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Some New Techniques and their Applications in the Theory of Distributions

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SOME NEW TECHNIQUES AND THEIR APPLICATIONS IN THE THEORY OF DISTRIBUTIONS

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Abstract

This dissertation is a compilation of three articles in the theory of distributions. Each essay focuses on a different technique or concept related to distributions.

The focus of the first essay is the concept of distributional point values. Distributions are sometimes called generalized functions, as they share many similarities with ordinary functions, with some key differences. Distributional point values, among other things, demonstrate that distributions are even more akin to ordinary functions than one might think.

The second essay concentrates on two major topics in analysis, namely asymptotic expansions and the concept of moments. There are many variations of moment problems and we demonstrate how techniques from asymptotic analysis and the theory of distributions can be used to study such problems.

The third essay presents several Tauberian theorems for smooth functions. Tauberian theorems are theorems in analysis of a particular form. The first theorem of this type was proved by Alfred Tauber in 1897. These results are then applied to the division problem for tempered distributions.

Introduction

Laurent Schwartz is generally credited with inventing distributions as a means to solve differential equations that did not admit ordinary solutions. Distributions are sometimes fittingly referred to as *generalized functions*, as they share many features with ordinary functions, but with some added benefit. For example, any distribution is differentiable; in fact, infinitely differentiable. However, there are some drawbacks, such as not being closed under multiplication.

Distributions are often used in applied mathematics or physics. They can also be studied for their own sake, which is the focus of this research. More specifically, we present several techniques that one can employ to study distributions.

The basic ideas about distributions can be found in the texts [1, 27, 54, 58]. Questions about the topological vector space structure of the spaces of distributions are available in those textbooks as well as in several texts on Functional Analysis [25, 55].

We shall employ the standard notation for the usual spaces of test functions and distributions, namely, $\mathcal{D}$ and $\mathcal{D}'$, $\mathcal{S}$ and $\mathcal{S}'$, or $\mathcal{E}$ and $\mathcal{E}'$, in one or several variables. There are other important spaces such as $\mathcal{O}_M$, which we will work with later.

0.0.1. Distributions of several variables

When considering distributions of several variables, $f \in \mathcal{D}'(\mathbb{R}^n)$, it is convenient to employ polar coordinates, $\mathbf{x} = r\omega$, $r > 0$, $\omega \in \mathbb{S}$, where $\mathbb{S}$ is the unit sphere. Indeed, suppose that $\mathbf{0} \notin \operatorname{supp} f$; then if $\varphi \in \mathcal{D}(\mathbb{S})$ is a test function on the unit sphere we define the distribution of one variable $\langle f(r\omega), \varphi(\omega)\rangle_\omega$ as

$$\left\langle \langle f(r\omega), \varphi(\omega)\rangle_\omega, r^{n-1}\rho(r)\right\rangle = \langle f(\mathbf{x}), \Phi(\mathbf{x})\rangle_{\mathbf{x}}\ , \qquad (0.0.1)$$

where $\Phi(\mathbf{x}) = \rho(r)\varphi(\omega)$ for test functions $\rho \in \mathcal{D}(\mathbb{R})$. One can also define $\langle f(r\omega), \varphi(\omega)\rangle_\omega$ in case $\mathbf{0}$ is in the support of f [14]. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ if and only if $\langle f(r\omega), \varphi(\omega)\rangle_\omega$ belongs to $\mathcal{S}'(\mathbb{R})$ for all $\varphi \in \mathcal{D}(\mathbb{S})$ (when $\mathbf{0} \in \operatorname{supp} f$ we just write $f = f_1 + f_2$ where f_1 has compact support, $f_1 \in \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R})$, and where $\mathbf{0} \notin \operatorname{supp} f_2$, since $f \in \mathcal{S}'(\mathbb{R}^n)$ if and only if $f_2 \in \mathcal{S}'(\mathbb{R}^n)$). We will elaborate more on distributions of several variables when necessary. Actually, as part of future research, I plan to expand on these methods in the style of this thesis.

0.0.2. The Cesàro Behavior of Distributions at Infinity

The behavior of distributions at infinity can be studied by considering parametric asymptotics or asymptotics in the Cesàro sense, as we now explain. See [6, 17, 48, 49, 59]. We shall start with the situation in one variable, at $+\infty$, the case at $-\infty$ being similar.

The Cesàro behavior of a distribution at infinity was first considered in [11], using the order symbols $O(x^\beta)$ and $o(x^\beta)$ in the Cesàro or *average* sense. By an N^{th} order *primitive* of a distribution $f \in \mathcal{D}'(\mathbb{R})$, we mean a distribution F such that $F^{(N)} = f$.

Definition 0.0.1. Let $f \in \mathcal{D}'(\mathbb{R})$ and $\beta \in \mathbb{R} \setminus \{-1, -2, -3, \ldots\}$. We say that

$$f(x) = O(x^\beta) \quad (\mathrm{C}) , \quad x \to \infty , \tag{0.0.2}$$

if there is a natural number N, an N^{th} order primitive F of f, and a polynomial p of degree $N-1$ such that F is locally integrable for large x and

$$F(x) = p(x) + O(x^{N+\beta}) , \quad x \to \infty , \tag{0.0.3}$$

holds. Here (1.1.7) is in the ordinary sense.

When $\beta = -1, -2, -3, \ldots$ then a slightly different definition should be used, replacing the function $x^{N+\beta}$ by a fixed primitive of x^β [6, 17, 48, 49, 59]. Usually we will consider the case $\beta \in \mathbb{R} \setminus \{-1, -2, -3, \ldots\}$.

Let $f \in \mathcal{D}'(\mathbb{R})$ be a distribution with support bounded on the left. Then the behavior of f in the Cesàro sense is closely related to the behavior of $f(\lambda x)$ as $\lambda \to +\infty$ in the distributional sense, that is, the asymptotics of $\langle f(\lambda x), \phi(x) \rangle$ for each test function $\phi \in \mathcal{D}(\mathbb{R})$. In fact, if $\beta > -1$ then $f(x) = O(x^\beta)$ (C) as $x \to \infty$ if and only if $f(\lambda x) = O(\lambda^\beta)$ as $\lambda \to \infty$, while when $-(k+1) > \beta > -(k+2)$ for some $k \in \mathbb{N}$, the Cesàro relation is equivalent to the existence of constants $\mu_0, \ldots, \mu_k$ such that as $\lambda \to \infty$

$$f(\lambda x) = \sum_{j=0}^{k} \frac{(-1)^j \mu_j \delta^{(j)}(x)}{j! \lambda^{j+1}} + O(\lambda^\beta) \ . \tag{0.0.4}$$

The constants μ_j are the generalized moments of f.

The Cesàro behavior allows us to characterize several spaces of distributions. For example [11, 17] a distribution of one variable f satisfies the *moment asymptotic expansion*, $f(\lambda x) \sim \sum_{j=0}^{\infty} (-1)^j \mu_j \delta^{(j)}(x) / j! \lambda^{j+1}$ as $\lambda \to \infty$ if and only if it is of rapid decay at infinity in the Cesàro sense, that is, $f(x) = O(|x|^\beta)$ (C) as $|x| \to \infty$ for all β, and this is in turn equivalent to $f \in \mathcal{K}'(\mathbb{R})$. A very important result of this kind is the following characterization of tempered distributions in terms of their average behavior at infinity [11, 17], a result that is useful throughout our analysis.

Proposition 0.0.2. *Let $f \in \mathcal{D}'(\mathbb{R})$. Then the following statements are equivalent:*

1. *f is a tempered distribution, i.e., $f \in \mathcal{S}'(\mathbb{R})$.*
2. *There exists $\alpha \in \mathbb{R}$ such that*

$$f(x) = O(x^\alpha) \ \ (C) \ , \ x \to \infty \, , \tag{0.0.5}$$

and

$$f(x) = O(|x^\alpha|) \ \ (C) \, , \ x \to -\infty \, . \tag{0.0.6}$$

3. There exists $\alpha \in \mathbb{R}$ *such that*

$$f(\lambda x) = O(\lambda^{\alpha}) \, , \, \lambda \to \infty \, , \tag{0.0.7}$$

distributionally[1]*.*

0.0.3. Distributional Regularizations

The basic idea of a regularization is as follows. Start with a function g that is not locally integrable; say g has a singularity at $x = x_0$. Then if $\mathcal{A}$ is a space of test functions and $\phi \in \mathcal{A}$, the integral

$$\int_{-\infty}^{\infty} g(x)\phi(x)\, dx$$

may or may not converge; in fact, in general, the integral will converge if there is an n such that $\phi^j(x_0) = 0$ for $j \leq n$, but not otherwise. This means that the expression

$$\langle g(x), \phi(x) \rangle = \int_{-\infty}^{\infty} g(x)\phi(x)\, dx$$

does not define a regular distribution in $\mathcal{A}'$, but it can be defined on a closed subspace of $\mathcal{A}$. By the Hanh-Banach Theorem, there exists an extension of g to all of $\mathcal{A}$; such an extension is called a *regularization* of g. Observe that regularizations may not exist and, even if they do, they are not unique. There are several methods for regularizations such as the Cauchy principal value, analytic continuation, or the method of finite parts. There are specific cases where a particular method works, but the others do not.

[1]Estimate (1.1.10) can be understood in the weak sense, $\langle f(\lambda x), \phi(x) \rangle = O(\lambda^{\alpha})$ for each $\phi \in \mathcal{D}$, or in the strong topology of $\mathcal{D}'$, since weak and strong convergence of sequences coincides in $\mathcal{D}'$ [25, 55].

Chapter 1. Distributional Point Values and Delta Sequences

1.1. Motivation and Background

Distributional point values were first defined in one variable by Łojasiewics [35]. His definition is given as a distributional limit over a *continuous* variable. In other words, if $f \in \mathcal{D}'(\mathbb{R})$ and $x_0 \in \mathbb{R}$ then we say that f has a distributional point value, equal to γ, at x_0 if

$$\lim_{\varepsilon \to 0} f(x_0 + \varepsilon x) = \gamma\,, \tag{1.1.1}$$

in the distributional sense, that is, if for all test functions $\phi \in \mathcal{D}(\mathbb{R})$ we have that

$$\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi(x) \rangle = \gamma \int_{-\infty}^{\infty} \phi(x)\, \mathrm{d}x\,. \tag{1.1.2}$$

Similarly, in [36] point values in several variables are defined as a distributional limit over a continuous variable. Point values have been studied extensively and are the first step in the study of distributional asymptotic analysis and of the study of local properties of distributions [6, 17, 46, 48, 49, 59].

It is possible to find in the literature other definitions of distributional point values, based on the use of *delta sequences.* For instance, in a recent study, Sasane [52] uses the alternative definition "$f(x_0) = \eta$" if for all positive and even test functions ϕ with $\int_{-\infty}^{\infty} \phi(x)\, \mathrm{d}x = 1$ one has

$$\lim_{n \to \infty} \langle f(x_0 + x), \phi_n(x) \rangle = \eta\,, \tag{1.1.3}$$

where $\{\phi_n\}_{n=1}^{\infty}$ is the standard delta sequence generated by ϕ, namely, $\phi_n(x) = n\phi(nx)$. Naturally the question arises if the two definitions are equivalent. More generally, if $\mathfrak{F}$ is a family of test functions, we would like to consider the relationship between the existence of the distributional point value and the existence of the limit (2.1.4) whenever the sequence

$\{\phi_n\}_{n=1}^{\infty}$ belongs to $\mathfrak{F}$. Interestingly, the two definitions are *not* equivalent for many classes of delta sequences, in particular for the family considered in [52]. Nevertheless, we are able to show that for *some* classes they are actually equivalent.

In order to study this problem, we start by studying a very general question about limits. Indeed, in a metric space X, given a function $f : X \setminus \{x_0\} \to \mathbb{R}$, then the limit

$$\lim_{x \to x_0} f(x) = L\,, \tag{1.1.4}$$

exists if and only if

$$\lim_{n \to \infty} f(x_n) = L\,, \tag{1.1.5}$$

for all sequences $\{x_n\}_{n=1}^{\infty}$ in $X \setminus \{x_0\}$ that converge to x_0. The question we would like to consider is whether the existence of the limit $\lim_{n\to\infty} f(x_n)$ for sequences $\{x_n\}_{n=1}^{\infty}$ of a certain family implies that (1.1.4) is satisfied. We consider the case where $X = (0, \infty]$, $x_0 = \infty$ and f continuous, showing that in such cases the existence of the limit

$$\lim_{n \to \infty} f(na) = F(a)\ , \tag{1.1.6}$$

for all $a > 0$ implies that, in fact, F is a constant function, $F(a) = L$, for all $a > 0$, and that $\lim_{x\to\infty} f(x) = L$. We give examples of other families of sequences for which $\lim_{x\to\infty} f(x) = L$ might not hold true. We are also able to present, in Section 1.3, a corresponding result when the function f in (1.1.6) is not necessarily continuous but just measurable and the limit holds almost everywhere.

As previously stated, if $f \in \mathcal{D}'(\mathbb{R})$ and $x_0 \in \mathbb{R}$ then [35] we say that f has a *distributional point value,* equal to γ, at x_0 if $\lim_{\varepsilon\to 0} f(x_0 + \varepsilon x) = \gamma$, in the strong topology of $\mathcal{D}'(\mathbb{R})$. Equivalently, since a sequence of distributions converges strongly if and only if it converges weakly [55], if for all test functions $\phi \in \mathcal{D}(\mathbb{R})$ we have that

$$\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi(x) \rangle = \gamma \int_{-\infty}^{\infty} \phi(x)\, \mathrm{d}x\,. \tag{1.1.7}$$

Interestingly, the existence of the distributional limit $\lim_{\varepsilon \to 0} f(x_0 + \varepsilon x)$ implies that this limit is a constant and that the point value exists. On the other hand, if the limit

$$\lim_{\varepsilon \to 0^+} f(x_0 + \varepsilon x) = g(x)\ , \tag{1.1.8}$$

exists, then g does not have to be a constant, but it will have the jump behavior [57], that is, g is of the form

$$g(x) = \gamma_- H(-x) + \gamma_+ H(x)\ , \tag{1.1.9}$$

where H is the Heaviside function and $\gamma_\pm$ are some constants. Distributions of the form (1.1.9) are the most general homogeneous distributions of degree 0 in one variable. Alternatively, (1.1.8) and (1.1.9) hold if and if the lateral limits $f(x_0 \pm 0) = \lim_{\varepsilon \to 0^+} f(x_0 \pm \varepsilon x) = \gamma_\pm$, exist in $\mathcal{D}'(0, \infty)$ and f does not have delta functions at x_0.

In several variables point values are defined similarly [36], namely, if $f \in \mathcal{D}'(\mathbb{R}^d)$, then the distributional point value $f(\mathbf{x}_0)$ exists and equals γ if $\lim_{\varepsilon \to 0} f(\mathbf{x}_0 + \varepsilon \mathbf{x}) = \gamma$, distributionally. In several variables the limit $\lim_{\varepsilon \to 0} f(\mathbf{x}_0 + \varepsilon \mathbf{x})$ could exist without being a constant. In fact, if

$$\lim_{\varepsilon \to 0^+} f(\mathbf{x}_0 + \varepsilon \mathbf{x}) = g(\mathbf{x})\ , \tag{1.1.10}$$

then g is homogeneous of degree 0. Homogeneous distributions of degree zero are given by a formula of the type

$$\langle g(\mathbf{x}), \phi(\mathbf{x}) \rangle = \int_0^{\infty} \langle \alpha(\mathbf{w}), \phi(r\mathbf{w}) \rangle_{\mathcal{D}'(\mathbb{S}) \times \mathcal{D}(\mathbb{S})} r^{d-1}\, \mathrm{d}r\,, \tag{1.1.11}$$

for a certain distribution $\alpha \in \mathcal{D}'(\mathbb{S})$ [17, Thm. 2.6.2]. Here $\mathbb{S}$ is the unit sphere, $\mathcal{D}(\mathbb{S})$ is the space of test functions on the sphere, and $\mathcal{D}'(\mathbb{S})$ is its dual. The distribution α is the *thick distributional value* [13] of f at $\mathbf{x}_0$, namely, f has no delta functions at $\mathbf{x}_0$ and α is the thick limit

$$\lim_{\varepsilon\to 0^+} f(\mathbf{x}_0 + r\varepsilon\mathbf{w}) = \alpha(\mathbf{w}) , \tag{1.1.12}$$

in the space $\mathcal{D}'((0,\infty), \mathcal{D}'(\mathbb{S}))$, that is, for all $\rho \in \mathcal{D}(0,\infty)$,

$$\left\langle \lim_{\varepsilon\to 0^+} f(\mathbf{x}_0 + r\varepsilon\mathbf{w}), \rho(r) \right\rangle_{\mathcal{D}'(0,\infty)\times\mathcal{D}(0,\infty)} = \left(\int_0^\infty \rho(r)\, \mathrm{d}r\right) \alpha(\mathbf{w}) , \tag{1.1.13}$$

1.2. The continuous case

We start with a general known result that will be useful in our analysis.

Proposition 1.2.1. *Let $f:(0,\infty)\to\mathbb{R}$ be continuous. Suppose that for each $a>0$ the sequence $\{f(an)\}_{n=1}^\infty$ converges, to $F(a)$. Then $F(a)$ does not depend on a, that is,*

$$F(a) = L \text{ for all } a > 0 , \tag{1.2.1}$$

for some L, and actually

$$\lim_{x\to\infty} f(x) = L . \tag{1.2.2}$$

Proof. Clearly the function F is constant in each class of the quotient space $\mathbb{R}/\mathbb{Q}$, $F(ra) = F(a)$ if $r \in \mathbb{Q}$. Also, F is continuous or of the first Baire class [21, 42], so that the set D of points of continuity of F is dense in $(0,\infty)$. Let $\alpha \in D$. Let $b > 0$. If $\varepsilon > 0$ then there exists $\delta > 0$ such that $|a-\alpha| < \delta$ implies $|F(a) - F(\alpha)| < \varepsilon$ and there exist r rational such that $|rb - \alpha| < \delta$. Therefore

$$|F(b) - F(\alpha)| = |F(rb) - F(\alpha)| < \varepsilon , \tag{1.2.3}$$

and since ε is arbitrary, $F(b) = F(\alpha)$.

In order to prove (3.4.2), observe that if $\varepsilon > 0$, then for each $a \in [1,2]$ there exists $n_0 = n_0(a)$ such that $|f(ka) - L| < \varepsilon$ for $k \geq n_0(a)$. This means that

$$[1,2] = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} \{a \in [1,2] : |f(ka) - L| < \varepsilon\} . \tag{1.2.4}$$

Therefore there exists n_0 such that $\bigcap_{k \geq n_0} \{a \in [1,2] : |f(ka) - L| < \varepsilon\}$ contains an interval $I = [\alpha, \beta]$ with $\alpha \neq \beta$. Observe now that $\bigcup_{k=n_0}^{\infty} kI$ contains a ray (B, ∞), since in fact $\bigcup_{k=n_1}^{\infty} kI$ is a closed ray if $n_1 > 1/(\beta - \alpha)$. Hence if $x > B$ then $x = ka$ for some $k \geq n_0$ and some $a \in I$ and, consequently, $|f(x) - L| = |f(ka) - L| < \varepsilon$. □

It is interesting that there are sequences $\{\xi_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} \xi_n = \infty$ such that for some continuous functions $f : (0,\infty) \to \mathbb{R}$ the limit

$$\lim_{n\to\infty} f(a\xi_n) = G(a) \tag{1.2.5}$$

exists for all $a > 0$, but the function G is not constant. Indeed, let $f(x) = \sin(2\pi \ln x)$ and $\xi_n = e^{(n+1/n)}$. The limit $\lim_{n\to\infty} f(a\xi_n) = \sin(2\pi \ln a)$, exists but it is not constant, of course.

1.3. The measurable case

We shall now consider an extension of the previous results to measurable functions.

Proposition 1.3.1. *Suppose $f : (0,\infty) \to \mathbb{R}$ is measurable. For $a > 0$, suppose that the function F defined by*

$$F(a) = \lim_{n\to\infty} f(an), \tag{1.3.1}$$

is well-defined almost everywhere. Then F is constant almost everywhere.

Proof. Let us first suppose that $f \in L^{\infty}(0,\infty)$. Let $\phi \in \mathcal{D}(0,\infty)$ be a test function. For $\lambda > 0$, let us define

$$G(\lambda) = \int_0^{\infty} f(\lambda x)\phi(x)\,\mathrm{d}x\,. \tag{1.3.2}$$

The function G is continuous because $\phi \in \mathcal{D}(0,\infty)$. For a fixed λ, let us consider the sequence $\{G(\lambda n)\}_{n=1}^{\infty}$. Since f is bounded, we can apply the dominated convergence theorem to see that

$$\lim_{n\to\infty} G(\lambda n) = \int_0^{\infty} \lim_{n\to\infty} f(\lambda n x)\phi(x)\,\mathrm{d}x = \int_0^{\infty} F(\lambda x)\phi(x)\,\mathrm{d}x\,,$$

exists. Proposition 3.4.3 then yields that $\int_0^{\infty} F(\lambda x)\phi(x)\,\mathrm{d}x$ does not depend on λ,

$$\int_0^{\infty} F(\lambda x)\phi(x)\,\mathrm{d}x = \int_0^{\infty} F(x)\phi(x)\,\mathrm{d}x\,. \tag{1.3.3}$$

Therefore, the regular distribution F is constant since $F(\lambda x) = F(x)$, $\lambda > 0$, and only the constants are homogeneous of degree 0 in the interval $(0,\infty)$ [17], that is, $F(x) = C$, as *distributions.* Notice now that the locally integrable function that gives a regular distribution is unique almost everywhere, so that $F(x) = C \quad (\text{a.e.})\,.$

Let us now consider the case of a general measurable function f for which the limit $\lim_{n\to\infty} f(an) = F(a)$ exists (a.e.). We can then define the bounded function

$$h(x) = \arctan f(x)\,. \tag{1.3.4}$$

Then $\lim_{n\to\infty} h(an) = \arctan F(a)$ exists (a.e.). Consequently, $\arctan F(a)$ is constant, and therefore so is $F(a)$. □

In Proposition 3.4.3, it is shown that in the continuous case not only is $F(a) = L$ for all $a > 0$, but actually $\lim_{x\to\infty} f(x) = L$. This is no longer true in the measurable case;

for example, if $f = \chi_B$, the characteristic function of a set B of measure zero such that $B \cap (x, \infty) \neq \emptyset$ for all $x > 0$, then $\lim_{x\to\infty} f(x)$ does not exist. Of course, this function f is equal almost everywhere to a function $\widetilde{f}$, the zero function, for which $\lim_{x\to\infty} \widetilde{f}(x)$ exists. An example where (1.3.1) exists for all $a > 0$ but $\lim_{x\to\infty} \widetilde{f}(x)$ does not exist for any function f such that $f(x) = \widetilde{f}(x)$ (a.e.) can be constructed as follows. The strategy will be to construct an unbounded open set, B, such that for almost all $x > 0$ there are only a finite number of integers n with $nx \in B$ and then take $f = \chi_B$, the characteristic function of B.

Example 1.3.2. Let $\{N_k\}_{k=1}^{\infty}$ be a sequence of positive integers such that

$$kN_k < N_{k+1} \, . \tag{1.3.5}$$

For each k let us choose a non empty open interval $B_k \subset (N_k - 1, N_k)$. Then if $j \in \mathbb{N}$, $j\left(\frac{1}{k}, 1\right) \cap B_k \neq \emptyset$ only if $N_k \leq j < N_{k+1}$. Therefore, if $x \in \left(\frac{1}{k}, 1\right)$, then

$$\chi_{B_k}(jx) = 0 \ \text{ for all } \ j \in \mathbb{N}\,, \quad x \notin A_k\,, \tag{1.3.6}$$

where $A_k = \bigcup_{j=N_k}^{N_{k+1}-1} \frac{1}{j} B_k$. Let now $\{\eta_k\}_{k=1}^{\infty}$ be a sequence of strictly positive numbers such that the series $\sum_{k=1}^{\infty} \eta_k$ converges and let us further restrict the sets B_k by requiring that $\mu(A_k) < \eta_k$, for all k. Let $A = \limsup_{k\to\infty} A_k = \bigcap_{k=1}^{\infty} \bigcup_{q=k}^{\infty} A_q$. Then $\mu(A) = 0$.

Let us now define the function $f : (0, \infty) \to \mathbb{R}$ by

$$f(x) = \sum_{k=1}^{\infty} \chi_{B_k}(x) \ . \tag{1.3.7}$$

If $\widetilde{f}(x) = f(x)$ almost everywhere, then $\lim_{x\to\infty} \widetilde{f}(x)$ does not exist. On the other hand, $\lim_{n\to\infty} f(nx)$ exists almost everywhere. Indeed, it is enough to show the existence almost everywhere in $(0, 1)$, and the limit of $f(nx)$ exists and equals 0 if $x \in (0, 1) \setminus A$ (since if

$x \notin A$ then there exists k_0 such that $x \notin A_k$ for $k \geq k_0$ and consequently, $f(nx) = 0$ whenever $n \geq N_{k_0}$).

An example involving continuous functions can be obtained by a slight modification.

Example 1.3.3. Let g be a continuous function in $(0, \infty)$ such that $0 \leq g(x) \leq f(x)$ and such that there exist points $\xi_k \in B_k$, for all k, such that $g(\xi_k) = 1$. Then $\lim_{n\to\infty} g(nx) = 0$ almost everywhere, *but not everywhere*, since $\lim_{x\to\infty} g(x)$ does not exist.

These examples show that it is possible for $\lim\limits_{n\to\infty} f(an)$ to be equal to a constant L almost everywhere but without $\lim_{x\to\infty} f(x)$ existing. We do have a convergence in measure type result.

Proposition 1.3.4. *Suppose $f : (0, \infty) \to \mathbb{R}$ is measurable. Suppose that*

$$\lim_{n\to\infty} f(an) = L \qquad (a.e.)\ . \tag{1.3.8}$$

Then for all $\varepsilon > 0$ and all $C > 1$,

$$\lim_{x\to\infty} \frac{\mu\left(\{t \in [x, Cx] : |f(t) - L| > \varepsilon\}\right)}{\mu\left([x, Cx]\right)} = 0\,, \tag{1.3.9}$$

where μ denotes the Lebesgue measure of a set.

Proof. Let us denote by $G(x)$ the quotient $\mu\left(\{t \in [x, cx] : |f(t) - L| > \varepsilon\}\right) / (1 - C)\, x$. Notice that G is a continuous function in $(0, \infty)$. Let $a > 0$ be fixed and consider the sequence of functions $f_n(x) = f(nx)$ in the interval $[a, Ca]$. Since f_n converges to L almost everywhere in this finite interval, it converges to L in measure. This means that for all $\varepsilon > 0$ the measure of the set $\{s \in [a, Ca] : |f_n(s) - L| > \varepsilon\}$ tends to zero. But the transformation $t = ns$ gives

$$\begin{aligned}\frac{\mu\left(\left\{s \in [a, Ca] : \left|f_n(s) - L\right| > \varepsilon\right\}\right)}{\mu\left([a, Ca]\right)} &= \frac{\mu\left(\left\{s \in [a, Ca] : \left|f(ns) - L\right| > \varepsilon\right\}\right)}{\mu\left([a, Ca]\right)} \\ &= \frac{\mu\left(\left\{t \in [na, Cna] : \left|f(t) - L\right| > \varepsilon\right\}\right)}{\mu\left([na, Cna]\right)} \\ &= G(na) \ ,\end{aligned}$$

so that $\lim_{n\to\infty} G(na) = 0$. Proposition 3.4.3 then yields (1.3.9). □

1.4. Delta sequences

A sequence $\{f_n\}_{n=1}^{\infty}$ of distributions is called a delta sequence if $f_n(\mathbf{x}) \to \delta(\mathbf{x})$ in either the strong or the weak topology of $\mathcal{D}'\left(\mathbb{R}^d\right)$, since the two notions are equivalent [55]. In other words, $\{f_n\}_{n=1}^{\infty}$ is a delta sequence if

$$\lim_{n\to\infty} \langle f, \phi\rangle = \phi(\mathbf{0}) \ , \tag{1.4.1}$$

for all $\phi \in \mathcal{D}\left(\mathbb{R}^d\right)$. In this article we will be interested mostly in the case when the distributions f_n are actually smooth functions, but general delta sequences are also of interest, of course. They have been employed in several problems [1], such as the definitions of point values [52] or the definition of products of distributions [30, 34, 44].

There are many ways to construct delta sequences. A simple one is the following. First, recall the multi-index notations $\mathbf{k}! = k_1!k_2!\cdots k_n!$ and

$$\left\langle \mathbf{D}^{\mathbf{k}} f, \phi\right\rangle = (-1)^{|\mathbf{k}|}\left\langle f, \mathbf{D}^{\mathbf{k}}\phi\right\rangle . \tag{1.4.2}$$

Now let f be a fixed distribution of rapid decay at infinity, that is, $f \in \mathcal{K}'\left(\mathbb{R}^d\right)$. Then all the moments $\mu_{\mathbf{k}} = \left\langle f(\mathbf{x}), \mathbf{x}^{\mathbf{k}}\right\rangle$, exist for $\mathbf{k} \in \mathbb{N}^d$ since all polynomials belong to $\mathcal{K}\left(\mathbb{R}^d\right)$, and the *moment asymptotic expansion*

$$f(\lambda \mathbf{x}) \sim \sum_{q=0}^{\infty} \sum_{|\mathbf{k}|=q} \frac{\mu_{\mathbf{k}} \mathbf{D}^{\mathbf{k}} \delta(\mathbf{x})}{\mathbf{k}!} \frac{1}{\lambda^{q+d}}, \quad \text{as } \lambda \to \infty, \tag{1.4.3}$$

holds in $\mathcal{K}'(\mathbb{R}^d)$ [17]. Therefore, when $\mu_{\mathbf{0}} \neq 0$ if $\{\xi_n\}_{n=1}^{\infty}$ is any sequence of positive numbers with $\lim_{n\to\infty} \xi_n = \infty$ then

$$g_n(\mathbf{x}) = \frac{\xi_n^d}{\mu_{\mathbf{0}}} f(\xi_n \mathbf{x}), \tag{1.4.4}$$

is a delta sequence, generated by f and $\{\xi_n\}_{n=1}^{\infty}$. When $\xi_n = n$ for all n, we call this sequence the *standard* delta sequence generated by f.

Another useful construction of delta sequences is provided by the ensuing well known result.

Lemma 1.4.1. *Suppose $\{\psi_n\}_{n=1}^{\infty}$ is a sequence of normalized positive test functions in $\mathcal{D}'(\mathbb{R}^d)$ such that* $\operatorname{supp}\psi_n \subset \{\mathbf{x} : |\mathbf{x}| < r_n\}$, *where* $\lim_{n\to\infty} r_n = 0$. *Then $\{\psi_n\}_{n=1}^{\infty}$ is a delta sequence.*

Proof. Let ϕ be any test function. Then by the first mean value theorem for integrals,

$$\langle \psi_n, \phi \rangle = \int_{\operatorname{supp}\psi_n} \psi_n(\mathbf{x}) \phi(\mathbf{x}) \, d\mathbf{x} = \phi(\mathbf{x}_n), \tag{1.4.5}$$

for some $\mathbf{x}_n \in \operatorname{supp}\psi_n$. Since $|\mathbf{x}_n| \leq r_n \to 0$, we obtain that $\mathbf{x}_n \to 0$, and consequently, $\phi(\mathbf{x}_n) \to \phi(\mathbf{0})$. Thus $\psi_n(\mathbf{x}) \to \delta(\mathbf{x})$. □

We now give a notion of point value of a distribution based on delta sequences. Our definition applies to several spaces of distributions, but the cases $\mathcal{A} = \mathcal{D}$, $\mathcal{E}$, or $\mathcal{S}$ seem the most relevant.

Definition 1.4.2. Let $\mathcal{A}(\mathbb{R}^d)$ be a space of test functions. Let $\mathfrak{F}$ be a family of delta sequences whose elements belong to $\mathcal{A}(\mathbb{R}^d)$. If $f \in \mathcal{A}'(\mathbb{R}^d)$ and $\mathbf{x}_0 \in \mathbb{R}^d$ we say that the value $f(\mathbf{x}_0)$ exists and equals γ with respect to $\mathfrak{F}$ if

$$\lim_{n\to\infty} \langle f(\mathbf{x}_0+\mathbf{x}), \phi_n \rangle = \gamma\,, \tag{1.4.6}$$

for all $\{\phi_n\}_{n=1}^{\infty} \in \mathfrak{F}$. When this holds we write

$$f(\mathbf{x}_0) = \gamma \quad (\mathfrak{F})\,. \tag{1.4.7}$$

The definition of point value employed by Sasane [52] corresponds to the case when $d = 1$, $f \in \mathcal{D}'(\mathbb{R})$, and $\mathfrak{F}$ is the family of all delta sequences whose elements are the standard sequences generated from a positive, normalized, and symmetric test function of $\mathcal{D}(\mathbb{R})$.

1.5. Several lemmas

In this section, we present several results on how positive test functions allow us to study many properties of distributions. In particular, we see how positive test functions tell us if a distribution is a regular distribution given by a bounded measurable function and give us the essential supremum and infimum of such a function. In this section, and only in this section, we will make a notational difference between a regular distribution $\mathsf{f} \in \mathcal{D}'(\mathbb{R}^d)$ and the locally integrable function f that generates it as

$$\langle \mathsf{f}(\mathbf{x}), \phi(\mathbf{x}) \rangle = \int_{\mathbb{R}^d} f(\mathbf{x})\,\phi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,, \qquad \phi \in \mathcal{D}(\mathbb{R}^d)\,. \tag{1.5.1}$$

In the rest of the article, we will use the same notation, f, for the distribution and the function.

Let us start with following simple result.

Lemma 1.5.1. *The set of functions of the form*

$$\phi = c_1\psi_1 - c_2\psi_2\,, \tag{1.5.2}$$

where c_1 and c_2 are constants and where ψ_1 and ψ_2 are normalized positive test functions is the whole space $\mathcal{D}\left(\mathbb{R}^d\right)$.

When $d = 1$, the corresponding space with ψ_1 and ψ_2 normalized positive symmetric test functions is the space of all even test functions.

Proof. It is enough to show that the real valued elements of $\mathcal{D}\left(\mathbb{R}^d\right)$ have the form (3.4.6) for some positive constants c_1 and c_2. Let $\zeta_1 \in \mathcal{D}\left(\mathbb{R}^d\right)$ be such that $\zeta_1(\mathbf{x}) \geq \max\{\phi(\mathbf{x}), 0\}$ and let $\zeta_2 = \zeta_1 - \phi$. Then we write $\zeta_j = c_j\psi_j$ where the ψ_j are normalized positive test functions and $c_j = \int_{\mathbb{R}^d} \zeta_j(\mathbf{x})\, \mathrm{d}\mathbf{x}$. In the symmetric case we just also ask ζ_1 to be even. □

Our first characterization using positive normalized test functions is the following.

Lemma 1.5.2. *Let $\mathsf{f} \in \mathcal{D}'\left(\mathbb{R}^d\right)$. Then f is a regular distribution in an open set $U \subset \mathbb{R}^d$, given by a bounded function $f \in L^\infty(U)$ if and only if there exists a constant $M > 0$ such that for all positive, normalized test functions $\phi \in \mathcal{D}(U)$ we have*

$$|\langle \mathsf{f}(\mathbf{x}), \phi(\mathbf{x})\rangle| \leq M. \tag{1.5.3}$$

Proof. If $f \in L^\infty(U)$. Then when $\phi \in \mathcal{D}(U)$,

$$|\langle \mathsf{f}(\mathbf{x}), \phi(\mathbf{x})\rangle| = \left|\int_U f(\mathbf{x})\,\phi(\mathbf{x})\, \mathrm{d}\mathbf{x}\right| \leq \|f\|_{L^\infty(U)} \|\phi\|_{L^1(U)},$$

so that if ϕ is normalized, $|\langle \mathsf{f}(x), \phi(x)\rangle| \leq \|f\|_{L^\infty(U)}$. Therefore (1.5.3) holds with $M = \|f\|_{L^\infty(U)}$.

Conversely, if (1.5.3) is satisfied for some $M > 0$ for all positive, normalized test functions of U then $|\langle \mathsf{f}(x), \psi(x)\rangle| \leq 2M\|\psi\|_{L^1(U)}$ for all real test functions $\psi \in \mathcal{D}(U)$, because of Lemma 1.5.1 (or $4M$ if complex). This means that f is continuous in $\mathcal{D}(U)$, a dense subspace of $L^1(U)$ with the topology induced by $L^1(U)$ in its subspace. Hence, f admits an extension $f \in (L^1(U))' \simeq L^\infty(U)$, and this means that

$$\langle \mathsf{f}(\mathbf{x}), \psi(\mathbf{x})\rangle = \int_U f(\mathbf{x})\,\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{1.5.4}$$

for all $\psi \in \mathcal{D}(U)$. Therefore, f is a regular distribution given by the bounded function f in the open set U. $\square$

In the proof we can see that $\inf\{M:\ \text{(1.5.3) holds}\} \leq \|f\|_{L^\infty(U)}$. In fact, we have more.

Lemma 1.5.3. *If $\mathsf{f} \in \mathcal{D}'(\mathbb{R}^d)$ is a regular distribution in U, given by a bounded function $f \in L^\infty(U)$ then*

$$\|f\|_{L^\infty(U)} = \inf\{M:\ \textit{(1.5.3) holds for all positive, normalized test functions}\}, \tag{1.5.5}$$

and

$$\|f\|_{L^\infty(U)} = \sup\{|\langle \mathsf{f}, \phi\rangle| :\ \phi \in \mathcal{D}(U)\ \textit{positive, normalized test function}\}. \tag{1.5.6}$$

Proof. Clearly $\inf\{M:\ \text{(1.5.3) holds for all positive, normalized test functions}\}$ is equal to $\sup\{|\langle \mathsf{f}, \phi\rangle| :\ \phi \in \mathcal{D}(U)\ \text{positive, normalized test function}\}$; let us call this K. We know that $K \leq \|f\|_{L^\infty(U)}$. To prove the converse inequality, let $s < \|f\|_{L^\infty(U)}$. Then there exists $\mathbf{x}_0 \in U$ such that the distributional point value $\mathsf{f}(\mathbf{x}_0)$ exists and $s < |\mathsf{f}(\mathbf{x}_0)|$. If ϕ is a positive normalized test function, then so are the test functions $\varphi_\lambda(\mathbf{x}) = \lambda^d \phi(\mathbf{x}_0 + \lambda\mathbf{x})$ for all $\lambda > 0$, and if λ is big enough, $\varphi_\lambda \in \mathcal{D}(U)$. Since $\lim_{\lambda\to\infty}\langle \mathsf{f}, \varphi_\lambda\rangle = \mathsf{f}(\mathbf{x}_0)$, we can find λ such that $|\langle \mathsf{f}, \varphi_\lambda\rangle| > s$. Consequently, $K > s$, and because $s < \|f\|_{L^\infty(U)}$ is arbitrary, $K \geq \|f\|_{L^\infty(U)}$. $\square$

In fact, the same argument in the proof of Lemma 1.5.3 allows us to obtain the following.

Lemma 1.5.4. *If* $\mathsf{f} \in \mathcal{D}'\left(\mathbb{R}^d\right)$ *is a real regular distribution in* U, *given by a function* $f \in L^1(U)$ *then the essential supremum and infimum of* f *are also given as*

$$\operatorname*{ess\,sup}_{\mathbf{x}\in U} f(x) = \sup_{\phi\in\mathcal{D}(U),\phi\geq 0,\int\phi=1} \langle \mathsf{f}(\mathbf{x}), \phi(\mathbf{x})\rangle , \tag{1.5.7}$$

and

$$\operatorname*{ess\,inf}_{\mathbf{x}\in U} f(\mathbf{x}) = \inf_{\phi\in\mathcal{D}(U),\phi\geq 0,\int\phi=1} \langle \mathsf{f}(x), \phi(x)\rangle . \tag{1.5.8}$$

We notice that when $f \in L^1(U)$ then (1.5.7) could be $+\infty$ and (1.5.8) could be $-\infty$.

1.6. Comparison of Definitions

We will now study whether the existence of the distributional point value $f(\mathbf{x}_0)$ is equivalent to the existence of $f(\mathbf{x}_0)\ \ (\mathfrak{F})$ for several families of delta sequences $\mathfrak{F}$.

1.6.1. Standard delta sequences generated by a positive normalized test function

In this section we consider the family $\mathfrak{F}$ of standard delta sequences generated by a positive normalized test function of $\mathcal{D}\left(\mathbb{R}^d\right)$.

Proposition 1.6.1. *Let* $f \in \mathcal{D}'\left(\mathbb{R}^d\right)$. *Then* f *has a thick distributional point value at* $\mathbf{x}_0$ *if and only if for all standard delta sequences generated by a positive normalized test function of* $\mathcal{D}\left(\mathbb{R}^d\right)$, $\{\phi_n\}_{n=1}^{\infty}$, *the limit*

$$\lim_{n\to\infty} \langle f(\mathbf{x}_0+\mathbf{x}), \phi_n(\mathbf{x})\rangle = \gamma_{\{\phi_n\}} , \tag{1.6.1}$$

exists.

Proof. A standard delta sequences generated by a normalized positive test function ϕ is of the form $\phi_n(\mathbf{x}) = n^d \phi(n\mathbf{x})$. If the distributional thick point value $f_{\mathbf{x}_0}(\mathbf{w}) = \gamma(\mathbf{w})$ exists, $\gamma \in \mathcal{D}'(\mathbb{S})$, then

$$\begin{aligned}\lim_{n\to\infty} \langle f(\mathbf{x}_0+\mathbf{x}), \phi_n(\mathbf{x})\rangle &= \lim_{n\to\infty} \left\langle f(\mathbf{x}_0+\mathbf{x}), n^d \phi(n\mathbf{x})\right\rangle \\ &= \lim_{n\to\infty} \langle f(\mathbf{x}_0+(1/n)\mathbf{x}), \phi(\mathbf{x})\rangle \\ &= \int_0^\infty \langle \gamma(\mathbf{w}), \phi(r\mathbf{w})\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})} r^{d-1}\,\mathrm{d}r\,,\end{aligned}$$

exists. Conversely, let ϕ be a normalized positive test function. If the limit (3.4.7) exists for all standard delta sequences generated by a positive normalized test function, it will exist for $\phi_n^{\{a\}}(\mathbf{x}) = n^d a^d \phi(na\mathbf{x})$ for all $a > 0$. Consequently, if the function Φ is defined as

$$\Phi(a) = \left\langle f(\mathbf{x}_0+\mathbf{x}), a^d\phi(a\mathbf{x})\right\rangle, \quad a>0\,, \tag{1.6.2}$$

then

$$\lim_{n\to\infty}\Phi(na) = \lim_{n\to\infty}\left\langle f(\mathbf{x}_0+\mathbf{x}), \phi_n^{\{a\}}(\mathbf{x})\right\rangle = \gamma_{\left\{\phi_n^{\{a\}}\right\}}, \tag{1.6.3}$$

exists for all a. Since Φ is continuous, Proposition 3.4.3 yields that $\gamma_{\left\{\phi_n^{\{a\}}\right\}} = \gamma_0(\phi)$ is independent of a and actually $\lim_{\lambda\to\infty}\Phi(\lambda) = \gamma_0(\phi)$. Hence,

$$\lim_{\varepsilon\to0^+}\langle f(\mathbf{x}_0+\varepsilon\mathbf{x}), \phi(\mathbf{x})\rangle = \gamma_0(\phi)\,, \tag{1.6.4}$$

for all normalized positive test functions. Therefore, Lemma 1.5.1 yields that the limit $\lim_{\varepsilon\to0^+}\langle f(x_0+\varepsilon x), \phi(x)\rangle = \gamma_0(\phi)$ exists whenever $\phi \in \mathcal{D}\left(\mathbb{R}^d\right)$. The formula $\langle\gamma_0,\phi\rangle = \gamma_0(\phi)$, defines a distribution $\gamma_0 \in \mathcal{D}'\left(\mathbb{R}^d\right)$, and γ_0 is homogeneous of degree 0, that is, $\gamma_0(t\mathbf{x}) = \gamma_0(\mathbf{x})$, $t > 0$. As explained in the background, using [17, Thm. 2.6.2] we conclude that γ_0 is obtained from a distribution $\gamma \in \mathcal{D}'(\mathbb{S})$ by the formula

$$\langle\gamma_0,\phi\rangle = \int_0^\infty \langle\alpha(\mathbf{w}), \phi(r\mathbf{w})\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})} r^{d-1}\,\mathrm{d}r\,, \tag{1.6.5}$$

and that α is the thick distributional value of f at $\mathbf{x}_0$. $\square$

Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of test functions. If T is an orthogonal transformation of $\mathbb{R}^d$, that is, with $|\det T| = 1$, then the sequence $\left\{\phi_n^T\right\}_{n=1}^{\infty}$, where $\phi^T(\mathbf{x}) = \phi(T\mathbf{x})$, is also a delta sequence. We have then the following result.

Proposition 1.6.2. *Let $f \in \mathcal{D}'\left(\mathbb{R}^d\right)$. Then the distributional point value $f(\mathbf{x}_0)$ exists if and only if for all standard delta sequences generated by a positive normalized test function of $\mathcal{D}\left(\mathbb{R}^d\right)$, $\{\phi_n\}_{n=1}^{\infty}$, the limit $\lim_{n\to\infty}\langle f(\mathbf{x}_0+\mathbf{x}), \phi_n(\mathbf{x})\rangle = \gamma_{\{\phi_n\}}$ exists and for all orthogonal transformations T of $\mathbb{R}^d$, $\gamma_{\{\phi_n^T\}} = \gamma_{\{\phi_n\}}$.*

Proof. This follows immediately from Proposition 1.6.1 if we observe that a homogeneous function or distribution of degree 0 is a constant if and only if it is invariant with respect to orthogonal transformations. $\square$

Notice that in one variable, Proposition 1.6.1 says that $\lim_{n\to\infty}\langle f(x_0+x), \phi_n(x)\rangle = \gamma_{\{\phi_n\}}$ exists for all standard delta sequences generated by a positive normalized test function if and only if there are constants γ_+ and γ_- such that

$$\lim_{\varepsilon\to 0^+}\langle f(x_0+\varepsilon x), \psi(x)\rangle = \gamma_-\int_{-\infty}^{0}\psi(x)\,\mathrm{d}x + \gamma_+\int_{0}^{\infty}\psi(x)\,\mathrm{d}x\,, \tag{1.6.6}$$

for all $\psi \in \mathcal{D}(\mathbb{R})$. On the other hand, since the only orthogonal transformations in dimension one are the identity and $x \rightsquigarrow -x$, Proposition 1.6.2 says that the distributional point value $f(x_0)$ exists if and only if for all standard delta sequences generated by a positive normalized test function of $\mathcal{D}(\mathbb{R})$, $\{\phi_n\}_{n=1}^{\infty}$, the limit $\lim_{n\to\infty}\langle f(x_0+x), \phi_n(x)\rangle = \gamma_{\{\phi_n\}}$ exists and $\gamma_{\{\phi_n(-x)\}} = \gamma_{\{\phi_n(x)\}}$.

Our results also give the ensuing equivalence.

Proposition 1.6.3. *Let $f \in \mathcal{D}'\left(\mathbb{R}^d\right)$. Then the distributional point value $f(\mathbf{x}_0)$ exists and equals γ if and only if for $\mathfrak{F}$ the family of standard delta sequences generated by a positive normalized test function*

$$f(\mathbf{x}_0) = \gamma \ \ (\mathfrak{F}) \,. \tag{1.6.7}$$

1.6.2. Standard delta sequences generated by an even positive normalized test function

We now consider the case of symmetric standard delta sequences, the family considered by Sasane [52].

We first need to explain the idea of symmetric point values. Let $f \in \mathcal{D}'(\mathbb{R})$ and $x_0 \in \mathbb{R}$. The symmetric distributional point value of f exists at x_0 and equals γ if

$$\lim_{\varepsilon \to 0} \frac{f(x_0 + \varepsilon x) + f(x_0 - \varepsilon x)}{2} = \gamma \,, \tag{1.6.8}$$

in $\mathcal{D}'(\mathbb{R})$. Each distribution can be written as the sum of an even one and an odd one,

$$g = g_{\mathrm{e}} + g_{\mathrm{o}} \,, \tag{1.6.9}$$

where

$$g_{\mathrm{e}}(x) = \frac{g(x) + g(-x)}{2} \,, \qquad g_{\mathrm{o}}(x) = \frac{g(x) - g(-x)}{2} \,. \tag{1.6.10}$$

Applying this to $g(x) = f(x_0 + x)$, we see that the distributional symmetric value $f(x_0)$ exists and equals γ if and only if the distributional value $g_{\mathrm{e}}(0)$ exists and equals γ.

Notice also that if ϕ is a test function and we write $\phi = \phi_{\mathrm{e}} + \phi_{\mathrm{o}}$, then

$$\langle g, \phi \rangle = \langle g_{\mathrm{e}}, \phi_{\mathrm{e}} \rangle + \langle g_{\mathrm{o}}, \phi_{\mathrm{o}} \rangle \,. \tag{1.6.11}$$

Therefore we have the following result.

Lemma 1.6.4. *A distribution $f \in \mathcal{D}'(\mathbb{R})$ has a symmetric distributional value γ at x_0 if and only if*

$$\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi_{\mathrm{e}}(x) \rangle = \gamma \int_{-\infty}^{\infty} \phi_{\mathrm{e}}(x)\, \mathrm{d}x\,, \tag{1.6.12}$$

for all even test functions ϕ_{e}.

Proof. Indeed, if (1.6.8) is satisfied, then

$$\begin{aligned}
\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi_{\mathrm{e}}(x) \rangle &= \lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x) - g_{\mathrm{o}}(\varepsilon x), \phi_{\mathrm{e}}(x) \rangle \\
&= \lim_{\varepsilon \to 0} \left\langle \frac{f(x_0 + \varepsilon x) + f(x_0 - \varepsilon x)}{2}, \phi_{\mathrm{e}}(x) \right\rangle \\
&= \gamma \int_{-\infty}^{\infty} \phi_{\mathrm{e}}(x)\, \mathrm{d}x\,.
\end{aligned}$$

Conversely, if (1.6.12) holds, then for any test function $\phi = \phi_{\mathrm{e}} + \phi_{\mathrm{o}}$,

$$\begin{aligned}
\lim_{\varepsilon \to 0} \langle g_{\mathrm{e}}(\varepsilon x), \phi(x) \rangle &= \lim_{\varepsilon \to 0} \langle g_{\mathrm{e}}(\varepsilon x), \phi_{\mathrm{e}}(x) \rangle \\
&= \lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi_{\mathrm{e}}(x) \rangle \\
&= \gamma \int_{-\infty}^{\infty} \phi_{\mathrm{e}}(x)\, \mathrm{d}x \\
&= \gamma \int_{-\infty}^{\infty} \phi(x)\, \mathrm{d}x\,.
\end{aligned}$$

Hence $g_{\mathrm{e}}(0) = \gamma$, so that the symmetric distributional value of f at x_0 equals γ. □

We can now give an equivalence to the existence of the point value $f(x_0) = \gamma\ (\mathfrak{F}_{\mathrm{sy}})$, where $\mathfrak{F}_{\mathrm{sy}}$ is the family of standard delta sequences generated by a positive normalized even test function of $\mathcal{D}(\mathbb{R})$.

Proposition 1.6.5. *Let $f \in \mathcal{D}'(\mathbb{R})$. Then the following are equivalent:*

1. If $\mathfrak{F}_{\mathrm{sy}}$ is the family of standard delta sequences generated by a positive normalized even test function then

$$f(x_0) = \gamma \quad (\mathfrak{F}_{\mathrm{sy}}) \,. \tag{1.6.13}$$

2. The symmetric distributional point value of f exists at x_0 and equals γ.

Proof. Indeed, if (1.6.13) holds then

$$\lim_{n\to\infty} \langle f(x_0 + x), \phi_n(x)\rangle = \gamma \,, \tag{1.6.14}$$

for all standard delta sequences $\{\phi_n\}_{n=1}^{\infty}$ generated by a positive normalized *even* test function ϕ_{e}, and use of Proposition 3.4.3 yields that

$$\lim_{\varepsilon\to 0} \langle f(x_0 + \varepsilon x), \phi_{\mathrm{e}}(x)\rangle = \gamma \,, \tag{1.6.15}$$

for such normalized even test functions ϕ_{e}. This last statement is equivalent to the fact that (1.6.12) holds for *all* even test functions because of Lemma 1.5.1, and Lemma 1.6.4 yields that in turn this is equivalent to the symmetric distributional point value being equal to γ. □

Actually, using the same ideas as in the proof of this Proposition we see that the limit $\lim_{n\to\infty} \langle f(x_0 + x), \phi_n(x)\rangle = \gamma_{\{\phi_n\}}$ exists for all standard delta sequences $\{\phi_n\}_{n=1}^{\infty}$ generated by a positive normalized even test function ϕ_{e} if and only if this limit is a constant γ and (1.6.13) is satisfied.

1.6.3. The family of standard delta sequences generated by a radial positive normalized test function

We now consider the family $\mathfrak{F}_{\mathrm{rad}}$ of standard sequences generated by a radial positive normalized test function.

Let us start with some notation. We denote $r = |\mathbf{x}|$ the radial variable in $\mathbb{R}^d$. A test function $\phi \in \mathcal{D}\left(\mathbb{R}^d\right)$ is called *radial* if it is a function of r, $\phi(\mathbf{x}) = \varphi(r)$, for some even function $\varphi \in \mathcal{D}(\mathbb{R})$; the space of all radial test functions of $\mathcal{D}\left(\mathbb{R}^d\right)$ is denoted as $\mathcal{D}_{\text{rad}}\left(\mathbb{R}^d\right)$. Similarly, we denote as $\mathcal{D}'_{\text{rad}}\left(\mathbb{R}^d\right)$ the space of all radial distributions; a distribution $f \in \mathcal{D}'\left(\mathbb{R}^d\right)$ is radial if $f(T\mathbf{x}) = f(\mathbf{x})$ for any orthogonal transformation of $\mathbb{R}^d$, and this actually means [14, 22] that $f(\mathbf{x}) = f_1(r)$ for some distribution of one variable f_1. Notice, however, that while φ is uniquely determined by ϕ, for a given f there are several possible distributions f_1.

When $d = 1$ then $\mathcal{D}_{\text{rad}}(\mathbb{R})$ and $\mathcal{D}'_{\text{rad}}(\mathbb{R})$ become the spaces of even test functions and distributions, respectively, and are also denoted as $\mathcal{D}_{\text{even}}(\mathbb{R})$ and $\mathcal{D}'_{\text{even}}(\mathbb{R})$. This was the situation considered in the previous subsection.

Observe that the space $\mathcal{D}'_{\text{rad}}\left(\mathbb{R}^d\right)$ is naturally isomorphic to the dual space $\left(\mathcal{D}_{\text{rad}}\left(\mathbb{R}^d\right)\right)'$, that is to say, if the action of a radial distribution is known in all radial test functions, then it can be obtained for arbitrary test functions. Indeed, if $f \in \mathcal{D}'_{\text{rad}}\left(\mathbb{R}^d\right)$ and $\phi \in \mathcal{D}\left(\mathbb{R}^d\right)$, then

$$\langle f(\mathbf{x}), \phi(\mathbf{x})\rangle = \left\langle f(\mathbf{x}), \widetilde{\phi}(\mathbf{x})\right\rangle, \tag{1.6.16}$$

where $\widetilde{\phi} \in \mathcal{D}_{\text{rad}}(\mathbb{R})$ is given as

$$\widetilde{\phi}(\mathbf{x}) = \phi^o(|\mathbf{x}|), \tag{1.6.17}$$

$\phi^o \in \mathcal{D}_{\text{even}}(\mathbb{R})$ being defined as

$$\phi^o(r) = \frac{1}{\omega}\int_{\mathbb{S}} \phi(r\theta)\, \mathrm{d}\sigma(\theta). \tag{1.6.18}$$

Here we denote by $\mathbb{S}$ the unit sphere of $\mathbb{R}^d$, $\mathrm{d}\sigma$ is the Lebesgue measure in $\mathbb{S}$ and $\omega = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the sphere.

Equations (1.6.17) and (1.6.18) define the *radial component of a test function.* We can also define the radial component of a distribution f, $\widetilde{f} \in \mathcal{D}'_{\text{rad}}\left(\mathbb{R}^d\right)$, as

$$\left\langle \widetilde{f}\left(\mathbf{x}\right), \phi\left(\mathbf{x}\right)\right\rangle = \left\langle f\left(\mathbf{x}\right), \widetilde{\phi}\left(\mathbf{x}\right)\right\rangle . \tag{1.6.19}$$

The distributional analog of (1.6.18) is not well defined, however [14, 22].

We say that a distribution f has a *radial distributional point value* at $\mathbf{x}_0$ equal to γ if

$$\widetilde{g}\left(\mathbf{0}\right) = \gamma\,, \tag{1.6.20}$$

where $\widetilde{g}$ is the radial component of $g\left(\mathbf{x}\right) = f\left(\mathbf{x}_0 + \mathbf{x}\right)$. Similar to Lemma 1.6.4, we have the following characterization.

Lemma 1.6.6. *A distribution $f \in \mathcal{D}'\left(\mathbb{R}^d\right)$ has a radial distributional value γ at $\mathbf{x}_0$ if and only if*

$$\lim_{\varepsilon\to 0}\left\langle f\left(\mathbf{x}_0 + \varepsilon\mathbf{x}\right), \phi_{\text{rad}}\left(\mathbf{x}\right)\right\rangle = \gamma\int_{\mathbb{R}^d}\phi_{\text{rad}}\left(\mathbf{x}\right)\,\mathrm{d}\mathbf{x}\,, \tag{1.6.21}$$

for all radial test functions ϕ_{rad}.

Proof. If $\widetilde{g}\left(\mathbf{0}\right) = \gamma$, then

$$\begin{aligned}\lim_{\varepsilon\to 0}\left\langle f\left(\mathbf{x}_0 + \varepsilon\mathbf{x}\right), \phi_{\text{rad}}\left(\mathbf{x}\right)\right\rangle &= \lim_{\varepsilon\to 0}\left\langle \widetilde{g}\left(\varepsilon\mathbf{x}\right), \phi_{\text{rad}}\left(\mathbf{x}\right)\right\rangle \\ &= \gamma\int_{\mathbb{R}^d}\phi_{\text{rad}}\left(\mathbf{x}\right)\,\mathrm{d}\mathbf{x}\,.\end{aligned}$$

On the other hand, if (1.6.21) holds, then for any test function ϕ,

$$\begin{aligned}\lim_{\varepsilon\to 0}\left\langle \widetilde{g}\left(\varepsilon\mathbf{x}\right),\phi\left(\mathbf{x}\right)\right\rangle &= \lim_{\varepsilon\to 0}\left\langle \widetilde{g}\left(\varepsilon\mathbf{x}\right),\widetilde{\phi}\left(\mathbf{x}\right)\right\rangle \\ &= \lim_{\varepsilon\to 0}\left\langle f\left(\mathbf{x}_0+\varepsilon\mathbf{x}\right),\widetilde{\phi}\left(\mathbf{x}\right)\right\rangle \\ &= \gamma\int_{\mathbb{R}^d}\widetilde{\phi}\left(\mathbf{x}\right)\,\mathrm{d}\mathbf{x} \\ &= \gamma\int_{\mathbb{R}^d}\phi\left(\mathbf{x}\right)\,\mathrm{d}\mathbf{x}\,.\end{aligned}$$

Hence $\widetilde{g}\left(\mathbf{0}\right)=\gamma$, that is, the radial distributional value of f at $\mathbf{x}_0$ equals γ. □

Therefore, we obtain the ensuing equivalence for the fact that $f\left(\mathbf{x}_0\right)=\gamma\ \ (\mathfrak{F}_{\mathrm{rad}})$.

Proposition 1.6.7. *Let $f\in\mathcal{D}'\left(\mathbb{R}^d\right)$. Then the following are equivalent:*

1. If $\mathfrak{F}_{\mathrm{rad}}$ is the family of standard delta sequences generated by a positive normalized radial test function then

$$f\left(\mathbf{x}_0\right)=\gamma \quad \left(\mathfrak{F}_{\mathrm{rad}}\right). \tag{1.6.22}$$

2. The radial distributional point value of f exists at $\mathbf{x}_0$ and equals γ.

Proof. Indeed, if (1.6.22) holds then

$$\lim_{n\to\infty}\left\langle f\left(\mathbf{x}_0+\varepsilon\mathbf{x}\right),\phi_n\left(\mathbf{x}\right)\right\rangle=\gamma\,, \tag{1.6.23}$$

for all standard delta sequences $\{\phi_n\}_{n=1}^{\infty}$ generated by a positive normalized *radial* test function ϕ_{rad}. Use of Proposition 3.4.3 yields that

$$\lim_{\varepsilon\to 0}\left\langle f\left(\mathbf{x}_0+\varepsilon\mathbf{x}\right),\phi_{\mathrm{rad}}\left(\mathbf{x}\right)\right\rangle=\gamma\,, \tag{1.6.24}$$

for such normalized radial test functions ϕ_{rad}. This last statement is equivalent to the fact that (1.6.12) holds for *all* radial test functions because of Lemma 1.5.1, and Lemma 1.6.6

yields that, in turn, this is equivalent to the radial distributional point value being equal to γ. □

We also have the next result, that is obtained from Lemma 1.5.1.

Proposition 1.6.8. *The limit* $\lim_{n\to\infty}\langle f(\mathbf{x}_0+\varepsilon\mathbf{x}),\phi_n(\mathbf{x})\rangle=\gamma_{\{\phi_n\}}$ *exists for all standard delta sequences* $\{\phi_n\}_{n=1}^{\infty}$ *generated by a positive normalized radial test function* ϕ_{rad} *if and only if this limit is a constant* γ *and* $f(\mathbf{x}_0)=\gamma\quad(\mathfrak{F}_{\text{rad}})$.

1.6.4. The family of all positive normalized test functions

We saw in Subsection 1.6.2 that Sasane's notion of point values was not equivalent to the standard definition, nor, in the next subsection, is the notion based on the family $\mathfrak{F}_{\text{rad}}$ of standard delta sequences generated by a positive normalized radial test function. Nevertheless, for the family $\mathfrak{F}$ of standard delta sequences generated by a positive normalized test function the point value definition is in fact equivalent to the standard Łojasiewicz definition. Of course, Sasane was considering the family of standard delta sequences generated by an *even* positive normalized test function $\mathfrak{F}_{\text{sy}}$. Both $\mathfrak{F}_{\text{sy}}$ and $\mathfrak{F}_{\text{rad}}$ are subfamilies of $\mathfrak{F}$. We can also consider families larger than $\mathfrak{F}$. For instance, we can consider the family $\mathfrak{F}_{\text{all}}$ of *all* delta sequences formed with positive normalized test functions. In this next example, we will see that the distributional point value $f(\mathbf{x}_0)=\gamma$ is not equivalent to $f(\mathbf{x}_0)=\gamma\quad(\mathfrak{F}_{\text{all}})$. Later on we shall find an equivalent formulation to $f(\mathbf{x}_0)=\gamma\quad(\mathfrak{F}_{\text{all}})$.

Example 1.6.9. Let f be the regular distribution given by $f(x)=\sin(1/x)$. Then $f(0)=0$ distributionally [35]. Let a_n be a positive sequence with $a_n\to 0$ and $f(a_n)=C>0$. For instance, we could take $a_n=1/(2\pi n+\pi/6)$. For a fixed n, let $\{\psi_{n,m}\}_{m=1}^{\infty}$ be a sequence of positive test functions such that $\psi_{n,m} \to \delta(x - a_n)$ as $m \to \infty$. Then as $n \to \infty$, we obtain a sequence $\delta_n(x) = \delta(x - a_n)$ that converges to $\delta(x)$. For each n, let m_n be large enough so that

$$\left| \int_{B_{1/n}(a_n)} f\left(x\right) \psi_{n,m}\left(x\right) \, \mathrm{d}x - f\left(a_n\right) \right| < C/2$$

and $\operatorname{supp} \psi_{n,m} \subset B_{1/n}\left(a_n\right)$ for $m \geq m_n$, where $B_r\left(x\right)$ is the ball of radius r centered about x. Then we can define the sequence $\phi_n\left(x\right) = \psi_{n,m_n}\left(x\right)$. By Lemma 1.4.1, this is a delta sequence and we have $\langle f(x), \phi_n\left(x\right)\rangle > C/2$ for all n and so $\lim_{n\to\infty} \langle f(x), \phi_n\left(x\right)\rangle$ cannot be equal to 0.

The next lemma will be used in the proof of Proposition 7.11.

Lemma 1.6.10. *If $\{\phi_n\}_{n=1}^{\infty}$ is a delta sequence of positive test functions then*

$$\lim_{n\to\infty} \|\phi_n\|_{L^1(B\setminus U)} = 0\,, \tag{1.6.25}$$

where B and U are both neighborhoods of the origin.

Proof. Choose $\psi \in \mathcal{D}\left(\mathbb{R}^d\right)$ such that $\psi \geq 0$,

$$\psi\left(\mathbf{x}\right) = 1\,, \quad \mathbf{x} \in B \setminus U\,, \quad \psi\left(\mathbf{0}\right) = 0\,, \tag{1.6.26}$$

which is possible because $\mathbf{0} \notin \overline{B \setminus U}$. Then

$$\begin{aligned} \|\phi_n\|_{L^1(B\setminus U)} &= \int_{B\setminus U} \left|\phi_n\left(\mathbf{x}\right)\right| \, \mathrm{d}\mathbf{x} = \int_{B\setminus U} \phi_n\left(\mathbf{x}\right) \, \mathrm{d}\mathbf{x} \\ &= \int_{B\setminus U} \psi\left(\mathbf{x}\right) \phi_n\left(\mathbf{x}\right) \, \mathrm{d}\mathbf{x} \\ &\leq \int_{B} \psi\left(\mathbf{x}\right) \phi_n\left(\mathbf{x}\right) \, \mathrm{d}\mathbf{x} \to \psi\left(\mathbf{0}\right) = 0\,, \end{aligned}$$

as $n \to \infty$. □

We are now ready to prove the main result of this section.

Proposition 1.6.11. *Suppose $f \in \mathcal{D}'\left(\mathbb{R}^d\right)$ and $\mathbf{x}_0 \in \mathbb{R}^d$. If*

$$\lim_{n\to\infty} \langle f\left(\mathbf{x}_0 + \mathbf{x}\right), \phi_n\left(\mathbf{x}\right)\rangle = \gamma\,, \tag{1.6.27}$$

for all positive delta sequences $\{\phi_n\}$, then the following two conditions hold:

1. *There is an $r^* > 0$ such that $f|_{B_{r^*}(\mathbf{x}_0)} \in L^\infty\left(B_{r^*}\left(assignment\mathbf{x}_0\right)\right)$.*

2. $\lim_{r\to 0} \left\| f|_{B_r(\mathbf{x}_0)} - \gamma\chi_{B_r(\mathbf{x}_0)} \right\|_\infty = 0.$

Here $\chi_{B_r(\mathbf{x}_0)}$ is the characteristic function of the ball $B_r\left(\mathbf{x}_0\right)$. Conversely, if (1) and (2) are satisfied, then (1.6.27) holds for all positive delta sequences with support contained in $B_{r^*}\left(\mathbf{0}\right)$.

Proof. Suppose that (1.6.27) holds. Notice that (1) follows from Lemma 6.2. To see that (2) is true, suppose instead that

$$\limsup_{r\to 0} \left\| f|_{B_r(\mathbf{x}_0)} - \gamma\chi_{B_r(\mathbf{x}_0)} \right\|_\infty = C > 0\,. \tag{1.6.28}$$

assignment Let r_n be a decreasing sequence of positive numbers with $r_n < r^*$ and $r_n \to 0$. For each n, there is a positive normalized test function supported in $B_{r_n}\left(\mathbf{0}\right)$, say ϕ_n, such that

$$\left|\langle f\left(\mathbf{x}_0 + \mathbf{x}\right), \phi_n\left(\mathbf{x}\right)\rangle - \gamma\right| > \frac{C}{2}\,. \tag{1.6.29}$$

By Lemma 1.4.1, $\{\phi_n\}_{n=1}^\infty$ forms a delta sequence and so $\lim_{n\to\infty} \langle f\left(\mathbf{x}_0 + \mathbf{x}\right), \phi_n\left(\mathbf{x}\right)\rangle = \gamma$, which contradicts (1.6.29).

For the converse, let $\{\psi_n\}$ be a delta sequence of positive normalized test functions supported in $B_{r^*}\left(\mathbf{0}\right)$. Since by (1) f is a regular distribution in $B_{r^*}\left(\mathbf{0}\right)$ we have

$$\langle f\left(\mathbf{x}_0 + \mathbf{x}\right), \psi_n\left(\mathbf{x}\right)\rangle - \gamma = \int_{B_{r^*}(\mathbf{0})} \left(f\left(\mathbf{x}_0 + \mathbf{x}\right) - \gamma\right)\psi_n\left(\mathbf{x}\right)\,\mathrm{d}\mathbf{x}\,. \tag{1.6.30}$$

Let $\varepsilon > 0$. By condition (2), we can find an open neighborhood V of the origin that is contained in $B_{r^*}(\mathbf{0})$ such that if $W = \mathbf{x}_0 + V$, then $\|f-\gamma\|_{L^\infty(W)} < \varepsilon$ and for this V we can find n_0 such that $\|\psi_n\|_{L_1(B_{r^*}(\mathbf{0})\setminus V)} < \varepsilon$ if $n \geq n_0$. If M is the constant $\|f-\gamma\|_{L^\infty(B_{r^*}(\mathbf{x}_0))}$, then we have

$$\begin{aligned}|\langle f(\mathbf{x}_0+\mathbf{x}), \psi_n(\mathbf{x})\rangle - \gamma| &= \left|\int_W (f(\mathbf{x})-\gamma)\,\psi_n(\mathbf{x}-\mathbf{x}_0)\,\mathrm{d}\mathbf{x}\right.\\ &\qquad \left. + \int_{B_{r^*}(\mathbf{0})\setminus V} (f(\mathbf{x}_0+\mathbf{x})-\gamma)\,\psi_n(\mathbf{x})\,\mathrm{d}\mathbf{x}\right| \\ &\leq \|f-\gamma\|_{L^\infty(W)}\,\|\psi_n\|_{L^1(V)} \\ &\qquad + \|f-\gamma\|_{L^\infty(B_{r^*}(\mathbf{x}_0)\setminus W)}\,\|\psi_n\|_{L^1(B_{r^*}(\mathbf{0})\setminus V)} \\ &< \varepsilon + M\varepsilon\,,\end{aligned}$$

and consequently $\lim_{n\to\infty}\langle f(\mathbf{x}_0+\mathbf{x}), \psi_n(\mathbf{x})\rangle = \gamma$. $\square$

Chapter 2. Asymptotic Expansion of Moments

2.1. Motivation and Background

In this essay, we investigate the asymptotic behavior of moments of the type

$$M_n(f) = \int_X (f(x))^n \, \mathrm{d}x \,, \tag{2.1.1}$$

as $n \to \infty$, where f is a bounded measurable function defined in a measurable space (X, μ) and with values in $\mathbb{R}$ or $\mathbb{C}$, by assigning to f a generalized function F in such a way that the moments $M_n(f)$ coincide with the moments[1] of F,

$$\mu_n(F) = \langle F(u), u^n \rangle \,, \tag{2.1.2}$$

for n large. In particular, if $X = \mathbb{N}$ with the counting measure, we may consider the behavior of moment series, $M_n(f) = M_p(\{\xi_q\}) = \sum_{q=0}^{\infty} \xi_q^p$, where $\{\xi_q\}_{q=0}^{\infty}$ is a bounded sequence of complex numbers.

Depending on the problem, the generalized function F could be a distribution, an analytic functional, or a hyperfunction. Interestingly, the results are quite different for each type of generalized function.

The study of the behavior of the moments $M_n(f)$ for n large is very important in many areas. We mention the recent work of Schlage-Puchta [53], who motivated by problems in signal analysis [5] obtained the asymptotic expansion

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^n \mathrm{d}x \sim \sqrt{\frac{3\pi}{2n}} \left(1 - \frac{3}{20n} - \frac{13}{1120n^2} + \cdots \right) . \tag{2.1.3}$$

This formula illustrates a basic but important point about the moments $M_n(f)$. Namely, if a function with real values can be both positive and negative, then there is no simple

[1] Unfortunately, the same term "moments" is applied to both (2.1.1) and (2.1.2) in the literature. We have tried to avoid any confusion caused by this practice.

or useful way to define powers of the type $(f(x))^{\lambda}$ if λ is not an integer, and thus integrals of the type $\int_X (f(x))^{\lambda}\,\mathrm{d}x$ do *not* make sense unless λ is an integer. Actually, when $f(x) > 0$ for all x, one can obtain the asymptotic expansion of the now well defined integral $\int_X (f(x))^{\lambda}\,\mathrm{d}x$ as the *real* number $\lambda \to \infty$ by using a change of variables in the Laplace asymptotic formula [17, (3.133)], so that in particular

$$\int_{-\infty}^{\infty} (f(x))^{\lambda}\,\phi(x)\,\mathrm{d}x \sim f(x_0)^{\lambda+1/2}\sqrt{\frac{-2\pi}{\lambda f''(x_0)}}\phi(x_0)\ , \tag{2.1.4}$$

as $\lambda \to \infty$ if the *positive* function f has a single maximum at the interior point x_0 where $f''(x_0) < 0$. However, (2.1.4) cannot be applied if f changes signs as in the case when $f(x) = \sin x/x$.

2.2. Distributions: a general construction

In this article we will need to employ a method of construction of distributions as regularizations of integrals of the type

$$\int_X \phi(f(x))\,\mathrm{d}\mu(x)\ , \tag{2.2.1}$$

where ϕ is a test function in $\mathbb{R}^d$ and where f is a bounded measurable function defined in a measurable space (X, μ) and with values in $\mathbb{R}^d$. In general the integrals in (3.3.1) are divergent; notice, in particular that if $\phi = 1$ and X has infinite measure the integral is divergent.

Similar considerations [15] allow us to construct analytic functionals from complex valued functions f.

2.2.1. One variable

We now will consider a general construction of distributions from divergent integrals of the type of (3.3.1); it is convenient to start in the case of real valued functions.

Let $\mathcal{A}$ be a space of test functions on $\mathbb{R}$ or on $[0,\infty)$, such as $\mathcal{A} = \mathcal{D}$, $\mathcal{S}$, or $\mathcal{E}$. Following Grafakos and Teschl [22] and Estrada [14], we shall denote as $\mathcal{R}_n = \mathcal{R}_n(\mathcal{A})$ the subspace

$$\mathcal{R}_n = \left\{\phi \in \mathcal{A} : \phi^{(j)}(0) = 0\ ,\ 0 \leq j \leq n-1\right\}\ , \tag{2.2.2}$$

for $n = 1, 2, \ldots$, or for $n = \infty$. Since $\mathcal{R}_n$ is a closed subspace of $\mathcal{A}$ for any n, it follows from the Hahn-Banach theorem that any distribution $f_n \in \mathcal{R}_n'$ has extensions $f \in \mathcal{A}'$. If f_* is an extension, then the general form of all extensions is

$$f(x) = f_*(x) + \sum_{j=0}^{n-1} c_j \delta^{(j)}(x)\ , \tag{2.2.3}$$

where the c_j are arbitrary constants. For $\mathcal{R}_\infty$ it would be (2.2.3) but for an arbitrary n.

Notice, however, that the extension from $\mathcal{R}_\infty$ can be done in two steps. Let us work in $\mathcal{R}_\infty(\mathcal{E})$ to fix the ideas; the analysis in other spaces of test functions being similar. First, because of the way the topology of the space of test functions $\mathcal{E}(\mathbb{R})$ is defined, in terms of the seminorms of the type

$$\|\phi\|_{a,n} = \max_{|x|\leq a,\, j\leq n} \left|\phi^{(j)}(x)\right|\ , \tag{2.2.4}$$

it follows that if $f_\infty \in \mathcal{R}_\infty'$ then there exists $n \in \mathbb{N}$ such that f_∞ admits a continuous extension to $\mathcal{R}_n'$, that is, f_∞ is continuous with respect to this seminorm:

$$\left|\langle f_\infty(x), \phi(x)\rangle\right| \leq M\,\|\phi\|_{a,n}\ , \tag{2.2.5}$$

whenever $\phi \in \mathcal{R}_\infty$ has support included in $[-a,a]$. Hence f_∞ can be extended (in a unique way) to this $\mathcal{R}_n$. Another extension where one actually uses the Hahn-Banach theorem gives extensions to $\mathcal{A}'$; this second extension contains n arbitrary constants.

We may employ these ideas for distributions defined by integrals. Indeed, we immediately obtain the following result.[2]

Lemma 2.2.1. *Suppose that* f *is a measurable function defined in* $\mathbb{R}$ *such that* $\mathsf{f}\chi_{\mathbb{R}\setminus(-a,a)}$ *is a regular distribution of* $\mathcal{A}'$ *for all* $a>0$, *and such that the integral*

$$\int_{-\infty}^{\infty}\mathsf{f}\left(x\right)\phi\left(x\right)\,\mathrm{d}x\,, \tag{2.2.6}$$

exists (as a $(\mathfrak{Z})$ *integral) for each* $\phi\in\mathcal{R}_{\infty}$. *Then there exists* $n\in\mathbb{N}$ *such that the integral will exist in the same sense for all* $\phi\in\mathcal{R}_n$ *and there exist distributions (not unique, depending on* n *arbitrary constants)* $f\in\mathcal{A}'$ *such that*

$$\left\langle f\left(x\right),\phi\left(x\right)\right\rangle=\int_{-\infty}^{\infty}\mathsf{f}\left(x\right)\phi\left(x\right)\,\mathrm{d}x\,,\quad\phi\in\mathcal{R}_n\,. \tag{2.2.7}$$

The $(\mathfrak{Z})$ sense could be absolute convergence at 0, that is Lebesgue integrals, or conditionally convergent at 0, that is, improper Riemann integrals at the origin, or others such as Denjoy integrals, etc. In general one needs to use the *same* integration sense when extending to $\mathcal{R}_n$, as the next example shows.

Example 2.2.2. Consider the function $\mathsf{f}\left(x\right)=e^{1/x^2}x^{-2}\cos\left(e^{1/x^2}\right)$. In this case the integrals (2.2.6) exist whenever $\phi\in\mathcal{R}_1\left(\mathcal{D}\right)$ and are conditionally convergent, in general, but they might not be absolutely convergent, even if $\phi\in\mathcal{R}_{\infty}$.

Our aim is to use these ideas to define distributions not from integrals of the form (2.2.6) but from integrals of the type $\int_X\phi\left(f\left(x\right)\right)\,\mathrm{d}\mu\left(x\right)$, where (X,μ) is a measure space and $f\in L^{\infty}\left(X\right)$ is real-valued. In the next result we can take $\mathcal{A}=\mathcal{E}$.

[2]In the following lemma we use the notation f for the measurable function and f for the probably not uniquely determined distribution. We will employ such notation only if there is a danger of confusion.

Lemma 2.2.3. *Suppose* $\phi \circ f \in L^1(X)$ *whenever* $\phi \in \mathcal{R}_\infty(\mathcal{E})$. *Then there exists* $n \in \mathbb{N}$ *such that* $\phi \circ f \in L^1(X)$ *if* $\phi \in \mathcal{R}_n(\mathcal{E})$. *If* $k > n$ *then* $f \in L^k(X)$, *that is,*

$$\int_X |f(x)|^k \, \mathrm{d}\mu(x) < \infty \,. \tag{2.2.8}$$

There exist distributions $F \in \mathcal{E}'$ *such that*

$$\langle F(u), \phi(u) \rangle = \int_X \phi(f(x)) \, \mathrm{d}\mu(x) \,, \quad \phi \in \mathcal{R}_n(\mathcal{E}) \,. \tag{2.2.9}$$

Proof. The proof follows from our analysis, but there is a little detail when establishing that $f \in L^k(X)$ if $k > n$. Indeed, if k is even, $k \geq n$, then x^k belongs to $\mathcal{R}_n(\mathcal{E})$ and thus (2.2.8) follows. Thus $f \in L^n(X)$ or $f \in L^{n+1}(X)$, and since if $f \in L^q(X)$ then $f \in L^k(X)$ for $k > q$, we obtain (2.2.8) whenever $k > n$. □

In the case X is $\mathbb{R}$ or $[0, \infty)$ we may also obtain a stronger result. Actually, for a general space X the Lemma 2.2.3 applies to Lebesgue integrals, which for $\mathbb{R}$ or $[0, \infty)$ means *absolutely* convergent integrals. However, in this case we may also consider *conditionally* convergent integrals.

Lemma 2.2.4. *Let* $f \in L^\infty([0, \infty))$ *and* g *measurable and non-negative in* $[0, \infty)$ *such that the integrals*

$$\int_0^\infty \phi(f(x)) \, g(x) \, \mathrm{d}x \,, \tag{2.2.10}$$

are (maybe conditionally) convergent at the origin for any $\phi \in \mathcal{R}_\infty(\mathcal{E})$. *Then for some* $n \in \mathbb{N}$ *the integrals are all absolutely convergent if* $\phi \in \mathcal{R}_n(\mathcal{E})$. *Also*

$$\int_0^\infty |f(x)|^k \, g(x) \, \mathrm{d}x < \infty \,, \tag{2.2.11}$$

if $k > n$. *There exist distributions* $F \in \mathcal{E}'$ *such that*

$$\langle F(u), \phi(u) \rangle = \int_0^\infty \phi(f(x)) \, \mathrm{d}x \,, \tag{2.2.12}$$

for $\phi \in \mathcal{R}_n(\mathcal{E})$.

Proof. Indeed, if we define the functional $F_\infty \in \mathcal{R}'_\infty(\mathcal{E})$ by the formula (2.2.10) then it will have a uniquely defined extension to $\mathcal{R}'_{n_0}(\mathcal{E})$ for some n_0. As the proof of the Lemma 2.2.3 shows, this implies that $f \in L^k([0,\infty); g(x)\,\mathrm{d}x)$ if $k > n_0$. Consequently, if $n = n_0 + 1$, then the integrals (2.2.10) will be absolutely convergent if $\phi \in \mathcal{R}_n(\mathcal{E})$. $\square$

Example 2.2.2 shows that a corresponding result on absolute convergence does not hold for integrals of the type (2.2.6), in general.

The next lemma complements these results.

Lemma 2.2.5. *Let* $f \in L^\infty([0,\infty))$ *and* g *measurable and non-negative in* $[0,\infty)$. *Then* $\int_0^\infty \phi(f(x))\,g(x)\,\mathrm{d}x$ *admits regularizations in* $\mathcal{E}$, *that is, there exist distributions* $F \in \mathcal{E}'$ *such that* $\langle F(u), \phi(u)\rangle = \int_0^\infty \phi(f(x))\,\mathrm{d}x$ *for* $\phi \in \mathcal{R}_\infty(\mathcal{E})$ *if and only if there exists* k *such that* $\int_0^\infty |f(x)|^k g(x)\,\mathrm{d}x < \infty$.

We can also consider the extension of distributions defined by series, that is, if $X = \mathbb{N}$ and μ is a positive multiple of the counting measure. The proof of the ensuing result is identical to that of the Lemma 2.2.4.

Lemma 2.2.6. *Let* $\{a_q\}_{q=0}^\infty$ *be a sequence of real numbers with* $\lim_{q\to\infty} a_q = 0$ *and* $\{\mu_q\}_{q=0}^\infty$ *another sequence with* $\mu_q > 0$ *for all* q. *Suppose that the series*

$$\sum_{q=0}^\infty \phi(a_q)\,\mu_q\,, \tag{2.2.13}$$

are (maybe conditionally) convergent if $\phi \in \mathcal{R}_\infty(\mathcal{E})$. *Then for some* $n \in \mathbb{N}$ *the series are absolutely convergent if* $\phi \in \mathcal{R}_n(\mathcal{E})$. *If* $k > n$ *then*

$$\sum_{q=0}^\infty |a_q|^k \mu_q < \infty\,. \tag{2.2.14}$$

There are distributions $F \in \mathcal{E}'$ such that

$$\langle F(u), \phi(u) \rangle = \sum_{q=0}^{\infty} \phi(a_q)\, \mu_q\,, \quad \phi \in \mathcal{R}_n(\mathcal{E}) \ . \tag{2.2.15}$$

Observe that a distribution F that satisfies (2.2.15) is a regularization of the series of delta functions

$$\sum_{q=0}^{\infty} \mu_q \delta(u - a_q) \ , \tag{2.2.16}$$

that is probably divergent in $\mathcal{E}'$. Notice also that the support of any such distribution F is the set $\operatorname{supp} F = \{a_q : q \geq 0\} \cup \{0\}$. In case $\mu_q = 1$ for all q, then the convergence of the series (2.2.13) for all $\phi \in \mathcal{R}_\infty(\mathcal{E})$ implies that $\lim_{q\to\infty} a_q = 0$. The series (2.2.16) admits a regularization in $\mathcal{E}$ if and only if (2.2.14) is satisfied for some k.

2.2.2. Several variables

Most of the ideas of the one variable case also apply to the multidimensional situation. In this article we would be interested mainly in the situation of functions into $\mathbb{C}$, that we shall identify with $\mathbb{R}^2$ in this analysis.

Indeed, if $\mathcal{A}$ is a space of test functions in $\mathbb{R}^d$ denote by $\mathcal{R}_n = \mathcal{R}_n(\mathcal{A})$ the subspace

$$\mathcal{R}_n = \left\{\phi \in \mathcal{A} : \nabla^{\mathbf{j}} \phi(\mathbf{0}) = 0 \ , \ 0 \leq |\mathbf{j}| \leq n-1\right\} \ , \tag{2.2.17}$$

for $n = 1, 2, \ldots$, or for $n = \infty$. Here ∇ is the gradient vector operator, $\mathbf{j} = (j_1, \ldots, j_d)$ is a multi-index and $|\mathbf{j}| = \sum_{q=1}^{d} j_q$. Since $\mathcal{R}_n$ is a closed subspace of $\mathcal{A}$ for any n, it follows from the Hahn-Banach theorem that any distribution $f_n \in \mathcal{R}_n'$ has extensions $f \in \mathcal{A}'$. If f_* is an extension, then the general form of all extensions is

$$f(x) = f_*(x) + \sum_{|\mathbf{j}| \leq n-1} c_{\mathbf{j}} \nabla^{\mathbf{j}} \delta(\mathbf{x}) \ , \tag{2.2.18}$$

where the $c_{\mathbf{j}}$ are arbitrary constants. (For $\mathcal{R}_\infty$ it would be (2.2.18) for an arbitrary n).

The extension from $\mathcal{R}_\infty$ can be done in two steps. First, because of the way the topology of spaces of test functions is defined, in terms of the seminorms of the type

$$\|\phi\|_{a,n} = \max_{|\mathbf{x}|\le a,\, |\mathbf{j}|\le n-1} \left|\nabla^{\mathbf{j}}\phi\left(\mathbf{x}\right)\right| \,, \tag{2.2.19}$$

(plus some other condition at infinity that is not important presently), it follows that if $f_\infty \in \mathcal{R}_\infty'$ then there exists $n \in \mathbb{N}$ such that f_∞ admits a continuous extension to $\mathcal{R}_n$, that is, f_∞ is continuous with respect to this seminorm:

$$\left|\left\langle f_\infty\left(x\right), \phi\left(x\right)\right\rangle\right| \le M \left\|\phi\right\|_{a,n} \,, \tag{2.2.20}$$

whenever $\phi \in \mathcal{R}_\infty$ has support included in the ball $B_a = \{\mathbf{x} : |\mathbf{x}| \le a\}$. Hence f_∞ can be extended (in a unique way) to this $\mathcal{R}_n$; another extension where one actually uses the Hahn-Banach theorem gives extensions to $\mathcal{A}'$, that contain arbitrary constants $c_{\mathbf{j}}$ for $|\mathbf{j}| < n$.

Repeating the arguments of the previous section, we obtain the following result for functions from a measure space (X, μ) to $\mathbb{R}^d$.

Lemma 2.2.7. *Let $\mathbf{f} : X \to \mathbb{R}^d$ be a bounded measurable function. Suppose $\phi \circ \mathbf{f} \in L^1(X)$ whenever $\phi \in \mathcal{R}_\infty(\mathcal{E})$. Then there exists $n \in \mathbb{N}$ such that $\phi \circ \mathbf{f} \in L^1(X)$ if $\phi \in \mathcal{R}_n(\mathcal{E})$. If $k > n$ then*

$$\int_X \left\|\mathbf{f}\left(x\right)\right\|^k \, \mathrm{d}\mu\left(x\right) < \infty \,. \tag{2.2.21}$$

There exist distributions $F \in \mathcal{E}'$ such that

$$\left\langle F\left(\mathbf{u}\right), \phi\left(\mathbf{u}\right)\right\rangle = \int_X \phi\left(\mathbf{f}\left(x\right)\right) \, \mathrm{d}\mu\left(x\right) \,, \quad \phi \in \mathcal{R}_n\left(\mathcal{E}\right) \,. \tag{2.2.22}$$

This lemma covers the case of Lebesgue integrals. When $X = [0, \infty)$ or $\mathbb{R}$ we may consider conditionally convergent integrals.

Lemma 2.2.8. *Let* $\mathbf{f} : [0, \infty) \to \mathbb{R}^d$ *be a bounded measurable function and* g *measurable and non-negative in* $[0, \infty)$ *such that the integrals*

$$\int_0^\infty \phi\left(\mathbf{f}\left(x\right)\right) g\left(x\right)\, \mathrm{d}x\,, \tag{2.2.23}$$

are (maybe conditionally) convergent at the origin for any $\phi \in \mathcal{R}_\infty\left(\mathcal{E}\right)$. *Then for some* $n \in \mathbb{N}$ *the integrals are all absolutely convergent if* $\phi \in \mathcal{R}_n\left(\mathcal{E}\right)$. *Also*

$$\int_0^\infty \left\|\mathbf{f}\left(x\right)\right\|^k g\left(x\right)\, \mathrm{d}x < \infty\,, \tag{2.2.24}$$

if $k > n$. *There are distributions* $F \in \mathcal{E}'$ *such that*

$$\left\langle F\left(\mathbf{u}\right), \phi\left(\mathbf{u}\right)\right\rangle = \int_0^\infty \phi\left(\mathbf{f}\left(x\right)\right) g\left(x\right)\, \mathrm{d}x\,, \quad \phi \in \mathcal{R}_n\left(\mathcal{E}\right)\,. \tag{2.2.25}$$

The existence of such regularizations F *is equivalent to the existence of* k *for which (2.2.24) holds.*

For conditionally convergent series we obtain the next result.

Lemma 2.2.9. *Let* $\{\mathbf{a}_q\}_{q=0}^\infty$ *be a sequence in* $\mathbb{R}^d$ *with* $\lim_{q\to\infty} \mathbf{a}_q = 0$ *and* $\{\mu_q\}_{q=0}^\infty$ *a sequence with* $\mu_q > 0$ *for all* q. *Suppose that the series*

$$\sum_{q=0}^\infty \phi\left(\mathbf{a}_q\right)\mu_q\,, \tag{2.2.26}$$

are (maybe conditionally) convergent if $\phi \in \mathcal{R}_\infty\left(\mathcal{E}\right)$. *Then for some* $n \in \mathbb{N}$ *the series are all absolutely convergent whenever* $\phi \in \mathcal{R}_n\left(\mathcal{E}\right)$. *If* $k > n$ *then*

$$\sum_{q=0}^\infty \left\|\mathbf{a}_q\right\|^k \mu_q < \infty\,. \tag{2.2.27}$$

There are distributions $F \in \mathcal{E}'$ such that

$$\langle F(\mathbf{u}), \phi(\mathbf{u}) \rangle = \sum_{q=0}^{\infty} \phi(\mathbf{a}_q)\, \mu_q\,, \quad \phi \in \mathcal{R}_n(\mathcal{E})\ . \tag{2.2.28}$$

The existence of such regularizations F is equivalent to the existence of k for which (2.2.27) holds.

2.3. The expansion of quotients of gamma functions

A key component in the study of the asymptotic behavior of moments is the expansion of quotients of gamma functions

$$\frac{\Gamma(z+\alpha)}{\Gamma(z)}\,, \tag{2.3.1}$$

as $z \to \infty$. This problem has been studied for many years [20, 56], and it admits a rather simple and elegant asymptotic formula in terms of the Stirling numbers.

The *form* of the expansion can be obtained from Stirling's formula [17, (3.88)]

$$\ln\Gamma(z) \sim \left(z - \frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + \sum_{k=0}^{\infty} \frac{B_{k+2}}{(k+1)(k+2)\, z^{k+1}}\,, \tag{2.3.2}$$

where the B_k are the Bernoulli numbers. Indeed, with a little work one obtains the expansion of $\ln(\Gamma(z+\alpha)/\Gamma(z)) = \ln\Gamma(z+\alpha) - \ln\Gamma(z)$ as

$$\ln\left(\frac{\Gamma(z+\alpha)}{\Gamma(z)}\right) \sim \ln z^{\alpha} + \sum_{k=1}^{\infty} \frac{a_k(\alpha)}{z^k}\,, \tag{2.3.3}$$

where the $a_k(\alpha)$ are some polynomial expressions of α. With a little more work one can find a formula for the first few a_k; for instance [32]

$$a_1(\alpha) = \frac{\alpha(\alpha-1)}{2}\,. \tag{2.3.4}$$

If we take exponentials in (2.3.3) we obtain that

$$\frac{\Gamma(z+\alpha)}{\Gamma(z)} \sim z^{\alpha}\left\{1 + \sum_{k=1}^{\infty} \frac{A_k(\alpha)}{z^k}\right\}\,, \tag{2.3.5}$$

where the A_k are polynomials in α. In fact, Tricomi and Erdélyi [56] give formulas for the first polynomials A_k by using binomial coefficients, namely,

$$A_1(\alpha) = \binom{\alpha}{2}, \quad A_2(\alpha) = \frac{3\alpha - 1}{4}\binom{\alpha}{3}, \quad A_3(\alpha) = \binom{\alpha}{2}\binom{\alpha}{4}. \tag{2.3.6}$$

While the form of the expansion (2.3.5) is simple, it looks as if a formula for the A_k would be very complicated; surprisingly, they can be written rather easily in terms of the Stirling numbers of the first kind, as we now explain.

The idea is the following: since the A_k are polynomials, it is enough to find their expression for *some* values of α, in particular, it is enough to find the formula when α is a positive integer. But if $\alpha = m$, a positive integer, we obtain

$$\frac{\Gamma(z+m)}{\Gamma(z)} = z(z+1)\cdots(z+m-1) . \tag{2.3.7}$$

The polynomial $z(z+1)\cdots(z+m-1)$ has degree m and no constant term, and thus it admits a development in terms of z, z^2, ..., z^m. The coefficients of this development are *exactly* the (signless) Stirling numbers of the first kind [26], that is

$$z(z+1)\cdots(z+m-1) = \begin{bmatrix} m \\ 1 \end{bmatrix} z + \begin{bmatrix} m \\ 2 \end{bmatrix} z^2 + \begin{bmatrix} m \\ 3 \end{bmatrix} z^3 + \cdots + \begin{bmatrix} m \\ m \end{bmatrix} z^m, \tag{2.3.8}$$

where we use the notation $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]$ [29] for the Stirling numbers. We remark that Jordan notation is S_m^k, for the signed version,

$$\begin{bmatrix} m \\ k \end{bmatrix} = (-1)^{m+k} S_m^k . \tag{2.3.9}$$

We can rewrite (2.3.8) as

$$\frac{\Gamma(z+m)}{\Gamma(z)} = z^m \sum_{k=0}^{\infty} \begin{bmatrix} m \\ m-k \end{bmatrix} z^{-k}, \tag{2.3.10}$$

if we set $\begin{bmatrix} m \\ k \end{bmatrix} = 0$ for $k \leq 0$. We now observe that Jordan [26, p. 149] establishes that $\begin{bmatrix} m \\ m-k \end{bmatrix}$ is a polynomial of degree $2k$ in m. Actually,

$$\begin{bmatrix} m \\ m \end{bmatrix} = 1\,, \tag{2.3.11}$$

$$\begin{bmatrix} m \\ m-1 \end{bmatrix} = \binom{m}{2} = \frac{m^2-m}{2}\,, \tag{2.3.12}$$

$$\begin{bmatrix} m \\ m-2 \end{bmatrix} = 3\binom{m}{4} + 2\binom{m}{3}\,, \tag{2.3.13}$$

$$\begin{bmatrix} m \\ m-3 \end{bmatrix} = 15\binom{m}{6} + 20\binom{m}{5} + 6\binom{m}{4}\,, \tag{2.3.14}$$

and more generally

$$\begin{bmatrix} m \\ m-k \end{bmatrix} = \sum_{j=0}^{k-1} C_{k,j}\binom{m}{2k-j}\,. \tag{2.3.15}$$

A table for the $C_{k,j}$ is given in [26, p. 152]. Therefore *we define* $\begin{bmatrix} \alpha \\ \alpha-k \end{bmatrix}$ for $\alpha \in \mathbb{C}$ as this polynomial of degree $2k$ evaluated at α. Since $A_k(m) = \begin{bmatrix} m \\ m-k \end{bmatrix}$ when m is a positive integer we immediately obtain the formula[3]

$$A_k(\alpha) = \begin{bmatrix} \alpha \\ \alpha-k \end{bmatrix}, \qquad \alpha \in \mathbb{C}\,. \tag{2.3.16}$$

Summarizing, we have the following expansion.

Proposition 2.3.1. *If* $\alpha \in \mathbb{C}$ *then*

$$\frac{\Gamma(z+\alpha)}{\Gamma(z)} \sim z^{\alpha} \sum_{k=0}^{\infty} \begin{bmatrix} \alpha \\ \alpha-k \end{bmatrix} z^{-k}, \tag{2.3.17}$$

as $z \to \infty$ *in any sector that does not contain* $\alpha - n$ *for* $n \in \mathbb{N}$ *large.*

We remark that the series (2.3.17) is always asymptotic, but sometimes it converges. Indeed, as we saw, if $\alpha \in \mathbb{N}$ then the series is actually a polynomial. On the other

[3]That this coincides with the Tricomi-Erdélyi formulas (2.3.6) is easy to see.

hand, if $\alpha = -1, -2, -3, \ldots$, i.e, $\alpha = -q$ then

$$\frac{\Gamma(z-q)}{\Gamma(z)} = \frac{1}{(z-1)\cdots(z-q)} = z^{-q}\sum_{k=0}^{\infty}\begin{bmatrix}-q\\ -q-k\end{bmatrix} z^{-k}, \tag{2.3.18}$$

is a series that converges for $|z| > q$; it reduces to the geometric series if $q = 1$. In fact, if the signless Stirling numbers of the second kind $\left\{ {k \atop m} \right\}$ are defined as [29]

$$\begin{Bmatrix}k\\ m\end{Bmatrix} = \begin{bmatrix}-m\\ -k\end{bmatrix}, \tag{2.3.19}$$

then (2.3.18) can be rewritten as

$$\frac{\Gamma(z-q)}{\Gamma(z)} = z^{-q}\sum_{k=0}^{\infty}\begin{Bmatrix}q+k\\ q\end{Bmatrix} z^{-k}. \tag{2.3.20}$$

Interestingly, when $\alpha \notin \mathbb{Z}$ then the series (2.3.17) is a *divergent* asymptotic series. Indeed, if (2.3.17) converges for some z_0 then it would converge for $|z| > |z_0|$ and thus $F(z) = \sum_{k=0}^{\infty}\left[{\alpha \atop \alpha-k} \right] z^{-k}$ would be an analytic function in the region $|z| > |z_0|$ and thus $z^{-\alpha}\Gamma(z+\alpha)/\Gamma(z)$ would likewise be analytic in that exterior disc, and this is never true if $\alpha \notin \mathbb{Z}$.

A similar analysis yields the expansion

$$\begin{aligned}\frac{\Gamma(z+1)}{\Gamma(z+\beta+1)} &\sim z^{-\beta}\sum_{k=0}^{\infty}\begin{bmatrix}-\beta\\ -\beta-k\end{bmatrix}(-1)^k z^{-k}\\ &\sim z^{-\beta}\sum_{k=0}^{\infty}\begin{Bmatrix}\beta+k\\ \beta\end{Bmatrix}(-1)^k z^{-k}\end{aligned} \tag{2.3.21}$$

as $z \to \infty$ inside sectors that do not contain the negative integers. Notice in particular the finite development

$$z(z-1)\cdots(z-m+1) = \frac{\Gamma(z+1)}{\Gamma(z-m+1)} = z^m\sum_{k=0}^{m-1}\begin{bmatrix}m\\ m-k\end{bmatrix}(-1)^k z^{-k}, \tag{2.3.22}$$

that Jordan [26, p. 142] employs to define the Stirling numbers of the first kind S_m^k.

2.4. Asymptotic behavior of the moments of distributions with compact support

Let $F \in \mathcal{E}'(\mathbb{R})$ be a distribution of compact support in one variable, with $\operatorname{supp} F = [a, b]$. We will now show how the asymptotic behavior of the moments

$$\mu_n = \mu_n(F) = \langle F(u), u^n \rangle , \tag{2.4.1}$$

can be obtained from the distributional behavior of F at the endpoints. We start with a simple observation.

Lemma 2.4.1. *If* $\operatorname{supp} F = [a, b]$ *then there exists* $k \in \mathbb{N}$ *such that*

$$\mu_n = O\left(n^k c^n\right) , \tag{2.4.2}$$

as $n \to \infty$ *if* $c = \max\{|a|, b\}$.

Proof. Indeed, there exists $M > 0$ and $k \in \mathbb{N}$ such that

$$|\langle F(u), \phi(u) \rangle| \leq M \max_{a \leq u \leq b} \left\{ \left|\phi^{(j)}(u)\right| : j \leq n \right\} , \tag{2.4.3}$$

and this yields (2.4.2). □

Notice, that the lemma gives that if $c_1 > c$ then $\mu_n = o(c_1^n)$ as $n \to \infty$.

In our analysis we will employ the following terminology. Let α be a real number that is not a negative integer and let $F \in \mathcal{D}'(\mathbb{R})$ be a distribution. We say that

$$F(x) = o(|x - a|^\alpha) , \quad \text{as} \ \ x \to a \ , \ \text{distributionally} , \tag{2.4.4}$$

if $F(a + \varepsilon x) = o(\varepsilon^\alpha)$ as $\varepsilon \to 0$ in $\mathcal{D}'(\mathbb{R})$, that is, if for all test functions $\phi \in \mathcal{D}(\mathbb{R})$ we have

$$\langle F(a + \varepsilon x), \phi(x) \rangle = o(\varepsilon^\alpha) , \quad \text{as } \varepsilon \to 0 . \tag{2.4.5}$$

When $\operatorname{supp} F \subset [a,\infty)$ we just write $F(x) = o((x-a)^{\alpha})$ as $x \to a^{+}$, while if $\operatorname{supp} F \subset (-\infty, a]$ we write $F(x) = o((a-x)^{\alpha})$ as $x \to a^{-}$. It is easy to see [17, 49] that if φ is a smooth function with $\varphi(a) = b$ and with $\varphi'(a) \neq 0$, then $F(x) = o(|x-b|^{\alpha})$ as $x \to b$ distributionally if and only if $F(\varphi(x)) = o(|x-a|^{\alpha})$ as $x \to a$ distributionally.

Lemma 2.4.2. *Let* $\alpha \in \mathbb{R} \setminus \{-1,-2,-3,\ldots\}$. *Suppose* $\operatorname{supp} F = [\varepsilon, b]$ *where* $\varepsilon > 0$ *and*

$$F(u) = o((b-u)^{\alpha}) \ , \tag{2.4.6}$$

distributionally as $u \to b^{-}$. *Then*

$$\mu_n = o\left(b^n n^{-\alpha-1}\right) \ , \quad as \quad n \to \infty \, . \tag{2.4.7}$$

Proof. Indeed, writing $u = be^{-t}$, $g(t) = f(be^{-t})$ we see that $g(t) = o(t^{\alpha})$ as $t \to 0^{+}$ distributionally. Therefore if $x = (n+1)t$, we see that

$$\begin{aligned} \mu_n &= \langle f(u), u^n \rangle_u = -b^{n+1} \left\langle f\left(be^{-t}\right), e^{-(n+1)t} \right\rangle_t \\ &= \frac{-b^{n+1}}{(n+1)} \left\langle g\left(\left(\frac{1}{n+1}\right)x\right), e^{-x} \right\rangle_x = \frac{b^{n+1}}{(n+1)} o\left(\frac{1}{(n+1)^{\alpha}}\right) \, , \end{aligned}$$

and (2.4.7) follows. □

We will denote as A_α a (fixed) function that has the asymptotic expansion

$$A_\alpha(n) \sim n^{-\alpha-1} \sum_{k=0}^{\infty} \begin{bmatrix} -\alpha-2 \\ \\ -\alpha-2-k \end{bmatrix} (-1)^k n^{-k}. \tag{2.4.8}$$

as $n \to \infty$. Clearly $A_\alpha(n) \sim n^{-\alpha-1}$ as $n \to \infty$. Employing the asymptotic formula (2.3.21) we obtain that a possible choice for the function A_α is the normalized moment function

$$\Gamma(\alpha+1) A_\alpha(n) = \langle H(u)\, H(1-u)\ (1-u)^{\alpha}, u^n \rangle = \int_0^1 u^n (1-u)^{\alpha} \, du \, ,$$

H being the Heaviside function, since

$$A_\alpha(n) = \frac{B(n+1,\alpha+1)}{\Gamma(\alpha+1)} = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+2)},$$

where $B(x,y)$ is the beta function [32].

Notice that the moments of the function $H(u)\,H(b-u)\,(b-u)^\alpha$ can then be written as

$$\langle H(u)\,H(1-u)\,(1-u)^\alpha, u^n\rangle = b^{n+\alpha+1}\Gamma(\alpha+1)\,A_\alpha(n)\ . \tag{2.4.9}$$

We thus immediately obtain the following.

Lemma 2.4.3. *Suppose* $\alpha_j \nearrow \infty$. *Suppose* $\operatorname{supp} F = [\varepsilon, b]$ *where* $\varepsilon > 0$ *and that for some constants* C_j

$$F(u) \sim \sum_{j=1}^{\infty} C_j\,(b-u)^{\alpha_j}\ , \tag{2.4.10}$$

distributionally as $u \to b^-$. *Then*

$$\mu_n \sim \sum_{j=1}^{\infty} b^{n+\alpha_j+1} C_j \Gamma(\alpha_j+1)\,A_{\alpha_j}(n)\ . \tag{2.4.11}$$

as $n \to \infty$.

Proof. The asymptotic expansion (2.4.10) means that as $u \to b^-$ for all m we have $F(u) = \sum_{j=1}^{m} C_j\,(b-u)^{\alpha_j} + o\left((b-u)^{\alpha_m}\right)$ distributionally. Hence

$$F(u) = \sum_{j=1}^{m} C_j H(u)\,H(b-u)\,(b-u)^{\alpha_j} + R_m(u)\ , \tag{2.4.12}$$

where the rest R_m satisfies $R_m(u) = o\left((b-u)^{\alpha_m}\right)$ distributionally as $u \to b^-$. Hence (2.4.9) yields

$$\mu_n = \sum_{j=1}^{m} b^{n+\alpha_j+1} C_j \Gamma(\alpha_j+1)\,A_{\alpha_j}(n) + o\left(b^n n^{-\alpha_m-1}\right)\ , \tag{2.4.13}$$

or,

$$\mu_n \sim \sum_{j=1}^{m} b^{n+\alpha_j+1} C_j \Gamma(\alpha_j+1) A_{\alpha_j}(n) + o(b^n A_{\alpha_m}(n)) \ , \tag{2.4.14}$$

since $A_{\alpha_m}(n) \sim n^{-\alpha_m-1}$. □

It should be pointed out that when $\alpha_1 = \alpha$ and $\alpha_{k+1} = \alpha_k + 1$, $k \geq 1$. Then (2.4.11) becomes the expansion

$$\begin{aligned} \mu_n =& \Gamma(\alpha+1)C_1 n^{-(\alpha+1)} + \sum_{k=1}^{\infty} (C_k \Gamma(\alpha+k+1) \\ & \sum_{j=1}^{k} (-1)^j C_{k-j} \Gamma(\alpha+k+1-j) \begin{bmatrix} -\alpha-(k-(j-1)) \\ -\alpha-(k+1) \end{bmatrix}) \, n^{-k-(\alpha+1)}. \end{aligned} \tag{2.4.15}$$

It is also easy to obtain the development of distributions of compact support in the negative half line. If $\operatorname{supp} F = [a, -\varepsilon]$ where $\varepsilon > 0$ then we consider the distribution $G(u) = F(-u)$ whose moments are related as $\mu_n(F) = (-1)^n \mu_n(G)$, and consider the behavior of G at $|a|$.

Lemma 2.4.4. *Suppose* $\beta_j \nearrow \infty$. *Suppose* $\operatorname{supp} F = [a, -\varepsilon]$ *where* $\varepsilon > 0$ *and*

$$F(u) \sim \sum_{j=1}^{\infty} D_j H(u-a)(u-a)^{\beta_j} \ , \tag{2.4.16}$$

distributionally as $u \to a^+$. *Then*

$$\mu_n \sim \sum_{j=1}^{\infty} (-1)^n |a|^{n+\beta_j+1} D_j \Gamma(\beta_j+1) A_{\beta_j}(n) \ . \tag{2.4.17}$$

Let us now consider the general case of a distribution F with $\operatorname{supp} F = [a, b]$ where $a < 0 < b$. We can decompose F as $F_a + F_\varepsilon + F_b$ where $\operatorname{supp} F_a = [a, -\varepsilon]$, $\operatorname{supp} F_\varepsilon = [-\varepsilon, \varepsilon]$, and $\operatorname{supp} F_b = [\varepsilon, b]$. We know how to find the asymptotic expansion of $\mu_n(F_a)$ and $\mu_n(F_b)$, while $\mu_n(F_\varepsilon) = o(c^n)$, where $c = \max\{|a|, b\}$. In case $|a| \neq b$ then the expansion of $\mu_n(F)$ will be equal to that of $\mu_n(F_a)$ if $|a| > b$ or of $\mu_n(F_b)$ if $b > |a|$. If $a = -b$ then the expansion of $\mu_n(F)$ will be the sum of the expansions of $\mu_n(F_a)$ and of $\mu_n(F_b)$; in such a case some cancellation is possible.

Naturally one can consider other types of asymptotic behavior of F at the endpoints, but our aim presently is to illustrate the possible *methods* of analysis.

2.5. Expansion of moments

In this section we will give the *form* of the expansion of moments sequences

$$M_n(f) = \int_{-\infty}^{\infty} (f(x))^n \, \mathrm{d}x \,, \tag{2.5.1}$$

as $n \to \infty$ where f is a bounded smooth function such that for some $n_0 \in \mathbb{N}$

$$\int_{-\infty}^{\infty} |f(x)|^n \, \mathrm{d}x < \infty \,, \qquad n \geq n_0 \,. \tag{2.5.2}$$

The results of Section 2.2 show that there are distributions of compact support F such that for some $m \in \mathbb{N}$

$$\langle F(u), \phi(u) \rangle = \int_{-\infty}^{\infty} \phi(f(x)) \, \mathrm{d}x \,, \tag{2.5.3}$$

for any test function $\phi \in \mathcal{R}_m(\mathcal{E}(\mathbb{R}))$ and such that

$$\mu_n = \langle F(u), u^n \rangle = M_n(f) \,, \qquad n \geq m \,. \tag{2.5.4}$$

We already know how to find the asymptotic expansion of the moments μ_n in terms of the behavior of the distribution F at the endpoints of its compact support. Furthermore, we can actually relate the endpoint asymptotic behavior of F to the behavior of the smooth function f at its absolute maxima and minima.

The following lemma will be useful momentarily. A corresponding result for absolute minimum values also holds.

Lemma 2.5.1. *Let $f \in \mathcal{C}^\infty(\mathbb{R})$. Suppose $f(0) = b$ is the absolute maximum value of f and $f''(0) < 0$. Then there exists a unique increasing smooth function ψ such that $f(x) = b - \psi^2(x)$ for all x in an interval I containing* 0.

Suppose, to fix the ideas, that we have the situation in the previous lemma, but that the smooth function f achieves its global maximum only once, say at $x = 0$. By the lemma, in a neighborhood of $x = 0$, we have $f = b - \psi^2$. Furthermore, the inverse function f^{-1} exists in a possibly smaller neighborhood of $x = 0$. Actually, the inverse function f^{-1} has two branches, f_-^{-1} and f_+^{-1}, which are well-defined to the left and right of $x = 0$, respectively. These two branches can be expanded as

$$f_+^{-1}(u) \sim \sum_{k=0}^{\infty} C_k (b-u)^{k/2}, \quad u \to b^-, \tag{2.5.5}$$

$$f_-^{-1}(u) \sim \sum_{k=0}^{\infty} (-1)^k C_k (b-u)^{k/2}, \quad u \to b^-, \tag{2.5.6}$$

where the C_j are the Taylor coefficients. It is important to observe that the coefficients of these two expansions are, except for the change in signs, basically the same.

We are now ready to consider the integral

$$\int_{-\infty}^{\infty} \phi(f(x))\, \mathrm{d}x\,, \tag{2.5.7}$$

where ϕ is a test function that vanishes of enough order at the origin to assure convergence. If we make the change of variables $u = f(x)$, we obtain

$$\langle F(u), \phi(u)\rangle = \int_{-\infty}^{\infty} \left(\frac{1}{f'\left(f_+^{-1}(u)\right)} + \frac{1}{f'\left(f_-^{-1}(u)\right)} \right) \phi(u)\, \mathrm{d}u\,, \tag{2.5.8}$$

so that for u near b,

$$F(u) = \frac{1}{f'\left(f_+^{-1}(u)\right)} + \frac{1}{f'\left(f_-^{-1}(u)\right)}\,. \tag{2.5.9}$$

Inversion of (2.5.5) and of (2.5.6) yields that as $u \to b^-$ the function $f_+^{-1}(u)$ has an expansion of the form

$$f_+^{-1}(u) \sim B_1(b-u)^{1/2} + B_2(b-u) + B_3(b-u)^{3/2} + \cdots , \qquad (2.5.10)$$

and similarly

$$f_-^{-1}(u) \sim -B_1(b-u)^{1/2} + B_2(b-u) - B_3(b-u)^{3/2} + \cdots . \qquad (2.5.11)$$

Notice, again, that the coefficients of the two expansions are obtained from the other by appropriate changes of sign. Next, by composing with $f'(x)$, we obtain expansions with the form

$$f'\left(f_+^{-1}(u)\right) \sim D_{1/2}(b-u)^{1/2} + D_1(b-u) + \cdots . \qquad (2.5.12)$$

and

$$\frac{1}{f'\left(f_+^{-1}(u)\right)} \sim E_{-1/2}(b-u)^{-1/2} + E_0 + E_{1/2}(b-u)^{1/2} + \cdots . \qquad (2.5.13)$$

Similarly

$$\frac{1}{f'\left(f_-^{-1}(u)\right)} = E_{-1/2}(b-u)^{-1/2} - E_0 + E_{1/2}(b-u)^{1/2} - \cdots . \qquad (2.5.14)$$

Thus, if we add (2.5.13) and (2.5.14), we can see that all integral powers of $(b-u)$ cancel. Therefore

$$F(u) \sim 2E_{-1/2}(b-u)^{-1/2} + 2E_{1/2}(b-u)^{1/2} + 2E_{1/2}(b-u)^{3/2} + \cdots , \qquad (2.5.15)$$

as $u \to b^-$. Naturally in any particular example one needs to find these coefficients $E_{-k/2}$, perhaps numerically.

We then obtain that the moments $M_n(f)$, that equal the moments μ_n of F, will have a development of the type given by (2.4.15), where we replace C_j by $2E_{1/2+j}$. Clearly, the first term has the form

$$M_n(f) \sim C \frac{b^{n+1/2}}{\sqrt{n}}, \tag{2.5.16}$$

where

$$C = \sqrt{\frac{-2\pi}{f''(0)}}. \tag{2.5.17}$$

It should be clear how one can obtain the expansion of the moments $M_n(f)$ if $|f|$ has a finite number of absolute maxima, by adding the expansions corresponding to (2.5.15) at each of the maxima of f and those of the type (2.4.17) at the absolute minima of f.

Interestingly, if one considers integrals over a finite interval,

$$\int_\sigma^\tau (f(x))^n \, \mathrm{d}x, \tag{2.5.18}$$

and the absolute maximum value b, or the absolute minimum value $-b$, is attained at one of the endpoints then the form of the expansion of the $M_n(f)$ would be different. Indeed, the reason that the distribution $F(u)$ has an expansion in terms of powers of the form $(b-u)^{-1/2}, (b-u)^{1/2}, (b-u)^{3/2}, \ldots$, is the cancellation in the expansions of functions of f_+^{-1} and f_-^{-1}. In the endpoint case this cancellation would not occur since we would not need to consider the branch f_-^{-1}. Consequently, the expansion of $F(u)$ will be given in terms of powers of the form $(b-u)^{-1/2}, (b-u)^0, (b-u)^{1/2}, (b-u), (b-u)^{3/2}, \ldots$. This, in turn, means that the form of the expansion of M_n will be given by (2.4.11) where the exponents are $\alpha_j = -\frac{1}{2} + j\frac{1}{2}$, $j = 0, 1, 2, \ldots$.

Chapter 3. Tauberian Theorems for Smooth Functions that Never Vanish

3.1. Motivation and Background

The aim of this study is to establish that under certain conditions smooth functions of several variables that have *distributional* slow growth at infinity have slow growth in the *ordinary sense.*

Tauberian theorems for smooth functions are quite useful, particularly in mathematical physics [16, Section 4]. A smooth function that shows some type of behavior in the distributional sense may or may not show the same behavior in the ordinary sense, but sometimes it is possible to give extra conditions that guarantee that this is the case. An interesting result was considered by Ortner and Wagner [43], who established that $\mathcal{O}_M \cap \mathcal{O}'_M = \mathcal{S}$; generalized functions that belong to $\mathcal{O}'_M$ are of rapid decay at infinity in a distributional sense [17], but clearly a smooth function $f \in \mathcal{O}'_M$ does not have to be of rapid decay at infinity in the ordinary sense, that is, it does not have to belong to $\mathcal{S}$, however the Tauberian condition $f \in \mathcal{O}_M$ gives that $f \in \mathcal{S}$. More generally, in [12] one can find conditions on the space of test functions $\mathcal{A}$ that guarantee that $\mathcal{A} \cap \mathcal{A}' \subset \mathcal{S}$. Tauberian theorems for generalized functions have received increasing attention in recent years, see [59], the comprehensive memoir [49], the references therein, as well as the more recent article [50].

The space $\mathcal{O}_M$ can alternatively be defined as the space of multipliers of $\mathcal{S}$ – smooth functions G for which $G\varphi \in \mathcal{S}$ for all $\varphi \in \mathcal{S}$ – or as the space of smooth functions G defined in $\mathbb{R}^n$ such that for each multiindex $\mathbf{k} \in \mathbb{N}^n$ there exists a constant $\alpha_{\mathbf{k}}$ such that $\nabla^{\alpha_{\mathbf{k}}} G(\mathbf{x}) = O(|\mathbf{x}|^{\alpha_{\mathbf{k}}})$ as $\mathbf{x} \to \infty$ [25]. There are smooth functions of slow growth, without zeros, that do not belong to $\mathcal{O}_M$, but as we show, if $F \in \mathcal{O}_M$ is strictly positive, then $G = 1/F$ belongs to $\mathcal{O}_M$ if and only if for some α we have $G(\mathbf{x}) = O(|\mathbf{x}|^{\alpha})$ as $\mathbf{x} \to \infty$.

The condition $G(\mathbf{x}) = O(|\mathbf{x}|^{\alpha})$ as $\mathbf{x} \to \infty$ has a distributional analog [17], namely $G(\mathbf{x}) = O(|\mathbf{x}|^{\alpha})$ (C) as $\mathbf{x} \to \infty$, boundedness in the Cesàro sense, which is equivalent, when $\alpha > -n$, to the fact that for each test function $\varphi \in \mathcal{D}$ we have that $\langle G(\lambda\mathbf{x}), \varphi(\mathbf{x})\rangle = O(\lambda^{\alpha})$ as the parameter $\lambda \to \infty$. It was proved in [11] that a distribution $G \in \mathcal{D}'$ belongs to $\mathcal{S}'$ if and only if there exists α such that $G(\mathbf{x}) = O(|\mathbf{x}|^{\alpha})$ (C). Therefore, the distributional version of the order relation $G(\mathbf{x}) = O(|\mathbf{x}|^{\alpha})$ as $\mathbf{x} \to \infty$ for some α is the fact that $G \in \mathcal{S}'$. We are able to show that in the one variable case, if $G(x) = O(1/x)$ (C) as $x \to +\infty$ and $F' = O(1)$ as $x \to +\infty$ then G must be of slow growth at infinity, but provide examples to show the limits of this result. Our main result is that if $F \in \mathcal{O}_M(\mathbb{R})$, $F > 0$, then $G = 1/F$ belongs to $\mathcal{O}_M(\mathbb{R})$ if and only G belongs to $\mathcal{S}'(\mathbb{R})$; in other words, if for some $\alpha > 0$ the average relation $G(x) = O(|x|^{\alpha})$ (C) as $|x| \to \infty$ holds, then $G \in \mathcal{O}_M(\mathbb{R})$.

Moreover, we show that for certain subsets $\mathfrak{Z} \subset \mathcal{S}'$ the smooth function $G = 1/F$, where $F \in \mathcal{O}_M$ is strictly positive, belongs to $\mathcal{O}_M$ if and only if for each $f \in \mathfrak{Z}$ there exists $\alpha = \alpha_f$ such that $G(\mathbf{x}) f(\mathbf{x}) = O(|\mathbf{x}|^{\alpha})$ (C). In particular, we prove that this is the case when $\mathfrak{Z}$ is a set of tempered trains of delta functions of special form.

Our results can be considered as a contribution to understanding the division problem for tempered distributions. Division problems for tempered distributions have received a lot of attention, starting with the famous work of Łojasiewics [37] and Hörmander [24]. A particularly important work is that of Bonet, Frerick, and Jordá [3], who were able to characterize the zeros and the rate of decrease of those multipliers that admit continuous right inverses in the one variable case. Our work considers the situation in several variables, but we restrict our analysis to the case of multipliers without zeros, so that the results of [3] and ours complement one another, with very little overlap.

3.2. Some Preliminaries

In general, if $f \in \mathcal{D}'$ is of slow growth distributionally at infinity, $f(x) = O(x^{\alpha})$ (C), as $x \to \infty$, then we cannot say much about the order of the primitive N for which (1.1.7) holds. However, when f satisfies some extra conditions it is possible to obtain precise results on this order. In fact, if f is a positive Radon measure for x large, then one can take $N = 1$, as the next very simple but useful Tauberian result gives.

Lemma 3.2.1. *Let μ be a positive Radon measure with support in the interval $[a, \infty)$. Let $\alpha > -1$. If $\mu(x) = O(x^{\alpha})$ (C), as $x \to \infty$, then*

$$\int_a^x \mathrm{d}\mu(t) = O\left(x^{\alpha+1}\right), \ \text{as } x \to \infty \ , \tag{3.2.1}$$

Proof. Let $\phi \in \mathcal{D}$ be a positive test function that satisfies $\phi(x) \geq 1$ for $0 \leq x \leq 1$. Then we may assume $a = 0$, since $\int_a^x \mathrm{d}\mu(t) = \int_a^0 \mathrm{d}\mu(t) + \int_0^x \mathrm{d}\mu(t) = \int_0^x \mathrm{d}\mu(t) + O(1)$. Now, if $\operatorname{supp}\mu \subset [0, \infty)$ then as $x \to \infty$,

$$\int_0^x \mathrm{d}\mu(t) \leq \int_0^x \phi(t/x)\ \mathrm{d}\mu(t) \leq \int_0^\infty \phi(t/x)\ \mathrm{d}\mu(t)$$
$$= \langle \mu(t), \phi(t/x) \rangle = x \langle \mu(xt), \phi(t) \rangle = xO(x^{\alpha}) \ ,$$

as required. □

An alternative proof of this result could also be given using the techniques employed to obtain asymptotics for derivatives [31, pg. 34-37]. Notice that the lemma yields that if f is *increasing* and $f(x) = O(x^{\alpha})$ (C), as $x \to \infty$, then actually the ordinary relation $f(x) = O(x^{\alpha})$ holds.

Also, when $\mu = \sum_{k=1}^{\infty} c_k \delta(x - x_k)$ is a *positive* train of delta functions, with $x_k \nearrow \infty$, then if $\mu(x) = O(x^{\alpha})$ (C), as $x \to \infty$, we obtain $\sum_{k=1}^{m} c_k \le x_m^{\alpha+1}$ and, consequently with $\beta = \alpha + 1$, $c_m = O\left(x_m^{\beta}\right)$ as $m \to \infty$. Therefore, for a positive train of deltas, $\mu \in \mathcal{S}'$ if and only if there exists β such that $c_m = O\left(x_m^{\beta}\right)$ as $m \to \infty$. While the condition $c_m = O\left(x_m^{\beta}\right)$ as $m \to \infty$ implies $\mu \in \mathcal{S}'$, even if μ is not positive, it is not true that $\mu \in \mathcal{S}'$ implies $c_m = O\left(x_m^{\beta}\right)$ as the following example of a tempered train of deltas $\mu \in \mathcal{S}'$ shows. Namely, assume that $\xi_k \nearrow \infty$ and also that $\xi_k + \xi_k^{-k} < \xi_{k+1}$, and take

$$\mu(x) = \sum_{k=1}^{\infty} \xi_k^k \left\{\delta(x - \xi_k) - \delta\left(x - \xi_k - \xi_k^{-k}\right)\right\} \ . \tag{3.2.2}$$

Interestingly, however, for *some* sequences $\{x_k\}_{k=1}^{\infty}$ a train of deltas $\mu = \sum_{k=1}^{\infty} c_k \delta(x - x_k)$ is tempered if and only if $c_m = O\left(x_m^{\beta}\right)$ as $m \to \infty$; in particular, that is the case if $x_k = k$ for all k.

Lemma 3.2.2. *A train of deltas of the form* $\mu = \sum_{k=1}^{\infty} c_k \delta(x - k)$ *belongs to* $\mathcal{S}'$ *if and only if there exists* β *such that* $c_k = O\left(k^{\beta}\right)$ *as* $k \to \infty$.

Proof. Indeed, the series $\sum_{k=1}^{\infty} c_k \delta(x - k)$ converges in $\mathcal{S}'$ if and only if its Fourier transform, $\sum_{k=1}^{\infty} c_k e^{iku}$, converges in $\mathcal{S}'$, and it is well known that a Fourier series $g(u) = \sum_{k=1}^{\infty} c_k e^{iku}$ converges in $\mathcal{S}'$ if and only if it converges in $\mathcal{D}'$, and in turn that holds if and only if there exists β such that $c_m = O\left(m^{\beta}\right)$ as $m \to \infty$. □

It is also convenient to consider the space s of rapidly decreasing sequences $\{\eta_k\}_{k=1}^{\infty}$, that is sequences that satisfy $\lim_{k\to\infty} \eta_k k^{\alpha} = 0$ for each $\alpha > 0$, with its natural topology. If $\phi \in \mathcal{S}$ then for each $a \in \mathbb{R}$ the sequence $\{\phi(ka)\}_{k=1}^{\infty}$ belongs to s, and (a particular case of Proposition 3.3.4) for each fixed $a \neq 0$, for each $\{\eta_k\} \in s$ we can find $\phi \in \mathcal{S}$ such that $\phi(ka) = \eta_k$ for $k \geq 1$. The dual space s' is the space of sequences $\{c_k\}_{k=1}^{\infty}$ such that there exists $\beta > 0$ such that $c_k = O\left(k^{\beta}\right)$ as $k \to \infty$, in other words, the train of deltas $\mu = \sum_{k=1}^{\infty} c_k \delta(x - ak)$ belongs to $\mathcal{S}'$ if and only if $\{c_k\} \in s'$, and actually

$$\left\langle \sum_{k=1}^{\infty} c_k \delta(x - ak), \phi(x) \right\rangle_{\mathcal{S}' \times \mathcal{S}} = \langle \{c_k\}_{k=1}^{\infty}, \{\phi(ka)\}_{k=1}^{\infty} \rangle_{s' \times s} \ . \tag{3.2.3}$$

We shall also require a multidimensional version of the Lemma 3.2.2.

Lemma 3.2.3. *Let $\{\mathbf{x}_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$ with $|\mathbf{x}_k| = ak$ for some constant $a > 0$ and all k. Then a train of deltas of the form $\mu = \sum_{k=1}^{\infty} c_k \delta(\mathbf{x} - \mathbf{x}_k)$ belongs to $\mathcal{S}'(\mathbb{R}^n)$ if and only if there exists β such that $c_k = O\left(k^{\beta}\right)$ as $k \to \infty$.*

Proof. Indeed, writing $\mathbf{x}_k = ak\theta_k$, we observe that $\sum_{k=1}^{\infty} c_k \delta(\mathbf{x} - \mathbf{x}_k)$ belongs to $\mathcal{S}'(\mathbb{R}^n)$ if and only if for all $\varphi \in \mathcal{D}(\mathbb{S})$ the train of deltas

$$\langle f(r\omega), \varphi(\omega) \rangle = \sum_{k=1}^{\infty} c_k \varphi(\theta_k) \delta(r - ak) \ , \tag{3.2.4}$$

belongs to $\mathcal{S}'(\mathbb{R})$, and this holds if only if for some $\beta_{\varphi} > 0$ we have $c_k \varphi(\theta_k) = O\left(k^{\beta_{\varphi}}\right)$ as $k \to \infty$, and clearly this is equivalent to $c_k = O\left(k^{\beta}\right)$ as $k \to \infty$. $\square$

Let now $f \in \mathcal{D}'(\mathbb{R}^n)$. We say that f is bounded in the Cesàro sense by $|\mathbf{x}|^{\beta}$, written as $f(\mathbf{x}) = O\left(|\mathbf{x}|^{\beta}\right)$ (C) as $|\mathbf{x}| \to \infty$ if for each test function $\varphi \in \mathcal{D}(\mathbb{S})$ in the unit sphere we have a corresponding Cesàro estimate for the distribution of one variable $\langle f(r\omega), \varphi(\omega) \rangle$ given by (0.0.1), namely,

$$\langle f(r\omega), \varphi(\omega) \rangle = O\left(r^{\beta+n-1}\right) \quad \text{(C)} \ , \tag{3.2.5}$$

as $r \to \infty$.

It follows from the Proposition 0.0.2 that tempered distributions in several variables can be characterized in a similar way.

Proposition 3.2.4. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Then the following statements are equivalent:*

1. *f is a tempered distribution, i.e., $f \in \mathcal{S}'(\mathbb{R}^n)$.*
2. *There exists $\beta > 0$ such that*
$$f(\mathbf{x}) = O\left(|\mathbf{x}|^{\beta}\right) \ (C) \ , \ \mathbf{x} \to \infty \,. \tag{3.2.6}$$
3. *There exists $\beta > 0$ such that*
$$f(\lambda \mathbf{x}) = O\left(\lambda^{\beta}\right) \ , \ \lambda \to \infty \,, \tag{3.2.7}$$
distributionally, (weakly or strongly).

In fact, there is a stronger characterization of tempered distributions.

Proposition 3.2.5. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. If for each $\varphi \in \mathcal{D}(\mathbb{S})$ there exists $\beta_\varphi > 0$ such that*

$$\langle f(r\omega), \varphi(\omega)\rangle = O\left(r^{\beta_\varphi + n - 1}\right) \quad (C) \ , \tag{3.2.8}$$

then $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof. In fact, if (3.2.8) holds, then $\langle f(r\omega), \varphi(\omega)\rangle$ is a tempered distribution of the variable r, and since this is true for all $\varphi \in \mathcal{D}(\mathbb{S})$ it follows that $f \in \mathcal{S}'(\mathbb{R}^n)$. Notice that it follows that there exists $\beta > 0$, independent of φ, such that for all φ (3.2.8) holds if we replace β_φ by β. □

It is interesting to observe that a corresponding statement for behavior along rays is not true. In fact, there exist continuous functions $f \in C(\mathbb{R}^n)$ such that for all $\omega \in \mathbb{S}$ there exists $\alpha_\omega > 0$ such that $f(r\omega) = O(r^{\alpha_\omega})$ (C) as $r \to \infty$, but such that $f \notin \mathcal{S}'(\mathbb{R}^n)$. Actually, it is possible that $\sup_{\omega \in \mathbb{S}} \alpha_\omega = \infty$, as the following example shows.

Example 3.2.6. Let $\omega_0 \in \mathbb{S}$. A smooth function that is given in polar coordinates as

$$f(r\omega) = |\omega - \omega_0|^2 r^{e(r,\omega)}, \quad r \geq 1, \tag{3.2.9}$$

where

$$e(r,\omega) = \frac{r^2}{1 + r^2 |\omega - \omega_0|^2}, \tag{3.2.10}$$

does not belong to $\mathcal{S}'(\mathbb{R}^n)$ although for each fixed ω the distribution of one variable $f(r\omega)$ is tempered.

3.3. Growth Conditions on Spaces of Multipliers

The order symbols can be used to define many spaces of functions. In this study we will be particularly interested in the space $\mathcal{O}_M$, which consists of those smooth functions defined in all $\mathbb{R}^n$ such that for each multi-index $\mathbf{k} \in \mathbb{N}^n$, there exists $\alpha_{\mathbf{k}} \in \mathbb{R}$ such that

$$\nabla^{\mathbf{k}} F(\mathbf{x}) = O\left(|\mathbf{x}|^{\alpha_{\mathbf{k}}}\right), \ \ \mathbf{x} \to \infty. \tag{3.3.1}$$

In other words, $F \in \mathcal{O}_M$ if it is smooth and F and all of its derivatives are of slow growth at infinity.

If a smooth function F belongs to $\mathcal{O}_M$ then it is of slow growth. However, the converse is not true, in general, as can be seen from the simple example $G_0(x) = \sin(\mathrm{e}^x)$; in fact the function $G_1(x) = 2 + \sin(\mathrm{e}^x)$ is a *strictly positive* smooth function of slow growth that does not belong to $\mathcal{O}_M$. Our first result shows that a strictly positive function of slow growth that has the form $G = 1/F$ for $F \in \mathcal{O}_M$ must belong to $\mathcal{O}_M$.

Proposition 3.3.1. *Suppose that $F > 0$, $F \in \mathcal{O}_M$, and $G = 1/F$. If G is of slow growth, that is, if there exists $\alpha \in \mathbb{R}$ such that*

$$G(\mathbf{x}) = O\left(|\mathbf{x}|^{\alpha}\right), \ \ \mathbf{x} \to \infty, \tag{3.3.2}$$

then $G \in \mathcal{O}_M$.

Proof. Since $F \in \mathcal{O}_M$, we have that for each $\mathbf{k} \in \mathbb{N}^n$ there exists $\gamma_{\mathbf{k}} \in \mathbb{R}$ such that

$$\nabla^{\mathbf{k}} F(\mathbf{x}) = O\left(|\mathbf{x}|^{\gamma_{\mathbf{k}}}\right), \quad \mathbf{x} \to \infty\,. \tag{3.3.3}$$

That (3.3.2) implies $G \in \mathcal{O}_M$ is easy to see via the calculation

$$\nabla_j G = \frac{-\nabla_j F}{F^2} = O(|\mathbf{x}|^{\gamma_{\mathbf{e}_j}+2\alpha})\,, \quad \mathbf{x} \to \infty\,,$$

where $\{\mathbf{e}_j\}$ is the canonical basis of $\mathbb{R}^n$. Similarly, we have

$$\nabla_k \nabla_j G = \frac{2\nabla_k \nabla_j F - F \nabla_k F \nabla_j F}{F^3} = O\left(|\mathbf{x}|^{\beta_{kj}}\right), \quad \mathbf{x} \to \infty\,,$$

where $\beta_{kj} = \max\left\{\gamma_{\mathbf{e}_k \mathbf{e}_j}, \gamma_0 \gamma_{\mathbf{e}_k} \gamma_{\mathbf{e}_j}\right\} + 3\alpha$. In general, it is clear that all derivatives of G are of slow growth. $\square$

Another interpretation of $\mathcal{O}_M$ is that $\mathcal{O}_M$ is the space of *multipliers* of $\mathcal{S}$. In other words, $F \in \mathcal{O}_M$ if and only if $F\phi \in \mathcal{S}$ for every $\phi \in \mathcal{S}$. In fact, the subindex M in the notation $\mathcal{O}_M$ refers to this fact. More generally, if X is a space of functions, then one can ask what the corresponding space of multipliers is. For example, suppose

$$\mathcal{X} = \left\{ f : f \in C\left(\mathbb{R}^n\right) \text{ and } \lim_{\mathbf{x}\to\infty} f(\mathbf{x})\,|\mathbf{x}|^{\beta} = 0 \text{ for all } \beta \right\}. \tag{3.3.4}$$

Then $H \in C\left(\mathbb{R}^n\right)$ is a multiplier of $\mathcal{X}$ if and only if there exists an $\alpha \in \mathbb{R}$ such that $H(\mathbf{x}) = O\left(|\mathbf{x}|^{\alpha}\right)$, as $\mathbf{x} \to \infty$. Using this terminology, Proposition 3.3.1 can be re-stated as follows.

Proposition 3.3.2. *Let* $F \in \mathcal{O}_M$, $F > 0$, *and* $G = 1/F$. *If* G *is a multiplier of* $\mathcal{X}$, *then* G *is a multiplier of* $\mathcal{S}$.

In fact, the following stronger result holds.

Proposition 3.3.3. *Let $F \in \mathcal{O}_M$, $F > 0$, and $G = 1/F$. Suppose that for all $\phi \in \mathcal{S}$, we have $G\phi \in \mathcal{X}$. Then G is a multiplier of $\mathcal{S}$, that is, $G \in \mathcal{O}_M$.*

Proof. Suppose that G is not a multiplier of $\mathcal{S}$. Then Proposition 3.3.1 yields that for each $\alpha \in \mathbb{R}$ the function $G(\mathbf{x})|\mathbf{x}|^{-\alpha}$ is not bounded in the region $|\mathbf{x}| \geq 1$, and thus for each $n \in \mathbb{N}$ and each $A_n > 0$ we can find $\mathbf{x}_n$ with $|\mathbf{x}_n| > A_n$ such that $G(\mathbf{x}_n) = |\mathbf{x}_n|^n$. In particular, if we take $A_1 = 1$ and $A_n = |\mathbf{x}_n| + 1$, for $n > 1$, we can choose the sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ in such a way that $|\mathbf{x}_{n+1}| > |\mathbf{x}_n| + 1$ for all n.

Our construction yields that if $z_n = |\mathbf{x}_n|^{-n/2}$ then $\{z_n\}_{n=1}^{\infty}$ is of rapid decay at infinity, that is, $\lim z_n n^{\alpha} = 0$ for all $\alpha > 0$. Therefore we can use the interpolation result given in Proposition 3.3.4 to find a smooth function $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\varphi(|\mathbf{x}_n|) = |\mathbf{x}_n|^{-n/2}$ for $n \geq 1$ and such that $\varphi(x)$ vanishes for $|x| < 1/2$. Then the function $\phi(\mathbf{x}) = \varphi(|\mathbf{x}|)$ belongs to $\mathcal{S}(\mathbb{R}^n)$ and $\phi(\mathbf{x}_n) = |\mathbf{x}_n|^{-n/2}$. Nevertheless this gives a contradiction because $G\phi$ cannot belong to $\mathcal{X}$ since $\lim_{n\to\infty} G(\mathbf{x}_n)\phi(\mathbf{x}_n) = \infty$. □

Actually the proof of this proposition gives a stronger result, namely, *if $G = 1/F$ with $F \in \mathcal{O}_M$, $F > 0$, has the property that $G\phi$ is of slow growth for each $\phi \in \mathcal{S}$ then G must belong to $\mathcal{O}_M$.*

3.3.1. Interpolation by Smooth Functions

We needed an interpolation result in the proof of the Proposition 3.3.3, which we now proceed to explain. Suppose we have a strictly increasing sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \nearrow \infty$. If $\{z_n\}$ is a given sequence of real numbers such that $\lim_{n\to\infty} z_n x_n^{\alpha} = 0$ for each $\alpha > 0$, then there exist a function $\phi \in \mathcal{X}$ such that $\phi(x_n) = z_n$ for all n. Actually the function ϕ can be chosen to be smooth. In general, one cannot ask that the interpolation function $\phi \in \mathcal{S}$; however, the question of whether ϕ could be chosen to belong to $\mathcal{S}$ can be answered in the affirmative if extra conditions are assumed.

For example, if we choose $x_{2n-1} = n - e^{-2n}$, $x_n = n$, $z_{2n-1} = e^{-n}$, and $z_{2n} = 2e^{-n}$, the interpolation problem has smooth solutions $\phi \in \mathcal{X}$, but no such solution belongs to $\mathcal{S}$ since the derivative of ϕ is not even of slow growth at infinity.

However, we have interpolation solutions in $\mathcal{S}$ if the terms of the sequence $\{x_n\}_{n=1}^{\infty}$ are separated.

Proposition 3.3.4. *If the sequence* $\{x_n\}_{n=1}^{\infty}$ *satisfies*

$$x_n > x_{n-1} + 1\,, \tag{3.3.5}$$

then the interpolation problem $\phi(x_n) = z_n$ *has a solution in* $\mathcal{S}$ *for each sequence* $\{z_n\}_{n=1}^{\infty}$ *with* $\lim_{n\to\infty} z_n x_n^{\alpha} = 0$ *for each* $\alpha > 0$.

Proof. Let $\varphi \in \mathcal{D}$ with $\operatorname{supp}\varphi \subset [-1/4, 1/4]$ and with $\varphi(0) = 1$. Then the series

$$\phi(x) = \sum_{n=1}^{\infty} z_n \varphi(x - x_n)\ , \tag{3.3.6}$$

converges to a smooth function, since for each x at most one term of the series does not vanish. That ϕ belongs to $\mathcal{S}$ can be seen by observing that $\phi(x) = 0$ unless $|x - x_n| < 1/4$ for some n, and when this holds then for each $\alpha > 0$,

$$\left|\phi^{(j)}(x)\right| \leq |z_n| \max\left\{\left|\phi^{(j)}(t)\right| : |t| \leq 1/4\right\} = O\left(x_n^{-\alpha}\right) = O\left(x^{-\alpha}\right)\ .$$

□

Notice that if $\{z_n\}_{n=1}^{\infty}$ is of rapid decay at infinity, that is, $\lim_{n\to\infty} z_n n^{\alpha} = 0$ for each $\alpha > 0$, and condition (3.3.5) holds then $\lim_{n\to\infty} z_n x_n^{\alpha} = 0$ for each $\alpha > 0$. Therefore, the conclusion of the Proposition remains valid if we assume that $\{z_n\}_{n=1}^{\infty}$ is of rapid decay at infinity.

It is interesting to observe that when $x_n = n$ and $\{z_n\}_{n=1}^{\infty}$ is of rapid decay at infinity, the well known Whittaker-Shannon interpolating function [60]

$$\psi(x) = \sum_{n=1}^{\infty} z_n \frac{\sin(n\pi x)}{\pi(x-n)}, \tag{3.3.7}$$

will belong to $\mathcal{O}_M$ but it never belongs to $\mathcal{S}$ if not all the z_n vanish. On the other hand, the series could be divergent in $\mathcal{D}'$ when $\{z_n\}_{n=1}^{\infty}$ is of slow growth at infinity [10].

3.4. Application to the Division Problem in $\mathcal{S}'$

In this section we will consider the following division problem for tempered distributions: Let F be a never vanishing multiplier of $\mathcal{S}'(\mathbb{R}^n)$, $F \in \mathcal{O}_M(\mathbb{R}^n)$, and let $h \in \mathcal{S}'(\mathbb{R}^n)$; can we find $g \in \mathcal{S}'(\mathbb{R}^n)$ such that $Fg = h$?. We will present several conditions that guarantee that for each $h \in \mathcal{S}'(\mathbb{R}^n)$, there exists such a $g \in \mathcal{S}'(\mathbb{R}^n)$, results that say that if we can solve the division problem in $\mathcal{S}'(\mathbb{R}^n)$ for all g that belong to a certain subset $\mathfrak{Z}$ of $\mathcal{S}'(\mathbb{R}^n)$ then we solve if for all $g \in \mathcal{S}'(\mathbb{R}^n)$. Starting with the famous work of Łojasiewics [37] and Hörmander [24], division problems for tempered distributions remain an important area of analysis [3].

Our first result is the case when $\mathfrak{Z} = \mathcal{X}'$, and is obtained by employing transpose operators in the Proposition 3.3.3, since the transpose of multiplication by a real smooth function is also multiplication by the same function. Notice that the dual space $\mathcal{X}'$ of the space of continuous functions $\mathcal{X}$ given by (3.3.4) is a subspace of $\mathcal{S}'(\mathbb{R}^n)$ since $\mathcal{S}$ is a dense subspace of $\mathcal{X}$.

Proposition 3.4.1. *Suppose* $F \in \mathcal{O}_M(\mathbb{R}^n)$, $F > 0$, *and suppose that the division problem*

$$Fg = h\,, \tag{3.4.1}$$

has a solution $g \in \mathcal{S}'$ *for all* $h \in \mathcal{X}'$. *Then the division problem has a solution* $g \in \mathcal{S}'$ *for all* $h \in \mathcal{S}'$.

Proof. Let $G = 1/F$. Then our hypothesis says that $Gh \in \mathcal{S}'$ for all $h \in \mathcal{X}'$. Therefore, G is a multiplier of $\mathcal{X}'$ into $\mathcal{S}'$ and, by transposition, it follows that for all $\phi \in \mathcal{S}$, we have that $G\phi \in \mathcal{X}$. Proposition 3.3.3 then yields that $G \in \mathcal{O}_M$, and consequently $Gh \in \mathcal{S}'$ for all $h \in \mathcal{S}'$. □

It should be observed that the Proposition 3.3.2 gives the weaker result that if (3.4.1) has a solution $g \in \mathcal{X}'$ for all $h \in \mathcal{X}'$ then it will have a solution $g \in \mathcal{S}'$ for all $h \in \mathcal{S}'$. In fact, we have a stronger result: we can take $\mathfrak{Z} = \mathcal{X}_1'$, where $\mathcal{X}_1$ is the space of continuous functions of slow growth at infinity.

Proposition 3.4.2. *Suppose* $F \in \mathcal{O}_M(\mathbb{R}^n)$, $F > 0$, *and suppose that the division problem (3.4.1) has a solution* $g \in \mathcal{S}'$ *for all* $h \in \mathcal{X}_1'$. *Then the division problem has a solution* $g \in \mathcal{S}'$ *for all* $h \in \mathcal{S}'$.

Proof. Indeed, we had already observed that $G \in \mathcal{O}_M$ if for all $\phi \in \mathcal{S}$, we have that $G\phi$ is a continuous function of slow growth at infinity. □

Our next conditions for the solvability of the division problem for tempered distributions are obtained by taking $\mathfrak{Z}$ as the set of trains of delta functions of a special kind. We start with some generalizations of several results given by us recently [18], especially the next one, whose proof can be found in that reference.

Proposition 3.4.3. *Let $f : (0, \infty) \longrightarrow \mathbb{R}$ be continuous. Suppose that for each $a > 0$ the sequence $\{f(an)\}_{n=1}^{\infty}$ converges, to $F(a)$. Then $F(a)$ does not depend on a, that is, $F(a) = L$ for all $a > 0$, for some L, and actually*

$$\lim_{x \to \infty} f(x) = L\,. \tag{3.4.2}$$

In particular, if

$$f(ak) = o(1)\,, \quad k \to \infty\,, \tag{3.4.3}$$

for all $a > 0$, then (3.4.2) holds, that is,

$$f(x) = o(1)\,, \quad x \to \infty\,. \tag{3.4.4}$$

The result holds when f is continuous, but not otherwise, in general, but corresponding results when f is just measurable can be found in [18].

We will now show that we can replace the little oh by a big oh in (3.4.3) and (3.4.4). Actually in such a case it is enough to assume that f is lower semicontinuous.

Proposition 3.4.4. *Let $f : (0, \infty) \longrightarrow \mathbb{R}$ be lower semicontinuous and positive. If $f(ak) = O(1)$ as $k \to \infty$ for all $a > 0$, then $f(x) = O(1)$ as $x \to \infty$.*

To prove Proposition 3.4.4, we use the next lemma.

Lemma 3.4.5. *Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ that increase to ∞ and satisfy $x_n < y_n < x_{n+1}$ for all n. Consider the open (disjoint) intervals*

$$I_n = (x_n, y_n)\,. \tag{3.4.5}$$

Then there exists $a > 0$ such that

$$ak \in \bigcup_{n=1}^{\infty} I_n\,, \tag{3.4.6}$$

for infinitely many $k \in \mathbb{N}$.

Proof. Let f be the function defined to be 0 in $(0,\infty) \setminus \bigcup_{n=1}^{\infty} I_n$ and equal to the triangle function with vertices $(x_n, 0), ((x_n + y_n)/2, 1)$, and $(y_n, 0)$ in I_n. Then f is continuous. If for each $a > 0$ only a finite number of terms of $\{ak\}_{k=1}^{\infty}$ belong to $\bigcup_{n=1}^{\infty} I_n$, then it follows that $f(ak) = 0$ for $k \geq k_0$ ($k_0 = k_0(a)$) and thus $\lim_{k\to\infty} f(ak) = 0$. Hence it would follow from the Proposition 3.4.3 that $\lim_{x\to\infty} f(x) = 0$, which is clearly not true. □

Returning now to the proof of Proposition 3.4.4:

Proof. Suppose that $f(x) \neq O(1)$ as $x \to \infty$. Since f is lower semicontinuous, for each n we can find intervals $I_n = (x_n, y_n)$ such that $f(x) > n$ for $x \in I_n$, and we can choose them in such a way that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ increase to ∞ and satisfy $x_n < y_n < x_{n+1}$ for all n. According to the Lemma 3.4.5, there is an $a > 0$ such that $ak \in \bigcup_{n=1}^{\infty} I_n$, for infinitely many $k \in \mathbb{N}$. For this choice of a, the sequence $\{f(ak)\}$ is not bounded, which is a contradiction. □

If we consider $f(x)/x^{\alpha}$, we have the ensuing corollary.

Corollary 3.4.6. *Let $f : (0,\infty) \longrightarrow \mathbb{R}$ be lower semicontinuous and positive. Let $\beta > 0$. If $f(ak) = O(k^{\beta})$ as $k \to \infty$ for all $a > 0$, then $f(x) = O(x^{\beta})$ as $x \to \infty$.*

We can actually obtain stronger results if we assume β depends on a.

Proposition 3.4.7. *Let $f : (0,\infty) \longrightarrow \mathbb{R}$, with $|f|$ lower semicontinuous. Suppose for each $a > 0$ there exists $\beta_a > 0$ such that $f(ak) = O(k^{\beta_a})$ as $k \to \infty$. Then there exists $\beta > 0$ such that $f(x) = O(x^{\beta})$ as $x \to \infty$.*

Proof. We may assume that $|f(x)| > 1$ for all x. From the hypothesis we have that for all

$a > 0$

$$|f(ka)| \leq M_a k^{\beta_a}, \tag{3.4.7}$$

for some constants M_a and so

$$\log|f(ka)| \leq \log M_a + \beta_a \log k\,. \tag{3.4.8}$$

This means for every $a > 0$, the sequence $\log|f(ka)|\,/\log k$ is bounded. Therefore, there is a $\beta > 0$ and $A > 0$ such that

$$\frac{\log|f(x)|}{\log x} \leq \beta\,, \qquad x > A\,, \tag{3.4.9}$$

which means

$$|f(x)| \leq x^{\beta}\,, \qquad x > A\,. \tag{3.4.10}$$

□

These results allow us to determine if a function is of slow growth by only investigating the behavior at $x = ka$. However, when combined with Proposition 3.3.1, they also imply the ensuing characterization.

Theorem 3.4.8. *Let $F \in \mathcal{O}_M(\mathbb{R})$, $F > 0$, and $G = 1/F$. Then $G \in \mathcal{O}_M(\mathbb{R})$ if and only if*

$$\sum_{k=1}^{\infty} G(ak)\delta(x - ak) \in \mathcal{S}'(\mathbb{R})\,, \tag{3.4.11}$$

for all $a \neq 0$.

In terms of the division problem, Theorem 3.4.8 can be re-stated as follows.

Theorem 3.4.9. *Let $F \in \mathcal{O}_M(\mathbb{R})$, $F > 0$. The division problem $Fg = h$ can be solved in $\mathcal{S}'(\mathbb{R})$ for all $h \in \mathcal{S}'(\mathbb{R})$ if and only if it can be solved for $h = \sum_{k=1}^{\infty}\delta(x - ka)$ for every $a \neq 0$.*

We can also give corresponding results in several variables if we consider trains of delta functions of the type $\sum_{k=1}^{\infty}\delta\left(\mathbf{x}-\mathbf{x}_k\right)$ where $\{\mathbf{x}_k\}_{k=1}^{\infty}$ is a sequence of points of $\mathbb{R}^n$ with $|\mathbf{x}_k|=ka$.

Theorem 3.4.10. *Let $F\in\mathcal{O}_M\left(\mathbb{R}^n\right)$, $F>0$, and $G=1/F$. Then $G\in\mathcal{O}_M\left(\mathbb{R}^n\right)$ if and only if*

$$\sum_{k=1}^{\infty}G(\mathbf{x}_k)\delta(\mathbf{x}-\mathbf{x}_k)\in\mathcal{S}'\left(\mathbb{R}^n\right), \tag{3.4.12}$$

for all $a>0$ and all sequences $\{\mathbf{x}_k\}_{k=1}^{\infty}$ of $\mathbb{R}^n$ with $|\mathbf{x}_k|=ka$.

The division problem $Fg=h$ can be solved in $\mathcal{S}'\left(\mathbb{R}^n\right)$ for all $h\in\mathcal{S}'\left(\mathbb{R}^n\right)$ if and only if it can be solved for $h=\sum_{k=1}^{\infty}\delta\left(\mathbf{x}-\mathbf{x}_k\right)$ for every $a>0$ and all sequences $\{\mathbf{x}_k\}_{k=1}^{\infty}$ of $\mathbb{R}^n$ with $|\mathbf{x}_k|=ka$.

Proof. Employing the Proposition 3.3.1, we just need to show that if (3.4.12) holds for all such sequences, then G is of slow growth at infinity. Furthermore, Lemma 3.2.3 tell us that (3.4.12) holds if and only if for each $a>0$ and each sequence $\{\mathbf{x}_k\}_{k=1}^{\infty}$ with $|\mathbf{x}_k|=ka$ we have $G\left(\mathbf{x}_k\right)=O\left(k^{\beta}\right)$ for some $\beta=\beta\left(\{\mathbf{x}_k\}\right)$. Let now $\mathbf{z}:(0,\infty)\longrightarrow\mathbb{R}^n$ be a function that is continuous except for finite jumps, at points that increase to infinity, and that satisfies $|\mathbf{z}\left(r\right)|=r$. At the points of discontinuity we choose the value of $\mathbf{z}$ in such a way that $G\left(\mathbf{z}\left(r\right)\right)$ is lower semicontinuous. Then the Proposition 3.4.7 yields that $G\left(\mathbf{z}\left(r\right)\right)$ is of slow growth at infinity. Let $\varepsilon>0$; we can choose one of these function $\mathbf{z}$ in such a way that for all $r>0$ we have that $G\left(\mathbf{z}\left(r\right)\right)>\max_{|\mathbf{x}|=r}G\left(\mathbf{x}\right)-\varepsilon$. Since $G\left(\mathbf{z}\left(r\right)\right)$ is of slow growth, so is $\max_{|\mathbf{x}|=r}G\left(\mathbf{x}\right)$, and, consequently, $G\left(\mathbf{x}\right)$ itself. □

3.5. Tauberian Theorems

In this section we give several Tauberian theorems in one variable. Here we present the main result of this article, namely, if $G = 1/F$, where $F \in \mathcal{O}_M(\mathbb{R})$, $F > 0$, and $G \in \mathcal{S}'(\mathbb{R})$ then necessarily $G \in \mathcal{O}_M(\mathbb{R})$.

Throughout this section in our proofs we shall consider the situation at $+\infty$ only. This is possible because if $F \in \mathcal{O}_M(\mathbb{R})$, $F > 0$ then we can write $F = F_1F_2F_3$ where $F_j \in \mathcal{O}_M(\mathbb{R})$, $F_j > 0$ for $j = 1, 2, 3$, and where $F_1(x) = 1$ for $x \leq 1$, $F_2(x) = 1$ for $|x| \geq 2$, and $F_3(x) = 1$ for $x \geq -1$. Thus in our analysis we may suppose that $F = F_1$.

We start with a Tauberian result for slowly increasing smooth functions.

Lemma 3.5.1. *Let $G = 1/F$, where F is smooth and strictly positive in $\mathbb{R}$. If $F'(x) = O(1)$, as $x \to +\infty$ and if*

$$G(x) = O\left(\frac{1}{x}\right) \quad (C) \quad \text{as } x \to \infty\,, \tag{3.5.1}$$

then G is of slow growth at infinity, that is, there exists $\alpha > 0$ such that $G(x) = O(x^{\alpha})$ as $x \to +\infty$.

Proof. We may suppose without lost of generality that $F(x) = 1$ for $x \leq 1$. Since G is positive, the Cesàro order relation (3.5.1) means that

$$\int_0^x G(t)\,\mathrm{d}t = O(\ln x)\,, \quad x \to +\infty\,. \tag{3.5.2}$$

Hence there is constant K such that $\int_0^x G(t)\,\mathrm{d}t \leq K \ln x$ for $x \geq 1$. Also, there is another constant M such that $F'(x) \leq M$ for $x \geq 1$. If G is not of slow growth, for any $n > 0$ we can find x_* such that $G(x_*) \geq x_*^n$. Let $\varepsilon = F(x_*)$. Then $F(x) \leq \varepsilon + M(x - x_*)$ for $x \geq x_*$.

Hence

$$\int_0^{x_*+1} G(t)\,\mathrm{d}t \geq \int_{x_*}^{x_*+1} G(t)\,\mathrm{d}t \geq \int_0^1 \frac{\mathrm{d}t}{\varepsilon + Mt}$$
$$\sim \frac{1}{M}\ln\left(\varepsilon^{-1}\right) \geq \frac{n}{M}\ln x_*\,,$$

so that $n/M \leq K$, a contradiction since n can be arbitrarily large. □

We can now make a slight generalization in the following way.

Proposition 3.5.2. *Let $G = 1/F$, where F is smooth and strictly positive in $\mathbb{R}$. If for some $\beta > -1$,*

$$F'(x) = O\left(x^{\beta}\right)\,,\ \textit{as}\ x \to +\infty\,, \tag{3.5.3}$$

and

$$\int_0^x \frac{G(t)}{t^{\beta}}\,\mathrm{d}t = O(\ln x)\,, x \to +\infty\,, \tag{3.5.4}$$

then G is of slow growth at infinity.

Proof. Let $\gamma = 1/(\beta+1)$ and define $H(x) = F(x^{\gamma})$; note that $H'(x) = O(1)$ as $x \to +\infty$. In terms of the function H the order relation (3.5.4) tell us that we have

$$\int_0^x \frac{dt}{H(t)} = O(\ln x)\,, x \to +\infty\,. \tag{3.5.5}$$

By Lemma 3.5.1, there exists $\alpha > 0$ such that $(1/H(x)) = O(x^{\alpha})$ as $x \to +\infty$. Consequently, $G(x) = O\left(x^{\alpha/\gamma}\right)$ as $x \to +\infty$. □

We now give Tauberian theorems that guarantee that a smooth function G belongs to $\mathcal{O}_M$.

Theorem 3.5.3. *Let $G = 1/F$, where F is smooth and $F > 0$. Suppose that F'' is of slow growth at infinity. If there exists $\beta > 0$ such that*

$$G(x) = O\left(x^{\beta}\right) \quad (C) \quad \text{as } x \to +\infty\,, \tag{3.5.6}$$

then there exists $\alpha > 0$ such that the ordinary relation

$$G(x) = O\left(x^{\alpha}\right) \quad \text{as } x \to +\infty\,, \tag{3.5.7}$$

holds.

Proof. There are constants M and κ such that $F''(x) \le Mx^{\kappa}$ for $x \ge 1$. If the function G is increasing for x large, then the Lemma 3.2.1 yield that if the Cesàro order relation (3.5.6) holds then it is also valid in the ordinary sense $G(x) = O\left(x^{\beta}\right)$ as $x \to +\infty$. Therefore, let us suppose that G is not increasing for x large and that G is not of slow growth at infinity and find a contradiction. Indeed, for any $n > 0$ we can find $x_* > 0$ such that $G(x_*) \ge x_*^n$ and such that x_* is a local maximum of G. It follows that $F'(x_*) = 0$ and $\varepsilon = F(x_*) \le x_*^{-n}$. Notice that for some constant K,

$$F''(x) \le Kx_*^{\kappa}\,, \quad x_* \le x \le x_* + 1\,. \tag{3.5.8}$$

Hence

$$F(x) \le \varepsilon + \frac{Kx_*^{\kappa}}{2}(x - x_*)^2\,, \quad x_* \le x \le x_* + 1\,. \tag{3.5.9}$$

It follows that

$$\begin{aligned}\int_0^{x_*+1} G(t)\,\mathrm{d}t \ge \int_{x_*}^{x_*+1} G(t)\,\mathrm{d}t \ge \int_0^1 \frac{\mathrm{d}t}{\varepsilon + Kx_*^{\kappa}t^2/2} \\ \ge \frac{2}{Kx_*^{\kappa}}\left(\frac{2\varepsilon}{Kx_*^{\kappa}}\right)^{-1/2} \arctan\left(\left(\frac{2\varepsilon}{Kx_*^{\kappa}}\right)^{-1/2}\right)\end{aligned}$$

so that if ε is small enough,

$$\int_0^{x_*+1} G(t)\,\mathrm{d}t \geq \frac{Q}{x_*^{\kappa/2}}\varepsilon^{-1/2} \geq Q x_*^{(n-\kappa)/2}, \tag{3.5.10}$$

where $Q = (2/K)^{1/2}\,\pi/4$ is another constant. This contradicts (3.5.6). □

The Theorem 3.5.3 allow us to immediately obtain the following Tauberian result.

Theorem 3.5.4. *Let $G = 1/F$, where $F \in \mathcal{O}_M(\mathbb{R})$, $F > 0$. If $G \in \mathcal{S}'(\mathbb{R})$, then $G \in \mathcal{O}_M(\mathbb{R})$.*

We can rewrite this theorem in terms of the division problem in $\mathcal{S}'(\mathbb{R})$.

Theorem 3.5.5. *Let $F \in \mathcal{O}_M(\mathbb{R})$, $F > 0$. The equation*

$$Fg = h\,, \tag{3.5.11}$$

will have a solution g in $\mathcal{S}'(\mathbb{R})$ for all $h \in \mathcal{S}'(\mathbb{R})$ if and only if $1/F \in \mathcal{S}'(\mathbb{R})$.

We finish by considering an example that shows the limits of our results.

Example 3.5.6. Let $\varepsilon \in (0,1)$ and consider the triangle function T, that equals 1 for $|x| \geq 1-\varepsilon$ and that follows the lower part of the triangle in $\mathbb{R}^2$ with vertices $(-1+\varepsilon,1),(0,\varepsilon)$, and $(1-\varepsilon,1)$. Consider the function ψ_ε, a smooth version of T, constructed as follows

$$\psi_\varepsilon(x) = \begin{cases} T(x) & x \notin (-\varepsilon/2,\varepsilon/2)\cup(-1+\varepsilon,-1+2\varepsilon)\cup(1-2\varepsilon,1-\varepsilon)\;, \\ \eta\,(x) & x \in [-1+\varepsilon,-1+2\varepsilon]\cup[1-2\varepsilon,1-\varepsilon]\;, \\ \lambda(x) & x \in [-\varepsilon/2,\varepsilon/2]\,, \end{cases} \tag{3.5.12}$$

where η and λ have the following properties. The function η is a smooth function in the union $[-1+\varepsilon,-1+2\varepsilon] \cup [1-2\varepsilon,1-\varepsilon]$ that coincides with T of infinite order at the four endpoints, it decreases in $[-1+\varepsilon, -1+2\varepsilon]$, increases in $[1-2\varepsilon, 1-\varepsilon]$, and satisfies $T(x) \leq \eta(x) \leq 1$. On the other hand, the function λ is a smooth function in the interval $[-\varepsilon/2, \varepsilon/2]$, that coincides with T of infinite order at the endpoints, it decreases in $[-\varepsilon/2, 0]$, increases in $[0, \varepsilon/2]$, so that λ has a local minimum at $x = 0$, and satisfies $T(x) \leq \lambda(x) \leq 3\varepsilon/2$. Then what we have is

$$\int_0^1 \frac{1}{\psi(x)}\, dx = O(\ln 1/\varepsilon) \ . \tag{3.5.13}$$

If $\{\varepsilon_k\}_{k=1}^{\infty}$ is a sequence of $(0,1)$ then the function

$$\Phi(x) = \prod_{k=1}^{\infty} \psi_{\varepsilon_k}(x - 3k) \ , \tag{3.5.14}$$

is a smooth function with $\Phi > 0$. Then if we take $\epsilon_n = 1/(2n)^n$, we have that $1/\Phi$ has slow growth in the average sense but $1/\Phi \notin \mathcal{O}_M$.

The function $\Theta = 1/\Phi$ actually provides counterexamples to several other results.

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Vita

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