The Generalized Kompaneets Equation.

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THE GENERALIZED KOMPANEETS EQUATION

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

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# TABLE OF CONTENTS

**ACKNOWLEDGEMENTS** ................................................................. ii

**ABSTRACT** ................................................................................. iv

**INTRODUCTION** ....................................................................... 1

**CHAPTER 1  DEGENERATE NONLINEAR PARABOLIC PROBLEMS** ........................................................................ 3
  1.1 Introduction ........................................................................... 3
  1.2 The Semigroup Approach ....................................................... 4
  1.3 The Degenerate Case ............................................................ 5

**CHAPTER 2  THE LINEAR KOMPANEETS EQUATION** ............. 13
  2.1 Introduction .......................................................................... 13
  2.2 Dissipativity .......................................................................... 15
  2.3 The Range Condition ........................................................... 17
  2.4 A Self-adjointness Result ...................................................... 27

**CHAPTER 3  LINEAR PARABOLIC PROBLEMS WITH DISCONTINOUS COEFFICIENTS AND THE WENTZELL BOUNDARY CONDITION** ............................................. 33
  3.1 Introduction .......................................................................... 33
  3.2 Dissipativity .......................................................................... 37
  3.3 The Range Condition ........................................................... 38

**CHAPTER 4  THE NONLINEAR KOMPANEETS EQUATION** ........ 48
  4.1 Introduction .......................................................................... 48
  4.2 Dissipativity .......................................................................... 49
  4.3 Perturbation Theory .............................................................. 50

**BIBLIOGRAPHY** ...................................................................... 53

**APPENDIX:LETTER OF PERMISSION** ..................................... 55

**VITA** ......................................................................................... 56
ABSTRACT

In the dissertation, the generalized Kompaneets equation

\[ \frac{\partial u}{\partial t} = \frac{1}{\beta(x)} [\alpha(x)(u_x + ku + F(x, u))]_x \]

(for \( x, t > 0 \)) is studied. For the linear case, when \( F \equiv 0 \), a complete theory is given. A brief discussion is carried for the nonlinear case when \( F(x, u) = f(x)g(u) \).

For the following equation,

\[ v_t = \varphi(y, v_y)v_{yy} + \psi(y, v, v_y), \]

Goldstein and Lin’s result is extended to degenerate case.

Also, for the following linear operator,

\[ Au = \alpha(x)u'' + \beta(x)u' \]

(for \( x \in [0, 1] \)), Clément and Timmermans' result is extended to the case of discontinuous coefficients \( \alpha \) and \( \beta \).
INTRODUCTION

A basic equation of plasma physics is the Kompaneets equation
\[
\frac{\partial u}{\partial t} = \frac{1}{x^2}[x^4(u_x + u + u^2)]_x
\]
(for \(x, t > 0\)). The "natural" space for this problem (based on physical considerations) is \(L^1((0, \infty); x^2 dx)\). For brevity we postpone discussion of the physically relevant boundary condition. The wellposedness of the corresponding initial value-boundary value problem has been open ever since (1) was introduced in 1957 [17]. There are two obvious ways to attack this by semigroup methods.

One is to look at a class of problems of the form
\[
\frac{\partial u}{\partial t} = \frac{1}{\beta(x)}[\alpha(x)(u_x + ku + F(x, u))]_x
\]
(for \(x, t > 0\)). We give a complete theory of the linear version of this (i.e. \(F \equiv 0\)) in Chapter 2. The nonlinear version is briefly discussed in Chapter 4. Another approach is to make (2) a problem for \(v(t, y) = u(t, x)\) with \(y = \frac{2}{\pi} \tan^{-1} x \in (0, 1)\). This takes the form
\[
v_t = \varphi(y, v_y)v_{yy} + \psi(y, v, v_y).
\]
Equations such as this were studied in a series of papers by J. A. Goldstein and Lin, but they required that \(\varphi \in C([0, 1] \times \mathbb{R})\) whereas in the case of (3) derived from (1), \(\varphi(y, v_y)\) must approach \(\infty\) as \(y \to 1\). In Chapter 1 we extend some of the work of J. A. Goldstein and Lin to this context.

A key feature of these studies is boundary degeneracy using the Wentzell boundary condition. A definitive result in this area was proved by Clément and Timmermans in the \(C[0, 1]\) context. Heavy use of this is made throughout this thesis.
Clément and Timmermans [2] studied linear operator of the form

\[ Au = \alpha(x)u'' + \beta(x)u' \]

where \( \alpha, \beta \in C(0, 1) \) with \( \alpha > 0 \) on \((0, 1)\). Chapter 3 is devoted to extending some of their work to the context in which \( \alpha \) and \( \beta \) are allowed to be discontinuous. The ideas of [2] also play a role in the proofs of Chapter 2.
CHAPTER 1

DEGENERATE NONLINEAR PARABOLIC PROBLEMS

1.1 Introduction

In this chapter, we consider the parabolic partial differential equation of the form

\[
\frac{\partial u}{\partial t} = \varphi(x, u') u'' + \psi(x, u, u')
\]

for \( t \geq 0 \) and \( x \in \Omega = (0, 1) \). Here ' denotes \( \partial/\partial x \). The boundary condition is the Wentzell boundary condition, i.e.,

\[
\varphi(x, u') u'' + \psi(x, u, u') \to 0, \quad \text{as } x \to 0, 1.
\]

J. A. Goldstein and Lin studied this problem in a series of papers. In [11], they studied (1) with \( \psi \equiv 0 \). They assumed \( \varphi(x, u') \to 0 \) slowly as \( x \to 0, 1 \). The boundary conditions could be of various types. It is assumed that \( \varphi(x, \xi) \geq \varphi_0(x) \) holds for all \( (x, \xi) \in [0, 1] \times \mathbb{R} \), where \( \varphi_0 \in C[0, 1], \varphi_0(x) > 0 \) on \( (0, 1) \), and \( \varphi_0(x) \to 0 \) slowly in the sense that \( \int_0^1 \varphi_0(x)^{-p} dx < \infty \) for a suitable \( p \geq 1 \). As G. R. Goldstein [6] showed \( p \) can always be chosen to be \( p = 1 \). In [13], again they assumed \( \varphi(x, \xi) \geq \varphi_0(x), \varphi_0 \in C[0, 1], \varphi_0(x) > 0 \) on \( (0, 1) \). No assumption is made on how fast \( \varphi_0(x) \to 0 \) as \( x \to 0, 1 \); the convergence can be arbitrarily rapid. In [14], they studied (1) with \( \psi \neq 0 \). They assumed

(H1) \( \phi \in C([0, 1] \times R); \varphi(x, \xi) \geq \varphi_0(x) \) holds for all \( x, \xi \in [0, 1] \times \mathbb{R} \); \( \varphi_0 \in C[0, 1], \varphi_0(x) > 0 \) on \( (0, 1) \);

(H2) \( \psi \in C^1([0, 1] \times R \times R); \xi \to \psi(x, \eta, \xi) \) is nonincreasing on \( R \) for all \( (x, \xi) \in [0, 1] \times \mathbb{R} \); \( \psi \) has at most linear growth in \( \xi \), i.e., for each \( a > 0 \) there is a constant
$K_a$ such that

$$|\psi(x, \eta, \xi)| \leq K_a(1 + |\xi|)$$

for all $x \in [0, 1]$, $\eta \in [-a, a]$, and $\xi \in R$; $\psi(x, 0, 0) = 0$ for all $x \in [0, 1]$.

In this chapter we extend J. A. Goldstein and Lin's result in the case that we assume $\varphi \in C((0, 1) \times R)$; $\varphi(x, \xi) \geq \varphi_0(x)$ holds for all $x, \xi \in (0, 1) \times R$; $\varphi_0(x) \in C(0, 1)$, $\varphi_0(x) > 0$ on $(0, 1)$ compared to (H1) in [14]. The point is that $\varphi$ and $\varphi_0$ need not be continuous at the endpoints $x = 0, 1$.

1.2 The Semigroup Approach

The problem (1) with boundary condition (2) can be written as a differential equation in Banach space of the form

$$\begin{cases}
\frac{du}{dt} = Au \\
u(0) = f,
\end{cases} \tag{3}$$

where the boundary condition is incorporated into the domain $\mathcal{D}(A)$ of the operator $A$. The semigroup method replaces (3) by the backward difference equation

$$\epsilon^{-1}[u_\epsilon(t) - u_\epsilon(t - \epsilon)] = Au_\epsilon(t),$$

where $\epsilon > 0$, whence $u_\epsilon(t) = (I - \epsilon A)^{-1}u_\epsilon(t - \epsilon)$. Thus for $\epsilon = t/n$ and $u_\epsilon(0) = u_0$,

$$u_\epsilon(t) = (I - \frac{t}{n}A)^{-n}u_0.$$  

In order to have $u_\epsilon$ converge to the desired solution $u$, it is convenient to assume that $A$ is *essentially m-dissipative*.

An operator $A$ is called *dissipative* if it satisfies

$$\|(I - \lambda A)^{-1}\|_{\text{Lip}} \leq 1 \tag{4}$$

for each $\lambda > 0$, and it is called *m-dissipative* if it satisfies (4) and

$$\mathcal{R}(I - \lambda A) = X \tag{5}$$
for each \( \lambda > 0 \). \( A \) is essentially \( m \)-dissipative if it satisfies (4) for each \( \lambda > 0 \), and \( \mathcal{R}(I - \mu A) \) is dense in \( X \) for some (or equivalently for all) \( \mu > 0 \).

Here \( \|S\|_{\text{Lip}} \) is the smallest constant \( L \) such that \( \|Sf - Sg\| \leq L\|f - g\| \) holds for all \( f, g \) in the domain of \( S \).

**Crandall-Liggett-Benilan Theorem.** Let \( A \) be essentially \( m \)-dissipative on \( X \). Let \( \overline{A} \) be the closure of (the graph of) \( A \) in \( X \times X \). Then for all \( f \in D := \overline{D(A)} \),

\[
T(t)f = \lim_{n \to \infty} (I - \frac{t}{n} A)^{-n} f
\]

exists. Moreover, \( T = \{T(t) : t \geq 0\} \) is a strongly continuous contraction semigroup on \( D \), i.e.,

\[
T(t) : D \to D \quad \text{for all } t \geq 0,
\]

\[
T(t + s) = T(t)T(s) \quad \text{for all } t, s \geq 0,
\]

\[
T(0)f = f \quad \text{for all } f \in D,
\]

\[
t \to T(t)f \in C([0, \infty); D) \quad \text{for each } f \in D,
\]

\[
\|T(t)\|_{\text{Lip}} \leq 1 \quad \text{for each } t \geq 0.
\]

Finally, for all \( u_0 \in D \), \( u(t) = T(t)u_0 \) is the unique mild solution of (3).

**1.3 The Degenerate Case**

Consider equation (1)

\[
\frac{\partial u}{\partial t} = \varphi(x, u')u'' + \psi(x, u, u')
\]

in the real Banach space \( X = C[0, 1] \) with the Wentzell boundary condition (2). We also make the following hypotheses.

(H1) \( \varphi \in C((0, 1) \times R); \varphi(x, \xi) \geq \varphi_0(x) \) holds for all \( (x, \xi) \in (0, 1) \times R; \varphi_0 \in C(0, 1), \)
\( \varphi_0(x) > 0 \) on \((0,1)\).

(H2) \( \psi \in C([0,1] \times R \times R); \xi \rightarrow \psi(x, \eta, \xi) \) is nonincreasing on \(R\) for all \((x, \xi) \in (0,1) \times R\).

Define an operator \(A\) by

\[
Au(x) = \varphi(x, u')u'' + \psi(x, u, u')
\]

for \(x \in [0,1]\) and \(u \in D(A) = \{ u \in C^2(0,1) \cap C[0,1]: Au(x) \rightarrow 0 \text{ as } x \rightarrow \{0,1}\}\).

\textbf{(1.3.1) Lemma.} Suppose (H1), (H2) hold. Assume in addition \(\varphi_0 \geq \epsilon > 0\) for some \(\epsilon > 0\) and all \(x \in (0,1)\) and

\[
\limsup_{|\eta| + |\xi| \rightarrow \infty} \sup_{x} \frac{|\psi(x, \eta, \xi)|}{|\eta| + |\xi|} = 0.
\]

Then \(A\) is \(m\)-dissipative.

Proof: Dissipativity. Let \(u, v \in D(A), w = u - v \neq 0\) and \(w(x_0) = \pm \|w\|\) for some \(x_0 \in [0,1]\). Take \(w(x_0) = \|w\|\); otherwise replace \(w\) by \(-w = v - u\).

If \(x_0 \in (0,1)\), then \(w'(x_0) = 0, w''(x_0) \leq 0\) and, using (H1), (H2) and the first and second derivative tests for weak maxima,

\[
\|u - v\| = (u - v)(x_0) \\
\leq (u - v)(x_0) - \lambda \cdot (x, u'(x_0))w''(x_0) - \lambda [\psi(x_0, u(x_0), u'(x_0)) - \psi(x_0, v(x_0), v'(x_0))] \\
= (u - v - \lambda (A \cdot \cdot - A v))(x_0) \\
\leq \|u - v - \lambda (A \cdot \cdot - A v)\|.
\]

If \(x_0 \in \{0,1\}\), then \(Au(x_0) = Av(x_0) = 0\). The above inequality continues to hold.
Range condition. Let $\lambda > 0$, $h \in C[0,1]$. We need to solve the equation

$$u - \lambda Au = h,$$

(6)
i.e.,

$$u - \lambda \varphi(x, u')u'' - \lambda \psi(x, u, u') = h$$
on $(0,1)$. The Wentzell boundary condition implies that $u$ satisfies $u(0) = h(0)$ and $u(1) = h(1)$.

Let $f$ be a linear function such that $f(0) = h(0)$, $f(1) = h(1)$. Let $u = v + f$, then $u' = v' + f'$ and $u'' = v''$. Note that $f'$ is a constant function.

So we have

$$\begin{cases}
v - \lambda \tilde{\varphi}(x, v')v'' - \lambda \tilde{\psi}(x, v, v') = \tilde{h}, \\
v(0) = v(1) = 0
\end{cases}$$

(7)

where $\tilde{\varphi}(x, v') = \varphi(x, v' + f')$, $\tilde{\psi}(x, v, v') = \psi(x, v + f, v' + f')$, $\tilde{h} = h - f$.

Let $L$ be the negative Dirichlet Laplacian on $[0,1]$; (7) is equivalent to

$$v(x) = \int_0^1 g(x, y) \frac{\tilde{h}(y) - v(y) + \lambda \tilde{\psi}(y, v(y), v'(y))}{\lambda \tilde{\varphi}(y, v'(y))} dy$$

where $g$ is the Green function of $L$.

Let

$$(S_\tilde{h}v)(x) = \int_0^1 g(x, y) \frac{\tilde{h}(y) - v(y) + \lambda \tilde{\psi}(y, v(y), v'(y))}{\lambda \tilde{\varphi}(y, v'(y))} dy.$$ 

We want to show that $S_\tilde{h}$ is a continuous compact self-map of some ball in $C^1[0,1]$.

The proof that $S_\tilde{h}$ is a continuous and compact is similar to the proof in the case of $\psi \equiv 0$. See [11] for details.

Here we show $S_\tilde{h}$ maps some ball into itself in $C^1[0,1]$. 
Let $\epsilon > 0$ be given.

Since
\[
\limsup_{|\eta| + |\xi| \to \infty} \sup_x \frac{\psi(x, \eta, \xi)}{|\eta| + |\xi|} = 0,
\]
then there exists $N > 0$ such that
\[
\|\psi(x, v, v')\| \leq \frac{\epsilon}{2\|g\|} \|v\|_C
\]
when $\|v\|_C > N$ and
\[
\|\psi(x, v, v')\| \leq \frac{\epsilon}{2\|g\|} N
\]
when $\|v\|_C \leq N$. Choose $\lambda > 0$ so large that
\[
\lambda^{-1} \epsilon^{-1} \max\{\|\partial_x g\|, \|g\|\} < 1/2
\]
and choose $N' \geq \frac{2\lambda^{-1} \epsilon^{-1} \|\tilde{h}\| \max\{\|\partial_x g\|, \|g\|\}}{1 - 2\lambda^{-1} \epsilon^{-1} \max\{\|\partial_x g\|, \|g\|\}}$.

Let $\tilde{N} = \max\{N, N'\}$. Then for $\|v\|_C \leq \tilde{N}$,
\[
\|S_h v\|
\leq \|g\| \lambda^{-1} \epsilon^{-1} \|\tilde{h}\| + \|v\| + \lambda \|\psi(x, v, v')\|
\leq \|g\| \lambda^{-1} \epsilon^{-1} \|\tilde{h}\| + \|v\| + \lambda \frac{\epsilon}{2\|g\|} \tilde{N}
\leq \tilde{N}.
\]
Similarly, $\|S_h v'\| \leq \tilde{N}$.

By the Schauder fixed point theorem, $S_h$ has a fixed point, i. e. , $A$ is m-
 dissipative. $\Box$

(1.3.2) Theorem Suppose (H1), (H2) hold. Assume in addition that
\[
\psi(x, \eta, \xi) = M_1(x, \eta, \xi) + M_0(x, \eta)\xi,
\]
where for all $a > 0$ there is a constant $K_a$ such that $|M_0(x, \eta)|, |M_1(x, \eta, \xi)| \leq K_a \sin(\pi x)$ for $x \in (0, 1), \eta \in [-a, a], \xi \in \mathbb{R}$. Moreover, $M_0(x, \eta) \leq \pi (\tan \pi x) \varphi_0(x)$ on $(0, 1/2) \times \mathbb{R}, M_0(x, \eta) \geq \pi (\tan \pi x) \varphi_0(x)$ on $(1/2, 1) \times \mathbb{R}$, $\psi(x, 0, 0) = 0$ on $(0, 1)$.

Then $A$ is essentially $m$-dissipative.

Proof: Note that $\lim_{z \to \pm \frac{1}{2}} \tan \pi z = \pm \infty$, so the inequalities on $M_0$ essentially restrict the behavior of $M_0$ only near the endpoints $x = 0, 1$. Let $(a_m, b_m)$ be an increasing sequence of open intervals with union $(0, 1)$ such that

$$\varphi^{-1}(x, \xi) \leq \varphi_0^{-1}(x) \leq m \text{ for } x \in (a_m, b_m),$$

$$\varphi_m(x, \xi) = \begin{cases} \varphi(x, \xi) & \text{for } x \in (a_m, b_m), \xi \in \mathbb{R}, \\ \max(\varphi(b_m, \xi), 1/m) & \text{for } x \in [b_m, 1), \xi \in \mathbb{R}, \\ \max(\varphi(a_m, \xi), 1/m) & \text{for } x \in (0, a_m], \xi \in \mathbb{R}. \end{cases}$$

If $\varphi_0(x) \to \infty$ as $x \to 1$, then we let

$$\varphi_m(x, \xi) = \begin{cases} \varphi(x, \xi) & \text{for } x \in (a_m, 1), \xi \in \mathbb{R}, \\ \max(\varphi(a_m, \xi), 1/m) & \text{for } x \in (0, a_m], \xi \in \mathbb{R}, \end{cases}$$

i.e. $b_m = 1$.

If $\varphi_0(x) \to \infty$ as $x \to 0$, then we let

$$\varphi_m(x, \xi) = \begin{cases} \varphi(x, \xi) & \text{for } x \in (0, b_m), \xi \in \mathbb{R}, \\ \max(\varphi(b_m, \xi), 1/m) & \text{for } x \in [b_m, 1), \xi \in \mathbb{R}, \end{cases}$$

i.e. $a_m = 0$.

Therefore, $\varphi_m$ can be chosen to have the following property:

$\varphi_m \geq \varphi$ and $\varphi_{m+1} \leq \varphi_m$.

Let

$$\psi_1(x, \eta, \xi) = \begin{cases} m & \text{when } \psi(x, \eta, \xi) > m, \\ \psi(x, \eta, \xi) & \text{when } |\psi(x, \eta, \xi)| \leq m, \\ -m & \text{when } \psi(x, \eta, \xi) < -m, \end{cases}$$
and
\[
\psi_m(x, \eta, \xi) = \begin{cases} 
\psi_1(x, \eta, \xi) & \text{for } x \in (a_m, b_m), \\
\min\{\frac{m \sin(\pi x)}{\sin(c_m \pi)}, \psi_1(x, \eta, \xi)\} & \text{when } \psi_1 \geq 0, \\
\max\{\frac{-m \sin(\pi x)}{\sin(c_m \pi)}, \psi_1(x, \eta, \xi)\} & \text{when } \psi_1 \leq 0,
\end{cases}
\]

for \( x \in [0, a_m] \cup [b_m, 1] \), where \( c_m = \min(a_m, 1 - b_m) \).

Since
\[
\lim \sup \sup_x \frac{\psi_m(x, \eta, \xi)}{|\eta| + |\xi|} = 0,
\]

then by Lemma 1.3.1, \( A_m \), defined with \( \varphi_m \) in place of \( \varphi \), \( \psi_m \) in place of \( \psi \), is \( m \)-dissipative.

Let \( h \in C^1[0,1], \lambda = 1 \).

To solve
\[
u_m - A_m u_m = h,
\]

let \( f \) be linear function on \([0,1]\) satisfying \( f(x) = h(x) \) for \( x \in \{0,1\} \). Then for \( \tilde{h} = h - f \), \( v_m = u_m - f \) satisfies
\[
\begin{cases} 
v_m - \bar{\varphi}_m(x, u'_m)v_m' - \bar{\psi}_m(x, u_m, u'_m) = \tilde{h}, \\
v_m(0) = v_m(1) = 0
\end{cases}
\]

where \( \bar{\varphi}_m(x, \xi) = \varphi_m(x, \xi + f') \), \( \bar{\psi}_m(x, \eta, \xi) = \psi_m(x, \eta + f, \xi + f') \).

For this problem, there exists a unique solution \( v_m \). Moreover,
\[
\|v_m\| \leq \|\tilde{h}\|.
\]

By the compactness and diagonalization method in [13], we can find a subsequence \( \{v_{m_k}\} \) of \( \{v_m\} \) converging to some \( v \) in \( C^2_{loc}(0,1) \). \( v \) satisfies
\[
v - \varphi(x, v' + f')v'' - \psi(x, v + f, v' + f') = \tilde{h}
\]
in \((0,1)\). Moreover

\[ \|v\| \leq \|\tilde{h}\|. \]

We wish to show that \(v_m \to v\) uniformly on \([0,1]\). In that case, \(v(0) = v(1) = 0\) and \(u = v + f\) satisfies

\[ u - Au = h. \]

Let \(v_m\) be the unique solution of

\[
\begin{cases}
  v_m - \varphi_m(x, v_m) v_m'' - \mathcal{M}_1(x, v_m, v_m') - M_0(x, v_m) v_m' = \tilde{h}, \\
  v_m(0) = v_m(1) = 0
\end{cases}
\]

where \(M_0(x, \eta) = M_0(x, \eta + f)\), \(M_1(x, \eta, \xi) = M_1(x, \eta + f, \xi + f') + f' M_0(x, \eta + f)\).

Since \(\tilde{h} \in C^1[0,1]\) and \(\tilde{h}(0) = \tilde{h}(1) = 0\), then

\[ |\tilde{h}(x)| \leq k_1 \sin(\pi x) \]

on \([0,1]\) for some \(k_1 > 0\).

Since \(\|v_m\| \leq \|\tilde{h}\|\), then there exists \(k_2 > 0\) such that

\[ |\mathcal{M}_1(x, v_m, v_m')| \leq k_2 \sin(\pi x). \]

Let \(k \geq k_1 + k_2\), \(l(x) = k \sin(\pi x)\), \(\gamma_m(x) = \varphi_m(x, v_m')^{-1}\). Since \(l'' = -k \pi^2 \sin(\pi x)\), \(l' = k \pi \cos(\pi x)\), \(M_0(x, \eta) \leq \pi (\tan \pi x) \varphi_0(x)\) on \((0,1/2) \times R\), \(M_0(x, \eta) \geq \pi (\tan \pi x) \varphi_0(x)\) on \((1/2,1) \times R\), then

\[
|v_m'' - \gamma [v_m - M_0(x, v_m) v_m']|
= \gamma_m|\tilde{h} + \mathcal{M}_1(x, v_m, v_m')|
\leq \gamma_m l
\leq \gamma_m l' - l'' - \gamma_m M_0(x, v_m) l'.
\]

So for \(z_m = l \pm v_m\), we have

\[
\begin{cases}
  z_m'' - \gamma_m [z_m - M_0(x, v_m) z_m'] \leq 0 \\
  z_m(0) = z_m(1) = 0.
\end{cases}
\]
By the maximal principle, we have

\[ z_m \geq 0 \]

on \([0,1]\). So

\[ |v_m(x)| \leq l(x) \]

for all \(x \in [0,1]\) and for all \(m \in \mathbb{N}\).

Thus the theorem follows. \(\square\)

**Example:** Let \( M_0 = -\sin(\pi x) \), \( M_1 = -x(1-x)u \), i.e., \( \psi(x,u,u') = -x(1-x)u - \sin(\pi x)u' \) and \( \varphi(x,u') = x(1-x) \). Then the operator \( A \) defined by

\[ Au(x) = \varphi(x,u')u'' + \psi(x,u,u') \]

is essentially \( m \)-dissipative on \( C[0,1] \).
CHAPTER 2
THE LINEAR KOMPANEETS EQUATION

2.1 Introduction

In this chapter, we consider the linear Kompaneets equation

\[ \frac{\partial u}{\partial t} = \frac{1}{\beta(x)}[\alpha(x)(u' + k(x)u)]' \]  

for \( 0 < x, t < \infty \). Here \( ' \) stands for \( \partial/\partial x \). The coefficients \( \alpha, \beta, k, \alpha', k' \) are continuous and \( \alpha, \beta \) are strictly positive on the open interval \( (0, \infty) \). No assumption is made about the coefficients at \( x = 0 \) or \( x = \infty \). The initial and boundary conditions are

\[ u(0, x) = f(x) \]

\[ \alpha(x)[u' + k(x)u] \to 0 \text{ as } x \to 0, \infty. \]

The model above comes from the generalized Kompaneets equation

\[ \frac{\partial u}{\partial t} = \frac{1}{\beta(x)}[\alpha(x)(u' + F(x, u))]' \]

where the most important case is considered by Kompaneets [17], namely, \( \alpha(x) = x^4, \beta(x) = x^2, F(x, u) = u + u^2 \). The context is plasma physics. The radiation density \( u(t, x) \) is nonnegative and gives the total photon number,

\[ N = \int_0^\infty u(t, x)x^2dx(< \infty). \]

Here \( x = h\nu/\theta \) is the normalized photon energy, \( h \) is Planck's constant, \( \nu \) is the frequency and \( \theta \) is the temperature. Also, the \( x^2 \) factor is a geometric factor

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expressing spherical symmetry. For more of the physical background, see Caflisch and Levermore [1]. For a discussion in the abstract context, see Goldstein [10].

Let $X_p = L^p((0, \infty); \beta(x)dx)$.

Define $A_p$ on $X_p$ as follows: $u \in Dom(A_p)$ iff $u \in X_p \cap W^{2,p}_{loc}(0, \infty)$, $\omega := \alpha(u' + k(x)u) \in W^{1,p}_{loc}(0, \infty)$, $\omega(x) \to 0$ as $x \to 0, \infty$, and $A_p u := \frac{1}{\beta(x)} \omega' \in X_p$.

As before, $' = \partial/\partial x$. We can formulate (1) - (3) as follows:

\[
\begin{align*}
\frac{du}{dt} &= Au \\
u(0) &= f.
\end{align*}
\]

Here $A = A_p$, with emphasis on the case $p = 1$.

To apply the Hille-Yosida Theorem, we need to check three things:
(i) Dissipativity. For all $\lambda > 0$ and $u - \lambda Au = h$, $\|u\| \leq \|h\|$ or $\|(I - \lambda A)^{-1}\| \leq 1$.
(ii) Range condition. There exists a dense set $D$ in the Banach space $X$ such that for any sufficient small $\lambda > 0$ and $h \in D$, the problem

\[u - \lambda Au = h\]

has a solution.
(iii) $Dom(A)$ is dense in $X$.

If (i)-(iii) hold, then by Hille-Yosida Theorem, (4) is governed by a semigroup $T$ given by

\[T(t)f = \lim_{n \to \infty} (I - \frac{t}{n} \overline{A})^{-n}f\]

where $\overline{A}$ is the closure of $A$, and the unique (mild) solution of (4) is given by $u(t) = T(t)f$, for all $f \in X$.

Our main results will be stated in Section 2.3. The difficult parts of the proof involve checking the range condition. This leads to various assumptions on $\alpha, \beta, k$. 
The dissipativity works very generally; this is verified in Section 2.2. Finally, the \( m \)-dissipativity results of Section 2.3 have some self-adjointness analogues; these are presented in Section 2.4.

2.2 Dissipativity

As before, \( \alpha, \beta, k \in C(0, \infty) \) with \( \alpha \) and \( \beta \) positive on \( (0, \infty) \).

\( (2.2.1) \) Lemma. [10] \( A_1 \) is dissipative on \( X_1 \).

For completeness, we include the proof.

Proof: Let \( \lambda > 0, h \in X_1 \) and let \( u \in Dom(A_1) \) be a solution of the equation

\[
 u - \lambda A_1 u = h.
\]

Then \( u \) is continuously differentiable and \( u' \) is locally absolutely continuous on \( (0, \infty) \).

Let

\[
 [u > 0] = \{ x \in (0, \infty) : u(x) > 0 \} = \bigcup_{n \in J} (a_n, b_n),
\]

\[
 [u < 0] = \{ x \in (0, \infty) : u(x) < 0 \} = \bigcup_{n \in K} (c_n, d_n);
\]

these are open sets by the continuity of \( u \). Next,

\[
 \|u\|_1 = \int_0^\infty u(x) \text{sign}_0(u) \beta(x) dx
\]

\[
 = \int_0^\infty u(x) \text{sign}_0(u) \beta(x) dx + \lambda \int_0^\infty A_1 u(x) \text{sign}_0(u) \beta(x) dx
\]

where \( \delta = \int_0^\infty A_1 u(x) \text{sign}_0(u) \beta(x) dx \). It suffices to prove \( \delta \leq 0 \).

\[
 \delta = \int_0^\infty = \int_{[u > 0]} + \int_{[u < 0]} + \int_{[u = 0]}
\]
Consider a term of the form $\int_{a_n}^{b_n}$. If $0 < a_n < b_n < \infty$, then $u(a_n) = u(b_n) = 0$ and $u'(a_n) \geq 0 \geq u'(b_n)$. Therefore we have

$$\int_{a_n}^{b_n} A_1 u(x) \text{sign}_0(u) \beta(x) dx$$

$$= \int_{a_n}^{b_n} [\alpha(x)(u'(x) + k(x)u(x))]' dx$$

$$= \left. \alpha(x)(u'(x) + k(x)u(x)) \right|_{a_n}^{b_n}$$

$$= \alpha(b_n)u'(b_n) - \alpha(a_n)u'(a_n)$$

$$\leq 0.$$

For the case of $(a_n, b_n) = (0, \infty)$ or $(a_n, b_n) = (0, b_n)$ with $b_n < \infty$ or $(a_n, b_n) = (a_n, \infty)$ with $a_n > 0$, the arguments are similar.

Let us just consider the case of $0 = a_n < b_n < \infty$.

$$\int_{a_n}^{b_n} A_1 u(x) \text{sign}_0(u) \beta(x) dx$$

$$= \left. \alpha(x)(u'(x) + k(x)u(x)) \right|_{0}^{b_n}$$

$$= I_1 + I_2.$$

By the argument above,

$$I_1 = \left. \alpha(x)(u'(x) + k(x)u(x)) \right|_{a_n}^{b_n} \leq 0.$$

Since $u \in \text{Dom}(A_1)$, by the boundary condition,

$$I_2 = - \lim_{x \to 0} \alpha(x)(u'(x) + k(x)u(x)) = 0;$$

thus $I_1 + I_2 \leq 0$. 

Similarly, we can show that each term \( \int_{c_0}^{\delta} \) is nonpositive. It follows that \( \delta \leq 0 \), and the lemma is proved. \( \blacksquare \)

2.3 The Range Condition

Let \( \lambda > 0, h \in C_c(0, \infty) \), where \( C_c(0, \infty) \) denotes the continuous compactly supported functions on the open interval \((0, \infty)\). We want to solve

\[
\begin{align*}
\{ & u - \lambda A_1 u = h, \\
& u \in D(A_1),
\end{align*}
\]

i.e.

\[
\begin{align*}
& \{ u - \lambda \frac{1}{\beta(x)}[\alpha(x)(u' + k(x)u)]' = h \in C_c(0, \infty), \\
& \beta(x)(u' + k(x)u) \to 0 \text{ as } x \to 0, \infty.
\end{align*}
\]

Instead of solving (5), we set

\[
v(x) = \int_0^x u(s) \beta(s) ds, \quad f(x) = \int_0^x h(s) \beta(s) ds.
\]

Here we require \( u \in X_1 \), therefore \( v \in C[0, \infty] \) necessarily. Hence \( v'(x) = u(x) \beta(x) \) or \( u(x) = \frac{v(x)}{\beta(x)} \).

Then (5) can be transformed to

\[
\begin{align*}
& \{ v(x) - \lambda \frac{\alpha(x)}{\beta(x)} v'' - \frac{\alpha(x)}{\beta(x)} v' + \frac{\alpha(x)k(x)}{\beta(x)} v' = f(x), \\
& \frac{\alpha(x)}{\beta(x)} v'' - \frac{\alpha(x)}{\beta(x)} v' + \frac{\alpha(x)k(x)}{\beta(x)} v' \to 0 \text{ as } x \to 0, \infty.
\end{align*}
\]

Let

\[
Bv = \frac{\alpha(x)}{\beta(x)} v'' - \frac{\alpha(x)}{\beta^2(x)} v' + \frac{\alpha(x)}{\beta(x)} k(x) v'.
\]

Then (6) can be written as

\[
\begin{align*}
& \{ v(x) - \lambda Bv(x) = f(x), \\
& Bv(x) \to 0 \text{ as } x \to 0, \infty.
\end{align*}
\]

Now change the interval from \((0, \infty)\) to \((0, 1)\) by letting \( y = \frac{2}{\pi} \tan^{-1} x \). Thus \( x \in (0, \infty) \) iff \( y \in (0, 1) \). Let \( \tilde{\alpha}(y) = \alpha(\tan \frac{\pi}{2} y), \tilde{\alpha}(y) = \alpha(\tan \frac{\pi}{2} y), \tilde{\beta}(y) = \beta(\tan \frac{\pi}{2} y), \)
\[ f(y) = f(\tan \frac{\pi}{2} y), \quad k(y) = k(\tan \frac{\pi}{2} y). \] Then equation (7) is transformed to
\[
\begin{align*}
\begin{cases}
\tilde{v}(y) - \lambda \tilde{B} \tilde{v}(y) = \tilde{f}(y), \\
\tilde{B} \tilde{v}(y) \to 0 \text{ as } y \to 0, 1,
\end{cases}
\end{align*}
\] (8)

where
\[
\tilde{B} \tilde{v}(y) = \frac{\tilde{\alpha}}{\beta} \left( \frac{4}{\pi^2} \cos^4 \frac{\pi}{2} y \right) \tilde{v}'' - \left( \frac{4}{\pi} \cos^2 \frac{\pi}{2} y \sin \frac{\pi}{2} y + \frac{4}{\beta^2} \left( \frac{4}{\pi^2} \cos^4 \frac{\pi}{2} y \right) - \frac{2}{\beta} \cos \frac{\pi}{2} y \right) \tilde{v}'.
\]

Clearly (7) and (8) are equivalent. They are equivalent to (5) when \( A_1 \) is replaced by its restriction \( A_2 \) to \( D(A_2) = D(A_1) \cap C^2(0, \infty) \). That is, \( D(A_2) \) is the same as \( D(A_1) \) except with \( W^{2,1}_{\text{loc}}(0, \infty) \) replaced by \( C^2(0, \infty) \) in the definition of \( D(A_1) \). (Note that \( A_2 \) has a different meaning here than it does in Section 2.1.)

To show \( A_1 \) is m-dissipative, it suffices to prove that \( \mathcal{R}(I - A_2) \) is dense (since \( A_1 \) is closed and dissipative).

To solve equation (8), we apply the Clément and Timmermans result [2] here. The Clément and Timmermans result is as follows.

Let \( \gamma_1, \gamma_2 \) be continuous on \( (0, 1) \) with \( \gamma_1 \) positive. Define \( A \) on \( X = C[0, 1] \) by
\[
Au = \gamma_1 u'' + \gamma_2 u',
\] (9)

\[ u \in \text{Dom}(A) = \{ u \in C[0, 1] \cap C^2(0, 1) : \text{ for } j = 0, 1, \lim_{x \to j} (\gamma_1 u'' + \gamma_2 u')(x) \to 0 \}. \]

The operator \( A \) is densely defined on \( X \).

(2.3.1) Lemma. [2] The operator \( A \) in (9) is m-dissipative iff \( \mathcal{R}(I - A) = X \).

The main result of [2] is:

(2.3.2) Lemma. [2] \( \mathcal{R}(I - A) = X \) iff for
\[
W(x) = \exp\left[-\int_{\frac{1}{2}}^{x} \gamma_2(s)\gamma_1(s)^{-1} ds\right],
\]
both \((H_0)\) and \((H_1)\) hold:

\((H_0)\) \(W \in L^1(0, \frac{1}{2})\) or \(\int_0^{\frac{1}{2}} W(x) \int_0^x \gamma_1(s)^{-1}W(s)^{-1} ds \, dx = \infty\) or both;

\((H_1)\) \(W \in L^1(\frac{1}{2}, 1)\) or \(\int_\frac{1}{2}^1 W(x) \int_x^1 \gamma_1(s)^{-1}W(s)^{-1} ds \, dx = \infty\) or both.

For the operator \(\mathcal{B} \tilde{v} = \gamma_1(y) \tilde{v}'' + \gamma_2(y) \tilde{v}'\) of (8), we have

\[
\gamma_1(y) = \frac{\tilde{\alpha}}{\beta} \left( \frac{4}{\pi^2} \cos^4 \frac{\pi}{2} y \right),
\]

\[
\gamma_2(y) = -\left[ \frac{\tilde{\alpha}}{\beta} \left( \frac{4}{\pi^2} \cos^3 \frac{\pi}{2} y \sin \frac{\pi}{2} y \right) + \frac{\tilde{\beta}'}{\beta^2} \left( \frac{4}{\pi^2} \cos^4 \frac{\pi}{2} y \right) - \frac{\tilde{\kappa}}{\beta \pi} \cos^2 \frac{\pi}{2} y \right],
\]

and \(\gamma_1(y) > 0\) on \((0, 1)\), \(\gamma_1(y), \gamma_2(y) \in C(0, 1)\).

\[
W(y) = \exp\left\{ -\int_{\frac{1}{2}}^y \gamma_2(s) \gamma_1(s)^{-1} ds \right\}
\]

\[
= \exp\left\{ -\int_{\frac{1}{2}}^y \left( \frac{\pi \sin \frac{\pi}{2} s}{\cos \frac{\pi}{2} s} \right) + \frac{\tilde{\beta}'}{\beta} - \frac{\pi \tilde{\kappa}}{2 \beta \pi} \right\} ds
\]

\[
= \frac{\tilde{\beta}(y)}{2\beta(\frac{1}{2}) \cos^2 \frac{\pi}{2} y} \exp\left\{ -\int_1^{\tan \frac{\pi}{2} y} k(s) ds \right\};
\]

\[
\int_0^{\frac{1}{2}} |W(y)| \, dy = \int_0^{\frac{1}{2}} \frac{\tilde{\beta}(y)}{2\beta(\frac{1}{2}) \cos^2 \frac{\pi}{2} y} \exp\left\{ -\int_1^{\tan \frac{\pi}{2} y} k(s) ds \right\} dy
\]

\[
= \frac{1}{\pi \beta(1)} \int_0^1 \beta(x) \exp\left\{ -\int_1^x k(s) ds \right\} dx;
\]

\[
\int_0^{\frac{1}{2}} W(y) \int_0^y \gamma_1(s)^{-1}W(s)^{-1} ds \, dy = \int_0^1 \beta(x) \exp\left\{ -\int_1^x k(s) ds \right\} \int_0^x \frac{\exp\left\{ \int_1^y k(s) ds \right\}}{\alpha(y)} \, dy \, dx.
\]

Similarly,

\[
\int_{\frac{1}{2}}^1 |W(y)| \, dy = \frac{1}{\pi \beta(1)} \int_1^\infty \beta(x) \exp\left\{ -\int_1^x k(s) ds \right\} dx;
\]

\[
\int_{\frac{1}{2}}^1 W(y) \int_y^1 \gamma_1(s)^{-1}W(s)^{-1} ds \, dy = \int_1^\infty \beta(x) \exp\left\{ -\int_1^x k(s) ds \right\} \int_x^\infty \frac{\exp\left\{ \int_1^y k(s) ds \right\}}{\alpha(y)} \, dy \, dx.
\]

(2.3.3) Lemma. \(\mathcal{R}(I - \mathcal{B}) = X\) iff both \((K_0)\) and \((K_1)\) hold:
(K₀) \( \beta(x) \exp[- \int_1^x k(s) ds] \in L^1(0, 1) \) or
\[
\int_0^1 \beta(x) \exp[- \int_1^x k(s) ds] \int_0^x \frac{\exp[\int_1^{x'} k(s) ds]}{a(y)} dy dx = \infty \text{ or both;}
\]
(K₁) \( \beta(x) \exp[- \int_1^x k(s) ds] \in L^1(1, \infty) \) or
\[
\int_1^\infty \beta(x) \exp[- \int_1^x k(s) ds] \int_x^\infty \frac{\exp[\int_1^{x'} k(s) ds]}{a(y)} dy dx = \infty \text{ or both.}
\]

(2.3.4) \textbf{Proposition.} Suppose
\[
\int_0^\infty \beta(x) \exp[- \int_1^x k(s) ds] dx < \infty,
\]
\text{i.e.,}
\[
\beta(x) \exp[- \int_1^x k(s) ds] \in L^1(0, \infty).
\]
Then \( \hat{B} \) defined in (8) is \( m \)-dissipative on \( X = C[0,1] \).

\textbf{Proof:} It follows directly from Lemma 2.3.3, which in turn follows from Lemmas 2.3.1 and 2.3.2. \( \square \)

By Proposition 2.3.4, equation (8) has a solution \( \tilde{v} \in C[0,1] \cap C^2(0,1) \). Then equation (7) has a solution \( v \in C[0, \infty) \cap C^2(0, \infty) \) with \( \lim_{x \to \infty} v(x) \) finite. Furthermore, there exists a solution \( u(x) \in C^1(0, \infty) \) for the equation (5) since \( u(x) = \frac{v'(x)}{\beta(x)} \), provided \( \beta \in C^1(0, \infty) \).

(2.3.5) \textbf{Proposition.} Let \( u \in C^1(0, \infty) \) be a solution of equation (5). Suppose
\[
\int_0^\infty \beta(x) \exp[- \int_1^x k(s) ds] dx < \infty.
\]
Then \( u(x) \geq 0 \) for \( x \in (0, \infty) \) if \( h \geq 0 \). \textbf{In other words, the resolvent operator governing (5) \textbf{(and hence also the semigroup governing (1)-(3)) is positive.}

\textbf{Proof:} (Compare \cite{10}). If \( u(x) < 0 \) for \( x \in (0, \infty) \), then for
\[
u - \lambda Au = h
\]
we have
\[
\frac{u - h}{\lambda} = Au = \frac{1}{\beta} u',
\]
where \( \omega = \alpha(u' + ku) \), and
\[
\int_0^\infty \frac{u-h}{\lambda} \beta dx < 0.
\]

But
\[
\int_0^\infty \frac{1}{\beta} \omega' \beta dx = \omega(x) \bigg|_0^\infty = 0,
\]
\[
\int_0^\infty \frac{1}{\beta} \omega' \beta dx = \omega(x) \bigg|_0^\infty = 0,
\]
by (5), which is a contradiction.

So \( \{x : u(x) \geq 0\} \neq \emptyset \) and \( \{x : u(x) < 0\} \) is open since \( u \in C^1(0, \infty) \).

Let \( \{x : u(x) < 0\} = \bigcup_{n \in J} (a_n, b_n) \), where \( J \) is an at most countable set and \( \{(a_n, b_n)\} \) are pairwise disjoint open intervals.

Pick \( n \in J \) (if \( J \neq \emptyset \)). If \( 0 < a_n < b_n < \infty \), then we have \( u(a) = u(b) = 0 \), \( u'(a) \leq 0 \leq u'(b) \), \( u(x) < 0 \) for \( x \in (a_n, b_n) \).

So
\[
\int_{a_n}^{b_n} \frac{u-h}{\lambda} \beta dx < 0
\]
and for \( \omega = \alpha(u' + ku) \) as above,
\[
\int_{a_n}^{b_n} \frac{1}{\beta} \omega' \beta dx = \omega(x) \bigg|_{a_n}^{b_n} = \omega(b_n) - \omega(a_n)
\]
\[
= \alpha(b_n)(u'(b_n) + k(b_n)u(b_n)) - \alpha(a_n)(u'(a_n) + k(a_n)u(a_n))
\]
\[
= \alpha(b_n)u'(b_n) - \alpha(a_n)u'(a_n)
\]
\[
\geq 0,
\]
again a contradiction.

If \( (a_n, b_n) = (0, b) \), \( b < \infty \), then \( u(b) = 0 \), \( u'(b) \geq 0 \),
\[
\int_0^b \frac{u-h}{\lambda} \beta dx < 0
\]
and
\[
\int_0^b \frac{1}{\beta} \omega' \beta dx = \omega(x) \bigg|_0^b \\
= \alpha(b)(u'(b) + k(b)u(b)) \\
= \alpha(b)u'(b) \\
\geq 0,
\]
again a contradiction.

If \((a_n, b_n) = (a, \infty), 0 < a\), then \(u(a) = 0, u'(a) \leq 0,\)
\[
\int_a^\infty \frac{u - h}{\lambda} \beta dx < 0
\]
and
\[
\int_\alpha^\infty \frac{1}{\beta} \omega' \beta dx = \omega(x) \bigg|_\alpha^\infty \\
= -\alpha(a)(u'(a) + k(a)u(a)) \\
= -\alpha(a)u'(a) \\
\geq 0,
\]
which again is a contradiction. Thus \(u(x) \geq 0\) for \(x \in (0, \infty)\) follows. □

(2.3.6) Proposition. Suppose
\[
\int_0^\infty \beta(x) \exp[- \int_1^x k(s) ds] dx < \infty.
\]
Then for \(h \in X_1 \cap C[0, \infty],\) there is a solution \(u \in X_1\) of
\[
u - \lambda A_2 u = h.
\]
Moreover, \(u(x) \geq 0\) for \(x \in (0, \infty)\) if \(h \geq 0.\)

Proof: By Proposition 2.3.4, there is \(u \in C^1(0, \infty)\) such that
\[
u - \lambda A_2 u = h.
\]
If \( h \geq 0 \), then \( u(x) \geq 0 \) by Proposition 2.3.5. In this case, since

\[
v(x) = \int_{0}^{x} u(s)\beta(s)ds = \int_{0}^{x} |u(s)|\beta(s)ds
\]

and \( \lim_{x \to \infty} v(x) \) is finite by Proposition 2.3.4, then \( u \in X_1 \).

For the general case of \( h = h_+ - h_- \), that \( u \) is in \( X_1 \) follows easily. \( \square \)

Following the three previous Propositions, we have the main result.

(2.3.7) Theorem. Suppose \( \alpha, \beta \in C(0, \infty), \alpha > 0, \beta > 0 \) on \( (0, \infty) \), \( \alpha, k \in C^1(0, \infty) \),

\[
\int_{0}^{\infty} \beta(x) \exp\left[-\int_{1}^{x} k(s)ds\right]dx < \infty.
\]

Then \( A_2 \) is essentially \( m \)-dissipative on \( X_1 = L^1((0, \infty), \beta(x)dx) \).

Remark 1: In stating Theorem 2.3.7, we kept the statement simple by using only the integrability part (or the first half) of conditions \((K_0), (K_1)\) in Lemma 2.3.3. A more general (but more complicated) theorem can be stated using the alternative mentioned in Lemma 2.3.3.

Remark 2: Let \( \{T(t) : t \geq 0\} \) be the contraction semigroup generated by \( \overline{A}_2 \). Then we have \( T(t)f \geq 0 \) if \( f \geq 0 \). This follows from Proposition 2.3.5 since

\[
T(t)f = \lim_{n \to \infty} (I - \frac{t}{n} \overline{A})^{-n}f
\]

if \( f \geq 0 \). Alternatively, it follows from an abstract result of Crandall and Tartar [3].

Remark 3: Consider \( \alpha \equiv x^4, \beta(x) \equiv x^2, k(x) \equiv k_0 \). If \( k_0 > 0 \), then

\[
\int_{0}^{\infty} \beta(x) \exp\left[-\int_{1}^{x} k(s)ds\right]dx = \int_{0}^{\infty} x^2 \exp[-k_0(x - 1)]dx < \infty
\]
and 

\[ A_2u = \frac{1}{x^2}[x^4(u' + k_0u)]' \]

is essentially m-dissipative on \( X_1 \). For \( k = 0 \), although

\[ \int_0^\infty \beta(x) \exp[-\int_1^x k(s)ds]dx = \int_0^\infty x^2dx = \infty, \]

\( A_2 \) is still essentially m-dissipative on \( X_1 \) because

\[ \int_0^1 \beta(x) \exp[-\int_1^x k(s)ds]dx = \int_0^1 x^2dx < \infty \]

and

\[ \int_1^\infty x^2 \int_x^\infty \frac{1}{y^4}dydx = \infty. \]

(See Lemma 2.3.3.)

For \( k_0 < 0, \beta = x^2, \alpha = x^4 \), \((K_1)\) in Lemma 2.3.3 fails since

\[ \int_1^\infty \beta(x) \exp[-k_0(x - 1)]dx \geq \int_1^\infty x^2dx = \infty \]

and

\[ \int_1^\infty \beta(x) \exp[-k_0(x - 1)] \int_x^\infty \exp[\int_y^\infty k_0] \frac{dy}{\alpha(y)} < \infty. \]

So \( A_2 \) is not essentially m-dissipative on \( X_1 \).

Define

\[ B_1u(x) = \frac{1}{\beta(x)}[\alpha(x)(u' + k(x)u)]', \]

\[ D(B_1) = \{ u \in C^1(0, \infty) \cap X_1 : u' \in AC_{loc}(0, \infty), \{ \alpha(u' + k(x)u) \}' \in L^1(0, \infty) \}
\]

and \( \alpha(u' + k(x)u) \to 0 \) as \( x \to 0, \infty \).

(2.3.8) Theorem. Suppose \( \alpha, \beta > 0 \) on \((0, \infty), \beta \in C(0, \infty), \alpha, k \in C^1(0, \infty), \)

and

\[ \int_0^\infty \beta(x) \exp[-\int_1^x k(s)ds]dx < \infty. \]
Then the following conditions are satisfied:

(i) $\mathcal{D}(A_2) \subset \mathcal{D}(B_1)$, $\mathcal{D}(B_1)$ is dense in $X_1$.

(ii) $B_1$ is closed.

(iii) $B_1$ is $m$-dissipative.

(iv) $B_1 = \overline{A_2} = A_1$.

Proof: (i) By the definitions of $A_2$ and $B_1$, it is easy to see that $\mathcal{D}(A_2) \subset \mathcal{D}(B_1) \subset X_1$. Since $C^2_c(0, \infty)$ is dense in $X_1$, then $\mathcal{D}(B_1)$ is dense in $X_1$.

(ii) Suppose $h_n \in \mathcal{D}(B_1)$, $h_n \to h_0$ and $B_1 h_n \to g_0$, i. e.,

\[
\frac{1}{\beta} [\alpha(h'_n + k(x)h_n)]' \to g_0;
\]

then

\[
\alpha(h'_n + k(x)h_n) \to \beta g_0 \in L^1(0, \infty). \quad (10)
\]

Since

\[
\alpha(h'_n + k(x)h_n) \to \alpha(h'_0 + k(x)h_0)
\]

and

\[
[\alpha(h'_n + k(x)h_n)]' \to [\alpha(h'_0 + k(x)h_0)]'
\]

in the sense of distributions and $\beta g_0$ is a function, then

\[
[\alpha(h'_0 + k(x)h_0)]' = \beta g_0.
\]

Therefore

\[
\alpha(h'_0 + k(x)h_0) \in AC_{loc}(0, \infty)
\]

and

\[
h'_0 \in L^1_{loc}(0, \infty).
\]

Hence $h_0 \in AC_{loc}(0, \infty)$. 
On the other hand,

\[
[\alpha(h_0' + k(x)h_0)]' = \alpha'(h_0' + k(x)h_0) + \alpha(h_0'' + k'h_0 + kh_0')
\]

\[= \beta g_0 \in L^1_{loc}\]

and \(\alpha \in C^1, k \in C^1, \alpha > 0\) on \((0, \infty)\) implies \(\alpha h_0'' \in L^1_{loc}\), i.e., \(h_0'' \in L^1_{loc}\). Hence \(h_0 \in C^1(0, \infty)\) and \(h_0' \in AC_{loc}(0, \infty)\).

For

\[
\alpha(h_n' + k(x)h_n) \to 0 \text{ as } x \to 0, \infty,
\]

let

\[q_n(x) := \alpha(h_n' + k(x)h_n)(x) = \int_0^x [\alpha(h_n' + k(s)h_n)]'(s)ds\]

for \(0 \leq x \leq \infty\).

For any \(x \in [0, \infty]\),

\[
|q_n(x) - q_m(x)| = |\int_0^x (q_n' - q_m')ds| \\
\leq \int_0^x |q_n' - q_m'|ds \\
\leq \|q_n' - q_m'\|_{L^1(0, \infty)} \\
< \epsilon
\]

for \(m, n \geq N_\epsilon, N_\epsilon\) some integer, by (10).

So \(q_n\) converges uniformly on \([0, \infty]\) and

\[q_n(x) \to q_0(x) = \int_0^x [\alpha(h_0' + kh_0)]'(s)ds.\]

Therefore \(q_0 \in C[0, \infty]\) and \(q_0(0) = q_0(\infty) = 0\), i.e., the boundary conditions hold for \(h_0\).
So $B_1$ is closed.

(iii) $B_1$ is dissipative by Lemma 2.2.1. For any $\lambda > 0$,

$$X_1 = \overline{\mathcal{R}(I - \lambda A_2)} \subset \overline{\mathcal{R}(I - \lambda B_1)} = \mathcal{R}(I - \lambda B_1) \subset X_1,$$

and so $\mathcal{R}(I - \lambda B_1) = X_1$, i.e., $B_1$ is m-dissipative.

(iv) Since $B_1$ and $A_1$ are closed dissipative extensions of $A_2$, then $B_1 = \overline{A_2} = A_1$.

□

2.4 A Self-adjointness Result

Consider

$$A_0u(x) = \frac{1}{\beta(x)}[\alpha(x)(u' + k(x)u)]',$$

$$\mathcal{D}(A_0) = \{u \in C^1(0, \infty) \cap Y : u' \in AC_{loc}(0, \infty), \frac{1}{\beta(x)}[\alpha(x)(u' + k(x)u)]' \in Y,$$

$$\alpha(u' + k u)(x) \rightarrow 0 \text{ as } x \rightarrow 0, \infty\}$$

where $Y = L^2((0, \infty); \beta(x)e^\int_0^x k(s)ds dx)$.

We assume: $\alpha > 0$, $\beta > 0$ on $(0, \infty)$, $\beta \in C(0, \infty)$, $\alpha, k \in C^1(0, \infty)$.

(2.4.1) Proposition. $A_0$ is symmetric and nonpositive on $Y$.

Proof: Since $C^2_c(0, \infty) \subset \mathcal{D}(A_0)$ and $C^2_c(0, \infty)$ is dense in $Y$, then $\mathcal{D}(A_0)$ is dense in $Y$.

Let $u, v \in \mathcal{D}(A_0)$. Then

$$< A_0u, v > = \int_0^\infty \frac{1}{\beta}[\alpha(u' + k u)]' \beta \beta e^{\int_0^x k(s)ds} dx$$

$$= \int_0^\infty [\alpha(u' + k u)]' \beta e^{\int_0^x k(s)ds} dx$$

$$= \alpha(u' + k u) \beta e^{\int_0^x k(s)ds} |_{0}^{\infty} - \int_0^\infty [\alpha(u' + k u)](\beta)' + k\beta e^{\int_0^x k(s)ds} dx$$
Also, taking \( u = v \) gives
\[
< A_0 u, u > = - \int_0^\infty \alpha u' + k u e^{\int_1^x k(s)ds} dx < 0
\]
unless \( u \equiv ce^{-\int_1^x k(s)ds}, c \in \mathbb{C} \). In any case, \( A_0 \leq 0 \) follows. \( \square \)

\textbf{(2.4.2) Proposition.} Suppose
\[
\int_0^\infty \beta(x) \exp[- \int_1^x k(s)ds] dx < \infty.
\]
Then for any \( \lambda > 0 \), \( \mathcal{R}(\lambda I - A_0) \) is dense in \( Y \).

Proof: Let \( h \in C_c(0, \infty) \), \( h = h_+ - h_- \) as usual. Then \( h_+, h_- \in C_c(0, \infty) \).

By Proposition 2.3.5, for any \( \lambda > 0 \), there exist \( u_1, u_2 \in C^1(0, \infty) \) such that
\[
u_1, u_2 \geq 0 \]
and
\[
u_1 - \lambda \frac{1}{\beta(x)} [\alpha(x)(u_1' + k(x)u_1)]' = h_+,
\]
\[
u_2 - \lambda \frac{1}{\beta(x)} [\alpha(x)(u_2' + k(x)u_2)]' = h_-.
\]
We must show \( u_1, u_2 \in Y \). Thus it is sufficient to assume \( h \geq 0 \) and \( u \equiv u_1 \).

Since \( h \in C_c(0, \infty) \), then there exists \( \delta > 0 \) such that \( \epsilon, \frac{1}{\epsilon} \) is a compact support of \( h \) for \( 0 < \epsilon < \delta \). Moreover,
\[
[u - \lambda \frac{1}{\beta(x)} [\alpha(x)(u' + k(x)u)]'] = h.
\]
Multiply both sides by \( u \beta e^{\int_1^x k(s)ds} \) and integrate on \( [\epsilon, 1] \); we have
\[
\int_\epsilon^1 (u - \lambda \frac{1}{\beta(x)} [\alpha(x)(u' + k(x)u)]') u \beta e^{\int_1^x k(s)ds} dx = \int_\epsilon^1 h u \beta e^{\int_1^x k(s)ds} dx
\]
or
\[
\int_\epsilon^1 u^2 \beta e^{\int_1^x k(s)ds} dx - \lambda I = \int_\epsilon^1 h u \beta e^{\int_1^x k(s)ds} dx
\]
where $I = \int_{\xi}^{t} [\alpha(x)(u' + k(x)u)'] u e^{\int_{\xi}^{s} k(s)ds} ds$.

Since $h \in C_{c}(0, \infty)$, then $\int_{\xi}^{t} h u \beta e^{s} \int_{s}^{x} k(s)ds ds \leq C_{1}$ for some constant $C_{1} > 0$ and $0 < \epsilon < \delta$.

Now we want to show $I \leq 0$.

Claim: $\alpha(x)(u' + k(x)u) \leq 0$ for $x \geq \frac{t}{\epsilon}$ and $\alpha(x)(u' + k(x)u) \geq 0$ for $x \leq \epsilon$.

Suppose $\alpha(x)(u' + k(x)u) > 0$ for some $x = x' > j$. Therefore

$$\alpha(x)(u' + k(x)u) - \int_{\xi}^{x} \alpha(x)(u' + k(x)u) u = \int_{\xi}^{x} \alpha(x)(u' + k(x)u) u \beta e^{s} \int_{s}^{x} k(s)ds ds > 0$$

for $x > x'$. This contradicts the boundary condition $\alpha(x)(u' + k(x)u) \rightarrow 0$ as $x \rightarrow \infty$.

Similarly, we can show that $\alpha(x)(u' + k(x)u) \geq 0$ for $x \leq \epsilon$.

Hence $I \leq 0$.

So

$$\int_{\xi}^{t} u^{2} \beta e^{s} \int_{s}^{x} k(s)ds ds < \infty$$
for $0 < \epsilon < \delta$, i.e., $u \in X$, $u \in D(A_0)$.

(2.4.3) Proposition. $A_0$ is closed.

Proof: The proof is similar to that of Theorem 2.3.8 (ii), so the details are omitted.

Combining Propositions 2.4.1, 2.4.2 and 2.4.3, we have the following self-adjointness result.

(2.4.4) Theorem. $A_0$ is self-adjoint on $Y$.

Remark: 1. By the Schwarz inequality, we have

$$u \in L^2((0, \infty); \beta(x) e^{\int_1^x k(s) ds} dx)$$

implies

$$u \in L^1((0, \infty); \beta(x) dx)$$

if

$$\int_0^\infty \beta(x) \exp[- \int_1^x k(s) ds] dx < \infty$$

since

$$\int_0^\infty |u(x)| \beta(x) dx = \int_0^\infty |u(x)| \beta(x)^{1/2} \exp[1/2 \int_1^x k(s) ds] \beta(x)^{1/2} \exp[-1/2 \int_1^x k(s) ds] dx \leq \left[ \int_0^\infty (|u(x)| \beta(x)^{1/2} \exp[1/2 \int_1^x k(s) ds])^2 dx \right]^{1/2} \left[ \int_0^\infty (\beta(x)^{1/2} \exp[-1/2 \int_1^x k(s) ds])^2 dx \right]^{1/2} = \left[ \int_0^\infty u(x)^2 \beta(x) \exp[1/2 \int_1^x k(s) ds] dx \right]^{1/2} \left[ \int_0^\infty \beta(x) \exp[- \int_1^x k(s) ds] dx \right]^{1/2}.$$
Remark 2: By classical interpolation theorems [19], for

\[ S \in \mathcal{B}(L^p(\Omega, k_i(x)d\mu(x))), \quad i = 0, 1, \]

\[ \frac{1}{p} = \frac{1 - \nu}{p_0} + \frac{\nu}{p_1}, \quad 0 < \nu < 1, \]

we have

\[ S \in \mathcal{B}(L^p(\Omega, k_\nu(x)d\mu(x))) \]

where

\[ k_\nu(x) = k_0(x)^{1-\nu} k_1(x)^\nu \]

and

\[ \|S\|_p \leq C \|S\|_{p_0}^{1-\nu} \|S\|_{p_1}^\nu \]

for some constant \( C \).

For

\[ Au(x) = \frac{1}{\beta(x)} [\alpha(x)(u' + k(x)u)]' \]

where

\[ \int_0^\infty \beta(x) \exp[- \int_1^x k(s)ds]dx < \infty, \]

we have

\[ T(t), (\lambda - A)^{-1} \in \mathcal{B}(L^1(R^+, \beta(x)dx)) \]

and

\[ T(t), (\lambda - A)^{-1} \in \mathcal{B}(L^2(R^+, \beta(x) \exp[\int_1^x k(s)ds]dx)) \]

for \( t \geq 0 \) and \( \lambda > 0 \).

Therefore

\[ T(t), (\lambda - A)^{-1} \in \mathcal{B}(L^p(R^+, \beta(x) \exp[\nu \int_1^x k(s)ds]dx)) \subseteq \mathcal{B}(X_p) \]

for \( 0 < \nu < 1 \) and \( p = \frac{2}{2-\nu} \in (1, 2) \).
Furthermore, we have
\[
\|(\lambda - A)^{-1}\|_{X_1} \leq \frac{1}{Re\lambda}
\]
\[
\|(\lambda - A)^{-1}\|_{Y} \leq \frac{1}{|\lambda|};
\]
thus for \(1 < p < 2\),
\[
\|(\lambda - A)^{-1}\|_{X_p} \leq \frac{C_3}{|\lambda|^{2-\frac{2}{p}}}
\]
\[
\|T(t)\|_{X_p} \leq M
\]
for some constant \(M\). Hence \(\{T(t) : t \geq 0\}\) is a differentiable semigroup on \(X_p\) for \(1 < p \leq 2\) since
\[
\limsup_{|\lambda| \to \infty} \log |Im\lambda|\|(\lambda - A)^{-1}\|_{p} = 0.
\]
(Cf. [18].)
CHAPTER 3

LINEAR PARABOLIC PROBLEMS WITH DISCONTINUOUS COEFFICIENTS AND THE WENTZELL BOUNDARY CONDITION

3.1 Introduction

In this chapter, we consider the following operator:

\[ Au = \alpha(x)u'' + \beta(x)u', \]

where \( \alpha, \beta \) are Lebesgue measurable and satisfy the following conditions: there exist \( \alpha_i, \beta_i \in C(0,1) \), \( \alpha_i > 0 \) on \((0,1)\) for \( i = 1, 2 \) with

\[
\alpha_1(x) \leq \alpha(x) \leq \alpha_2(x), \tag{2}
\]

\[
\beta_1(x) \leq \beta(x) \leq \beta_2(x), \tag{3}
\]

for any \( x \in (0,1) \).

Let \( X = L^\infty(0,1) \) and

\[
D(A) = \{ u \in C^1[0,1] : u' \in AC[0,1], \alpha(x)u'' + \beta(x)u' \in L^\infty(0,1), \quad \text{for } j = 0, 1, \lim_{z \to y}(\alpha u'' + \beta u')(x) \to 0 \}. \]

Then \( X_0 = C[0,1] \subset X \).

Then

\[ X_0^* \cong \mathcal{M}[0,1] = \text{all finite signed measures on } [0,1], \text{ with total variation norm} ; \]

\[ X^* \cong ba[0,1] = \text{all bounded additive measures on } [0,1], \text{ with total variation norm} . \]
Here \( \cong \) denotes "isometrically isomorphic" (in a canonical way).

By Dunford & Schwartz [4], we know \( \mathcal{M}[0,1] \subset \mathcal{M} ba[0,1] \). Also, \( \delta_a \in \mathcal{M} \setminus X \) \((0 \leq a \leq 1)\), where \( \delta_a \) is the Dirac mass at \( a \). Let \( \omega_a \) be any Hahn-Banach extension of \( \delta_a \), so that \( \omega_a \in X^* \).

Let
\[
\mathcal{B} = \text{Borel sets of } [0,1],
\]
\[
\mathcal{L} = \text{Lebesgue sets of } [0,1].
\]

**Definition:** We say \( \omega \) is a \( 0-1 \) measure if \( \omega \in ba[0,1] \) and \( \omega : \mathcal{B} \to \{0,1\} \).

This definition makes sense for \( \omega \in \mathcal{M}[0,1] \).

**Lemma (3.1.1).** Let \( \mu \in \mathcal{M}[0,1] \) be a \( 0-1 \) measure, and \( \mathcal{B}(\mu) = \{ \mu \in \mathcal{B} : \mu(E) = 1 \} \). Then \( \mu = \delta_a \) for some \( a \in [0,1] \) and \( \bigcap_{E \in \mathcal{B}(\delta_a)} E = \{a\} \).

**Proof:** Since \( \mu : \mathcal{B} \to \{0,1\} \) is countably additive and \( \mu([0,1]) = 1 \), then for any \( b \in [0,1] \), let \( I_n = (b - \frac{1}{n}, b + \frac{1}{n}) \cap [0,1] \); then \( \mu(I_n(b)) = 1 \) or \( 0 \).

Let \( \mathcal{B} = \{ b \in [0,1] : \mu(I_n(b)) = 0 \text{ for } n \text{ large enough} \} \). \( b \in \mathcal{B} \) implies that there exists \( n_0 \) such that \( \mu(I_{n_0}(b)) = 0 \), i.e., \( \mu(\{(b - \frac{1}{n_0}, b + \frac{1}{n_0})\}) = 0 \).

Let \( \tilde{B} = \bigcup\{(b - \frac{1}{n_0}, b + \frac{1}{n_0}) : b \in \mathcal{B}, n_0 = n_0(b) \text{ as above}\} \).

If \( \mu \) is not a \( \delta \)-function, then \( \tilde{B} \supset [0,1] \). Therefore, by the Heine-Borel Theorem, there exist \( b_1, \ldots, b_N \) and corresponding \( n_{01}, \ldots, n_{0N} \) such that \( [0,1] \in \bigcup_{i=1}^{N}(b_i - \frac{1}{n_{0i}}, b_i + \frac{1}{n_{0i}}) \).

Hence, \( \mu \equiv 0 \), which is a contradiction.
So, there exists \( a \in [0,1] \setminus \tilde{B} \) such that

\[
\mu(\{a\}) = \lim_{n \to \infty} \mu((a - 1/n, a + 1/n)) = 1
\]

by \( \sigma \)-additivity of \( \mu \) and \( a \in \tilde{B} \).

Therefore, \( \mu = \delta_a \) and \( a \) is unique since \( \mu([0,1]) = 1 \). Furthermore, \( \cap_{E \in \tilde{B}(\delta_a)} E = \{a\} \).

(3.1.2) Lemma. Let \( \omega \) be a 0-1 measure, \( \omega \in ba[0,1] \) with \( \omega \) not a Dirac measure. Let \( \tilde{\mathcal{L}}(\omega) = \{E \in \mathcal{L} : \omega(E) = 1\} \). Then \( \cap_{E \in \tilde{\mathcal{L}}(\omega)} E = \emptyset \), \( \omega \) is purely finitely additive, and there exists a unique \( a \in [0,1] \) such that for any open neighborhood \( U_a \) of \( a \) in \([0,1] \), \( \omega(U_a) = 1 \). (We say \( \omega \) is concentrated at \( a \).)

Proof: Since \( \omega : \mathcal{L} \to \{0,1\} \), then \( \omega(a) = 0 \) for any \( a \in [0,1] \) and \( \omega([0,1]) = 1 \).

Since \( \{a\}^c = [0,1] \setminus \{a\} \in \tilde{\mathcal{L}}(\omega) \), then \( \omega(\{a\}^c) = 1 \).

Therefore, \( \cap_{E \in \tilde{\mathcal{L}}(\omega)} E \subseteq \cap_{a \in [0,1]} \{a\}^c = \emptyset \).

To show that \( \omega \) is purely finitely additive, we need to show that for any \( \sigma \in \mathcal{M}[0,1] \) with \( 0 \leq \sigma(E) \leq \omega(E) \) for any \( E \in \mathcal{L} \), then \( \sigma \equiv 0 \).

Claim: \( \sigma \) is a 0-1 measure.

Suppose \( \sigma(A) = \alpha \in (0,1) \). Then \( \sigma(A^c) = 1 - \alpha \in (0,1) \), and

\[
0 \leq \sigma(A^c) \leq \omega(A^c).
\]

So, \( \omega(A^c) = 1 \), i.e., \( \omega(A) = 0 \). But \( \omega(A) = 0 \) implies \( \sigma(A) = 0 \) which is a contradiction. Therefore, \( \sigma \) is a 0-1 measure.

Suppose \( \sigma \neq 0 \); then there exists a sequence of (relatively) open intervals in
[0, 1] A_n \downarrow \{a\} for some a such that \(\sigma(A_n) = 1\) for any \(n\) by Lemma 3.1.1, and so \(\sigma(\{a\}) = 1\) by countable additivity. But \(1 = \sigma(\{a\}) \leq \omega(\{a\}) = 0\), again, a contradiction. Thus \(\sigma = 0\).

Claim: there exists \(a \in [0, 1]\) such that for any neighborhood \(U_a, \omega(U_a \cap [0, 1]) = 1\).

If not, then for any \(a \in [0, 1]\), there exists a neighborhood \(U_a\) such that \(\omega(U_a) = 0\). So we have an open cover of \([0, 1]\). By the Heine-Borel Theorem, there exists a finite open subcover of \(\{U_a, i = 1, ..., N\}\). Then \(\omega = 0\) since \(\omega\) is finitely additive. This contradiction establishes the claim. \(\square\)

(3.1.3) Lemma. Let \(\omega\) be as in Lemma 3.1.2. Then \(\omega|_{X_0 = C[0,1]} = \delta_a\) for some \(a \in [0, 1]\), i. e., for any 0–1 measure \(\omega \in ba[0,1]\), there exists a unique \(a \in [0,1]\) such that \(\omega|_{X_0} = \delta_a\), where we view \(\omega\) as being in \(X^* \supset X_0^*\).

Proof: Since \(\omega|_{X_0}\) is a 0–1 measure in \(\mathcal{M}[0,1]\), then by Lemma 3.1.1 there exists a unique \(a \in [0, 1]\) such that \(\omega|_{X_0} = \delta_a\). \(\square\)

Remark: \(\omega\) is always multiplicative, i. e.,

\(<uv, \omega> = <u, \omega><v, \omega>\)

for any \(u, v \in X\), where \(<u, \omega> = \int_{[0,1]} u(s)\omega(ds)\).

In the following, we will state the Generalized Maximum Principle without proof. It was given by Hashimoto and Oharu [16] and will be used in our dissipativity proof.

(3.1.4) Generalized Maximum Principle: Let \(u \in C^1[0,1]\) be such that \(u' \in AC_{loc}(0,1)\). Assume there exists \(a \in (0,1)\) such that \(u(a) = \max_{[0,1]} u\), \(u'(a) = 0\)
and \( u'' \in L^\infty(U_a) \), \( u' \in \text{Lip}(U_a) \) where \( U_a \) is a neighborhood of \( a \) in \([0,1]\). Then there exists \( \omega \in ba[0,1] \), a finitely additive \( 0-1 \) measure "concentrated on a" such that 
\[ < u'', \omega_a > \leq 0. \]

### 3.2 Dissipativity

**Lemma.** Let \( A \) be defined as in Section 3.1. Then \( A \) is dissipative on \( L^\infty(0,1) \).

Proof: The idea of this proof is due to G. Goldstein, J. Goldstein and Oharu [7]. Let \( u \in D(A) \subset X = L^\infty(0,1) \). Choose \( x_0 \in [0,1] \) such that \( u(x_0) = \pm \|\omega\|_\infty \). Without loss of generality, assume \( u(x_0) = \|\omega\|_\infty \). If \( u \equiv 0 \), then there is nothing to prove.

So suppose \( \|\omega\|_\infty > 0 \). If \( 0 < x_0 < 1 \), then there exists a \( 0-1 \) finitely additive measure \( \omega_{x_0} \) concentrated at \( x_0 \) such that \( < u', \omega_{x_0} > = 0 \) and \( < u'', \omega_{x_0} > \leq 0 \) by the Generalized Maximum Principle.

Since

\[
< Au, \omega_{x_0} > = \int_{[0,1]} (\alpha(x)u''(x) + \beta(x)u'(x))\omega_{x_0}(dx)
\]

\[
= < \alpha u'', \omega_{x_0} > + < \beta u', \omega_{x_0} >
\]

\[
= < \tilde{\alpha}, \omega_{x_0} > < u'', \omega_{x_0} > + < \tilde{\beta}, \omega_{x_0} > < u', \omega_{x_0} >
\]

where \( \epsilon > 0, \tilde{\alpha} = \alpha \) on \((\epsilon, 1-\epsilon)\), \( \tilde{\alpha} \in L^\infty[0,1], x_0 \in (\epsilon, 1-\epsilon) \), \( \tilde{\beta} = \beta \) on \((\epsilon, 1-\epsilon)\), \( \tilde{\beta} \in L^\infty[0,1] \). Here \( \epsilon \) is so small that \( x_0 \in (\epsilon, 1-\epsilon) \). Then \( < \tilde{\alpha}, \omega_{x_0} > > 0 \) since \( \alpha \geq \epsilon_0 > 0 \) in \((\epsilon, 1-\epsilon)\), \( < u'', \omega_{x_0} > \leq 0, < u', \omega_{x_0} > = 0 \). Therefore \( < Au, \omega_{x_0} > \leq 0. \)
If \( x_0 = 0 \), then by the boundary condition,
\[
< Au, \omega_{x_0} > = \lim_{x \to 0} (Au)(x) = 0 \leq 0,
\]
where \( \omega_{x_0} = \omega_0 \) can be viewed as \( \delta_0 \) or any \( 0-1 \) extension of it.

Similarly, if \( x_0 = 1 \), we have
\[
< Au, \omega_{x_0} > = 0.
\]

Since \( \omega_{x_0} \in \mathcal{J}(\|u\|_{\infty}) \) in all cases, therefore we conclude that \( A \) is dissipative on \( X = L^\infty \). \( \square \)

3.3 The Range Condition

Let \( h \in C^1[0,1] \). We want to solve the equation
\[
u - (\alpha(x)u''(x) + \beta(x)u'(x)) = h.
\]
We can choose \( \{\alpha_n, \ n \geq 3\} \), \( \{\beta_n, \ n \geq 3\} \) such that
\[
\alpha_1 \leq \alpha_n \leq \alpha_2,
\]
\[
\beta_1 \leq \beta_n \leq \beta_2,
\]
\( \alpha_n, \beta_n \in C(0,1), \alpha_n > 0 \) in \( (0,1) \), for \( n \geq 3 \),
\[
\alpha_n \to \alpha \ a.e.,
\]
\[
\beta_n \to \beta \ a.e..
\]

Case 1: \( \beta = 0 \).

Define \( A_n u = \alpha_n(x)u'' \).
For \( h \in C^1[0,1] \), we want to solve

\[
\begin{aligned}
\begin{cases}
u - \alpha(x)u'' = h, \\
\alpha(x)u'' \to 0, \text{ as } x \to 0, 1.
\end{cases}
\end{aligned}
\]

Let \( f \) be the unique linear function on \([0,1]\) satisfying \( f'' = 0, f(0) = -h(0), f(1) = -h(1) \).

For \( v = u + f \), \( v \) satisfies (if \( v \) exists)

\[
\begin{aligned}
\begin{cases}
u - \alpha(x)v'' = g, \\
v(0) = v(1) = 0
\end{cases}
\end{aligned}
\]

where \( g = h + f \).

Replacing \( \alpha(x) \) by \( \alpha_n(x) \), according to Goldstein and Lin[11],

\[
\begin{aligned}
\begin{cases}
v_n - \alpha_n(x)v_n'' = g, \\
v_n(0) = v_n(1) = 0
\end{cases}
\end{aligned}
\]

has a unique solution \( v_n \) and

\[
\|v_n\|_\infty \leq \|g\|_\infty,
\]

\[
\|\alpha_n v_n''\|_\infty \leq 2\|g\|_\infty,
\]

for \( n \geq 3 \).

Let \( 0 < \varepsilon < 1/2 \) and \( J = [\varepsilon, 1 - \varepsilon] \). Since \( \alpha_n(x) \geq \alpha_1(x) \geq 0 \) in \((0,1)\) and \( \alpha_1 \in C(0,1) \), then \( 1/\alpha_n \leq 1/\alpha_1 \leq C_\varepsilon \) in \( J \).

So,

\[
\|v_n''\|_{L^\infty[\varepsilon,1-\varepsilon]} \leq C'_\varepsilon
\]

for some constant \( C'_\varepsilon \) depending on \( \varepsilon \) and all \( n \geq 3 \).

By the compactness arguments of [11], \( \{v_n\} \) has a subsequence \( \{v_{n_k}\} \) which converges to a function \( v \) locally uniformly in \((0,1)\). Furthermore, by passing to another subsequence if necessary, \( \{v'_{n_k}\} \) converges to \( v' \) locally uniformly in \((0,1)\).
Since $\alpha_{n_k} \to \alpha$ a.e., $v_{n_k} \to v$, $v'_{n_k} \to v'$ and $v_{n_k} - \alpha_{n_k} v''_{n_k} = g$, then we have

$$v - \alpha v'' = g$$
a.e. in $(0, 1)$.

For $v$ to be our solution, we still need to show that $v \in C[0,1]$ and $v(0) = v(1) = 0$.

To that end, choose a function $J$ in $C^2[0,1]$ such that

$$J \geq 0 \text{ in } [0,1],$$

$$J(0) = J(1) = 0,$$

$$J''(x) \leq 0 \text{ for all } x \text{ in } [0,1],$$

$$J(x) \geq |g(x)| \text{ for all } x \text{ in } [0,1].$$

This can be done since $g \in C^1[0,1]$. The idea is to make $J'(0)$ and $-J'(1)$ very large and proceed from there. Noting $g \in C^1[0,1]$, one may take $J(x) = Lx(1-x)$ for large enough $L$. Cf. [13].

Then

$$|v''_{n_k} - \alpha_{n_k}^{-1} v_{n_k}| = \alpha_{n_k}^{-1} |g| \leq -J'' + \alpha_{n_k}^{-1} J$$

which implies

$$(J \pm v_{n_k})'' - \alpha_{n_k}^{-1} (J \pm v_{n_k}) \leq 0$$

in $(0, 1)$.

But also $(J \pm v_{n_k})(0) = (J \pm v_{n_k})(1) = 0$.

By the Generalized Maximum Principle,

$$J \pm v_{n_k} \geq 0 \text{ on } [0,1],$$
i. e.,

$$|v_n| \leq J(x)$$

for all $x$ in $[0, 1]$.

Letting $n \to \infty$, we conclude that $v \in C[0, 1]$, $v(0) = v(1) = 0$.

Hence $u - \alpha(x)u'' = h$ has a solution $u \in D(A)$ for $h \in C^1[0, 1]$.

**Theorem 3.3.1** The closure of the operator $A$ defined by $Au = \alpha(x)u''$ with Wentzell boundary conditions satisfies the conditions of the Crandall-Liggett theorem on $L^\infty[0, 1]$.

**Proof:** By the argument above,

$$u - \lambda\alpha(x)u'' = h$$

has a solution $u \in D(A)$ for any $h \in C^1[0, 1]$ and any $\lambda > 0$. And by Lemma 3.2.1, $A$ is dissipative on $L^\infty[0, 1]$. Since $C^1[0, 1]$ is dense in $\overline{D(A)}$, the closure of $A$ satisfies the Crandall-Liggett hypotheses. We remark that the semigroup determined by $A$ acts on $L^\infty(0, 1)$, but the generator is not densely defined. We have $\overline{D(A)} = C[0, 1]$ in this case. We are using the linear case of the Crandall-Liggett theorem. □

**Case 2:** $\beta \neq 0$.

For $h \in C^1[0, 1]$, we want to solve

$$\begin{cases} u - (\alpha(x)u'' + \beta(x)u') = h, \\ \alpha(x)u'' + \beta(x)u' \to 0, \text{ as } x \to 0, 1. \end{cases}$$

We make the following assumptions:

(i) For $W(x) = \exp[-\int_{0}^{x} \beta(s)/\alpha(s)ds]$, $W(x) \in L^1(0, 1)$;

(ii) There exists a $\delta > 0$ such that $\alpha(x) + (x - 1/2 \pm \delta)\beta(x) \geq 0$ in $(0, 1)$. 

We can choose \( \{\alpha_n, n \geq 3\}, \{\beta_n, n \geq 3\} \) such that

\[
\alpha_1 \leq \alpha_n \leq \alpha_2, \\
\beta_1 \leq \beta_n \leq \beta_2,
\]

\( \alpha_n, \beta_n \in C(0,1), \alpha_n > 0 \text{ in } (0,1), \) for \( n \geq 3, \)

\[
\alpha_n \rightarrow \alpha \text{ a.e.,} \\
\beta_n \rightarrow \beta \text{ a.e.,}
\]

and \( \alpha_n(x), \beta_n(x) \) satisfy (i) and (ii), i.e.,

(i') For \( W_n(x) = \exp[-\int_{x/2}^{x} \beta_n(s)/\alpha_n(s)ds], W_n(x) \in L^1(0,1); \)

(ii') There exists a \( \delta_n > 0 \) such that \( \alpha_n(x) + (x - 1/2 \pm \delta_n)\beta_n(x) \geq 0 \) in \( (0,1). \)

Replacing \( \alpha(x), \beta(x) \) by \( \alpha_n(x), \beta_n(x) \), then by Clément and Timmermans [2], (i') implies

\[
\begin{cases}
  u_n - (\alpha_n(x)u_n'' + \beta_n(x)u_n') = h, \\
  \alpha_n(x)u_n'' + \beta_n(x)u_n' \rightarrow 0, \text{ as } x \rightarrow 0,1
\end{cases}
\]

has a unique solution \( u_n \) and

\[
\|u_n\|_{\infty} \leq \|h\|_{\infty},
\]

\[
\|\alpha_n u_n'' + \beta_n u_n'\|_{\infty} \leq 2\|h\|_{\infty} = C
\]

for \( n \geq 3. \)

Let \( Y_n(x) = \alpha_n(x)W_n(x), \gamma_n(x) = u_n'(x)/W_n(x). \) Then

\[
Y_n \gamma_n' = \alpha_n u_n'' + \beta_n u_n'.
\]

So, \( \|Y_n \gamma_n'\|_{\infty} \leq C. \)

Since \( Y_n(x) \geq \epsilon_0^* > 0 \) for \( 0 < \epsilon < x < 1 - \epsilon < 1, \) where \( \epsilon_0^* \) depends on \( \epsilon, \) then

\[
\|\gamma_n'\|_{L^\infty(\epsilon,1-\epsilon)} \leq C^* \quad (4)
\]
for some constant $C^*$. 

Since $W_n^{-1}(x) \geq \epsilon_1 > 0$ for $0 < \epsilon < x < 1 - \epsilon < 1$, then

$$\|u_n'\|_{L^\infty(\epsilon, 1-\epsilon)} \leq \frac{1}{\epsilon_1}\|\gamma_n\|_{L^\infty(\epsilon, 1-\epsilon)}.$$

Claim: $\gamma_n(1/2)$ is uniformly bounded.

Suppose not, i.e., there exists some subsequence of $\gamma_n(1/2)$ which converges to $+\infty$ or $-\infty$. We may suppose $\gamma_n(1/2) \to \infty$ to simplify the exposition.

By (4),

$$\gamma_n(x) = \gamma_n(1/2) + \int_{1/2}^{x} \gamma_n'(s)ds \to \infty$$

as $n \to \infty$, for any $x \in (\epsilon, 1 - \epsilon)$.

Then

$$u_n(x) = u_n(1/2) + \int_{1/2}^{x} \gamma_n(s)W_n(s)ds \to \infty$$

for $0 < \epsilon < x < 1 - \epsilon < 1$ and $x \neq 1/2$ which contradicts $\|u_n\|_{L^\infty} \leq C$.

So $\gamma_n(1/2)$ is uniformly bounded.

Therefore

$$\|u_n'\|_{L^\infty(\epsilon, 1-\epsilon)} \leq C_1$$

for some constant $C_1$.

From the equation for $u_n$, $\|u_n''\|_{L^\infty(\epsilon, 1-\epsilon)} \leq C_2$ holds for some constant $C_2$ depending on $\epsilon$. By compactness and diagonal arguments, $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ such that

$$\{u_{n_k}\} \to u \text{ locally uniformly in } (0, 1),$$

$$\{u'_{n_k}\} \to u' \text{ locally uniformly in } (0, 1).$$
\{u''_{n_k}\} \to q \text{ a.e. in } (0,1)

for some function \(q\) and \(u\) satisfies

\[ u - (\alpha u'' + \beta u') = h \text{ a.e.} \]

and \(u'' = q\) a.e..

It remains to show that

\[ \alpha u'' + \beta u' \to 0 \text{ as } x \to 0,1 \]

i.e. \(u(0) = h(0), u(1) = h(1)\) (by the equation).

Let \(f\) be the unique linear function on \([0,1]\) satisfying \(f'' = 0\), and \(f(0) = -h(0)\) and \(f(1) = -h(1)\).

For \(v = u + f\), \(v\) satisfies

\[ v - (\alpha(x)v'' + \beta(x)v') = g \]

where \(g = h + f - f'\beta(x)\).

Therefore, we need to show \(v(0) = v(1) = 0\).

Since \(v_n = u_n + f\) satisfies

\[ v_n - (\alpha_n(x)v_n'' + \beta_n(x)v_n') = g_n \]

where \(g_n = h + f - f'\beta_n(x)\), \(f + h \in C^1\), \((f + h)(0) = (f + h)(1) = 0\), then there exists \(k_1 > 0\) such that \(|f + h| \leq k_1 x(1 - x)\) for \(x \in [0,1]\). Next,

\[ v_n'' + \frac{\beta_n(x)}{\alpha_n(x)} v_n' - \frac{v_n}{\alpha_n(x)} = -\frac{g_n(x)}{\alpha_n(x)} \].
and

\[ |v'' + \frac{\beta_n(x)}{\alpha_n(x)} v' - \frac{v_n}{\alpha_n(x)}| \]

\[ = |g_n(x)| \]

\[ = \frac{|f + h - f'\beta_n(x)|}{\alpha_n(x)} \]

\[ \leq \frac{|f + h| + |f'||\beta_n(x)|}{\alpha_n(x)} \]

\[ \leq \frac{k_1 x(1 - x)}{\alpha_n(x)} + |f'| \frac{|\beta_n(x)|}{\alpha_n(x)}. \]

By (ii'), $\alpha_n(x) + (x - 1/2 \pm \delta_n) \beta_n(x) \geq 0$ in $(0,1)$.

Let $J_1 = k_2 x(1 - x)$. Since

\[ -\frac{\beta_n(x)}{\alpha_n(x)} J_1' - J_1'' \]

\[ = -\frac{\beta_n(x)}{\alpha_n(x)} k_2 (1 - 2x) + 2k_2 \]

\[ = \frac{2k_2}{\alpha_n} [\alpha_n + (x - 1/2) \beta_n] \]

\[ \geq \frac{2k_2 \delta_n}{\alpha_n} |\beta_n| \]

\[ \geq |f'| \frac{|\beta_n|}{\alpha_n} \]

for large $k_2 > 0$ such that $2k_2 \delta_n \geq |f'|$.

Pick $k = \max(k_1, k_2)$. Let $J = k x(1 - x)$. Then

\[ |v'' + \frac{\beta_n(x)}{\alpha_n(x)} v' - \frac{v_n}{\alpha_n(x)}| \leq \frac{J}{\alpha_n} - \frac{\beta_n(x)}{\alpha_n(x)} J' - J'' \]

i.e.

\[ (J \pm v_n)'' + \frac{\beta_n(x)}{\alpha_n(x)} (J \pm v_n)' - \frac{(J \pm v_n)}{\alpha_n(x)} \leq 0 \]

in $(0,1)$.
Since

\[(J \pm v_n)(0) = (J \pm v_n)(1) = 0,\]

by the Generalized Maximum Principle, \(J \pm v_n \geq 0\) on \((0, 1)\), i.e.,

\[|v_n(x)| \leq J(x)\]

for all \(x\) in \([0, 1]\).

This proves that \(v \in C[0, 1]\) and \(v(0) = v(1) = 0\).

Hence

\[u - (\alpha(x)u''(x) + \beta(x)u'(x)) = h\]

has a solution \(u \in \mathcal{D}(A)\) for \(h \in C^1[0, 1]\).

\((3.3.2)\) Theorem. The closure of the operator \(A\) defined by \(Au = \alpha(x)u''(x) + \beta(x)u'(x)\) with Wentzell boundary condition satisfies the hypotheses of the Crandall-Liggett theorem if \(\alpha(x)\) and \(\beta(x)\) are measurable and satisfy

(i)

\[\alpha_1(x) \leq \alpha(x) \leq \alpha_2(x),\]

\[\beta_1(x) \leq \beta(x) \leq \beta_2(x),\]

where \(\alpha_i, \beta_i \in C(0, 1), \alpha_i > 0\) for any \(x \in (0, 1)\);

(ii) For \(W(x) = \exp[-\int_{1/2}^x \beta(s)/(\alpha(s))ds], W(x) \in L^1(0, 1)\);

(iii) There exists a \(\delta > 0\) such that \(\alpha(x) + (x - 1/2 \pm \delta)\beta(x) \geq 0\) in \((0, 1)\).

Proof: It follows immediately from the argument above. \(\square\)

Example: Let \(\alpha(x) = x^k(1 - x)^k, k \geq 1\) and \(\beta(x) = x^{k+1}(1 - x)^{k+1}\). Then

\[W(x) = \exp[-\int_{1/2}^x \beta(s)/(\alpha(s))ds] = \exp[-\frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{12}].\]

Since \(W(x) \in L^1(0, 1)\) and (iii) in Theorem 3.3.2 holds for \(\delta = 1/2\), therefore the operator \(A\) defined
by $Au = \alpha(x)u''(x) + \beta(x)u'(x)$ with Wentzell boundary condition is essentially m-dissipative on $C[0, 1]$. 
CHAPTER 4
THE NONLINEAR KOMPANEETS EQUATION

4.1 Introduction

In Chapter 2, we consider the linear Kompaneets equation

\[ \frac{\partial u}{\partial t} = \frac{1}{\beta(x)}[\alpha(x)(u' + k(x)u)]' \tag{1} \]

for \(0 < x, t < \infty\). Here \(\cdot\) stands for \(\partial / \partial x\). The model above comes from the generalized Kompaneets equation

\[ \frac{\partial u}{\partial t} = \frac{1}{\beta(x)}[\alpha(x)(u' + F(x,u))]' \]

where the most important case is considered by Kompaneets [17], namely, \(\alpha(x) = x^4, \beta(x) = x^2, F(x,u) = u + u^2\). Under certain assumptions on \(\alpha, \beta, k\), the operator \(Au = \frac{1}{\beta(x)}[\alpha(x)(u' + k(x)u)]'\) was shown in earlier chapter to be m-dissipative on \(X_1 = L^1((0,\infty);\beta(x)dx)\).

In this chapter, we further discuss the nonlinear Kompaneets equation

\[ \frac{\partial u}{\partial t} = \frac{1}{\beta(x)}[\alpha(x)(u' + k(x)u) + F(x)g(u)]' \tag{2} \]

for \(0 < x, t < \infty\). For convenience, we choose \(k(x) \equiv 1\).

Define

\[ A_1 u = \frac{1}{\beta(x)}[F(x)g(u)]', \]

\[ D(A_1) = \{u \in X_1 \cap W^{1,1}_{loc}(0,\infty) : F(x)g(u) \in W^{1,1}_{loc}(0,\infty), \]

\[ F(x)g(u) \to 0, \text{ as } x \to 0, \infty, \text{ and } A_1 u \in X_1 \}.\]
We consider $A_1$ as a perturbing operator of the linear operator $A$. In 4.2, we establish the dissipativity of $A_1$; in 4.3, we discuss perturbation theory of the operator $A_1$.

4.2 Dissipativity

Suppose $F$ and $g$ are continuous.

(4.2.1) Lemma. $A_1$ is dissipative on $X_1$.

Proof: For any $u, v \in D(A_1)$, as in Lemma 2.2.1, we define $w = u - v$ and

$$[w > 0] = \{x \in (0, \infty) : w(x) > 0\} = \cup_{n \in J}(a_n, b_n),$$

$$[w < 0] = \{x \in (0, \infty) : w(x) < 0\} = \cup_{n \in K}(c_n, d_n);$$

these are open sets by the continuity of $u - v$. Then we have

$$< A_1 u - A_1 v, J(u - v) >$$

$$= \int_0^\infty [F(x)(g(u) - g(v))]' \text{sign}_0(u - v) dx$$

$$= \sum_{n \in J} \int_{a_n}^{b_n} [F(x)(g(u) - g(v))]' \text{sign}_0(u - v) dx + \sum_{n \in K} \int_{c_n}^{d_n} [F(x)(g(u) - g(v))]' \text{sign}_0(u - v) dx.$$ 

Consider a term of the form $\int_{a_n}^{b_n}$.

For $0 < a_n < b_n < \infty$, since $F$ and $g$ are continuous, then $(u - v)(a_n)$

$$(u - v)(b_n) = 0, u - v > 0 \text{ on } (a_n, b_n)$$

and

$$\int_{a_n}^{b_n} [F(x)(g(u) - g(v))]' \text{sign}_0(u - v) dx$$

$$= \int_{a_n}^{b_n} [F(x)(g(u) - g(v))]' dx$$

$$= F(x)(g(u(x)) - g(v(x))) \big|_{a_n}^{b_n}$$

$$= F(b_n)(g(u(b_n)) - g(u(b_n))) - F(a_n)(g(u(a_n)) - g(u(a_n)))$$

$$= 0.$$
Similarly, we can show that
\[ \int_{a_n}^{b_n} [F(x)(g(u) - g(v))]' \text{sign}_0(u - v) dx = 0 \]
for \( 0 \leq a < b < \infty \) or \( 0 < a < b \leq \infty \) or \( 0 \leq a < b \leq \infty \) by boundary condition. (Compare with Lemma 2.2.1.)

Also, we can show that each term \( \int_{c_n}^{d_n} \) is zero.

Therefore, \( A_1 \) is dissipative on \( X_1 \). (So is \( -A_1 \).)

4.3 Perturbation Theory

We write the operator \( A_1 \) as
\[
A_1u = \frac{1}{\beta(x)} [F(x)g(u)]' \\
= \frac{1}{\beta(x)} F'(x)g(u) + \frac{1}{\beta(x)} F(x)[g(u)]'
\]
\[ = A_2u + A_3u, \]
where \( A_2u = \frac{1}{\beta(x)} F'(x)g(u) \), \( A_3u = \frac{1}{\beta(x)} F(x)[g(u)]' \).

(4.3.1) Lemma. [9] Let \( A \) be linear and generate a \( (C_0) \) contraction semigroup. Let \( B \) be dissipative with \( \mathcal{D}(A) \subset \mathcal{D}(B) \). Assume there are constants \( 0 < a < 1, b \geq 0 \) such that
\[ \|Bf - Bg\| \leq a\|A(f - g)\| + b\|f - g\| \]
for all \( f, g \in \mathcal{D}(A) \). Then \( A + B \) [with domain \( \mathcal{D}(A) \)] is \( m \)-dissipative.

We want to have
\[ \|A_1u - A_1v\|_{X_1} \leq a\|Au - Av\|_{X_1} + b\|u - v\|_{X_1} \]
for \( a < 1, b \in \mathbb{R} \), where \( Au = \frac{1}{\beta(x)} [\alpha(x)(u' + u)]' \). Next,
\[ \|A_1u - A_1v\|_{X_1} \leq \|A_2u - A_2v\|_{X_1} + \|A_3u - A_3v\|_{X_1}, \]
and under the following assumptions on $F, g$: $F \in C^1$, $|F'(x)| \leq a_1 \beta(x)$ and $\|g\|_{\text{Lip}} < \infty$, we have a good estimation on $\|A_2u - A_2v\|_{X_1}$, namely,

$$\|A_2u - A_2v\|_{X_1} = \int_0^\infty \frac{1}{\beta(x)} |F'(x)||g(u) - g(v)| \, dx \leq \int_0^\infty \frac{1}{\beta(x)} |F'(x)||g|_{\text{Lip}} |u - v| \, dx \leq a_1 \|g\|_{\text{Lip}} \|u - v\|_{X_1}.$$ 

However, it is difficult to estimate the term $\|A_3u - A_3v\|_{X_1}$. Due to difficulties on the operator $A_3$, we study the operator $A_3^h$ instead of $A_3$. The operator $A_3^h$ is defined as

$$A_3^h u = \frac{1}{\beta(x)} F(x) \left( \frac{g(u_h) - g(u)}{h} \right)$$

where $u_h(x) = u(x + h)$, $h > 0$, $x > 0$.

(4.3.2) Lemma. [9] Let $A$ generate a $(C_0)$ semigroup and let $B \in \text{Lip}(X)$. Then $A + B - \omega I$ is $m$-dissipative for a suitable real number $\omega$.

(4.3.3) Corollary. Assume the following conditions:

(a) $\|g\|_{\text{Lip}} < \infty$;
(b) $|F'(x)| + |F(x)| \leq C_1 \beta(x)$ for some constant $C_1 > 0$;
(c) $|F(x - h)| \leq C_2 \beta(x)$ for some constant $C_2 > 0$ and any $h \in (0, \delta), \delta > 0$.

Then $A + A_2 + A_3^h - \omega I$ is $m$-dissipative and determines a semigroup $S$ on $X$ satisfying

$$\|S(t)\|_{\text{Lip}} \leq e^{\omega t}$$

for $t \geq 0$, for some real $\omega$.

Proof: Since

$$\|A_2u - A_2v\|_{X_1}$$
\[ = \left\| \frac{F'(x)}{\beta(x)} (g(u) - g(v)) \right\|_{x_i} \]
\[ = \int_0^\infty \frac{F'(x)}{\beta(x)} \left| g(u) - g(v) \right| \beta(x) dx \]
\[ \leq \int_0^\infty \frac{F'(x)}{\beta(x)} \left\| g \right\|_{Lip} \left| u - v \right| \beta(x) dx \]
\[ \leq C_1 \left\| g \right\|_{Lip} \left\| u - v \right\|_{x_i} \]

by (b), and

\[ \left\| A_3^h u - A_3^h v \right\|_{x_i} \]
\[ = \left\| \frac{F(x)}{h \beta(x)} \left[ (g(u(x + h)) - g(v(x + h))) - (g(u(x)) - g(v(x))) \right] \right\|_{x_i} \]
\[ \leq \int_0^\infty \frac{F(x)}{h \beta(x)} \left\| g \right\|_{Lip} \left| u(x + h) - v(x + h) \right| \beta(x) dx \]
\[ \leq \int_0^\infty \frac{F(x)}{h \beta(x)} \left\| g \right\|_{Lip} \left| u(x + h) - v(x + h) \right| \beta(x) dx \]
\[ + \int_0^\infty \frac{F(x)}{h \beta(x)} \left\| g \right\|_{Lip} \left| u(x) - v(x) \right| \beta(x) dx \]
\[ \leq \int_0^\infty \frac{F(y - h)}{h \beta(y)} \left\| g \right\|_{Lip} \left| u(y) - v(y) \right| \beta(y) dy \]
\[ + \int_0^\infty \frac{F(x)}{h \beta(x)} \left\| g \right\|_{Lip} \left| u(x) - v(x) \right| \beta(x) dx \]
\[ \leq \frac{C_2}{h} \left\| g \right\|_{Lip} \left\| u - v \right\|_{x_1} + \frac{C_1}{h} \left\| g \right\|_{Lip} \left\| u - v \right\|_{x_1} , \]

then by Lemma 4.3.2, \( A + A_2 + A_3^h \) satisfies the desired conclusion. \( \square \)

**Example.** Let \( \beta = x^2 \), \( F(x) = cx^2 \sin x \) and \( g(u) = \sin u \). Then (a), (b) and (c) in Corollary 4.3.3 are satisfied. Hence \( A + A_2 + A_3^h - \omega I \) is \( m \)-dissipative for some real \( \omega \).

**Remark:** The nonlinear theory of generalized Kompaneets equation is far from completed. We hope that we will make great progress in the near future and solve it eventually.
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To whom it may concern:

My paper "The Linear Kompaneets Equation" was accepted for publication by JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS. The manuscript number is R012, JMAA 4810.

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January 29, 1996

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