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Szasz-Muntz Theorems for Nilpotent Lie Groups.

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SZASZ-MÜNTZ THEOREMS FOR NILPOTENT LIE GROUPS

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ABSTRACT

The classical Szasz-Müntz theorem says that for $f \in L^2([0,1])$ and $\{n_k\}_{k=1}^{\infty}$ a strictly increasing sequence of positive integers ,

$$\int_0^1 x^{n_j} f(x)dx = 0 \ \forall j \Rightarrow f = 0 \iff \sum_{j=1}^{\infty} \frac{1}{n_j} = \infty.$$

We have generalized this theorem to compactly supported functions on $\Re^n$ and to an interesting class of nilpotent Lie groups. On $\Re^n$ we rephrased the condition above on an integral against a monomial as a condition on the derivative of the Fourier transform $\hat{f}$. For compactly supported f this transform has an entire extension to complex n-space, and these derivatives are coefficients in a Taylor series expansion of $\hat{f}$.

In the nilpotent Lie groups case there are several possible choices for the equivalent of a Fourier transform: the operator valued transform, the matrix coefficients for the operator transform relative to a basis of an infinite dimensional Hilbert space, and finally the trace transform. We have proven a Szasz-Müntz theorem for the matrix coefficients on groups that have a fixed polarizer for the representations in general position. For groups with flat orbits and a fixed radical for the representations in general position, we have proven a Szasz-Müntz theorem for the trace transform.

Our work here is inspired by recent work on Paley-Wiener theorems on nilpotent Lie groups. Moss [1] proved a Paley-Wiener theorem on groups with a fixed

polarizer for the generic representations. Park [2] extended these results to two and three step groups. Lipsman and Rosenberg [3] have proven a Paley-Wiener theorem for the matrix coefficients on any simply connected nilpotent Lie group.

As part of the proof of the Szasz-Müntz theorem for matrix coefficients we construct a new basis in a nilpotent Lie algebra, which we call an *almost strong Malcev basis.* This new basis has many of the features of a strong Malcev basis, although it can be used to pass through subalgebras which are not ideals. Almost strong Malcev basis have the nice property that they are unique up to the original strong Malcev basis.

CHAPTER 1

BACKGROUND MATERIAL

In this chapter we present, without proof, background material for nilpotent representation theory needed for this dissertation. An excellent reference for further reading on this subject is [7]. The first section deals with nilpotent Lie groups, the Campbell-Baker-Hausdorff formula (which is not peculiar to nilpotent Lie groups but has a stronger statement in this context), and Malcev bases. The second section covers topics related to representation theory such as unitary equivalence, irreducible representations and the unitary dual of a group. The third section deals with the elements of nilpotent representation theory: polarizing subalgebras, Kirillov's wonderful work on the equivalence classes of unitary, irreducible representations, and other topics related to coadjoint orbits. The third section will deal with the more specialized material germain to this paper. Topics such as matrix coefficients, the trace transform, and parameterizing representations, all of which are integral parts of this paper, will be discussed there.

1.1 Nilpotent Lie Groups

We will start out with the Campbell-Baker-Hausdorff formula. So let $\mathfrak{G}$ be a Lie group with corresponding Lie algebra $\mathfrak{g}$. The exponential map $exp : \mathfrak{g} \rightarrow \mathfrak{G}$ is the correspondence between tangent vectors to $\mathfrak{G}$ and one parameter subgroups of $\mathfrak{G}$. The Campbell-Baker-Hausdorff formula is a formula for computing the group multiplication using the Lie algebra bracket relations:

$$exp(X)exp(Y) = exp(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X[X,Y]] - \frac{1}{12}[Y[X,Y]] + \ldots).$$

We will make extensive use of this formula in chapter 3 when we are computing the action of represention using right translation.

In any Lie algebra $\mathfrak{g}$ we may define the descending central series $\mathfrak{g}^{(1)}, \mathfrak{g}^{(2)}, \mathfrak{g}^{(3)}, \ldots$ by the relations:

$$\mathfrak{g}^{(1)} = \mathfrak{g}$$

$$\mathfrak{g}^{(j)} = [\mathfrak{g}^{(j-1)}, \mathfrak{g}]$$

where $[\mathfrak{g}^{(j-1)}, \mathfrak{g}]$ is taken to be the span of all the vectors of the form $[X,Y]$ with $X \in \mathfrak{g}^{(j-1)}$ and $Y \in \mathfrak{g}$. A *nilpotent Lie algebra* is a Lie algebra whose descending central series terminates at $\{0\}$ after a finite number of *steps.* If m is the smallest number such that $\mathfrak{g}^{(m)} \neq \{0\}$ then we call $\mathfrak{g}$ m-step nilpotent. A *nilpotent Lie group* will be a group whose Lie algebra is nilpotent. It turns out that for a Lie group this is equivalent to the standard definition from group theory of a nilpotent group. Simply connected nilpotent Lie groups have the nice property that the exponential map is a diffeomorphism onto the whole group $\mathfrak{G}$.

An *ideal* in $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ such that for any $X \in \mathfrak{h}, [X,Y] \in \mathfrak{h}$ for any $Y \in \mathfrak{g}$. Now let $\{X_1, \ldots, X_n\}$ be a vector space basis for $\mathfrak{g}$. This basis will be called *weak Malcev* if for each $1 \leq j \leq n$ the vector space $\Re$−span$\{X_1, \ldots, X_j\}$ is a subalgebra of $\mathfrak{g}$ (Here $\Re$−span$\{X_1, \ldots, X_j\}$ means the real span of the vectors $X_1, \ldots, X_j$). If in addition each such subalgebra is an ideal, the basis is called *strong Malcev.* A basis X for $\mathfrak{g}$ will be said to pass through a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ if the first k vectors of the basis form a basis for $\mathfrak{h}$ (where k is the dimension of $\mathfrak{h}$). When talking about nilpotent Lie groups a basis is almost always taken to be at least weak Malcev. Strong Malcev bases are generally preferable to weak Malcev bases, however if one is forced to take a basis passing through a subalgebra that is not an ideal, a strong Malcev basis cannot be used. Also notice that Malcev bases are ordered bases. In chapter 3 we construct a basis, which we call almost strong Malcev, through subalgebras(not necessarily ideals) that have a lot of the merits of a strong Malcev basis including a nice multiplication that comes from the Campbell-Baker-Hausdorff formula. These basis are unique up to the choice of original strong Malcev basis.

1.2 Representation Theory

Before we can talk about the Kirillov model, which is the core of nilpotent representation theory, we need some definitions. A *representation* π is a homomorpism of a group G into the space of bounded linear operators on a Hilbert space $\mathbf{H}_\pi$. $\mathbf{H}_\pi$ is called the *modeling space* for π. In our case G will be a separable nilpotent Lie group and $\mathbf{H}_\pi$ will be a separable Hilbert space. A continuous representation π is a representation such that:

$$||\pi(x_n)(\xi) - \pi(x)(\xi)|| \to 0$$

as $x_n \to x$ for all $\xi \in \mathbf{H}_\pi$. The representation π will be called unitary if π is a homomorphism into $U(\mathbf{H}_\pi)$, the space of unitary operators on $\mathbf{H}_\pi$. $\mathbf{H}_\pi$ does not have to be infinite dimensional: the *characters*

$$\chi_\ell(s) = e^{2\pi i \ell s}, \qquad \ell, s \in \Re^n$$

are (continuous)representations of $\Re^n$ in the space of unitary operators on $\mathbf{C}$, for any ℓ. Here ℓs means the dot product of the vectors ℓ, s. Note that if you use complex ℓ you lose the unitary condition. The characters are important in the theory of inducing representations on nilpotent Lie groups, and they play a central role in Fourier transforms. This is no coincidence, as we will soon see.

Two representations π and σ on the same group G will be said to be *unitarily equivalent* if there is a unitary operator $A\colon \mathbf{H}_\pi \to \mathbf{H}_\sigma$ such that

$$A(\pi(x)) = \sigma(x)(A), \qquad \forall x \in G.$$

The operator A is called an *intertwining operator.*

Let π be a representation on the group G acting on the space $\mathbf{H}_\pi$. A subspace K of $\mathbf{H}_\pi$ is said to be *invariant* under the action of π if

$$\pi(x)\colon K \to K, \qquad \forall x \in G.$$

The representation π is said to be *irreducible* if the only closed invariant subspaces of $\mathbf{H}_\pi$ are the zero subspace or $\mathbf{H}_\pi$ itself. *Reducible representations* (representations that are not irreducible) can be thought of as representations that are acting on a space that is too big for them.

The *dual object* $\hat{G}$ of the group G is the collection of equivalence classes, under unitary equivalence, of irreducible unitary representations on the group G. For the n-dimensional Euclidean space this collection is the collection of characters $\{\chi_\ell | \ell \in \Re^n\}$. So $\widehat{\Re^n} \cong \Re^n$.

From a representation theory point of view the Euclidean Fourier transform

$$\hat{f}(\ell) = \int_{\Re^n} e^{2\pi i \ell s} f(s) ds$$

is an integral over the group $\mathfrak{G} = \Re^n$. The inverse transform

$$f(s) = \int_{\Re^n} e^{-2\pi i \ell s} \hat{f}(\ell) d\ell$$

is really an integral over the dual object of $\Re^n$. This observation is very helpful in understanding Fourier transforms on Lie groups.

1.3 Kirillov Theory

In this section we will present a brief overview of nilpotent representation theory leading up to Kirillov's classification of $\hat{\mathfrak{G}}$ as being isomorphic to $\mathfrak{g}^*/Ad^*(\mathfrak{G})$. We will start with radicals and polarizing subalgebras, proceed to co-adjoint orbits, and finish up with the Kirillov model.

Let $\ell \in \mathfrak{g}^*$ (here $\mathfrak{g}^*$ is the linear dual of $\mathfrak{g}$ as a vector space). The *radical* of ℓ, denoted r_ℓ, is the subalgebra of all vectors X such that $\ell[X, Y] = 0$ for every Y in $\mathfrak{g}$. A *polarizing* subalgebra m_ℓ for ℓ must satisfy two properties:

1) For any $X, Y \in m_\ell$, we must have $\ell[X, Y] = 0$.

2) m_ℓ must have the maximum possible dimension, among all subalgebras satisfying property 1. This dimension is $\frac{1}{2}(\dim(\mathfrak{g}) + \dim(r_\ell))$.

It is known from linear algebra that if we treat $\mathfrak{g}$ as a vector space polarizing subspaces of $\mathfrak{g}$ exist [11]. It is a theorem of Vergne's [6] that polarizing subalgebras of $\mathfrak{g}$ exist as a Lie algebra.

Polarizing subalgebras are not unique. For example let $\mathfrak{h} = \Re - \mathrm{span}\{Z, Y, X\}$ be the Heisenberg algebra, with all nonzero bracket relations generated by the condition $[X, Y] = Z$. Then the subalgebras spanned by the vectors $\{Z, Y\}$ and $\{Z, X\}$ respectively are both (abelian) polarizing subalgebras for the dual vector $\ell = Z^*$ from the standard dual basis.

Polarizing subalgebras are important in the construction of unitary irreducible representations on a nilpotent Lie group. To see why we need to introduce the idea of inducing representations. So let $\mathfrak{H}$ be a closed subgroup of $\mathfrak{G}$, and let $\pi: \mathfrak{H} \to \mathbf{H}_\pi$ be a unitary representation of $\mathfrak{H}$. We can pump π up to a representation $\sigma = Ind_{\mathfrak{H}}^{\mathfrak{G}}(\pi)$ of $\mathfrak{G}$ by the following method:

Let dx be a Haar measure on $\mathfrak{G}$, and let $\mathfrak{H} \setminus \mathfrak{G}$ denote the set of all left cosets $\mathfrak{H}g$. Let $d\dot{x}$ be any right $\mathfrak{G}$ – invariant measure on $\mathfrak{H} \setminus \mathfrak{G}$. We will not require that $\mathfrak{H}$ is normal in $\mathfrak{G}$.

Model σ on the Hilbert space

$$\mathbf{H}_\sigma = \{f: \mathfrak{G} \to \mathbf{H}_\pi | f(hg) = \pi(h)f(g) \forall h \in \mathfrak{H}, \int_{\mathfrak{H}\setminus\mathfrak{G}} ||f(x)||^2 d\dot{x} < \infty\}$$

by defining $\big(\sigma(x)(f)\big)(y) = f(yx)$ for $f \in \mathbf{H}_\sigma$ and $x, y \in \mathfrak{G}$.

Remarks. 1) Since π is unitary the map

$$\phi: \mathfrak{H} \setminus \mathfrak{G} \to \mathbf{C}$$

defined by

$$\phi(x) = \langle f(x), g(x) \rangle$$

is well defined on $\mathfrak{H} \setminus \mathfrak{G}$ cosets. Here $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathbf{H}_\pi}$ is the inner product on $\mathbf{H}_\pi$. Notice that $\langle f(hx), g(hx) \rangle = \langle \pi(h) f(x), \pi(h) g(x) \rangle = \langle f(x), g(x) \rangle$.

2) By property 1 we may define an inner product on $\mathbf{H}_\sigma$ by

$$\langle f, g \rangle = \int_{\mathfrak{H} \setminus \mathfrak{G}} \langle f(x), g(x) \rangle d\dot{x}.$$

With this inner product $\mathbf{H}_\sigma$ is a Hilbert space.

Part of the Kirillov theory is that all of the irreducible unitary representations on a simply connected nilpotent Lie group are obtained up to unitary equivalence by inducing characters of subgroups. The characters arise as follows. Let $\chi_\ell(x) = e^{2\pi i \ell(log(x))}$ for $\ell \in \mathfrak{g}^*$. The trick is that you must induce from a polarizing subgroup M (the subgroup corresponding to a polarizing subalgebra m). The polarizing subalgebras are used because of the two properties of polarizers outlined above. The first property ensures that χ_ℓ will be a homomorphism on M, and hence a character. Property 2, the maximal dimension property, ensures that resulting representation will be irreducible.

Above we mentioned that polarizing subalgebras are not unique. Another part of the Kirillov theory says if you induce from two different polarizers for the same ℓ

the resulting representions are equivalent. For the final aspect of the Kirillov theory we need some information about co-adjoint orbits.

Define the adjoint map $Ad: \mathfrak{G} \to \mathrm{Aut}(\mathfrak{g})$ as the derivative of inner automorphism, α_x, at the identity of $\mathfrak{G}$. That is for each $x \in \mathfrak{G}$ we define $Ad(x)$ by the equation

$$exp(Ad(x)Y) = xexp(Y)x^{-1} = \alpha_x(exp(Y)).$$

The co-adjoint map is defined in terms of the adjoint map: $Ad^*(x): \mathfrak{g}^* \to \mathfrak{g}^*$ is defined by

$$(Ad^*(x)\ell)(Y) = \ell(Ad(x^{-1})(Y))$$

for each $x \in \mathfrak{G}$.

Kirillov theory says that for $\ell, \ell' \in \mathfrak{g}^*$ the representations π_ℓ and $\pi_{\ell'}$, induced by the characters $\chi_\ell, \chi_{\ell'}$ respectively, are equivalent if and only if ℓ and ℓ' come from the same $Ad^*(\mathfrak{G})$ orbit in $\mathfrak{g}$. Thus there exists a one to one correspondence between $\hat{\mathfrak{G}}$ and $\mathfrak{g}^*/Ad^*(\mathfrak{G})$. Naturally coadjoint orbits are an important topic in represention theory on nilpotent Lie groups. It's a topic we explore in more detail in the next section.

1.4 More Advanced Topics

The main goal of this paper is to prove a Szasz-Müntz theorem for matrix coefficients and for the trace transform (on certain classes of nilpotent Lie groups). It is the purpose of this section to introduce matrix coefficients and the trace transform.

Fix $\ell \in \mathfrak{g}^*$, and let π_ℓ be the unitary irreducible representation corresponding to ℓ, modeled in the Hilbert space $\mathbf{H}_{\pi_\ell}$. For $f, g \in \mathbf{H}_{\pi_\ell}$ and for $\varphi \in L^2(\mathfrak{G}) \bigcap L^1(\mathfrak{G})$ consider the bilinear map

$$B_\varphi(f,g) = \int_{\mathfrak{G}} \varphi(x)\langle \pi_\ell(x) f, g\rangle dx.$$

Since this map is linear and continuous in g, the Riesz representation theorem guarantees that there is a vector $\pi_\ell(\varphi)(f) \in \mathbf{H}_{\pi_\ell}$ such that

$$\langle \pi_\ell(\varphi)(f), g\rangle = \int_{\mathfrak{G}} \varphi(x)\langle \pi_\ell(x) f, g\rangle dx.$$

That is $\pi_\ell(\varphi) : \mathbf{H}_{\pi_\ell} \to \mathbf{H}_{\pi_\ell}$ is a continuous linear operator which we will call the operator valued Fourier transform of φ. The inner product $\langle \pi_\ell(\varphi)(f), g\rangle$ is a matrix coefficient of this transform. It is the purpose of this paper to investigate how these coefficients, and hence the transform itself, depend on ℓ.

The trace transform is defined as the trace of the operator $\pi_\ell(\varphi)$. So

$$Tr(\pi_\ell(\varphi)) = \sum_{n=1}^{\infty} \langle \pi_\ell(\varphi)\psi_n, \psi_n\rangle$$

where ψ_n is any orthonormal basis of $\mathbf{H}_{\pi_\ell}$. Again it is our intent to study the behavior of the trace transform as ℓ varies.

We will see that we can always model the representations π_ℓ in the same Hilbert space. This will make our study of the matrix coefficients and the trace transform easier, since we do not need to change our modeling space as ℓ changes, however this may be at the cost of 'nice' behavior as an example in chapter 4 points out.

Before we finish up we need to investigate the co-adjoint orbits further. Define $ad : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ as the derivative of the adjoint map at the identity. So $ad(X)Y$ is defined by the equation

$$Ad(exp(X))Y = exp(ad(X)(Y)).$$

It turns out that ad is an old friend: $ad(X)(Y) = [X,Y]$. The corresponding derivative of the coadjoint map is the map $ad^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ defined by the equation

$$(ad^*(X)(\ell))(Y) = \ell(ad(-X)(Y)) = \ell([Y,X]).$$

With these definitions we can talk about generic orbits, jump indices, parameterizing represention, etc. Fix a strong Malcev basis $X = \{X_1, \ldots, X_n\}$ for $\mathfrak{g}$. Let $\mathfrak{g}_j = \Re - \text{span}\{X_1, \ldots, X_j\}$. The basis X is a *Jordan-Hölder* basis for the action of Ad. That is for each j and $X \in \mathfrak{g}$ we have $Ad(exp(X)(g_j) \subseteq g_j$. Therefore $\ell_n, \ldots, \ell_1$ is a Jordan-Hölder basis for the action of Ad^*, where $\ell_j = X_j^*$. That is for all $X \in \mathfrak{g}$ and $1 \leq j \leq n$, $Ad^*(exp(X))(A_j) \subseteq A_j$, where $A_j = \Re\text{–span}\{\ell_n, \ldots, \ell_j\}$.

So Ad^* can be defined as a quotient action on $\mathfrak{g}^*/A_j$. Let $d_j(\ell)$ be the orbit dimension for the equivalence class of ℓ under this action. The *generic* ℓ are those ℓ whose orbits have the maximal possible dimension $d_j(\ell)$ for each j. A *generic orbit* is an $Ad^*(\mathfrak{G})$-orbit in $\mathfrak{g}^*$ that contains a generic ℓ. On the other hand a generic orbit is comprised of only generic ℓ. The Chevalley-Rosenlicht theorem shows that the collection of generic orbits U is a Zariski-open set in $\mathfrak{g}^*$, and in particular it is a set of full Euclidean measure.

For a generic ℓ let $d_j = d_j(\ell)$ (d_j is well defined by the definition of the generic ℓ). A *jump index* s_j is an index for which the orbit dimension increases, i.e. $d_{s_j} = d_{s_j-1} + 1$. The *non − jump* indices are the remaining indices, those indices for which the orbits do not increase in dimension.

The generic orbits, jump indices, and non-jump indices all depend on the original strong Malcev basis.

There is a nice equivalent condition for an index to be a non-jump index [7].

Lemma *An index i is a non-jump index if and only if for every $\ell \in U$ there is a vector $y \in \Re - \mathrm{span}\{X_1, \ldots, X_{i-1}\}$ such that $X + Y \in r_\ell$.*

We have quoted this lemma specifically because we will be making extensive use of it.

Remember that for any two members ℓ, ℓ' of a co-adjoint orbit, their corresponding representations $\pi_\ell, \pi_{\ell'}$ on $\mathfrak{G}$ are equivalent. We want to study the operator valued transform and the trace transform which depends not on ℓ but really on the equivalence class of the representation π_ℓ. So for two different members of a co-adjoint orbit the actions of these transforms will be equivalent. What we need is a convenient way to choose one representative from each orbit. If we restrict ourselves to the generic orbits there is a way to do this.

Let S, T be the collection of jump and non-jump indices respectively and $V_S = \Re - \mathrm{span}\{\ell_{s_i} | s_i \in S\}, V_T = \Re - \mathrm{span}\{\ell_{t_i} | t_i \in T\}$. Then V_S has the same dimension as the generic orbits, and in fact for a generic orbit O_ℓ, projection maps

O_ℓ diffeomorphically onto V_S. From this it can be seen that a convenient way to choose orbit representatives is to take members of the Zariski open set $U \bigcap V_T$.

In fact there is a parameterizing map $\psi(\ell_T, \ell_S) : (U \bigcap V_T) \times V_S \to U$ such that ψ is rational in ℓ_T and for fixed $\ell_T, \psi(\ell_T, \cdot)$ is a polynomial diffeomorphism onto the orbit O_{ℓ_T} of ℓ_T. The point $\psi(\ell_T, \ell_S)$ is the unique point in the orbit O_{ℓ_T} such that $p(\psi(\ell_T, \ell_S)) = \ell_S$, where p is projection onto V_S. The functionals from $U \bigcap V_T$ are called the *parameterizing functionals.*

CHAPTER 2

SZASZ-MÜNTZ THEOREMS FOR N-DIMENSIONAL EUCLIDEAN SPACE

The classical Szasz-Müntz theorem [8] for $L^2([0,1])$ says that the collection of monomials $\{x^{n_k}\}_{k=1}^{\infty}$,where n_k is a sequence of integers such that $1 < n_1 < n_2 < \ldots$, spans $L^2([0,1])$ if and only if $\sum_{k=1}^{\infty} \frac{1}{n_k} = \infty$. Our goal is to prove an analogous (in an appropriate sense) theorem for $L_c^2(\Re^n)$ and then for $L_c^2(G)$, where G comes from a class of simply connected nilpotent Lie groups that we shall define later. Here L_c^2 means the compactly supported L^2 functions on the group G. There has been a lot of work done on Szasz-Müntz theorems for continuous functions in $[0,\infty)$, see [4].

So the job at hand is to first state, and then prove a Szasz-Müntz theorem for $L_c^2(\Re^n)$. In this direction we first extend the classical Szasz-Müntz theorem to a theorem on $L^2([0,b])$. Throughout this section we will take b to be a positive real number, and we will make use of the following theorem from functional analysis [9]: *In a Hilbert space H, the collection of vectors f_n spans H if and only if $\langle f_n, g\rangle = 0$ for every n implies that $g = 0$.*

Theorem 2.1. *The collection of monomials $\{x^{n_k}\}_{k=1}^{\infty}$, for integers $1 < n_1 < n_2 < \ldots$, spans $L^2([0,b])$ if and only if $\sum_{k=1}^{\infty} \frac{1}{n_k} = \infty$.*

Proof. First we will show that if $\{x^{n_k}\}_{k=1}^{\infty}$ spans $L^2([0,b])$, then we must have $\sum_{k=1}^{\infty} \frac{1}{n_k} = \infty$. Suppose instead that $\sum_{k=1}^{\infty} \frac{1}{n_k} < \infty$. Our job then, is to construct a function in $L^2([0,b])$ that is not in the span of $\{x^{n_k}\}_{k=1}^{\infty}$.

Since the $\{n_k\}_{k=1}^{\infty}$ does not satisfy the conditions of the classical Szasz-Müntz theorem we may choose a function f in $L^2([0,1])$ such that f is nonzero, and

$$0 = \int_{[0,1]} x^{n_k} f(x)dx$$

for every k = 1,2,....

Now define the function g on $[0, b]$ by $g(x) = f(\frac{x}{b})$. We will show that although g is nonzero, the integral of g against any of the monomials x^{n_k} is zero, and hence g is not in the span of the $\{x^{n_k}\}_{k=1}^{\infty}$.

By the definition of g, for each k we have:

$$\begin{aligned}\int_{[0,b]} x^{n_k} g(x)dx &= b \int_{[0,1]} (bu)^{n_k} g(bu)du \quad \text{where} \quad u = bx \\ &= b^{n_k+1} \int_{[0,1]} u^{n_k} f(u)du \\ &= 0.\end{aligned}$$

So the monomials $\{x^{n_k}\}_{k=1}^{\infty}$ cannot span $L^2([0,b])$.

Now we will work on the other direction. Assume that $\sum_{k=1}^{\infty} \frac{1}{n_k} = \infty$, we must show that $\{x^{n_k}\}_{k=1}^{\infty}$ spans $L^2([0,b])$. Choose $f \in L^2([0,b])$, and suppose that

$$0 = \int_{[0,b]} x^{n_k} f(x)dx$$

for every $k = 1, 2, \ldots$.

We will show that this can happen only if $f = 0$, by making use of the Szasz-Müntz theorem for $L^2([0,1])$. In that direction define the function f_b on $[0,1]$ by $f_b(x)$

We will show that this can happen only if $f = 0$, by making use of the Szasz-Müntz theorem for $L^2([0,1])$. In that direction define the function f_b on $[0,1]$ by

$f_b(x) = f(bx)$.

Next we change variables:

$$\begin{aligned} 0 &= \int_{[0,b]} x^{n_k} f(x)dx \\ &= b \int_{[0,1]} (bx)^{n_k} f_b(x)dx \\ &= b^{n_k+1} \int_{[0,1]} x^{n_k} f_b(x)dx. \end{aligned}$$

Now we may apply the classical Szasz-Müntz theorem for $L^2([0,1])$ to the function f_b, and by the above argument we conclude that $f_b = 0$. It follows from the definition of f_b that $f = 0$ as desired. □

Now that we have a Szasz-Müntz theorem for $L^2([0,b])$ the next step in the process is to prove a version of the classical Szasz-Müntz theorem for $L^2([-b,b])$. It turns out that $\sum_{k=1}^{\infty} \frac{1}{n_k} = \infty$ is not a sufficient condition for the monomials $\{x^{n_k}\}_{k=1}^{\infty}$ to have their closed linear span be all of $L^2([-b,b])$. For example consider the sequence $n_k = 2k, \quad k = 1, 2, \ldots$. This sequence satisfies the conditions of the classical Szasz-Müntz theorem, and yet any linear combination of the monomials $\{x^{2k}\}_{k=1}^{\infty}$ will be an even function. So it would be impossible for an odd function on the interval $[-b,b]$ to be in their closed linear span. In order to avoid this situation we place a stronger condition on the the powers $\{n_k\}_{k=1}^{\infty}$.

Definition 2.2. *A Szasz-Müntz sequence is a sequence of integers* $1 < n_1 < n_2 <$ *... such that* $\sum_{k=1}^{\infty} \frac{1}{n_{e_k}} = \sum_{k=1}^{\infty} \frac{1}{n_{o_k}} = \infty$. *Here* $\{n_{e_k}\}_{k=1}^{\infty}$ *and* $\{n_{o_k}\}_{k=1}^{\infty}$ *are the subsequences of even and odd terms of* $\{n_k\}_{k=1}^{\infty}$ *respectively.*

Notice that the sequence of terms $n_k = 2k, \quad k = 1, 2, \ldots$, does not satisfy this definition.

With this definition in hand we are able to prove the following theorem.

Theorem 2.3. *The collection of monomials $\{x^{n_k}\}_{k=1}^{\infty}$ spans $L^2([-b,b])$ if and only if $\{n_k\}_{k=1}^{\infty}$ is a Szasz-Müntz sequence.*

Proof. First we will show that if the monomials $\{x^{n_k}\}_{k=1}^{\infty}$ span $L^2([-b,b])$ then $\{n_k\}_{k=1}^{\infty}$ is as a Szasz-Müntz sequence. In fact what we will show is that if $\{n_k\}_{k=1}^{\infty}$ is not a Szasz-Müntz sequence, then the monomials $\{x^{n_k}\}_{k=1}^{\infty}$ cannot span $L^2([-b,b])$.

Assume, for instance, that $\sum_{k=1}^{\infty} \frac{1}{n_{e_k}} < \infty$. By theorem 2.1, we may choose an $f \in L^2([0,b])$ such that f is nonzero, and

$$\int_{[0,b]} x^{n_k} f(x) = 0$$

for every k. Now extend f evenly to the function g on $[-b,b]$. That is define g in terms of f as

$$g(x) = \begin{cases} f(x), & \text{if } x \geq 0; \\ f(-x), & \text{if } x \leq 0. \end{cases}$$

Then $g \in L^2([-b,b])$, g is nonzero, and for the even n_k we have

$$\int_{[-b,b]} x^{n_k} g(x)dx = \int_{[-b,0]} x^{n_k} g(x)dx + \int_{[0,b]} x^{n_k} g(x)dx$$

$$= \int_{[0,b]} (-x)^{nk} g(-x)dx + \int_{[0,b]} x^{n_k} g(x)dx$$
$$= 2\int_{[0,b]} x^{n_k} f(x)dx$$
$$= 0$$

by a change of variables, and the definition of f. On the other hand for the odd n_k we have

$$\int_{[-b,b]} x^{n_k} g(x)dx = \int_{[-b,0]} x^{n_k} g(x)dx + \int_{[0,b]} x^{n_k} g(x)dx$$
$$= \int_{[0,b]} (-x)^{n_k} g(-x)dx + \int_{[0,b]} x^{n_k} g(x)dx$$
$$= -\int_{[0,b]} x^{n_k} f(x)dx + \int_{[0,b]} x^{n_k} f(x)dx$$
$$= 0.$$

In either case the inner product of g with x^{n_k} is zero. So our function g is not in the span of the $\{x^{n_k}\}_{k=1}^{\infty}$. The case where $\sum_{k=1}^{\infty} \frac{1}{n_{o_k}} < \infty$ is handled similarly.

Now let's assume that the $\{n_k\}_{k=1}^{\infty}$ is a Szasz-Müntz sequence. We want to show that the closed linear span of the $\{x^{n_k}\}_{k=1}^{\infty}$ is all of $L^2([-b,b])$. To do this we will show that if the inner product of a function $f \in L^2([-b,b])$ against each of the monomials x^{n_k} is zero, then the function must be zero. So assume that

$$0 = \int_{[-b,b]} x^{n_k} f(x)dx$$

for every k.

Let $f_1, f_2 \in L^2([0,b])$ be defined by $f_1(x) = f(x)$ and $f_2(x) = f(-x)$. Then for each k:

$$\begin{aligned} 0 = \int_{[-b,b]} x^{n_k} f(x)dx &= \int_{[-b,0]} x^{n_k} f(x)dx + \int_{[0,b]} x^{n_k} f(x)dx \\ &= \int_{[0,b]} (-x)^{n_k} f(-x)dx + \int_{[0,b]} x^{n_k} f(x)dx \\ &= \int_{[0,b]} (-x)^{n_k} f_2(x)dx + \int_{[0,b]} x^{n_k} f_1(x)dx \\ &= \int_{[0,b]} [x^{n_k} f_1(x) + (-x)^{n_k} f_2(x)]dx. \end{aligned}$$

If n_k is even then we have

$$0 = \int_{[0,b]} x^{n_k}[f_1(x) + f_2(x)]dx,$$

otherwise if n_k is odd we have

$$0 = \int_{[0,b]} x^{n_k}[f_1(x) - f_2(x)]dx.$$

By assumption the subsequences $\{n_{e_k}\}_{k=1}^{\infty}$ and $\{n_{o_k}\}_{k=1}^{\infty}$ of even and odd terms of n_k are both Szasz-Müntz sequences, and so the Szasz-Müntz theorem for $L^2([0,b])$, says that

$$f_1(x) + f_2(x) = 0$$

$$f_1(x) - f_2(x) = 0.$$

Thus $f_1(x) = 0$ and $f_2(x) = 0$, and it follows that $f = 0$. □

We are now in a position to extend our theorem to a box $\prod_{i=1}^{n}[-b_i, b_i]$ in n-dimensional Euclidean space where $b_i, i = 1, 2, \ldots$,n, are fixed positive real numbers.

Theorem 2.4. *Let $\mu_1(k_1), \ldots, \mu_n(k_n)$ be n sequences of positive integers such that $1 < \mu_i(1) < \mu_i(2) < \ldots$ for every i. Then the collection of monomials*

$$\{x_1^{\mu_1(k_1)} \cdots x_n^{\mu_n(k_n)}\}|_{(k_1,\ldots,k_n)\in N^n}$$

spans $L^2(\prod_{i=1}^{n}[-b_i, b_i])$ if and only if each sequence $\mu_i(k)$ is a Szasz-Müntz sequence.

Proof. Suppose that one of the $\mu_i(k)$ is not a Szasz-Müntz sequence. In fact we may as well assume that $i = 1$. So we may choose an $f \in L^2([-b_1, b_1])$ such that f is nonzero and

$$0 = \int\limits_{[-b_1,b_1]} x_1^{\mu_1(k)} f(x)dx$$

for every k by theorem 2.3. Now let the function $g \in L^2(\prod_{i=1}^{n}[-b_i, b_i])$ be defined by

$$g(x_1, \ldots, x_n) = f(x_1)h(x_2, \ldots, x_n)$$

where $h \in L^2(\prod_{i=2}^{n}[-b_i, b_i])$ is any nonzero function. We want to show that the inner product of g with any of the monomials $x_1^{\mu_1(k_1)} \cdots x_n^{\mu_n(k_n)}$ is zero, i.e. that g is not in the span of these monomials. By our choice of f:

$$\int\limits_{\prod_{i=1}^{n}[-b_i,b_i]} x_1^{\mu_1(k_1)} \cdots x_n^{\mu_n(k_n)} g(x_1, \ldots, x_n)dx_1 \ldots dx_n$$

$$= \int\limits_{[-b_1,b_1]} x_1^{\mu_1(k_1)} f(x_1)dx_1 \int\limits_{\prod_{i=2}^{n}[-b_i,b_i]} x_2^{\mu_2(k_2)} \cdots x_n^{\mu_n(k_n)} h(x_2, \ldots, x_n)dx_2 \ldots dx_n$$

$$= 0$$

Therefore g cannot be in the span of the $x_1^{\mu_1(k_1)} \cdots x_n^{\mu_n(k_n)}$.

Now assume that each of the $\mu_i(k)$, $k = 1, 2, \ldots$, is a Szasz-Müntz sequence. Choose $f \in L^2(\prod_{i=1}^{n}[-b_i, b_i])$. We need to show that if the inner product of f against all of the monomials $x_1^{\mu_1(k_1)} \cdots x_n^{\mu_n(k_n)}$ is zero, then f must be zero. We will integrate f against these monomials, and then make repeated use of our one dimensional Szasz-Müntz theorem.

So assume that for each $k_1, \ldots, k_n$

$$\begin{aligned} 0 &= \int_{\prod_{i=1}^{n}[-b_i,b_i]} x_1^{\mu_1(k_1)} \cdots x_n^{\mu_n(k_n)} f(x_1, \ldots, x_n) dx_1 \ldots dx_n \\ &= \int_{[-b_1,b_1]} x_1^{\mu_1(k_1)} \Big(\int_{\prod_{i=2}^{n}[-b_i,b_i]} x_2^{\mu_2(k_2)} \cdots x_n^{\mu_n(k_n)} f(x_1, \ldots, x_n) dx_2 \ldots dx_n \Big) dx_1 \\ &= \int_{[-b_1,b_1]} x_1^{\mu_1(k_1)} F(x_1) \end{aligned}$$

where

$$F(x_1) = \int_{\prod_{i=2}^{n}[-b_i,b_i]} x_2^{\mu_2(k_2)} \cdots x_n^{\mu_n(k_n)} f(x_1, \ldots, x_n) dx_2 \ldots dx_n.$$

We may use Fubini in the second line since

$$f \in L^2(\prod_{i=1}^{n}[-b_i, b_i]) \subset L^1(\prod_{i=1}^{n}[-b_i, b_i]).$$

Notice that F has support contained in $[-b_1, b_1]$ since f does. Also notice that F is an L^2 function. In fact $F \in L^2$ if and only if $\hat{F} \in L^2$. But

$$\hat{F}(s) = \int_{\prod_{i=1}^{n}[-b_i,b_i]} e^{2\pi i s x_1} x_2^{\mu_2(k_2)} \cdots x_n^{\mu_n(k_n)} f(x_1, \ldots, x_n) dx_2 \ldots dx_n dx_1$$

$$= \int_{\prod_{i=1}^{n}[-b_i,b_i]} e^{2\pi i s x_1} h(x_1,\ldots,x_n)dx_2\ldots dx_n dx_1$$

$$= \hat{h}(s,0,\ldots,0) \in L^2(\Re)$$

since $h(x_1,\ldots,x_n) = x_2^{\mu_2(k_2)}\cdots x_n^{\mu_n(k_n)} f(x_1,\ldots,x_n) \in L_c^2(\prod_{i=1}^{n}[-b_i,b_i])$.

Now we may apply the one-dimensional Szasz-Müntz theorem to F. Then we must have $F = 0$. Now proceed inductively, peeling away an outer integral at each step. We conclude that $f = 0$, which is what we needed to show. □

Care must be taken in formulating a theorem like the above for $L_c^2(\Re^n)$, since this is not a Hilbert space. In fact although we haven't said it explicitly, in theorem 2.4 when we are considering the monomials $x_1^{\mu_1(k_1)}\cdots x_n^{\mu_n(k_n)}$, we really mean $x_1^{\mu_1(k_1)}\cdots x_n^{\mu_n(k_n)}\Big|_{\prod_{i=1}^{n}[-b_i,b_i]} \in L^2(\prod_{i=1}^{n}[-b_i,b_i])$. However for any $f \in L_c^2(\Re^n)$ there are $b_1,\ldots,b_n$ such that $f \in L_c^2(\prod_{i=1}^{n}[-b_i,b_i])$. This remark enables us to prove the following.

Theorem 2.5. *Let $\mu_1,\ldots,\mu_n$ be n sequences of strictly increasing natural numbers. For $f \in L_c^2(\Re^n)$, the inner product of f against all of the monomials $x_1^{\mu_1(k_1)}\cdot\cdot x_n^{\mu_n(k_n)}$ being zero implies that f is zero if and only if each of the sequences μ_i is a Szasz-Müntz sequence.*

Proof. ($\Leftarrow$) Given $f \in L_c^2(\Re^n)$, choose $b_1,\ldots,b_n$ such that $f \in L^2(\prod_{i=1}^{n}[-b_i,b_i])$. Now apply theorem 2.4.

($\Rightarrow$) Assume that for compactly supported f, $\int x_1^{\mu_1(k_1)} \cdots x_n^{\mu_n(k_n)} f(x) = 0$ implies that $f = 0$. In particular this works for all $f \in L^2(\prod_{i=1}^{n}[-b_i, b_i])$ for any postive real numbers b_i. So we may apply theorem 2.4 for some fixed b_i. □

We now would like to rephrase the preceding result about functions f in $L_c^2(\Re^n)$ to a theorem on Fourier Tranforms. For such f, the Paley-Wiener theorem [10] states that the Fourier Transform $\hat{f}$ of f has an entire extension to n-dimensional complex space. In particular, all partial derivatives of $\hat{f}$ exist, and in fact

$$\begin{aligned}\frac{\partial^{k_1+\ldots+k_n}}{\partial s_1^{k_1}\cdots\partial s_n^{k_n}}\hat{f}(s_1,\ldots,s_n) &= c\int_{\Re^n} x_1^{k_1}\cdots x_n^{k_n} e^{2\pi\imath\vec{s}\cdot\vec{x}} f(x_1,\ldots,x_n)dx_1\ldots dx_n \\ &= c\int_{\Re^n} x_1^{k_1}\cdots x_n^{k_n} h_s(x_1,\ldots,x_n)dx_1\ldots dx_n\end{aligned}$$

where $c = (2\pi\imath)^{k_1+\ldots+k_n}$ and $h_s(x_1,\ldots,x_n) = e^{2\pi\imath\vec{s}\cdot\vec{x}} f(x_1,\ldots,x_n) \in L_c^2(\Re^n)$. It follows from theorem 2.5 that if for some fixed s

$$\frac{\partial^{\mu_1(k_1)+\ldots+\mu_n(k_n)}}{\partial s_1^{\mu_1(k_1)}\cdots\partial s_n^{\mu_n(k_n)}}\hat{f}(s_1,\ldots,s_n) = 0$$

for Szasz-Müntz sequences $\mu_1(k_1),\ldots,\mu_n(k_n)$ then $f = 0$. These comments make up the content of following theorem.

Theorem 2.6. *Let $\mu_1,\ldots,\mu_n$ be n Szasz-Müntz sequences. Suppose that $f \in L_c^2(\Re^n)$, and s is a fixed element of $\mathbf{C^n}$. Then*

$$\frac{\partial^{\mu_1(k_1)+\ldots+\mu_n(k_n)}}{\partial s_1^{\mu_1(k_1)}\cdots\partial s_n^{\mu_n(k_n)}}\hat{f}(s_1,\ldots,s_n) = 0$$

for every $k_1,\ldots,k_n \in \mathbf{N^n}$ implies $f = 0$ if and only if each μ_i is a Szasz-Müntz sequence.

Proof. Here $PW(\Re^n)$ is the image under the Fourier Transform of $L^2_c(\Re^n)$, in other words the L^2 functions with an entire extension to $\mathbf{C}^n$ of exponential type.

Remark Theorem 2.6 can be rephrased in terms of power series as follows: Suppose that $f \in L^2_c(\Re^n)$. Then $\hat{f}$ has an entire extension to $\mathbf{C}^n$, and hence is representable by a power series

$$\hat{f}(z_1, \ldots, z_n) = \sum_{j_1,\ldots,j_n=0}^{\infty} c_{j_1,\ldots,j_n}\, z^1 \cdots z^n$$

where $c_{j_1,\ldots,j_n}$ is a constant multiple of $\frac{\partial^{c_1+\cdots+c_n}}{\partial s_1^{c_1}\cdots\partial s_n^{c_n})}\hat{f}(0,\ldots,0)$. Theorem 2.6 says that if $c_{\mu_1(k_1),\ldots,\mu_n(k_n)} = 0$ for Szasz-Müntz sequences $\mu_1, \ldots, \mu_n$, then $f = 0$.

The advantage to using the coefficients from a power series is that in some cases these ideas can be extended to functions with a Laurent series. Consider the example where $f \in L^2_c(\Re)$. $\hat{f}$ has an entire extension to $\mathbf{C}$, so we may write $\hat{f}$ as a power series $\hat{f}(z) = \sum_{j=0}^{\infty} c_j z^j$, where $c_j = \frac{1}{j!}\hat{f}^{(j)}(0)$. Therefore if $c_{\mu(k)} = 0$ for a Szasz-Müntz sequence μ, then $f = 0$.

Now consider the function $g(z) = \hat{f}\left(\frac{1}{z}\right)$. g has a Laurent series expansion about zero: $g(z) = \sum_{j=-\infty}^{0} d_j z^j$ where $d_j = c_{-j}$ for $j = 0, -1, -2, \ldots$. By the above remarks if $d_{-\mu(k)} = 0$ for a Szasz-Müntz sequence $\mu(k)$, then $g = 0$. So in some cases we can prove a Szasz-Müntz theorem for $\hat{f}$ composed with a rational function.

CHAPTER 3

ALMOST STRONG MALCEV BASIS

In the previous chapter we were able to prove Szasz-Müntz theorems for the compactly supported functions in $L^2(\Re^n)$, and for their transforms: functions of Paley-Wiener type. We would like to extend this theorem in a natural way to connected, simply connected nilpotent Lie groups. In this case the role of the Fourier Transform can be played by the trace transform or the matrix coefficients of the operator valued transform, as functions of the parameterizing functionals. In order to make this extension we will show that in certain classes of nilpotent Lie groups the equivalent of the Fourier Transform, appropriately chosen depending on the class of group being treated, has an entire extension to the complexification of the dual.

One of the goals of this chapter will be to derive a formula for the action of the generic representations on a Hilbert space. In fact what we will show is that the action of π_ℓ has the form

$$(\pi_\ell(x)(f))(y) = e^{2\pi i\left(\sum_{j=1}^{a} \alpha_{i_j}(x,y,\omega)\ell(M_j)\right)} f(\beta_{e_1}(x,y,\omega),\cdots,\beta_{e_c}(x,y,\omega))$$

where a is the dimension of a polarizer for generic ℓ, c is the dimension of a cross-section for the polarizer, and the ω are coefficients of the basis vectors M_j, for the polarizer, in terms of a fixed strong Malcev basis. What's nice about the construction is the form of the polynomials α_{i_j} and β_j. In the first section of this chapter we will construct what we call an *almost strong Malcev basis.* The benefit of using an almost strong Malcev basis through our polarizer is that in most ways this basis acts like a strong Malcev basis, even though the polarizer may not be an ideal. The net effect is that the polynomials α and β that result from factoring the right translation by x above have nice triangularity conditions:

$$\alpha_{i_k}(x, y, \omega) = x_{i_k} + A_{i_k}(x_{i_k+1}, \ldots, x_n, y_{e_s} \text{ with } e_s \geq i_k, \omega)$$

$$\beta_{e_k}(x, y, \omega) = x_{e_k} + y_{e_k} + B_{e_k}(x_{e_k+1}, \ldots, x_n, y_{e_{k+1}}, \ldots, y_{e_c}, \omega)$$

where A and B are polynomials in x, y and ω. The indices i_k and e_k will be explained in the next section.

3.1 Constructing an almost strong Malcev basis

It turns out that the existence of an entire extension is highly dependent on the choice of polarizer and the basis through the polarizer. That brings us to the idea of an almost strong Malcev basis. In this section we will show that given a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$ we can construct a basis for $\mathfrak{g}$ passing through $\mathfrak{h}$ such that the basis acts like a strong Malcev basis in $\mathfrak{h}$ and also as a basis for the cross section $\mathfrak{g}/\mathfrak{h}$.

Throughout this chapter $\mathfrak{G}$ will be a connected, simply connected nilpotent Lie group with corresponding Lie algebra $\mathfrak{g}$. We will start this section with a theorem that is useful in the construction of Malcev bases.

Theorem 3.1.1 *Let $\mathfrak{h}$ be an m-dimensional subalgebra of $\mathfrak{g}$, and $W_1, \ldots, W_n$ be a weak (strong) Malcev basis for $\mathfrak{g}$. Assume that $a_1 < \ldots < a_m$ are m distinct indices* *such that*

$$Y_{a_i} = \sum_{j=1}^{a_i} \alpha^i_j W_j \in \mathfrak{h} \text{ where } (\alpha^i_{a_i} \neq 0).$$

Then $Y_{a_1}, \ldots, Y_{a_m}$ *is a weak(strong) Malcev basis for* $\mathfrak{h}$.

Remark. In our notation, the vector W_1 is the first central vector in the basis $W_1, \ldots, W_n$

Proof. Because $\alpha^i_{a_i} \neq 0$, the Y_{a_i} are linearly independent. That is $\{Y_{a_1}, \ldots, Y_{a_m}\}$ forms a vector space basis for $\mathfrak{h}$. It remains to check that this basis is weak(strong) Malcev. In fact it suffices to check that

$$[Y_{a_i}, Y_{a_j}] \in \Re\text{–span}\{Y_{a_1}, \ldots, Y_{a_d}\}$$

where d=max(min)$\{i, j\}$. We know two things about $[Y_{a_i}, Y_{a_j}]$:

1)$[Y_{a_i}, Y_{a_j}] \in \Re$–span$\{Y_{a_1}, \ldots, Y_{a_m}\}$, since $\{Y_{a_1}, \ldots, Y_{a_m}\}$ is a vector space basis for the Lie algebra $\mathfrak{h}$.

2)$[Y_{a_i}, Y_{a_j}] \in \Re$–span$\{W_1, \ldots, W_p\}$ where p = max (min)$\{a_i, a_j\}$, since we are assuming that $W_1, \ldots, W_n$ is a weak(strong) Malcev basis for $\mathfrak{g}$, and by the definition of the Y_{a_i}.

Now pick coefficients γ_k such that

$$\begin{aligned}[Y_{a_i}, Y_{a_j}] &= \sum_{k=1}^{m} \gamma_k Y_{a_k} \\ &= \sum_{k=1}^{m} \gamma_k (\sum_{s=1}^{a_k} \alpha^k_s W_s).\end{aligned}$$

Since $a_1 < \ldots < a_m$, the coefficient on W_{a_m} is $\gamma_m \alpha^m_{a_m}$. By 2) this coefficent is zero if $a_m > p$, or equivalently if $m > d = \max(\min)\{i,j\}$, . By assumption $\alpha^m_{a_m}$ is nonzero, so we must have $\gamma_m = 0$ if $m > d$. Now proceed inductively. So in the next step the coefficient on $W_{a_{m-1}}$ is $\gamma_{m-1}\alpha^{m-1}_{a_{m-1}}$, and by 2) this coefficent is zero if $a_{m-1} > p$, or equivalently if $m-1 > d$. Conclude that $\gamma_{m-1} = 0$ if $m-1 > d$. From induction we get that $\gamma_k = 0$ if $k > d$. So we have

$$[Y_{a_i}, Y_{a_j}] = \sum_{k \leq d} \gamma_k Y_{a_k}$$

which is what we needed to show. □

Now that we have the preliminary theorem we are ready to start constructing an almost strong Malcev basis for $\mathfrak{g}$ through $\mathfrak{h}$, which we will define when the construction is done. We start with a fixed, but arbitrary, strong Malcev basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$. Choose a (vector space) basis $\{M_1, \ldots, M_a\}$ of $\mathfrak{h}$. Until we explicitly say so we are considering this basis as a linear basis. Now choose highest weight indices $i_1, \ldots, i_a$ such that

$$M_k = \sum_{j=1}^{i_k} \omega^k_j X_j$$

with $\omega^k_{i_k} \neq 0$. Since we are considering our basis as a linear basis we may assume, by rearrangement if necessary, that the collection $\{i_1, \ldots, i_a\}$ is linearly ordered.

In fact we may assume that $i_k < i_{k+1}$, for if $i_k = i_{k+1}$ we may redefine M_k as the new vector

$$\tilde{M_k} = \omega^{k+1}_{i_{k+1}} M_k - \omega^k_{i_k} M_{k+1}.$$

The linear independence of our original basis, and the fact that $\omega^j_{i_j} \neq 0$ for each j, ensures that our newly constructed vectors will be linearly independent. In fact by multiplying by an appropriate scalar we may assume that we have a distinct indices $i_1 < \ldots < i_a$ such that

$$M_k = X_{i_k} + \sum_{j=1}^{i_k - 1} \omega_j^k X_j.$$

Denote by I the indices $I = \{i_1, \ldots, i_a\}$ and by $E = \sim I$ the linearly ordered set, $\{e_1, \ldots, e_c\}$, of indices not in I. Moss [1] calls these indices the *internal* and *external* indices respectively.

We have one final adjustment to our basis, which is handled by the following lemma.

Lemma 3.1.2. *We can choose M_k so that*

$$M_k = X_{i_k} + \sum_{\substack{j=1 \\ j \notin I}}^{i_k - 1} \omega_j^{i_k} X_j = X_{i_k} + \sum_{\substack{j=1 \\ j \in E}}^{i_k - 1} \omega_j^{i_k} X_j = X_{i_k} + \sum_{e_p < i_k} \omega_{e_p}^{i_k} X_{e_p}.$$

Proof. The proof is done by induction. Notice that M_1 is central, and by the way M_1 was chosen above we must have $M_1 = X_1$.

Now assume that $M_1, \ldots, M_{k-1}$ have the form we desire. Then

$$X_{i_j} - M_j \in \Re\text{-span}\{X_{e_s} | e_s \in E\}$$

for $j = 1, \ldots, k-1$, and

$$
\begin{aligned}
M_k &= X_{i_k} + \sum_{j=1}^{i_k-1} \omega_j^k X_j \\
&= X_{i_k} + \sum_{\substack{j=1 \\ j\in I}}^{i_k-1} \omega_j^k X_j + \sum_{\substack{j=1 \\ j\in E}}^{i_k-1} \omega_j^k X_j \\
&= X_{i_k} + \sum_{s=1}^{k-1} \omega_{i_s}^k X_{i_s} + \sum_{\substack{j=1 \\ j\in E}}^{i_k-1} \omega_j^k X_j \\
&= X_{i_k} + \sum_{s=1}^{k-1} \omega_{i_s}^k M_s + \sum_{s=1}^{k-1} \omega_{i_s}^k (X_{i_s} - M_s) + \sum_{\substack{j=1 \\ j\in E}}^{i_k-1} \omega_j^k X_j.
\end{aligned}
$$

So we can take $M_k = X_{i_k} + \sum_{s=1}^{k-1} \omega_{i_s}^k (X_{i_s} - M_s) + \sum_{\substack{j=1 \\ j\in E}}^{i_k-1} \omega_j^k X_j$ without affecting the span of the M_k, and by the remark above

$$
\sum_{s=1}^{k-1} \omega_{i_s}^k (X_{i_s} - M_s) + \sum_{\substack{j=1 \\ j\in E}}^{i_k-1} \omega_j^k X_j \in \Re - \mathrm{span}\{X_{e_s} | e_s \in E\}
$$

which is what we needed. □

We are now in a position to extend our basis to a weak Malcev basis of $\mathfrak{g}$. By theorem 3.1.1 we already know that $\{M_1, \ldots, M_a\}$ is a strong Malcev basis of $\mathfrak{h}$. Notice that in extending to all of $\mathfrak{g}$ weak Malcev is the best we can do since $\mathfrak{h}$ is not necessarily an ideal.

Theorem 3.1.3. *$B = \{M_1, \ldots, M_a, X_{e_1}, \ldots, X_{e_c}\}$ is a weak Malcev basis for $\mathfrak{g}$ passing through $\mathfrak{h}$.*

Proof. The fact that the i_j and e_k are all distinct shows that B is a vector space basis for $\mathfrak{g}$, since $\{X_1, \ldots, X_n\}$ is a basis. It remains to show that the basis is weak Malcev. In fact we already know that $\{M_1, \ldots, M_a\}$ is strong Malcev, so it remains to check that

1)$[X_{e_j}, X_{e_k}] \in \Re\text{–span}\{M_1, \ldots, M_a, X_{e_1}, \ldots, X_{e_d}\}$ where $\mathrm{d} = \max\{\mathrm{j}, \mathrm{k}\}$

2)$[X_{e_j}, M_k] \in \Re\text{–span}\{M_1, \ldots, M_a, X_{e_1}, \ldots, X_{e_j}\}$

We will cover both cases with the following lemma, which actually shows more than we need.

Lemma 3.1.4. *Let $V_k = \sum_{s=1}^{p} \nu_s X_s$ be an arbitrary vector in $\mathfrak{g}$ with p chosen such that $\nu_p \neq 0$. Then*

$$[X_{e_j}, V_k] \in \Re\text{–span}\{M_s \text{ with } i_s \leq \min\{e_j - 1, p-1\} = A,$$

$$X_{e_s} \text{ with } e_s \leq \min\{e_{j-1}, p-1\} = A_1\}.$$

Proof. Since $\{X_1, \ldots, X_n\}$ is a strong Malcev basis, we know that

$$[X_{e_j}, V_k] \in \Re\text{–span}\{X_s \text{ with } s \leq \min\{e_j - 1, p-1\}\}$$

$$= \Re\text{–span}\{X_s \text{ with } s \leq A\}. \tag{1}$$

We also know that B is a linear basis of $\mathfrak{g}$. So we may choose coefficients α_s, β_s so that

$$[X_{e_j}, V_k] = \sum_{s=1}^{a} \alpha_s M_s + \sum_{t=1}^{c} \beta_t X_{e_t}$$

$$= \sum_{s=1}^{a} \alpha_s (X_{i_s} + \sum_{e_p < i_s} \omega_{e_p}^{i_s} X_{e_p}) + \sum_{t=1}^{c} \beta_t X_{e_t}.$$

Now the coefficient on X_{i_s} is α_s. By 1, if $i_s > A$ we must have $\alpha_s = 0$. So

$$[X_{e_j}, V_k] = \sum_{i_s \leq A} \alpha_s M_s + \sum_{t=1}^{c} \beta_t X_{e_t}.$$

Now for $t > A$, the coefficient on X_{e_t} is β_t. Again by 1, for such t we must have $\beta_t = 0$. Thus

$$\begin{aligned}[X_{e_j}, V_k] &= \sum_{i_s \le A}^{a} \alpha_s M_s + \sum_{e_t \le A}^{c} \beta_t X_{e_t} \\ &= \sum_{i_s \le A}^{a} \alpha_s M_s + \sum_{e_t \le A_1}^{c} \beta_t X_{e_t}.\end{aligned}$$

In the last line we may change A to A_1 since $e_s \le e_j - 1$ if and only if $s \le j - 1$. This is what we wanted to show. □

We have proven the following theorem.

Theorem 3.1.5 *Let $\{X_1, \dots, X_n\}$ be a strong Malcev basis for the Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be an a dimensional subalgebra of $\mathfrak{g}$. Then there exist two disjoint collections of indices $I = \{i_1, \dots, i_a\}$ and $E = \{e_1, \dots, e_c\}$ along with vectors $\{M_1, \dots, M_a\}$ such that*

1)$a + c = n$.

2)$M_s = X_{i_s} + \sum_{e_p < i_s} \omega_{e_p}^{i_s} X_{e_p}$.

3)$\{M_1, \dots, M_a\}$ is a strong Malcev basis for $\mathfrak{h}$.

4)$\{M_1, \dots, M_a, X_{e_1}, \dots, X_{e_c}\}$ is a weak Malcev basis for $\mathfrak{g}$ through $\mathfrak{h}$.

5)$[X_{e_s}, X_{e_t}] \in \Re\text{–span}\{M_1, \dots, M_a, X_{e_w} \text{ with } e_w \le \min\{e_{s-1}, e_{t-1}\}\}$.

Definition 3.1.6. *An almost strong Malcev basis for $\mathfrak{g}$ passing through $\mathfrak{h}$ will be a basis satisfying the conditions of theorem 3.1.5.*

Theorem 3.1.7. *Let $\mathfrak{h}$ be an a-dimensional subalgebra of $\mathfrak{g}$. The almost strong Malcev basis in Definition 3.1.6 is unique up to the choice of strong Malcev basis $\{X_1, \dots, X_n\}$.*

Proof. Suppose that there are two bases $P = \{M_1^P, \ldots, M_a^P, X_{e_1}^P, \ldots, X_{e_c}^P\}$ and $Q = \{M_1^Q, \ldots, M_a^Q, X_{e_1}^Q, \ldots, X_{e_c}^Q\}$ satisfying the hypothesis of theorem 3.1.5.

Claim. To show that $P = Q$ it suffices to show that the internal indices, $I^P = \{i_1^P, \ldots, i_a^P\}$, $I^Q = \{i_1^Q, \ldots, i_a^Q\}$, for these two bases are the same.

Proof of claim. Suppose that $I^P = I^Q$. By taking set theoretic complements $E^P = E^Q$. Therefore $X_{e_j}^P = X_{e_j}^Q$ for $j = 1, \ldots, c$, so we may drop the superscript on these vectors. Since the internal indices are the same, by property 2 of almost strong bases $M_k^P - M_k^Q \in \Re\text{–span}\{X_{e_j} | e_j \in E\}$. Of course we also have $M_k^P - M_k^Q \in \mathfrak{h}$, and by the linear independence of the vectors $\{M_1^P, \ldots, M_a^P, X_{e_1}^P, \ldots, X_{e_c}^P\}$ we must have $M_k^P - M_k^Q = 0$. □

The proof that $I^P = I^Q$ proceeds by induction. So consider i_1^P, i_1^Q. We may as well assume that $i_1^P < i_1^Q$. By the definition of almost strong Malcev basis, $i_1^P < i_1^Q < i_2^Q < \ldots < i_a^Q$. If this were true the vectors $\{M_1^P, M_1^Q, \ldots, M_a^Q\} \subset \mathfrak{h}$ would be linearly independent, which contradicts $\dim(\mathfrak{h}) = a$. Now proceed with the induction step.

Assume that $i_k^P = i_k^Q$ for $k < j$, we must show that $i_j^P = i_j^Q$. Suppose, for instance, that $i_j^P < i_j^Q$. We know that $i_j^P > i_{j-1}^P = i_{j-1}^Q$ by the definition of almost strong Malcev basis and by the induction hypothesis. So $i_1^Q < \ldots < i_{j-1}^Q < i_j^P < i_j^Q < \ldots < i_a^Q$. It follows from property 2 of almost strong Malcev basis that $\{M_1^Q, \ldots, M_{j-1}^Q, M_j^P, M_j^Q, \ldots, M_a^Q\}$ are linearly independent, which is impossible since the dimension of $\mathfrak{h}$ is a. □

Corollary 3.1.8. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with strong Malcev basis $\{X_1, \ldots, X_n\}$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ and $\{M_1, \ldots, M_a, X_{e_1}, \ldots, X_{e_c}\}$ the almost strong Malcev basis for $\mathfrak{g}$ through $\mathfrak{h}$. If a vector $X \in \mathfrak{h}$ has the form*

$$X = X_j + \sum_{e_p < j} \alpha_{e_p} X_{e_p},$$

then $X = M_k$ for some k.

Proof. For each $1 \leq k \leq a$ express M_k in terms of the original strong Malcev basis,

$$M_k = X_{i_k} + \sum_{e_p < i_k} \omega_{e_p}^{i_k} X_{e_p}.$$

We must have $j = i_k$ for some k, for otherwise X would be linearly independent from all of the M_k. Therefore $X - M_k \in \mathfrak{h} \bigcap \Re - \mathrm{span}\{X_{e_p} | e_p \in E\} = \{0\}$. □

Example 3.1.9. Consider the 4-step chain algebra $\mathfrak{k}_4$ spanned by the vectors $\{X_1, X_2, X_3, X_4, X_5\}$ with nonzero brackets generated by the relations:

$$[X_5, X_4] = X_3$$

$$[X_5, X_3] = X_2$$

$$[X_5, X_2] = X_1.$$

Let $\mathfrak{h}$ be the three dimensional abelian subalgebra spanned by the vectors

$$\{M_1 = X_1, M_2 = X_4 + X_2 + X_1, M_3 = X_4 + X_3 - X_2 + X_1\}.$$

Notice that this is not part of an almost strong Malcev basis since M_2 and M_3 both have highest weight index 4. The almost strong Malcev basis for $\mathfrak{g}$ through $\mathfrak{h}$ is

$$\{\tilde{M}_1, \tilde{M}_2, \tilde{M}_3, X_{e_1}, X_{e_2}\} = \{X_1, X_3 - 2X_2, X_4 + X_2, X_2, X_5\}$$

with internal indices $I = \{1, 3, 4\}$ and external indices $E = \{2, 5\}$.

3.2 Calculating the unitary irreducible representations

Now that we have almost strong Malcev bases we would like to calculate the action of a unitary irreducible representation induced from a polarizing subalgebra in these bases. In this section we will show that the action of such a representation has the form outlined in the beginning of this chapter.

So fix $\ell = \sum_{j=1}^{n} \ell_j X_j^* \in \mathfrak{g}^*$, a polarizer m for ℓ, and let $M = exp(m)$, the Lie subgroup of $\mathfrak{G}$ corresponding to m. Then π_ℓ acts on the Hilbert space H_{π_ℓ} by right translation:

$$\pi_\ell(x)f(y) = f(yx)$$

for $x, y \in \mathfrak{G}$. Here H_{π_ℓ} is the collection of functions f with $|f| \in L^2(M \setminus \mathfrak{G})$ such that

$$f(ab) = \chi_\ell(a)f(b) = e^{2\pi i\ell(log(a))}f(b)$$

for every $a \in M$ and $b \in \mathfrak{G}$. Notice that it is sufficient to calculate $\pi_\ell(x)f(y)$ for y coming from a cross-section of $M \setminus \mathfrak{G}$.

The calculation of π_ℓ amounts to factoring an arbitrary product yx, $x \in \mathfrak{G}$, y in a cross section for $M \setminus \mathfrak{G}$, into a product $\alpha\beta$ where $\alpha \in M$ and β comes from the cross-section for $M \setminus \mathfrak{G}$. This calculation is made easier using an almost strong Malcev basis.

Let $\{M_1, \ldots, M_a, X_{e_1}, \ldots, X_{e_c}\}$ be the almost strong Malcev basis for $\mathfrak{g}$ passing through m with respect to the fixed strong Malcev basis $\{X_1, \ldots, X_n\}$. Here

$$M_k = X_{i_k} + \sum_{e_p < i_k} \omega_{e_p}^{i_k} X_{e_p}$$

as in theorem 3.1.5. We will use $\{exp(t_{e_1}X_{e_1})\cdots exp(t_{e_c}X_{e_c})|t \in \Re^c\}$ for the cross-section of $M \setminus \mathfrak{G}$.

Let $x = exp(x_1X_1)\cdots exp(x_nX_n) \in \mathfrak{G}$ and $y = exp(y_{e_1}X_{e_1})\cdots exp(y_{e_c}X_{e_c})$ in the cross-section. Choose the unique $\alpha = exp(\alpha_{i_1}M_1)\cdots(\alpha_{i_a}M_a) \in M$ and $\beta = exp(\beta_{e_1}X_{e_1})\cdots exp(\beta_{e_c}X_{e_c})$ in the cross-section such that $yx = \alpha\beta$.

We need to calculate α_{i_j} and β_{e_k} in terms of the coordinates x_s and y_t. First we calculate yx. $\{X_1, \ldots, X_n\}$ is a strong Malcev basis, so there are polynomials P_j and $\tilde{P}_j$ such that

$$\begin{aligned} x &= exp(x_1X_1)\cdots exp(x_nX_n) \\ &= exp\left(\sum_{j=1}^{n} P_j(x_j, \ldots, x_n)X_j\right) \end{aligned}$$

and

$$\begin{aligned} y &= exp(y_{e_1}X_{e_1})\cdots exp(y_{e_c}X_{e_c}) \\ &= exp\left(\sum_{k=1}^{n} P_k(y_{e_j} \text{ with } \mathrm{e_j} \geq \mathrm{k})\mathrm{X_k}\right) \end{aligned}$$

where $P_k(z_k, \ldots, z_n) = z_k + \tilde{P}_k(z_{k+1}, \ldots, z_n)$. *It is convenient to abuse notation:* $P_k(y_{e_j} \text{ with } \mathrm{e_j} \geq \mathrm{k})$ *is understood to have* $n - e_k + 1$ *variables, with the coordinates corresponding to internal indices fixed at 0.* Putting this together with the Campbell-Baker-Hausdorff formula we have

$$yx = exp\left(\sum_{k=1}^{n} P_k(y_{e_j} \text{ with } \mathrm{e_j} \geq \mathrm{k})\mathrm{X_k}\right) * exp\left(\sum_{j=1}^{n} P_j(x_j, \ldots, x_n)X_j\right)$$

$$
\begin{aligned}
=&exp\Bigg(\sum_{k=1}^{n} P_k(y_{e_j} \text{ with } e_j \geq k)X_k + \sum_{j=1}^{n} P_j(x_j,\ldots,x_n)X_j + \\
&\qquad \frac{1}{2}\Big[\sum_{k=1}^{n} P_k(y_{e_j} \text{ with } e_j \geq k)X_k, \sum_{j=1}^{n} P_j(x_j,\ldots,x_n)X_j\Big] + \ldots\Bigg) \\
=&exp\left(\sum_{k=1}^{n} p_k(x,y)X_k\right)
\end{aligned}
$$

where

$$
p_k(x,y) = \begin{cases} x_{i_j} + \tilde{p}_{i_j}(x_{i_j+1},\ldots,x_n,y_{e_s} \text{ with } e_s \geq i_j) & \text{if } k = i_j \text{ for some j;} \\ x_{e_j} + y_{e_j} + \tilde{p}_k(x_{e_j+1},\ldots,x_n,y_{e_{j+1}},\ldots,y_{e_c}) & \text{if } k = e_j \text{ for some j.} \end{cases}
$$

Now that we have computed yx, we need to compute $\alpha\beta$. Since $\{M_1,\ldots,M_a\}$ is a strong Malcev basis for M, there are polynomials K_j and $\tilde{K}_j$ such that

$$
\begin{aligned}
\alpha &= exp\,(\alpha_{i_1}M_1)\cdots exp\,(\alpha_{i_a}M_a) \\
&= exp\left(\sum_{j=1}^{a} K_{i_j}(\alpha_{i_j},\ldots,\alpha_{i_a})M_j\right)
\end{aligned}
$$

with $K_{i_j}(\alpha_{i_j},\ldots,\alpha_a) = \alpha_{i_j} + \tilde{K}_{i_j}(\alpha_{i_{j+1}},\ldots,\alpha_{i_a})$. Since β comes from the cross-section

$$
\begin{aligned}
\beta &= exp\,(\beta_{e_1}X_{e_1})\cdots exp\,(\beta_{e_c}X_{e_c}) \\
&= exp\left(\sum_{k=1}^{n} P_k(\beta_{e_j} \text{ with } e_j \geq k)X_k\right).
\end{aligned}
$$

Fix α and β for the moment and let $K_{i_j} = K_{i_j}(\alpha_{i_j},\ldots,\alpha_{i_a})$ and

$P_k = P_k(\beta_{e_j} \text{ with } e_j \geq k)$. Then

$$
\begin{aligned}
\alpha\beta =& exp\left(\sum_{j=1}^{a} K_{i_j}M_j\right) * exp\left(\sum_{k=1}^{n} P_kX_k\right) \\
=& exp\left(\sum_{j=1}^{a} K_{i_j}M_j + \sum_{k=1}^{n} P_kX_k + \frac{1}{2}\Big[\sum_{j=1}^{a} K_{i_j}M_j, \sum_{k=1}^{n} P_kX_k\Big] + \ldots\right)
\end{aligned}
$$

$$
\begin{aligned}
&= exp\Bigg(\sum_{j=1}^{a} K_{i_j}(X_{i_j} + \sum_{e_p<i_j} \omega_{e_p}^{i_j} X_{e_p}) + \sum_{k=1}^{n} P_k X_k + \\
&\qquad \frac{1}{2}\Big[\sum_{j=1}^{a} K_{i_j}(X_{i_j} + \sum_{e_p<i_j} \omega_{e_p}^{i_j} X_{e_p}), \sum_{k=1}^{n} P_k X_k\Big] + \dots\Bigg) \\
&= exp\Bigg(\sum_{j=1}^{a} K_{i_j} X_{i_j} + \sum_{k=1}^{n} P_k X_k + \sum_{j=1}^{a} K_{i_j}\Big(\sum_{e_p<i_j} \omega_{e_p}^{i_j} X_{e_p}\Big) + \\
&\qquad \frac{1}{2}\Big[\sum_{j=1}^{a} K_{i_j}(X_{i_j} + \sum_{e_p<i_j} \omega_{e_p}^{i_j} X_{e_p}), \sum_{k=1}^{n} P_k X_k\Big] + \dots\Bigg) \\
&= exp\Bigg(\sum_{j=1}^{n} Q_j(K, P, \omega) X_j\Bigg).
\end{aligned}
$$

Since $\{X_1, \dots, X_n\}$ is a strong Malcev basis, by the Campbell-Baker-Hausdorff formula we know that Q_j is polynomial in K,P and ω, and that there exist polynomials $\tilde{Q}_s$ such that

$$
Q_j(K,P,\omega) = \begin{cases} K_{i_k} + P_{i_k} + \tilde{Q}_{i_k}(K_{i_{k+1}}, \dots, K_{i_a}, P_{i_k+1}, \dots, P_n, \\ \qquad \omega_{e_p}^{i_q} \text{ where } \mathrm{i_q > e_p > i_k}) & \text{if } j = i_k \text{ for some k;} \\ P_{e_k} + \tilde{Q}_{e_k}(K_{i_p} \text{with } \mathrm{i_p > e_k, P_{e_k+1}, \dots, P_n}, \\ \qquad \omega_{e_p}^{i_q} \text{ where } \mathrm{i_q > e_p \geq e_k}) & \text{if } j = e_k \text{ for some k.} \end{cases}
$$

Define the polynomials $q_j(\alpha,\beta,\omega), \tilde{q}_j(\alpha,\beta,\omega)$ by the equations

$$
\begin{aligned}
&q_j(\alpha,\beta,\omega) \\
&= Q_j(K(\alpha), P(\beta), \omega) \\
&= \begin{cases} K_{i_k}(\alpha) + P_{i_k}(\beta) + \tilde{Q}_{i_k}(K_{i_{k+1}}(\alpha), \dots, K_{i_a}(\alpha), P_{i_k+1}(\beta), \dots, P_n(\beta), \\ \qquad \omega_{e_p}^{i_q} \text{ where } \mathrm{i_q > e_p > i_k}) & \text{if } j = i_k \text{ for some k;} \\ P_{e_k}(\beta) + \tilde{Q}_{e_k}(K_{i_p}(\alpha) \text{with } \mathrm{i_p > e_k, P_{e_k+1}}(\beta), \dots, \mathrm{P_n}(\beta), \\ \qquad \omega_{e_p}^{i_q} \text{ where } \mathrm{i_q > e_p \geq e_k}) & \text{if } j = e_k \text{ for some k;} \end{cases}
\end{aligned}
$$

$$
= \begin{cases} \alpha_{i_k} + \tilde{q}_{i_k}(\alpha_{i_{k+1}}, \dots, \alpha_{i_a}, \beta_{i_k+1}, \dots, \beta_n, \omega_{e_p}^{i_q} \text{ where } \mathrm{i_q > e_p > i_k}) \\ \qquad \text{if } j = i_k \text{ for some k;} \\ \beta_{e_k} + \tilde{q}_{e_k}(\alpha_{i_p} \text{with } \mathrm{i_p > e_k}, \beta_{e_k+1}, \dots, \beta_n, \omega_{e_p}^{i_q} \text{ where } \mathrm{i_q > e_p \geq e_k}) \\ \qquad \text{if } j = e_k \text{ for some k.} \end{cases}
$$

The last equality is from what we already know about K and P. Since α and β where chosen so that $yx = \alpha\beta$, we must have

$$p_k(x,y) = q_k(\alpha,\beta,\omega)$$

for $k = 1,\ldots,n$. Use this to solve for α and β.

$$\alpha_{i_k} = x_{i_k} + \tilde{p}_{i_k}(x_{i_k+1},\ldots,x_n,y_{e_s} \text{ with } e_s \geq i_k) -$$

$$\tilde{q}_{i_k}(\alpha_{i_{k+1}},\ldots,\alpha_{i_a},\beta_{i_k+1},\ldots,\beta_n,\omega_{e_p}^{i_q} \text{ where } i_q > e_p > i_k)$$

$$\beta_{e_k} = x_{e_k} + y_{e_k} + \tilde{p}_k(x_{e_k+1},\ldots,x_n,y_{e_{k+1}},\ldots,y_{e_c}) -$$

$$\tilde{q}_{e_k}(\alpha_{i_p} \text{ with } i_p > e_k, \beta_{e_k+1},\ldots,\beta_n,\omega_{e_p}^{i_q} \text{ where } i_q > e_p \geq e_k).$$

By the triangular dependencies of the $\tilde{q}$ there are polynomials A and B such that

$$\alpha_{i_k}(x,y,\omega) = x_{i_k} + A_{i_k}(x_{i_k+1},\ldots,x_n,y_{e_s} \text{ with } e_s \geq i_k, \omega_{e_p}^{i_q} \text{ where } i_q > e_p > i_k)$$

$$\beta_{e_k}(x,y,\omega) = x_{e_k} + y_{e_k} + B_{e_k}(x_{e_k+1},\ldots,x_n,y_{e_{k+1}},\ldots,y_{e_c},\omega_{e_p}^{i_q} \text{ where } i_q > e_p \geq e_k).$$

Now that we have this calculation

$$\begin{aligned}
\pi_\ell(x)f(y) =& f(yx)\\
=&\chi_\ell(\alpha(x,y,\omega))f(\beta(x,y,\omega))\\
=&\chi_\ell\big(exp(\alpha_{i_1}(x,y,\omega)M_1),\ldots,exp(\alpha_{i_a}(x,y,\omega)M_a)\big)\\
&f\big(exp(\beta_{e_1}(x,y,\omega)X_{e_1}),\ldots,exp(\beta_{e_c}(x,y,\omega)X_{e_c})\big)\\
=&\chi_\ell\big(exp(\alpha_{i_1}(x,y,\omega)M_1)\big)\cdots\chi_\ell\big(exp(\alpha_{i_a}(x,y,\omega)M_a)\big)\\
&f\big(exp(\beta_{e_1}(x,y,\omega)X_{e_1}),\ldots,exp(\beta_{e_c}(x,y,\omega)X_{e_c})\big)\\
=&e^{2\pi i\left(\sum_{j=1}^{a}\alpha_{i_j}(x,y,\omega)\ell(M_j)\right)} f\big(exp(\beta_{e_1}(x,y,\omega)X_{e_1}),\ldots,exp(\beta_{e_c}(x,y,\omega)X_{e_c})\big).
\end{aligned}$$

We have proven the following:

Theorem 3.2.1. *Let $\mathfrak{g}$ be a connected, simply connected nilpotent Lie algebra with strong Malcev basis $\{X_1, \ldots, X_n\}$. Let $\ell \in \mathfrak{g}^*$ and let m be a polarizer for ℓ. Finally let $\{M_1, \ldots, M_a, X_{e_1}, \ldots, X_{e_c}\}$ be the almost strong Malcev basis for $\mathfrak{g}$ through m. If we use this basis to calculate the action of π_ℓ on $L^2(\Re^c)$, then that action is given by the following formula:*

$$\pi_\ell(x)f(y)$$
$$= e^{2\pi i\left(\sum_{j=1}^{a} \alpha_{i_j}(x,y,\omega)\ell(M_j)\right)} f\left(exp(\beta_{e_1}(x,y,\omega)X_{e_1}), \ldots, exp(\beta_{e_c}(x,y,\omega)X_{e_c})\right)$$

where the polynomials α_{i_j} and β_{e_j} have the form

$$\alpha_{i_k}(x,y,\omega) = x_{i_k} + A_{i_k}(x_{i_k+1}, \ldots, x_n, y_{e_s} \text{ with } e_s > i_k, \omega_{e_p}^{i_q} \text{ where } i_q > e_p > i_k)$$

$$\beta_{e_k}(x,y,\omega) = x_{e_k} + y_{e_k} + B_{e_k}(x_{e_k+1}, \ldots, x_n, y_{e_{k+1}}, \ldots, y_{e_c}, \omega_{e_p}^{i_q} \text{ where } i_q > e_p \geq e_k)$$

for some polynomials A, B.

Notice that this equation for π_ℓ has the form outlined in the beginning of this chapter. It is known that the coefficients ω for the basis vectors of m can be chosen so that they are rational functions of ℓ. Theorem 3.2.1 shows then that the action of π_ℓ depends rationally on ℓ. In the next section we will make use of theorem 3.2.1 when $\ell \in U \bigcap V_T$ the set of parameterizing functionals, and there is one fixed polarizer m for all such ℓ. In this case we will not need to keep track of the coefficients ω.

In the next example we compute the action of π_ℓ in two different polarizers. One polarizer is fixed, and the other one rotates with ℓ. We will be using this example in several places to compare the resulting representations.

Example 3.2.2. Consider the 9-dimensional algebra $\mathfrak{g} = \Re\text{–span}\{X_1, \ldots, X_9\}$ with non-zero bracket relations

$$[X_9, X_5] = X_3 \qquad [X_9, X_4] = X_3 \qquad [X_9, X_8] = -X_6$$

$$[X_9, X_7] = -X_2 \quad [X_8, X_3] = -X_2 \quad [X_5, X_4] = -X_1$$

$$[X_6, X_5] = -X_2 \quad [X_6, X_4] = -X_2$$

and center $\mathfrak{z}(\mathfrak{g}) = \Re\text{–span}\{X_1, X_2\}$. Let $\ell = \sum_{i=1}^{9} \ell_i X_i^*$. Then $U = \{\ell \,|\, \ell_1, \ell_2 \neq 0\}$ are the generic ℓ, and for generic ℓ the radical, r_ℓ, of ℓ is

$$r_l = \Re\text{–span}\{X_1, X_2, \ell_1 X_6 - \ell_2 X_5 + \ell_2 X_4\}.$$

We will make use of two different polarizers for this example:

$$m = \Re\text{–span}\{X_1, X_2, X_3, X_5 - X_4, X_6, X_7\}$$

and

$$m_\ell = \Re\text{–span}\{X_1, X_2, X_3, X_4, \ell_1 X_6 - \ell_2 X_5, X_7\}$$

which rotates with ℓ. For the fixed polarizer the external vectors are $\{X_4, X_8, X_8\}$, and for the rotating polarizers the external vectors are $\{X_5, X_8, X_9\}$.

We will use π_ℓ and $\bar{\pi}_\ell$ to denote the actions of the representations in the fixed polarizer and rotating polarizer respectively. As might be expected the action of π_ℓ is nicer than the action of $\bar{\pi}_\ell$, in the sense that π_ℓ depends polynomially on ℓ while $\bar{\pi}_\ell$ depends rationally on ℓ. The dependencies are made clearer by the following formulas for the actions of π_ℓ and $\bar{\pi}_\ell$:

$$
\begin{aligned}
&\pi_\ell(y)f(x_4,x_8,x_9)\\
&=\pi_\ell\Big(exp\,(y_1X_1)\cdots exp\,(y_9X_9)\Big)f\,(exp\,(x_4X_4)\,exp\,(x_8X_8)\,exp\,(x_9X_9))\\
&=e^{2\pi i\ell_1\left(y_1+y_5(x_4+y_4+\frac{1}{2}y_5)\right)}\\
&\cdot e^{2\pi i\ell_2\left(y_2-x_8y_3+y_6(y_4+y_5+x_4)-x_9\left(y_7+x_4y_8+(x_8+y_8)(y_4+y_5)\right)\right)}\\
&\cdot e^{2\pi i\left(\ell_3\left(y_3+x_9(y_4+y_5)\right)-\ell_4y_5+\ell_5y_5+\ell_6\left(y_6-x_9y_8\right)+\ell_7y_7\right)}\\
&\cdot f\left(x_4+y_4+y_5,x_8+y_8,x_9+y_9\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\bar\pi_\ell(y)f(x_5,x_8,x_9)\\
&=\bar\pi_\ell\Big(exp\,(y_1X_1)\cdots exp\,(y_9X_9)\Big)f\,(exp\,(x_5X_5)\,exp\,(x_8X_8)\,exp\,(x_9X_9))\\
&=e^{2\pi i\ell_1\left(y_1-x_5y_4\right)}\\
&\cdot e^{2\pi i\ell_2\left(y_2-x_8y_3-x_9y_7+x_5y_6+y_5y_6-x_8x_9y_4-x_8x_9y_5-x_9y_5y_8-x_5x_9y_8\right)}\\
&\cdot e^{2\pi i\left(\ell_3\left(y_3+x_9(y_4+y_5)\right)+\ell_4y_4+\ell_6\left(y_6-x_9y_8\right)+\ell_7y_7\right)}\\
&\cdot e^{2\pi i\frac{\ell_2}{\ell_1}(y_6-x_9y_8)\left(\frac{\ell_2}{2}(y_6-x_9y_8)-\ell_5\right)}\\
&\cdot f\big(\frac{\ell_2}{\ell_1}(y_6-x_9y_8)+x_5+y_5,x_8+y_8,x_9+y_9\big).
\end{aligned}
$$

3.3 The parameterizing representations

As in section 3.1 we will start with a fixed strong Malcev basis $\{X_1,\ldots,X_n\}$ of the Lie algebra $\mathfrak{g}$. With respect to this basis we will let U denote the collection of generic orbits and $S=\{s_1,\ldots,s_o\}$, $T=\{t_1,\ldots,t_r\}$ will denote the collection of jump and non-jump indices respectively. Then o is the dimension of the generic orbits and r is the dimension of the radical r_ℓ for generic $\ell \in \mathfrak{g}^*$. If we let $V_T = \{X^*_{t_i} | t_i \in T\}$, then $U \bigcap V_T$ is the collection of parameterizing functionals.

In the previous section we showed that

$$\pi_{\ell_T}(x)f(y)$$
$$= e^{2\pi i\left(\sum_{j=1}^{a} \alpha_{i_j}(x,y,\omega)\ell_T(M_j)\right)} f\left(exp(\beta_{e_1}(x,y,\omega)X_{e_1}, \ldots, exp(\beta_{e_c}(x,y,\omega)X_{e_c}\right).$$

Here $\{M_1, \ldots, M_c, X_{e_1}, \ldots, X_{e_c}\}$ is an almost strong Malcev basis for $\mathfrak{g}$ through a polarizer m for ℓ, and the ω are coefficients for the M_j in terms of the original strong Malcev basis: $M_j = X_{i_j} + \sum_{e_p < i_j} \omega^{i_j}_{e_p} X_{e_p}$.

We will assume that there is one fixed polarizer m for all $\ell_T \in U \bigcap V_T$. In this case the coefficients ω are fixed, and so we may rewrite the equation for π_{ℓ_T} as

$$\pi_{\ell_T}(x)f(y)$$
$$= e^{2\pi i\left(\sum_{j=1}^{a} \alpha_{i_j}(x,y)\ell_T(M_j)\right)} f\left(exp(\beta_{e_1}(x,y)X_{e_1}), \ldots, exp(\beta_{e_c}(x,y)X_{e_c})\right).$$

It turns out that when you pass an almost strong Malcev basis through a polarizer for a generic representation, then the set T of non-jump indices must be contained in the set of internal indices I, which is what the next lemma shows.

Lemma 3.3.1. *For each $t_i \in T$ there is a vector $Y_{t_i} \in \Re$–span$\{X_{e_k} | e_k < t_i\}$ such that $X_{t_i} + Y_{t_i}$ is an element of our almost strong Malcev basis B through m.*

Proof. Let $B = \{M_1, \ldots, M_a, X_{e_1}, \ldots, X_{e_c}\}$ be the almost strong Malcev basis for $\mathfrak{g}$ through m and fix a parameterizing functional $\ell \in U \bigcap V_T$. Then for each $t_j \in T$

we may choose a vector $Y_{t_j} \in \Re\text{-span}\{X_1, \ldots, X_{t_j-1}\}$ such that $X_{t_j} + Y_{t_j} \in r_\ell$. We may do this since t_j is a non-jump index and ℓ is generic.

Define the projection map $P : \mathfrak{g} \to \Re\text{-span}\{X_{e_1}, \ldots, X_{e_c}\}$ with respect to the almost strong Malcev basis B.

Finally let $\tilde{Y}_{t_j} = P(Y_{t_j})$, so that $Y_{t_j} = \tilde{Y}_{t_j} + M$ for some vector $M \in m$. That is we still have

$$X_{t_j} + \tilde{Y}_{t_j} \in m$$

and this vector has the correct form to be part of a strong Malcev basis. By uniqueness of almost strong Malcev basis, and Corollary 3.1.8, $X_{t_j} + \tilde{Y}_{t_j} = M_k$ for some k. □

Remark. Lemma 3.3.1 shows that the non-jump indices are contained in the internal indices for the almost strong Malcev basis. By taking set theoretic compliments we can see that the external indices for the basis are always jump indices. Notice that Lemma 3.3.1 does not depend on the polarizer being fixed. So the external indices will always be jump indices, and the non-jump indices will always be internal indices, even if the polarizers rotate with ℓ.

These comments along with our calculated action of π_{ℓ_T} allows us to prove.

Theorem 3.3.2 *Suppose that there is one fixed polarizer for the parameterizing functionals $\ell_T = \sum_{j=1}^{r} \ell_{t_j} X_{t_j}^* \in U \bigcap V_T$. Then π_{ℓ_T} can be modeled in a fixed* *modeling space* $L^2(\Re^c) \cong L^2(exp(\Re X_{e_1}), \cdots, exp(\Re X_{e_c}))$*, where the action of* π_{ℓ_T} *is given by:*

$$\begin{aligned}
&\pi_{\ell_T}(x)f(y)\\
&= \pi_{\ell_T}\big(exp(x_1X_1), \ldots, exp(x_nX_n)\big)f\big(exp(y_{e_1}X_{e_1}), \cdots, exp(y_{e_c}X_{e_c})\big)\\
&= e^{2\pi i\left(\sum_{j=1}^{r} \ell_{t_j}\left(x_{t_j}+A_{t_j}(x_{t_j+1},\ldots,x_n,y_k \text{ with } e_k>t_j)\right)\right)} f\big(\beta_{e_1}(x,y), \ldots, \beta_{e_c}(x,y)\big)
\end{aligned}$$

where $\beta_{e_k}(x,y) = x_{e_k} + y_{e_k} + B_{e_k}(x_{e_k+1}, \ldots, x_n; y_{e_{k+1}}, \ldots, y_{e_c})$ *and* A_{t_j}, B_{e_k} *are polynomial in* x, y.

Proof. Notice that

$$\begin{aligned}
\ell_T(M_j) =&\ell_T\Big(X_{i_j} + \sum_{e_p<i_j} \omega^{i_j}_{e_p} X_{e_p}\Big)\\
=&\ell_T\Big(X_{i_j}\Big) + \ell_T\Big(\sum_{e_p<i_j} \omega^{i_j}_{e_p} X_{e_p}\Big).
\end{aligned}$$

Now from the remarks following lemma 3.3.1, $E \subset S$. Since ℓ_T is in the span of the vectors $\{X^*_{t_i} | t_i \in T\}$, $\ell_T\Big(\sum_{e_p<i_j} \omega^{i_j}_{e_p} X_{e_p}\Big) = 0$.

So

$$\begin{aligned}
\ell_T(M_j) =&\ell_T\Big(X_{i_j} + \sum_{e_p<i_j} \omega^{i_j}_{e_p} X_{e_p}\Big)\\
=&\ell_T\Big(X_{i_j}\Big)\\
=&\begin{cases} 0 & \text{if } i_j \notin T;\\ \ell_{t_k} & \text{if } i_j = t_k \text{ for some k.}\end{cases}
\end{aligned}$$

If we combine this with theorem 3.2.1 and lemma 3.3.1 we get:

$$\begin{aligned}
\pi_\ell(x)f(y) =&e^{2\pi i\left(\sum_{j=1}^{a} \alpha_{i_j}(x,y)\ell(M_j)\right)} f\big(exp(\beta_{e_1}(x,y)X_{e_1}, \ldots, exp(\beta_{e_c}(x,y)X_{e_c}\big)\\
=&e^{2\pi i\left(\sum_{k=1}^{r} \ell_{t_k}\alpha_{t_k}(x,y)\right)} f\big(exp(\beta_{e_1}(x,y)X_{e_1}, \ldots, exp(\beta_{e_c}(x,y)X_{e_c}\big).
\end{aligned}$$

This combined with our knowledge of the polynomials α and β from theorem 3.2.1 proves the theorem. This is the form we would expect from a representation induced from a polarizer that is also an ideal, so our use of almost strong Malcev basis paid off. □

Example 3.2.2 continued. The non-jump indices are those indices t_k, $k \in T$, such that for each generic ℓ there exists a vector $Y_{t_k}(\ell) \in \Re\text{-span}\{X_1, \ldots, X_{t_k-1}\}$ with $X_{t_k} + Y_{t_k}(\ell) \in r_\ell$. From our previous calculations we see that the non-jump indices are $T = \{t_1, t_2, t_3\} = \{1, 2, 6\}$, and the jump indices are $S = \sim T = \{s_1, \ldots, s_6\} = \{3, 4, 5, 7, 8, 9\}$.

Therefore parameterizing functionals are of the form $\ell_T = \ell_1 X_1^* + \ell_2 X_2^* + \ell_6 X_6^*$ with $\ell_1, \ell_2 \neq 0$. The actions of π_{ℓ_T} and $\bar{\pi}_{\ell_T}$ are given by

$$
\begin{aligned}
&\pi_{\ell_T}(y) f(x_4, x_8, x_9) \\
=&\pi_{\ell_T}\Big(exp\,(y_1 X_1) \cdots exp\,(y_9 X_9)\Big) f\,(exp\,(x_4 X_4)\, exp\,(x_8 X_8)\, exp\,(x_9 X_9)) \\
=&e^{2\pi i \ell_1 \left(y_1 + y_5(x_4 + y_4 + \frac{1}{2} y_5)\right)} \\
&\cdot e^{2\pi i \ell_2 \left(y_2 - x_8 y_3 + y_6(y_4 + y_5 + x_4) - x_9\left(y_7 + x_4 y_8 + (x_8 + y_8)(y_4 + y_5)\right)\right)} \\
&\cdot e^{2\pi i \ell_6 (y_6 - x_9 y_8)} f\,(x_4 + y_4 + y_5, x_8 + y_8, x_9 + y_9)
\end{aligned}
$$

and

$$
\begin{aligned}
&\bar{\pi}_{\ell_T}(y) f(x_5, x_8, x_9) \\
=&\bar{\pi}_{\ell_T}\Big(exp\,(y_1 X_1) \cdots exp\,(y_9 X_9)\Big) f\,(exp\,(x_5 X_5)\, exp\,(x_8 X_8)\, exp\,(x_9 X_9)) \\
=&e^{2\pi i \ell_1 \left(y_1 - x_5 y_4\right)}
\end{aligned}
$$

$$\cdot e^{2\pi i \ell_2 \left(y_2 - x_8 y_3 - x_9 y_7 + x_5 y_6 + y_5 y_6 - x_8 x_9 y_4 - x_8 x_9 y_5 - x_9 y_5 y_8 - x_5 x_9 y_8 \right)}$$

$$\cdot e^{2\pi i \left(\ell_6 \left(y_6 - x_9 y_8 \right) + \frac{\ell_2^2}{2\ell_1} (y_6 - x_9 y_8)^2 \right)} f\left(\frac{\ell_2}{\ell_1}(y_6 - x_9 y_8) + x_5 + y_5, x_8 + y_8, x_9 + y_9\right).$$

CHAPTER 4

A SZASZ-MÜNTZ THEOREM FOR MATRIX COEFFICIENTS

In Chapter 2 we proved a Szasz-Müntz theorem for compactly supported L^2 functions in $\Re^n$. In this chapter we will prove a similar theorem for compactly supported L^2 functions on a certain class of simply connected nilpotent Lie groups. The class of groups we will deal with throughout this chapter is the class mentioned in the previous chapter: *the groups that have one fixed polarizer for all of the parameterizing functionals* $\ell_T \in U \bigcap V_T$. For example this includes all of the n-step chain groups.

Fix a group $\mathfrak{G}$ and a strong Malcev basis $\{X_1, \ldots, X_n\}$ so that $\mathfrak{G}$ satisfies the conditions above. Let m be the fixed polarizer for the parameterizing functionals, and $\{M_1, \ldots, M_a, X_{e_1}, \ldots, X_{e_c}\}$ be the almost strong Malcev basis of $\mathfrak{g}$ passing through m. From the previous chapter we know that for each ℓ_T, π_{ℓ_T} can be modeled in the fixed modeling space $L^2(\Re^c)$.

For $\varphi \in L^2(\mathfrak{G}) \bigcap L^1(\mathfrak{G})$ and $f, g \in L^2(\Re^c)$ we define the matrix coefficient of the operator valued transform $\hat{\varphi}_{op}(\ell_T) = \pi_{\ell_T}(\varphi)$ by

$$\langle \pi_{\ell_T}(\varphi) f, g \rangle = \int_{\mathfrak{G}} \varphi(x) \langle \pi_{\ell_T}(x) f, g \rangle dx.$$

In fact this relation also defines $\pi_{\ell_T}(\varphi)$. The next theorem shows that in the scenario outlined above these matrix coefficients are actually Euclidean Fourier Transforms of certain $L^2 \bigcap L^1$ functions. Then we will show that if φ is compactly supported, we can use the Euclidean Szasz-Müntz theorem.

Before we proceed we need to fix some notation. Denote by $T = \{t_1, \ldots, t_r\}$ the non-jump indices associated with the generic orbits, where r is the dimension of the radical for generic ℓ. As we have shown in the previous chapter, $T \subseteq I$, where I is the collection of internal indices for the almost strong Malcev basis $\{M_1, \ldots, M_a, X_{e_1}, \ldots, X_{e_c}\}$.

Theorem 4.1. *Assume $\varphi \circ exp \in L^2(\Re^n) \bigcap L^1(\Re^n)$ and $f, g \in L^2(\Re^c)$. Then there exist polynomials q and β such that*

$$\begin{aligned}\langle \pi_{\ell_T}(\varphi) f, g\rangle &= \int_{\Re^{n+c}} e^{2\pi i \sum_{j=1}^{r} l_{t_j} x_{t_j}} \tilde{\varphi}(q(x,y)) f(\beta(x,y)) g(y) dx dy \\ &= \hat{\Phi}_{f,g}(\ell_T)\end{aligned}$$

where

$$1) \tilde{\varphi}(z_1, \ldots, z_n) = \varphi\big(exp(z_1 X_1) \cdots exp(z_n X_n)\big), (z_1, \ldots, z_n) \in \Re^n$$

$$2) q(x,y) = (q_1(x,y), \ldots, q_n(x,y)), \quad \beta(x,y) = (\beta_{e_1}(x,y), \ldots, \beta_{e_c}(x,y))$$

$$3) q_j(x,y) = \begin{cases} x_{t_i} + \tilde{q}_{t_i}(x_{t_i+1}, \ldots, x_n; y_{e_k} \text{ with } e_k > t_i) & \text{if } j = t_i \text{ for some } i; \\ x_{s_i} & \text{if } j = s_i \text{ for some } i; \end{cases}$$

$$4) \beta_{e_j}(x,y) = x_{e_j} + y_{e_j} + \tilde{\beta}_{e_j}(x_{e_j+1}, \ldots, x_n; y_{e_{j+1}}, \ldots, y_{e_c})$$

$$5) \Phi_{f,g}(x_T) = \int_{\Re^{3c}} \tilde{\varphi}(q(x,y)) f(\beta(x,y)) g(y) dx_S dy \in L^1(\Re^r) \bigcap L^2(\Re^r)$$

Remarks. Here c is the co-dimension of the polarizer. $x_T = (x_{t_1}, \ldots, x_{t_r}), x_S = (x_{s_1}, \ldots, x_{s_o})$ where o is the dimension of the generic orbits and $S = \{s_1, \ldots, s_o\}$ is the collection of jump indices. In 3), 3c is the correct dimension since $n + c - r = (n - r) + c = 2c + c = 3c$.

Proof. From chapter 3 we know that the action of π_{ℓ_T} is given by

$$\pi_{\ell_T}(x)f(y) =$$
$$e^{2\pi i\left(\sum_{j=1}^{r}\ell_{t_j}\left(x_{t_j}+A_{t_j}(x_{t_j+1},\ldots,x_n,y_k \text{ with } e_k>t_j)\right)\right)} f\left(\beta_{e_1}(x,y),\ldots,\beta_{e_c}(x,y)\right)$$

where $\beta_{e_k}(x,y) = x_{e_k} + y_{e_k} + \tilde{\beta}_{e_k}(x_{e_k+1},\ldots,x_n; y_{e_{k+1}},\ldots,y_{e_c})$, $\ell_T = \sum_{t_j\in T}\ell_{t_j}X^*_{t_j}$ and $A_{t_j}, \tilde{\beta}_{e_k}$ are polynomial in x, y.

Therefore

$$\begin{aligned}
&\langle \pi_{\ell_T}(\varphi)f, g\rangle \\
&= \int_{\mathfrak{G}} \varphi(h)\langle \pi_{\ell_T}(h)f, g\rangle dh \\
&= \int_{\Re^n} \tilde{\varphi}(z)\Big(\int_{\Re^c} \pi_{\ell_T}(z)f(y)g(y)dy\Big)dz \\
&= \int_{\Re^{n+c}} e^{2\pi i\left(\sum_{j=1}^{r}\ell_{t_j}\left(z_{t_j}+A_{t_j}(z_{t_j+1},\ldots,z_n,y_k \text{ with } e_k>t_j)\right)\right)} \\
&\qquad \tilde{\varphi}(z)f\left(\beta_{e_1}(z,y),\ldots,\beta_{e_c}(z,y)\right)g(y)dydz \\
&= \int_{\Re^{n+c}} e^{2\pi i\left(\sum_{j=1}^{r}\ell_{t_j}\left(z_{t_j}+A_{t_j}(z_{t_j+1},\ldots,z_n,y_k \text{ with } e_k>t_j)\right)\right)} \\
&\qquad \tilde{\varphi}(z)f\left(\beta_{e_1}(z,y),\ldots,\beta_{e_c}(z,y)\right)g(y)dz_Tdz_Sdy \\
&= \int_{\Re^{n+c}} e^{2\pi i\sum_{j=1}^{r}\ell_{t_j}x_{t_j}} \tilde{\varphi}(q(x,y))f(\beta(x,y))g(y)dxdy
\end{aligned}$$

where $q(x,y)$ comes from the change of variables

$$x_j = \begin{cases} z_{t_i} + A_{t_i}(z_{t_i+1},\ldots,z_n,y_k \text{ with } e_k > t_i) & \text{if } j = t_i \text{ for some i;} \\ z_{s_i} & \text{if } j = s_i \text{ for some i.} \end{cases}$$

Remark. We may use Fubini in the previous calculations since,

$$\int_{\Re^{n+c}} \left| e^{2\pi i\left(\sum_{j=1}^{r} \ell_{t_j}\left(z_{t_j}+A_{t_j}(z_{t_j+1},\ldots,z_n,y_k \text{ with } e_k>t_j)\right)\right)} \right.$$

$$\left. \tilde{\varphi}(z) f\left(\beta_{e_1}(z,y),\ldots,\beta_{e_c}(z,y)\right) g(y) \right| dydz$$

$$= \int_{\Re^n} |\tilde{\varphi}(z)\langle \pi_{\ell_T}(z)f,g\rangle| dh$$

$$\leq \|g\|_2^2 \|\varphi\|_1 \|f\|_2^2$$

by the Schwarz inequality and the fact that π_{ℓ_T} is unitary.

So

$$\langle \pi_{\ell_T}(\varphi)f,g\rangle$$

$$= \int_{\Re^{n+c}} e^{2\pi i \sum_{j=1}^{r} \ell_{t_j} x_{t_j}} \tilde{\varphi}\left(q(x,y)\right) f\left(\beta(x,y)\right) g(y) dxdy$$

$$= \int_{\Re^r} e^{2\pi i \sum_{j=1}^{r} \ell_{t_j} x_{t_j}} \left(\int_{\Re^{3c}} \tilde{\varphi}\left(q(x,y)\right) f\left(\beta(x,y)\right) g(y) dx_S dy \right) dx_T$$

$$= \int_{\Re^r} e^{2\pi i \sum_{j=1}^{r} \ell_{t_j} x_{t_j}} \Phi_{f,g}(x_T) dx_T$$

$$= \hat{\Phi}_{f,g}(\ell_T)$$

where the use of Fubini may be justified as above, and in fact this shows that $\Phi_{f,g} \in L^1(\Re^r)$. It remains to show that $\Phi_{f,g} \in L^2(\Re^r)$.

We will show that $\hat{\Phi}_{f,g} \in L^2(\Re^r)$.

First assume that f and g are part of an orthonormal basis for $L^2(\Re^c)$. Then by the Plancherel theorem and the definition of the Hilbert-Schmidt norm

$$\left\|\hat{\Phi}_{f,g}\right\|_2^2 = \int\limits_{\Re^r} \left|\hat{\Phi}_{f,g}(\ell_T)\right|^2 d\ell_T$$
$$= \int\limits_{\Re^r} \left|\langle \pi_{\ell_T}(\varphi) f, g\rangle\right|^2 d\ell_T$$
$$\leq \int\limits_{\Re^r} \left\|\pi_{\ell_T}(\varphi)\right\|_{H-S}^2 d\ell_T$$
$$= \|\varphi\|_2^2 < \infty.$$

So $\hat{\Phi}_{f,g} \in L^2(\Re^r)$ in this case. For arbitrary f and g, the Gram-Scmidt process shows that f, g are a linear combination of at most two basis vectors from some orthonormal basis. Hence by the linearity of $\hat{\Phi}$ in f and g, $\hat{\Phi}$ is a finite linear combination of L^2 functions, and hence L^2. □

Example 4.2. We will continue with the use of example 3.2.2 from the previous chapter. In this case we will restrict our attention to the representations π_{ℓ_T} that come from the fixed polarizer

$$m = \Re\text{–span}\{X_1, X_2, X_3, X_5 - X_4, X_6, X_7\}.$$

For $\varphi \in L^2(\mathfrak{G}) \bigcap L^1(\mathfrak{G})$ and $f, g \in L^2(\Re^3)$ we have

$$\langle \pi_{\ell_T}(\varphi) f, g\rangle$$
$$= \int\limits_{\Re^9} \tilde{\varphi}(y) \langle \pi_{\ell_T}(y)(\varphi) f, g\rangle dy$$
$$= \int\limits_{\Re^{12}} e^{2\pi i\left(\ell_1\left(y_1 + A_1(x,y)\right) + \ell_2\left(y_2 + A_2(x,y)\right) + \ell_6\left(y_6 + A_6(x,y)\right)\right)} \tilde{\varphi}(y)$$
$$f(x_4 + y_4 + y_5, x_8 + y_8, x_9 + y_9) g(x_4, x_8, x_9) dx_4 dx_8 dx_9 dy$$

where

$$A_1(x,y) = y_5(x_4 + y_4 + \frac{1}{2}y_5)$$

$$A_2(x,y) = -x_8y_3 + y_6(y_4 + y_5 + x_4) - x_9\big(y_7 + x_4y_8 + (x_8 + y_8)(y_4 + y_5)\big)$$

$$A_3(x,y) = -x_9y_8$$

Make the change of variables

$$\bar{y}_6 = y_6 + A_3(x,\bar{y})$$

$$\bar{z}_1 = z_1 + A_1(x,\bar{y})$$

$$\bar{z}_2 = z_2 + A_2(x,\bar{y})$$

where, for ease of notation, $\bar{y}_i = y_i$ for $i \neq 6$. Then

$$\begin{aligned}&\langle \pi_{\ell_T}(\varphi)f,g\rangle \\ &= \int_{\Re^{12}} e^{2\pi i\left(\sum_{i=1}^{3} \ell_{t_i}\bar{y}_{t_i}\right)} \tilde{\varphi}(q(x,\bar{y}))f(x_4 + \bar{y}_4 + \bar{y}_5, x_8 + \bar{y}_8, x_9 + \bar{y}_9)g(x)dxd\bar{y} \\ &= \int_{\Re^3} e^{2\pi i\left(\sum_{i=1}^{3} \ell_{t_i}\bar{y}_{t_i}\right)} \Phi(\bar{y}_T)d\bar{y}_T\end{aligned}$$

where $\{t_1,t_2,t_3\} = \{1,2,6\}$ are the non-jump indices,

$$q_i(x,\bar{y}) = \begin{cases} \bar{y}_i & \text{i}\neq 1,2,6 \\ \bar{y}_6 + x_9\bar{y}_8 & \text{i=6} \\ \bar{y}_2 + x_8\bar{y}_3 - (\bar{y}_6 + x_9\bar{y}_8)(\bar{y}_4 + \bar{y}_5 + x_4) + \\ \qquad x_9\left(\bar{y}_7 + x_4\bar{y}_8 + (x_8 + \bar{y}_8)(\bar{y}_4 + \bar{y}_5 + x_9\bar{y}_8)\right) & \text{i=2} \\ \bar{y}_1 - \bar{y}_5\left(x_4 + \bar{y}_4 + \frac{1}{2}\bar{y}_5\right) & \text{i=1} \end{cases}$$

and

$$\begin{aligned}&\Phi(\bar{y}_T) \\ &= \Phi(\bar{y}_1, \bar{y}_2, \bar{y}_6) \\ &= \int_{\Re^9} \tilde{\varphi}(q(x,\bar{y}_S))f(x_4 + \bar{y}_4 + \bar{y}_5, x_8 + \bar{y}_8, x_9 + \bar{y}_9)g(x)dxd\bar{y}_S\end{aligned}$$

is $L^2(\Re^3)\bigcap L^1(\Re^3)$. Notice that if in addition φ and g are compactly supported, then so is Φ. The compact support of the variables x and y_S forces the compact support of the variables y_T. □

Corollary 4.3. *If $\varphi \in L^1(\mathfrak{G})\bigcap L^2(\mathfrak{G})$ and $f, g \in L^2(\Re^c)$ then*

$$\lim_{|\ell_T|\to\infty} \langle \pi_{\ell_T}(\varphi)f, g\rangle = 0.$$

Proof. By theorem 4.1 we may use the Riemann-Lebesgue lemma for Euclidean L^1 functions. □

We are now in a position to prove a Szasz-Müntz theorem for these matrix coefficients. For a differentiable function f of k variables, let

$$f^{(n)}(s) = f^{(n_1,\ldots,n_k)}(s_1,\ldots,s_k) = \frac{\partial^{n_1+\ldots+n_k}}{\partial s_1^{n_1}\cdots\partial s_k^{n_k}} f(s_1,\ldots,s_k).$$

Theorem 4.4. *Suppose that in addition to the conditions of theorem 4.1 we also know that φ and g have compact support. Then $\hat{\Phi}_{f,g}$ has an entire extension to $\mathbf{C}^{\mathbf{r}}$. In addition if $\mu = (\mu_1, \ldots, \mu_r)$ is an r-tuple of Szasz-Müntz sequences, ℓ_T is a fixed element of the complexified dual, and*

$$\hat{\Phi}_{f,g}^{(\mu_1(k_1),\ldots,\mu_r(k_r))}(\ell_T) = 0$$

for every $k_1, \ldots, k_r$ then $\hat{\Phi}_{f,g}(\ell_T) = 0$ for all ℓ_T.

Proof. By the Euclidean Szasz-Müntz theorem and theorem 4.1 it remains to check that $\Phi_{f,g}$ is compactly supported. By definition

$$\Phi_{f,g}(x_T) = \int\limits_{\Re^{3c}} \tilde{\varphi}(q(x,y)) f(\beta(x,y)) g(y) dx_S dy.$$

So it suffices to check that $\tilde{\varphi}(q(x,y)) f(\beta(x,y)) g(y)$ is compactly supported. From the compact support of g we can see that y is compactly supported. By the compact support of $\tilde{\varphi}$ and by property 3) from theorem 4.1 we can see that x_S is compactly supported. So it remains to see that x_T is compactly supported. We will work inductively starting with x_{t_r}.

Notice by the triangularity conditions on the q_j that

$$q_r(x,y) = x_{t_r} + \tilde{q}_{t_r}(y_{e_k} \text{ with } \mathrm{e_k} > \mathrm{t_r}).$$

Since the y are compactly supported and so is $\tilde{\varphi}$, we must have x_{t_r} compactly supported. Now assume that $x_{t_r}, \ldots, x_{t_{i+1}}$ are compactly supported, and consider x_{t_i}. Notice that $q_{t_i}(x,y) = x_{t_i} + \tilde{q}_{t_i}(x_{t_i+1}, \ldots, x_n; y_{e_k} \text{ with } \mathrm{e_k} > \mathrm{t_i})$. $\tilde{q}_{t_i}$ is a continuous function on a compact set, and hence bounded. Therefore by the compact support of $\tilde{\varphi}$, x_{t_i} must be compactly supported. □

Remark. In definition 2.2 of a Szasz-Müntz sequence μ we require that the reciprocals of both the even and odd terms in μ sum to infinity. Recall that this was necessary in order to have Szasz-Müntz theorems for functions with arbitrary support (functions with support not necessarily in $\Re^{n^+}$). Notice that this kind of general theorem is necessary in theorem 4.4. Even if we assume that the function φ has support in $\Re^{n^+}$, the function $\varphi(q)$ may have it's support anywhere.

Example 4.2 continued. Theorem 4.3 shows that in our example if φ and g have compact support then the matrix coefficients for φ satisfy a Szasz-Müntz theorem. This assumes we are using the representations π_ℓ that come from the fixed polarizer m. However when we started this example in chapter 3 we computed the generic representations using two different basis, one fixed, and one that rotated with ℓ. So what happens if we use the representations $\bar{\pi}_\ell$ that come from the rotating polarizers to compute the matrix coefficients for φ? The answer is not quite as nice:

$$\left\langle \bar{\pi}_{\ell_T}(\varphi)f, g \right\rangle$$

$$= \int_{\Re^9} \tilde{\varphi}(y)\langle \bar{\pi}_{\ell_T}(y)(\varphi)f, g\rangle dy$$

$$= \int_{\Re^{12}} e^{2\pi i\left(\ell_1\left(y_1+A_1(x,y)\right)+\ell_2\left(y_2+A_2(x,y)\right)+\ell_6\left(y_6+A_6(x,y)\right)\right)}$$

$$e^{2\pi i\frac{\ell_2^2}{2\ell_1}(y_6-x_9y_8)^2}\tilde{\varphi}(y)$$

$$f(\frac{\ell_2}{\ell_1}(y_6-x_9y_8)+x_5+y_5+y_6, x_8+y_8, x_9+y_9)g(x_4, x_8, x_9)dx_5dx_8dx_9dy$$

where

$$A_1(x,y) = -x_5y_4$$

$$A_2(x,y) = -x_8y_3 - x_9y_7 + x_5y_6 + y_5y_6 - x_8x_9y_4 - x_8x_9y_5 - x_9y_5y_8 - x_5x_9y_8$$

$$A_3(x,y) = -x_9y_8.$$

The problem here is two-fold: first we now have rational components in ℓ_1, and second we have introduced components of ℓ inside f. If we add the assumption that f has an entire extension to $\mathbf{C}^3$, i.e. f is the Fourier transform of a compactly supported function, then Lipsman and Rosenberg [3] have shown that the resulting matrix coefficients have a Laurent expansion about $\ell_1 = 0$.

However we also know that the representations from the two different bases are unitarily equivalent. So for each ℓ_T there is a unitary operator $U(\ell_T) : L^2(\Re^3) \to L^2(\Re^3)$ such that

$$U(\ell_T)\pi_{\ell_T} = \bar{\pi}_{\ell_T} U(\ell_T).$$

In this case the operator $U(\ell_T)$ is given by

$$\Big(U(\ell_T)f\Big)(x,y,z) = e^{2\pi i \frac{\ell_1}{2} x^2} \int_{\Re} e^{2\pi i \ell_1 x t} f(t,y,z)dt.$$

So

$$\begin{aligned}
&\Big\langle \pi_{\ell_T}(\varphi)f, g\Big\rangle \\
=&\int_{\Re^9} \varphi(y)\langle \pi_{\ell_T}(y)f, g\rangle dy \\
=&\int_{\Re^9} \varphi(y)\langle U(\ell_T)\big(\pi_{\ell_T}(y)f\big), U(\ell_T)g\rangle dy \\
=&\int_{\Re^9} \varphi(y)\langle \bar{\pi}_{\ell_T}(y)\big(U(\ell_T)f\big), U(\ell_T)g\rangle dy \\
=&\Big\langle \bar{\pi}_{\ell_T}(\varphi)\big(U(\ell_T)f\big), U(\ell_T)g\Big\rangle.
\end{aligned}$$

The moral is that if you are willing to let the Hilbert space in which you model your representations rotate with the polarizers, then you can have the nice behavior of the represention coming from a fixed polarizer. □

For $\mathfrak{G}$ as specified at the beginning of this chapter, we are ready to state a Szasz-Müntz theorem for the operator valued Fourier transform. We make use of what we know about the matrix coefficients of compactly supported functions from our previous work. We would like to write a partial derivative of the operator valued transform in terms of sums of derivatives of matrix coefficients, however there is no guarantee that these sums will converge. We discuss this further in the remarks following the theorem.

Theorem 4.5(SZASZ-MÜNTZ). *Let* $\varphi \in L^2_c(\mathfrak{G})$ *and* $\{\xi_i\}_{i=1}^{\infty} \subset L^2_c(\Re^c)$ *be a fixed orthonormal basis of* $L^2(\Re^c)$. *Then*

1) The operator valued Fourier Transform of $\hat{\varphi}_{op}(\ell_T) : L^2(\Re^c) \to L^2(\Re^c)$ *of* φ *has an entire extension to* $\mathbf{C}^r$ *in the sense that*

$$\hat{\varphi}_{op}(\ell_T)(f) = \sum_{j=1}^{\infty} \hat{\Phi}_{f,\xi_j}(\ell_T)\xi_j$$

and each matrix coefficient $\hat{\Phi}_{f,\xi_j}(\ell_T)$ *has such an extension.*

2)If for each pair of basis vectors $\{\xi_i, \xi_j\}$ *there exists a parameterizing functional* ℓ_T *and an r-tuple of Szasz-Müntz sequences* $\mu_{i,j}$ *such that* $\hat{\Phi}^{\mu_{i,j}}_{\xi_i,\xi_j}(\ell_T) = 0$, *then* $\varphi = 0$.

Proof. Notice that $L^2_c(\Re^c) \subset L^1(\Re^c)$ so that 1) follows from Theorem 4.3. In 2), the existence of the Ssasz-Müntz sequence $\mu_{i,j}$ ensures that $\hat{\Phi}_{\xi_i,\xi_j}(\ell_T) = 0$ for all ℓ_T and i, j, by theorem 4.4. Finally

$$\left\|\hat{\varphi}_{op}(\ell_T)\right\|^2_{H-S} = \sum_{i,j=1}^{\infty} |\hat{\Phi}_{\xi_i,\xi_j}(\ell_T)|^2 = 0$$

and

$$\|\varphi\|^2_2 = \int_{V_T} \left\|\hat{\varphi}_{op}(\ell_T)\right\|^2_{H-S} d\ell_T = 0.$$

by the nilpotent Plancherel theorem. □

Remark. Let $\mu = (\mu_1, \dots, \mu_r)$ be an r-tuple of Szasz-Müntz sequences. Formally define a partial derivative of $\hat{\varphi}_{op}(\ell_T)$ by the formula

$$\hat{\varphi}^{\mu}_{op}(\ell_T)(f) = \sum_{j=1}^{\infty} \hat{\Phi}^{\mu}_{f,\xi_j}(\ell_T)\xi_j.$$

As remarked before the theorem these sums do not necessarily converge, however for this formal definition theorem 4.5 shows that if enough of these formal derivatives vanish, for all f coming from the orthonormal basis and a fixed ℓ_T, then φ itself must vanish. In this sense theorem 4.5 is the equivalent of a Szasz-Müntz theorem for the operator valued transform on a nilpotent Lie group.

CHAPTER 5

MATRIX COEFFICIENTS FOR A GROUP WITH ROTATING POLARIZERS

In chapter 4 we considered matrix coefficients on groups that had a fixed polarizer m for all of the parameterizing functionals $\ell_T \in U \bigcap V_T$. We were able to show in this instance that these matrix coefficients had an entire extension to the complexified dual space, and that they satisfied a Szasz-Müntz theorem.

We now want to consider an example where the polarizers rotate with ℓ_T. Much of nilpotent Harmonic analysis depends on making choices, choosing the correct polarizer, choosing a coordinate system, choosing a basis, etc. In order to avoid an example where we simply made a poor choice for a polarizer we will work with an example where there is no one fixed polarizer for all of the parameterizing representations. The group is the free abelian group on four generators, two step, which we will call $F_{4,2}$. The corresponding algebra will be called $f_{4,2}$.

We will proceed as follows: First we will show that polarizers for the parameterizing representations must rotate, then we will fix a family of polarizers that rotate with parameterizing ℓ_T and calculate the matrix coefficients for this example. When we do that we will find that the matrix coefficients do not have an entire extension, however they have a Laurent expansion. From there we will consider possible Szasz-Müntz theorems in terms of the coefficients in the Laurent expansion of the matrix coefficient, and we will prove a partial Szasz-Müntz theorem for $f_{4,2}$.

The Szasz-Müntz theorem we prove is not as strong as the Szasz-Müntz theorems for groups with a fixed polarizer. This example is important because it illustrates the difficulty when you go outside the class of groups we have proven Szasz-Müntz theorems for.

We start with a strong Malcev basis $Y = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Y_1, Y_2, Y_3, Y_4\}$ of $f_{4,2}$,with nontrivial bracket relations:

$$[Y_1, Y_2] = -Z_1 \qquad [Y_2, Y_3] = Z_3$$

$$[Y_1, Y_3] = Z_4 \qquad [Y_2, Y_4] = Z_5$$

$$[Y_1, Y_4] = Z_6 \qquad [Y_3, Y_4] = -Z_2.$$

The collection of generic orbits is given by the Zariski open conditions

$$U = \{\ell | \ell_1 \neq 0, \ell_1\ell_2 - \ell_4\ell_5 + \ell_3\ell_6 \neq 0\}$$

(where $\ell = \ell_1 Z_1 + \ldots + \ell_6 Z_6 + \ell_7 Y_1 + \ldots + \ell_{10} Y_4$) and the non-jump and jump indices are the indices $T = \{1, \ldots, 6\}$ and $S = \{7, 8, 9, 10\}$.

The parameterizing functionals then are the functionals with the coefficients in the direction of S set to zero. That is

$$U \bigcap V_T = \{\ell_T | \ell_1 \neq 0, \ell_1\ell_2 - \ell_4\ell_5 + \ell_3\ell_6 \neq 0; \ell_7 = \ell_8 = \ell_9 = \ell_{10} = 0\}.$$

Notice that for generic ℓ, dim $r_\ell = |T| = 6$, which is the dimension of the center of $f_{4,2}$. So this group has square integrable representations mod the center. In the next section we show that any choice of polarizers for parameterizing ℓ must rotate.

5.1 There is no fixed polarizer for the parameterizing functionals in $f_{4,2}$

Notice that $m_{\ell_T} = \{Z_1, \ldots, Z_6, Y_1, \ell_1 Y_3 + \ell_4 Y_2\}$ is a polarizer for each of the parameterizing functionals.

Let B_{ℓ_T} be the antisymmetric bilinear form $B_{\ell_T}(X, V) = \ell_T([X, V])$. We will use $B^Y_{\ell_T}$ to denote the matrix for B_{ℓ_T} in the original strong Malcev basis. In this case $B^Y_{\ell_T}$ is the 10x10 matrix

$$B^Y_{\ell_T} = \begin{bmatrix} 0 & 0 \\ 0 & b^Y_{\ell_T} \end{bmatrix}$$

where $b^Y_{\ell_T}$ is the 4x4 matrix

$$b^Y_{\ell_T} = \begin{bmatrix} 0 & -\ell_1 & \ell_4 & \ell_6 \\ \ell_1 & 0 & \ell_3 & \ell_5 \\ -\ell_4 & -\ell_3 & 0 & -\ell_2 \\ -\ell_6 & -\ell_5 & \ell_2 & 0 \end{bmatrix}.$$

Suppose that there were one polarizer m for all of the parameterizing functionals. Let $W = \{W_1, \ldots, W_{10}\}$ be a weak Malcev basis for $f_{4,2}$ passing through m. Let $B^W_{\ell_T}$ be the matrix for B_{ℓ_T} in this new basis. Again $B^W_{\ell_T}$ has the form

$$B^W_{\ell_T} = \begin{bmatrix} 0 & 0 \\ 0 & b^W_{\ell_T} \end{bmatrix}.$$

In this case, since the basis W passes through the 8-dimensional polarizer m, the matrix $b^W_{\ell_T}$ contains a 2x2 submatrix of zeroes in the upper left corner. That is

$$B^W_{\ell_T} = \begin{bmatrix} 0 & 0 & p_{1,3}(\ell_T) & p_{1,4}(\ell_T) \\ 0 & 0 & p_{2,3}(\ell_T) & p_{2,4}(\ell_T) \\ -p_{1,3}(\ell_T) & -p_{2,3}(\ell_T) & 0 & p_{3,4}(\ell_T) \\ -p_{1,4}(\ell_T) & -p_{2,4}(\ell_T) & -p_{3,4}(\ell_T) & 0 \end{bmatrix}$$

where $p_{i,j}$ is polynomial in ℓ_T. Let A be the change of basis matrix taking the original strong Malcev basis to the weak Malcev basis W(by hypothesis A is independent of ℓ_T). So the i^{th} row of A contains the coefficients of the vector W_i

in terms of the original strong Malcev basis. Then $A^T B^Y_{\ell_T} A = B^W_{\ell_T}$. The first 6 vectors in the new basis W are central, so we may as well assume that they are the original central vectors from the strong Malcev basis. That is to say that A can be represented as

$$A = \begin{bmatrix} I_6 & 0 \\ 0 & A' \end{bmatrix}$$

where $A' = (a_{i,j})_{i,j=1}^4$ is a 4x4 matrix, with rank 4, such that $A'^T b^Y_{\ell_T} A' = b^W_{\ell_T}$. Writing out the multiplication we have

$$\begin{bmatrix} a_{1,1} & a_{2,1} & a_{3,1} & a_{4,1} \\ a_{1,2} & a_{2,2} & a_{3,2} & a_{4,2} \\ a_{1,3} & a_{2,3} & a_{3,3} & a_{4,3} \\ a_{1,4} & a_{2,4} & a_{3,4} & a_{4,4} \end{bmatrix} \begin{bmatrix} 0 & -\ell_1 & \ell_4 & \ell_6 \\ \ell_1 & 0 & \ell_3 & \ell_5 \\ -\ell_4 & -\ell_3 & 0 & -\ell_2 \\ -\ell_6 & -\ell_5 & \ell_2 & 0 \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & p_{1,3}(\ell_T) & p_{1,4}(\ell_T) \\ 0 & 0 & p_{2,3}(\ell_T) & p_{2,4}(\ell_T) \\ -p_{1,3}(\ell_T) & -p_{2,3}(\ell_T) & 0 & p_{3,4}(\ell_T) \\ -p_{1,4}(\ell_T) & -p_{2,4}(\ell_T) & -p_{3,4}(\ell_T) & 0 \end{bmatrix}.$$

Consider the equation that produces the 0 in the (1,2) entry of the matrix $b^W_{\ell_T}$. Computing the corresponding entry from $A'^T b^Y_{\ell_T} A'$ shows that we must have

$$\ell_1(-a_{1,1}a_{2,2} + a_{2,1}a_{1,1}) + \ell_2(-a_{3,1}a_{4,2} + a_{4,1}a_{3,2}) + \ell_3(a_{2,1}a_{3,2} - a_{3,1}a_{2,2})$$

$$+ \ell_4(a_{1,1}a_{3,2} - a_{3,1}a_{1,2}) + \ell_5(a_{2,1}a_{4,2} - a_{4,1}a_{2,2}) + \ell_6(a_{1,1}a_{4,2} - a_{4,1}a_{1,2}) = 0.$$

Here is where we can use the fact that we assumed m was a fixed polarizer for all of the parameterizing ℓ_T. First the coefficients $a_{i,j}$ are independent of ℓ_T as we have already stated. Since the above equation must hold for all of the parameterizing ℓ_T, we get 6 independent equations

$$\begin{array}{rrr} -a_{1,1}a_{2,2} + a_{2,1}a_{1,1} = 0 & -a_{3,1}a_{4,2} + a_{4,1}a_{3,2} = 0 & a_{2,1}a_{3,2} - a_{3,1}a_{2,2} = 0 \\ a_{1,1}a_{3,2} - a_{3,1}a_{1,2} = 0 & a_{2,1}a_{4,2} - a_{4,1}a_{2,2} = 0 & a_{1,1}a_{4,2} - a_{4,1}a_{1,2} = 0. \end{array}$$

Notice that these are the determinants of all of the 4x4 minors of the matrix

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \\ a_{4,1} & a_{4,2} \end{bmatrix}$$

which contradicts the fact that the matrix A' must have rank 4. □

5.2 Laurent series for matrix coefficients

Since any choice of polarizers for the parameterizing functionals must rotate, we will use the vectors $\{Z_1, \ldots, Z_6, Y_1, Y_3 + \frac{\ell_4}{\ell_1} Y_2\}$ as an almost strong Malcev basis for our rotating family of polarizers m_{ℓ_T}, and we will use exponential coordinates. The vectors $\{Y_2, Y_4\}$ will serve as a basis for our cross-section.

We are going to calculate the action of representations in this basis, and the matrix coefficients. We will see that the matrix coefficients will not be entire, but they will have Laurent expansions. We are going to investigate what happens when some of the coefficients in the Laurent expansion of these matrix coefficients are zero, and we will state and prove a weak analogue of the Szasz-Müntz theorem for this case. Let

$$x = exp(\alpha_1 Z_1) \cdots exp(\alpha_6 Z_6) exp(\alpha_7 Y_1) \cdots exp(\alpha_{10} Y_4)$$

$$y = exp(\beta_1 Y_2) exp(\beta_2 Y_4).$$

Then for parameterizing ℓ_T and $f \in L^2(\Re^2)$, Theorem 3.3.1 reduces in this example to

$$\pi_{\ell_T}(x) f(y)$$
$$= e^{2\pi i \left(\ell_1 \left(\alpha_1 + \alpha_7 \beta_1 \right) + \ell_2 \left(\alpha_2 + \alpha_9 \beta_2 \right) + \ell_3 \left(\alpha_3 + \alpha_9 \left(\alpha_8 + \beta_1 - \frac{\ell_4}{2\ell_1} \alpha_9 \right) \right) \right)}$$

$$\cdot e^{2\pi i\left(\ell_4\alpha_4+\ell_5\left(\alpha_5-\alpha_8\beta_2\right)+\ell_6\left(\alpha_6-\alpha_7\beta_2\right)\right)}$$

$$\cdot f(\alpha_8+\beta_1-\frac{\ell_4}{\ell_1}\alpha_9,\alpha_{10}+\beta_2).$$

For a function φ on the group, let

$$\tilde{\varphi}(\alpha)=\tilde{\varphi}(\alpha_1,\ldots,\alpha_{10})=\varphi(exp(\alpha_1 Z_1)\cdots exp(\alpha_6 Z_6)exp(\alpha_7 Y_1)\cdots exp(\alpha_{10}Y_4)).$$

For $\varphi\in L^1(F_{4,2})\bigcap L^2(F_{4,2})$ and $f,g\in L^2(\Re^2)$ we have

$$\begin{aligned}
\hat{\Phi}_{f,g}(\ell_T) =&\langle\pi_{\ell_T}(\varphi)f,g\rangle\\
=&\int_{\Re^{10}}\int_{\Re^2}\tilde{\varphi}(\alpha)\big(\pi_{\ell_T}(\alpha)f\big)(\beta)\bar{g}(\beta)d\beta d\alpha\\
=&\int_{\Re^{12}} e^{2\pi i\left(\ell_1\left(\alpha_1+\alpha_7\beta_1\right)+\ell_2\left(\alpha_2+\alpha_9\beta_2\right)+\ell_3\left(\alpha_3+\alpha_9\left(\alpha_8+\beta_1-\frac{\ell_4}{2\ell_1}\alpha_9\right)\right)\right)}\\
&\cdot e^{2\pi i\left(\ell_4\alpha_4+\ell_5\left(\alpha_5-\alpha_8\beta_2\right)+\ell_6\left(\alpha_6-\alpha_7\beta_2\right)\right)}\\
&\cdot\tilde{\varphi}(\alpha)f(\alpha_8+\beta_1-\frac{\ell_4}{\ell_1}\alpha_9,\alpha_{10}+\beta_2)\bar{g}(\beta)d\beta d\alpha. \qquad (1)
\end{aligned}$$

If we assume that φ, g have compact support and that f has an entire extension to complex 2-space we can show that the matrix coefficient $\hat{\Phi}_{f,g}(\ell_T)$ is holomorphic in the region $\{\ell_T\in\mathbf{C}^6|\ell_1\neq 0\}$. Before we prove this we give a name to the $L^2(\Re^n)$ functions that have an entire extension to complex n-space.

Definition 5.2.1. Let

$$L^2_H(\Re^n)=\{f\in L^2(\Re^n)|f \text{ has an entire extension to } \mathbf{C}^n\}$$

Remark. $L^2_H(\Re^n)$ includes the Paley-Wiener class of functions: functions that are Fourier transforms of compactly supported L^2 functions.

Lemma 5.2.2 If $\tilde{\varphi} \in L^2_c(\Re^{10})$, $g \in L^2_c(\Re^2)$, and $f \in L^2_H(\Re^2)$ then $\hat{\Phi}_{f,g}(\ell_T)$ has a holomorphic extension to $\mathbf{C_0} = \{\ell_T \in \mathbf{C}^6 | \ell_1 \neq 0\}$.

Proof. It suffices to prove that the function is holomorphic in the indicated region in each variable separately. To do this we will use Morera's theorem. Let

$$\psi(\alpha,\beta) = \tilde{\varphi}(\alpha)\bar{g}(\beta) \in L^2_c(\Re^{12}) \subset L^1_c(\Re^{12})$$

and

$$\begin{aligned} h(\ell_T,\alpha,\beta) =& e^{2\pi i\left(\ell_1\left(\alpha_1+\alpha_7\beta_1\right)+\ell_2\left(\alpha_2+\alpha_9\beta_2\right)+\ell_3\left(\alpha_3+\alpha_9(\alpha_8+\beta_1-\frac{\ell_4}{2\ell_1}\alpha_9)\right)\right)} \\ & \cdot e^{2\pi i\left(\ell_4\alpha_4+\ell_5\left(\alpha_5-\alpha_8\beta_2\right)+\ell_6\left(\alpha_6-\alpha_7\beta_2\right)\right)} \\ & \cdot f(\alpha_8+\beta_1-\frac{\ell_4}{\ell_1}\alpha_9, \alpha_{10}+\beta_2) \end{aligned}$$

so that $\hat{\Phi}_{f,g}(\ell_T) = \int_{\Re^{12}} h(\ell_T,\alpha,\beta)\psi(\alpha,\beta)d\alpha d\beta$. Notice that for each fixed α,β the function $h(\alpha,\beta,\cdot)$ is holomorphic in the region $\mathbf{C_0}$. Also notice that for each fixed $\ell_T, h(\cdot,\cdot,\ell_T)$ is continuous in α,β. In particular $h(\cdot,\cdot,\ell_T)$ is bounded on the support of ψ. Let γ_i be a simple closed curve in the i^{th} coordinate plane. For γ_0 assume that the origin is not on nor in the interior of this curve. Finally let $\gamma = \gamma_1 \times \ldots \times \gamma_6$. By the Cauchy integral formula $\int_\gamma h(\alpha,\beta,\ell_T)d\ell_T = 0$ for each α,β so that

$$\int_\gamma \hat{\Phi}_{f,g}(\ell_T)d\ell_T = \int_\gamma \int_{\Re^{12}} h(\ell_T,\alpha,\beta)\psi(\alpha,\beta)d\alpha d\beta d\ell_T$$

$$= \int_{\Re^{12}} \int_{\gamma} h(\ell_T, \alpha, \beta)\psi(\alpha, \beta) d\ell_T d\alpha d\beta$$
$$= 0$$

Here we have made use of the fact that a function in several variables is holomorphic if and only if it is holomorphic in each variable separately. The main point in the proof of the lemma is ensuring that the Fubini argument will work. □

To get a (weak) Szasz-Müntz theorem for this group we look at the coefficients in the Laurent expansion of the matrix coefficient $\hat{\Phi}_{f,g}$ about zero:

$$\hat{\Phi}_{f,g}(\ell_T) = \langle \pi_{\ell_T}(\varphi)f, g\rangle = \sum_{j_1=-\infty}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_6=0}^{\infty} c^{f,g}_{j_1,\ldots,j_6} \ell_1^{j_1} \ldots \ell_6^{j_6}.$$

What we want to know is how many of these coefficients must be zero for all such f, g before we can say that φ must be zero. These coefficients are going to involve partial derivatives in the $\ell_2, \ldots, \ell_6$ directions. These derivatives are complicated by the appearance of the ℓ_4 term in several places.

Let $\mathbf{C}_1, \ldots, \mathbf{C}_6$ be six circles about the origin, one in each coordinate complex plane. Since $\hat{\Phi}_{f,g}$ is entire in $\ell_2, \ldots, \ell_6$ the coefficient $c^{f,g}_{n_1,\ldots,n_6}$ is given by the formula:

$$\begin{aligned}
c^{f,g}_{n_1,\ldots,n_6} &= \frac{1}{(2\pi i)^6} \int_{\mathbf{C}_1}\int_{\mathbf{C}_2} \cdots \int_{\mathbf{C}_6} \frac{1}{\ell_1^{n_1+1} \cdots \ell_6^{n_6+1}} \hat{\Phi}_{f,g}(\ell_1, \ldots, \ell_6) d\ell_6 \ldots d\ell_1 \\
&= \frac{1}{2\pi i} \frac{1}{n_2! \cdots n_6!} \int_{\mathbf{C}_1} \frac{1}{\ell_1^{n_1+1}} D_2^{n_2} \cdots D_6^{n_6} \hat{\Phi}_{f,g}(\ell_1, 0, \ldots, 0) d\ell_1 \\
&= \frac{1}{2\pi i} \frac{1}{n_2! \cdots n_6!} \int_{\mathbf{C}_1} \frac{1}{\ell_1^{n_1+1}} \hat{\Phi}^{(n_2,\ldots,n_6)}_{f,g}(\ell_1, 0, \ldots, 0) d\ell_1
\end{aligned}$$

where D_i is the partial differential operator in the i^{th} coordinate and

$$D_2^{n_2} \cdots D_6^{n_6} \hat{\Phi}_{f,g}(\ell_1, \ldots, \ell_6) = \hat{\Phi}_{f,g}^{(n_2,\ldots,n_6)}(\ell_1, \ldots, \ell_6).$$

We will use of a modified equation (1) to compute these coefficients. In equation (1) make the following change of variables:

$$\begin{aligned} a_1 &= \alpha_1 + \alpha_7\beta_1 & a_5 &= \alpha_5 - \alpha_8\beta_2 \\ a_2 &= \alpha_2 + \alpha_9\beta_2 & a_6 &= \alpha_6 - \alpha_7\beta_2 \\ a_3 &= \alpha_3 + \alpha_9(\alpha_8 + \beta_1) & a_i &= \alpha_i, \quad i \geq 7, i = 4. \end{aligned}$$

Then

$$\hat{\Phi}_{f,g}(\ell_T)$$

$$= \int_{\Re^{12}} e^{2\pi i\left(\sum_{j=1}^{6} \ell_i a_i - \frac{\ell_3 \ell_4 a_9^2}{2\ell_1}\right)} \tilde{\varphi}(p(a,\beta)) f(a_8 + \beta_1 - \frac{\ell_4}{2\ell_1} a_9, a_{10} + \beta_2) \bar{g}(\beta) d\beta da \quad (2)$$

where p is the polynomial change of variables outlined above:

$$\begin{aligned} p_i(a,\beta) &= a_i, \quad i \geq 7, i = 4 & p_3(a,\beta) &= a_3 - \alpha_9(\alpha_8 + \beta_1) \\ p_6(a,\beta) &= a_6 + a_7\beta_2 & p_2(a,\beta) &= a_2 - a_9\beta_2 \\ p_5(a,\beta) &= a_5 + a_8\beta_2 & p_1(a,\beta) &= a_1 - \alpha_7\beta_1. \end{aligned}$$

Now we are ready to compute $\hat{\Phi}_{f,g}^{(n_2,\ldots,n_6)}(\ell_1, 0, \ldots, 0)$. From (2) we see that, by passing differentiation through the integral

$$\hat{\Phi}_{f,g}^{(n_2,n_5,n_6)}(\ell_1, \ldots, \ell_6)$$

$$= (2\pi i)^{n_2+n_5+n_6} \int_{\Re^{12}} e^{2\pi i\left(\sum_{j=1}^{6} \ell_i a_i - \frac{\ell_3 \ell_4 a_9^2}{2\ell_1}\right)} a_2^{n_2} a_5^{n_5} a_6^{n_6} \tilde{\varphi}(p(a,\beta))$$

$$f(a_8 + \beta_1 - \frac{\ell_4}{2\ell_1} a_9, a_{10} + \beta_2) \bar{g}(\beta) d\beta da.$$

In the ℓ_3 direction, notice that the coefficient on ℓ_3 in the complex exponential is $a_3 - \frac{\ell_4 a_9^2}{2\ell_1}$. So

$$\hat{\Phi}_{f,g}^{(n_2,n_3,n_5,n_6)}(\ell_1,\ldots,\ell_6)$$

$$= (2\pi i)^{n_2+n_3+n_5+n_6} \int_{\Re^{12}} e^{2\pi i\left(\sum_{j=1}^{6} \ell_i a_i - \frac{\ell_3 \ell_4 a_9^2}{2\ell_1}\right)} a_2^{n_2}(a_3 - \frac{\ell_4 a_9^2}{2\ell_1})^{n_3} a_5^{n_5} a_6^{n_6} \bar{\varphi}(p(a,\beta))$$

$$f(a_8 + \beta_1 - \frac{\ell_4}{2\ell_1} a_9, a_{10} + \beta_2)\bar{g}(\beta) d\beta da. \quad (3)$$

The derivative in the ℓ_4 direction is complicated by the appearance of this term in three places. Because of this we use the rule for the derivative of a product of three functions:

$$(\psi\eta\lambda)^{(n)} = \sum_{k=0}^{n}\sum_{j=0}^{j} \binom{n}{k}\binom{k}{j} \psi^{(n-k)}\eta^{(k-j)}\lambda^{j}.$$

This formula applied to (3) with $n = n_4$ and

$$\psi(\ell_4) = e^{2\pi i \ell_4 (a_4 - \frac{\ell_3 a_9^2}{2\ell_1})}, \eta(\ell_4) = (a_3 - \frac{\ell_4 a_9^2}{2\ell_1})^{n_3}, \lambda(\ell_4) = f(a_8 + \beta_1 - \frac{\ell_4}{2\ell_1} a_9, a_{10} + \beta_2)$$

yields:

$$\hat{\Phi}_{f,g}^{(n_2,n_3,n_4,n_5,n_6)}(\ell_1,\ldots,\ell_6)$$

$$= \sum_{k_4=0}^{n_4}\sum_{j_4=0}^{k_4} (2\pi i)^{n_2+\ldots+n_6-k_4} \binom{n_4}{k_4}\binom{k_4}{j_4} \int_{\Re^{12}} e^{2\pi i\left(\sum_{j=1}^{6} \ell_i a_i - \frac{\ell_3 \ell_4 a_9^2}{2\ell_1}\right)} a_2^{n_2} \cdot$$

$$(a_3 - \frac{\ell_4 a_9^2}{2\ell_1})^{n_3 - j_4} (\frac{-a_9^2}{2\ell_1})^{j_4} (a_4 - \frac{\ell_3 a_9^2}{2\ell_1})^{n_4-k_4} a_5^{n_5} a_6^{n_6} \bar{\varphi}(p(a,\beta)) \cdot$$

$$f^{(k_4-j_4)}(a_8 + \beta_1 - \frac{\ell_4}{2\ell_1} a_9, a_{10} + \beta_2)(\frac{-a_9}{\ell_1})^{k_4-j_4} \bar{g}(\beta) d\beta da$$

where $f^{(n)}$ is the n^{th} partial derivative of f in the first variable.

To compute the coefficients we need the derivatives at the points $(\ell_1, 0, \ldots, 0)$:

$$\hat{\Phi}_{f,g}^{(n_2,n_3,n_4,n_5,n_6)}(\ell_1,0,\ldots,0)$$

$$= \sum_{k_4=0}^{n_4}\sum_{j_4=0}^{k_4}(2\pi i)^{n_2+\ldots+n_6-k_4}\binom{n_4}{k_4}\binom{k_4}{j_4}\int_{\Re^{12}} e^{2\pi i\ell_1 a_1}a_2^{n_2}a_3^{n_3-j_4}\left(\frac{-a_9^2}{2\ell_1}\right)^{j_4}a_4^{n_4-k_4}a_5^{n_5}a_6^{n_6}\cdot$$

$$\tilde{\varphi}(p(a,\beta))f^{(k_4-j_4)}(a_8+\beta_1,a_{10}+\beta_2)\left(\frac{-a_9}{\ell_1}\right)^{k_4-j_4}\bar{g}(\beta)d\beta da$$

$$= \sum_{k_4=0}^{n_4}\sum_{j_4=0}^{k_4}(2\pi i)^{n_2+\ldots+n_6-k_4}\binom{n_4}{k_4}\binom{k_4}{j_4}\frac{1}{2^{j_4}}(-1)^{k_4}\int_{\Re^{12}} e^{2\pi i\ell_1 a_1}\left(\frac{1}{\ell_1}\right)^{k_4}a_2^{n_2}a_3^{n_3-j_4}\cdot$$

$$a_4^{n_4-k_4}a_5^{n_5}a_6^{n_6}a_9^{k_4+j_4}\tilde{\varphi}(p(a,\beta))f^{(k_4-j_4)}(a_8+\beta_1,a_{10}+\beta_2)\bar{g}(\beta)d\beta da$$

$$= \sum_{k_4=0}^{n_4}\sum_{j_4=0}^{k_4}d(n_2,\ldots,n_6)\int_{\Re^{12}} e^{2\pi i\ell_1 a_1}\left(\frac{1}{\ell_1}\right)^{k_4}a_2^{n_2}a_3^{n_3-j_4}a_4^{n_4-k_4}a_5^{n_5}a_6^{n_6}a_9^{k_4+j_4}\cdot$$

$$\tilde{\varphi}(p(a,\beta))f^{(k_4-j_4)}(a_8+\beta_1,a_{10}+\beta_2)\bar{g}(\beta)d\beta da$$

where $d(n_2,\ldots,n_6) = (2\pi i)^{n_2+\ldots+n_6-k_4}\binom{n_4}{k_4}\binom{k_4}{j_4}\frac{1}{2^{j_4}}(-1)^{k_4}$. So

$$c_{n_1,\ldots,n_6}^{f,g}$$

$$= \frac{1}{2\pi i}\frac{1}{n_2!\cdots n_6!}\int_{C_1}\frac{1}{\ell_1^{n_1+1}}\hat{\Phi}_{f,g}^{(n_2,\ldots,n_6)}(\ell_1,0,\ldots,0)d\ell_1$$

$$= \frac{1}{2\pi i}\frac{1}{n_2!\cdots n_6!}\int_{C_1}\frac{1}{\ell_1^{n_1+1}}\left(\sum_{k_4=0}^{n_4}\sum_{j_4=0}^{k_4}d(n_2,\ldots,n_6)\int_{\Re^{12}} e^{2\pi i\ell_1 a_1}\left(\frac{1}{\ell_1}\right)^{k_4}a_2^{n_2}a_3^{n_3-j_4}a_4^{n_4-k_4}a_5^{n_5}\cdot\right.$$

$$\left.a_6^{n_6}a_9^{k_4+j_4}\tilde{\varphi}(p(a,\beta))f^{(k_4-j_4)}(a_8+\beta_1,a_{10}+\beta_2)\bar{g}(\beta)d\beta da\right)d\ell_1$$

$$= \frac{1}{n_2!\cdots n_6!}\sum_{k_4=0}^{n_4}\sum_{j_4=0}^{k_4}d(n_2,\ldots,n_6)\int_{\Re^{12}}\left(\frac{1}{2\pi i}\int_{C_1}\frac{1}{\ell_1^{n_1+1}}e^{2\pi i\ell_1 a_1}\left(\frac{1}{\ell_1}\right)^{k_4}d\ell_1\right)\cdot$$

$$a_2^{n_2}a_3^{n_3-j_4}a_4^{n_4-k_4}a_5^{n_5}a_6^{n_6}a_9^{k_4+j_4}\tilde{\varphi}(p(a,\beta))\cdot$$

$$f^{(k_4-j_4)}(a_8+\beta_1,a_{10}+\beta_2)\bar{g}(\beta)d\beta da. \qquad (4)$$

where

$$\frac{1}{2\pi i}\int_{C_1}\frac{1}{\ell_1^{n_1+1}}e^{2\pi i\ell_1 a_1}\left(\frac{1}{\ell_1}\right)^{k_4}d\ell_1$$
$$=\frac{1}{2\pi i}\int_{C_1}\frac{1}{\ell_1^{n_1+k_4+1}}e^{2\pi i\ell_1 a_1}d\ell_1$$

is the n_1+k_4 coefficient in the Laurent expansion (note that n_1 can be negative) of the function

$$e^{2\pi i\ell_1 a_1}=\sum_{j=0}^{\infty}\frac{(2\pi i a_1)^j}{j!}\ell_1^j.$$

Therefore

$$\frac{1}{2\pi i}\int_{C_1}\frac{1}{\ell_1^{n_1+k_4+1}}e^{2\pi i\ell_1 a_1}d\ell_1=\begin{cases}\frac{(2\pi i a_1)^{n_1+k_4}}{(n_1+k_4)!} & \text{if } n_1+k_4\geq 0;\\ 0 & \text{otherwise.}\end{cases}$$

If we combine this with (4) we have

$$c^{f,g}_{n_1,\ldots,n_6}$$
$$=\frac{1}{n_2!\cdots n_6!}\sum_{k_4=0}^{n_4}\sum_{j_4=0}^{k_4}d(n_2,\ldots,n_6)\int_{\Re^{12}}\left(\frac{1}{2\pi i}\int_{C_1}\frac{1}{\ell_1^{n_1+1}}e^{2\pi i\ell_1 a_1}\left(-\frac{1}{\ell_1}\right)^{k_4}d\ell_1\right)\cdot$$
$$a_2^{n_2}a_3^{n_3-j_4}a_4^{n_4-k_4}a_5^{n_5}a_6^{n_6}a_9^{k_4+j_4}\cdot$$
$$\tilde{\varphi}(p(a,\beta))f^{(k_4-j_4)}(a_8+\beta_1,a_{10}+\beta_2)\bar{g}(\beta)d\beta da$$
$$=\sum_{k_4=0}^{n_4}\sum_{j_4=0}^{k_4}d(n,k_4)\int_{\Re^{12}}a_1^{n_1+k_4}a_2^{n_2}a_3^{n_3-j_4}a_4^{n_4-k_4}a_5^{n_5}a_6^{n_6}a_9^{k_4+j_4}\cdot$$
$$\tilde{\varphi}(p(a,\beta))f^{(k_4-j_4)}(a_8+\beta_1,a_{10}+\beta_2)\bar{g}(\beta)d\beta da \qquad (5)$$

where

$$d(n,k_4)=\begin{cases}\frac{(2\pi i)^{n_1+k_4}}{n_2!\cdots n_6!(n_1+k_4)!}d(n_2,\ldots,n_6) & \text{if } n_1+k_4\geq 0;\\ 0 & \text{otherwise.}\end{cases}$$

5.3 A Szasz-Müntz theorem for $F_{4,2}$

Now that we have calculated the coefficients $c^{f,g}_{n_1,\dots,n_6}$ we will state a weak Szasz-Müntz theorem for the matrix coefficients on $F_{4,2}$. The theorem is Szasz-Müntz in nature because will not require all of the coefficients in the Laurent series to vanish in order to show that φ vanishes. On the other hand the theorem is not as strong as a full Szasz-Müntz theorem because we do require that for Szasz-Müntz sequences n_1, n_2, n_3, n_5, n_6 and for each pair f, g the coefficient $c^{f,g}_{n_1,\dots,n_6}$ vanish for all n_4, instead of a Szasz-Müntz sequence n_4.

It is interesting to note that although n_1 can assume negative values, we do not use these. We will only consider a positive Szasz-Müntz sequence n_1.

Theorem 5.3.1 *Let $\varphi \in L^2_c(\mathfrak{G}), g \in L^2_c(\Re^2), f \in L^2_H(\Re^2)$ and for each integer n_1 and positive integers $n_2, \dots, n_6$ let $c^{f,g}_{n_1,\dots,n_6}$ be the $(n_1, \dots, n_6)$ coefficient in the Laurent expansion for the matrix coefficient $\langle \pi_{\ell_T}(\varphi) f, g \rangle$. Let each variable n_1, n_2, n_3, n_5, n_6 vary over a Szasz-Müntz sequence, while n_4 varies over* $\mathbf{N}$. *If $c^{f,g}_{n_1,\dots,n_6} = 0$ for all such $n_1, \dots, n_6$ and for all the specified f, g then $\varphi = 0$.*

Proof. We are going to proceed by induction on n_4. In fact we will show that for every n_4

$$0 = \int_{\Re} \alpha_4^{n_4} \varphi(\alpha) d\alpha_4$$

for $\alpha_1, \dots, \alpha_3, \alpha_5, \dots, \alpha_{10}$ almost everywhere, which proves the theorem.

We would like to be able to use one of the Euclidean Szasz-Müntz theorems. In order to do that we need to know that $\tilde{\varphi}(\alpha, \beta) g(\beta)$ is compactly supported.

Claim. $\psi(\alpha,\beta) = \tilde{\varphi}(p(\alpha,\beta))g(\beta)$ is compactly supported on $\Re^{12}$.

Proof. By the definition of the change of variables p,

$$\tilde{\varphi}(p(\alpha,\beta))g(\beta)$$

$$=\tilde{\varphi}(\alpha_1-\alpha_7\beta_1,\alpha_2-\alpha_9\beta_2,\alpha_3-\alpha_9(\alpha_8+\beta_1),\alpha_4,\alpha_5+\alpha_8\beta_2,\alpha_6+\alpha_7\beta_2,\alpha_8,\alpha_9,\alpha_{10})g(\beta).$$

The compact support of g guarantees the compact support of β. The compact support of φ guarantees the compact support of $\alpha_{10},\ldots,\alpha_7$ and α_4. In the other variables we have a translation by a polynomial which is defined on a compact set, and hence is bounded. This guarantees the compact support of the other variables $\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_6$. □

Now we proceed with the proof of the theorem. Start with the case $n_4 = 0$. By (5)

$$\begin{aligned} 0 &= \int_{\Re^{12}} a_1^{n_1}a_2^{n_2}a_3^{n_3}a_5^{n_5}a_6^{n_6}\tilde{\varphi}(p(a,\beta))f(a_8+\beta_1,a_{10}+\beta_2)\bar{g}(\beta)d\beta da \\ &= \int_{\Re^{12}} a_1^{n_1}a_2^{n_2}a_3^{n_3}a_5^{n_5}a_6^{n_6}\tilde{\varphi}(p(a,\beta))f(a_8+\beta_1,a_{10}+\beta_2)\bar{g}(\beta)dad\beta \end{aligned}$$

for Szasz-Müntz sequences n_1,n_2,n_3,n_5,n_6 and for all of the specified f,g. Fubini is justified here since $\tilde{\varphi}(p)g$ has compact support in L^2, and hence $a_1^{n_1}a_2^{n_2}a_3^{n_3}a_5^{n_5}a_6^{n_6}\cdot$ $\tilde{\varphi}(p(a,\beta))\bar{g}(\beta)$ is L^1. Since this integral is zero for every $g\in L_c^2(\Re^2)$, for almost every β we have

$$0 = \int_{\Re^{10}} a_1^{n_1}a_2^{n_2}a_3^{n_3}a_5^{n_5}a_6^{n_6}\tilde{\varphi}(p(a,\beta))f(a_8+\beta_1,a_{10}+\beta_2)da.$$

Since the class $L^2_H(\Re^2)$ contains the Paley-Wiener class, it is dense in $L^2(\Re^2)$. Therefore for almost every a_8, a_{10}, β

$$0 = \int_{\Re^8} a_1^{n_1} a_2^{n_2} a_3^{n_3} a_5^{n_5} a_6^{n_6} \tilde{\varphi}(p(a,\beta)) da_1 \ldots da_7 da_9.$$

By the Szasz-Müntz theorem 2.4, applied to the sequences n_1, n_2, n_3, n_5, n_6 we have

$$0 = \int_{\Re^3} \tilde{\varphi}(p(a,\beta)) da_4 da_7 da_9 \tag{6}$$

for almost every $a_1, a_2, a_3, a_5, a_6, a_8, a_{10}, \beta$.

At this point it becomes clear that the variable a_7 is going to be hard to handle. Notice that in formula for the coefficients $c^{f,g}_{n_1,\ldots,n_6}$ the only place that a_7 appears is in φ. In other words we don't have a monomial in a_7 to help us 'eliminate' this variable. It is here that the change of variables p comes to the rescue. Really we are integrating over surfaces that are parameterized by β_1, β_2. To take advantage of this we take a Fourier transform in several of the variables of (6).

More specifically for almost every x_1, x_2, x_3, x_6 we have:

$$0 = \int_{\Re^7} e^{2\pi i(x_1 a_1 + x_2 a_2 + x_3 a_3 + x_6 a_6)} \tilde{\varphi}(p(a,\beta)) da_1 da_2 da_3 da_4 da_6 da_7 da_9.$$

Now undo the change of variables p in the a_1, a_2, a_5, a_6 directions and partially in a_3. We take

$$\alpha_1 = a_1 - a_7\beta_1 \qquad \alpha_5 = a_5 + a_8\beta_2$$

$$\alpha_2 = a_2 - a_9\beta_2 \qquad \alpha_6 = a_6 + a_7\beta_2$$

$$\alpha_3 = a_3 + a_9\beta_1 \qquad \alpha_i = a_i, \text{otherwise}.$$

Then

$$0 = \int_{\Re^7} e^{2\pi i(x_1(\alpha_1+\alpha_7\beta_1)+x_2(\alpha_2+\alpha_9\beta_2)+x_3(\alpha_3+\alpha_9\beta_1)+x_6(\alpha_6-\alpha_7\beta_2))}$$

$$\tilde{\varphi}(\alpha_1, \alpha_2, \alpha_3 + \alpha_9\alpha_8, \alpha_4, \ldots, \alpha_{10}) d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_6 d\alpha_7 d\alpha_9$$

$$= \int_{\Re^6} e^{2\pi i(\alpha_1 x_1+\alpha_2 x_2+\alpha_3 x_3+\alpha_6 x_6+\alpha_7(\beta_1 x_1-\alpha_7\beta_2 x_6)+\alpha_9(\beta_2 x_2+\beta_1 x_3))}$$

$$\Big(\int_{\Re} \tilde{\varphi}(\alpha_1, \alpha_2, \alpha_3 + \alpha_9\alpha_8, \alpha_4, \ldots, \alpha_{10}) d\alpha_4\Big) d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_6 d\alpha_7 d\alpha_9$$

$$= \hat{\varphi}_{(\alpha_5,\alpha_8,\alpha_{10})}(x_1, x_2, x_3, x_6, \beta_1 x_1 - \beta_2 x_6, \beta_2 x_2 + \beta_1 x_3)$$

for almost every $x_1, x_2, x_3, x_6, \alpha_5, \alpha_8, \alpha_{10}, \beta$.

Here $\hat{\varphi}_{(\alpha_5,\alpha_8,\alpha_{10})}$ is the Euclidean Fourier transform of the function

$$\varphi_{(\alpha_5,\alpha_8,\alpha_{10})}(\alpha) = \int_{\Re} \tilde{\varphi}(\alpha_1, \alpha_2, \alpha_3 + \alpha_9\alpha_8, \alpha_4, \ldots, \alpha_{10}) d\alpha_4.$$

On a set of full measure in $\Re^4$ the matrix

$$\begin{bmatrix} x_1 & -x_6 \\ x_3 & x_2 \end{bmatrix}$$

has a nonzero determinant. On this set of full measure

$$\begin{bmatrix} x_1 & -x_6 \\ x_3 & x_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

has a solution for any y_1, y_2. Combined with what we know from above we have

$$0 = \hat{\varphi}_{(\alpha_5,\alpha_8,\alpha_{10})}(x_1, x_2, x_3, x_6, y_1, y_2)$$

for almost every $x_1, x_2, x_3, x_6, \alpha_5, \alpha_8, \alpha_{10}, y$. Therefore

$$0 = \varphi_{(\alpha_5,\alpha_8,\alpha_{10})}(\alpha_1, \alpha_2, \alpha_3, \alpha_6, \alpha_7, \alpha_9)$$

for almost every $s_1, s_2, s_3, s_6, \alpha_5, \alpha_8, \alpha_{10}, t$. From the definition of $\varphi_{(\alpha_5,\alpha_8,\alpha_{10})}$ we can conclude that for almost every $\alpha_1, \dots, \alpha_{10}$

$$0 = \int_{\Re} \tilde{\varphi}(\alpha) d\alpha_4.$$

Let's sum up what we have done here as a lemma:

Lemma 5.3.2. For $\varphi \in L_c^2(\mathfrak{G})$, if

$$\int_{\Re^{12}} a_1^{n_1} a_2^{n_2} a_3^{n_3} a_5^{n_5} a_6^{n_6} \tilde{\varphi}(p(a,\beta)) f(a_8 + \beta_1, a_{10} + \beta_2) \bar{g}(\beta) d\beta da = 0$$

for every $g \in L_c^2(\Re^2), f \in L_H^2(\Re^2)$ and for n_1, n_2, n_3, n_5, n_6 members of five Szasz-Müntz sequences then

$$0 = \int_{\Re} \tilde{\varphi}(\alpha) d\alpha_4 = \int_{\Re} \tilde{\varphi}(p(\alpha,\beta)) d\alpha_4$$

for $\alpha_1, \dots, \alpha_3, \alpha_5, \dots, \alpha_{10}, \beta$ almost everywhere.

Remark. Notice that we cannot do any better than this, since we have only used $n_4 = 0$.

Proof. All that remains to show is that

$$0 = \int_{\Re} \varphi(p(\alpha,\beta)) d\alpha_4$$

for $\alpha_1, \dots, \alpha_3, \alpha_5, \dots, \alpha_{10}, \beta$ almost everywhere.

This follows since $p(\alpha_4) = \alpha_4$. □

Remark. Lemma 5.3.2 is related to Radon transforms. We started with

$$0 = \int_{\Re^3} \tilde{\varphi}(p(a,\beta))da_4 da_7 da_9$$

for almost every $a_1, a_2, a_3, a_5, a_6, a_8, a_{10}, \beta$

It turns that the change of variables p means we are integrating over surfaces that depend on β. We are able to use this to reduce this integral to an integral over $\Re$.

To finish the proof of Theorem 5.3.1 we will use induction on n_4 and Lemma 5.3.2. Now consider the case $n_4 = 1$. Then

$$0 = \sum_{k_4=0}^{1} \sum_{j_4=0}^{k_4} d(n,k_4) \int_{\Re^{12}} a_1^{n_1-k_4} a_2^{n_2} a_3^{n_3-j_4} a_4^{n_4-k_4} a_5^{n_5} a_6^{n_6} a_9^{k_4+j_4} \cdot$$
$$\tilde{\varphi}(p(a,\beta)) f^{(k_4-j_4)}(a_8+\beta_1, a_{10}+\beta_2)\bar{g}(\beta)d\beta da$$
$$= d(n,0) \int_{\Re^{12}} a_1^{n_1} a_2^{n_2} a_3^{n_3-j_4} a_4 a_5^{n_5} a_6^{n_6} \cdot$$
$$\tilde{\varphi}(p(a,\beta)) f(a_8+\beta_1, a_{10}+\beta_2)\bar{g}(\beta)d\beta da$$
$$+ \sum_{j_4=0}^{1} d(n,1) \int_{\Re^{12}} a_1^{n_1-1} a_2^{n_2} a_3^{n_3-j_4} a_5^{n_5} a_6^{n_6} a_9^{1+j_4} \cdot$$
$$\tilde{\varphi}(p(a,\beta)) f^{(1-j_4)}(a_8+\beta_1, a_{10}+\beta_2)\bar{g}(\beta)d\beta da.$$

By Lemma 5.3.2 we know that

$$0 = \int_{\Re} \varphi(\alpha)d\alpha_4 = \int_{\Re} \varphi(p(a,\beta))da_4$$

for almost every $a_1, \ldots, a_3, a_5, \ldots, a_{10}, \beta$. Therefore

$$0 = \sum_{j_4=0}^{1} d(n,1) \int_{\Re^{12}} a_1^{n_1+k_4} a_2^{n_2} a_3^{n_3-j_4} a_5^{n_5} a_6^{n_6} a_9^{1+j_4} \cdot$$
$$\tilde{\varphi}(p(a,\beta)) f^{(1-j_4)}(a_8+\beta_1, a_{10}+\beta_2)\bar{g}(\beta)d\beta da$$

for every f, g. We conclude that

$$0 = \int_{\Re^{12}} a_1^{n_1} a_2^{n_2} a_3^{n_3} a_4 a_5^{n_5} a_6^{n_6} \tilde{\varphi}(p(a,\beta)) f(a_8 + \beta_1, a_{10} + \beta_2)\bar{g}(\beta) d\beta da$$

for every f, g and for n_1, n_2, n_3, n_5, n_6 terms in a Szasz-Müntz sequence.

Now we may apply Lemma 5.3.2 to the function $\psi(a) = a_4\tilde{\varphi}(a)$, since $\psi(p(a,\beta)) = a_4\tilde{\varphi}(p(a,\beta))$. We can conclude that

$$0 = \int_{\Re} \alpha_4 \tilde{\varphi}(\alpha, \beta) d\alpha_4.$$

For the induction step we may assume that $0 = \int_{\Re} \alpha_4^j \varphi(\alpha) d\alpha_4$ for every $j < n_4$, and we must prove this for n_4. By the assumption of the theorem

$$0 = \sum_{k_4=0}^{n_4} \sum_{j_4=0}^{k_4} d(n, k_4) \int_{\Re^{12}} a_1^{n_1+k_4} a_2^{n_2} a_3^{n_3-j_4} a_4^{n_4-k_4} a_5^{n_5} a_6^{n_6} a_9^{k_4+j_4} \cdot$$
$$\tilde{\varphi}(p(a,\beta)) f^{(k_4-j_4)}(a_8 + \beta_1, a_{10} + \beta_2)\bar{g}(\beta) d\beta da. \quad (7)$$

By the induction hypothesis if $n_4 - k_4 < n_4$ then

$$0 = \int_{\Re^{12}} a_1^{n_1+k_4} a_2^{n_2} a_3^{n_3-j_4} a_4^{n_4-k_4} a_5^{n_5} a_6^{n_6} a_9^{k_4+j_4} \cdot$$
$$\tilde{\varphi}(p(a,\beta)) f^{(k_4-j_4)}(a_8 + \beta_1, a_{10} + \beta_2)\bar{g}(\beta) d\beta da$$

for every f, g and for n_1, n_2, n_3, n_5, n_6 terms in a Szasz-Müntz sequence. Therefore we must have $k_4 = 0$. Combine this with (7) to see that

$$0 = \int_{\Re^{12}} a_1^{n_1} a_2^{n_2} a_3^{n_3} a_4^{n_4} a_5^{n_5} a_6^{n_6} \tilde{\varphi}(p(a,\beta)) f(a_8 + \beta_1, a_{10} + \beta_2)\bar{g}(\beta) d\beta da$$

for every f, g and for n_1, n_2, n_3, n_5, n_6 terms in a Szasz-Müntz sequence.

Now apply Lemma 5.3.2 to this integral to see that

$$0 = \int\limits_{\Re} \alpha_4^{n_4} \varphi(\alpha) d\alpha_4$$

for $\alpha_1, \alpha_2, \alpha_3, \alpha_5, \ldots, \alpha_{10}$ almost everywhere. □

Remark. We will show in the next chapter that in groups with representations that are square integrable mod the center the trace transform (times the Pfaffian) has an entire extension to the complexified dual and satisfies a weak Szasz-Müntz theorem. It's interesting to note the seemingly different behavior between these two transforms, especially since the trace transform can be expressed as a sum (albeit infinite) of matrix coefficients.

CHAPTER 6

THE TRACE TRANSFORM

Now that we have considered the operator valued transform and its matrix coefficients in detail we want to turn our attention to the trace transform. We want to prove a Szasz-Müntz type theorem for the trace transform on groups that have flat orbits and a fixed radical for all of the parameterizing representations (this includes the groups that have square integrable representations mod the center [5]).

Let $\mathfrak{G}$ be the nilpotent Lie group under consideration, and $\varphi \in C_c^\infty(\mathfrak{G})$. The trace transform of φ is defined in terms of the operator valued transform

$$Tr\,\pi_\ell(\varphi) = \sum_{n=1}^{\infty} \left\langle \pi_\ell(\varphi)\xi_n, \xi_n \right\rangle.$$

It turns out that as in Euclidean space the trace transform has nicer properties when suitably normalized. So we will consider not the trace transform but the absolute value of the Pfaffian times the trace transform,

$$\hat{\varphi}_{T_r}(\ell) = |Pf(\ell)| Tr\pi_\ell(\varphi).$$

Generally the trace transform does not have an entire extension to the complexified dual space, even when φ has compact support [3]. But when the group is as specified above the transform $\hat{\varphi}_{T_r}(\ell)$ will have entire extension, if φ is compactly supported, and in fact it will satisfy a Szasz-Müntz theorem.

The proof centers around the parameterizing map ψ for the collection of generic orbits:

$$\psi : (U \bigcap V_T) \times V_S \to U$$

is the map defined by the the condition that $\psi(\ell_T, \ell_S)$ is the unique point in the orbit O_{ℓ_T} such that $p_S(\psi(\ell_T, \ell_S)) = \ell_S$, where $p_S : \mathfrak{g}^* \to V_S$ is projection. Here $V_T = \Re\text{--span}\{X^*_{t_i} | t_i \in T\}, V_S = \Re\text{--span}\{X^*_{s_i} | s_i \in S\}$

Let $\varphi_{\text{o}} = \varphi \circ exp$. It is known then that

$$Tr\pi_{\ell_T}(\varphi) = \frac{1}{|Pf(\ell_T)|} \int\limits_{V_S} \hat{\varphi}_{\text{o}}(\psi(\ell_T, \ell_S)) d\ell_S$$

or

$$\hat{\varphi}_{T_r}(\ell_T) = \int\limits_{V_S} \hat{\varphi}_{\text{o}}(\psi(\ell_T, \ell_S)) d\ell_S,$$

where $d\ell_S$ is normalized Lebesgue measure on the Euclidean space V_S.

Consider a group where the the orbits for the generic functionals are flat, and there is one fixed radical for all the parameterizing functionals ℓ_T. So for each ℓ_T, $r_{\ell_T} = r$ is an ideal and $O_{\ell_T} = \ell + r^{\perp}$ (where $\mathfrak{h}^{\perp} = \{\ell \in \mathfrak{g}^* | \ell(\mathfrak{h}) = 0\}$). Since r is an ideal we may take a strong Malcev basis $\{X_1, \ldots, X_n\}$ passing through r.

Therefore $r = \Re\text{--span}\{X_{t_i} | t_i \in T\}$, since t_i is a non-jump index if and only if there is a vector $Y_{t_i} \in \Re\text{--span}\{X_1, \ldots, X_{t_i - 1}\}$ such that $X_{t_i} + Y_{t_i} \in r$.

In particular for parameterizing ℓ_T we have $O_{\ell_T} = \ell_T + r^{\perp} = \ell_T + V_S$. So the parameterizing map $\psi(\ell_T, \ell_S)$ is the map $\psi(\ell_T, \ell_S) = \ell_T + \ell_S$. We have

$$\begin{aligned}\hat{\varphi}_{T_r}(\ell_T) &= \int\limits_{V_S} \hat{\varphi}_o(\psi(\ell_T, \ell_S))d\ell_S \\ &= \int\limits_{V_S} \hat{\varphi}_o(\ell_T + \ell_S)d\ell_S \\ &= \varphi_o(x_{t_1}, \ldots, x_{t_r}, 0, \ldots, 0)\hat{}(\ell_T).\end{aligned}$$

In this case the trace transform turns out to be a partial Fourier transform in the radical variables.

Unfortunately the trace transform can vanish on a set of positive measure in $U \bigcap V_T$ while $\varphi \neq 0$. So there is no hope of proving a full fledged Szasz-Müntz theorem as we did for the matrix coefficients. However we can prove the following variations.

Theorem 6.1 *If $\varphi \in C^\infty(\mathfrak{G})$ and compactly supported on r then $\hat{\varphi}_{T_r}(\ell) : V_T \to \mathbf{C}$ has entire extension to $V_T^{\mathbf{C}}$ and satisfies a Szasz-Müntz theorem. In particular if $\mu_1, \ldots, \mu_r$ are Szasz-Müntz sequences, ℓ_T is a fixed parameterizing functional, and*

$$\frac{\partial^{\mu_1(k_1)+\ldots+\mu_r(k_r)}}{\partial\ell_{t_1}^{\mu_1(k_1)} \ldots \partial\ell_{t_r}^{\mu_r(k_r)}} \hat{\varphi}_{T_r}(\ell_T) = 0$$

for every $k_1, \ldots, k_n \in \mathbf{N}^n$ then $\hat{\varphi}_{T_r}(\ell_T) = 0$.

Proof. The theorem follows from the above arguments and the Euclidean Szasz-Müntz theorem, Theorem 2.6. □

Theorem 6.2 *If $\varphi \in C_c^\infty(\mathfrak{G})$ then $\widehat{(\varphi^* * \varphi)}_{T_r}(\ell) : V_T \to \mathbf{C}$ has entire extension to $V_T^{\mathbf{C}}$ and satisfies a Szasz-Müntz theorem. In particular if $\mu_1, \ldots, \mu_r$ are Szasz-Müntz sequences, ℓ_T is a fixed parameterizing functional, and*

$$\frac{\partial^{\mu_1(k_1)+\ldots+\mu_r(k_r)}}{\partial \ell_{t_1}^{\mu_1(k_1)} \ldots \partial \ell_{t_r}^{\mu_r(k_r)}} \widehat{(\varphi^* * \varphi)}_{T_r}(\ell_T) = 0$$

for every $k_1, \ldots, k_n \in \mathbf{N}^n$ then $\varphi = 0$.

Proof. If $\varphi \in C_c^\infty(\mathfrak{G})$ then so $\varphi^* * \varphi$. The above theorem shows that

$$Tr\pi_{\ell_T}(\varphi^* * \varphi) = \|\pi_{\ell_T}(\varphi)\|_{H-S}^2 = 0$$

for all parameterizing ℓ_T. The Plancherel theorem for nilpotent Lie groups now shows that φ vanishes. □

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