infinitely many paths of $\mathcal{J}$ of type 1, then the lemma holds.*

Since paths in $\mathcal{J}$ are disjoint, we can find infinitely many paths $J_1, J_2, \ldots$ in $\mathcal{J}$ such that with respect to A, J_i is on the left of J_{i+1} for $i = 1, 2, \ldots$. For each J_i, let a_i, b_i be its two endpoints on A. Observe that every $A[a_ib_i]$ contains endpoints of finitely many rungs because G is locally finite. First, if an $A[a_ib_i]$ contains endpoints of more than one rung, then we delete edges of all but one rung with an endpoint in $A[a_ib_i]$. Hence, we may assume every $A[a_ib_i]$ contains endpoint of at most one rung. Next, suppose a rung has an endpoint r in an $A[a_ib_i]$. If $r \in A(a_ib_i)$, then we delete edges of $A[a_ir]$. Otherwise, $r \in \{a_i, b_i\}$, and we delete edges of $A[a_ib_i]$. By repeating this process, we obtained a subdivision of $\mathcal{LL}_1$ containing infinitely many elements of X. By Lemma 4.1.14, it contains an X-rich LL. This proves the claim.

Claim 4.1.18.2. *If there exist infinitely many paths of $\mathcal{J}$ of type 2, then the lemma holds.*

Let $\mathcal{J}'$ be an infinite subset of $\mathcal{J}$ such that every path in $\mathcal{J}'$ is of type 2. Then $\mathcal{J}'$ has an infinite subset $\mathcal{J}''$ such that every path in $\mathcal{J}''$ is an AB-path or every path in $\mathcal{J}''$ is an AC-path. Let J'' be the union of all paths in $\mathcal{J}''$. The subgraph $A \cup B \cup C \cup S_2 \cup J''$ is a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely elements of X. By Lemma 4.1.14, it contains an X-rich LL. This proves the claim.

Claim 4.1.18.3. *If there exist infinitely many paths of $\mathcal{J}$ of type 3 or type 6, then the lemma holds.*

Let $\mathcal{J}'$ be an infinite subset of $\mathcal{J}$ such that every path in $\mathcal{J}'$ is of type 3 or every path in $\mathcal{J}'$ is of type 6. Since G is locally finite, $\mathcal{J}'$ has an infinite subset $\mathcal{J}'' = \{P_1, P_2, \ldots\}$ such that every path in $\mathcal{S}_1$ contains endpoints of at most one path in $\mathcal{J}''$. For each i, let Q_i be the unique path in $\mathcal{S}_1$ such that $\mathring{Q}_i$ contains an endpoint of P_i. If every path in $\mathcal{J}'$ is of type 3, then let x_i be the endpoint of Q_i on B. Otherwise, every path in $\mathcal{J}'$ is of type 6 and we let x_i be the endpoint of Q_i on A. Let y_i be the endpoint of P_i on $\mathring{Q}_i$. Let M_i be the x_iy_i-subpath of Q_i. Now $Q'_i = P_i \cup M_i$ is an AB-path. Let $S'_1 = \bigcup_{i=1}^{\infty} Q'_i$. The subgraph $A \cup B \cup C \cup S'_1 \cup S_2$ is a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely elements of X. By Lemma 4.1.14, it contains an X-rich LL. This proves the claim.

Claim 4.1.18.4. *If there exist infinitely many paths of $\mathcal{J}$ of type 4, then the lemma holds.*

Let $\mathcal{J}'$ be an infinite subset of $\mathcal{J}$ such that every path in $\mathcal{J}'$ is of type 4. Since G is locally finite, $\mathcal{J}'$ has an infinite subset $\mathcal{J}'' = \{P_1, P_2, \ldots\}$ such that every path in $\mathcal{S}_2$ contains endpoints of at most one path in $\mathcal{J}''$. For each i, let Q_i be the unique path in $\mathcal{S}_2$ such that $\mathring{Q}_i$ contains an endpoint of P_i. Let x_i be the endpoint of Q_i on C. Let y_i be the endpoint of P_i on $\mathring{Q}_i$. Let M_i be the x_iy_i-subpath of Q_i. Now $Q'_i = P_i \cup M_i$ is an AC-path. Let $S'_2 = \bigcup_{i=1}^{\infty} Q'_i$. The subgraph $A \cup B \cup C \cup S_1 \cup S'_2$ is a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely elements of X. By Lemma 4.1.14, it contains an X-rich LL. This proves the claim.

Claim 4.1.18.5. *If there exist infinitely many paths of $\mathcal{J}$ of type 5, then the lemma holds.*

Since paths in $\mathcal{J}$ are disjoint, we can find infinitely many paths $J_1, J_2, \ldots$ in $\mathcal{J}$ such that with respect to B, J_i is on the left of J_{i+1} for $i = 1, 2, \ldots$. For each J_i, let x_i, y_i be its two endpoints on B. Observe that every $B[x_iy_i]$ contains endpoints of finitely many rungs because G is locally finite. If a $B[x_iy_i]$ contains endpoints of more than one rung in $\mathcal{S}_1$, then we delete edges of all but one rung in $\mathcal{S}_1$ with an endpoint in $B[x_iy_i]$. Similarly, if a $B[x_iy_i]$ contains endpoints of more than one rung in $\mathcal{S}_2$, then we delete edges of all but one rung in $\mathcal{S}_2$ with an endpoint in $B[x_iy_i]$. Hence, we may assume every $B[x_iy_i]$ contains endpoints of at most one rung in $\mathcal{S}_1$ and at most one rung in $\mathcal{S}_2$. We consider a path J_i to be type 5a if $B[x_iy_i]$ contains no endpoint of rungs in $\mathcal{S}_1$ and no endpoint of rungs in $\mathcal{S}_2$ and to be type 5b otherwise. Let $I = \{i \mid J_i \text{ is of type 5a}\}$ and let $I' = \{i \mid J_i \text{ is of type 5b}\}$. Then either I or I' is infinite. First, suppose I is infinite. By replacing $B[x_iy_i]$ with J_i for every $i \in I$, we obtain a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely elements of X. By Lemma 4.1.14, it contains an X-rich LL. Now suppose I' is infinite. Let I_1 be the subset of I' such that if $i \in I_1$, then $B[x_iy_i]$ contains an endpoint of a rung in $\mathcal{S}_1$ and let I_2 be the subset of I' such that if $i \in I_2$, then $B[x_iy_i]$ contains an endpoint of a rung in $\mathcal{S}_2$. Note that I_1 and I_2 may have common elements. Since I' is infinite, at least one of the I_1 or I_2 is infinite. We divide the remain of this claim into two cases.

Case 1: Both I_1 and I_2 are infinite. By Lemma 3.1.2, there exist two infinite sets $I_3 \subseteq I_1$ and $I_4 \subseteq I_2$ such that $I_3 \cap I_4 = \emptyset$. For $i \in I_3$, let M_i be the rung in $\mathcal{S}_1$ with an endpoint m_i in $B[x_iy_i]$. If $m_i \in B(x_iy_i)$, then we delete edges of $B[x_im_i]$. Otherwise, $m_i \in \{x_i, y_i\}$, and we delete edges of $B[x_iy_i]$. For $j \in I_4$, let N_j be the rung in $\mathcal{S}_2$ with an endpoint n_j in $B[x_jy_j]$. If $n_j \in B(x_jy_j)$, then we delete edges of $B[x_jn_j]$. Otherwise, $n_j \in \{x_j, y_j\}$, and we delete edges of $B[x_jy_j]$. By repeating this process for all $i \in I_3$ and all $j \in I_4$, we obtain a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely elements of X. By Lemma 4.1.14, it contains an X-rich LL.

Case 2: Exactly one of the I_1 or I_2 is infinite. Without loss of generality, we may assume I_1 is infinite while I_2 is finite. Hence, $I_3 = I_1 - I_2$ is infinite. For $i \in I_3$, let M_i be the rung in $\mathcal{S}_1$ with an endpoint m_i in $B[x_iy_i]$. If $m_i \in B(x_iy_i)$, then we delete edges of

$B[x_i m_i]$. Otherwise, $m_i \in \{x_i, y_i\}$, and we delete edges of $B[x_i y_i]$. By repeating this process for every $i \in I_3$, we obtain a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely elements of X. By Lemma 4.1.14, it contains an X-rich LL. This proves the claim.

Claim 4.1.18.6. *If there exist infinitely many paths of $\mathcal{J}$ of type 7, then the lemma holds.*

Let $\mathcal{J}'$ be an infinite subset of $\mathcal{J}$ such that every path in $\mathcal{J}'$ is of type 7. Since G is locally finite, $\mathcal{J}'$ has an infinite subset $\mathcal{J}''$ such that every path in $\mathcal{S}_1$ contains endpoints of at most one path in $\mathcal{J}''$. To see this, we start with $\mathcal{J}'' = \emptyset$. Let $J \in \mathcal{J}' - \mathcal{J}''$. Then J has endpoints in $\mathring{P}, \mathring{Q}$ for some $P, Q \in \mathcal{S}_1$ where $P = Q$ is possible. Next, we delete edges of all paths in $\mathcal{J}'$, except for J, with an endpoint in $\mathring{P} \cup \mathring{Q}$ and then we add J into $\mathcal{J}''$. Note that after doing this, $\mathcal{J}'$ is still infinite as we only delete finitely many paths in $\mathcal{J}'$. We then pick a $J' \in \mathcal{J}' - \mathcal{J}''$ and repeat the process. This yields the desired $\mathcal{J}''$. A path in $\mathcal{J}''$ is called type 7a if its two endpoints belong to $\mathring{Q}$ for some $Q \in \mathcal{S}_1$ and is called type 7b otherwise. First, suppose there are infinitely many paths $P_1, P_2, \ldots$ of type 7a in $\mathcal{J}''$. Then each P_i has endpoints $x_i, y_i \in \mathring{Q}_i$ for some $Q_i \in \mathcal{S}_1$. Let Q_i' be the path obtained by replacing $Q_i[x_i y_i]$ by P_i and let $S_1' = \bigcup_{i=1}^{\infty} Q_i'$. The subgraph $A \cup B \cup C \cup S_1' \cup S_2$ is a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely elements of X. By Lemma 4.1.14, it contains an X-rich LL. Now suppose there are infinitely many paths $P_1, P_2, \ldots$ of type 7b in $\mathcal{J}''$. Then each P_i has endpoints $x_i \in \mathring{Q}_i$ and $y_i \in \mathring{R}_i$ for some distinct $Q_i, R_i \in \mathcal{S}_1$. Note that both Q_i, R_i contain only endpoints of P_i by the choice of $\mathcal{J}''$. Let q_i be the endpoint of Q_i on A and let r_i be the endpoint of R_i on B. Let Q_i' be the $q_i x_i$-subpath of Q_i and let R_i' be the $y_i r_i$-subpath of R_i. Let $M_i = Q_i' \cup P_i \cup R_i'$ and let $S_1' = \bigcup_{i=1}^{\infty} M_i$. The subgraph $A \cup B \cup C \cup S_1' \cup S_2$ is a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely elements of X. By Lemma 4.1.14, it contains an X-rich LL. This proves the claim.

Claim 4.1.18.7. *If there exist infinitely many paths of $\mathcal{J}$ of type 8, then the lemma holds.*

Let $\mathcal{J}'$ be an infinite subset of $\mathcal{J}$ such that every path in $\mathcal{J}'$ is of type 8. Since G is locally finite, $\mathcal{J}'$ has an infinite subset $\mathcal{J}'' = \{J_1, J_2, \ldots\}$ such that every path in $\mathcal{S}_1$ and every path in $\mathcal{S}_2$ contain endpoints of at most one path in $\mathcal{J}''$. To see this, we start with $\mathcal{J}'' = \emptyset$. Let $J \in \mathcal{J}' - \mathcal{J}''$. Then J has endpoints in $\mathring{P}, \mathring{Q}$ for some $P \in \mathcal{S}_1$ and some $Q \in \mathcal{S}_2$. Next, we delete edges of all paths in $\mathcal{J}'$, except for J, with an endpoint in $\mathring{P} \cup \mathring{Q}$ and then we add J into $\mathcal{J}''$. Note that after doing this, $\mathcal{J}'$ is still infinite as we only delete finitely many paths in $\mathcal{J}'$. We then pick a $J' \in \mathcal{J}' - \mathcal{J}''$ and repeat the process. This yields the desired $\mathcal{J}''$. Now every J_i has its two endpoints on $\mathring{P}_i, \mathring{Q}_i$ where P_i is a path in $\mathcal{S}_1$ and Q_i is a path in $\mathcal{S}_2$. Let x_i be the endpoint of P_i on A and let y_i be the endpoint of J_i on $\mathring{P}_i$. Let $M_i = J_i \cup Q_i \cup P_i[x_i y_i]$ and let $M = \bigcup_{i=1}^{\infty} M_i$. The subgraph $A \cup B \cup C \cup M$ is a subdivision of a graph in $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL. This proves the claim.

We have shown that if infinitely many paths of $\mathcal{J}$ are of type i, for any $i \in \{1, 2, \ldots, 8\}$, then the lemma holds. This completes the proof. $\square$

Lemma 4.1.19. *Let H be a subdivision of a graph in $\mathcal{LL}_2$ and let $\mathcal{J}$ be an infinite set of disjoint H-paths. Let G be the union of H and all paths in $\mathcal{J}$ and let X be an infinite subset of $V(G)$ such that every path in $\mathcal{J}$ contains at least one element of X. Then G contains an X-rich LL.*

Proof. First, observe that G is locally finite since H, being a subdivision of a graph in $\mathcal{LL}_2$, is locally finite and paths in $\mathcal{J}$ are disjoint. In H, let A, B, C be its rails and let $\mathcal{S}_1$ be the set of rungs between A, B and let $\mathcal{S}_2$ be the set of spokes. Let S_i be the union of all paths in $\mathcal{S}_i$ for $i = 1, 2$. By definition, every path in $\mathcal{J}$ has its two endpoints on H. Up to symmetry, we may assume that each path in $\mathcal{J}$ is exactly one of the following types

- type 1: both endpoints belong to A,
- type 2: one endpoint belongs to A and the other endpoint belongs to B,
- type 3: one endpoint belongs to A and the other endpoint belongs to C,
- type 4: one endpoint belongs to A and the other endpoint belongs to $S_1 - (A \cup B)$,
- type 5: one endpoint belongs to A and the other endpoint belongs to $S_2 - (C \cup S_1)$,
- type 6: both endpoints belong to C,
- type 7: one endpoint belongs to C and the other endpoint belongs to $S_1 - (A \cup B)$,
- type 8: one endpoint belongs to C and the other endpoint belongs to $S_2 - (C \cup S_1)$,
- type 9: both endpoints belong to $S_1 - (A \cup B)$,
- type 10: both endpoints belong to $S_2 - (C \cup S_1)$,
- type 11: one endpoint belongs to $S_1 - (A \cup B)$ and the other endpoint belongs to $S_2 - (C \cup S_1)$.

  Note that infinitely many paths in $\mathcal{J}$ are of one type.

Claim 4.1.19.1. *If there exist infinitely many paths of $\mathcal{J}$ of type 1, then the lemma holds.*

Since paths in $\mathcal{J}$ are disjoint, we can find infinitely many paths $J_1, J_2, \ldots$ in $\mathcal{J}$ such that with respect to A, J_i is on the left of J_{i+1} for $i = 1, 2, \ldots$. For each J_i, let a_i, b_i be its two endpoints on A. Observe that every $A[a_ib_i]$ contains endpoints of finitely many rungs because G is locally finite. First, if an $A[a_ib_i]$ contains endpoints of more than one rung, then we delete edges of all but one rung with an endpoint in $A[a_ib_i]$. Hence, we may assume every $A[a_ib_i]$ contains endpoint of at most one rung. Next, suppose a rung has an endpoint r in an $A[a_ib_i]$. If $r \in A(a_ib_i)$, then we delete edges of $A[a_ir]$. Otherwise, $r \in \{a_i, b_i\}$, and we delete edges of $A[a_ib_i]$. By repeating this process, we obtained a subdivision of $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL. This proves the claim.

Claim 4.1.19.2. *If there exist infinitely many paths of $\mathcal{J}$ of type 2, then the lemma holds.*

Let $\mathcal{J}'$ be an infinite subset of $\mathcal{J}$ such that every path in $\mathcal{J}'$ is of type 2. Let J' be the union of all paths in $\mathcal{J}'$. The subgraph $H \cup J'$ is a subdivision of a graph in $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL. This proves the claim.

Claim 4.1.19.3. *If there exist infinitely many paths of $\mathcal{J}$ of type 3, then the lemma holds.*

Let $\mathcal{J}'$ be an infinite subset of $\mathcal{J}$ such that every path in $\mathcal{J}'$ is of type 3. Let J' be the union of all paths in $\mathcal{J}'$. The subgraph $A \cup B \cup C \cup S_1 \cup J'$ is a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely many elements of X. By Lemma 4.1.14, it contains an X-rich LL. This proves the claim.

Claim 4.1.19.4. *If there exist infinitely many paths of $\mathcal{J}$ of type 4, then the lemma holds.*

Let $\mathcal{J}'$ be an infinite subset of $\mathcal{J}$ such that every path in $\mathcal{J}'$ is of type 4. Since G is locally finite, $\mathcal{J}'$ has an infinite subset $\mathcal{J}''$ such that every path in $\mathcal{S}_1$ contains endpoints of at most one path in $\mathcal{J}''$. Let $\mathcal{S}_1'$ be an infinite subset of $\mathcal{S}_1$ such that every path in $\mathcal{S}_1'$ contains endpoints of a path in $\mathcal{J}''$. A path in $\mathcal{S}_1'$ is called type 4a if it is hooked by a spoke and is called type 4b otherwise. We divide the remain of this claim into two cases.

Case 1: There exist infinitely many paths in $\mathcal{S}_1'$ of type 4a. Since G is locally finite, $\mathcal{S}_1'$ has an infinite subset $\mathcal{S}_1'' = \{P_1, P_2, \ldots\}$ such that paths in $\mathcal{S}_1''$ are pairwise disjoint. For each i, let Q_i be the path in $\mathcal{J}''$ whose one of the endpoints is in $\mathring{P}_i$ and let S_i be the spoke that is hooked to P_i. Let x_i be the endpoint of S_i on $\mathring{P}_i$ and let y_i be the endpoint of Q_i on $\mathring{P}_i$. Let p_i be the endpoint of P_i on B. Let M_i be the x_ip_i-subpath of P_i and let N_i be the y_ip_i-subpath of P_i. Let $A_i = Q_i \cup M_i \cup N_i \cup S_i$ and let $A' = \bigcup_{i=1}^{\infty} A_i$. The subgraph $A \cup B \cup C \cup A'$ is a subdivision of a graph in $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL.

Case 2: There exist infinitely many paths $P_1, P_2, \ldots$ in $\mathcal{S}_1'$ of type 4b. For each i, let Q_i be the path in $\mathcal{J}''$ whose one of the endpoints is in $\mathring{P}_i$. Let x_i be the endpoint of P_i on A and let y_i be the endpoint of Q_i on $\mathring{P}_i$. We then remove the edges of $P_i[x_iy_i]$. By repeating this process, we obtain a subdivision of a graph in $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL. This proves the claim.

Claim 4.1.19.5. *If there exist infinitely many paths of $\mathcal{J}$ of type 5, then the lemma holds.*

Let $\mathcal{J}'$ be an infinite subset of $\mathcal{J}$ such that every path in $\mathcal{J}'$ is of type 5. Since G is locally finite, $\mathcal{J}'$ has an infinite subset $\mathcal{J}''$ such that every path in $\mathcal{S}_2$ contains endpoints of at most one path in $\mathcal{J}''$. Let $\mathcal{S}_2' = \{P_1, P_2, \ldots\}$ be an infinite subset of $\mathcal{S}_2$ such that every P_i contains endpoints of a path in $\mathcal{J}''$. For each i, let Q_i be the path in $\mathcal{J}''$ whose one of the endpoints is in $\mathring{P}_i$. Let x_i be the endpoint of P_i on C and let y_i be the endpoint of Q_i on $\mathring{P}_i$. Let M_i be the $x_i y_i$-subpath of P_i and let $N_i = Q_i \cup M_i$. Then N_i is an AC-path. Let $N = \bigcup_{i=1}^{\infty} N_i$. The subgraph $A \cup B \cup C \cup S_1 \cup N$ is a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely many elements of X. By Lemma 4.1.14, it contains an X-rich LL. This proves the claim.

Claim 4.1.19.6. *If there exist infinitely many paths of $\mathcal{J}$ of type 6, then the lemma holds.*

Since paths in $\mathcal{J}$ are disjoint, we can find infinitely many paths $J_1, J_2, \ldots$ in $\mathcal{J}$ such that with respect to C, J_i is on the left of J_{i+1} for $i = 1, 2, \ldots$. For each J_i, let a_i, b_i be its two endpoints on C. Observe that every $C[a_i b_i]$ contains endpoints of finitely many spokes because G is locally finite. First, if a $C[a_i b_i]$ contains endpoints of more than one spoke, then we delete edges of all but one spoke with an endpoint in $C[a_i b_i]$. Hence, we may assume every $C[a_i b_i]$ contains endpoint of at most one spoke. Next, suppose a spoke has an endpoint s in a $C[a_i b_i]$. If $s \in C(a_i b_i)$, then we delete edges of $C[a_i s]$. Otherwise, $s \in \{a_i, b_i\}$, and we delete edges of $C[a_i b_i]$. By repeating this process, we obtained a subdivision of $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL. This proves the claim.

Claim 4.1.19.7. *If there exist infinitely many paths of $\mathcal{J}$ of type 7, then the lemma holds.*

Let $\mathcal{J}'$ be an infinite subset of $\mathcal{J}$ such that every path in $\mathcal{J}'$ is of type 7. Since G is locally finite, $\mathcal{J}'$ has an infinite subset $\mathcal{J}''$ such that every path in $\mathcal{S}_1$ contains endpoints of at most one path in $\mathcal{J}''$. Let J'' be the union of all paths in $\mathcal{J}''$. The subgraph $A \cup B \cup C \cup S_1 \cup J''$ is a subdivision of a graph in $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL. This proves the claim.

Claim 4.1.19.8. *If there exist infinitely many paths of $\mathcal{J}$ of type 8, then the lemma holds.*

Let $\mathcal{J}'$ be an infinite subset of $\mathcal{J}$ such that every path in $\mathcal{J}'$ is of type 8. Since G is locally finite, $\mathcal{J}'$ has an infinite subset $\mathcal{J}'' = \{P_1, P_2, \ldots\}$ such that every path in $\mathcal{S}_2$ contains endpoints of at most one path in $\mathcal{J}''$. For each i, let S_i be the path in $\mathcal{S}_2$ that contains an endpoint of P_i and let R_i be the path in $\mathcal{S}_1$ for which S_i is hooked to. Let x_i be the endpoint of P_i on $\mathring{S}_i$ and let y_i be the endpoint of S_i on $\mathring{R}_i$. Let M_i be the $x_i y_i$-subpath of S_i and let $N_i = P_i \cup M_i$. Let $N = \bigcup_{i=1}^{\infty} N_i$. The subgraph $A \cup B \cup C \cup S_1 \cup N$ is a subdivision of a graph in $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL. This proves the claim.

Claim 4.1.19.9. *If there exist infinitely many paths of $\mathcal{J}$ of type 9, then the lemma holds.*

Let $\mathcal{J}'$ be an infinite subset of $\mathcal{J}$ such that every path in $\mathcal{J}'$ is of type 9. Since G is locally finite, $\mathcal{J}'$ has an infinite subset $\mathcal{J}''$ such that every path in $\mathcal{S}_1$ contains endpoints of at most one path in $\mathcal{J}''$. To see this, we start with $\mathcal{J}'' = \emptyset$. Let $J \in \mathcal{J}' - \mathcal{J}''$. Then J has endpoints in $\mathring{P}, \mathring{Q}$ for some $P, Q \in \mathcal{S}_1$ where $P = Q$ is possible. Next, we delete edges of all paths in $\mathcal{J}'$, except for J, with an endpoint in $\mathring{P} \cup \mathring{Q}$ together with edges of all spokes incident with those paths, and then we add J into $\mathcal{J}''$. Note that after doing this, $\mathcal{J}'$ is still infinite as we only delete finitely many paths in $\mathcal{J}'$. We then pick a $J' \in \mathcal{J}' - \mathcal{J}''$ and repeat the process. This yields the desired $\mathcal{J}''$. A path in $\mathcal{J}''$ is called type 9a if both of is endpoints belong to $\mathring{P}$ for some $P \in \mathcal{S}_1$ and is called type 9b otherwise. We divide the remain of this claim into two cases.

Case 1: There exist infinitely many paths $\{P_1, P_2, \ldots\}$ in $\mathcal{J}''$ of type 9a. For each i, let Q_i be the path in $\mathcal{S}_1$ that contains both endpoints of P_i. Now each Q_i is either hooked or not hooked. Suppose there exist infinitely many Q_i that are not hooked; we label them $Q'_1, Q'_2, \ldots$. Let P'_i be the path in $\mathcal{J}''$ whose endpoints x'_i, y'_i are in $\mathring{Q}'_i$. We then delete edges of $Q'_i[x'_i y'_i]$. By repeating this process, we obtain a subdivision of a graph in $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL. Now suppose there exist infinitely many Q_i that are hooked; we label them $Q''_1, Q''_2, \ldots$. Let P''_i be the path in $\mathcal{J}''$ whose endpoints x''_i, y''_i are in $\mathring{Q}''_i$. Let R''_i be the spoke in $\mathcal{S}_2$ that is hooked to Q''_i; let r''_i be the endpoint of R''_i in $\mathring{Q}''_i$. If $r''_i \notin Q''_i(x''_i, y''_i)$, then we delete edges of $Q''_i[x''_i y''_i]$. Otherwise, $r''_i \in Q''_i(x''_i y''_i)$, and we delete edges of $Q''_i[r''_i y''_i]$. By repeating this process, we obtain a subdivision of a graph in $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL.

Case 2: There exist infinitely many paths $\{P_1, P_2, \ldots\}$ in $\mathcal{J}''$ of type 9b. For each i, there exist distinct $A_i, B_i \in \mathcal{S}_1$ such that $\mathring{A}_i$ contains an endpoint x_i of P_i and $\mathring{B}_i$ contains the other endpoint y_i of P_i. We call P_i type 9b1 if neither A_i nor B_i is hooked and type 9b2 otherwise. First, suppose infinitely many paths in $\mathcal{J}''$ is of type 9b1. For each such P_j of type 9b1, let a_j be the endpoint of A_j on A and let b_j be the endpoint of B_j on B. Let $M_j = A_j[a_j x_j] \cup P_j \cup B_j[y_j b_j]$ and let $M = \bigcup_{j=1}^{\infty} M_j$. Let S'_1 be the union of all rungs that are hooked. The subgraph $A \cup B \cup C \cup S_1' \cup S_2 \cup M$ is a subdivision of a graph in $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL. Now suppose infinitely many paths in $\mathcal{J}''$ is of type 9b2. For each such P_j, we may assume that A_j is hooked by a spoke S_j. This means that S_j has an endpoint $s_j \in \mathring{A}_j$. Without loss of generality, we may assume that $s_j \in A_j(a_j x_j]$ where a_j is the endpoint of A_j on A. Let b_j be the endpoint of B_j on B. Let $M_j = A_j[a_j x_j] \cup P_j \cup B_j[y_j b_j]$ and let $M = \bigcup_{j=1}^{\infty} S_j \cup M_j$. The subgraph $A \cup B \cup C \cup M$ is a subdivision of a graph in $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL. This proves the claim.

Claim 4.1.19.10. *If there exist infinitely many paths of $\mathcal{J}$ of type 10, then the lemma holds.*

Let $\mathcal{J}'$ be an infinite subset of $\mathcal{J}$ such that every path in $\mathcal{J}'$ is of type 10. Since G is locally finite, $\mathcal{J}'$ has an infinite subset $\mathcal{J}''$ such that every path in $\mathcal{S}_2$ contains endpoints of at most one path in $\mathcal{J}''$. To see this, we start with $\mathcal{J}'' = \emptyset$. Let $J \in \mathcal{J}' - \mathcal{J}''$. Then J has endpoints in $\mathring{P}, \mathring{Q}$ for some $P, Q \in \mathcal{S}_2$ where $P = Q$ is possible. Next, we delete edges of all paths in $\mathcal{J}'$, except for J, with an endpoint in $\mathring{P} \cup \mathring{Q}$ and then we add J into $\mathcal{J}''$. Note that after doing this, $\mathcal{J}'$ is still infinite as we only delete finitely many paths in $\mathcal{J}'$. We then pick a $J' \in \mathcal{J}' - \mathcal{J}''$ and repeat the process. This yields the desired $\mathcal{J}''$. A path in $\mathcal{J}''$ is called type 10a if both its endpoints belong to $\mathring{P}$ for some $P \in \mathcal{S}_2$ and is called type 10b otherwise. First, suppose there exist infinitely many paths $\{P_1, P_2, \ldots\}$ in $\mathcal{J}''$ of type 10a. For each i, let Q_i be the spoke in $\mathcal{S}_2$ containing the endpoints x_i, y_i of P_i. We then delete edges of $Q_i[x_i y_i]$. By repeating this process, we obtain a subdivision of a graph in $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL. Now suppose there exist infinitely many paths $\{P'_1, P'_2, \ldots\}$ in $\mathcal{J}''$ of type 10b. Then each P'_i has endpoints $x'_i \in \mathring{Q}'_i$ and $y'_i \in \mathring{R}'_i$ for some distinct $Q'_i, R'_i \in \mathcal{S}_2$. Note that both Q'_i, R'_i contain endpoints of only P'_i by the choice of $\mathcal{J}''$. Let q'_i be the endpoint of Q'_i on C and let r'_i be the endpoint of R'_i on $\mathring{R}$ for some $R \in \mathcal{S}_1$. Let $M_i = \bigcup_{i=1}^{\infty} Q'_i[q'_i x'_i] \cup P'_i \cup R'_i[y'_i r'_i]$ and let $M = \bigcup_{i=1}^{\infty} M_i$. The subgraph $A \cup B \cup C \cup S_1 \cup M$ is a subdivision of a graph in $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL. This proves the claim.

Claim 4.1.19.11. *If there exist infinitely many paths of $\mathcal{J}$ of type 11, then the lemma holds.*

Let $\mathcal{J}'$ be an infinite subset of $\mathcal{J}$ such that every path in $\mathcal{J}'$ is of type 11. Since G is locally finite, $\mathcal{J}'$ has an infinite subset $\mathcal{J}'' = \{P_1, P_2, \ldots\}$ such that every path in $\mathcal{S}_1$ and every path in $\mathcal{S}_2$ contain endpoints of at most one path in $\mathcal{J}''$. To see this, we start with $\mathcal{J}'' = \emptyset$. Let $J \in \mathcal{J}' - \mathcal{J}''$. Then J has endpoints in $\mathring{P}, \mathring{Q}$ for some $P \in \mathcal{S}_1$ and some $Q \in \mathcal{S}_2$. Next, we delete edges of all paths in $\mathcal{J}'$, except for J, with an endpoint in $\mathring{P} \cup \mathring{Q}$ and then we add J into $\mathcal{J}''$. (If we delete edges of a rung that is hooked, then we also delete edges of the spoke that is incident with that rung.) Note that after doing this, $\mathcal{J}'$ is still infinite as we only delete finitely many paths in $\mathcal{J}'$. We then pick a $J' \in \mathcal{J}' - \mathcal{J}''$ and repeat the process. This yields the desired $\mathcal{J}''$. For each i, let $S_i \in \mathcal{S}_2$ be the spoke that contains an endpoint x_i of P_i. Let s_i be the endpoint of S_i on $V(C)$ and let $M_i = S_i[s_i x_i] \cup P_i$. Let $M = \bigcup_{i=1}^{\infty} M_i$. The subgraph $A \cup B \cup C \cup S_1 \cup M$ is a subdivision of a graph in $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL. This proves the claim.

We have shown that if infinitely many paths of $\mathcal{J}$ are of type i, for any $i \in \{1, 2, \ldots, 11\}$, then the lemma holds. This completes the proof. $\square$

To finish this section, we discuss the crossing property of bridges and paths.

Lemma 4.1.20. *Let S be an infinite path and let $B_1, B_2, \dots$ be distinct S-bridges, each having finitely many feet such that with respect to S, B_{i+1} crosses B_i but does not cross B_j for any $j < i$. Then there exist infinitely many S-paths $Q_1, Q_2, \dots$ satisfying the following*

1. *two distinct Q_i and Q_j are internally-disjoint,*
2. *with respect to S, Q_{i+1} crosses Q_i but does not cross Q_j for any $j < i$.*

Proof. Throughout this proof, every crossing and left, right positions are with respect to S.

Claim 4.1.20.1. *No two bridges have the same set of feet.*

Suppose there exist two distinct bridges B_i and B_j where $i < j$ with the same set of feet. Then B_{j+1} crosses B_i and this is contradictory. This proves the claim.

Back to our proof, we proceed using induction with our induction hypothesis $P(n)$, for a positive integer n, as:

There exist $Q_1^n, Q_2^n, \dots, Q_n^n$, where Q_i^n is the ith S-path obtained at step n, such that

1. Q_i^n is a subgraph of B_i for every i,
2. for every $i \leq n-1$, Q_{i+1}^n crosses Q_i^n,
3. $Q_i^{n-1} = Q_i^n$ for $i = 1, 2, \dots, n-2$.

We first prove $P(n)$ for $n = 1$. Since B_1 crosses B_2, it has two feet x_1, y_1 that crosses two feet of B_2. Thus, we can find an x_1y_1-path in B_1 and let Q_1^1 be this path. Clearly, Q_1^1 satisfies statements 1, 2, and 3 in the induction hypothesis. Now suppose $P(n)$ is true for some $n = k$. We show that $P(n)$ holds for $n = k + 1$. Let $B_1, B_2, \dots$ be given and let

$Q_1^k, Q_2^k, \ldots, Q_k^k$ be obtained from $P(k)$. Since B_k crosses B_{k-1}, we may assume, without loss of generality, that B_{k-1} has two feet x_{k-1}, y_{k-1} and B_k has two feet x_k, y_k such that x_{k-1} is on the left of y_{k-1}, $x_k \in S(x_{k-1}y_{k-1})$, and y_k is on the right of y_{k-1}. This also implies that x_{k-1} is on the left of x_k and y_{k-1} is on the right of x_k. For each B_i, let $m_i, n_i \in S$ be two distinct feet such that all feet of B_i are contained in $S[m_i n_i]$, denoted as S_i. Such m_i, n_i exist since B_i has finitely many feet.

Claim 4.1.20.2. *No foot of B_{k+1} is in $S(x_{k-1}y_{k-1})$.*

Suppose for contradiction that there exists a foot of B_{k+1} in $S(x_{k-1}y_{k-1})$. Since B_{k+1} does not cross B_{k-1}, all feet of B_{k+1} are in $S[x_{k-1}y_{k-1}]$. Hence, $S_{k+1} \subseteq S_{k-1}$. Next, if $S_j \subseteq S_{k-1}$ for some $j \geq k+1$, then $S_{j+1} \subseteq S_{k-1}$ since B_{j+1} does not cross B_{k-1}. Since $S_{k+1} \subseteq S_{k-1}$, it follows that $S_j \subseteq S_{k-1}$ for all $j \geq k+1$. Since S_{k-1} is finite, there exist infinitely many bridges with the same set of feet, contradicting Claim 4.1.20.1. This proves the claim.

Back to our proof, we divide the proof into two cases.

Case 1: No foot of B_{k+1} is in $S[y_{k-1}n_k)$.

From the previous claim, it follows that no foot of B_{k+1} is in $S(x_{k-1}n_k)$. First, suppose $m_k = x_{k-1}$ or m_k is on the right of x_{k-1}. Then $S(m_k n_k) \subseteq S(x_{k-1}n_k)$. Since B_{k+1} crosses B_k, there exists a foot of B_{k+1} in $S(m_k n_k)$. But $S(m_k n_k) \subseteq S(x_{k-1}n_k)$, so there exists a foot of B_{k+1} in $S(x_{k-1}n_k)$ and this is not possible. Thus, m_k is on the left of x_{k-1}. Since B_{k+1} crosses B_k, there exists a foot x_{k+1} of B_{k+1} in $S(m_k n_k)$. This foot x_{k+1} must be in $S(m_k x_{k-1}]$ since $x_{k+1} \notin S(x_{k-1}n_k)$. We now have two further cases to consider.

Case 1a: There exists a foot y_{k+1} of B_{k+1} not in $S[m_k x_{k-1}]$.

This implies that $y_{k+1} \notin S[m_k n_k)$ since $y_{k+1} \notin S(x_{k-1} n_k)$. Let $Q_i^{k+1} = Q_i^k$ for $i = 1, 2, \ldots, k-1$. Let Q_k^{k+1} be the $m_k x_k$-path in B_k and let Q_{k+1}^{k+1} be the $x_{k+1} y_{k+1}$-path in B_{k+1}. Now Q_{k+1}^{k+1} crosses Q_k^{k+1} because $x_{k+1} \in S(m_k x_{k-1}] \subset S(m_k x_k)$ and $y_{k+1} \notin S[m_k x_k]$ since $y_{k+1} \notin S[m_k n_k)$ and $S[m_k n_k) \supset S[m_k x_k]$. In addition, Q_k^{k+1} crosses Q_{k-1}^{k+1} because $x_k \in S(x_{k-1} y_{k-1})$ and m_k, being on the left of x_{k-1}, is not contained in $S[x_{k-1} y_{k-1}]$.

Case 1b: Every foot of B_{k+1} is in $S[m_k x_{k-1}]$.

Since B_{k+1} crosses B_k, there exists a foot z_k of B_k in $S(m_{k+1} n_{k+1})$. Let $Q_i^{k+1} = Q_i^k$ for $i = 1, 2, \ldots, k-1$. Let Q_k^{k+1} be the $z_k x_k$-path in B_k and let Q_{k+1}^{k+1} be the $m_{k+1} n_{k+1}$-path in B_{k+1}. Now Q_{k+1}^{k+1} crosses Q_k^{k+1} because $z_k \in S(m_{k+1} n_{k+1})$ and $x_k \notin S[m_{k+1} n_{k+1}]$ since $n_{k+1} \in S[m_k x_{k-1}]$ and x_{k-1} is on the left of x_k. In addition, Q_k^{k+1} crosses Q_{k-1}^{k+1} because $x_k \in S(x_{k-1} y_{k-1})$ and $z_k \notin S[x_{k-1} y_{k-1}]$ since $z_k \in S(m_{k+1} n_{k+1}) \subseteq S(m_{k+1} x_{k-1})$.

Case 2: There exists a foot x_{k+1} of B_{k+1} in $S[y_{k-1} n_k)$.

We now have two further cases to consider.

Case 2a: There exists a foot y_{k+1} of B_{k+1} that is not in $S[y_{k-1} n_k]$.

From the previous claim, we deduce that $y_{k+1} \notin S(x_{k-1} n_k]$. Let $Q_i^{k+1} = Q_i^k$ for $i = 1, 2, \ldots, k-1$. Let Q_k^{k+1} be the $x_k n_k$-path in B_k and let Q_{k+1}^{k+1} be the $x_{k+1} y_{k+1}$-path in B_{k+1}. Now Q_k^{k+1} crosses Q_{k-1}^{k+1} because $x_k \in S(x_{k-1} y_{k-1})$ and $n_k \notin S[x_{k-1} y_{k-1}]$ since $n_k = y_k$ or n_k is on the right of y_k and y_k is on the right of y_{k-1}. In addition, Q_{k+1}^{k+1} crosses Q_k^{k+1} because $x_{k+1} \in S[y_{k-1} n_k) \subseteq S(x_k n_k)$ and $y_{k+1} \notin S[x_k n_k]$ since $y_{k+1} \notin S(x_{k-1} n_k]$ and $S(x_{k-1} n_k] \supset S[x_k n_k]$.

Case 2b: Every foot of B_{k+1} is in $S[y_{k-1}n_k]$.

Since B_{k+1} crosses B_k, there exists a foot z_k of B_k in $S(m_{k+1}n_{k+1})$. Let $Q_i^{k+1} = Q_i^k$ for $i = 1, 2, \ldots, k-1$. Let Q_k^{k+1} be the $x_k z_k$-path in B_k and let Q_{k+1}^{k+1} be the $m_{k+1}n_{k+1}$-path in B_{k+1}. Now Q_{k+1}^{k+1} crosses Q_k^{k+1} because $z_k \in S(m_{k+1}n_{k+1})$ and $x_k \notin S[m_{k+1}n_{k+1}]$ since $m_{k+1} = y_{k-1}$ or m_{k+1} is on the right of y_{k-1} and y_{k-1} is on the right of x_k. In addition, Q_k^{k+1} crosses Q_{k-1}^{k+1} because $x_k \in S(x_{k-1}y_{k-1})$ and $z_k \notin S[x_{k-1}y_{k-1}]$ since z_k is on the right of m_{k+1} and $m_{k+1} = y_{k-1}$ or m_{k+1} is on the right of y_{k-1}.

By setting $Q_i^{k+1} = Q_i^k$ for $i = 1, 2, \ldots, k-1$ and Q_k^{k+1} and Q_{k+1}^{k+1} as described above, $P(k+1)$ is true. We have shown that $P(k+1)$ is true when $P(k)$ is true, so $P(n)$ holds for all positive integers n. We define the S-paths $Q_1, Q_2, \ldots$ by taking $Q_i = Q_i^{i+2}$ in the induction hypothesis $P(i+2)$ for $i = 1, 2, \ldots$. Observe that statement 1 in the lemma is satisfied since Q_i is a subgraph of B_i for every i. For statement 2, Q_{i+1} crosses Q_i from the induction hypothesis and Q_{i+1} does not cross Q_j for any $j < i$ by the definition of crossing of $B_1, B_2, \ldots$. This completes the proof. □

Lemma 4.1.21. *Let H be the union of a double ray S and infinitely many internally disjoint S-paths $Q_1, Q_2, \ldots$ such that with respect to S, Q_{i+1} crosses Q_i but does not cross Q_j for any $j < i$. Then H contains three disjoint rays R_1, R_2, R_3 such that every Q_i is contained in some R_j. In addition, if $R = R_1 \cup R_2 \cup R_3$ and $J_1, J_2, \ldots$ are all R-bridges, then each J_i is a subpath of S with endpoints in different R_j and we call it a jump. Finally, if all but finitely many jumps are between R_i and R_j, then R_k contains a subray of S.*

Proof. Throughout this proof, every crossing is with respect to S. For each Q_i, let x_i, y_i be its two endpoints and let $S_i = S[x_i y_i]$. Since Q_{i+1} crosses Q_i, exactly one endpoint of Q_{i+1} belongs to $\mathring{S}_i$. Let x_{i+1} be that endpoint and let y_{i+1} be the other endpoint. This uniquely determines the endpoints of $Q_2, Q_3, \ldots$. For Q_1, let y_1 be the endpoint belonging to $\mathring{S}_2$ and let x_1 be the other endpoint. Let y_{-1}, y_0 be the two neighbors of x_1 on S. The definition of y_{-1}, y_0 implies that for every $n \geq 1$, $S(x_1 y_n)$ contains a vertex of the form y_i. Among all such vertices in $S(x_1 y_n)$, let $y_{n'}$ be the one whose distance to y_n on S is smallest. Let $H_i = (\bigcup_{j=1}^{i} Q_j \cup S_j) \cup S[y_0 y_{-1}]$.

Back to our proof, we proceed using induction with our induction hypothesis $P(n)$, for a positive integer n, as

1. H_n is the union of $Q_1, Q_2, \ldots, Q_n$ and $S[y_n y_{n''}]$ for some n''.

2. H_n contains three disjoint paths R_1^n, R_2^n, R_3^n between x_1, y_0, y_{-1} and $y_n, y_{n'}, y_{n''}$ such that each Q_j, for $j \leq n$, is contained in some R_i^n. In addition, R_i^{n-1} is a subpath of R_i^n for $i = 1, 2, 3$.

3. Let $R_n = R_1^n \cup R_2^n \cup R_3^n$. Then $S[y_n y_{n'}]$ is an R_n-bridge. In addition, $x_{n+1} \in S(y_n y_{n'}]$ and $y_{n+1} \in S - S[y_n y_{n''}]$.

4. Let $J_1^n, J_2^n, \ldots, J_t^n$ be the R_n-bridges of H_n that are not $S[y_n y_{n'}]$, which we call jumps. Then each jump is a subpath of S with endpoints in two different R_i^n. In addition, every jump of H_{n-1} is a jump of H_n.

First, we prove $P(1)$ is true. Let us consider H_1, so that $n' \in \{-1, 0\}$. Let $n'' = \{-1, 0\} - \{n'\}$.

1. By definition, $H_1 = Q_1 \cup S_1 \cup S[y_0y_{-1}]$. Hence, $H_1 = Q_1 \cup S[y_1y_{n''}]$.

2. It is easy to verify statement 2 in the induction hypothesis by letting $R_1^1 = Q_1,\ R_2^1 = y_{-1},\ R_3^1 = y_0$.

3. Let $R_1 = R_1^1 \cup R_2^1 \cup R_3^1$. Clearly, $S[y_1y_{n'}]$ is an R_1-bridge. In addition, $x_2 \in S(y_1y_{n'}]$ because $x_2 \in \overset{\circ}{S}_1$ and $y_2 \in S - S[y_1y_{n''}]$ because Q_2 crosses Q_1.

4. The two R_1-bridges that are not $S[y_1y_{n'}]$ are $S[y_{n'}x_1]$ and $S[y_{n''}x_1]$. Each is a subpath of S with endpoints in two different R_i^1.

Thus, $P(1)$ is true. Now suppose $P(n)$ is true for some $n = m$. We show that $P(n)$ holds for $n = m + 1$. Observe that H_{m+1} is obtained from H_m by adding Q_{m+1} and one of the $S[y_{m+1}y_m]$ or $S[y_{m+1}y_{m''}]$. In H_{m+1}, let $R_1^{m+1} = R_1^m,\ R_2^{m+1} = R_2^m,\ R_3^{m+1} = R_3^m \cup S[y_{m'}x_{m+1}] \cup Q_{m+1}$. Observe that $R_1^{m+1}, R_2^{m+1}, R_3^{m+1}$ starts at $x_1,\ y_0,\ y_{-1}$ and ends at $y_{m+1}, y_m, y_{m''}$. In addition, every Q_i, for $i \leq m+1$, is contained in some R_j^{m+1}. Furthermore, R_i^m is a subpath of R_i^{m+1} for $i = 1, 2, 3$ by our construction.

Next, let $R_{m+1} = R_1^{m+1} \cup R_2^{m+1} \cup R_3^{m+1}$. Then $S[y_{m+1}y_m]$ is an R_{m+1}-bridge. Since Q_{m+2} crosses $Q_{m+1},\ x_{m+2} \in S(x_{m+1}y_{m+1})$.

Claim 4.1.21.1. $x_{m+2} \notin S(x_{m+1}y_m)$.

Assume for contradiction that $x_{m+2} \in S(x_{m+1}y_m)$. Then $x_{m+2} \in S(x_my_m)$. Since Q_{m+2} does not cross Q_m with respect to S, $y_{m+2} \in S(x_my_m)$. Hence, $S_{m+2} \subseteq S_m$. Next, if $S_j \subseteq S_m$ for some $j \geq m + 2$, then $S_{j+1} \subseteq S_m$ since with respect to S, Q_{j+1} crosses Q_j but does not cross Q_m. Since $S_{m+2} \subseteq S_m$, it follows that $S_j \subseteq S_m$ for all $j \geq m + 2$. Since S_m

is finite, there exist infinitely many Q_j with the same set of endpoints. But this means that there exist $j > i \geq m+2$ such that Q_{j+1} crosses Q_i with respect to S, a contradiction. This proves the claim.

The previous claim implies that $x_{m+2} \in S[y_m y_{m+1})$. Also, $y_{m+2} \in S - S[y_{m+1} y_{m''}]$ because Q_{m+2} does not cross Q_j for any $j < m+1$.

Finally, let $J_1^m, J_2^m, \ldots, J_t^m$ be the jumps of H_m. Then the jumps of H_{m+1} are

$$J_1^m, J_2^m, \ldots, J_t^m, S[x_{m+1} y_m].$$

Clearly, each jump is a subpath of S with endpoints in two different R_i^{m+1}. In addition, every jump of H_m is a jump of H_{m+1}. We have shown that $P(m+1)$ is true when $P(m)$ is true, so $P(n)$ is true for all $n \geq 1$.

To finish the proof, let $R_1 = \bigcup_{i=1}^{\infty} R_1^i$, $R_2 = \bigcup_{i=1}^{\infty} R_2^i$, and $R_3 = \bigcup_{i=1}^{\infty} R_3^i$. We will show that if all but finitely many jumps are between R_i and R_j, then R_k contains a subray of S.

Claim 4.1.21.2. *R_k contains finitely many Q_i.*

Note that every endpoint of any Q_i has degree at least 3 in H and every Q_i is contained in some R_j. Thus, every endpoint of any Q_i is incident with a jump. Therefore, if R_k contains finitely many jumps, then it contains finitely many Q_i. This proves the claim.

Let Q be the union of all Q_i that are contained in R_k. Then Q is a finite graph by the previous claim. Now $R_k \subseteq S \cup Q$, so $R_k - Q \subseteq S$. Since Q is finite, R_k contains a subray of S. □

4.2. Vertex Version

As mentioned at the beginning of this chapter, we will prove a stronger result than Theorem 1.2.5. We formally state the theorem below.

Theorem 4.2.1. *Let G be a weakly 3-connected graph and let X be an infinite subset of $V(G)$. Then G contains an X-rich H for some H in $\{K_{3,\infty}, FF, FL, LL\}$.*

Now every weakly 3-connected graph is 2-connected. Hence, by Theorem 1.2.3, G contains an X-rich H for some H in $\{K_{2,\infty}, F_\infty, L_\infty\}$. We divide the proof Theorem 4.2.1 into three lemmas, each considers a separate case for H.

Lemma 4.2.2. *Let G be a weakly 3-connected graph and let X be an infinite subset of $V(G)$. Assume G contains an X-rich $K_{2,\infty}$. Then G contains an X-rich $K_{3,\infty}$ or an X-rich FF.*

Proof. Let H be the subdivided $K_{2,\infty}$ in G and let x, y be the infinite-degree vertices of H. Since H contains infinitely many elements of X, it contains a subgraph H' that is also a subdivided $K_{2,\infty}$ and for every xy-path P in H', $\mathring{P}$ contains at least one element of X.

Claim 4.2.2.1. *There does not exist two distinct xy-paths S_1, S_2 in H' such that S_1 is a path xmy and S_2 is a path xny where m, n has degree 2 in G.*

If such xy-paths S_1, S_2 exist in H', then the underlying 3-connected graph of G, denoted by G', has parallel edges between x and y, which contradicts the assumption that G' is simple. This proves the claim.

From the previous claim, we may choose an xy-path S of H' containing at least three vertices of degree at least 3 and we will consider S-bridges of G. First, suppose there exists an S-bridge B containing infinitely many xy-paths of H', denoted by $P_1, P_2, \ldots$. Note that every P_i has length at least 2, so $\mathring{P}_i$ is non-empty. By applying Lemma 4.1.5 to the connected graph $B - S$ and disjoint subgraphs $\mathring{P}_1, \mathring{P}_2, \ldots$, one of the following is true in $B - S$

1. There exists an infinite subset $Y = \{Y_1, Y_2, \ldots\}$ of $\{\mathring{P}_1, \mathring{P}_2, \ldots\}$ and internally disjoint paths $Q_1, Q_2, \ldots$ where Q_i is between Y_i and Y_{i+1} for $i = 1, 2, \ldots$. Now $\bigcup_{i=1}^{\infty} Y_i \cup Q_i \cup \{x, y\}$ is a subdivision of a graph in $\mathcal{FF}_1$ containing infinitely many elements of X, so by Lemma 4.1.10, it contains an X-rich FF.

2. $B - S$ contains K, which is a subdivided $K_{1,\infty}$, where each leaf belongs to a $\mathring{P}_i$ and this $\mathring{P}_i$ does not contain any other vertices of K. In addition, every non-leaf vertex in K does not belong to $\mathring{P}_1 \cup \mathring{P}_2 \cup \ldots$. In K, let z be the vertex of infinite degree, $Y = \{y_1, y_2, \ldots\}$ be the set of its leaves, and Q_i be the zy_i-path for $i = 1, 2, \ldots$. Let $\mathcal{Q} = \bigcup_{i=1}^{\infty} Q_i$. Let $\mathcal{P} \subseteq \{P_1, P_2, \ldots\}$ be the set of paths P_k that contains a y_j and let $\mathcal{P}' = \bigcup_{P \in \mathcal{P}} P$. Now $\mathcal{P}' \cup \mathcal{Q}$ is a subdivided $K_{3,\infty}$, with x, y, z as its infinite-degree vertices, containing infinitely many elements of X.

3. $B - S$ contains K, which is a subdivided comb, where each leaf belongs to a $\mathring{P}_i$ and this $\mathring{P}_i$ does not contain any other vertices of K. In addition, every non-leaf vertex in K does not belong to $\mathring{P}_1 \cup \mathring{P}_2 \cup \ldots$. In K, let P be the spine, $Y = \{y_1, y_2, \ldots\}$ be the set of its leaves, and Q_i be the Py_i-path for $i = 1, 2, \ldots$. Let $\mathcal{Q} = \bigcup_{i=1}^{\infty} Q_i$. Let $\mathcal{P} \subseteq \{P_1, P_2, \ldots\}$ be the set of paths P_k that contains a y_j and let $\mathcal{P}' = \bigcup_{P \in \mathcal{P}} P$. Now $\mathcal{P}' \cup \mathcal{Q} \cup P$ is a subdivision of a graph in $\mathcal{FF}_2$ containing infinitely many elements of

X, so by Lemma 4.1.11, it contains an X-rich FF.

Now suppose every S-bridge contains finitely many xy-paths of H'. This means that there are infinitely many such S-bridges, each contains at least one xy-paths of H', denoted by $B_1, B_2, \ldots$. We define the set $\mathcal{B}$ as following: if there exists an S-bridge $B_i \in \{B_1, B_2, \ldots\}$ that is a path xmy where m has degree 2 in G, then such a B_i is unique by Claim 4.2.2.1 and in this case, $\mathcal{B} = \{B_1, B_2, \ldots\} - B_i$. Otherwise, no such B_i exists, and in this case, $\mathcal{B} = \{B_1, B_2, \ldots\}$.

Claim 4.2.2.2. *Every $B_i \in \mathcal{B}$ contains a vertex $u \notin V(S)$ and three weakly disjoint uS-paths in B_i whose union contains an element of X.*

Let $B_i \in \mathcal{B}$. Observe that $B_i - S$ contains a vertex $u \in X$ since B_i contains an xy-path P where $\mathring{P}$ is disjoint from S and $\mathring{P}$ contains an element of X. If u has degree at least 3 in G, then by Lemma 4.1.4, u cannot be separated from S by fewer than 3 vertices. By Corollary 1.3.2, there exist three weakly disjoint uS-paths whose union contains the vertex u of X. In addition, the union of those three paths is a subgraph of B_i since B_i is an S-bridge. In this case, the claim is done. Otherwise, u has degree 2 in G. Let a, b be the neighbors of u in G. If both $a, b \in V(S)$, then B_i is the path aub. Since B_i contains at least one xy-path of H', it follows that $\{a, b\} = \{x, y\}$. This is not possible by the construction of $\mathcal{B}$. Thus, we may assume that u has a neighbor a that is not on S. Now a has degree at least 3 in G, so by Lemma 4.1.4, a cannot be separated from S by fewer than 3 vertices. By Corollary 1.3.2, there exist three weakly disjoint aS-paths P_a, Q_a, R_a in G. In addition, the union of those three paths is a subgraph of B_i since B_i is an S-bridge. If $u \in V(P_a \cup Q_a \cup R_a)$, then the claim is done. Otherwise, $u \notin V(P_a \cup Q_a \cup R_a)$. Since G is 2-connected and u has a neighbor a, by applying Corollary 1.3.3 to u and $P_a \cup Q_a \cup R_a \cup S$ in G, there exist two weakly disjoint $u(P_a \cup Q_a \cup R_a \cup S)$-paths P_u, Q_u. Since u has degree 2 in G, one of those paths P_u must be the edge ua. If Q_u has an endpoint on S, then in B_i, we have three weakly disjoint paths aS-paths $P_a, Q_a, ua \cup Q_u$ whose union contains the vertex u of X, which proves the claim. Otherwise, we may assume, without loss of generality, that Q_u has an endpoint v on P_a. Let t be the endpoint of P_a on S. Let P_a' be the vt-subpath of P_a. We now have three weakly disjoint paths aS-paths $Q_a, R_a, ua \cup Q_u \cup P_a'$ whose union contains the vertex u of X, which proves the claim.

From the previous claim, each B_i in $\mathcal{B}$ has a vertex u and three weakly disjoint uS-paths in B_i whose union contains an element of X. Let a_i, b_i, c_i be the three vertices on S of those paths. Since $V(S)$ is finite, there exist infinitely many B_i whose corresponding vertices a_i, b_i, c_i on S coincide. This yields a subdivided $K_{3,\infty}$, with a_i, b_i, c_i as its infinite-degree vertices that contains infinitely many elements of X. □

Lemma 4.2.3. *Let G be a weakly 3-connected graph and let X be an infinite subset of $V(G)$. Assume G contains a subdivided F_∞ with infinitely many elements of X. Then G contains an X-rich FF or an X-rich FL.*

Proof. Let H be the subdivided F_∞. Since H contains infinitely many elements of X, it contains a subgraph H' that is also a subdivided F_∞, satisfying one of the following

1. the rail of H' contains infinitely many elements of X,

2. every spoke of H' contains at least one element of X.

In H', let R be its rail and let Z^* be its first spoke, namely the spoke that is incident with the endpoint of R. Let $S = R \cup Z^*$, so S is a ray and we will consider S-bridges of G.

Claim 4.2.3.1. *Every spoke of H', except the first one, is contained in some S-bridge of G.*

This is because every spoke of H' is connected and its two endpoints are on S. This proves the claim.

Claim 4.2.3.2. *If G has an S-bridge B with infinitely many feet, then the lemma holds.*

From the previous claim, every spoke of H' is contained in an S-bridge. First, suppose B contains finitely many spokes of H'. Let $\mathcal{A}$ be the set of spokes of H' not contained in B and let A be the union of all spokes in $\mathcal{A}$. Note that $\mathcal{A}$ is an infinite set. Let Y be the set of feet of B. By Corollary 3.1.3, B contains one of the following subgraphs

1. A subdivided $K_{1,\infty}$, call it K, whose leaves belong to Y. Observe that A and K only have common vertices on S. Since K has infinitely many leaves on S, it contains a subdivided $K_{1,\infty}$, denoted by K', whose leaves are on R. The subgraph $K' \cup S \cup A$ is a subdivision of a graph in $\mathcal{FF}_1$ containing infinitely many elements of X. By Lemma 4.1.10, it contains an X-rich FF.

2. A subdivided comb, call it K, whose leaves belong to Y. Observe that A and K only have common vertices on S. Since K has infinitely many leaves on S, it contains a subdivided comb, denoted by K', whose leaves are on R. The subgraph $K' \cup S \cup A$ is a subdivision of a graph in $\mathcal{FL}_1$ containing infinitely many elements of X. By Lemma

4.1.12, it an X-rich FL.

Now suppose B contains infinitely many spokes of H'. Let $\mathcal{A}'$ be the set of spokes that are contained in B. Then $\mathcal{A}'$ is an infinite set. In addition, every spoke in $\mathcal{A}'$ has length at least 2 for otherwise, it would be a trivial bridge and thus cannot be contained in B. Thus, every spoke in $\mathcal{A}'$ has a nonempty interior and also, two distinct spokes in $\mathcal{A}'$ have disjoint interiors. Let $\mathcal{A}'' = \{\mathring{A} \mid A \in \mathcal{A}'\}$. By applying Lemma 4.1.5 to the connected graph $B - S$ and all paths in $\mathcal{A}''$, one of the following is true in $B - S$

1. There exists an infinite subset $Z = \{Z_1, Z_2, \ldots\}$ of $\mathcal{A}''$ and internally disjoint $(Z_1 \cup Z_2 \cup \ldots)$-paths $Q_1, Q_2, \ldots$ of $B - S$ where Q_i is between Z_i and Z_{i+1} for $i = 1, 2, \ldots$. Let T_i be the Z_{i+1}-subpath between Q_i and Q_{i+1} and let $M = \bigcup_{i=1}^{\infty} T_i \cup Q_i$. Then M is a ray. Let S_i be the spoke of H' that contains Z_i. The subgraph $(\bigcup_{i=1}^{\infty} S_i) \cup M \cup R$ is a subdivision of a graph in $\mathcal{FL}_1$ containing infinitely many elements of X. By Lemma 4.1.12, it contains an X-rich FL.

2. $B - S$ contains K, a subdivided $K_{1,\infty}$, where each leaf belongs to an $\mathring{A} \in \mathcal{A}''$ and this $\mathring{A}$ does not contain any other vertices of K. In addition, every non-leaf vertex in K does not belong to $\bigcup_{\mathring{A} \in \mathcal{A}''} \mathring{A}$. Let $A_1, A_2, \ldots$ be the spokes in $\mathcal{A}'$ such that each $\mathring{A}_i \in \mathcal{A}''$ contains a leaf of K. The subgraph $(\bigcup_{i=1}^{\infty} A_i) \cup K \cup S$ is a subdivision of a graph in $\mathcal{FF}_2$ containing infinitely many elements of X. By Lemma 4.1.11, it contains an X-rich FF.

3. $B - S$ contains K, a subdivided comb, where each leaf belongs to an $\mathring{A} \in \mathcal{A}''$ and

this $\mathring{A}$ does not contain any other vertices of K. In addition, every non-leaf vertex in K does not belong to $\bigcup_{\mathring{A} \in \mathcal{A}''} \mathring{A}$. Let $A_1, A_2, \ldots$ be the spokes in $\mathcal{A}'$ such that each $\mathring{A}_i \in \mathcal{A}''$ contains a leaf of K. The subgraph $(\bigcup_{i=1}^{\infty} A_i) \cup K \cup S$ is a subdivision of a graph in $\mathcal{FL}_2$ containing infinitely many elements of X. By Lemma 4.1.13, it contains an X-rich FL.

This proves the claim.

From the previous claim, we may assume that every S-bridge of G has finitely many feet. To analyze the connection between the S-bridges, we introduce the auxiliary graph Γ. We first partition the S-bridges of G into groups such that two bridges belong to the same group if they have the same set of feet.

Claim 4.2.3.3. *There are infinitely many groups.*

Suppose this is not the case. Since every S-bridge has finitely many feet and there are finitely many groups, the set $\{x \in S \mid x \text{ is a foot of a bridge}\}$ is finite. But this implies that there exists a vertex in S with degree at least 3 in G that does not belong to any bridge and this is not possible. This proves the claim.

We define the graph Γ whose vertex set consists of the groups and two vertices are adjacent in Γ if the two corresponding bridges chosen from the two groups cross with respect to S. This definition of Γ is well-defined because bridges in the same group have the same set of feet. By the previous claim, Γ is an infinite graph.

Claim 4.2.3.4. *The graph Γ has no finite component.*

Suppose for contradiction that Γ has a finite component with vertices $x_1, x_2, \ldots, x_k$. Let $B_1, B_2, \ldots, B_k$ be a set of corresponding S-bridges chosen from the k groups. Observe that each B_i has at least two feet because G is weakly 3-connected. Since $B_1 \cup B_2 \cup \cdots \cup B_k$ is a finite union of bridges, each has finitely many feet, there exist two distinct $x, y \in S$ where x, y are feet of some bridges in $\{B_1, B_2, \ldots, B_k\}$ such that every foot of a bridge in $\{B_1, B_2, \ldots, B_k\}$ belongs to $S[xy]$. If $S(xy)$ does not contain a vertex of degree at least 3 in G, then $S[xy]$ is either the edge xy or a path xzy where z has degree 2 in G. In either case, note that $B_1 - S$ contains a vertex u of degree at least 3 in G. Now $\{x, y\}$ separates u from S in G, contradicting Lemma 4.1.4. Thus, $S(xy)$ contains a vertex w of degree at least 3 in G. By Lemma 4.1.4, $\{x, y\}$ does not separate w from $S - S[xy]$. Thus, there exists a $w(S - S[xy])$-path in $G - \{x, y\}$. This means that there exists an S-bridge B^* with a foot t on $S(xy)$ and another foot on $S - S[xy]$. This bridge $B^* \notin \{B_1, B_2, \ldots, B_k\}$ because it has a foot on $S - S[xy]$. Additionally, B^* does not cross any B_i with respect to S because B^* is not in the component $\{B_1, B_2, \ldots, B_k\}$ of Γ. Now in $\{B_1, B_2, \ldots, B_k\}$, there is a bridge B_i with x as a foot and a bridge B_j with y as a foot. Since none of the bridges in $\{B_1, B_2, \ldots, B_k\}$ crosses B^* with respect to S, every bridge in $\{B_1, B_2, \ldots, B_k\}$ has all feet either on $S[xt]$ or $S[yt]$. But this means the component $\{B_1, B_2, \ldots, B_k\}$ of Γ is not connected, a contradiction. This proves the claim.

We have shown that Γ has no finite component, so it has an infinite component. In this infinite component, there exists a vertex of infinite degree or an induced ray by Theorem 3.1.4. Suppose Γ has a vertex of infinite degree. Then there exist S-bridges B and $B_1, B_2, \ldots$

such that B crosses B_i with respect to S for every i. Since B has finitely many feet, there exist two feet x, y of B that cross infinitely many B_i. By the definition of crossing, every B_i has a foot in $S(xy)$. Since $S(xy)$ is finite, it has a vertex z that is a foot of infinitely many B_i. Let Q_{xy} be an xy-subpath of B and let S' be obtained from S by replacing $S[xy]$ with Q_{xy}. Then S' is a ray and we have an S'-bridge with infinitely many feet on S'. Let F' be the union of S' and all spokes of H' with both endpoints in S'. Then F' is a subdivided F_∞. Furthermore, its rail contains infinitely many elements of X or every of its spokes contains at least one element of X. By Claim 4.2.3.2, G contains a subdivision of FF or FL, each contains infinitely many elements of X.

Now suppose Γ has an infinite induced path. Then there exist S-bridges $B_1, B_2, \ldots$ such that with respect to S, B_{i+1} crosses B_i but does not cross B_j for any $j < i$. By Lemma 4.1.20, there exist infinitely many S-paths $Q_1, Q_2, \ldots$ satisfying the following

1. two distinct Q_i and Q_j are internally-disjoint,

2. with respect to S, Q_{i+1} crosses Q_i but does not cross Q_j for any $j < i$.

Let $K = S \cup Q_1 \cup Q_2 \cup \ldots$. By Lemma 3.1.8, $K = H_1 \cup H_2$ where H_1 is a finite graph and H_2 is the union of two disjoint rays A, B and a set $\mathcal{R}$ of infinitely many internally disjoint AB-paths. In addition, H_1 and H_2 are edge-disjoint and K is locally finite. Let R_1 be the union of all paths in $\mathcal{R}$. Let u be the infinite degree vertex and let $S_1, S_2, \ldots$ be the spokes of H'. For each i, let S_i' be the subpath of S_i such that S_i' has u as an endpoint and the other endpoint is the first time S_i intersects $H' - u$. Let $K' = (\bigcup_{i=1}^{\infty} S_i') \cup K$. We divide the last part of the proof into two cases.

Case 1: K' contains infinitely many elements of X. Since $u \in K$, one of the following is true

- $u \in H_1$,
- $u \in (A \cup B) - H_1$,
- $u \in R_1 - (A \cup B \cup H_1)$.

In each case, K contains a subgraph satisfying Lemma 4.1.16, so it contains an X-rich FL.

Case 2: K' contains finitely many elements of X. This implies that S contains finitely many elements of X since S is a subgraph of K'. Hence, we may assume that every spoke of H' contains at least one element of X. In addition, by deleting edges of finitely many spokes with an element of X in K', we may assume that no S'_i contains any element of X. Since every S_i contains at least one element of X not in K, every S_i has a shortest subpath N_i that is a K-path and contains at least one element of X not in K. Let $\mathcal{N}$ be the set of those such subpaths N_i. Then $\mathcal{N}$ has an infinite subset $\mathcal{N}'$ such that every path in $\mathcal{N}'$ is a $(A \cup B \cup R_1)$-path. In addition, by deleting finitely many edges, all of the following are true

- A, B have a subrays A', B' respectively such that $u \notin A' \cup B'$,
- $\mathcal{R}$ has an infinite subset $\mathcal{R}'$ such that no path in $\mathcal{R}'$ contains u. Let R'_1 be the union of all paths in $\mathcal{R}'$.
- $\mathcal{N}'$ has an infinite subset $\mathcal{N}''$ such that every path in $\mathcal{N}''$ is an $(A' \cup B' \cup R'_1)$-path. Let N'' be the union of all paths in $\mathcal{N}''$.

  The subgraph $A' \cup B' \cup R'_1 \cup N''$ satisfies the conditions in Lemma 4.1.17, so it contains an X-rich L_∞, call it L. Now L contains infinitely many elements of X, each of them belongs to an $S_i - u$, and $u \notin L$. Thus, there exist infinitely many weakly disjoint uL-paths. Let L' be the union of L and those weakly disjoint uL-paths. Now L' contains a subdivided $\mathcal{FL}_1$ or a subdivided $\mathcal{FL}_2$, each contains many elements of X. By Lemma 4.1.12 or Lemma 4.1.13, it contains an X-rich FL. □

Lemma 4.2.4. *Let G be a weakly 3-connected graph and let X be an infinite subset of $V(G)$. Assume G contains a subdivided L_∞ with infinitely many elements of X. Then G contains an X-rich FL or an X-rich LL.*

Proof. Let H be the subdivided L_∞. Since H contains infinitely many elements of X, it contains a subgraph H' that is also a subdivided L_∞ satisfying one of the following

1. the rails of H' contain infinitely many elements of X,
2. every rung of H' contains at least one element of X.

In H', let P, Q be its rails and let Z^* be its first rung, namely the rung that is incident with the endpoints of P, Q. Let $S = P \cup Q \cup Z^*$, so S is a double ray and we will consider S-bridges of G.

Claim 4.2.4.1. *Every rung of H', except the first one, is contained in some S-bridge of G.*

This is because every rung of H' is connected and its two endpoints are on S. This proves the claim.

Claim 4.2.4.2. *If G has an S-bridge B with infinitely many feet, then the lemma holds.*

From the previous claim, every rung of H' is contained in an S-bridge. First, suppose B contains finitely many rungs of H'. Let $\mathcal{A}$ be the set of rungs of H' not contained in B and let A be the union of all rungs in $\mathcal{A}$. Note that $\mathcal{A}$ is an infinite set. Let Y be the set of feet of B. By Corollary 3.1.3, B contains one of the following subgraphs

1. A subdivided $K_{1,\infty}$, call it K, whose leaves belong to Y. Observe that A and K only have common vertices on S. Since K has infinitely many leaves on S, it contains a subdivided $K_{1,\infty}$, denoted by K', such that all leaves of K' are on P or all leaves of K' are on Q. The subgraph $K' \cup S \cup A$ is a subdivision of a graph in $\mathcal{FL}_1$ containing infinitely many elements of X. By Lemma 4.1.12, it contains an X-rich FL.

2. A subdivided comb, call it K, whose leaves belong to Y. Observe that A and K only have common vertices on S. Since K has infinitely many leaves on S, it contains a subdivided comb, denoted by K', such that all leaves of K' are on P or all leaves of K' are on Q. The subgraph $K' \cup S \cup A$ is a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely many elements of X. By Lemma 4.1.14, it contains an X-rich LL.

Now suppose B contains infinitely many rungs of H'. Let $\mathcal{A}'$ be the set of rungs that are contained in B. Then $\mathcal{A}'$ is an infinite set. In addition, every rung in $\mathcal{A}'$ has length at least 2 for otherwise, it would be a trivial bridge and thus cannot be contained in B. Thus, every rung in $\mathcal{A}'$ has a nonempty interior and also, two distinct rungs in $\mathcal{A}'$ have disjoint interiors. Let $\mathcal{A}'' = \{\mathring{A} \mid A \in \mathcal{A}'\}$. By applying Lemma 4.1.5 to the connected graph $B - S$ and all paths in $\mathcal{A}''$, one of the following is true in $B - S$

1. There exists an infinite subset $Z = \{Z_1, Z_2, \ldots\}$ of $\mathcal{A}''$ and internally disjoint $(Z_1 \cup Z_2 \cup \ldots)$-paths $Q_1, Q_2, \ldots$ of $B - S$ where Q_i is between Z_i and Z_{i+1} for $i = 1, 2, \ldots$. Let T_i be the Z_{i+1}-subpath between Q_i and Q_{i+1} and let $M = \bigcup_{i=1}^{\infty} T_i \cup Q_i$. Then M is a ray. Let R_i be the rung of H' that contains Z_i. The subgraph $(\bigcup_{i=1}^{\infty} R_i) \cup M \cup P \cup Q$ is a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely many elements of X. By Lemma 4.1.14, it contains an X-rich LL.

2. $B - S$ contains K, a subdivided $K_{1,\infty}$, where each leaf belongs to an $\mathring{A} \in \mathcal{A}''$ and this $\mathring{A}$ does not contain any other vertices of K. Let $A_1, A_2, \ldots$ be the rungs in $\mathcal{A}'$ such that each $\mathring{A}_i$ contains a leaf of K. The subgraph $(\bigcup_{i=1}^{\infty} A_i) \cup K \cup S$ is a subdivision of a graph in $\mathcal{FL}_2$ containing infinitely many elements of X. By Lemma 4.1.13, it contains an X-rich FL.

3. $B - S$ contains K, a subdivided comb, where each leaf belongs to an $\mathring{A} \in \mathcal{A}''$ and this $\mathring{A}$ does not contain any other vertices of K. Let $A_1, A_2, \ldots$ be the rungs in $\mathcal{A}'$ such that each $\mathring{A}_i$ contains a leaf of K. The subgraph $(\bigcup_{i=1}^{\infty} A_i) \cup K \cup S$ is a subdivision of a graph in $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL.

This proves the claim.

From the previous claim, we may assume that every S-bridge of G has finitely many feet. To analyze the connection between the S-bridges, we introduce the auxiliary graph Γ. We first partition the S-bridges of G into groups such that two bridges belong to the same group if they have the same set of feet.

Claim 4.2.4.3. *There are infinitely many groups.*

Suppose this is not the case. Since every S-bridge has finitely many feet and there are finitely many groups, the set $\{x \in S \mid x \text{ is a foot of a bridge}\}$ is finite. But this implies that there exists a vertex in S with degree at least 3 in G that does not belong to any bridge and this is not possible. This proves the claim.

We define the graph Γ whose vertex set consists of the groups and two vertices are adjacent in Γ if the two corresponding bridges chosen from the two groups cross with respect to S. This definition of Γ is well-defined because bridges in the same group have the same set of feet. By the previous claim, Γ is an infinite graph.

Claim 4.2.4.4. *The graph Γ has no finite component.*

Suppose for contradiction that Γ has a finite component with vertices $x_1, x_2, \ldots, x_k$. Let $B_1, B_2, \ldots, B_k$ be a set of corresponding S-bridges chosen from the k groups. Observe that each B_i has at least two feet because G is weakly 3-connected. Since $B_1 \cup B_2 \cup \cdots \cup B_k$ is a finite union of bridges, each has finitely many feet, there exist two distinct $x, y \in S$ where x, y are feet of some bridges in $\{B_1, B_2, \ldots, B_k\}$ such that every foot of a bridge in $\{B_1, B_2, \ldots, B_k\}$ belongs to $S[xy]$. If $S(xy)$ does not contain a vertex of degree at least 3 in G, then $S[xy]$ is either the edge xy or a path xzy where z has degree 2 in G. In either case, note that $B_1 - S$ contains a vertex u of degree at least 3 in G. Now $\{x, y\}$ separates u from S in G, contradicting Lemma 4.1.4. Thus, $S(xy)$ contains a vertex w of degree at least 3 in G. By Lemma 4.1.4, $\{x, y\}$ does not separate w from $S - S[xy]$. Thus, there exists a

$w(S-S[xy])$-path in $G-\{x,y\}$. This means that there exists an S-bridge B^* with a foot t on $S(xy)$ and another foot on $S-S[xy]$. This bridge $B^* \notin \{B_1, B_2, \ldots, B_k\}$ because it has a foot on $S-S[xy]$. Additionally, B^* does not cross any B_i with respect to S because B^* is not in the component $\{B_1, B_2, \ldots, B_k\}$ of Γ. Now in $\{B_1, B_2, \ldots, B_k\}$, there is a bridge B_i with x as a foot and a bridge B_j with y as a foot. Since none of the bridges in $\{B_1, B_2, \ldots, B_k\}$ crosses B^* with respect to S, every bridge in $\{B_1, B_2, \ldots, B_k\}$ has all feet either on $S[xt]$ or $S[yt]$. But this means the component $\{B_1, B_2, \ldots, B_k\}$ of Γ is not connected, a contradiction. This proves the claim.

We have shown that Γ has no finite component, so it has an infinite component. In this infinite component, there exists a vertex of infinite degree or an induced ray by Theorem 3.1.4. Suppose Γ has a vertex of infinite degree. Then there exist S-bridges B and $B_1, B_2, \ldots$ such that B crosses B_i with respect to S for every i. Since B has finitely many feet, there exist two feet x, y of B that cross infinitely many B_i. By the definition of crossing, every B_i has a foot in $S(xy)$. Since $S(xy)$ is finite, it has a vertex z that is a foot of infinitely many B_i. Let Q_{xy} be an xy-subpath of B and let S' be obtained from S by replacing $S[xy]$ with Q_{xy}. Then S' is a double ray and we have an S'-bridge with infinitely many feet on S'. Let L' be the union of S' and all rungs of H' with both endpoints in S'. Then L' is a subdivided L_∞. Furthermore, its rails contains infinitely many elements of X or every of its rungs contains at least one element of X. By Claim 4.2.4.2, G contains an X-rich FL or an X-rich LL.

Now suppose Γ has an infinite induced path. Then there exist S-bridges $B_1, B_2, \ldots$

such that with respect to S, B_{i+1} crosses B_i but does not cross B_j for any $j < i$. By Lemma 4.1.20, there exist infinitely many S-paths $Q_1, Q_2, \ldots$ satisfying the following

1. two distinct Q_i and Q_j are internally-disjoint,
2. with respect to S, Q_{i+1} crosses Q_i but does not cross Q_j for any $j < i$.

By Lemma 4.1.21, the graph $K = S \cup Q_1 \cup Q_2 \cup \ldots$ contains three disjoint rays R_1, R_2, R_3 such that every Q_i is contained in some R_j. In addition, if $R = R_1 \cup R_2 \cup R_3$ and $J_1, J_2, \ldots$ are all R-bridges of K, then each J_i is a subpath of S with endpoints in different R_j and we call it a jump. Finally, if all but finitely many jumps are between R_i and R_j, then R_k contains a subray of S. We divide the last part of the proof into two cases.

Case 1: K contains infinitely many elements of X.

We further divide this case into two subcases.

Case 1a: There are infinitely many jumps between at least two pairs of $\{R_1, R_2, R_3\}$.

Since K contains infinitely many elements of X, either $R_1 \cup R_2 \cup R_3$ contains infinitely many elements of X or without loss of generality, there exist infinitely many jumps between R_1, R_2 each contains at least one element of X. Let A be the union of all jumps between R_1, R_2. Since there are infinitely many jumps between at least two pairs of $\{R_1, R_2, R_3\}$, we may assume without loss of generality that there are infinitely many jumps between R_2, R_3. Let B be the union of all jumps between R_2, R_3. The subgraph $R_1 \cup R_2 \cup R_3 \cup A \cup B$ is a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely many elements of X. By Lemma 4.1.14, it contains an X-rich LL.

Case 1b: There are infinitely many jumps between only one pair of $\{R_1, R_2, R_3\}$.

Without loss of generality, we may assume that there is a set $\mathcal{A}$ of infinitely many R_1R_2-jumps, but there are only finitely many R_2R_3-jumps and only finitely many R_1R_3-jumps. Then R_3 contains a subray of S, so we may assume it contains a subray P' of P. Let A be the union of all jumps in $\mathcal{A}$. Now there exist infinitely many rungs $R'_1, R'_2, \ldots$ of H' with one endpoint in P' and the other endpoint in $R_1 \cup R_2 \cup A$. For each R'_i, let R''_i be the subpath of R'_i with one endpoint in P' and the other endpoint is the first time R'_i intersects $R_1 \cup R_2 \cup A$. Let $M = \bigcup_{i=1}^{\infty} R''_i$. If there exist infinitely many R''_i with an endpoint in $R_1 \cup R_2$, then the subgraph $R_1 \cup R_2 \cup R_3 \cup A \cup M$ contains a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely many elements of X. By Lemma 4.1.14, it contains an X-rich LL. Otherwise, there exist infinitely many R''_i with an endpoint in $A - (R_1 \cup R_2)$. The subgraph $R_1 \cup R_2 \cup R_3 \cup A \cup M$ contains a subdivision of a graph in $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL.

Case 2: K contains finitely many elements of X. This implies that S contains finitely many elements of X since S is a subgraph of K. Hence, we may assume that every rung of H' contains at least one element of X.

We further divide this case into two subcases.

Case 2a: There are infinitely many jumps between at least two pairs of $\{R_1, R_2, R_3\}$.

Without loss of generality, we may assume that there is a set $\mathcal{A}$ of infinitely many R_1R_2-jumps and a set $\mathcal{B}$ of infinitely many R_2R_3-jumps. Let A be the union of all jumps in $\mathcal{A}$ and let B be the union of all jumps in $\mathcal{B}$. Since every rung of H' contains at least one element of X and K contains only finitely many elements of X, K contains finitely many rungs of H'. Thus, there exist an infinite subset $\{R'_1, R'_2, \ldots\}$ of the rungs such that each R'_i contains an element of X that is not in K. Since every rung in $\{R'_1, R'_2, \ldots\}$ contains at least one element of X, every R'_i has a shortest subpath N_i that is a K-path and contains at least one element of X that is not in K. Let $\mathcal{N}$ be the set of those such subpaths N_i. Suppose there exist only finitely many R_1R_3-jumps. Then $\mathcal{N}$ has an infinite subset $\mathcal{N}'$ such that every path in $\mathcal{N}'$ is an $(R_1 \cup R_2 \cup R_3 \cup A \cup B)$-path. Let N' be the union of all path in $\mathcal{N}'$. The subgraph $R_1 \cup R_2 \cup R_3 \cup A \cup B \cup N'$ satisfies the hypotheses in Lemma 4.1.18, so it contains an X-rich LL. Now suppose there exist infinitely many R_1R_3-jumps. Let C be the union of all R_1R_3-jumps. Let $K_1 = R_1 \cup R_2 \cup R_3 \cup A \cup B$, $K_2 = R_1 \cup R_2 \cup R_3 \cup A \cup C$, and $K_3 = R_1 \cup R_2 \cup R_3 \cup B \cup C$. Then $\mathcal{N}$ has an infinite subset $\mathcal{N}''$ such that every path in $\mathcal{N}''$ is a K_i-path for some $i \in \{1, 2, 3\}$. Let N'' be the union of all path in $\mathcal{N}''$. The subgraph $K_i \cup N''$ satisfies the hypotheses in Lemma 4.1.18, so it contains an X-rich LL.

Case 2b: There are infinitely many jumps between only one pair of $\{R_1, R_2, R_3\}$.

Without loss of generality, we may assume that there is a set $\mathcal{A}$ of infinitely many R_1R_2-jumps, but there are only finitely many R_2R_3-jumps and only finitely many R_1R_3-jumps. Then R_3 contains a subray of S, so we may assume it contains a subray P' of P. Let A be the union of all jumps in $\mathcal{A}$. Now there exist infinitely many rungs $R'_1, R'_2, \ldots$ of H' with one endpoint in P' and the other endpoint in $R_1 \cup R_2 \cup A$. For each R'_i, let R''_i be the subpath of R'_i with one endpoint in P' and the other endpoint is the first time R'_i intersects $R_1 \cup R_2 \cup A$. We further divide this case into two subcases.

Case 2b1: There exist infinitely many R''_i with an endpoint in $R_1 \cup R_2$.

Without loss of generality, we may assume that infinitely many R_i'' has an endpoint in R_1. Let M be the union of all such R_i''. If there exist infinitely many elements of X in M, then $R_1 \cup R_2 \cup R_3 \cup A \cup M$ is a subdivision of a graph in $\mathcal{LL}_1$ containing infinitely many elements of X. By Lemma 4.1.14, it contains an X-rich LL. Otherwise, let R_i' be the rung of H' that contains R_i'' where R_i'' has an endpoint in R_1. Since every R_i' contains at least one element of X not in K, every R_i' has a shortest subpath N_i that is a K-path and contains at least one element of X not in K. Let $\mathcal{N}$ be the set of those such subpaths N_i. Then $\mathcal{N}$ has an infinite subset $\mathcal{N}'$ such that every path in $\mathcal{N}'$ is a $(R_1 \cup R_2 \cup R_3 \cup A \cup M)$-path. Let N' be the union of all paths in $\mathcal{N}'$. The subgraph $R_1 \cup R_2 \cup R_3 \cup A \cup M \cup N'$ satisfies the hypotheses in Lemma 4.1.18, so it contains an X-rich LL.

Case 2b2: There exist infinitely many R_i'' with an endpoint in $A - (R_1 \cup R_2)$.

Let $\mathcal{M}$ be the set of all such R_i''. Then $\mathcal{M}$ has an infinite subset $\mathcal{M}'$ such that every jump in $\mathcal{A}$ is incident with at most one R_i'' in $\mathcal{M}'$ because the rungs of H' are disjoint. Let M' be the union of all paths in $\mathcal{M}'$. If there exist infinitely many elements of X in M', then $R_1 \cup R_2 \cup R_3 \cup A \cup M'$ is a subdivision of a graph in $\mathcal{LL}_2$ containing infinitely many elements of X. By Lemma 4.1.15, it contains an X-rich LL. Otherwise, for each R_i'' in $\mathcal{M}'$, let R_i' be the rung of H' that contains R_i''. Since every R_i' contains at least one element of X not in K, every R_i' has a shortest subpath N_i that is a K-path and contains at least one element of X not in K. Let $\mathcal{N}$ be the set of those such subpaths N_i. Then $\mathcal{N}$ has an infinite subset $\mathcal{N}'$ such that every path in $\mathcal{N}'$ is a $(R_1 \cup R_2 \cup R_3 \cup A \cup M')$-path. Let N' be the union of all paths in $\mathcal{N}'$. The subgraph $R_1 \cup R_2 \cup R_3 \cup A \cup M' \cup N'$ satisfies the hypotheses in Lemma

4.1.19, so it contains an X-rich LL. □

Proof of Theorem 4.2.1. Since G is weakly 3-connected, it is 2-connected. By Theorem 1.2.3, G contains an X-rich H for some H in $\{K_{2,\infty}, F_\infty, L_\infty\}$. The theorem then follows from Lemma 4.2.2, Lemma 4.2.3, and Lemma 4.2.4. □

We will now prove the vertex version.

Proof of Theorem 1.2.5. Since G is 3-connected, it is also weakly 3-connected. The theorem then follows from Theorem 4.2.1. □

4.3. Edge Version

We conclude this chapter with the proof of the edge version.

Proof of Theorem 1.2.6. Let G' be obtained from G by subdividing each edge in X exactly once. Then G' is weakly 3-connected. Let Y be the set of subdividing vertices of G'. Then Y is infinite because X is infinite. In addition, every vertex of Y has degree 2 in G'. By Theorem 4.2.1, G' contains an Y-rich H' for some H' in $\{K_{3,\infty}, FF, FL, LL\}$. Consequently, G contains a subdivided H containing infinitely many edges of X for some H in $\{K_{3,\infty}, FF, FL, LL\}$. □

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