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A POLYNOMIAL INVARIANT OF LINKS IN A SOLID TORUS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

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B.S., Chungbuk National University, Korea, 1977
M.S., Louisiana State University, 1992
December 1996

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TABLE OF CONTENTS

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LIST OF FIGURES

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ABSTRACT

A polynomial invariant of links in a solid torus is defined through an algebra $H_n(\frac{1}{2})$. $H_n(\frac{1}{2})$ modulo by an ideal is the type-B Hecke algebra. This invariant satisfies the S_3-skein relation as in the 1-trivial links case of dicromatic link invariant discovered by J. Hoste and M. Kidwell.

A link in the solid torus is isotopic to a closed braid which is a braid in the braid group of the annulus. We find an invariant of links through a represention π of the braid group of the annulus to the algebra $H_n(\frac{1}{2})$. A trace map X is defined on a basis

$$\mathcal{B} = \{(t_1')^{s_1} \cdots (t_n')^{s_n}\beta \,|\, s_i \in \mathbb{Z},\ \beta \in H(A_{n-1}), \text{ in normal form }\}$$

of $H_n(\frac{1}{2})$. Then, there is a map Z from $\cup B_n(Ann)$ (braid group of annulus) to $C(q, \sqrt{\lambda})[\tau_i]_{i\in\mathbb{Z}}$ defined by $Z(\alpha) = (\sqrt{\lambda}z)^{1-n}\sqrt{\lambda}^e X(\pi(\alpha))$. The invariant $Z(\alpha)$ is an ambient isotopy invariant for the links in the isotopy class that α represents. Therefore, this is a computational approach to the S_3-skein module for solid torus.

An invariant of links in a solid torus was discovered by S. Lambropoulou through the type-B Hecke algebra. It can be recovered from $Z(\alpha)$.

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CHAPTER 1. INTRODUCTION AND BACKGROUND

1.1. Polynomial invariants of links

A link in 3-manifold is a smooth submanifold consisting of disjoint simple closed curves. A knot is a link with one component. Two links K and L are ambient isotopic if and only if a link diagram of K can be obtained from that of L by a sequence of Reidemeister moves (see Figure 1). An ambient isotopy invariant of oriented links is an invariant under the Reidemeister moves. Let L_+, L_-, and L_0 denote links that are identical except in one crossing of a link diagram L, conventionally with L_+ a single right handed crossing, as in Figure 2.

After the Alexander, the Conway, and the Jones polynomials, a two-variable twisted Alexander polynomial invariant of oriented links in 3-space was published in 1985. A combined paper due to the coincidence of the research announcements by four groups, each describing the same result (see [F]) is as follows:

Theorem 1.1.1. *[HOMFLY-PT] There is a unique function P from the set of isotopy classes of tame oriented links to the set of homogeneous Laurent polynomials of degree 0 in x, y, z such that*

(1) $xP_{L_+}(x,y,z) + yP_{L_-}(x,y,z) + zP_{L_0}(x,y,z) = 0$,

(2) $P_L(x,y,z) = 1$ *if* L *consists of a single unknotted component.*

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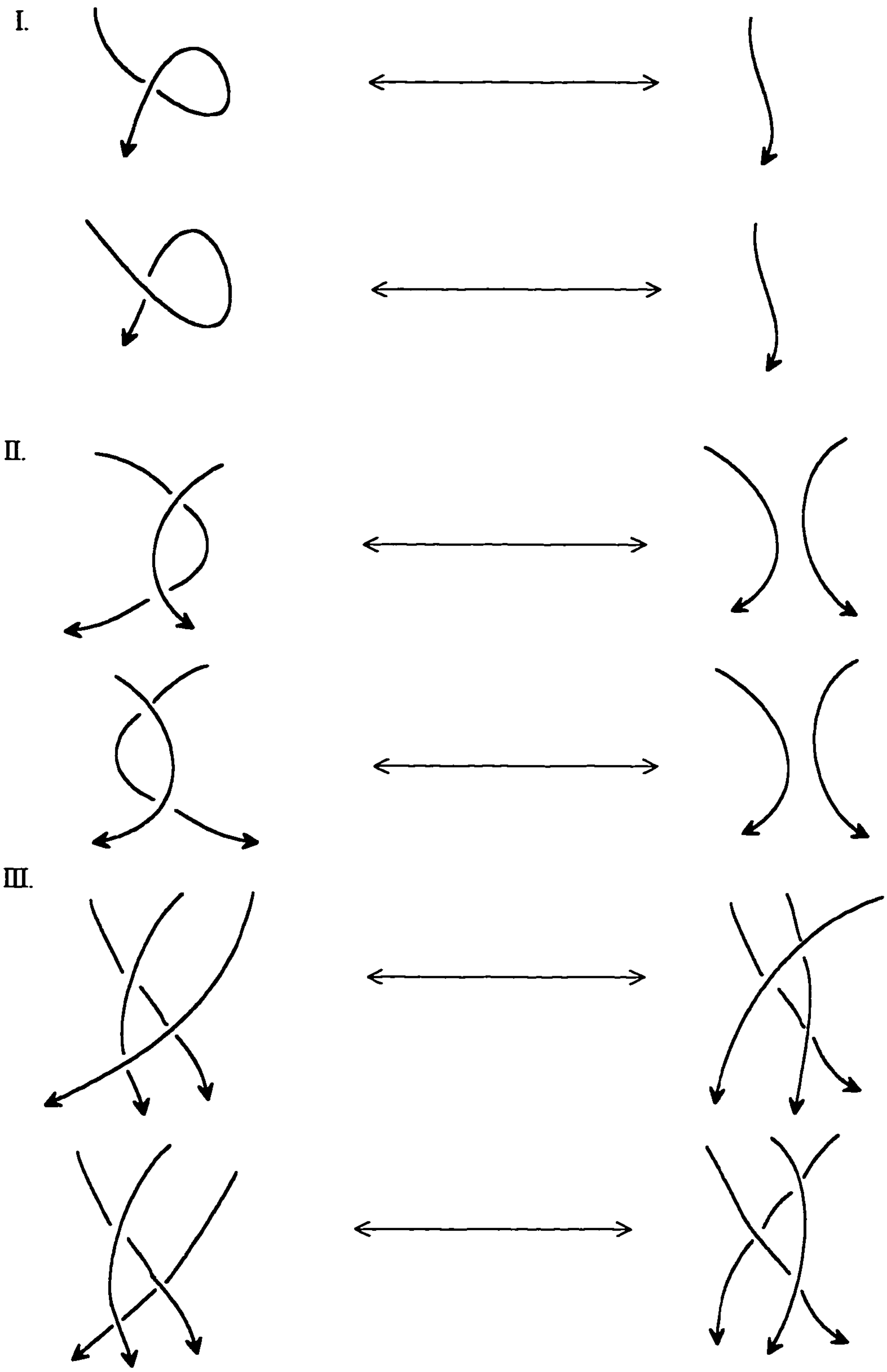

Figure 1. Reidemeister moves I, II, and III in an oriented link diagram

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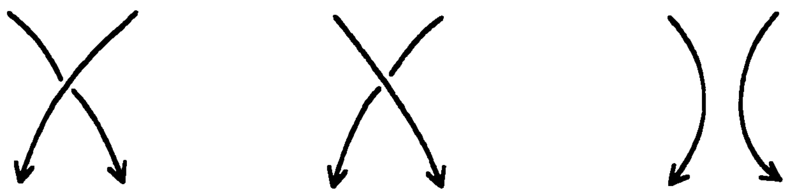

Figure 2. Signed crossings in an oriented link diagram

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Then the Jones polynomial can be expressed as a special case by $V_L(t) = P_L(t, -t^{-1}, t^{\frac{1}{2}} - t^{-\frac{1}{2}})$. A relation as in the part(1) in theorem 1.1.1 is called a skein relation. For the skein relation given by $v^{-1}P_{L_+} - vP_{L_-} = zP_{L_0}$, the Alexander polynomial and the Jones polynomial $V_L(t)$ occur as the special cases $v = 1$, and $v = t, z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$, respectively (see [HP2]).

To describe a series of further work, we introduce the type-A Hecke algebra $H(A_{n-1})$. It is an algebra with generators $g_1, g_2, \ldots, g_{n-1}$ and relations:

(1) $g_i g_j = g_j g_i$ if $|i - j| > 1$,

(2) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$, for $i = 1 \cdots n-2$,

(3) $g_i^2 = (q-1)g_i + q$ for all i, and q is a complex number as a parameter.

A basis for the type-A Hecke algebra is furnished by a system of reduced words as

$$\{(g_{i_1} g_{i_1-1} \cdots g_{i_1-k_1})(g_{i_2} \cdots g_{i_2-k_2}) \cdots (g_{i_m} \cdots g_{i_m-k_m})\}$$

where $1 \leq i_1 < i_2 < \cdots < i_m \leq n-1$. We see that the dimension of $H(A_{n-1})$ is $n!$. Using the basis of the type-A Hecke algebra, V.F.R. Jones again constructed the HOMFLY-PT polynomial invariant (see [J]) using Ocneanu's work as follows:

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Theorem 1.1.2. *[Ocneanu] For every $z \in C$ there is a linear trace*

tr on $\bigcup_{n=1}^{\infty} H(A_{n-1})$ uniquely defined by

(1) tr(ab) = tr(ba) a, $b \in H(A_{n-1})$;

(2) tr(1) =1 ;

(3) $tr(ag_n b) = z\, tr(ab)$ if a, $b \in H(A_{n-1})$.

Theorem 1.1.3. *[J] To each oriented link L (up to isotopy) there is a Laurent polynomial $X_L(\sqrt{q}, \sqrt{\lambda})$ satisfying:*

$(\sqrt{\lambda}\sqrt{q})^{-1}X_{L_+} - (\sqrt{\lambda}\sqrt{q})X_{L_-} = (\sqrt{q} - \frac{1}{\sqrt{q}})X_{L_0}.$

This work was done through the representation π of Artin's braid group to the type-A Hecke algebra. A connection of $H(B_n)$ to $B_n(Ann)$ was observed by A. McDaniel and L. Smolinsky (see [MS]), and X. Lin ([LI]). They noted that the Brieskorn braid group of type B_n is the braid group of the annulus, $B_n(Ann)$.

S. Lambropoulou produced an invariant for $S^1 \times D^2$ (see [LA]) by use of the braid group $B_n(Ann)$ and the type-B Hecke algebra.

$B_n(Ann)$ has generators $t, \sigma_1, \ldots, \sigma_{n-1}$ (see Figure 3) and relations:

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$;

(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, \ldots, n-2$;

(3) $t\,\sigma_1\,t\,\sigma_1 = \sigma_1\,t\,\sigma_1\,t$.

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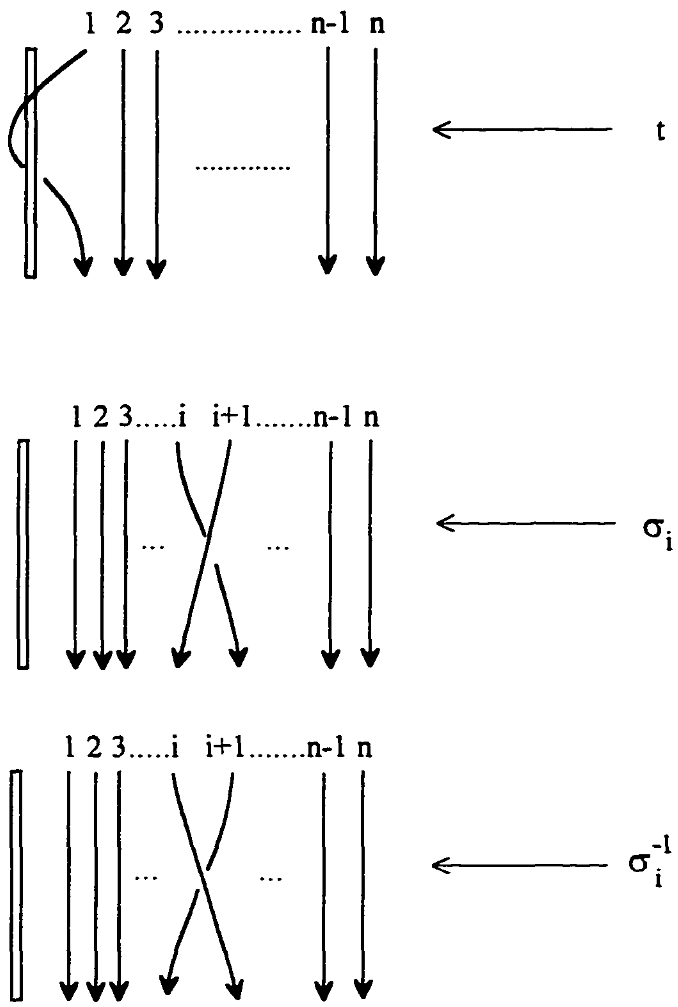

Figure 3. Generators of the braid group $B_n(Ann)$

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A presentation of the type-B Hecke algebra $H(B_n)$ is as follows:

$$\langle t_1, g_1, g_2, \ldots, g_{n-1} \mid t_1 g_1 t_1 g_1 = g_1 t_1 g_1 t_1,$$
$$g_i g_j = g_j g_i \quad \text{if } |i-j| > 1,$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1},$$
$$g_i^2 = (q-1)g_i + q \text{ for all } i\},$$
$$t_1^2 = (Q-1)t_1 + Q\rangle$$

and Q may be equal to q.

This work was also done through the representation π, where $\pi(\sigma_i) = g_i$ and $\pi(t) = t_1$. $\sqrt{\lambda}$ is also computed, as did Jones, to satisfy $tr(\sqrt{\lambda}\,\pi(\sigma_i)) = tr(\sqrt{\lambda}^{-1}\pi(\sigma_i^{-1}))$. Thus the modified representation π_λ is defined as $\pi_\lambda(\sigma_i) = \sqrt{\lambda} g_i$. t_i and t_i' are defined by $t_i = g_{i-1} \cdots g_1 t_1 g_1 \cdots g_{i-1}$ and $t_i' = g_{i-1} \cdots g_1 t_1 g_1^{-1} \cdots g_{i-1}^{-1}$, respectively. In Figure 4, there are illustrated two braids, the images of which under π are t_3, and t_3', respectively, in $H(B_n)$ and in the algebra $H_n(\frac{1}{2})$ that is introduced in chapter 2. This invariant through type-B Hecke algebra will be discussed again in section 1.3.

1.2. Markov moves

We review some background definitions and facts. A fibered knot or link in S^3 is a collection of disjointly embedded circles $L = L_1 \cup L_2 \cup \cdots \cup L_k$ such that $S^3 - L$ is the total space of a fiber bundle over S^1, and the meridians map to the S^1 by degree 1 maps (see [R]). The unknot is the simplest fibered knot in S^3 and its fiber is a disk D^2. This unknot is called an axis and a closed braid in S^3 is defined relative to this axis.

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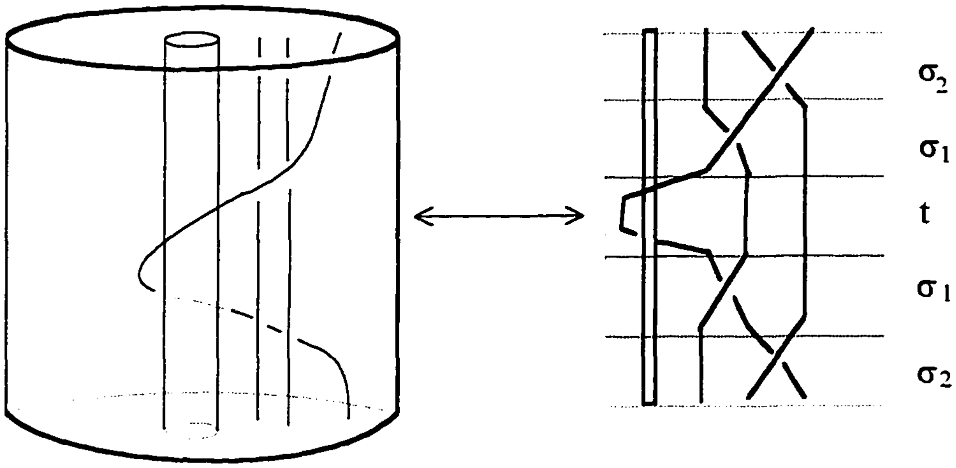
A.



B.

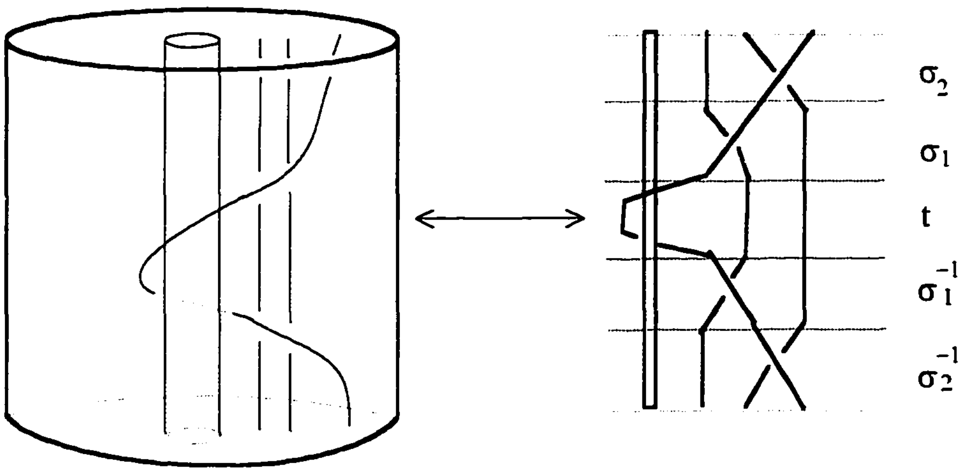


Figure 4. (A.) $t_3 = \pi(\sigma_2\sigma_1 t\sigma_1\sigma_2)$ (B.) $t_3' = \pi(\sigma_2\sigma_1 t\sigma_1^{-1}\sigma_2^{-1})$

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Alexander showed that every oriented link in S^3 is isotopic to a closed braid. This closed braid is never unique and the exact non-uniqueness is explained by the Markov Theorem. It says two closed braids are equivalent as oriented links if and only if one closed braid may be deformed to the other through horizontal, $\mathcal{H}$, and stabilizing, $\mathcal{W}$, deformations, which reflect Reidemeister moves II and I, respectively. A complete proof of the Markov Theorem was published by Birman (see [B]).

A link in M^3 is a closed braid in the braid group of the fiber if it is transverse to each fiber and its orientation agrees with the transverse orientation of the fiber. The height of a piecewise transverse link L with decomposition $s_1, \dots, s_k$ is the number of negative oriented segments. In other words a piecewise transverse link of height zero is a closed braid.

Using the known fact that every closed 3-manifold contains a fibered knot or link (see [A]), Skora showed the generalization of the Alexander Theorem that every link in any closed 3-manifold is isotopic to a closed braid for a fixed fibered knot or link which is called an axis which we denote A (see [S]). Let D be a disk in M and let β, β' be arcs in ∂D with disjoint interior and union equal to ∂D . Let L, L' be links in M. If $\beta = L \cap D$, $\beta' = L' \cap D$ and $L' = (L - \beta) \cup \beta'$, then say L' is obtained from L by an elementary deformation through D. Two links are combinatorially equivalent if there is a sequence of links $L = L_0, L_1, \dots, L_n = L'$ such that for each k, L_{k+1} is obtained from L_k by an elementary deformation.

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Skora also proved a generalization of the Markov Theorem. Let L, L' be closed braids. Then L, L' are equivalent if and only if there is a sequence of piecewise transverse links $L = L_0, L_1, \ldots, L_k = L'$, where each L_{i+1} results from L_i by an $\mathcal{H}$ or $\mathcal{W}$ deformation. An $\mathcal{H}$ deformation corresponds to genuine conjugation. A $\mathcal{W}$ deformation is defined as follows. Let L, L' be piecewise transverse links. Suppose a disk D meets the axis A transversely in one point and it meets each fiber transversely except exactly one fiber it meets in a saddle. If L' results from L by an elementary deformation through D where $\partial D = s \cup s'$, $L \cap D = s$, $L' \cap D = s'$, then say L' results from L by a $\mathcal{W}$ deformation through D. The Markov theorem is restated in terms of braid representative β of a closed braid $\hat{\beta}$ as a link (see [B]) as follows:

Theorem 1.2.1. *Let $\hat{\beta}$ and $\hat{\beta}'$ be two closed braids, with braid representatives β, β'. Then $\hat{\beta}$ is combinatorially equivalent to $\hat{\beta}'$ if and only if there is a deformation chain $\beta = \beta_1 \to \cdots \to \beta_s = \beta'$ such that each braid β_{i+1} in the chain can be obtained from $\beta_i \in B_n$ with n strings by the following moves:*

$\mathcal{H}: \beta_i \mapsto \alpha^{-1}\beta_i\alpha$ where α is a braid word in the same braid group;

$\mathcal{W}: \beta_i \mapsto \beta_i\sigma_n^{\pm 1}$ where $\beta_i \in B_n$, $\sigma_n \in B_{n+1}$.

A closed braid in a solid torus $S^1 \times D^2$ can be viewed as follows ([HK], [LA]). In S^3, the axis A is an unknot, and its fiber is a disk D^2. A 1-trivial dicromatic link in S^3 is a link colored with two colors $\{1, 2\}$ where the color 1 is used only to color a single unknotted component and the color 2 is used to

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color all the remaining of the link. Placing the unknotted component meeting every fiber of the axis A transversely in S^3, we can obtain a dicromatic closed braid. Then, we obtain a solid torus by removing a tubular neighborhood of the unknotted component of color 1 from S^3. The remaining closed braid with color 2 inside the solid torus can be viewed as a monocromatic closed braid. Thus, a closed braid in a solid torus is defined relative to the axis A, which serves for S^3 and which is a longitude with framing zero in the solid torus.(See Figure 4. and 5.) Therefore, the Markov moves for a solid torus are the same as for S^3.

1.3 Invariants through Hecke algebras

In general, a Hecke algebra is associated with each type of the Weyl group. It is known that the type-B Weyl group W is generated by $\omega_0, \omega_1, \ldots, \omega_l$ and the relations:

(1) $\omega_i^2 = 1$,

(2) $\omega_i \omega_j = \omega_j \omega_i$ if $|i-j| > 1$,

(3) $\omega_i \omega_{i+1} \omega_i = \omega_{i+1} \omega_i \omega_{i+1}$ for $i = 1, \ldots, l-1$

(4) $\omega_0 \omega_1 \omega_0 \omega_1 = \omega_1 \omega_0 \omega_1 \omega_0$.

With $H(B_n)$ introduced in section 1.1, if we define ϕ by $\phi(\omega_i) = g_i$, $\phi(\omega_0) = t_1$, then the images of the reduced words, $t_1^{\varepsilon_1} t_2^{\varepsilon_2} \cdots t_n^{\varepsilon_n} \alpha$ where $\varepsilon_i = 0$ or 1 and α is a normal form of type-A, under ϕ form a basis of the type-B Hecke algebra $H(B_n)$.

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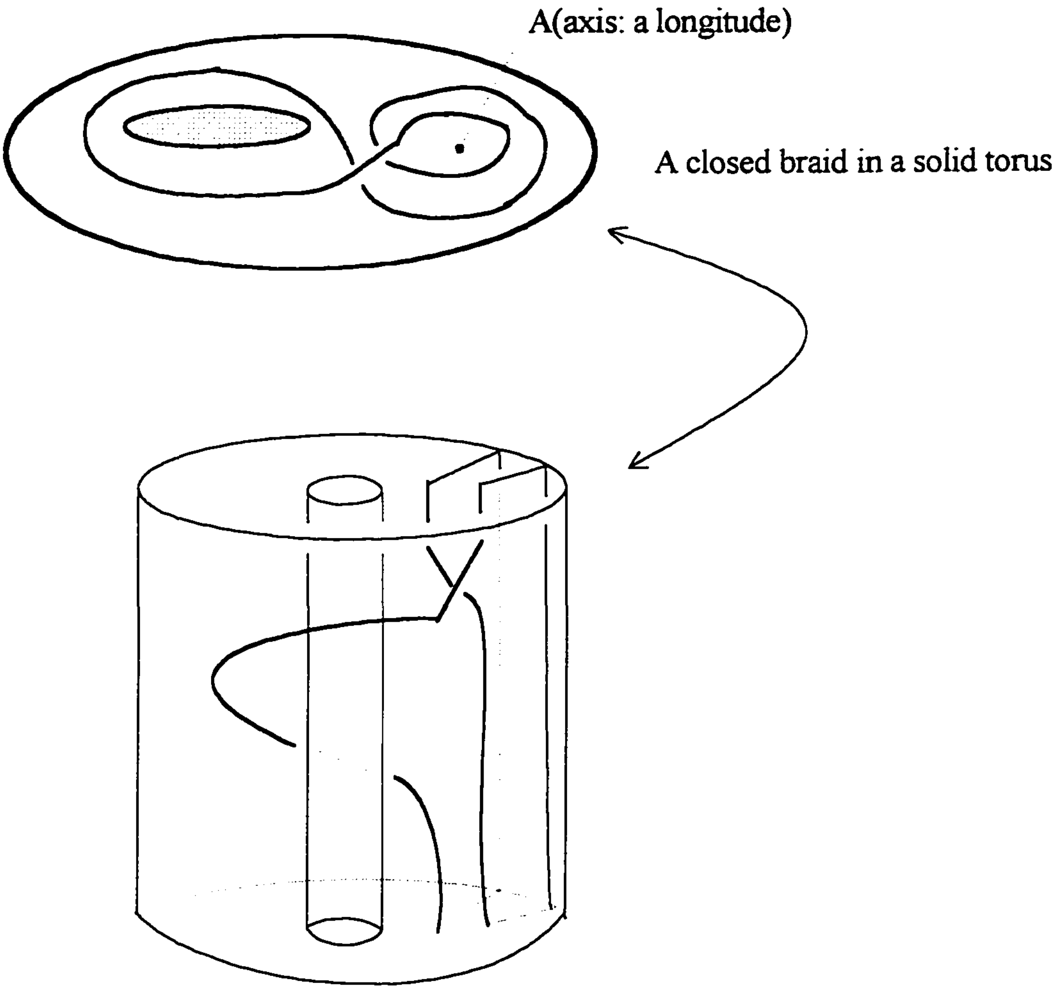

Figure 5. A closed braid in a solid torus

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We restrict the general case of the theorem to the type-B case.

Theorem 1.3.1. *(N. Iwahori[I]) Let W be the Weyl group for B_n, then*

(i) $g(\omega)$, $\omega \in W$ form a basis of the free $\mathbb{Z}$- module $H(B_n)$, the type-B Hecke algebra;

(ii) if $\omega = \omega_{i_1} \cdots \omega_{i_r}$ is a reduced expression for $\omega \in W$ then

$g(\omega) = g_{i_1} \cdots g_{i_r}$.

Therefore, the rank of the free $\mathbb{Z}$-module $H(B_n)$ is equal to the cardinal number of the Weyl group W by part (i) of the theorem above. The type-B Weyl group is known to be isomorphic to the semidirect product $\mathbb{Z}_2^n \rtimes \Sigma_n$, Σ_n the symmetric group.

Let $t_i = \omega_{i-1}\omega_{i-2}\cdots\omega_1\omega_0\omega_1\cdots\omega_{i-2}\omega_{i-1}$. Then $t_i^2 = 1$ and $t_it_j = t_jt_i$, so $\langle t_1, \cdots t_n\rangle$ is isomorphic to $\mathbb{Z}_2^n$. Since the type-A Weyl group is generated by $\{w_1, \ldots, w_{n-1}\}$ together with the relations $\{w_i^2 = 1,\ \omega_i\omega_j = \omega_j\omega_i$ if $|i-j| > 1$, $\omega_i\omega_{i+1}\omega_i = \omega_{i+1}\omega_i\omega_{i+1}$ for $i = 1, \ldots, n-2\}$ and it is isomorphic to the symmetric group Σ_n, a word ω in type-A Weyl group can be used as a word in Σ_n. One can also check that $\omega_it_i\omega_i = t_{i+1}$, $\omega_it_{i+1}\omega_i = t_i$, and $\omega_jt_i\omega_j = t_i$ if $j \neq i, i+1$. Thus $W \cong \mathbb{Z}_2^n \rtimes \Sigma_n$. A word in the semidirect product $\mathbb{Z}_2^n \rtimes \Sigma_n$ can be written as $t_1^{\varepsilon_1}t_2^{\varepsilon_2}\cdots t_n^{\varepsilon_n}\ \omega$, $\varepsilon_i = 0$ or 1, $t_it_j = t_jt_i$ for all i, j. Then $\omega \in \Sigma_n$ acts on $\mathbb{Z}_2^n$ as $\omega(\alpha) = \omega\alpha\omega^{-1}$ in $\mathbb{Z}_2^n$ where $\alpha \in \mathbb{Z}_2^n$.

Sofia S.F. Lambropoulou defined an analogue to the HOMFLY-PT polynomial for the links in solid torus. By representing the braid group of the annulus into type-B Hecke algebra, $H(B_n)$ (or $H_n(q, Q)$) in the

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following theorem, the invariant contains one more variable τ representing the longitude. The representation π of the braid of annulus into $H_n(q,Q)$ is defined by $\pi(t) = t_1$, $\pi(\sigma_i) = g_i$.

The unique trace function X_B was defined as follows.

Theorem 1.3.2. *(S. Lambropoulou [LA]) Given z and s in C, there exists a unique linear function $X_B : H := \cup_{n=1}^{\infty} H_n(q,Q) \to C$ such that the following hold:*

(1) $X_B(ab) = X_B(ba)$, $a, b \in H$

(2) $X_B(1) = 1$ for all $H_n(q,Q)$

(3) $X_B(ag_n) = z\, X_B(a)$, $a \in H_n(q,Q)$

(4) $X_B(at'_n) = \tau\, X_B(a)$, $a \in H_{n-1}(q,Q)$ where

$$t'_n = g_{n-1} \cdots g_1 t_1 g_1^{-1} \cdots g_{n-1}^{-1}$$

Then a 4-variable invariant $I(q,Q,\lambda,\tau)$ was defined as:

$$I_\alpha = I(q,Q,\lambda,\tau) = \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1} (\sqrt{\lambda})^e X_B(\pi(\alpha))$$

where e is the exponent sum of the g_is that appear in α. Thus, an invariant I was defined that satisfies the skein relations:

$\frac{1}{\sqrt{q}\sqrt{\lambda}} I_{L_+} - \sqrt{q}\sqrt{\lambda} I_{L_-} = (\sqrt{q} - \frac{1}{\sqrt{q}}) I_{L_0}$ and $\frac{1}{\sqrt{Q}} I_{L'_+} - \sqrt{Q} I_{L'_-} = (\sqrt{Q} - \frac{1}{\sqrt{Q}}) I_{L'_0}$
where $a\sigma_i^2 b$ is a braid presentation for L_+, ab for L_-, $a\sigma_i b$ for L_0, atb for L'_+, $at^{-1}b$ for L'_-, and ab for L'_0.

Skein module $S_3(M)$ has been computed for $M = S^3$, $M = S^1 \times D^2$, and $M = H_n$ a handlebody (see [HP2], [P]). For $M = S^1 \times D^2$, links in a solid

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torus was interpreted as the second colored components where the first one is a single unknotted component in dicromatic links (see [HK]). The following theorem from [HK] is restated in [P].

Theorem 1.3.3. *([HK]) $S_3(S^1 \times D^2)$ is a free $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ module with basis consisting of a trivial circle and families of layered torus links of type $(k,1)$, $k \neq 0$ satisfying the following properties: $v^{-1}L_+ - vL_- = zL_0$.*

In chapter 2, we define an invariant of links in a solid torus through an algebra $H_n(\frac{1}{2})$ (we adopted the notation for the algebra in a private communication from J. H. Przytycki). Our approach to the invariant is similar to the ones by Jones and Lambropoulou. Our invariant is an invariant in $C(q, \sqrt{\lambda})[\cdots, \tau_{-2}, \tau_{-1}, \tau_1, \tau_2, \cdots]$ while Lambropoulou's is in $C(q, Q, \sqrt{\lambda})[\tau]$. In chapter 3, we recover the invariant of links in S^3 by Jones and the invariant of links in the solid torus by S.F Lambropoulou from this invariant.

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CHAPTER 2. A POLYNOMIAL INVARIANT OF LINKS IN A SOLID TORUS

Here, we define an algebra $H_n(\frac{1}{2})$ with a trace map. Using the trace, we define a polynomial invariant of links in $S^1 \times D^2$.

2.1. An algebra $H_n(\frac{1}{2})$ as a vector space

$H_n(\frac{1}{2})$ is an algebra with a presentation:

$$
\begin{aligned}
H_n(\tfrac{1}{2}) = \langle t_1, t_1^{-1}, g_1, g_2, \cdots, g_{n-1} \mid\ & t_1 t_1^{-1} = t_1^{-1} t_1 = 1, \\
& t_1 g_1 t_1 g_1 = g_1 t_1 g_1 t_1, \\
& g_i g_j = g_j g_i \text{ if } |i-j| > 1, \\
& g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \text{ for } i = 1, \cdots, n-2, \\
& g_i^2 = (q-1) g_i + q \text{ for } i = 1 \cdots, n-1 \rangle.
\end{aligned}
$$

$H_n(\frac{1}{2})$ is an algebra over the ring $k[q, q^{-1}]$ where k is a commutative ring with 1. We obtain $H_n(\frac{1}{2})$ through an exact sequence,

$$0 \to T_1 \to H_n(\tfrac{1}{2}) \to H(B_{n-1}) \to 0$$

where T_1 is the ideal generated by $\{t \mid t = t_1^2 - (Q-1)t_1 - Q\}$. Let p be the map from $H_n(\frac{1}{2})$ to $H(B_{n-1})$, then $T_1 = ker(p)$. Furthermore, $\pi : B_n \to H_n(\frac{1}{2})$ defined by $\pi(\sigma_i) = g_i\ \forall i$, and $\pi(t) = t_1$ is a representation of the braid group of the annulus into $H_n(\frac{1}{2})$.

Note that $g_i^{-1} \in H_n(\frac{1}{2})$ as $g_i^{-1} = q^{-1} g_i + (q^{-1} - 1)$ from the relation $g_i^2 = (q-1)g_i + q$. In this section, we define and show a basis for $H_n(\frac{1}{2})$ as an infinite dimensional vector space.

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Definition 2.1.1. *We define* t_i^k *by* $t_i^k = (g_{i-1}g_{i-2}\cdots g_1t_1g_1g_2\cdots g_{i-2}g_{i-1})^k$, *for* $i = 1, \cdots, n$, *and* $(t_i')^h$ *by* $(t_i')^h = g_{i-1}g_{i-2}\cdots g_1t_1^hg_1^{-1}g_2^{-1}\cdots g_{i-2}^{-1}g_{i-1}^{-1}$, *for* $i = 1, \cdots, n$.

Throughout the sections 2.1 and 2.2, we will make use of the following formulas which are derived from the relations in $H_n(\frac{1}{2})$.

Lemma 2.1.2. *For arbitrary* n, i, $j \leq n$ *and* m, k, $s \in \mathbb{Z}$,

$(a1)$ $t_it_j = t_jt_i$ *for any* i, j;

$(a1')$ $t_1^mg_1t_1g_1 = g_1t_1g_1t_1^m$;

$(a1'')$ $g_1t_1^mg_1t_1 = g_1^2t_1g_1t_1^mg_1^{-1}$;

$(a2)$ $t_1g_1t_1g_1^{-1} = g_1^{-1}t_1g_1t_1$;

$(a3)$ $g_1t_1^kg_1t_1^sg_1^{-1} = (q-1)\,g_1t_1^{k-1}g_1^{-1}t_1^{s+1} - (q-1)\,g_1t_1^{k+s-1}g_1^{-1}t_1$ $+ g_1t_1^{k-1}g_1t_1^sg_1^{-1}t_1$, *which is a recursive formula;*

and

$$(a3')\quad g_1t_1^kg_1t_1^sg_1^{-1} = \left\{\begin{array}{c} (q-1)\sum_{i=1}^{k-1} g_1t_1^{k-i}g_1^{-1}t_1^{s+i} \\ +\quad t_1^sg_1t_1^k \quad + \\ (1-q)\sum_{i=1}^{k-1} g_1t_1^{k+s-i}g_1^{-1}t_1^i \end{array}\right\}$$

Proof:

$(a1)$: We see $t_it_j = t_jt_i$ for any i, j as following. Assume $i < j$, then either $i = j-1$ or $i < j-1$. Since $t_it_j = t_ig_{j-1}t_{j-1}g_{j-1} = g_{j-1}t_it_{j-1}g_{j-1}$ for $i < j-1$, $t_it_j = t_jt_i$ if $t_it_{i+1} = t_{i+1}t_i$. For $i = 1$, $t_1t_2 = t_1g_1t_1g_1 = g_1t_1g_1t_1 = t_2t_1$ by the relation in $H_n(\frac{1}{2})$.

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Assuming it is true for $1, 2, \cdots i-1$,

$$t_i t_{i+1} = g_{i-1} t_{i-1} g_{i-1} g_i g_{i-1} t_{i-1} g_{i-1} g_i$$

$= g_{i-1} t_{i-1} g_i g_{i-1} g_i t_{i-1} g_{i-1} g_i$, since $g_{i-1} g_i g_{i-1} = g_i g_{i-1} g_i$,

$= g_{i-1} g_i t_{i-1} g_{i-1} t_{i-1} g_i g_{i-1} g_i$, since $t_{i-1} g_i = g_i t_{i-1}$,

$= g_{i-1} g_i t_{i-1} g_{i-1} t_{i-1} g_{i-1} g_i g_{i-1}$

$= g_{i-1} g_i t_{i-1} t_i g_i g_{i-1}$

$= g_{i-1} g_i t_i g_i t_{i-1} g_{i-1}$ from induction hypothesis, and $t_{i-1} g_i = g_i t_{i-1}$,

$= g_{i-1} t_{i+1} t_{i-1} g_{i-1}$, since $g_{i-1} t_{i+1} = t_{i+1} g_{i-1}$

$= t_{i+1} g_{i-1} t_{i-1} g_{i-1}$

$= t_{i+1} t_i$, by definition of t_i.

$(a1')$: It holds by $(a1)$ since $t_1^m g_1 t_1 g_1 = t_1^m t_2$, and

$g_1 t_1 g_1 t_1^m = t_2 t_1^m$ by definition 2.1.1.

$(a1'')$: $g_1 t_1^m g_1 t_1 = g_1 t_1^m g_1 t_1 g_1 g_1^{-1}$

$= g_1^2 t_1 g_1 t_1^m g_1^{-1}$, by $(a1')$.

$(a2)$: $t_1 g_1 t_1 g_1^{-1} = g_1^{-1} t_1 g_1 t_1$;

$$\begin{aligned} t_1 g_1 t_1 g_1^{-1} &= q^{-1} t_1 g_1 t_1 g_1 + (q^{-1} - 1) t_1 g_1 t_1 \\ &= q^{-1} g_1 t_1 g_1 t_1 + (q^{-1} - 1) t_1 g_1 t_1 \\ &= [q^{-1} g_1 + (q^{-1} - 1)] t_1 g_1 t_1 \\ &= g_1^{-1} t_1 g_1 t_1. \end{aligned}$$

$(a3)$:

$g_1 t_1^k g_1 t_1^s g_1^{-1} = g_1 t_1^{k-1} t_1 g_1 t_1^s g_1^{-1}$

$= g_1 t_1^{k-1} g_1^{-1} g_1 t_1 g_1 t_1^s g_1^{-1}$

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$= g_1 t_1^{k-1} g_1^{-1} t_1^s g_1 t_1 g_1 g_1^{-1}$, by $(a1')$,

$= g_1 t_1^{k-1} g_1^{-1} t_1^s g_1 t_1$

$= q^{-1}\, g_1 t_1^{k-1} g_1 t_1^s g_1 t_1 + (q^{-1} - 1)\, g_1 t_1^{k-1} t_1^s g_1 t_1$ since $g_1^{-1} = q^{-1} g_1 + (q^{-1} - 1)$,

$= q^{-1}(q-1)\, g_1 t_1^{k-1} g_1 t_1^s t_1 + q^{-1} q\, g_1 t_1^{k-1} g_1 t_1^s g_1^{-1} t_1$

$+(q^{-1} - 1)(q-1)\, g_1 t_1^{k-1} t_1^s t_1 + (q^{-1} - 1) q\, g_1 t_1^{k-1} t_1^s g_1^{-1} t_1$

since $g_1 = (q-1) + q g_1^{-1}$,

$= q^{-1}(q-1)^2\, g_1 t_1^{k-1} t_1^s t_1 + q^{-1}(q-1) q\, g_1 t_1^{k-1} g_1^{-1} t_1^s t_1$

$+q^{-1} q\, g_1 t_1^{k-1} g_1 t_1^s g_1^{-1} t_1 + (q^{-1} - 1)(q-1)\, g_1 t_1^{k-1} t_1^s t_1$

$+(q^{-1} - 1) q\, g_1 t_1^{k-1} t_1^s g_1^{-1} t_1$

since $g_1 = (q-1) + q g_1^{-1}$,

$= (q-1)\, g_1 t_1^{k-1} g_1^{-1} t_1^{s+1} - (q-1)\, g_1 t_1^{k+s-1} g_1^{-1} t_1$

$+ g_1 t_1^{k-1} g_1 t_1^s g_1^{-1} t_1$

$(a3')$: Applying $(a3)$ repeatedly to the last part of the recursive formula $(a3)$ itself , we obtain

$g_1 t_1^k g_1 t_1^s g_1^{-1}$

$= (q-1)\, g_1 t_1^{k-1} g_1^{-1} t_1^{s+1} - (q-1)\, g_1 t_1^{k+s-1} g_1^{-1} t_1$

$+(q-1)\, g_1 t_1^{k-2} g_1^{-1} t_1^{s+2} - (q-1)\, g_1 t_1^{k+s-2} g_1^{-1} t_1^2 + g_1 t_1^{k-2} g_1 t_1^s g_1^{-1} t_1^2$

$+\cdots$

$= (q-1)\, g_1 t_1^{k-1} g_1^{-1} t_1^{s+1} - (q-1)\, g_1 t_1^{k+s-1} g_1^{-1} t_1$

$+(q-1)\, g_1 t_1^{k-2} g_1^{-1} t_1^{s+2} - (q-1)\, g_1 t_1^{k+s-2} g_1^{-1} t_1^2$

$+(q-1)\, g_1 t_1^{k-3} g_1^{-1} t_1^{s+3} - (q-1)\, g_1 t_1^{k+s-3} g_1^{-1} t_1^3$

$+\cdots$

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$$+(q-1)\, g_1 t_1^{k-(k-1)} g_1^{-1} t_1^{s+(k-1)} - (q-1)\, g_1 t_1^{k+s-(k-1)} g_1^{-1} t_1^{(} k-1)$$

$$+g_1 t_1^1 g_1 t_1^s g_1^{-1} t_1^{k-1}$$

$$= \left\{ \begin{array}{c} (q-1) \sum_{i=1}^{k-1} g_1 t_1^{k-i} g_1^{-1} t_1^{s+i} \\ + \quad t_1^s g_1 t_1^k \quad + \\ (1-q) \sum_{i=1}^{k-1} g_1 t_1^{k+s-i} g_1^{-1} t_1^{i} \end{array} \right\}$$

since $g_1 t_1 g_1 t_1^s g_1^{-1} t_1^{k-1} = t_1^s g_1 t_1 g_1 g_1^{-1} t_1^{k-1} = t_1^s g_1 t_1^k$. □

Lemma 2.1.3. *For arbitrary n,*

$(b1)$ $(g_{n-1} \cdots g_j) g_i^{\pm 1} = g_{i-1}^{\pm 1} (g_{n-1} \cdots g_j)$ *for* $j+1 \le i \le n-1$.

$(b1')$ $(g_{n-1}^{-1} \cdots g_j^{-1}) g_i^{\pm 1} = g_{i-1}^{\pm 1} (g_{n-1}^{-1} \cdots g_j^{-1})$ *for* $j+1 \le i \le n-1$.

$(b2)$ $(g_1 \cdots g_{n-1}) g_i^{\pm 1} = g_{i+1}^{\pm 1} (g_1 \cdots g_{n-1})$ *for* $1 \le i \le n-2$.

$(b2')$ $(g_1^{-1} \cdots g_{n-1}^{-1}) g_i^{\pm 1} = g_{i+1}^{\pm 1} (g_1^{-1} \cdots g_{n-1}^{-1})$ *for* $1 \le i \le n-2$.

$(b3)$ $g_i (g_{i+1}^{-1} \cdots g_{n-1}^{-1})(g_i^{-1} \cdots g_{n-2}^{-1}) = (g_{i+1}^{-1} \cdots g_{n-1}^{-1})(g_i^{-1} \cdots g_{n-2}^{-1}) g_{n-1}$

for $1 \le i \le n-2$.

Proof:

The formulas are derived from the relations: in $H_n(\frac{1}{2})$,

$g_i g_j = g_j g_i$ if $|i-j| > 1$;

$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for $i = 1, \cdots, n-2$.

Also, we have $g_i g_{i+1} g_i^{-1} = g_{i+1}^{-1} g_i g_{i+1}$ since $g_i g_{i+1} g_i^{-1} =$

$q^{-1} g_i g_{i+1} g_i + (q^{-1} - 1) g_i g_{i+1} = q^{-1} g_{i+1} g_i g_{i+1} + (q^{-1} - 1) g_i g_{i+1}$

$= [q^{-1} g_{i+1} + (q^{-1} - 1)] g_i g_{i+1} = g_{i+1}^{-1} g_i g_{i+1}$.

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Here, we will show only $(b1)$, and $(b3)$. The proofs for the others are similar.

$(b1)$: $(g_{n-1}\cdots g_j)g_i^{\pm 1} = g_{n-1}\cdots g_{i+1}g_ig_{i-1}g_i^{\pm 1}g_{i-2}\cdots g_j$

$= g_{n-1}\cdots g_{i+1}g_{i-1}^{\pm 1}g_ig_{i-1}g_{i-2}\cdots g_j$

$= g_{i-1}^{\pm 1}g_{n-1}\cdots g_{i+1}g_i\cdots g_j = g_{i-1}^{\pm 1}(g_{n-1}\cdots g_j)$

$(b3)$: $g_i(g_{i+1}^{-1}\cdots g_{n-1}^{-1})(g_i^{-1}\cdots g_{n-2}^{-1})$

$= g_ig_{i+1}^{-1}g_i^{-1}g_{i+2}^{-1}\cdots g_{n-1}^{-1}(g_{i+1}^{-1}\cdots g_{n-2}^{-1})$

$= g_{i+1}^{-1}g_i^{-1}g_{i+1}g_{i+2}^{-1}\cdots g_{n-1}^{-1}(g_{i+1}^{-1}\cdots g_{n-2}^{-1})$

$= g_{i+1}^{-1}g_i^{-1}g_{i+1}g_{i+2}^{-1}g_{i+1}^{-1}g_{i+3}^{-1}\cdots g_{n-1}^{-1}(g_{i+2}^{-1}\cdots g_{n-2}^{-1})$

$= g_{i+1}^{-1}g_i^{-1}g_{i+2}^{-1}g_{i+1}^{-1}g_{i+2}g_{i+3}^{-1}\cdots g_{n-1}^{-1}(g_{i+2}^{-1}\cdots g_{n-2}^{-1})$

$= (g_{i+1}^{-1}g_{i+2}^{-1})g_i^{-1}g_{i+1}^{-1}g_{i+2}g_{i+3}^{-1}\cdots g_{n-1}^{-1}(g_{i+2}^{-1}\cdots g_{n-2}^{-1})$

repeating this process,

$= (g_{i+1}^{-1}\cdots g_{n-1}^{-1})(g_i^{-1}\cdots g_{n-2}^{-1})g_{n-1}$ □

We will make use of the following formula for existence part of theorem 2.1.9, and theorem 2.2.2 in next section.

Lemma 2.1.4. *For $i \le n-1$, m positive*

(c1) $(t'_n)^m t'_i = t'_i(t'_n)^m$

$-(q^{-1}-1)(t'_i)^m(g_i^{-1}\cdots g_{n-2}^{-1})g_{n-1}(g_{n-2}\cdots g_1)t_1(g_1^{-1}\cdots g_{i-1}^{-1})$

$+(q^{-1}-1)(g_i^{-1}\cdots g_{n-2}^{-1})g_{n-1}(g_{n-2}\cdots g_1)t_1^{m+1}(g_1^{-1}\cdots g_{i-1}^{-1})$

(c2) $(t'_n)^m(t'_i)^{-1} = (t'_i)^{-1}(t'_n)^m$

$-(q^{-1}-1)(t'_i)^{-1}(g_i^{-1}\cdots g_{n-2}^{-1})g_{n-1}(g_{n-2}\cdots g_1)t_1^m(g_1^{-1}\cdots g_{i-1}^{-1})$

$+(q^{-1}-1)(t'_i)^{m-1}(g_i^{-1}\cdots g_{n-2}^{-1})(g_{n-1}\cdots g_i)$

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(c3) $(t'_n)^{-m}(t'_i) = (t'_i)(t'_n)^{-m}$

$-(q^{-1}-1)(t'_i)^{-m}(g_i^{-1}\cdots g_{n-2}^{-1})g_{n-1}(g_{n-2}\cdots g_1)t_1(g_1^{-1}\cdots g_{i-1}^{-1})$

$+(q^{-1}-1)(g_i^{-1}\cdots g_{n-2}^{-1})g_{n-1}(g_{n-2}\cdots g_1)t_1^{-m+1}(g_1^{-1}\cdots g_{i-1}^{-1}))$

(c4) $(t'_n)^{-m}(t'_i)^{-1} = (t'_i)^{-1}(t'_n)^{-m}$

$-(q^{-1}-1)(t'_i)^{-1}(g_i^{-1}\cdots g_{n-2}^{-1})g_{n-1}(g_{n-2}\cdots g_1)t_1^{-m}(g_1^{-1}\cdots g_{i-1}^{-1})$

$+(q^{-1}-1)(t'_i)^{-1-m}(g_i^{-1}\cdots g_{n-2}^{-1})g_{n-1}\cdots g_i.$

Proof:

Here is a computation for the case $(c1)$, and computations for the rest are similar. Commutativity relations are used without comment.

For $m \geq 1$, $\quad (t'_n)^m t'_i$

$= (g_{n-1}\cdots g_1)t_1^m(g_1^{-1}\cdots g_{n-1}^{-1})(g_{i-1}\cdots g_1 t_1 g_1^{-1}\cdots g_{i-1}^{-1})$

by the definition of t'_n,

$= (g_{n-1}\cdots g_1)t_1^m(g_1^{-1}\cdots g_i^{-1})(g_{i-1}\cdots g_1 t_1 g_1^{-1}\cdots g_{i-1}^{-1})(g_{i+1}^{-1}\cdots g_{n-1}^{-1})$

$= (g_{n-1}\cdots g_1)t_1^m(g_i\cdots g_2)(g_1^{-1}\cdots g_i^{-1})t_1(g_1^{-1}\cdots g_{i-1}^{-1})(g_{i+1}^{-1}\cdots g_{n-1}^{-1})$

by $(b2')$,

$= (g_{i-1}\cdots g_1)(g_{n-1}\cdots g_1)t_1^m(g_1^{-1}\cdots g_i^{-1})t_1(g_1^{-1}\cdots g_{i-1}^{-1}g_{i+1}^{-1}\cdots g_{n-1}^{-1})$

by repeated use of $(b1)$,

$= (g_{i-1}\cdots g_1)g_{n-1}\cdots g_2 g_1 t_1^m g_1^{-1} t_1(g_2^{-1}\cdots g_i^{-1})(g_1^{-1}\cdots g_{i-1}^{-1})(g_{i+1}^{-1}\cdots g_{n-1}^{-1})$

$= q^{-1}(g_{i-1}\cdots g_1)g_{n-1}\cdots g_2 g_1 t_1^m g_1 t_1(g_2^{-1}\cdots g_i^{-1})(g_1^{-1}\cdots g_{i-1}^{-1})(g_{i+1}^{-1}\cdots g_{n-1}^{-1})$

$+(q^{-1}-1)(g_{i-1}\cdots g_1)g_{n-1}\cdots g_2 g_1 t_1^{m+1}(g_2^{-1}\cdots g_i^{-1})$

$\cdot(g_1^{-1}\cdots g_{i-1}^{-1})(g_{i+1}^{-1}\cdots g_{n-1}^{-1})$

since $g_1^{-1} = q^{-1}g_1 + (q^{-1}-1)$,

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$= q^{-1}(g_{i-1}\cdots g_1)g_{n-1}\cdots g_2 g_1^2 t_1 g_1 t_1^m g_1^{-1}\cdots g_i^{-1}(g_1^{-1}\cdots g_{i-1}^{-1}g_{i+1}^{-1}\cdots g_{n-1}^{-1})$

$+(q^{-1}-1)(g_{i-1}\cdots g_1)(g_{n-1}\cdots g_2)g_1(g_2^{-1}\cdots g_i^{-1})t_1^{m+1}(g_1^{-1}\cdots g_{i-1}^{-1})$

$\cdot(g_{i+1}^{-1}\cdots g_{n-1}^{-1})$ by $(a1'')$,

$= -(q^{-1}-1)(g_{i-1}\cdots g_1)g_{n-1}\cdots g_1 t_1 g_1 t_1^m g_1^{-1}\cdots g_i^{-1}(g_1^{-1}\cdots g_{i-1}^{-1}g_{i+1}^{-1}\cdots g_{n-1}^{-1})$

$+(g_{i-1}\cdots g_1)(g_{n-1}\cdots g_2)t_1 g_1 t_1^m g_1^{-1}(g_2^{-1}\cdots g_i^{-1})(g_1^{-1}\cdots g_{i-1}^{-1})(g_{i+1}^{-1}\cdots g_{n-1}^{-1})$

$+(q^{-1}-1)(g_{n-1}\cdots g_2 g_1)(g_{i+1}^{-1}\cdots g_{n-1}^{-1})t_1^{m+1}(g_1^{-1}\cdots g_{i-1}^{-1})$

by expanding g_i^2, repeated use of $(b1)$, and cancellation,

$= -(q^{-1}-1)(g_{i-1}\cdots g_1)(g_{n-1}\cdots g_2)t_1^m g_1 t_1(g_2^{-1}\cdots g_i^{-1})$

$\cdot(g_1^{-1}\cdots g_{i-1}^{-1}g_{i+1}^{-1}\cdots g_{n-1}^{-1})$

$+(g_{i-1}\cdots g_1)t_1(g_{n-1}\cdots g_2)g_1 t_1^m(g_2^{-1}\cdots g_i^{-1})(g_1^{-1}\cdots g_{i-1}^{-1})g_i^{-1}(g_{i+1}^{-1}\cdots g_{n-1}^{-1})$

$+(q^{-1}-1)(g_{n-1}\cdots g_2 g_1)(g_{i+1}^{-1}\cdots g_{n-1}^{-1})t_1^{m+1}(g_1^{-1}\cdots g_{i-1}^{-1})$

by $(a1')$ in the first term and $(b3)$,

$= -(q^{-1}-1)(g_{i-1}\cdots g_1)t_1^m(g_1^{-1}\cdots g_{i-1}^{-1})g_{n-1}\cdots g_1 t_1(g_1^{-1}\cdots g_{i-1}^{-1}g_{i+1}^{-1}\cdots g_{n-1}^{-1})$

$+(g_{i-1}\cdots g_1)t_1(g_1^{-1}\cdots g_{i-1}^{-1})(g_{n-1}\cdots g_2)g_1 t_1^m(g_1^{-1}\cdots g_{i-1}^{-1})g_i^{-1}(g_{i+1}^{-1}\cdots g_{n-1}^{-1})$

$+(q^{-1}-1)(g_{n-1}\cdots g_2 g_1)(g_{i+1}^{-1}\cdots g_{n-1}^{-1})t_1^{m+1}(g_1^{-1}\cdots g_{i-1}^{-1})$

by repeated use of $(b1)$ in the first two terms,

$= -(q^{-1}-1)(t_i')^m(g_i^{-1}\cdots g_{n-2}^{-1})(g_{n-1})(g_{n-2}\cdots g_2 g_1)t_1(g_1^{-1}\cdots g_{i-1}^{-1})$

$+(t_i')(t_n')^m$

$+(q^{-1}-1)(g_i^{-1}\cdots g_{n-2}^{-1})(g_{n-1})(g_{n-2}\cdots g_2 g_1)t_1^{m+1}(g_1^{-1}\cdots g_{i-1}^{-1})$

by use of $(b1)$ and the definition of t_i'.

This completes the demonstration of formula $(c1)$.

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Theorem 2.1.5. *Let $\mathcal{H}$ be the set of elements of the form $(t_1)^{s_1}(t_2)^{s_2}\cdots(t_n)^{s_n}\alpha$, where $(t_i)^{s_i} = (g_{i-1}g_{i-2}\cdots g_1t_1g_1g_2\cdots g_{i-2}g_{i-1})^{s_i}$, for $i = 1,\cdots,n$, α is a word in normal form in $H(A_{n-1})$ and $s_i \in \mathbb{Z}$. Then $\mathcal{H}$ is a basis of $H_n(\frac{1}{2})$.*

Proof:

Let w be a word and let t_1 occur in w, then $w = \alpha_1 t_1^{\pm 1} w_1$ where $\alpha_1 = \sum_{i\in I}^{I:finite} \alpha_{2i}$ and α_{2i} is written in normal form of $H(A_{n-1})$, *i.e.*, $\alpha_{2i} = g_{i_1}g_2g_{i_2}\cdots g_{n-1}\cdots g_{i_{n-1}}$ where $1 \le i_k \le k$. So each term of w is

$\alpha_{2i}\, t_1^{\pm 1}w_1 = g_{i_1}g_2g_{i_2}\cdots g_kg_{k-1}\cdots g_1g_{k+1}\cdots g_{i_k}\cdots g_{n-1}\cdots g_{i_{n-1}}t_1^{\pm 1}w_1$.

If no g_1 occurs in α_{2i}, then $\alpha_{2i}t_1^{\pm}w_1 = t_1^{\pm}\alpha_{2i}w_1$. Suppose $i_k = 1,\ i_{k+l} \neq 1,\cdots, i_{n-1} \neq 1$ for some k. By the formula $g_h(g_kg_{k-1}\cdots g_1) = (g_kg_{k-1}\cdots g_1)g_{h+1}$ for $1 \le h < k$, by $b(1)$

$\alpha_{2i}t_1^{\pm 1}w_1 =$

$(g_k\cdots g_1)t_1^{\pm 1}(g_{i_1+1}g_3g_{i_2+1}\cdots g_k\cdots g_{i_{k-1}+1})(g_{k+1}\cdots g_{i_k}\cdots g_{n-1}\cdots g_{i_{n-1}})w_1$

$= (g_k\cdots g_1)t_1^{\pm 1}(g_1\cdots g_k)(g_k^{-1}\cdots g_1^{-1})w_2$ where

$w_2 = (g_{i_1+1}g_3g_{i_2+1}\cdots g_k\cdots g_{i_{k-1}+1})(g_{k+1}\cdots g_{i_k}\cdots g_{n-1}\cdots g_{i_{n-1}})w_1$,

$=t_{k+1}^{\pm 1}w_3$ where $w_3 = (g_k^{-1}\cdots g_1^{-1})w_2$. Let's denote this process (I).

For w_3, perform the process (I) as in w.

Eventually, we obtain sums of elements

$(t_{k_1})^{l_1}(t_{k_2})^{l_2}\cdots(t_{k_m})^{l_m}\gamma$ where $l_i \in \mathbb{Z}$, $k_i \in \{1,2,\cdots,n\}$ and $\gamma \in H(A_{n-1})$.

Up to this point we did not use that $t_1t_1^{-1} = 1$.

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Since $t_it_j = t_jt_i$ for all i, j by $(a1)$, the first part $(t_{k_1})^{l_1}(t_{k_2})^{l_2}\cdots(t_{k_m})^{l_m}$ can be rewritten easily as $(t_1)^{s_1}(t_2)^{s_2}\cdots(t_n)^{s_n}$ for some $s_i \in \mathbb{Z}$. □

For the linear independence of $\mathcal{H}$, we need some definitions and lemmas. Let $H_n = H(A_{n-1})$, $H_n(\frac{1}{2}) = \hat{H}_n$, and $\Lambda_n = k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. H_n is known as a free module over $k[q, q^{-1}]$. Λ_n is isomorphic to $k\mathbb{Z}^n$, so it is free. Therefore, the tensor product $\Lambda_n \otimes H_n$ is a free module with a basis $\{\tau \otimes h\}$ where τ is a base element of Λ and h is a base element of $H(A_{n-1})$.

Let $\Lambda_n \otimes H_n \xrightarrow{\phi} \hat{H}_n$ be $\phi(\tau \otimes h) = \tau h$ which is a $k[q, q^{-1}]$-module homomorphism. ϕ is surjective. We show the injectivity of ϕ using the following procedure modeled on the method for Hecke algebra (see [H]).
Let $\mathcal{E}_n = End_n(\Lambda_n \otimes H_n)$. We define $\chi : \hat{H}_n \to \mathcal{E}_n$ so that χ is an algebra homomorphism. There is a subalgebra $\mathcal{L}$ generated by
$\{G_i, T_i | G_i = \chi(g_i), T_i = \chi(t_i), \}$.

Let $\mathcal{L} \xrightarrow{\psi} \Lambda_n \otimes H_n$ be defined by $\psi(L) = L(1 \otimes 1)$. Then $\psi \circ \chi \circ \phi = id$ on $\Lambda_n \otimes H_n$. Thus, ϕ is one-to-one and $\{\tau h \mid \tau \in \Lambda, h \in H_n$ is in normal form $\}$ is linearly independent in $\hat{H}$. Here is the overall map:

$$\Lambda_n \otimes H_n \xrightarrow{\phi} \hat{H}_n \xrightarrow{\chi} \mathcal{E}_n \xrightarrow{\psi} \Lambda_n \otimes H_n.$$

First, we want to define and show χ is an algebra homomorphism. $\Lambda \otimes_k H_n$ is an algebra where multiplication is defined by $(a \otimes b) \cdot (c \otimes d) = ac \otimes bd$. The motivation for the following definition is the formulae for left multiplication by g_i and t_1 in $\hat{H}_n$, *i.e.*, if we replace the symbols G_i and T_1 by g_1 and t_1 and drop all $\otimes$ symbols, then we get true equations in $\hat{H}_n$.

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Definition 2.1.6. *Define $G_i \in End_k(\Lambda_n \otimes H_n)$ inductively as follows:*

$G_i(t_1^{s_1} t_2^{s_2} \cdots t_i^{s_i} t_{i+1}^{s_{i+1}} \cdots t_n^{s_n} \otimes \alpha)$

(1.) $= t_1^{s_1} t_2^{s_2} \cdots t_n^{s_n} \otimes g_i \alpha \quad$ *if* $s_i = s_{i+1} = 0$

(2.) $= (q-1) t_1^{s_1} t_2^{s_2} \cdots t_i^{s_i} t_{i+1}^{s_{i+1}} \cdots t_n^{s_n} \otimes \alpha$

$\quad + q(t_i \otimes 1) \cdot G_i(t_1^{s_1} \cdots t_{i+1}^{s_{i+1}-1} \cdots t_n^{s_n} \otimes \alpha) \quad$ *if* $s_i = 0,\ s_{i+1} > 0$

(3.) $= (q^{-1} - 1) t_1^{s_1} \cdots t_i^{s_i - 1} t_{i+1}^{s_{i+1}+1} \cdots t_n^{s_n} \otimes \alpha$

$\quad + q^{-1}(t_i^{-1} \otimes 1) \cdot G_i(t_1^{s_1} \cdots t_{i+1}^{s_{i+1}+1} \cdots t_n^{s_n} \otimes \alpha) \quad$ *if* $s_i = 0,\ s_{i+1} < 0$

(4.) $= (q^{-1} - 1) t_1^{s_1} \cdots t_i^{s_i - 1} t_{i+1}^{s_{i+1}+1} \cdots t_n^{s_n} \otimes \alpha$

$\quad + q^{-1}(t_{i+1} \otimes 1) \cdot G_i(t_1^{s_1} \cdots t_i^{s_i - 1} t_{i+1}^{s_{i+1}} \cdots t_n^{s_n} \otimes \alpha) \quad$ *if* $s_i > 0$

(5.) $= (q-1) t_1^{s_1} \cdots t_i^{s_i} t_{i+1}^{s_{i+1}} \cdots t_n^{s_n} \otimes \alpha$

$\quad + q(t_{i+1}^{-1} \otimes 1) \cdot G_i(t_1^{s_1} \cdots t_i^{s_i+1} t_{i+1}^{s_{i+1}} \cdots t_n^{s_n} \otimes \alpha) \quad$ *if* $s_i < 0$

(6.) Define T_1 *by* $T(t_1^{s_1} t_2^{s_2} \cdots t_n^{s_n} \otimes \alpha) = (t_1 \otimes 1)(t_1^{s_1} t_2^{s_2} \cdots t_n^{s_n} \otimes \alpha)$.

Define $\lambda_i, \mu_i : \Lambda \to \Lambda$ such that $G_i(\tau \otimes \alpha) = \lambda_i(\tau) \otimes g_i \alpha + \mu_i(\tau) \otimes \alpha$, then we know that $g_i \tau \alpha = \lambda_i(\tau) g_i \alpha + \mu_i(\tau) \alpha$. The above definition of G_i can be written as $G_i(t_1^{s_1} \cdots t_i^{s_i} t_{i+1}^{s_{i+1}} \cdots t_n^{s_n} \otimes \alpha)$

$= \lambda_i(t_1^{s_1} \cdots t_i^{s_i} t_{i+1}^{s_{i+1}} \cdots t_n^{s_n}) \otimes g_i \alpha + \mu_i(t_1^{s_1} \cdots t_i^{s_i} t_{i+1}^{s_{i+1}} \cdots t_n^{s_n}) \otimes \alpha$

We can, also, write

$\lambda_i(t_1^{s_1} \cdots t_n^{s_n}) = t_1^{s_1} \cdots \lambda_i(t_i^a t_{i+1}^b) \cdots t_n^{s_n}$

$\mu_i(t_1^{s_1} \cdots t_n^{s_n}) = t_1^{s_1} \cdots \mu_i(t_i^a t_{i+1}^b) \cdots t_n^{s_n}, \qquad$ since

$G_i(t_1^{s_1} \cdots t_n^{s_n} \otimes \alpha) = (t_1^{s_1} \cdots t_{i-1}^{s_{i-1}} t_{i+2}^{s_{i+2}} \cdots t_n^{s_n} \otimes 1) \cdot G_i(t_i^{s_i} t_{i+1}^{s_{i+1}} \otimes \alpha)$

following from the Definition 2.1.2.

Next, we will have inductive formula on the exponents of t_i, t_{i+1}.

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Lemma 2.1.7. *The following properties hold:*

(1) $T_1(t_1^a t_2^b \otimes \alpha) = t_1^{a+1} t_2^b \otimes \alpha;$

(2) $G_i(t_1^{s_1} \cdots t_{i-1}^{s_{i-1}} t_i^{a+1} t_{i+1}^b t_{i+2}^{s_{i+2}} \cdots t_n^{s_n} \otimes \alpha)$

$= q^{-1} t_{i+1} \lambda_i(t_1^{s_1} \cdots t_{i-1}^{s_{i-1}} t_i^a t_{i+1}^b t_{i+2}^{s_{i+2}} \cdots t_n^{s_n}) \otimes g_i \alpha$

$+ [q^{-1} t_{i+1} \mu_i(t_1^{s_1} \cdots t_{i-1}^{s_{i-1}} t_i^a t_{i+1}^b t_{i+2}^{s_{i+2}} \cdots t_n^{s_n})$

$+ (q^{-1} - 1) t_1^{s_1} \cdots t_{i-1}^{s_{i-1}} t_i^a t_{i+1}^{b+1} t_{i+2}^{s_{i+2}} \cdots t_n^{s_n}] \otimes \alpha;$

(3) $G_i(t_1^{s_1} \cdots t_{i-1}^{s_{i-1}} t_i^a t_{i+1}^{b+1} t_{i+2}^{s_{i+2}} \cdots t_n^{s_n} \otimes \alpha)$

$= q t_i \lambda_i(t_1^{s_1} \cdots t_{i-1}^{s_{i-1}} t_i^a t_{i+1}^b t_{i+2}^{s_{i+2}} \cdots t_n^{s_n}) \otimes g_i \alpha$

$+ [q t_i \mu_i(t_1^{s_1} \cdots t_{i-1}^{s_{i-1}} t_i^a t_{i+1}^b t_{i+2}^{s_{i+2}} \cdots t_n^{s_n})$

$+ (q - 1) t_1^{s_1} \cdots t_{i-1}^{s_{i-1}} t_i^a t_{i+1}^{b+1} t_{i+2}^{s_{i+2}} \cdots t_n^{s_n}] \otimes \alpha;$

(4) $G_i(t_1^{s_1} \cdots t_{i-1}^{s_{i-1}} t_i^a t_{i+1}^a t_{i+2}^{s_{i+2}} \cdots t_n^{s_n} \otimes \alpha) = t_1^{s_1} \cdots t_{i-1}^{s_{i-1}} t_i^a t_{i+1}^a t_{i+2}^{s_{i+2}} \cdots t_n^{s_n} \otimes g_i \alpha.$

Proof:

For T_1, it is trivial. As we see that G_i acts only on $t_i^{s_i} t_{i+1}^{s_{i+1}}$, for every i, G_i is just a copy of the case of $i = 1$ for $n = 2$. So we show the case of $i = 1$ for $n = 2$. Let $\lambda = \lambda_1, \mu = \mu_1$. Then, the cases (2), (3), and (4) are rewritten as follows:

$(2')$ $G_1(t_1^{a+1} t_2^b) \otimes \alpha) = q^{-1} t_2 \lambda(t_1^a t_2^b) \otimes g_1 \alpha + (q^{-1} t_2 \mu(t_1^a t_2^b) + (q^{-1} - 1) t_1^a t_2^{b+1}) \otimes \alpha$

$(3')$ $G_1(t_1^a t_2^{b+1} \otimes \alpha) = q t_1 \lambda(t_1^a t_2^b) \otimes g_1 \alpha + (q t_1 \mu(t_1^a t_2^b) + (q - 1) t_1^a t_2^{b+1}) \otimes \alpha$

$(4')$ $G_1(t_1^a t_2^a \otimes 1) = (t_1^a t_2^a \otimes 1) G_1(1 \otimes 1)$

Now, $G_1(t_1^{a+1} t_2^b \otimes \alpha) = \lambda(t_1^{a+1} t_2^b) \otimes g_1 \alpha + \mu(t_1^{a+1} t_2^b) \otimes \alpha.$

For $a \geq 0$, $G_1(t_1^{a+1} t_2^b \otimes \alpha)$

$= (q^{-1} - 1) t_1^a t_2^{b+1} \otimes \alpha + q^{-1}(t_2 \otimes 1) G_1(t_1^a t_2^b \otimes \alpha)$ by 2.1.6(4),

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$= (q^{-1}-1)t_1^at_2^{b+1} \otimes \alpha + q^{-1}(t_2 \otimes 1)(\lambda(t_1^at_2^b) \otimes g_1\alpha + \mu(t_1^at_2^b) \otimes \alpha)$

$= q^{-1}t_2\lambda(t_1^at_2^b) \otimes g_1\alpha + q^{-1}t_2\mu(t_1^at_2^b) \otimes \alpha + (q^{-1}-1)t_1^at_2^{b+1} \otimes \alpha.$

Thus, $\lambda(t_1^{a+1}t_2^b) = q^{-1}t_2\lambda(t_1^at_2^b)$ and

$\mu(t_1^{a+1}t_2^b) = q^{-1}t_2\mu(t_1^at_2^b) + (q^{-1}-1)t_1^at_2^{b+1}$ for $a \geq 0$.

Let $a < 0$, then

$G_1(t_1^at_2^b \otimes \alpha) = (q-1)t_1^at_2^b \otimes \alpha + q(t_2^{-1} \otimes 1)G_1(t_1^{a+1}t_2^b)$ by 2.1.6(5).

$\lambda(t_1^at_2^b) \otimes g_1\alpha + \mu(t_1^at_2^b) \otimes \alpha$

$= (q-1)t_1^at_2^b \otimes \alpha + qt_2^{-1}(\lambda(t_1^{a+1}t_2^b) \otimes g_1\alpha + \mu(t_1^{a+1}t_2^b) \otimes \alpha)$

$= (q-1)t_1^at_2^b \otimes \alpha + qt_2^{-1}\lambda(t_1^{a+1}t_2^b) \otimes g_1\alpha + qt_2^{-1}\mu(t_1^{a+1}t_2^b) \otimes \alpha.$

Thus, $\lambda(t_1^at_2^b) = qt_2^{-1}\lambda(t_1^{a+1}t_2^b)$, $\mu(t_1^at_2^b) = (q-1)t_1^at_2^b + qt_2^{-1}\mu(t_1^{a+1}t_2^b)$ which imply $\lambda(t_1^{a+1}t_2^b) = q^{-1}t_2\lambda(t_1^at_2^b)$, $\mu(t_1^{a+1}t_2^b) = (q^{-1}-1)t_1^at_2^{b+1} + q^{-1}t_2\mu(t_1^at_2^b)$, respectively.

We show the case (3) inductively.

Let $G_1(t_1^at_2^{b+1} \otimes \alpha) = \lambda(t_1^at_2^{b+1}) \otimes g_1\alpha + \mu(t_1^at_2^{b+1}) \otimes \alpha$.

$G_1(t_1^at_2^{b+1} \otimes \alpha)$, for $a = 0, b = 0$,

$= (q-1)t_1^at_2^{b+1} \otimes \alpha + q(t_1 \otimes 1)G_1(t_1^at_2^b \otimes \alpha)$ by 2.1.6(2),

$= (q-1)t_1^at_2^{b+1} \otimes \alpha + q(t_1 \otimes 1)(t_1^at_2^b \otimes g_1\alpha)$ by 2.1.6(1),

$= qt_1^{a+1}t_2^b \otimes g_1\alpha + [qt_1\mu(t_1^at_2^b) + (q-1)t_1^at_2^{b+1}] \otimes \alpha$

since $\mu(t_1^at_2^b) = 0$.

For the cases $a = 0, b > 0$, and $a = 0, b < 0$, similar arguments can be done through the definition 2.1.6. So, assume the stated formula is true for $1, 2, \cdots, a-1$, and $a > 0$.

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$G_1(t_1^a t_2^{b+1} \otimes \alpha)$

$= (q^{-1}-1)t_1^{a-1}t_2^{b+2} \otimes \alpha) + q^{-1}(t_2 \otimes 1)G_1(t_1^{a-1}t_2^{b+1} \otimes \alpha)$ by 2.1.6(4),

$= (q^{-1}-1)t_1^{a-1}t_2^{b+2} \otimes \alpha) + q^{-1}(t_2 \otimes 1)[qt_1\lambda(t_1^{a-1}t_2^b) \otimes g_1\alpha$

$+ (qt_1\mu(t_1^{a-1}t_2^b) + (q-1)t_1^{a-1}t_2^{b+1}) \otimes \alpha]$, by induction hypothesis,

$= t_1t_2\lambda(t_1^{a-1}t_2^b) \otimes g_1\alpha + t_1t_2\mu(t_1^{a-1}t_2^b) \otimes \alpha$ by cancellation,

$= qt_1\lambda(t_1^a t_2^b) \otimes g_1\alpha + [qt_1\mu(t_1^a t_2^b) + (q-1)t_1^a t_2^{b+1}] \otimes \alpha$

since, by case (2),

$\lambda(t_1^a t_2^b) = q^{-1}t_2\lambda(t_1^{a-1}t_2^b)$, and

$\mu(t_1^a t_2^b) = q^{-1}t_2\mu(t_1^{a-1}t_2^b) + (q^{-1}-1)t_1^{a-1}t_2^{b+1}$

give $t_2\lambda(t_1^{a-1}t_2^b) = q\lambda(t_1^a t_2^b)$, and

$t_2\mu(t_1^{a-1}t_2^b) = q\mu(t_1^a t_2^b) + (q-1)t_1^{a-1}t_2^{b+1}$, respectively.

Thus, $\lambda(t_1^a t_2^{b+1}) = qt_1\lambda(t_1^a t_2^b)$ and $\mu(t_1^a t_2^{b+1}) = qt_1\mu(t_1^a t_2^b) + (q-1)t_1^a t_2^{b+1}$.

A similar computation solves the $a < 0$ case.

For $a = b$, we use induction again.

Let $a = 0$, then $G_1(t_1^a t_2^a \otimes \alpha) = t_1^a t_2^a \otimes g_1\alpha$ by the definition 2.1.6(1).

Assume $a \geq 1$ and the case (4) is true for $1, 2, \cdots, a-1$.

$G_1(t_1^a t_2^a \otimes \alpha)$

$= q^{-1}t_2\lambda(t_1^{a-1}t_2^a) \otimes g_1\alpha + [q^{-1}t_2\mu(t_1^{a-1}t_2^a) + (q^{-1}-1)t_1^{a-1}t_2^{a+1}] \otimes \alpha$

bt case (2), $= q^{-1}t_2(qt_1\lambda(t_1^{a-1}t_2^{a-1})) \otimes g_1\alpha$

$+[q^{-1}t_2\{qt_1\mu(t_1^{a-1}t_2^{a-1}) + (q-1)t_1^{a-1}t_2^a\} + (q^{-1}-1)t_1^{a-1}t_2^{a+1}] \otimes \alpha$

by case (3),

$= t_1t_2t_1^{a-1}t_2^{a-1} \otimes g_1\alpha + [-(q^{-1}-1)t_1^{a-1}t_2^{a+1} + (q^{-1}-1)t_1^{a-1}t_2^{a+1}] \otimes \alpha$

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since $\lambda(t_1^{a-1}t_2^{a-1}) = t_1^{a-1}t_2^{a-1}$ and $\mu(t_1^{a-1}t_2^{a-1}) = 0$ by induction hypothesis,

$= t_1^a t_2^a \otimes g_1 \alpha$.

Thus, $\lambda(t_1^a t_2^a) = t_1^a t_2^a$ and $\mu(t_1^a t_2^a) = 0$. □

Remark:

To summarize cases (2) and (3), for any form A, $\lambda(t_1 A) = q^{-1}t_2\lambda(A)$ and $\lambda(t_2 A) = qt_1\lambda(A)$ since the power of t_1 in $t_1 A$ is one higher than that in A, and similarly for $\lambda(t_2 A)$).

$\mu(t_1 A) = q^{-1}t_2\mu(A) + (q^{-1} - 1)t_2 A$, and

$\mu(t_2 A) = qt_1\mu(A) + (q-1)t_2 A$, as well.

In fact, $\lambda(t_1^a t_2^b) = (q^{-1})^a t_2^a \lambda(t^b) = (q^{-1})^a t_2^a q^b t_1^b = q^{b-a} t_1^b t_2^a$.

Now, we have inductive formulas for $T_1 = \chi(t_1)$, $G_i = \chi(g_i)$ for all i.

Lemma 2.1.8. *χ is an algebra homomorphism, i.e. , the following relations hold:*

(1) $G_i^2 = (q-1)G_i + q$,

(2) $G_iG_j = G_jG_i$ if $|i-j| \geq 2$,

(3) $G_iG_{i+1}G_i = G_{i+1}G_iG_{i+1}$ for $i = 1, \ldots, n-2$, and

(4) $T_1G_1T_1G_1 = G_1T_1G_1T_1$.

Proof: Case 1.

Without loss of generality, we show $G_1^2 = (q-1)G_1 + q$. For general cases, we replace to get G_i the subscript index of G_1 by i. Let $\lambda_1 = \lambda, \mu_1 = \mu$.

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$G_1^2(t_1^a t_2^b \otimes \alpha)$

$= G_1(\lambda(t_1^a t_2^b) \otimes g_1\alpha + \mu(t_1^a t_2^b) \otimes \alpha)$

$= \lambda^2(t_1^a t_2^b) \otimes g_1^2\alpha + \mu\lambda(t_1^a t_2^b) \otimes g_1\alpha$

$+\lambda\mu(t_1^a t_2^b) \otimes g_1\alpha + \mu^2(t_1^a t_2^b) \otimes \alpha$

$= [(q-1)\lambda^2(t_1^a t_2^b) + \mu\lambda(t_1^a t_2^b) + \lambda\mu(t_1^a t_2^b)] \otimes g_1\alpha + [q\,\lambda^2(t_1^a t_2^b) + \mu^2(t_1^a t_2^b)] \otimes \alpha$

On the other hand, $((q-1)G_1 + q)(t_1^a t_2^b \otimes \alpha)$

$= (q-1)\lambda(t_1^a t_2^b) \otimes g_1\alpha + (q-1)\mu(t_1^a t_2^b) \otimes \alpha + qt_1^a t_2^b \otimes \alpha$

$= (q-1)\lambda(t_1^a t_2^b) \otimes g_1\alpha + [(q-1)\mu(t_1^a t_2^b) + qt_1^a t_2^b] \otimes \alpha$

Therefore, we need to check

$$(q-1)\lambda^2 + \mu\lambda + \lambda\mu = (q-1)\lambda$$

$$q\lambda^2 + \mu^2 = (q-1)\mu + q$$

We will check $(q-1)\lambda^2 + \mu\lambda + \lambda\mu - (q-1)\lambda = 0$

and $q\lambda^2 + \mu^2 - (q-1)\mu - q = 0$ by use of induction twice. Induction to show the formula holds for $t_1^{s_1} t_2^{s_2}$ is done by induction on $s_1 - s_2 \in N$ and $s_2 - s_1 \in N$. Both start with $t_1^a t_2^a$. We then check $t_1^{a+1} t_2^b$ and $t_1^a t_2^{b+1}$ assuming $t_1^a t_2^a$.

Consider $[(q-1)\lambda^2 + \mu\lambda + \lambda\mu - (q-1)\lambda](t_1^a t_2^a)$

$= (q-1)t_1^a t_2^a + 0 + 0 - (q-1)t_1^a t_2^a = 0$, by the case (4) in lemma 2.1.3.

And, $[q\lambda^2 + \mu^2 - (q-1)\mu + q](t_1^a t_2^a)$

$= qt_1^a t_2^a + 0 - 0 - qt_1^a t_2^a$

$= 0$

Suppose $((q-1)\lambda^2 + \mu\lambda + \lambda\mu - (q-1)\lambda)(t_1^a t_2^b) = 0$ for some $a, b \in \mathbb{Z}$. Then, using the formulas from lemma 2.1.7, (also, see the remark below the lemma),

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$[(q-1)\lambda^2+\mu\lambda+\lambda\mu-(q-1)\lambda](t_1^{a+1}t_2^b)$

$=((q-1)\lambda+\mu-(q-1))\{q^{-1}t_2\lambda(t_1^at_2^b)\}+\lambda\{q^{-1}t_2\mu(t_1^at_2^b)+(q^{-1}-1)t_1^at_2^{b+1}\}$

$=(q-1)q^{-1}qt_1\lambda^2(t_1^at_2^b)+q^{-1}qt_1\mu\lambda(t_1^at_2^b)+q^{-1}(q-1)t_2\lambda(t_1^at_2^b)$

$+(q^{-1}-1)t_2\lambda(t_1^at_2^b)+q^{-1}qt_1\lambda\mu(t_1^at_2^b)+(q^{-1}-1)qt_1\lambda(t_1^at_2^b)$

$=t_1((q-1)\lambda^2+\mu\lambda+\lambda\mu-(q-1)\lambda)(t_1^at_2^b)=0$

by the induction hypothesis.

Similarly, $[q\lambda^2+\mu^2-(q-1)\mu-q](t_1^{a+1}t_2^b)=0$

For $t_1^at_2^{b+1}$,

$[(q-1)\lambda^2+\mu\lambda+\lambda\mu-(q-1)\lambda](t_1^at_2^{b+1})$

$=((q-1)\lambda+\mu-(q-1))(qt_1\lambda(t_1^at_2^b))$

$+\lambda(qt_1\mu(t_1^at_2^b)+(q-1)t_1^at_2^{b+1})$

$=(q-1)t_2\lambda^2(t_1^at_2^b)+t_2\mu\lambda(t_1^at_2^b)$

$+q(q^{-1}-1)t_2\lambda(t_1^at_2^b)-(q-1)qt_1\lambda(t_1^at_2^b)$

$+t_2\lambda\mu(t_1^at_2^b)+(q-1)qt_1\lambda(t_1^at_2^b)$

$=[(q-1)t_2\lambda^2+t_2\mu\lambda-(q-1)t_2\lambda-(q-1)qt_1\lambda+t_2\lambda\mu+(q-1)qt_1\lambda](t_1^at_2^b)$

$=t_2[(q-1)\lambda^2+\mu\lambda+\lambda\mu-(q-1)\lambda](t_1^at_2^b)$

$=0$ by the induction hypothesis.

Similarly,

$[q\lambda^2+\mu^2-(q-1)\mu-q](t_1^at_2^{b+1})=t_2[q\lambda^2+\mu^2-(q-1)\mu-q](t_1^at_2^b)=0.$

Case 2.

Recall $\lambda_i(t_1^{s_1}\cdots t_n^{s_n})=t_1^{s_1}\cdots\lambda_i(t_i^at_{i+1}^b)\cdots t_n^{s_n}$, and

$\mu_i(t_1^{s_1}\cdots t_n^{s_n})=t_1^{s_1}\cdots\mu_i(t_i^at_{i+1}^b)\cdots t_n^{s_n}$. Thus, if $|i-j|\geq 2$,

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$$\begin{aligned}\lambda_i\lambda_j(t_1^{s_1}\cdots t_n^{s_n}) &= \lambda_i(t_1^{s_1}\cdots\lambda_j(t_j^a t_{j+1}^b)\cdots t_n^{s_n})\\ &= t_1^{s_1}\cdots\lambda_i(t_i^a t_{i+1}^b)\cdots\lambda_j(t_j^a t_{j+1}^b)\cdots t_n^{s_n}\\ &= \lambda_j(t_1^{s_1}\cdots\lambda_i(t_i^a t_{i+1}^b)\cdots t_n^{s_n}\\ &= \lambda_j\lambda_i(t_1^{s_1}\cdots t_n^{s_n}).\end{aligned}$$

Similarly, it is also shown that $\lambda_i\mu_j$, $\mu_i\lambda_j$, $\mu_i\mu_j$ commute.

Then, $G_iG_j(t_1^{s_1}\cdots t_n^{s_n}\otimes\alpha)$

$= \lambda_i\lambda_j(t_1^{s_1}\cdots t_n^{s_n})\otimes g_ig_j\alpha + \lambda_i\mu_j(t_1^{s_1}\cdots t_n^{s_n})\otimes g_i\alpha$

$+\mu_i\lambda_j(t_1^{s_1}\cdots t_n^{s_n})\otimes g_j\alpha + \mu_i\mu_j(t_1^{s_1}\cdots t_n^{s_n})\otimes\alpha$

$= \lambda_j\lambda_i(t_1^{s_1}\cdots t_n^{s_n})\otimes g_jg_i\alpha + \lambda_j\mu_i(t_1^{s_1}\cdots t_n^{s_n})\otimes g_j\alpha$

$+\mu_j\lambda_i(t_1^{s_1}\cdots t_n^{s_n})\otimes g_i\alpha + \mu_j\mu_i(t_1^{s_1}\cdots t_n^{s_n})\otimes\alpha$

$= G_jG_i(t_1^{s_1}\cdots t_n^{s_n}\otimes\alpha)$, as required.

Case 3. As before, we will show $G_1G_2G_1 = G_2G_1G_2$, without loss of generality.

$G_1G_2G_1(t_1^a t_2^b t_3^c\otimes\alpha)$

$= G_1G_2(\lambda_1(t_1^a t_2^b t_3^c)\otimes g_1\alpha + \mu_1(t_1^a t_2^b t_3^c)\otimes\alpha)$

$= G_1(\lambda_2\lambda_1(t_1^a t_2^b t_3^c)\otimes g_2g_1\alpha + \lambda_2\mu_1(t_1^a t_2^b t_3^c)\otimes g_2\alpha)$

$+G_1(\mu_2\lambda_1(t_1^a t_2^b t_3^c)\otimes g_1\alpha + \mu_2\mu_1(t_1^a t_2^b t_3^c)\otimes\alpha)$

$= \lambda_1\lambda_2\lambda_1(t_1^a t_2^b t_3^c)\otimes g_1g_2g_1\alpha + \lambda_1\lambda_2\mu_1(t_1^a t_2^b t_3^c)\otimes g_1g_2\alpha$

$+\mu_1\lambda_2\lambda_1(t_1^a t_2^b t_3^c)\otimes g_2g_1\alpha + \mu_1\lambda_2\mu_1(t_1^a t_2^b t_3^c)\otimes g_2\alpha$

$+\lambda_1\mu_2\lambda_1(t_1^a t_2^b t_3^c)\otimes g_1^2\alpha + \lambda_1\mu_2\mu_1(t_1^a t_2^b t_3^c)\otimes g_1\alpha$

$+\mu_1\mu_2\lambda_1(t_1^a t_2^b t_3^c)\otimes g_1\alpha + \mu_1\mu_2\mu_1(t_1^a t_2^b t_3^c)\otimes\alpha$

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$= \lambda_1\lambda_2\lambda_1(t_1^a t_2^b t_3^c) \otimes g_1g_2g_1\alpha + \lambda_1\lambda_2\mu_1(t_1^a t_2^b t_3^c) \otimes g_1g_2\alpha$

$+\mu_1\lambda_2\lambda_1(t_1^a t_2^b t_3^c) \otimes g_2g_1\alpha + \mu_1\lambda_2\mu_1(t_1^a t_2^b t_3^c) \otimes g_2\alpha$

$+[(q-1)\lambda_1\mu_2\lambda_1 + \lambda_1\mu_2\mu_1 + \mu_1\mu_2\lambda_1](t_1^a t_2^b t_3^c) \otimes g_1\alpha$

$+[q\lambda_1\mu_2\lambda_1 + \mu_1\mu_2\mu_1](t_1^a t_2^b t_3^c) \otimes \alpha$

And, $G_2G_1G_2(t_1^a t_2^b t_3^c \otimes \alpha)$

$= G_2G_1(\lambda_2(t_1^a t_2^b t_3^c) \otimes g_2\alpha + \mu_2(t_1^a t_2^b t_3^c) \otimes \alpha$

$= G_2(\lambda_1\lambda_2(t_1^a t_2^b t_3^c) \otimes g_1g_2\alpha + \lambda_1\mu_2(t_1^a t_2^b t_3^c) \otimes g_1\alpha)$

$+G_2(\mu_1\lambda_2(t_1^a t_2^b t_3^c) \otimes g_2\alpha + \mu_1\mu_2(t_1^a t_2^b t_3^c) \otimes \alpha)$

$= \lambda_2\lambda_1\lambda_2(t_1^a t_2^b t_3^c) \otimes g_2g_1g_2\alpha + \lambda_2\lambda_1\mu_2(t_1^a t_2^b t_3^c) \otimes g_2g_1\alpha$

$+\mu_2\lambda_1\lambda_2(t_1^a t_2^b t_3^c) \otimes g_1g_2\alpha + \mu_2\lambda_1\mu_2(t_1^a t_2^b t_3^c) \otimes g_1\alpha$

$+\lambda_2\mu_1\lambda_2(t_1^a t_2^b t_3^c) \otimes g_2^2\alpha + \lambda_2\mu_1\mu_2(t_1^a t_2^b t_3^c) \otimes g_2\alpha$

$+\mu_2\mu_1\lambda_2(t_1^a t_2^b t_3^c) \otimes g_2\alpha + \mu_2\mu_1\mu_2(t_1^a t_2^b t_3^c) \otimes \alpha$

$= \lambda_2\lambda_1\lambda_2(t_1^a t_2^b t_3^c) \otimes g_2g_1g_2\alpha + \lambda_2\lambda_1\mu_2(t_1^a t_2^b t_3^c) \otimes g_2g_1\alpha$

$+\mu_2\lambda_1\lambda_2(t_1^a t_2^b t_3^c) \otimes g_1g_2\alpha + \mu_2\lambda_1\mu_2(t_1^a t_2^b t_3^c) \otimes g_1\alpha$

$+[(q-1)\lambda_2\mu_1\lambda_2 + \lambda_2\mu_1\mu_2 + \mu_2\mu_1\lambda_2](t_1^a t_2^b t_3^c) \otimes g_2\alpha$

$+[q\lambda_2\mu_1\lambda_2 + \mu_2\mu_1\mu_2](t_1^a t_2^b t_3^c) \otimes \alpha$

Therefore, we need to verify the following equalities:

$\lambda_1\lambda_2\lambda_1 = \lambda_2\lambda_1\lambda_2,$

$\mu_1\lambda_2\lambda_1 = \lambda_2\lambda_1\mu_2, \qquad \lambda_1\lambda_2\mu_1 = \mu_2\lambda_1\lambda_2,$

$(q-1)\lambda_1\mu_2\lambda_1 + \lambda_1\mu_2\mu_1 + \mu_1\mu_2\lambda_1 = \mu_2\lambda_1\mu_2,$

$\mu_1\lambda_2\mu_1 = (q-1)\lambda_2\mu_1\lambda_2 + \lambda_2\mu_1\mu_2 + \mu_2\mu_1\lambda_2,$ and

$q\lambda_1\mu_2\lambda_1 + \mu_1\mu_2\mu_1 = q\lambda_2\mu_1\lambda_2 + \mu_2\mu_1\mu_2.$

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By the previous lemma, we obtain for c, d, integers

$$\lambda_1(t_1^c t_2^d) = q^{d-c} t_1^d t_2^c$$

$$\mu_1(t_1^c t_2^d) = (q-1)[-\sum_{i=0}^{c-1} q^{d-c+i} t_1^{d+i} t_2^{c-i} + \sum_{j=1}^{d} q^{d-j} t_1^{c+d-j} t_2^j]$$

Here we verify two of the six inequalities mentioned above. The others are similar.

$$\begin{aligned}\lambda_1\lambda_2\lambda_1(t_1^a t_2^b t_3^c) &= q^{b-a}\lambda_1\lambda_2(t_1^b t_2^a t_3^c)\\ &= q^{b-a+c-a}\lambda_1(t_1^b t_2^c t_3^a)\\ &= q^{b-a+c-a+c-b}(t_1^c t_2^b t_3^a)\end{aligned}$$

And, $\lambda_2\lambda_1\lambda_2(t_1^a t_2^b t_3^c) = q^{c-b}\lambda_2\lambda_1(t_1^a t_2^c t_3^b)$

$$\begin{aligned}&= q^{c-b+c-a}\lambda_2(t_1^c t_2^a t_3^b)\\ &= q^{c-b+c-a+b-a}(t_1^c t_2^b t_3^a)\end{aligned}$$

Thus, $\lambda_1\lambda_2\lambda_1 = \lambda_2\lambda_1\lambda_2$.

To verify $\lambda_1\lambda_2\mu_1 = \mu_2\lambda_1\lambda_2$,

$\lambda_1\lambda_2\mu_1(t_1^a t_2^b t_3^c)$

$= (q-1)\lambda_1\lambda_2[\{-\sum_{i=0}^{a-1} q^{b-a+i} t_1^{b+i} t_2^{a-i} + \sum_{j=1}^{b} q^{b-j} t_1^{a+b-j} t_2^j\} t_3^c]$

$= (q-1)\lambda_1[-\sum_{i=0}^{a-1} q^{c+b-2a+2i} t_1^{b+i} t_2^c t_3^{a-i} + \sum_{j=1}^{b} q^{c+b-2j} t_1^{a+b-j} t_2^c t_3^j]$

$= (q-1)[-\sum_{i=0}^{a-1} q^{2c-2a+i} t_1^c t_2^{b+i} t_3^{a-i} + \sum_{j=1}^{b} q^{2c-a-j} t_1^c t_2^{a+b-j} t_3^j]$.

On the other hand,

$\mu_2\lambda_1\lambda_2(t_1^a t_2^b t_3^c) = q^{c-b}\mu_2\lambda_1(t_1^a t_2^c t_3^b) = q^{c-b+c-a}\mu_2(t_1^c t_2^a t_3^b)$

$= (q-1)[-\sum_{i=0}^{a-1} q^{2c-2a+i} t_1^c t_2^{b+i} t_3^{a-i} + \sum_{j=1}^{b} q^{2c-a-j} t_1^c t_2^{a+b-j} t_3^j]$.

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Case 4.

If we suppose $G_1T_1G_1(t_1^at_2^b \otimes \alpha) = (t_1^at_2^{b+1} \otimes \alpha)$ for any $a, b \in \mathbb{Z}$, then,

$T_1G_1T_1G_1(t_1^at_2^b \otimes \alpha) = T_1(t_1^at_2^{b+1} \otimes \alpha)$

$= t_1^{a+1}t_2^{b+1} \otimes \alpha = G_1T_1G_1(t_1^{a+1}t_2^b \otimes \alpha)$

$= G_1T_1G_1(t_1^{a+1}t_2^b \otimes \alpha) = G_1T_1G_1T_1(t_1^at_2^b \otimes \alpha)$

Claim: we claim $G_1T_1G_1(t_1^at_2^b \otimes \alpha) = (t_1^at_2^{b+1} \otimes \alpha)$ for any $a, b \in \mathbb{Z}$.

Proof of claim:

Case 1: we want to show $G_1T_1G_1(t_2^b \otimes \alpha) = (t_2^{b+1} \otimes \alpha)$ for $b \geq 0$ inductively. Let $b = 0$, then we expect $G_1T_1G_1(1 \otimes \alpha) = (t_2 \otimes \alpha)$. Recall that all these calculations follow from definition 2.1.6 and lemma 2.1.7. Lemma 2.1.7(2) and (3) are rewritten in a slightly different form as follows. With these, we use the remark following the proof of lemma 2.1.7 without further comment.

For any a, $b \in \mathbb{Z}$,

2.1.7(*ii*) $G_1(t_1^{a+1}t_2^b \otimes \alpha) = q^{-1}(t_2 \otimes 1)G_1(t_1^at_2^b \otimes \alpha) + (q^{-1} - 1)t_1^at_2^{b+1} \otimes \alpha$

2.1.7(*iii*) $G_1(t_1^at_2^{b+1} \otimes \alpha) = q(t_1 \otimes 1)G_1(t_1^at_2^b \otimes \alpha) + (q - 1)t_1^at_2^{b+1} \otimes \alpha$

$G_1T_1G_1(1 \otimes \alpha) = G_1T_1(1 \otimes g_1\alpha)$ by the definition 2.1.6(1),

$= G_1(t_1 \otimes g_1\alpha)$ by the definition 2.1.6(6),

$= (q^{-1} - 1)(t_2 \otimes g_1\alpha) + q^{-1}(t_2 \otimes 1)G_1(1 \otimes g_1\alpha)$

by the definition 2.1.6(4),

$= (q^{-1} - 1)(t_2 \otimes g_1\alpha) + q^{-1}(t_2 \otimes 1)(1 \otimes g_1^2\alpha)$

by 2.1.6(1) and $g_1^2 = (q - 1)g_1 + q$,

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$$= (q^{-1}-1)(t_2 \otimes g_1\alpha) + q^{-1}(q-1)(t_2 \otimes g_1\alpha) + (t_2 \otimes \alpha)$$

$$= (t_2 \otimes \alpha)$$

Assume $G_1T_1G_1(t_2^{b-1} \otimes \alpha) = (t_2^b \otimes \alpha)$. Then,

$G_1T_1G_1(t_2^b \otimes \alpha)$

$= G_1T_1((q-1)(t_2^b \otimes \alpha) + q(t_1 \otimes 1)G_1(t_2^{b-1} \otimes \alpha))$ by 2.1.6(2),

$= G_1((q-1)(t_1t_2^b \otimes \alpha) + q(t_1^2 \otimes 1)G_1(t_2^{b-1} \otimes \alpha))$ by 2.1.6(6),

$= (q-1)(q^{-1}-1)(t_2^{b+1} \otimes \alpha) + q^{-1}(q-1)(t_2 \otimes 1)G_1(t_2^b \otimes \alpha)$

$+q(q^{-1}-1)(t_1t_2 \otimes 1)G_1(t_2^{b-1} \otimes \alpha) + qq^{-1}(t_2 \otimes 1)G_1((t_1 \otimes 1)G_1(t_2^{b-1} \otimes \alpha))$

using 2.1.6(4) for the first term and 2.1.7(ii) for the second term,

$= (q-1)(q^{-1}-1)(t_2^{b+1} \otimes \alpha)$

$-(q^{-1}-1)(t_2 \otimes 1)[(q-1)(t_2^b \otimes \alpha) + q(t_1 \otimes 1)G_1(t_2^{b-1} \otimes \alpha)]$

$-(q-1)(t_1t_2 \otimes 1)G_1(t_2^{b-1} \otimes \alpha) + (t_2 \otimes 1)G_1((t_1 \otimes 1)G_1(t_2^{b-1} \otimes \alpha))$

by 2.1.6(2),

$= (q-1)(q^{-1}-1)(t_2^{b+1} \otimes \alpha) - (q^{-1}-1)(q-1)(t_2^{b+1} \otimes \alpha)$

$+(q-1)(t_1t_2 \otimes 1)G_1(t_2^{b-1} \otimes \alpha)) \quad -(q-1)(t_1t_2 \otimes 1)G_1(t_2^{b-1} \otimes \alpha)$

$+(t_2 \otimes 1)G_1T_1(G_1(t_2^{b-1} \otimes \alpha))$

$= (t_2 \otimes 1)G_1T_1(G_1(t_2^{b-1} \otimes \alpha))$

$= (t_2 \otimes 1)(t_2^b \otimes \alpha)$ by the induction hypothesis,

$= (t_2^{b+1} \otimes \alpha)$.

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Case 2: $a = 0, b < 0$

$G_1T_1G_1(t_2^b \otimes \alpha)$

$= G_1T_1((q^{-1}-1)(t_1^{-1}t_2^{b+1} \otimes \alpha) + q^{-1}(t_1^{-1} \otimes 1)G_1(t_2^{b+1} \otimes \alpha))$

by the definition 2.1.6(3),

$= G_1((q^{-1}-1)(t_2^{b+1} \otimes \alpha) + q^{-1}G_1(t_2^{b+1} \otimes \alpha))$ by 2.1.6(6),

$= (q^{-1}-1)G_1(t_2^{b+1} \otimes \alpha) + q^{-1}G_1^2(t_2^{b+1} \otimes \alpha)$

$= (q^{-1}-1)G_1(t_2^{b+1} \otimes \alpha) + q^{-1}(q-1)G_1(t_2^{b+1} \otimes \alpha) + qq^{-1}(t_2^{b+1} \otimes \alpha)$

by 2.1.8(1),

$= t_2^{b+1} \otimes \alpha$, as required.

Case 3: We first examine a computation for all a, b. $G_1(t_1^a t_2^{b+1} \otimes \alpha)$

$= (q^{-1}-1)t_1^{a-1}t_2^{b+2} \otimes \alpha + q^{-1}(t_2 \otimes 1)G_1(t_1^{a-1}t_2^{b+1} \otimes \alpha)$ by 2.1.7(ii),

$= (q^{-1}-1)t_1^{a-1}t_2^{b+2} \otimes \alpha$

$+q^{-1}(t_2 \otimes 1)[q(t_1 \otimes 1)G_1(t_1^{a-1}t_2^b \otimes \alpha) + (q-1)t_1^{a-1}t_2^{b+1} \otimes \alpha]$ by 2.1.7(iii),

$= (q^{-1}-1)t_1^{a-1}t_2^{b+2} \otimes \alpha$

$+(t_2 \otimes 1)(t_1 \otimes 1)G_1(t_1^{a-1}t_2^b \otimes \alpha) - (q^{-1}-1)t_1^{a-1}t_2^{b+2} \otimes \alpha$

$= (t_1t_2 \otimes 1)G_1(t_1^{a-1}t_2^b \otimes \alpha)$ (♯)

Now for $a > 0$, and for all b,

$G_1T_1G_1(t_1^a t_2^b \otimes \alpha)$

$= G_1T_1((t_1t_2 \otimes 1)G_1(t_1^{a-1}t_2^{b-1} \otimes \alpha)$ by (♯),

$= G_1((t_1^2t_2 \otimes 1)G_1(t_1^{a-1}t_2^{b-1} \otimes \alpha)$ by 2.1.6(6),

$= (t_1t_2 \otimes 1)G_1((t_1 \otimes 1)G_1(t_1^{a-1}t_2^{b-1} \otimes \alpha))$ by (♯),

$= (t_1t_2 \otimes 1)G_1T_1G_1(t_1^{a-1}t_2^{b-1} \otimes \alpha))$ by 2.1.6(6),

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$= (t_1t_2 \otimes 1)(t_1^{a-1}t_2^b \otimes \alpha)$ by induction hypothesis,

$= (t_1^at_2^{b+1} \otimes \alpha)$ as required.

Case 4: For the last, let $a < 0$.

$G_1T_1G_1(t_1^at_2^b \otimes \alpha)$

$= G_1T_1((q-1)(t_1^at_2^b \otimes \alpha) + q(t_2^{-1} \otimes 1)G_1(t_1^{a+1}t_2^b \otimes \alpha))$ by 2.1.6(5),

$= (q-1)G_1(t_1^{a+1}t_2^b \otimes \alpha) + qG_1((t_1t_2^{-1} \otimes 1)G_1(t_1^{a+1}t_2^b \otimes \alpha))$ by 2.1.6(6),

$= (q-1)G_1(t_1^{a+1}t_2^b \otimes \alpha)$

$+qq^{-1}(t_2 \otimes 1)G_1((t_2^{-1} \otimes 1)G_1(t_1^{a+1}t_2^b \otimes \alpha))$

$+q(q^{-1}-1)G_1(t_1^{a+1}t_2^b \otimes \alpha)$ by 2.1.7(ii),

$= (t_2 \otimes 1)G_1((t_2^{-1} \otimes 1)G_1(t_1^{a+1}t_2^b \otimes \alpha))$ by cancellation,

Multiplying 2.1.7(ii) by t_2^{-1} and applying G_1, we obtain the following,

$G_1((t_2^{-1} \otimes 1)G_1(t_1^{a+1}t_2^b \otimes \alpha))$

$$=q^{-1}G_1^2(t_1^at_2^b \otimes \alpha) + (q^{-1}-1)G_1(t_1^at_2^b \otimes \alpha).$$

So, continuing the computation using this identity,

$= (t_2 \otimes 1)[q^{-1}G_1^2(t_1^at_2^b \otimes \alpha) + (q^{-1}-1)G_1(t_1^at_2^b \otimes \alpha)]$

$= (t_2 \otimes 1)[q^{-1}(q-1)G_1(t_1^at_2^b \otimes \alpha) + q^{-1}qt_1^at_2^b \otimes \alpha + (q^{-1}-1)G_1(t_1^at_2^b \otimes \alpha)]$

by 2.1.8(1),

$= (t_2 \otimes 1)(t_1^at_2^b \otimes \alpha)$ by cancellation,

$= (t_1^at_2^{b+1} \otimes \alpha)$

This completes the proof of the claim. □

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Now, we return to the proof of the theorem. The map $\phi : \Lambda_n \otimes H_n \to \hat{H}_n$ is well defined since it is a map from a free module defined on a basis. The map $\psi : End_n(\Lambda_n \otimes H_n) \to \Lambda_n \otimes H_n$ is the evaluation map. The map $\chi : \hat{H}_n \to End_n(\Lambda_n \otimes H_n)$ was defined on the generators of $\hat{H}_n$ and shown to extend to a well-defined map in Lemma 2.1.8. Since ψ is defined by $\psi(G_i) = G_i(1\otimes 1) = 1\otimes g_i$, if $\alpha \in H(A_{n-1})$, then $\psi(\chi(\alpha)) = \chi(\alpha)(1\otimes 1) = 1\otimes\alpha$. If $\alpha = w(g_1 \cdots g_{n-1})$ is a word, then let $G_\alpha = w(G_1, G_2, \cdots, G_{n-1})$. Now,

$\psi \circ \chi \circ \phi(t_1^{s_1} \cdots t_n^{s_n} \otimes \alpha)$

$= \psi \circ \chi(\phi(t_1^{s_1} \cdots t_n^{s_n} \otimes \alpha))$

$= \psi \circ \chi(t_1^{s_1} \cdots t_n^{s_n} \alpha)$

$= \psi(T_1^{s_1} \cdots T_n^{s_n} G_\alpha)$

$= t_1^{s_1} \cdots t_n^{s_n} \otimes \alpha$

Thus $\psi \circ \chi \circ \phi$ is the identity on $\Lambda_n \otimes H_n$. Therefore, ϕ is injective as required. Hence $\mathcal{H}$ is a basis of $H_n(\frac{1}{2})$. □

Let $\mathcal{B}$ be the set of elements of the form $(t_1')^{s_1}(t_2')^{s_2} \cdots (t_n')^{s_n}\alpha$, where $s_i \in \mathbb{Z}$ for all i, $t_i' = g_{i-1}g_{i-2} \cdots g_1 t_1 g_1^{-1} \cdots g_{i-2}^{-1} g_{i-1}^{-1}$, and $\alpha \in H(A_{n-1})$ is in normal form. Then we claim that $\mathcal{B}$ is a basis of $H_n(\frac{1}{2})$.

Theorem 2.1.9. *Let $\mathcal{B}$ be the set of elements of the form $(t_1')^{s_1}(t_2')^{s_2} \cdots (t_n')^{s_n}\alpha$, where $t_i' = g_{i-1}g_{i-2} \cdots g_1 t_1 g_1^{-1} \cdots g_{i-2}^{-1} g_{i-1}^{-1}$, for arbitrary i and α is a word in normal form in $H(A_{n-1})$ and $s_i \in \mathbb{Z}$. Then $\mathcal{B}$ is a basis of $H_n(\frac{1}{2})$.*

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Proof:

The first part of the proof that $\mathcal{B}$ is a generating set is similar to that of the normal form $\mathcal{H}$ in theorem 2.1.5. Let w be a word and let t_1 occur in w, then $w = \alpha_1 t_1^{\pm 1} w_1$ where $\alpha_1 = \sum_{i \in I}^{I:finite} \alpha_{2i}$ and α_{2i} is written in normal form of $H(A_{n-1})$, *i.e.*, $\alpha_{2i} = g_{i_1} g_2 g_{i_2} \cdots g_{n-1} \cdots g_{i_{n-1}}$ where $1 \le i_k \le k$.

So each term of w is

$\alpha_{2i}\, t_1^{\pm 1} w_1 = g_{i_1} g_2 g_{i_2} \cdots g_k g_{k-1} \cdots g_1 g_{k+1} \cdots g_{i_k} \cdots g_{n-1} \cdots g_{i_{n-1}} t_1^{\pm 1} w_1.$

If no g_1 occurs in α_{2i}, then $\alpha_{2i} t_1^{\pm} w_1 = t_1^{\pm} \alpha_{2i} w_1$. By the formula

$g_h (g_k g_{k-1} \cdots g_1) = (g_k g_{k-1} \cdots g_1) g_{h+1}$ for $1 \le h < k$ by $b(1)$,

$\alpha_{2i} t_1^{\pm 1} w_1 =$

$(g_k \cdots g_1) t_1^{\pm 1} (g_{i_1+1} g_3 g_{i_2+1} \cdots g_k \cdots g_{i_{k-1}+1})(g_{k+1} \cdots g_{i_k} \cdots g_{n-1} \cdots g_{i_{n-1}}) w_1$

$= (g_k \cdots g_1) t_1^{\pm 1} (g_1^{-1} \cdots g_k^{-1})(g_k \cdots g_1) w_2$ where

$w_2 = (g_{i_1+1} g_3 g_{i_2+1} \cdots g_k \cdots g_{i_{k-1}+1})(g_{k+1} \cdots g_{i_k} \cdots g_{n-1} \cdots g_{i_{n-1}}) w_1$

$=(t'_{k+1})^{\pm 1} w_3$ where $w_3 = (g_k \cdots g_1) w_2$.

Let's denote this process (I').

For w_3, perform the process (I') as in w. Eventually, we obtain sums of elements $(t'_{k_1})^{l_1} (t'_{k_2})^{l_2} \cdots (t'_{k_m})^{l_m} \gamma$ where $l_i \in \mathbb{Z}$,, $k_i \in \{1, 2, \cdots, n\}$ and $\gamma \in H(A_{n-1})$.

Now we reorder the indices of t'. To do this, we will make use of the formulas in lemma 2.1.4.

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For $(c1)$, if we let $b_{n-1,i} = g_{n-1}\cdots g_{i+1}g_i g_{i+1}^{-1}\cdots g_{n-1}^{-1}$ for $i \le n-1$, the formula $(c1)$ can be written into the following form, say $(c1')$:

$(c1')$ $(t'_n)^m t'_i = t'_i(t'_n)^m - (q^{-1}-1)(t'_i)^m t'_n b_{n-1,i} + (q^{-1}-1)(t'_n)^{m+1} b_{n-1,i}$,

by repeated use of formula (b1), commutativity and cancellation on the formula (c1), where

$$\begin{aligned}(c1)\quad (t'_n)^m t'_i &= t'_i(t'_n)^m \\ &-(q^{-1}-1)(t'_i)^m(g_i^{-1}\cdots g_{n-2}^{-1})g_{n-1}(g_{n-2}\cdots g_1)t_1(g_1^{-1}\cdots g_{i-1}^{-1}) \\ &+(q^{-1}-1)(g_i^{-1}\cdots g_{n-2}^{-1})g_{n-1}(g_{n-2}\cdots g_1)t_1^{m+1}(g_1^{-1}\cdots g_{i-1}^{-1})\end{aligned}$$

Similarly, the other formulas in Lemma 2.1.4 can be written as below.

$$\begin{aligned}(c2')\quad (t'_n)^m(t'_i)^{-1} &= (t'_i)^{-1}(t'_n)^m \\ &-(q^{-1}-1)(t'_i)^{-1}(t'_n)^m b_{n-1,i} + (q^{-1}-1)(t'_i)^{m-1} b_{n-1,i}\end{aligned}$$

$$\begin{aligned}(c3')\quad (t'_n)^{-m}(t'_i) &= (t'_i)(t'_n)^{-m} \\ &-(q^{-1}-1)(t'_i)^{-m} t'_n b_{n-1,i} + (q^{-1}-1)(t'_n)^{-m+1} b_{n-1,i}\end{aligned}$$

$$\begin{aligned}(c4')\quad (t'_n)^{-m}(t'_i)^{-1} &= (t'_i)^{-1}(t'_n)^{-m} \\ &-(q^{-1}-1)(t'_i)^{-1}(t'_n)^{-m} b_{n-1,i} + (q^{-1}-1)(t'_i)^{-1-m} b_{n-1,i}.\end{aligned}$$

We use induction on n to show that the first part $(t'_{k_1})^{l_1}(t'_{k_2})^{l_2}\cdots(t'_{k_m})^{l_m}$ can be deformed by a finite number of applications of the formulas involved in $H_n(\frac{1}{2})$ into a sum of the form $(t'_1)^{s_1}\cdots(t'_{n-2})^{s_{n-2}}(t'_{n-1})^{s_{n-1}}(t_n)^{\prime s_n}\beta$ with coefficients in k, where β is an element of the given basis in $H(A_{n-1})$.

Let F be $F = (t'_{k_1})^{l_1}(t'_{k_2})^{l_2}\cdots(t'_{k_m})^{l_m}$. The case $n=1$ is trivial since $k_i = k_j = 1$ for all i,j in F. Assume it is true for $F \in H_{n-1}(\frac{1}{2})$. Suppose $F \in H_n(\frac{1}{2})$, then let i be the smallest index such that $k_{i-1} = n$.

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We assume l_i, l_{i+1} are positive to use (c1) in this demonstration. The cases where l_i, l_{i+1} are not positive, require one of (c2),(c3), or (c4) from lemma 2.1.4. In $F = (t'_{k_1})^{l_1}(t'_{k_2})^{l_2}\cdots(t'_n)^{l_{i-1}}(t'_{k_i})^{l_i}\cdots(t'_{k_m})^{l_m}$, the absolute exponent sum $e = \sum_{j=k_i}^{m}|l_j|$ of t' in the right hand part of the $t_n'^{l_{i-1}}$ in F is finite. Applying $(c1')$ on $(t'_n)^{l_{i-1}}(t'_{k_i})^{l_i}$, we have

$$F = (t'_{k_1})^{l_1}(t'_{k_2})^{l_2}\cdots(t'_{k_{i-2}})^{l_{i-2}}t'_{k_i}(t'_n)^m(t'_{k_i})^{l_i-1}\cdots(t'_{k_m})^{l_m}$$
$$-(q^{-1}-1)(t'_{k_1})^{l_1}(t'_{k_2})^{l_2}\cdots(t'_{k_{i-2}})^{l_{i-2}}(t'_{k_i})^m t'_n b_{n-1,i}(t'_{k_i})^{l_i-1}\cdots(t'_{k_m})^{l_m}$$
$$+(q^{-1}-1)(t'_{k_1})^{l_1}(t'_{k_2})^{l_2}\cdots(t'_{k_{i-2}})^{l_{i-2}}(t'_n)^{m+1}b_{n-1,i}(t'_{k_i})^{l_i-1}\cdots(t'_{k_m})^{l_m}$$

In each term of F, the absolute exponent sum of t' in the corresponding part is strictly less than e. Thus a finite number of applications of process (I'), and (c1), (c2), (c3), and (c4) will ultimately reduces the corresponding exponent sum to 0. Therefore, each term of F can be written as a sum of words $F'(t'_n)^s\alpha$ where $s \in \mathbb{Z}$, $F' \in H_{n-1}(\frac{1}{2})$, and $\alpha \in H(A_{n-1})$. By the induction hypothesis, $F' = (t'_1)^{s_1}(t'_2)^{s_2}\cdots(t'_{n-1})^{s_{n-1}}\gamma$ where $\gamma \in H(A_{n-2})$ and $s_i \in \mathbb{Z}$. Since $\gamma \in H(A_{n-2})$, $\gamma t'_n = t'_n\gamma$. Therefore, $F'(t'_n)^s\alpha = (t'_1)^{s_1}(t'_2)^{s_2}\cdots(t'_{n-1})^{s_{n-1}}\gamma(t'_n)^s\alpha =$ $(t'_1)^{s_1}(t'_2)^{s_2}\cdots(t'_{n-1})^{s_{n-1}}(t'_n)^s\gamma\alpha$. $\gamma\alpha$ will be deformed into a sum of words in normal form in $H(A_{n-1})$. Letting $s_n = s$, we have each term in the form of $(t'_1)^{s_1}(t'_2)^{s_2}\cdots(t'_{n-1})^{s_{n-1}}(t'_n)^{s_n}\beta$ as required to be in $\mathcal{B}$.

Now, we return to the uniqueness of the normal form $\mathcal{B}$ in the theorem. We consider $H_n(\frac{1}{2})$ as a free module with basis $\mathcal{H} = \{t_1^{s_1}t_2^{s_2}\cdots t_{n-1}^{s_{n-1}}t_n^{s_n}\alpha\}$ for all $s_i \in \mathbb{Z}$.

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Suppose that $w(g_1, \cdots, g_{n-1}, t_1, \cdots, t_n)$ is a word in $g_1, \cdots, g_{n-1}$, $t_1, \cdots, t_n$. Let $P(w)$ be the sum of the positive exponents of the t_i and $N(w)$ be the sum of the negative exponents of the t_i. Note that P and N are not defined on monomials of $H_n(\frac{1}{2})$ as $t_1 t_1^{-1} = 1$ is a relation, it is only defined on abstract words. If $w(g_1, \cdots, g_{n-1}, t'_1, \cdots, t'_n)$ is a word in $g_1, \cdots, g_{n-1}, t'_1, \cdots, t'_n$, let $P(w)$ be the sum of the positive exponents of the t'_i and $N(w)$ be the sum of the negative exponents of the t'_i. Also note that $(P+N)(w)$ is well-defined on monomials of $H_n(\frac{1}{2})$, it is the exponential sum of t_1 (if the monomial is written in $g_1, \cdots, g_{n-1}, t_1$).

Let $\mathcal{M}_e = \{\sum a_i w_i \mid (P+N)(w_i) = e\}$. This is a submodule as the relations that define $H_n(\frac{1}{2})$ all preserve the submodule. We also have $H_n(\frac{1}{2}) = \oplus_{e \in \mathbb{Z}} \mathcal{M}_e$.

For $p \in \mathbb{Z}^+$, $m \in \mathbb{Z}^-$,

let $\mathcal{H}(p,m) = \langle w = t_1^{s_1} t_2^{s_2} \cdots t_{n-1}^{s_{n-1}} t_n^{s_n} \alpha \in \mathcal{H} \mid P(w) \le p \text{ and } p+m = \sum s_i \rangle$.

Let $\mathcal{B}(p,m) = \langle w = t_1'^{s_1} \cdots t_n'^{s_n} \alpha \in \mathcal{B} \mid P(w) \le p \text{ and } p+m = \sum s_i \rangle$. Note that $\mathcal{H}(p,m)$ and $\mathcal{B}(p,m)$ are finitely generated free $k[q,q^{-1}]$-modules.

Lemma 2.1.10. $\mathcal{H}(p,m) = \mathcal{B}(p,m)$.

Proof: We first show that $\mathcal{B}(p,m) \subset \mathcal{H}(p,m)$.

Let $R_i = g_{i-1} g_{i-2} \cdots g_2 g_1^2 g_2 \cdots g_{i-2} g_{i-1}$, so $t_i = t'_i R_i$.

Let $p+m = M$. Take $t_1'^{s_1} \cdots t_n'^{s_n} \alpha$, a generator of $\mathcal{B}(p,m)$, so $\sum s_i = M$.

Now, $t_1'^{s_1} \cdots t_n'^{s_n} \alpha = t_1^{s_1} (t_2 R_2^{-1})^{s_2} \cdots (t_n R_n^{-1})^{s_n} \alpha$. These words have the same P values and N values. We now apply the rewriting process used in

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the beginning of the proof of Theorem 2.1.5. Let $l = p - m = \sum |s_i|$. This process yields a sum of words of the form $t_{k_1}^{\pm 1} t_{k_2}^{\pm 1} \cdots t_{k_l}^{\pm 1} \alpha'$ where the k_i may have repetitions and again the P and N values are unchanged. Now use commutativity to rearrange the t_i's and relabel, so $k_1 \leq k_2 \leq \cdots \leq k_l$. Cancelling adjacent $t_i^{\pm 1}$ when possible may reduce the P value but leaves $M = P + N$ unchanged. We have that each term is in $\mathcal{H}(p, m)$ and so we have rewritten a generator of $\mathcal{B}(p, m)$ in elements in $\mathcal{H}(p, m)$.

We now show that $\mathcal{H}(p, m) \subset \mathcal{B}(p, m)$.

Take $t_1^{s_1} t_2^{s_2} \cdots t_{n-1}^{s_{n-1}} t_n^{s_n} \alpha$, a generator of $\mathcal{H}(p, m)$, so $\sum s_i = M$. Now $t_1^{s_1} t_2^{s_2} \cdots t_{n-1}^{s_{n-1}} t_n^{s_n} \alpha = t_1'^{s_1} (t_2' R_2)^{s_2} \cdots (t_n' R_n)^{s_n} \alpha$. These words have the same P values and N values. We now apply the rewriting process used in the beginning of the proof of theorem 2.1.9. Let $l = p - m = \sum |s_i|$. This process yields a sum of elements of the form $t_{k_1}'^{\pm 1} t_{k_2}'^{\pm 1} \cdots t_{k_l}'^{\pm 1} \alpha'$ where the k_i may have repetitions and again the P and N values are unchanged. Since the t_i''s do not commute, we use the relations $(c1'), (c2'), (c3'), (c4')$, and combine the exponents to get a sum of elements of the form $t_1'^{r_1} \cdots t_n'^{r_n} \alpha''$ where the P value is less than or equal to p and $M = P + N$ is unchanged. We have each term in $\mathcal{B}(p, m)$ and so we have rewritten a generator of $\mathcal{H}(p, m)$ into elements in $\mathcal{B}(p, m)$. □

We now complete the proof of 2.1.9. We saw that $H_n(\frac{1}{2}) = \oplus_{e \in \mathbf{Z}} \mathcal{M}_e$. Also note that $\mathcal{M}_e = \bigcup_{p+m=e} \mathcal{H}(p, m)$, a nested union.

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By the lemma, $S = \{w = (t'_1)^{s_1}(t'_2)^{s_2}\cdots(t'_n)^{s_n}\alpha \mid \alpha$ is in normal form, $P(w) \le p,\ P(w) + N(w) = p + m\}$ is a generating set for $\mathcal{H}(p, m)$ which has the same order as the basis $\{w = t_1^{s_1}t_2^{s_2}\cdots t_{n-1}^{s_{n-1}}t_n^{s_n}\alpha \mid \alpha$ is in normal form, $P(w) \le p, P(w) + N(w) = p + m\}$.

Since $k[q, q^{-1}]$ is a commutative ring with unit and $\mathcal{H}(p, m)$ is a finitely generated free module over $k[q, q^{-1}]$, a generating set with the same order as a basis is a basis (Corollary 4.4 in [E]). Therefore, S is a basis of $\mathcal{H}(p, m)$, $\{w \in \mathcal{B} \mid P(w) + N(w) = e\}$ is a basis of $\mathcal{M}_e$ and $\mathcal{B}$ is a basis of $H_n(\frac{1}{2})$. □

2.2 A trace map on the algebra $H_n(\frac{1}{2})$

In the previous section we have a basis $\mathcal{B}$ of $H_n(\frac{1}{2})$,

$$\mathcal{B} = \{(t'_1)^{s_1}\cdots(t'_{n-2})^{s_{n-2}}(t'_{n-1})^{s_{n-1}}(t'_n)^{s_n}\beta \mid \quad s_i \in \mathbb{Z}\}$$

where β is a basis element of $H(A_{n-1})$. Recall the basis of $H(A_{n-1})$ is inductively constructed as $\alpha g_{n-1}g_{n-2}\cdots g_{n-k}$ where $1 \le k \le n-1, \alpha \in H(A_{n-2})$ (see J). Now we want to find a trace function uniquely defined on the infinite union of $H_n(\frac{1}{2})$ for all n. Let X_n be the restriction of X on $H_n(\frac{1}{2})$ throughout this section.

Definition 2.2.1. *There is a linear map X,*

$X : \cup_{n=1}^{\infty} H_n(\frac{1}{2}) \to k[z, z^{-1}, q, q^{-1}, \ \cdots, \tau_{-2}, \tau_{-1}, \tau_1, \tau_2, \cdots]$ *defined by*

1. $X_n(1) = 1$
2. $X_n((t'_1)^{s_1}\cdots(t'_{n-2})^{s_{n-2}}(t'_{n-1})^{s_{n-1}}(t'_n)^{s}\beta)$

   $= \tau_s X_{n-1}((t'_1)^{s_1}\cdots(t'_{n-2})^{s_{n-2}}(t'_{n-1})^{s_{n-1}}\beta)$,

   if β is a basis element of $H(A_{n-2})$

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3. $X_n(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^{s_{n-1}} (t_n')^{s_n} \beta g_{n-1} \gamma)$

$= z\ X_{n-1}((t_1')^{s_1} \cdots (t_{n-2}')^{s_{n-2}} (t_{n-1}')^{s_{n-1}} \beta (t_{n-1}')^{s_n} \gamma)$

if $\beta g_{n-1} \gamma$ is a basis element of $H(A_{n-1})$.

We see the map X is well-defined since it is defined on the basis of the algebra. In the following theorem we have some properties that will make computations of X easier.

Theorem 2.2.2. *The linear map X on $H = \cup_{n=1}^{\infty} H_n(\frac{1}{2})$ satisfies the following properities:*

(1) $X(1) = 1$

(2) $X(\alpha g_{n-1} \beta) = z\, X(\alpha\beta)$, *where* $\alpha\beta \in H_{n-1}(\frac{1}{2})$

(3) $X(\alpha(t_n')^s (t_i')^r \beta) = X(\alpha(t_i')^r (t_n')^s \beta)$ *where* $\alpha\beta \in H_{n-1}(\frac{1}{2})$, s, r *any integers, and* $i \leq n-1$.

(4) $X(\alpha(t_n')^s \beta) = \tau_s\, X(\alpha\beta)$, *where* $\alpha\beta \in H_{n-1}(\frac{1}{2})$

(5) $X|_{H(A_{n-1})}$ *is Ocneanu trace function.*

Proof:

Since the first case is just one of the definition, we consider only the other cases. For $\beta \in H_{n-1}(\frac{1}{2})$, there exist $r_1, r_2, \cdots r_{n-1}$, for $r_i \in \mathbb{Z}$, $\beta_j \in H(A_{n-2})$ such that $\beta = \sum_j a_j (t_1')^{r_1} \cdots (t_{n-1}')^{r_{n-1}} \beta_j$, a linear combination of elements in $\mathcal{B}$ where $a_j \in k[q, q^{-1}]$. Therefore, we may begin with β a basis element of $H_{n-1}(\frac{1}{2})$ with $b \in H(A_{n-2})$.

$X(\alpha g_{n-1} (t_1')^{r_1} \cdots (t_{n-1}')^{r_{n-1}} b)$

$= X(\alpha (t_1')^{r_1} \cdots g_{n-1} (t_{n-1}')^{r_{n-1}} b)$, since g_{n-1}, $(t_1')^{r_1} \cdots (t_{n-2}')^{r_{n-2}}$ commutes,

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$= X(\alpha(t_1')^{r_1} \cdots g_{n-1}(t_{n-1}')^{r_{n-1}} g_{n-1}^{-1} g_{n-1} b)$ since $g_{n-1}^{-1} g_{n-1} = 1$

$= X(\alpha(t_1')^{r_1} \cdots (t_{n-2}')^{r_{n-2}} (t_n')^{r_{n-1}} g_{n-1} b)$ by defintion of t_n'

$= X(\alpha(t_1')^{r_1} \cdots (t_{n-2}')^{r_{n-2}} (t_n')^{r_{n-1}} b_1 g_{n-1} g_{n-2}^r b_2)$

since b is a basis element of $H(A_{n-2})$ where $b = b_1 g_{n-2}^r b_2$, $r = 0, 1$, $b_1 \in H(A_{n-3})$, and if $r = 0$, then $b_2 = 1$, *i.e.*, $b = b_1$,

$= X(\alpha(t_1')^{r_1} \cdots (t_{n-2}')^{r_{n-2}} b_1 (t_n')^{r_{n-1}} g_{n-1} g_{n-2}^r b_2)$,

$= \sum_m c_m X((t_1')^{s_{1,m}} \cdots (t_{n-2}')^{s_{n-2,m}} \gamma_m (t_n')^{r_{n-1}} g_{n-1} g_{n-2}^r b_2)$

since there exist $s_{1,m}, \cdots, s_{n-2,m}$ such that $\alpha(t_1')^{r_1} \cdots (t_{n-2}')^{r_{n-2}} b_1 = \sum_m c_m (t_1')^{s_{1,m}} \cdots (t_{n-2}')^{s_{n-2,m}} \gamma_m, \quad \gamma_m \in H(A_{n-3})$.

Continuing the calculation,

$= \sum_m c_m X((t_1')^{s_{1,m}} \cdots (t_{n-2}')^{s_{n-2,m}} (t_n')^{r_{n-1}} \gamma_m g_{n-1} g_{n-2}^r b_2)$

$= z \sum_m c_m X((t_1')^{s_{1,m}} \cdots (t_{n-2}')^{s_{n-2,m}} \gamma_m (t_{n-1}')^{r_{n-1}} g_{n-2}^r b_2)$

by definition of X since $\gamma_m g_{n-1} g_{n-2}^r b_2$ is a basis element of $H(A_{n-1})$

$= z \sum_m c_m X((t_1')^{s_{1,m}} \cdots (t_{n-2}')^{s_{n-2,m}} \gamma_m (t_{n-1}')^{r_{n-1}} g_{n-2}^r b_2)$

$= z X(\alpha(t_1')^{r_1} \cdots (t_{n-2}')^{r_{n-2}} b_1 (t_{n-1}')^{r_{n-1}} g_{n-2}^r b_2)$

$= z X(\alpha(t_1')^{r_1} \cdots (t_{n-2}')^{r_{n-2}} (t_{n-1}')^{r_{n-1}} b_1 g_{n-2}^r b_2)$

$= z X(\alpha(t_1')^{r_1} \cdots (t_{n-2}')^{r_{n-2}} (t_{n-1}')^{r_{n-1}} b)$ as required.

We claim that $X(\alpha(t_n')^s (t_i')^{\pm 1} \beta) = X(\alpha(t_i')^{\pm 1} (t_n')^s \beta)$ where $\alpha\beta \in H_{n-1}(\frac{1}{2})$, s any integer, and $i \leq n-1$.

Consequently, we can conclude $X(\alpha(t_n')^m (t_i')^r \beta) = X(\alpha(t_i')^r (t_n')^m \beta)$ for any $r \in \mathbb{Z}$ by applying that property r times. Applying the equality $(c1)$, assuming $m \geq 0$, of lemma 2.1.4 to $\alpha(t_n')^m t_i' \beta$,

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we have $X(\alpha(t'_n)^m t'_i \beta)$

$= -(q^{-1}-1)X(\alpha(t'_i)^m(g_i^{-1}\cdots g_{n-2}^{-1})(g_{n-1})(g_{n-2}\cdots g_2 g_1)t_1(g_1^{-1}\cdots g_{i-1}^{-1})\beta)$

$+X(\alpha(t'_i)(t'_n)^m\beta)$

$+(q^{-1}-1)X(\alpha(g_i^{-1}\cdots g_{n-2}^{-1})(g_{n-1})(g_{n-2}\cdots g_2 g_1)t_1^{m+1}(g_1^{-1}\cdots g_{i-1}^{-1})\beta)$

$= -(q^{-1}-1)z\,X(\alpha(t'_i)^m(g_i^{-1}\cdots g_{n-2}^{-1})(g_{n-2}\cdots g_2 g_1)t_1(g_1^{-1}\cdots g_{i-1}^{-1})\beta)$

$+(q^{-1}-1)z\,X(\alpha(g_i^{-1}\cdots g_{n-2}^{-1})(g_{n-2}\cdots g_2 g_1)t_1^{m+1}(g_1^{-1}\cdots g_{i-1}^{-1})\beta)$

$+X(\alpha(t'_i)(t'_n)^m\beta)$, by the second property of this theorem,

$= -(q^{-1}-1)z\,X(\alpha(t'_i)^m(g_{i-1}\cdots g_2 g_1)t_1(g_1^{-1}\cdots g_{i-1}^{-1})\beta)$

$+(q^{-1}-1)z\,X(\alpha(g_{i-1}^{-1}\cdots g_2 g_1)t_1^{m+1}(g_1^{-1}\cdots g_{i-1}^{-1})\beta)$

$+X(\alpha(t'_i)(t'_n)^m\beta)$

$= -(q^{-1}-1)z\,X(\alpha(t'_i)^m(t'_i)\beta) + (q^{-1}-1)zX(\alpha(t'_i)^{m+1}\beta) + X(\alpha(t'_i)(t'_n)^m\beta)$

$= -(q^{-1}-1)z\,X(\alpha(t'_i)^{m+1}\beta) + (q^{-1}-1)zX(\alpha(t'_i)^{m+1}\beta) + X(\alpha(t'_i)(t'_n)^m\beta)$

$= X(\alpha(t'_i)(t'_n)^m\beta)$ as we claimed.

The next property is a consequence of the third property as follows: Since β is a finite sum of the basis elements, we may begin with a basis element in place of β.

$X(\alpha(t'_n)^s(t'_1)^{r_1}\cdots(t'_{n-1})^{r_{n-1}}b)$ where $b\in H(A_{n-2})$,

$= X(\alpha(t'_1)^{r_1}\cdots(t'_{n-1})^{r_{n-1}}b(t'_n)^s)$ by the third property of this theorem,

$= \sum_k a_k X((t'_1)^{s_{1,k}}\cdots(t'_{n-1})^{s_{n-1,k}}b_k(t'_n)^s)$

where $\alpha(t'_1)^{r_1}\cdots(t'_{n-1})^{r_{n-1}}b = \sum_k a_k(t'_1)^{s_{1,k}}\cdots(t'_{n-1})^{s_{n-1,k}}b_k$ and $b_k\in H(A_{n-2})$,

$= \sum_k a_k X((t'_1)^{s_{1,k}}\cdots(t'_{n-1})^{s_{n-1,k}}(t'_n)^s b_k)$

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$= \tau_s \sum_k b_k X((t_1')^{s_{1,k}} \cdots (t_{n-1}')^{s_{n-1,k}} b_k)$ by definition 2.2.1,

$= \tau_s X(\alpha (t_1')^{r_1} \cdots (t_{n-1}')^{r_{n-1}} b)$

$= \tau_s X(\alpha (t_1')^{r_1} \cdots (t_{n-1}')^{r_{n-1}} b)$, which shows property (4) of the theorem.

Property (5) follows from (1) and (2) (see [J]). □

Now, we claim X is indeed a trace map in the following theorem.

Theorem 2.2.3. *The function X defined above is a trace map on $H = \cup_{n=1}^{\infty} H_n(\frac{1}{2})$, that is, it satisfies the property, $X(ab) = X(ba)$.*

Proof:

We will show this inductively on n of $H_n(\frac{1}{2})$. For $n = 1$, $X(t_1^k t_1) = X(t_1 t_1^k)$ is obviously true since t_1 is the only generator in $H_1(\frac{1}{2})$. Assume the assertion is true for $a, b \in H_{n-1}(\frac{1}{2})$.

We show the assertion for $a, b \in H_n(\frac{1}{2})$. For a monomial a in $H_n(\frac{1}{2})$, each factor of a is either t_1 or g_i for $i = 1, \ldots, n-1$. b is a sum of the form $(t_1')^{s_1} \cdots (t_{n-2}')^{s_{n-2}} (t_{n-1}')^{s_{n-1}} (t_n')^{s} \beta$, and $\beta \in H(A_{n-1})$ contains at most one g_{n-1}. So we consider the following cases as:

Case 1) $a = g_i$ for $i \leq n-2$ and $\beta \in H(A_{n-2})$

Case 2) $a = g_i$ for $i \leq n-2$ and g_{n-1} occurs in β

Case 3) $a = t_1$, and $\beta \in H(A_{n-2})$

Case 4) $a = t_1$, and g_{n-1} occurs in β

Case 5) $a = g_{n-1}$ and $\beta \in H(A_{n-2})$

Case 6) $a = g_{n-1}$ and g_{n-1} occurs in β

For the cases 1) and 3), and we use the preceding theorem as follows.

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$X(ab) = X(at_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^{s_{n-1}} (t_n')^s \beta)$

$= \tau_s X(at_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^{s_{n-1}} \beta)$

by theorem 2.2.2(4),

$= \tau_s X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^{s_{n-1}} \beta a)$

by the induction hypothesis,

$= X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^{s_{n-1}} (t_n')^s \beta a)$ by 2.2.2(4),

$= X(ba)$.

For the cases 2) and 4), β can be written as $\beta = \alpha g_{n-1} \gamma$ in normal form in $H(A_{n-1})$.

$X(ab) = X(at_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^{s_{n-1}} (t_n')^{s_n} \alpha g_{n-1} \gamma)$

$= X(at_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^{s_{n-1}} \alpha (t_n')^{s_n} g_{n-1} \gamma)$

$= X(at_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^{s_{n-1}} \alpha g_{n-1} (t_{n-1}')^{s_n} \gamma)$ as $t_n' g_{n-1} = g_{n-1} t_{n-1}'$

$= z\, X(at_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^{s_{n-1}} \alpha (t_{n-1}')^{s_n} \gamma)$ by theorem 2.2.2.,

$= z\, X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^{s_{n-1}} \alpha (t_{n-1}')^{s_n} \gamma a)$ by the inductive assumption,

$= X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^{s_{n-1}} \alpha g_{n-1} (t_{n-1}')^{s_n} \gamma a)$ by theorem 2.2.2.

$= X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^{s_{n-1}} \alpha (t_n')^{s_n} g_{n-1} \gamma a)$

$= X(ba)$

Therefore, we need only show the last two cases, (5) and (6).

Case(5) $\beta \in H(A_{n-2})$. We want to show

$$X(g_{n-1} t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^{s_{n-1}} (t_n')^s \beta) = X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^{s_{n-1}} (t_n')^s \beta g_{n-1}) \qquad (*)$$

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Since $\beta \in H(A_{n-2})$, $\beta = \alpha_1 g_{n-2}\alpha_2$ or $\beta = \alpha_1$ with $\alpha_1, \alpha_2 \in H(A_{n-3})$.

We will use induction on s_{n-1} for $\beta = \alpha_1 g_{n-2}\alpha_2$.

The proof for $\beta = \alpha_1 \in H(A_{n-2})$ is similar.

LHS and RHS denote the left and right hand sides of $(*)$.

If $s_{n-1} = 0$, then LHS $= X(g_{n-1}t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}(t_{n-1}')^0(t_n')^s\alpha_1 g_{n-2}\alpha_2)$

$= (q-1)X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}(t_n')^s\alpha_1 g_{n-2}\alpha_2)$

$+qX(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}(t_{n-1}')^s g_{n-1}^{-1}\alpha_1 g_{n-2}\alpha_2)$

since $g_{n-1}(t_n')^s = (q-1)(t_n')^s + q(t_{n-1}')^s g_{n-1}^{-1}$,

$= (q-1)\tau_s X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}\alpha_1 g_{n-2}\alpha_2)$

$+qq^{-1}X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}(t_{n-1}')^s g_{n-1}\alpha_1 g_{n-2}\alpha_2)$

$+q(q^{-1}-1)X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}(t_{n-1}')^s\alpha_1 g_{n-2}\alpha_2)$

by 2.2.2(4) and expanding g_{n-1}^{-1},

$= z(q-1)\tau_s \quad X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}\alpha_1\alpha_2)$

$+z \quad X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}\alpha_1(t_{n-1}')^s g_{n-2}\alpha_2)$

$+(1-q)X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}\alpha_1 g_{n-2}(t_{n-2}')^s\alpha_2)$

by 2.2.1(3) and 2.2.2.(2),

$= z(q-1)\tau_s \quad X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}\alpha_1\alpha_2)$

$+z^2 \quad X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}\alpha_1(t_{n-2}')^s\alpha_2)$

$-z(q-1) \quad X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}\alpha_1(t_{n-2}')^s\alpha_2)$

as $t_{n-1}'g_{n-2} = g_{n-2}t_{n-2}'$ and by use of 2.2.2(3) and (2),

RHS $= X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}t_{n-1}^0(t_n')^s\alpha_1 g_{n-2}\alpha_2 g_{n-1})$

$= X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}\alpha_1 g_{n-2}\alpha_2 g_{n-1}(t_{n-1}')^s)$

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$=z \quad X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 g_{n-2} \alpha_2 (t'_{n-1})^s)$ by 2.2.2(2),

$=z \quad X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 g_{n-2} (t'_{n-1})^s \alpha_2)$

$= z(q-1) \quad X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 (t'_{n-1})^s \alpha_2)$

$+qz \quad X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 (t'_{n-2})^s g_{n-2}^{-1} \alpha_2)$ as $g_{n-2} = (q-1) + q g_{n-2}^{-1}$,

$=z(q-1) \quad X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 (t'_{n-1})^s \alpha_2)$

$+zqq^{-1} \quad X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 (t'_{n-2})^s g_{n-2} \alpha_2)$

$+zq(q^{-1}-1) \quad X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 (t'_{n-2})^s \alpha_2)$

by expanding g_{n-2}^{-1} and using 2.2.2(2),

$= z(q-1)\tau_s \quad X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 \alpha_2)$

$+z^2 \quad X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 (t'_{n-2})^s \alpha_2)$

$-z(q-1) \quad X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 (t'_{n-2})^s \alpha_2)$ by 2.2.2(4) and (2).

Thus, the assertion is true for $s_{n-1} = 0$.

Assume it is true for $s_{n-1} \le k-1$.

What we now want to show is

$$X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} g_{n-1} (t'_{n-1})^k (t'_n)^s \alpha_1 g_{n-2} \alpha_2)$$
$$= X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} (t'_{n-1})^k (t'_n)^s \alpha_1 g_{n-2} \alpha_2 g_{n-1}) \qquad (\star)$$

Again LHS and RHS are the left and right hand sides of $(\star)$.

$\text{LHS} = X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} g_{n-1} (t'_{n-1})^k (t'_n)^s \alpha_1 g_{n-2} \alpha_2)$

$= X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 g_{n-1} (t'_{n-1})^k g_{n-2} (t'_n)^s \alpha_2)$ by using $\alpha_1 \in H(A_{n-2})$,

$= X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 g_{n-1} \cdots g_1 t_1^k g_1^{-1} \cdots g_{n-3}^{-1} (t'_n)^s \alpha_2)$

by definition of t'_{n-1},

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$$= X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 g_{n-2}(g_{n-1} \cdots g_1) t_1^k g_1^{-1} \cdots g_{n-3}^{-1} (t'_{n-1})^s g_{n-1}^{-1} \alpha_2)$$

by using $g_{n-1}(t'_{n-1})^s g_{n-1}^{-1} = (t'_n)^s$ and $(b1)$,

$$= X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 g_{n-2}(g_{n-1} \cdots g_1) t_1^k g_1^{-1} \cdots g_{n-3}^{-1} g_{n-2} \cdots g_1 t_1^s g_1^{-1} \cdots g_{n-1}^{-1} \alpha_2)$$

by definition of $(t'_{n-1})^s$,

$$= X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 g_{n-2}(g_{n-1} \cdots g_1) t_1^k g_{n-2} \cdots g_1 (g_2^{-1} \cdots g_{n-2}^{-1}) t_1^s g_1^{-1} \cdots$$

$$\cdots g_{n-1}^{-1} \alpha_2) \quad \text{by } (b1),$$

$$= X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 g_{n-2} \cdots g_1 (g_{n-1} \cdots g_1) t_1^k g_1 (g_2^{-1} \cdots g_{n-2}^{-1}) t_1^s g_1^{-1} \cdots g_{n-1}^{-1} \alpha_2)$$

by $(b1)$,

$$= X(t_1'^{s_1} \cdots t_{n-2}'^{s_{n-2}} \alpha_1 g_{n-2} \cdots g_1 (g_{n-1} \cdots g_1) t_1^k g_1 t_1^s (g_1^{-1} \cdots g_{n-1}^{-1}) g_1^{-1} \cdots g_{n-3}^{-1} \alpha_2)$$

by $(b2')$.

Since, by $(a3')$, $(g_{n-1} \cdots g_1) t_1^k g_1 t_1^s (g_1^{-1} \cdots g_{n-1}^{-1})$ is

$$(g_{n-1} \cdots g_2) \left\{ \begin{array}{c} (q-1) \sum_{i=1}^{k-1} g_1 t_1^{k-i} g_1^{-1} t_1^{s+i} \\ + \quad t_1^s g_1 t_1^k \quad + \\ (1-q) \sum_{i=1}^{k-1} g_1 t_1^{k+s-i} g_1^{-1} t_1^i \end{array} \right\} (g_2^{-1} \cdots g_{n-1}^{-1})$$

$$= (q-1) \textstyle\sum_{i=1}^{k-1} g_{n-1} \cdots g_2 g_1 t_1^{k-i} g_1^{-1} g_2^{-1} \cdots g_{n-1}^{-1} t_1^{s+i}$$

$$+ \quad t_1^s g_{n-1} \cdots g_2 g_1 g_2^{-1} \cdots g_{n-1}^{-1} t_1^k$$

$$+ (1-q) \textstyle\sum_{i=1}^{k-1} g_{n-1} \cdots g_2 g_1 t_1^{k+s-i} g_1^{-1} g_2^{-1} \cdots g_{n-1}^{-1} t_1^i$$

$$= (q-1) \textstyle\sum_{i=1}^{k-1} t_n'^{k-i} t_1^{s+i} + \; t_1^s g_1^{-1} \cdots g_{n-2}^{-1} g_{n-1} \cdots g_1 t_1^k$$

$$+ (1-q) \textstyle\sum_{i=1}^{k-1} t_n'^{k+s-i} t_1^i,$$

by the definition of t'_n and using (b1).

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Therefore, continuing the computation of LHS, we obtain

$= (q-1)\sum_{i=1}^{k-1}\tau_{k-i}X(t_1^{\prime s_1}\cdots t_{n-2}^{\prime s_{n-2}}\alpha_1(g_{n-2}\cdots g_1)t_1^{s+i}(g_1^{-1}\cdots g_{n-3}^{-1})\alpha_2)$

$+ z \quad X(t_1^{\prime s_1}\cdots t_{n-2}^{\prime s_{n-2}}\alpha_1(g_{n-2}\cdots g_1)t_1^{s+k}(g_1^{-1}\cdots g_{n-3}^{-1})\alpha_2)$

$-(q-1)\sum_{i=1}^{k-1}\tau_{k+s-i}X(t_1^{\prime s_1}\cdots t_{n-2}^{\prime s_{n-2}}\alpha_1(g_{n-2}\cdots g_1)t_1^{i}g_1^{-1}\cdots g_{n-3}^{-1}\alpha_2)$

by 2.2.2(4) and (2), and the above,

$= z(q-1)\sum_{i=1}^{k-1}\tau_{k-i}X(t_1^{\prime s_1}\cdots t_{n-2}^{\prime s_{n-2}}\alpha_1(g_{n-3}\cdots g_1)t_1^{s+i}(g_1^{-1}\cdots g_{n-3}^{-1})\alpha_2)$

$+ z^2 \quad X(t_1^{\prime s_1}\cdots t_{n-2}^{\prime s_{n-2}}\alpha_1(g_{n-3}\cdots g_1)t_1^{s+k}(g_1^{-1}\cdots g_{n-3}^{-1})\alpha_2)$

$-z(q-1)\sum_{i=1}^{k-1}\tau_{k+s-i}X(t_1^{\prime s_1}\cdots t_{n-2}^{\prime s_{n-2}}\alpha_1(g_{n-3}\cdots g_1)t_1^{i}g_1^{-1}\cdots g_{n-3}^{-1}\alpha_2)$

by 2.2.2(2),

$= z(q-1)\sum_{i=1}^{k-1}\tau_{k-i}X(t_1^{\prime s_1}\cdots t_{n-2}^{\prime s_{n-2}}\alpha_1 t_{n-2}^{\prime s+i}\alpha_2)$

$+ z^2 \quad X(t_1^{\prime s_1}\cdots t_{n-2}^{\prime s_{n-2}}\alpha_1 t_{n-2}^{\prime s+k}\alpha_2)$

$-z(q-1)\sum_{i=1}^{k-1}\tau_{k+s-i}X(t_1^{\prime s_1}\cdots t_{n-2}^{\prime s_{n-2}}\alpha_1 t_{n-2}^{\prime i}\alpha_2)$

by definition of t'_{n-2}.

RHS $= X(t_1^{\prime s_1}\cdots t_{n-2}^{\prime s_{n-2}}(t'_{n-1})^k(t'_n)^s\alpha_1 g_{n-2}\alpha_2 g_{n-1})$ in $(\star)$

$= X(t_1^{\prime s_1}\cdots t_{n-2}^{\prime s_{n-2}}\alpha_1(t'_{n-1})^k g_{n-2}(t'_n)^s g_{n-1}\alpha_2)$ by commutativity,

$= X(t_1^{\prime s_1}\cdots t_{n-2}^{\prime s_{n-2}}\alpha_1(g_{n-2}\cdots g_1)t_1^k(g_1^{-1}\cdots g_{n-2}^{-1})g_{n-2}(t'_n)^s g_{n-1}\alpha_2)$

by definition of $t_{n-1}^{\prime k}$,

$= X(t_1^{\prime s_1}\cdots t_{n-2}^{\prime s_{n-2}}\alpha_1 g_{n-2}\cdots g_1 t_1^k g_1^{-1}\cdots g_{n-3}^{-1}g_{n-1}(t'_{n-1})^s\alpha_2)$

by cancellation and $(t'_n)^s = g_{n-1}(t'_{n-1})^s g_{n-1}^{-1}$,

$= zX(t_1^{\prime s_1}\cdots t_{n-2}^{\prime s_{n-2}}\alpha_1 g_{n-2}\cdots g_1 t_1^k g_1^{-1}\cdots g_{n-3}^{-1}g_{n-2}\cdots g_1 t_1^s g_1^{-1}\cdots g_{n-2}^{-1}\alpha_2)$

by 2.2.2(2),

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$$= zX(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}\alpha_1 g_{n-2}\cdots g_1 t_1^k g_{n-2}\cdots g_1(g_2^{-1}\cdots g_{n-2}^{-1})t_1^s g_1^{-1}\cdots g_{n-2}^{-1}\alpha_2)$$

by $(b1)$,

$$= zX(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}\alpha_1 g_{n-3}\cdots g_1(g_{n-2}\cdots g_1)t_1^k g_1 t_1^s g_1^{-1}\cdots g_{n-2}^{-1} g_1^{-1}\cdots g_{n-3}^{-1}\alpha_2)$$

by $(b1)$ and $(b2')$.

By $(a3')$, $(g_{n-2}\cdots g_2)g_1 t_1^k g_1 t_1^s g_1^{-1}(g_2^{-1}\cdots g_{n-2}^{-1})$ is

$$(g_{n-2}\cdots g_2)\left\{\begin{array}{c}(q-1)\sum_{i=1}^{k-1} g_1 t_1^{k-i} g_1^{-1} t_1^{s+i}\\ +\quad t_1^s g_1 t_1^k \quad +\\ (1-q)\sum_{i=1}^{k-1} g_1 t_1^{k+s-i} g_1^{-1} t_1^i\end{array}\right\}(g_2^{-1}\cdots g_{n-2}^{-1}).$$

RHS

$$= z(q-1)\textstyle\sum_{i=1}^{k-1}\tau_{k-i}X((t_1')^{s_1}\cdots(t_{n-2}')^{s_{n-2}}\alpha_1(g_{n-3}\cdots g_1)t_1^{s+i}(g_1^{-1}\cdots g_{n-3}^{-1})\alpha_2)$$

$$+\, z^2\quad X((t_1')^{s_1}\cdots(t_{n-2}')^{s_{n-2}}\alpha_1(g_{n-3}\cdots g_1)t_1^{s+k}(g_1^{-1}\cdots g_{n-3}^{-1})\alpha_2)$$

$$-z(q-1)\textstyle\sum_{i=1}^{k-1}\tau_{k+s-i}X((t_1')^{s_1}\cdots(t_{n-2}')^{s_{n-2}}\alpha_1(g_{n-3}\cdots g_1)t_1^{i}(g_1^{-1}\cdots g_{n-3}^{-1})\alpha_2)$$

by 2.2.2(4), or $(b1)$ and 2.2.2(2),

$$= z(q-1)\textstyle\sum_{i=1}^{k-1}\tau_{k-i}X((t_1')^{s_1}\cdots(t_{n-2}')^{s_{n-2}}\alpha_1 t_{n-2}'^{s+i}\alpha_2)$$

$$+\, z^2\quad X((t_1')^{s_1}\cdots(t_{n-2}')^{s_{n-2}}\alpha_1 t_{n-2}'^{s+k}\alpha_2)$$

$$-z(q-1)\textstyle\sum_{i=1}^{k-1}\tau_{k+s-i}X((t_1')^{s_1}\cdots(t_{n-2}')^{s_{n-2}}\alpha_1 t_{n-2}'^{i}\alpha_2)$$

This completes the case (5).

Case (6) g_{n-1} occurs in β.

Let $\beta = \alpha_1 g_{n-1}\alpha_2$ where $\alpha_1\ \alpha_2 \in H(A_{n-2})$, then we want to show the following:

$$X(g_{n-1}t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}(t_{n-1}')^k(t_n')^s\alpha_1 g_{n-1}\alpha_2)$$

$$= X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}(t_{n-1}')^k(t_n')^s\alpha_1 g_{n-1}\alpha_2 g_{n-1}) \qquad (**)$$

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LHS and RHS denote the left and right hand sides of (**).

$$\text{LHS} = X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} g_{n-1}(t_{n-1}')^k (t_n')^s \alpha_1 g_{n-1}\alpha_2$$

$$= X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} g_{n-1}(t_{n-1}')^k \alpha_1 g_{n-1}(t_{n-1}')^s \alpha_2$$

since $\alpha_1 \in H(A_{n-2})$, and $(t_n')^s = g_{n-1}(t_{n-1}')^s g_{n-1}^{-1}$,

$$= (q-1)\ \ X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} g_{n-1}(t_{n-1}')^k \alpha_1 (t_{n-1}')^s \alpha_2)$$

$$+q\ \ X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} g_{n-1}(t_{n-1}')^k \alpha_1 g_{n-1}^{-1}(t_{n-1}')^s \alpha_2)$$

by using $g_{n-1} = (q-1) + qg_{n-1}^{-1}$,

$$= z(q-1)\ \ X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^k \alpha_1 (t_{n-1}')^s \alpha_2)$$

$$+q\ \ X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} g_{n-1}(t_{n-1}')^k \alpha_1 g_{n-1}^{-1}(t_{n-1}')^s \alpha_2) \qquad \text{(a)}$$

by 2.2.2(2)

$$\text{RHS} = X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^k (t_n')^s \alpha_1 g_{n-1}\alpha_2 g_{n-1})$$

$= X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^k \alpha_1 g_{n-1}(t_{n-1}')^s \alpha_2 g_{n-1})$ by cancellation,

$$=(q-1)X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^k \alpha_1 (t_{n-1}')^s \alpha_2 g_{n-1})$$

$$+qX(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^k \alpha_1 g_{n-1}^{-1}(t_{n-1}')^s \alpha_2 g_{n-1})$$

by expanding g_{n-1},

$$= z(q-1)X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^k \alpha_1 (t_{n-1}')^s \alpha_2)$$

$$+qX(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^k \alpha_1 g_{n-1}^{-1}(t_{n-1}')^s \alpha_2 g_{n-1}) \qquad \text{(b)}$$

by 2.2.2(2),

Since the first terms of (a) and (b) are the same, we reduce the assertion to

$$X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} g_{n-1}(t_{n-1}')^{s_{n-1}} \alpha_1 g_{n-1}^{-1}(t_{n-1}')^s \alpha_2)$$

$$= X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} (t_{n-1}')^{s_{n-1}} \alpha_1 g_{n-1}^{-1}(t_{n-1}')^s \alpha_2 g_{n-1}) \qquad (\star\star)$$

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Since $\beta = \alpha_1 g_{n-1}\alpha_2$ in normal form of $H(A_{n-1})$,

$\alpha_2 = g_{n-2}\cdots g_{n-p}$ for some p.

We prove here the case in which $\alpha_1 = \alpha_3 g_{n-2}\alpha_4$ and $\alpha_2 = g_{n-2}\alpha_5$ in normal form, by induction on s_{n-1}.

LHS and RHS are again the left and right hand sides of ($\star\star$).

If $s_{n-1} = 0$.

LHS $= X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} g_{n-1}\alpha_1 g_{n-1}^{-1}(t'_{n-1})^s\alpha_2$

$= X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} g_{n-1}\alpha_3 g_{n-2}\alpha_4 g_{n-1}^{-1}(t'_{n-1})^s g_{n-2}\alpha_5)$

$= X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} \alpha_3 g_{n-1} g_{n-2} g_{n-1}^{-1}\alpha_4 g_{n-2}(t'_{n-2})^s\alpha_5)$ by cancellation,

$= X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} \alpha_3 g_{n-2}^{-1} g_{n-1} g_{n-2}\alpha_4 g_{n-2}(t'_{n-2})^s\alpha_5)$ by $(b1)$,

$= z^2\ X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} \alpha_3\alpha_4(t'_{n-2})^s\alpha_5)$ (c)

by repeated use of 2.2.2(2).

RHS $= X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} \alpha_1 g_{n-1}^{-1}(t'_{n-1})^s g_{n-2}\alpha_5 g_{n-1})$

$= X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} \alpha_1 g_{n-1}^{-1} g_{n-2}(t'_{n-2})^s g_{n-1}\alpha_5)$ by cancellation,

$= X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} \alpha_1 g_{n-2} g_{n-1} g_{n-2}^{-1}(t'_{n-2})^s\alpha_5)$ by $(b1)$,

$= z\ X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} \alpha_3 g_{n-2}\alpha_4(t'_{n-2})^s\alpha_5)$ by 2.2.2(2) and α_1

$= z^2\ X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} \alpha_3\alpha_4(t'_{n-2})^s\alpha_5)$ by 2.2.2(2) again. (d)

Thus (c)=(d) proves the case $s_{n-1} = 0$. Assume the assertion is true for $s_{n-1} \leq k-1$. Let $s_{n-1} = k$. We use an argument similar to the previous case.

LHS

$=X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} g_{n-1}\alpha_3(t'_{n-1})^k g_{n-2}\alpha_4 g_{n-1}^{-1}(t'_{n-1})^s g_{n-2}\alpha_5)$

$=X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}} \alpha_3 g_{n-1} g_{n-2} t'^k_{n-2} g_{n-1}^{-1}\alpha_4 g_{n-2}(t'_{n-2})^s\alpha_5)$ by cancellation,

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$=X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}\alpha_3 g_{n-2}^{-1}g_{n-1}g_{n-2}(t'_{n-2})^k)\alpha_4 g_{n-2}(t'_{n-2})^s\alpha_5)$ by $(b1)$,

$= z^2\ X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}\alpha_3(t'_{n-2})^k)\alpha_4(t'_{n-2})^s\alpha_5)$

by repeated use of 2.2.2(2) and cancellation.

RHS

$=X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}\alpha_3(t'_{n-1})^k g_{n-2}\alpha_4 g_{n-1}^{-1}(t'_{n-1})^s g_{n-2}\alpha_5 g_{n-1})$

$=X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}\alpha_3 g_{n-2}t'^k_{n_2}\alpha_4 g_{n-1}^{-1}g_{n-2}(t'_{n-2})^s g_{n-1}\alpha_5)$ by cancellation

$=X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}\alpha_3 g_{n-2}t'^k_{n_2}\alpha_4 g_{n-2}g_{n-1}g_{n-2}^{-1}(t'_{n-2})^s\alpha_5)$

since $g_{n-1}^{-1}g_{n-2}g_{n-1} = g_{n-2}g_{n-1}g_{n-2}^{-1}$,

$= z^2\ X(t_1'^{s_1}\cdots t_{n-2}'^{s_{n-2}}\alpha_3(t'_{n-2})^k)\alpha_4(t'_{n-2})^s\alpha_5)$

by repeated use of 2.2.2(2) and cancellation.

This proves the last case (6). □

2.3 A polynomial invariant of links in a solid torus

We define an invariant of links in the solid torus using the trace map in section 2.2. We represent an oriented link in $S^1 \times D^2$ as a closed braid in $B_n(Ann)$. In order to be well-defined, two braids that differ by a Markov move must give the same result. We adjust the representation π of the braid group into $H_n(\frac{1}{2})$ as did Jones, and S. Lambropoulou.

We can find $\theta \in R$ such that $tr(a(\theta g_{n-1})b) = tr(a((\theta g_{n-1})^{-1})b)$ for $ab \in H_{n-1}(\frac{1}{2})$ where $\theta \in R$, the coefficent ring.

We calculate as follows:

$\theta^2 tr(ag_{n-1})b) = tr(ag_{n-1}^{-1}b) = q^{-1}tr(ag_{n-1}b) + (q^{-1}-1)tr(ab)$

$\theta^2 ztr(ab) = q^{-1}ztr(ab) + (q^{-1}-1)tr(ab)$

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$\theta^2 = (q^{-1}z + (q^{-1} - 1))z^{-1}$

Let $\lambda = \theta^2 = (z + 1 - q)q^{-1}z^{-1}$ where $z = -\frac{1-q}{1-\lambda q}$

Thus the adjusted representation π_λ on σ_i is $\pi_\lambda(\sigma_i) = \sqrt{\lambda}g_i$ and $\pi_\lambda(t) = t_1$, while $\pi(\sigma_i) = g_i$ and $\pi(t) = t_1$. Thus, $\pi_\lambda(\alpha) = \sqrt{\lambda}^e\pi(\alpha)$, and t is not counted in the exponent sum e of α. Let $B_n(Ann)$ be the braid group of annulus.

Theorem 2.3.1. *Let $Z : \bigcup B_n(Ann) \to C(\sqrt{\lambda}, q)[\tau_i]_{i\in\mathbf{Z}}$ be defined by $Z(\alpha) = TX(\pi_\lambda(\alpha)) = (-(1-\lambda q)/\sqrt{\lambda}(1-q))^{n-1}(\sqrt{\lambda})^e X(\pi(\alpha))$ where $T = (\sqrt{\lambda}z)^{1-n} = (-(1-\lambda q)/\sqrt{\lambda}(1-q))^{n-1}$, e to be the exponential sum of σ_i's in α. If $\alpha \in B_n(Ann)$ is a braid representative of the closed braid $\hat{\alpha} = L$ in $S^1 \times D^2$, then $Z(L) = Z(\alpha)$ is a link invariant of links in $S^1 \times D^2$.*

Proof: Since any two representations in $B_n(Ann)$ of a link differ by conjugations and Markov moves (see [LA]), it is enough to see that

(a) $Z(\alpha\beta\alpha^{-1}) = Z(\beta)$

(b) $Z(\alpha\sigma_n) = Z(\alpha)$ if $\alpha \in B_n(Ann)$

(c) $Z(\alpha\sigma_n^{-1}) = Z(\alpha)$ if $\alpha \in B_n(Ann)$

(a) follows by $X(ab) = X(ba)$, Theorem 2.2.3.

$$\begin{aligned} Z(\alpha\sigma_n) &= (\sqrt{\lambda}z)^{1-(n+1)}\sqrt{\lambda}^{e(\alpha\sigma_n)}X(\pi(\alpha\sigma_n)) \\ &= (\sqrt{\lambda}z)^{1-(n+1)}\sqrt{\lambda}^{e(\alpha\sigma_n)}X(\pi(\alpha)g_n) \text{ with } \pi(\alpha) \in H_{n-1}(\tfrac{1}{2}), \\ &= (\sqrt{\lambda}z)^{1-(n+1)}\sqrt{\lambda}^{e(\alpha)}\sqrt{\lambda}z\, X(\pi(\alpha)) \text{ by 2.2.2(2)}, \\ &= (\sqrt{\lambda}z)^{1-n}\sqrt{\lambda}^{e(\alpha)}X(\pi(\alpha)) \\ &= Z(\alpha) \text{ which shows (b).} \end{aligned}$$

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$$
\begin{aligned}
Z(\alpha\sigma_n^{-1}) &= (\sqrt{\lambda}z)^{1-(n+1)}\sqrt{\lambda}^{e(\alpha\sigma_n^{-1})}X(\pi(\alpha\sigma_n^{-1}))\\
&= (\sqrt{\lambda}z)^{1-(n+1)}\sqrt{\lambda}^{e(\alpha)}\sqrt{\lambda}^{-1}X(\pi(\alpha)g_n^{-1}))\\
&= (\sqrt{\lambda}z)^{1-(n+1)}\sqrt{\lambda}^{e(\alpha)}\sqrt{\lambda}^{-1}(q^{-1}z+(q^{-1}-1))X(\pi(\alpha))
\end{aligned}
$$

as $g_n^{-1} = q^{-1}g_n + (q^{-1}-1)$, and 2.2.2(2) applies,

$$= (\sqrt{\lambda}z)^{1-(n+1)}\sqrt{\lambda}^{e(\alpha)}\sqrt{\lambda}zX(\pi(\alpha))$$

as $\sqrt{\lambda}^{-1}(q^{-1}z+(q^{-1}-1)) = \frac{z+1-q}{\sqrt{\lambda}q} = \frac{-(1-q)\lambda q}{\sqrt{\lambda}q(1-\lambda q)} = \sqrt{\lambda}z$

where $z = -\frac{1-q}{1-\lambda q}$,

$$
\begin{aligned}
&= (\sqrt{\lambda}z)^{1-n}\sqrt{\lambda}^{e(\alpha)}X(\pi(\alpha))\\
&= Z(\alpha).
\end{aligned}
$$

Hence Z is indeed an invariant of links in solid torus. □

Theorem 2.3.2. *The map Z satisfies the skein relation:*

$$\frac{1}{\sqrt{q}\sqrt{\lambda}}Z_{L_+} - \sqrt{q}\sqrt{\lambda}Z_{L_-} = (\sqrt{q} - \frac{1}{\sqrt{q}})Z_{L_0}$$

Proof:

Let L be an oriented link diagram. For a single crossing in L, let L_+, L_- and L_0 be the three diagrams that are identical except in a ball containing only the crossing(see Figure 2). By a sequence of applications of elementary deformation (section 1.2), the links turn into closed braids without changing inside the ball [S], [LA]. Then, the crossing depending on the sign becomes either σ_i or σ_i^{-1} for some i in the braid representatives of the closed braids. We may express the braid representatives of the links as $a\sigma_i b, a\sigma_i^{-1}b$, and ab, respectively.

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By the relation $g_i = (q-1) + qg_i^{-1}$ in $H_n(\frac{1}{2})$, we obtain the following identity, $X(\pi(a\sigma_i b)) - q\,X(\pi(a\sigma_i^{-1}b)) = (q-1)X(\pi(ab))$.

Let e be the exponent sum of ab, then that of $a\sigma_i b$ is $e+1$ and that of $a\sigma_i^{-1}b$ is $e-1$.

Multiplying the identity by $T\frac{\sqrt{\lambda}^e}{\sqrt{q}}$, we get

$$\frac{1}{\sqrt{q}\sqrt{\lambda}}T(\sqrt{\lambda})^{e+1}X(\pi(a\sigma_i b)) - \sqrt{q}\sqrt{\lambda}T\sqrt{\lambda}^{e-1}X(\pi(a\sigma_i^{-1}b))$$
$$= (\sqrt{q} - \frac{1}{\sqrt{q}})T\sqrt{\lambda}^e X(\pi(ab))$$

Then, $\frac{1}{\sqrt{q}\sqrt{\lambda}}T\,X(\pi_\lambda(a\sigma_i b)) - \sqrt{q}\sqrt{\lambda}T\,X(\pi_\lambda(a\sigma_i^{-1}b))$

$$= (\sqrt{q} - \frac{1}{\sqrt{q}})T\,X(\pi_\lambda(ab))$$

since $\pi_\lambda(\alpha) = \sqrt{\lambda}^e\pi(\alpha)$.

By the defintion of $Z(\alpha)$,

$\frac{1}{\sqrt{q}\sqrt{\lambda}}Z(a\sigma_i b) - \sqrt{q}\sqrt{\lambda}Z(a\sigma_i^{-1}b) = (\sqrt{q} - \frac{1}{\sqrt{q}})Z(ab)$.

Thus, $\frac{1}{\sqrt{q}\sqrt{\lambda}}Z_{L_+} - \sqrt{q}\sqrt{\lambda}Z_{L_-} = (\sqrt{q} - \frac{1}{\sqrt{q}})Z_{L_0}$.

This shows that Z_L satisfies the skein relation. □

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CHAPTER 3. RECORVERING OTHER INVARIANTS

3.1 Recovering other invariants

We take $S^1 \times D^2$ to be the standard unknoted torus in S^3 and consider the map from links in $S^1 \times D^2$ to links in S^3. We first want to recover the HOMFLY-PT link polynomials defined by Jones in S^3, and Lambropoulou's polynomial invariant of links in $S^1 \times D^2$ from Z_L defined in section 2.3. Recall that for $\sigma_i, t \in B_n(Ann)$, the braid group of annulus, $\pi : B_n(Ann) \to H_n(\frac{1}{2})$ is defined by $\pi(t) = t_1, \quad \pi(\sigma_i) = g_i$, while $\pi_\lambda : B_n(Ann) \to H_n(\frac{1}{2})$ defined by $\pi_\lambda(t) = t_1, \quad \pi_\lambda(\sigma_i) = \sqrt{\lambda}\, g_i$. We denote the trace maps depending on the algebras as tr on $H(A_{n-1})$, X_B on $H(B_n)$, and X on $H_n(\frac{1}{2})$.

Theorem 3.1.1. *Let* $Z_L = \sum_k^{K:finite} h_k(\sqrt{\lambda}, q) v_k(\{\tau_i\}_{i \in \mathbf{Z}})$ *where* v_k *is a finite product of* τ_i*'s, then the HOMFLY-PT polynomial* X_L *in* S^3 *is* $X_L(\sqrt{\lambda}, q) = \sum_k^{K:finite} h_k(\sqrt{\lambda}, q)$, *i.e.,* X_L *is* Z_L *with* $\tau_i = 1$ *for all* i.

Proof:

Let $\phi : H_n(\frac{1}{2}) \to H(A_{n-1})$ be defined by $\phi(t_1) = 1$ and $\phi(g_i) = g_i$. Then ϕ is an algebra homomorphism such that $\phi(\sum_i a_i t_1'^{s_{1,i}} \cdots t_n'^{s_{n,i}} \alpha_i)$ $= \sum_i a_i \alpha_i$ where $t_1'^{s_{1,i}} \cdots t_n'^{s_{n,i}} \alpha_i$ is a basis element of $H_n(\frac{1}{2})$ and α_i is a basis element of $H(A_{n-1})$. Since both trace functions defined on $H_n(\frac{1}{2})$ and $H(A_{n-1})$ are linear maps, it suffices to show the following: if $X((t_1')^{s_1} \cdots (t_{n-1}')^{s_{n-1}} (t_n')^{s_n} \alpha) = \sum_k f_k(q,z) v_k$, then $tr(\alpha) = \sum_k f_k(q,z)$ where X is the trace map defined on $H_n(\frac{1}{2})$, while tr is the trace map on $H(A_{n-1})$.

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We prove it by induction on n.

For $n = 1$, the algebras are $H_1(\frac{1}{2})$ and $H(A_0)$, it is true since $X((t_1')^s\alpha) = \tau_s$ where $\alpha = 1$, and $tr(\alpha) = tr(1) = 1$. Assume that the assertion is true for every $j \leq n-1$. For $j = n$, $\alpha \in H(A_{n-2})$ or $\alpha = \alpha_1 g_{n-1}\alpha_2$ as a basis element of $H(A_{n-1})$.

If $\alpha \in H(A_{n-2})$,

$X((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}}(t_n')^{s_n}\alpha)$

$= \tau_{s_n}X((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}}\alpha)$ by 2.2.1(2)

$= \tau_{s_n}\sum_k f_k(q,z)v_k$ by the induction hypothesis,

$= \sum_k f_k(q,z)\tau_{s_n}v_k$

By the induction hypothesis, $tr(\alpha) = \sum_k f_k(q,z)$.

If $\alpha = \alpha_1 g_{n-1}\alpha_2$,

$X((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}}(t_n')^{s_n}\alpha)$

$= X((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}}(t_n')^{s_n}\alpha_1 g_{n-1}\alpha_2)$

$= z\,X((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}}\alpha_1(t_{n-1}')^{s_n}\alpha_2)$

by 2.2.1(3),

$= z\sum_k f_k(q,z)v_k = \sum_k z\,f_k(q,z)v_k$

We have $tr(\alpha_1\alpha_2) = \sum_k f_k(q,z)$ by the induction hypothesis, thus, $tr(\alpha) = tr(\alpha_1 g_{n-1}\alpha_2) = ztr(\alpha_1\alpha_2) = z\sum_k f_k(q,z) = \sum_k zf_k(q,z)$, as required. Therefore,

$Z(t_1'^{s_1}\cdots t_n'^{s_n}\alpha) = (\sqrt{\lambda}z)^{1-n}\sqrt{\lambda}^e X(t_1'^{s_1}\cdots t_n'^{s_n}\alpha)$

$= (\sqrt{\lambda}z)^{1-n}\sqrt{\lambda}^e\sum_k f_k(q,z)v_k,$

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$= \sum_k h_k(\sqrt{\lambda}, q)v_k$ by letting $h_k(\sqrt{\lambda}, q) = (\sqrt{\lambda}z)^{1-n}\sqrt{\lambda}^e f_k(q, z)$.

Since $\phi(t_1'^{s_1} \cdots t_n'^{s_n}\alpha) = \alpha$, and

$X_\alpha = (\sqrt{\lambda}z)^{1-n}\sqrt{\lambda}^e tr(\alpha)$, the polynomial of $\alpha \in H(A_{n-1})$,

$= (\sqrt{\lambda}z)^{1-n}\sqrt{\lambda}^e \sum_k f_k(q, z)$

$= \sum_k h_k(\sqrt{\lambda}, q)$ since $h_k(\sqrt{\lambda}, q) = (\sqrt{\lambda}z)^{1-n}\sqrt{\lambda}^e f_k(q, z)$.

□

Recall, from section 2.1, $p : H_n(\frac{1}{2}) \to H(B_n)$ is defined by $p(g_i) = g_i$ and $p(t_1) = t_1$. Then $p(\alpha t_1^2\beta) = (Q-1)\alpha t_1\beta + Q\alpha\beta$, for any α, β (see [LA]). Recall from the introduction that $I_L = (\sqrt{\lambda}z)^{1-n}X_B \circ p \circ \pi_\lambda(\alpha)$ is the invariant of Lambropoulou where π is a representation of $B_n(Ann)$ to $H(B_n)$, the type-B Hecke algebra, and $\pi_\lambda(\alpha) = \sqrt{\lambda}^e\pi(\alpha)$.

Theorem 3.1.2. *Suppose L is a link in $S^1 \times D^2$. Then $I_L = Z_L(\{a_i\tau + b_i\}_{i\in\mathbf{Z}})$ where $a_i, b_i \in C[Q]$, and formulas for a_i, b_i are given below.*

Proof: A basis element of $H_n(\frac{1}{2})$ is of the form

$A = (t_1')^{s_1} \cdots (t_{n-1}')^{s_{n-1}}(t_n')^{s_n}\beta$, for β a basis element of $H(A_{n-1})$. In $H(B_n)$, for any integer s, $p(t_i'^s) = a_s t_i' + b_s$, with a_s, b_s as follows:

$a_s = Q^{s-1} - Q^{s-2} + \cdots - 1$, $b_s = Q^{s-1} - Q^{s-2} + \cdots + Q$ for $s \geq 2$ even,

$a_s = Q^{s-1} - Q^{s-2} + \cdots + 1$, $b_s = Q^{s-1} - Q^{s-2} + \cdots - Q$ for $s \geq 2$ odd.

$a_1 = 1$, $b_1 = 0$

$a_0 = 0$, $b_0 = 1$

$a_s = Q^s - Q^{s+1} + \cdots - Q^{-1}$, $b_s = Q^s - Q^{s+1} + \cdots + 1$ for s negative even,

$a_s = Q^s - Q^{s+1} + \cdots + Q^{-1}$, $b_s = Q^s - Q^{s+1} + \cdots - 1$ for s negative odd.

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The above formulas are seen for t_1 by noting that $p(t_1^s) = p(t_1^2)p(t_1^{s-2}) = ((Q-1)t_1 + Q)(a_{s-2}t_1 + b_{s-2})$. One can check that the formulas satisfy the recursion. Then note that $p(t_i'^s) = p(g_{i-1}\cdots g_1 t_1^s g_1^{-1}\cdots g_{i-1}^{-1}) = p(a_s t_i' + b_s)$. Thus, for $A = (t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}}(t_n')^{s_n}\beta$,

$p(A) = (a_{s_1}t_1' + b_{s_1})(a_{s_2}t_2' + b_{s_2})\cdots(a_{s_n}t_n' + b_{s_n})\beta$

with a_{s_i}, b_{s_i} in $C(q,\ Q,\ \sqrt{\lambda})$. Thus $p(A)$ is expanded as:

$p(A) = \sum(\prod_{e_i=1} a_{s_i})(\prod_{e_i=0} b_{s_i})(t_1')^{e_1}\cdots(t_n')^{e_n}\beta$, a sum over the set

$\{(e_1, e_2, \cdots, e_n) | e_i = 0 \text{ or } 1,\ s_i \neq 0\}$. Each term is a basis element of $H(B_n)$, the basis of which is $\{(t_1')^{e_1}\cdots(t_n')^{e_n}\beta | e_i = 0 \text{ or } 1\}$, β is a basis element of $H(A_{n-1})$. Suppose L is the closed braid $\hat{\alpha}$. Since $p\circ\pi_\lambda = \sqrt{\lambda}^e p\circ\pi$, $(\sqrt{\lambda}z)^{1-n}(X_B\circ p\circ\pi_\lambda(\alpha)) = (\sqrt{\lambda}z)^{1-n}\sqrt{\lambda}^e(X_B\circ p\circ\pi(\alpha))$ is consequently the polynomial invariant I applied to L.

Below we define a map p_*. Consider the diagram below. The two maps π are different. One is the representation into $H_n(\frac{1}{2})$, the other S. Lambropoulou's into $H(B_n)$.

$$\begin{array}{ccccccc}
B_n & \xrightarrow{\pi} & H_n & \xrightarrow{X} & C(q,\sqrt{\lambda})[\tau_i]_{i\in\mathbf{Z}} & \xrightarrow{(\sqrt{\lambda}z)^{1-n}} & C(q,\sqrt{\lambda})[\tau_i]_{i\in\mathbf{Z}} \\
\downarrow id & & p\downarrow & & p_*\downarrow & & p_*\downarrow \\
B_n & \xrightarrow{\pi} & H(B_n) & \xrightarrow{X_B} & C(q,\ Q,\ \sqrt{\lambda})[\tau] & \xrightarrow{(\sqrt{\lambda}z)^{1-n}} & C(q,\ Q,\ \sqrt{\lambda})[\tau]
\end{array}$$

Let's define $p_* : C(q,\ \sqrt{\lambda})[\tau_i]_{i\in\mathbf{Z}} \to C(q,\ Q,\ \sqrt{\lambda})[\tau]$ by $p_*(\tau_i) = (a_{s_i}\tau + b_{s_i})$. So $p_*(\sum_i h_i(\sqrt{\lambda}, q)\prod_{s_i}\tau_{s_i}^{m_i}) = \sum_i h_i(\sqrt{\lambda}, q)\prod_{s_i}(a_{s_i}\tau + b_{s_i})^{m_i}$ for all $s_i \in \mathbb{Z}$. Except for the common factor of $\sqrt{\lambda}^e$, the bottom row gives the invariant I, and the top row gives the invariant Z.

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Hence the proof of the theorem will be complete when we show the diagram commutes. The first square commutes by the definition of the maps.

We show that p_* satisties $p_* \circ X = X_B \circ p$ by induction on n.

For $k = 1$, $p_* \circ X((t_1')^s) = p_*(\tau_s) = a_s\tau + b_s$ and

$X_B \circ p((t_1')^s) = X_B(a_s t_1' + b_s) = a_s\tau + b_s$.

Assume it is true for every $k \leq n - 1$. For $k = n$, let $(t_1')^{s_1} \cdots (t_{n-1}')^{s_{n-1}} (t_n')^{s_n} \alpha$ be a basis element with $\alpha \in H(A_{n-2})$ or $\alpha = \alpha_1 g_{n-1} \alpha_2$ as a basis element of $H(A_{n-1})$. If $\alpha \in H(A_{n-2})$, then

$p_* \circ X((t_1')^{s_1} \cdots (t_{n-1}')^{s_{n-1}} (t_n')^{s_n} \alpha)$

$= p_*(\tau_{s_n} X((t_1')^{s_1} \cdots (t_{n-1}')^{s_{n-1}} \alpha))$ by 2.2.2(4),

$= (a_{s_n}\tau + b_{s_n}) p_*(X((t_1')^{s_1} \cdots (t_{n-1}')^{s_{n-1}} \alpha))$

by definition of τ.

On the other hand, $X_B \circ p((t_1')^{s_1} \cdots (t_{n-1}')^{s_{n-1}} (t_n')^{s_n} \alpha)$

$= X_B(p((t_n')^{s_n})\, p((t_1')^{s_1} \cdots (t_{n-1}')^{s_{n-1}} \alpha))$

$= X_B((a_{s_n} t_n' + b_{s_n})\, p((t_1')^{s_1} \cdots (t_{n-1}')^{s_{n-1}} \alpha))$

$= a_{s_n} X_B(t_n'\, p((t_1')^{s_1} \cdots (t_{n-1}')^{s_{n-1}} \alpha)) + b_{s_n} X_B \circ p((t_1')^{s_1} \cdots (t_{n-1}')^{s_{n-1}} \alpha))$

$= a_{s_n} \tau X_B(p((t_1')^{s_1} \cdots (t_{n-1}')^{s_{n-1}} \alpha))$

$+ b_{s_n} X_B \circ p((t_1')^{s_1} \cdots (t_{n-1}')^{s_{n-1}} \alpha))$

by definition of X_B,

$= (a_{s_n}\tau + b_{s_n}) X_B(p((t_1')^{s_1} \cdots (t_{n-1}')^{s_{n-1}} \alpha))$

$= (a_{s_n}\tau + b_{s_n}) p_*(X((t_1')^{s_1} \cdots (t_{n-1}')^{s_{n-1}} \alpha))$

by the induction hypothesis.

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If $\alpha = \alpha_1 g_{n-1}\alpha_2$, then

$p_* \circ X((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}}(t_n')^{s_n}\alpha)$

$= p_*(X((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}}(t_n')^{s_n}\alpha_1 g_{n-1}\alpha_2))$

$= p_*(zX((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}}\alpha_1(t_{n-1}')^{s_n}\alpha_2))$ by definition 2.2.1,

$= p_*(z)p_*(X((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}}\alpha_1(t_{n-1}')^{s_n}\alpha_2))$

$= z\ p_* \circ X((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}}\alpha_1(t_{n-1}')^{s_n}\alpha_2)$; since $p_*(z) = z$.

And $X_B \circ p((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}}(t_n')^{s_n}\alpha)$

$= X_B(p((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}}(t_n')^{s_n}\alpha_1 g_{n-1}\alpha_2)$

$= X_B(p((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}})p((t_n')^{s_n})\alpha_1 g_{n-1}\alpha_2)$

$= X_B(p((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}})(a_{s_n}t_n' + b_{s_n})\alpha_1 g_{n-1}\alpha_2)$

$= a_{s_n}X_B(p((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}})t_n'\alpha_1 g_{n-1}\alpha_2)$

$+b_{s_n}X_B(p((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}})\alpha_1 g_{n-1}\alpha_2)$

$= a_{s_n}z\,X_B(p((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}})\alpha_1 t_{n-1}'\alpha_2)$

$+b_{s_n}z\,X_B(p((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}})\alpha_1\alpha_2)$ by definition of X_B,

$= z\,X_B(p((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}})\alpha_1(a_{s_n}t_{n-1}' + b_{s_n})\alpha_2)$

$= z\,X_B \circ p((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}}\alpha_1 t'^{s_n}_{n-1}\alpha_2)$;

$= z\ p_* \circ X((t_1')^{s_1}\cdots(t_{n-1}')^{s_{n-1}}\alpha_1 t'^{s_n}_{n-1}\alpha_2)$; by the induction hypothesis.

Hence, $p_* \circ X = X_B \circ p$ as required.

The last rectangle is commutative since the maps from $C(\sqrt{\lambda}, q)[\tau_i]_{i\in\mathbf{Z}}$ to $C(q,\ Q,\ \sqrt{\lambda})[\tau_i]_{i\in\mathbf{Z}}$ and $C(q, Q, \sqrt{\lambda})[\tau]$ to $C(q, Q, \sqrt{\lambda})[\tau]$ are just multiplication by the constant $T = (\sqrt{\lambda}z)^{1-n}$. This completes the proof. □

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Remark. *The polynomial $X(\pi_\lambda(\alpha))$ is a regular isotopy invariant of links in a solid torus. Moreover, it satisfies a skein relation:*

$R_1 : \frac{1}{\sqrt{q}\sqrt{\lambda}}X(\pi_\lambda(\alpha_+)) - \sqrt{q}\sqrt{\lambda}X(\pi_\lambda(\alpha_-)) = (\sqrt{q} - \frac{1}{\sqrt{q}})X(\pi_\lambda(\alpha_0)).$

Proof: The relations , $\sigma_i\sigma_i^{-1} = 1$, and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, in $B_n(Ann)$ and the property $X(a^{-1}ba) = X(b)$ in Theorem 2.2.5 show that it is a regular isotopy. For the skein relation, it is much the same as the case Z_α proved in section 2.3 omitting multiplication by $T := (\sqrt{\lambda}z)^{1-n}$ (see Theorem 2.3.2).

□

Remark. *The three trace maps, tr on $H(A_{n-1})$, X_B on $H(B_n)$, and X on $H_n(\frac{1}{2})$, all satisfy the following list of properties:*

(1) $tr'(\beta(t'_{n-1})^s g_{n-1}) = z_s tr'(\beta)$ *if* $\beta \in H_{n-2}(\frac{1}{2})$;

(2) $tr'(\alpha g_{n-1}) = ztr'(\alpha)$ *if* $\alpha \in H_{n-1}(\frac{1}{2})$ *and*

α *is not of the form* $\beta(t'_{n-1})^s$ *as in (1).*

(3) $tr'(\alpha(t'_n)^s) = l_s tr'(\alpha)$ *if* $\alpha \in H_{n-1}(\frac{1}{2})$;

(4) $tr'(1) = 1$.

If $z_s = z$ and $l_s = 1$, then $tr' = tr$ for type-A Hecke algebra.

If $z_s = z \cdot (a_s\tau + b_s)$ and $l_s = (a_s\tau + b_s)$, then $tr' = X_B$ for type-B Hecke algebra.

If $z_s = z\tau_s$ and $l_s = \tau_s$, then $tr' = X$ for algebra $H_n(\frac{1}{2})$.

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DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Jaehoo Park Kim

Major Field: Mathematics

Title of Dissertation: A Polynomial Invariant of Links in a Solid Torus

Approved:

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

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