4-4-2023

Matroid Generalizations of Some Graph Results

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MATROID GENERALIZATIONS OF SOME GRAPH RESULTS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
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B.S., Virginia Commonwealth University, 2013
M.S., Virginia Commonwealth University, 2017
May 2023

Dedicated to my parents, to my brother, and to Marie.

This is either madness, or brilliance.

—Will Turner

Pirates of the Caribbean: Curse of the Black Pearl

Acknowledgments

Before I even attended school, Dr. Gerry Dzura was a kind, intelligent advocate for my education. Vilma Cole and Janie Singleton were my earliest teachers, and built the foundation of my education.

At VCU, I'm lucky to have met Ben Mackinnon and Mitch Phillips; they've each been excellent friends to me in and out of the world of math. Richard Hammack is my favorite professor, and I'm fortunate to have had him as my advisor for my Master's. He is the model I unconsciously emulate when teaching, and I hope to embody his exceptional patience and clarity. In research, I am inspired by his creativity for solving problems.

At LSU, my friends and I grew up together as mathematicians. Lucas Meyers, Damien Wright, Prerna Agarwal, Vishnu Sivaprasad, Sterling Campbell, Samuel Wilson, Sean Eustace, Sean Bibby, and Lee Fisher were my family, and though many of them have moved on, I'm grateful for their camaraderie. I'm also thankful for my academic siblings Tara Fife, Jagdeep Singh, Dylan Douthitt, Matthew Mizell, and Chris Cho, each of whom I'm lucky to count as a friend.

Finally, my advisor James Oxley is a powerful force of generosity, kindness, patience, and intelligence. The writer, teacher, and mathematician that I am today are the results of his hard work on my behalf. He impacts the life of every graduate student who passes through the Math Department at LSU, and I more than most owe him my gratitude.

Thank you, James.

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Abstract

The edges of a graph have natural cyclic orderings. We investigate the matroids for which a similar cyclic ordering of the circuits is possible. A full characterization of the non-binary matroids with this property is given. Evidence of the difficulty of this problem for binary matroids is presented, along with a partial result for binary orderable matroids.

For a graph G, the ratio of $|E(G)|$ to the minimum degree of G has a natural lower bound. For a matroid M that is representable over a finite field, we generalize this to a lower bound on the ratio of $|E(M)|$ to the size of a smallest cocircuit of M. Further, we characterize the matroids that achieve equality in this bound.

Jamison and Mulder defined a graph G to be Θ_3-closed if, whenever vertices x and y of G are joined by three internally disjoint paths, x and y are adjacent. They found that graphs with this property can be built from cycles and complete graphs. We generalize this result to binary matroids, showing that the Θ_3-closed binary matroids can be built in a similar fashion from circuits, cycle matroids of complete graphs, and projective geometries.

Chapter 1. Introduction

Every graph represents a matroid, but not every matroid can be represented by a graph. In this way, matroids can be seen as generalizations of graphs. As Tutte [18, p. 497] wrote, “If a theorem about graphs can be stated in terms of edges and circuits only it probably exemplifies a more general theorem about matroids.” Each chapter that follows presents a result or observation about graphs that we generalize to matroids.

The reader is assumed to have a basic familiarity with matroids. The terminology and notation for matroids used in this document follow Oxley [11] with the following additions. We call a matroid N a *series extension* of a matroid M if N can be obtained from M by a (possibly empty) sequence of single-element series extensions; a *parallel extension* is defined analogously. We will also frequently use P_r and A_r to denote $PG(r-1, q)$ and $AG(r-1, q)$, respectively, where q should be clear from the context.

The cycles of a graph have an inherent ordering, but the circuits of a matroid do not. Chapter 2 explores the idea of ordering the circuits of a matroid in such a way that these orderings are consistent. The main result of the chapter characterizes all non-binary orderable matroids.

Examples of orderable binary matroids that are not graphic are presented, however, no such 3-connected examples are known. We conjecture that 3-connected orderable binary matroids are graphic, and we present a partial result in this direction.

In Chapter 3, we find a bound on the ratio of the number of elements in a matroid to the size of a smallest cocircuit in the matroid. A common first result in graph theory says that the degree-sum of a graph G equals twice the number of edges in G. This implies

$$\frac{|E(G)|}{\delta(G)} \geq \frac{1}{2}|V(G)|,$$

where $\delta(G)$ is the minimum degree of G. The main result of the chapter states that, for a binary matroid M that does not simplify to a projective geometry,

$$\frac{|E(M)|}{g^*(M)} \geq 2,$$

where $g^*(M)$ is the size of a smallest cocircuit of M. We also characterize the matroids that achieve equality in this bound. These are both special cases of results for matroids that are representable over arbitrary finite fields, and they have parallel results for matroids that do simplify to a projective geometry.

Chapter 4 generalizes a graph result of Jamison and Mulder [9]. They defined a graph G to be Θ_3-*closed* if, whenever vertices x and y of G are joined by three internally disjoint paths, x and y are adjacent. Their main result states that the Θ_3-closed graphs can be built from cycles and complete graphs by gluing these smaller graphs together along a vertex or along an edge.

We generalize this result to binary matroids, first by defining a matroid analogue of Θ_3-closure. We then prove that the Θ_3-closed binary matroids can be built from circuits, cycle matroids of complete graphs, and projective geometries using operations that parallel the corresponding gluing operations for graphs.

1.1. Background

This section presents some basic concepts that will be needed in the dissertation. A *theta-graph* is a graph that consists of two distinct vertices and three internally disjoint paths between them. A *theta-graph in a matroid* M is a restriction of M that is isomorphic to the cycle matroid of a theta-graph. Equivalently, it is a restriction of M that is isomorphic to a series extension of $U_{1,3}$. The series classes of a theta-graph are called its *theta-arcs*.

It is well known that a matroid that is not 2-connected has a unique decomposition as a direct sum of connected matroids. For matroids that are 2-connected, Cunningham and Edmonds [6] described a decomposition into 3-connected matroids, circuits, and cocircuits. This decomposition is described in terms of a certain tree, which we now define.

If $\{M_1, M_2, \dots, M_n\}$ is a set of a matroids, then a *matroid-labelled tree* with vertex set $\{M_1, M_2, \dots, M_n\}$ is a tree T such that

(i) if e is an edge of T with endpoints M_i and M_j, then $E(M_i) \cap E(M_j) = \{e\}$, and $\{e\}$ is not a separator of M_i or M_j; and

(ii) $E(M_i) \cap E(M_j)$ is empty if M_i and M_j are non-adjacent.

The matroids $M_1, M_2, \dots, M_n$ are called the *vertex labels* of T. Now suppose e is an edge of T with endpoints M_1 and M_2. We obtain a new matroid-labelled tree T/e by contracting e and relabelling the resulting vertex with $M_1 \oplus_2 M_2$. As 2-sum is associative, T/X is well defined for all subsets X of $E(T)$.

Let T be a matroid-labelled tree with $V(T) = \{M_1, M_2, \dots, M_n\}$ and with $E(T) = \{e_1, e_2, \dots, e_{n-1}\}$. Then T is a *tree decomposition* of a 2-connected matroid M if

(i) $E(M) = (E(M_1) \cup E(M_2) \cup \cdots \cup E(M_n)) - \{e_1, e_2, \dots, e_{n-1}\}$;

(ii) $|E(M_i)| \geq 3$ for all i unless $|E(M)| < 3$, in which case $n = 1$ and $M = M_1$; and

(iii) M labels the single vertex of $T/E(T)$.

In this case, the elements $\{e_1, e_2, \dots, e_{n-1}\}$ are the *edge labels* of T. The next theorem of Cunningham and Edmonds [6] (see also [11, Theorem 8.3.10]) tells us that M has a *canon-*

ical tree decomposition, unique to within relabelling of the edges.

Theorem 1.1.1. *Let M be a 2-connected matroid. Then M has a tree decomposition T in which every vertex label is 3-connected, a circuit, or a cocircuit, and there are no two adjacent vertices that are both labelled by circuits or are both labelled by cocircuits. Moreover, T is unique to within relabelling of its edges.*

Chapter 2. Ordering Circuits of Matroids

2.1. Introduction

In a graph, the edges of each cycle have an ordering on them. But this is not true for the circuits of a matroid. The goal of this chapter is to see to what extent we can distinguish graphic matroids by an ordering condition that mimics the ordering condition on the edges of the cycles of a graph.

A *reversible cyclic ordering* of a finite set X is an arrangement of the elements of X on the vertices of an n-gon with one element at each vertex. Elements x_1 and x_2 of X are *adjacent* in the ordering when the corresponding vertices of the n-gon lie on a common edge. Figure 2.1 shows an example of such an ordering $(x_1\ x_2\ \ldots\ x_n)$. The same ordering can also be denoted, for example, by $(x_3\ x_2\ x_1\ x_n\ \ldots\ x_4)$. Throughout this chapter, all orderings are assumed to be reversible cyclic orderings unless stated otherwise.

In a graph, there is an associated ordering on the edge set of each cycle. These orderings have the property that two edges are adjacent in an ordering of a given cycle if and only if they are adjacent in the ordering of every cycle in which the edges appear together.

Unlike the cycles of a graph, the circuits of a matroid are sets without inherent order. We give a matroid M an ordering by imposing an ordering on each of its circuits. Such an ordering of M is *consistent* if, for every pair $\{e, f\}$ of distinct elements of $E(M)$ and every pair $\{C, C'\}$ of circuits of M with $\{e, f\} \subseteq C \cap C'$, if e and f are adjacent in the ordering of C, then e and f are adjacent in the ordering of C'. A matroid is called *order-*

The material in this chapter was published in *The Electronic Journal of Combinatorics*, Volume 29 (iss. 4), Cameron Crenshaw and James Oxley, "Ordering circuits of matroids," (2022).

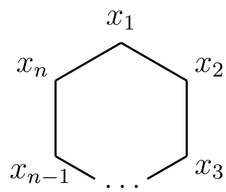

Figure 2.1. A reversible cyclic ordering.

able if it has a consistent ordering.

The primary goal of this work is characterizing orderable matroids. As noted above, our first examples of orderable matroids are graphic matroids.

Proposition 2.1.1. *If M is a graphic matroid, then M is orderable.*

However, orderability is not enough to distinguish graphic matroids from non-graphic matroids. Our main result specifies all non-binary orderable matroids. The infinitely many such matroids are all built from $U_{2,n}$ for some $n \geq 4$ by using two operations, which we now describe.

For a matroid M without coloops, a series extension of M is *balanced* if, for some integer k exceeding one, each element of M is replaced by k elements in series. We call k the *order* of the balanced series extension. The second operation is a generalization of the operation of adding an element in parallel to another. Let P be a nonempty subset of a series class of a matroid M. Fix an element t of P, contract $P - t$, and relabel t as t' to obtain M'. Let N be the cycle matroid of a theta-graph with theta-arcs $\{t'\}$, P, and P', where $|P'| = |P|$. Finally, let M'' be the 2-sum of M' and N with basepoint t'. The operation transforming M into M'' is called *parallel-path addition*. The *size* of this addition is $|P|$; we call P and P' *parallel paths* of M'', and say that M'' is obtained from M by adding P' *in parallel* to P. The following theorem is the main result of the chapter.

Theorem 2.1.2. *Let M be a connected non-binary matroid. Then M is orderable if and only if it can be obtained from a balanced series extension of $U_{2,n}$ for some $n \geq 4$ by a sequence of parallel-path additions.*

When we come to consider binary orderable matroids, we encounter considerable difficulty. For example, as we show in the next section, F_7^* and $M^*(K_5)$ are not orderable, yet each has an orderable series extension. In view of this, it is natural to consider additional conditions that one can add to orderability in order to distinguish graphic matroids within binary matroids. The next theorem gives three equivalent such additional conditions.

Theorem 2.1.3. *The following are equivalent for a binary matroid M:*

(i) *M is graphic.*

(ii) *every minor of M is orderable.*

(iii) *every series minor of M is orderable.*

(iv) *every parallel minor of M is orderable.*

Although, as noted above, there are orderable binary matroids that are not graphic, we know of no counterexample to the following.

Conjecture 2.1.4. *A 3-connected orderable binary matroid is graphic.*

We have, however, made the following progress.

Theorem 2.1.5. *A 4-connected regular orderable matroid is graphic.*

Another condition one can add to orderability to distinguish graphic matroids within binary matroids involves the theta-graphs in a matroid M. A subset B of a circuit C is a *block* if there is a listing $b_1, b_2, \ldots, b_k$ of the elements of B such that b_i and b_{i+1} are adjacent for all i in $[k-1]$. A consistent ordering of a matroid M is a *theta-ordering* if every theta-arc of every theta-graph of M is a block in the ordering; M is *theta-orderable* if it has a theta-ordering.

Theta-orderability turns out to be equivalent to a concept introduced by Wagner [20]. For distinct circuits C and D of a matroid M, an *arc* of C is a minimal non-empty subset A of C such that $A \cup D$ contains at least two circuits. A set $\{A_1, A_2, A_3\}$ of arcs of a common circuit is *incompatible* if $A_1 \cap A_2 \cap A_3 \neq \emptyset$ and $A_i - (A_j \cup A_k) \neq \emptyset$ for all i, j, and k such that $\{i, j, k\} = \{1, 2, 3\}$. In Section 2.4, we prove the following characterization of theta-orderable binary matroids. The equivalence of (i) and (ii) is Wagner's main result [20].

Theorem 2.1.6. *The following are equivalent for a binary matroid M:*

(i) M is graphic;

(ii) M has no set of incompatible arcs; and

(iii) M is theta-orderable.

The following characterization of theta-orderable non-binary matroids will also be proved in Section 2.4.

Theorem 2.1.7. *Let M be a connected non-binary matroid. Then M is theta-orderable if and only if M is a parallel extension of a balanced series extension of $U_{2,n}$ for some $n \geq 4$.*

In Section 2.2, after some preliminaries, we prove Theorem 2.1.3. The proof of our main result, Theorem 2.1.2, is in Section 2.3, and Theorem 2.1.5 is proved in Section 2.5.

2.2. Preliminaries

Our first proposition collects some basic properties of orderability. These properties will be used frequently and often implicitly. We omit their straightforward proofs.

Proposition 2.2.1. *Let M be a matroid.*

(i) If M is orderable, then $M\backslash e$ is orderable for all $e \in E(M)$.

(ii) If $r(M) \leq 2$, then M is orderable.

(iii) M is orderable if and only if the connected components of M are orderable.

(iv) M is orderable if and only if $\mathrm{si}(M)$ is orderable.

Next, we note a partial converse to Proposition 2.1.1.

Proposition 2.2.2. *If M is an orderable binary matroid with a spanning circuit, then M is graphic.*

Proof. Let C be a spanning circuit of M and e be an element in C. Fix a consistent ordering of M, and take a standard binary representation of M with respect to the basis $C - e$. Now construct a graph G beginning with a cycle having edge set C, ordered consistently with the fixed ordering of M. Now, for each element f of $E(M) - C$, let C_f be the fundamental circuit of f with respect to $C - e$. Because $C_f - f$ is a block in the ordering, we may add an edge f to G as a chord of C so that it forms a cycle with edge set C_f. The result is a graph whose cycle matroid has ground set $E(M)$, has $C - e$ as a basis, and has the same fundamental circuits with respect to this basis as M. Since M and $M(G)$ are binary, we deduce that $M = M(G)$. □

We now note a necessary condition for a matroid to be orderable, along with some consequences of this condition.

Proposition 2.2.3. *Let M be a simple matroid and X be a subset of $E(M)$ with $|X| \geq 3$. If there are elements e and f in $E(M) - X$ such that $X \cup e$ and $X \cup f$ are both circuits of M, then M is not orderable.*

Proof. Assume to the contrary that M has a consistent ordering. Notice that the ordering of $X \cup f$ is obtained from that of $X \cup e$ by replacing f with e. Let a and b be the elements in X that are adjacent to e. Using strong circuit elimination on $X \cup e$ and $X \cup f$, we obtain a circuit $C \subseteq X \cup \{e, f\}$ containing e but not a, and another $C' \subseteq X \cup \{e, f\}$ containing f but not b.

As C is not properly contained in either $X \cup e$ or $X \cup f$, it must contain both e and f. Further, M is simple, so $C \cap X$ is nonempty. Since a and b are the only elements in X adjacent to e or f, it follows that $C = \{e, f, b\}$. By symmetry, $C' = \{e, f, a\}$.

Circuit elimination applied to C and C' now yields a circuit D that does not contain e. Then $D \subseteq \{a, b, f\}$. Since $|X| \geq 3$, it follows that D is a proper subset of $X \cup f$, a contradiction. $\square$

Corollary 2.2.4. *Let M be a matroid of rank at least three and X be a circuit-hyperplane of M. If $E(M) - X$ is not a parallel class of M, then the matroid obtained from M by relaxing X is not orderable.*

Corollary 2.2.5. *The only orderable whirl is $U_{2,4}$.*

We now prove Theorem 2.1.3, whose proof relies on the next lemma and its corollary. The following technical property facilitates the statements of these results. A matroid M has the (e, f, g)-*property* if

(i) M has a circuit containing $\{e, f, g\}$;

(ii) e, f, and g are distinct; and

(iii) M has a circuit D containing f but neither e nor g and, with the exception of at most one d in D, there is a circuit of M containing $\{e, f, g, d\}$.

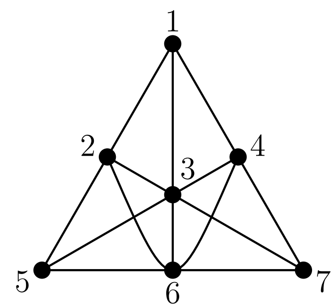

Figure 2.2. F_7 in the proof of Theorem 2.1.3.

Lemma 2.2.6. *If a matroid M has the (e, f, g)-property, then f is not adjacent to both e and g in a consistent ordering of M.*

Proof. Suppose M has the (e, f, g)-property and f is adjacent to both e and g. Then, in the circuit D of condition (iii), f is adjacent to elements d_1 and d_2 of $D - f$. But M has a circuit containing $\{e, f, g, d_i\}$ for some i in $\{1, 2\}$, a contradiction. □

Corollary 2.2.7. *Let C be a circuit of a matroid M. Suppose there is an element c of C so that M has the (e, c, g)-property for every choice of e and g in $C - c$. Then M does not have a consistent ordering.*

Proof of Theorem 2.1.3. Since graphic matroids are orderable and the class of graphic matroids is minor-closed, (i) implies (ii)-(iv). Let $\mathcal{S}$ be the set

$$\{F_7, F_7^*, M^*(K_5), M^*(K_{3,3}), M^*(K'_{3,3}), M^*(K''_{3,3}), M^*(K'''_{3,3}), R_{10}\}.$$

By results of Tutte [17] and Bixby [2, 3], $\mathcal{S}$ contains all binary matroids that are excluded minors, excluded series minors, or excluded parallel minors for the class of graphic matroids. Thus we can prove that (i) follows from each of (ii)-(iv) by showing that none of the matroids in $\mathcal{S}$ is orderable.

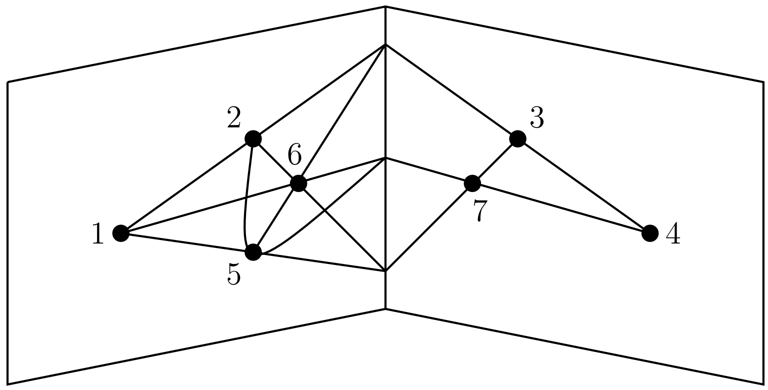

Figure 2.3. The matroid F_7^*.

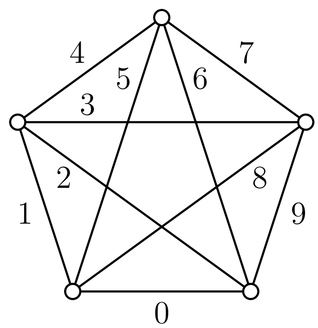

Figure 2.4. The graph K_5.

Let F_7 be labelled as in Figure 2.2. Using the element 1 in the circuit $\{1, 2, 3, 4\}$, Corollary 2.2.7 gives that F_7 has no consistent ordering.

Let F_7^* be labelled as in Figure 2.3. Consider the circuits $C_1 = \{1, 2, 3, 4\}$, $C_2 = \{1, 3, 5, 7\}$, and $C_3 = \{2, 4, 5, 7\}$. The ordering of a four-element circuit is uniquely determined by a single pair of non-adjacent elements, and the automorphism group of F_7^* is doubly transitive. Thus we may assume that C_1 has the ordering (1 2 3 4).

Since 1 and 3 are not adjacent in C_1, it follows that C_2 has the ordering (1 5 3 7). Thus 5 and 7 are non-adjacent, so C_3 has the ordering (2 5 4 7). However, the elements of the set $\{1, 2, 5\}$ are now pairwise adjacent, so the circuit $\{1, 2, 5, 6\}$ cannot be ordered. Thus F_7^* has no consistent ordering.

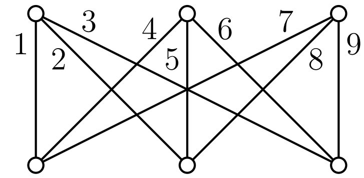

Figure 2.5. $K_{3,3}$ in the proof of Theorem 2.1.3.

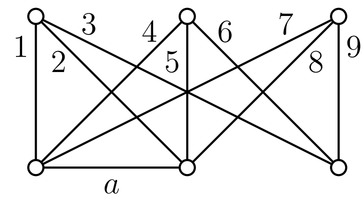

Figure 2.6. $K'_{3,3}$ in the proof of Theorem 2.1.3.

Let $M^*(K_5)$ be labelled as in Figure 2.4, and assume that $M^*(K_5)$ has a consistent ordering. Let C be the circuit $\{1, 2, 3, 4\}$. By symmetry, we may assume its ordering is (1 2 3 4). This ordering and the circuit $\{1, 2, 4, 7, 8, 9\}$ give that 1 and 8 are not adjacent, so the circuit $\{0, 1, 5, 8\}$ must be ordered (0 1 5 8). Similarly, the circuit $\{1, 2, 3, 5, 6, 7\}$ gives that 2 and 6 are not adjacent, so $\{0, 2, 6, 9\}$ must be ordered (0 2 9 6). Now 0 is adjacent to 1, 6, and 8 in the circuit $\{0, 1, 4, 6, 7, 8\}$, a contradiction.

Let $M^*(K_{3,3})$ be labelled as in Figure 2.5. We shall use Corollary 2.2.7 letting c be the element 1 in the circuit $C = \{1, 3, 5, 8\}$ of $M^*(K_{3,3})$. The cases $\{e, g\} = \{3, 5\}$ and $\{e, g\} = \{3, 8\}$ are symmetric, and the circuits C and $\{1, 3, 5, 7, 9\}$ certify that M has the $(3, 1, 5)$-property with $D = \{1, 4, 8, 9\}$. The circuit $\{1, 5, 6, 8, 9\}$ certifies that M has the $(5, 1, 8)$-property with $D = \{1, 2, 6, 9\}$. Corollary 2.2.7 now implies that $M^*(K_{3,3})$ has no consistent ordering.

The next two cases will also use Corollary 2.2.7. Let $M^*(K'_{3,3})$ be labelled as in Figure 2.6, and let c be the element 3 in the circuit $C = \{3, 6, 7, 8\}$ of $M^*(K'_{3,3})$. The cases $\{e, g\} = \{6, 7\}$ and $\{e, g\} = \{6, 8\}$ are symmetric, and the circuit $\{2, 3, 5, 6, 7, 8\}$ certifies

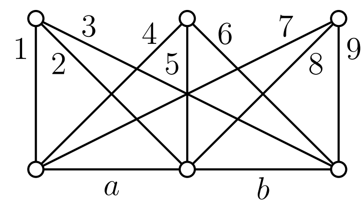

Figure 2.7. $K''_{3,3}$ in the proof of Theorem 2.1.3.

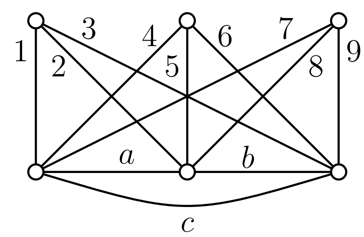

Figure 2.8. $K'''_{3,3}$ in the proof of Theorem 2.1.3.

that M has the $(6,3,7)$-property with $D = \{1,2,3\}$. The circuit C certifies that M has the $(7,3,8)$-property with $D = \{3,6,9\}$. Corollary 2.2.7 now implies that $M^*(K'_{3,3})$ has no consistent ordering.

Let $M^*(K''_{3,3})$ be labelled as in Figure 2.7, and let c be the element 1 in the circuit $C = \{1,3,5,8,a,b\}$ of $M^*(K''_{3,3})$. When $\{e,g\} \subseteq C - 3$, the circuit C certifies that M has the $(e,1,g)$-property with $D = \{1,2,3\}$. Each of the remaining cases uses $D = \{1,4,7,a\}$. The cases $\{e,g\} = \{3,5\}$ and $\{e,g\} = \{3,8\}$ are symmetric, and the circuit $\{1,3,5,7,9,a,b\}$ certifies that M has the $(3,1,5)$ property. The circuits $\{1,3,4,6,8,a,b\}$ and $\{1,3,5,7,9,a,b\}$ certify the $(3,1,a)$-property. Finally, the circuit $\{1,3,4,6,8,a,b\}$ certifies the $(3,1,b)$-property. Corollary 2.2.7 now implies that $M^*(K''_{3,3})$ has no consistent ordering.

Let $M^*(K'''_{3,3})$ be labelled as in Figure 2.8. We begin by noting that there must be at least one adjacent pair in the set $\{1,4,7\}$ due to the circuit $\{1,4,7,a,c\}$. By symmetry, we may assume that 1 and 4 are adjacent.

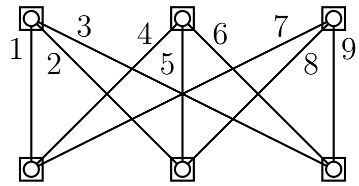

Figure 2.9. A graft corresponding to R_{10}.

Combining this adjacent pair with the three-element circuits, we get that 2145 is a block in the circuit $\{1, 2, 4, 5, 9, b, c\}$. Therefore 4 is not adjacent to 9, b, or c. This means that, in the circuit $\{3, 4, 5, 9, b, c\}$, we must have 4 adjacent to 3. Using the three-element circuit $\{4, 5, 6\}$, we now have that 4 is adjacent to 1, 3, and 6. Therefore the circuit $\{1, 3, 4, 6, 8, a, b\}$ cannot be ordered consistently, and $M^*(K'''_{3,3})$ has no consistent ordering.

Let M be the graft matroid of $K_{3,3}$ where the graft element e_γ corresponds to the set of boxed vertices in Figure 2.9. Then $M \cong R_{10}$. Using Corollary 2.2.7 again, let c be the element 1 in the circuit $C = \{1, 2, 4, 5\}$ of M. When $\{e, g\} = \{2, 4\}$, the circuit $\{1, 2, 4, 6, 8, 9\}$ certifies the $(2, 1, 4)$-property when $D = \{1, 3, 4, 6\}$. When $\{e, g\} = \{2, 5\}$, the circuit $\{1, 2, 5, 6, 7, 9\}$ certifies the $(2, 1, 5)$-property with $D = \{1, 3, 7, 9\}$. Finally, when $\{e, g\} = \{4, 5\}$, the circuit $\{1, 4, 5, 6, 7, e_\gamma\}$ certifies the $(4, 1, 5)$-property with $D = \{1, 6, 8, e_\gamma\}$. Corollary 2.2.7 now implies that R_{10} has no consistent ordering. □

We conclude this section with a pair of examples that indicate the potential difficulty of characterizing orderable binary matroids.

Example 2.2.8. This example describes a 12-element orderable series extension of F_7^*, which we refer to as O_1. Thus, the pair O_1 and F_7^* demonstrates that the class of binary orderable matroids is not closed under the taking of series minors. Let F_7^* be labelled as

$(1\ 5\ 1'\ 2'\ 6\ 2)$ $(1\ 5\ 1'\ 7'\ 3\ 7)$ $(2\ 6\ 2'\ 7'\ 3\ 7)$ $(3\ 4\ 5\ 4'\ 6\ 4'')$

$(1\ 4'\ 2'\ 1'\ 4\ 3\ 4''\ 2)$ $(1\ 7\ 4\ 1'\ 7'\ 4''\ 6\ 4')$ $(2\ 7\ 4\ 5\ 4'\ 2'\ 7'\ 4'')$

Figure 2.10. A consistent ordering of O_1.

$(4\ 6\ 5\ 7)$ $(2'\ 1\ 2\ 6\ 5\ 8\ 9)$ $(0'\ 1\ 0\ 9\ 3\ 4\ 6)$ $(2'\ 1\ 2\ 4\ 7\ 8\ 9)$

$(3\ 7\ 8\ 9)$ $(2\ 1\ 2'\ 3\ 7\ 5\ 6)$ $(0'\ 1\ 0\ 8\ 7\ 4\ 6)$ $(0'\ 1\ 0\ 9\ 3\ 7\ 5)$

$(0'\ 2'\ 3\ 4\ 2\ 0\ 8\ 5)$ $(0'\ 2'\ 9\ 0\ 2\ 4\ 7\ 5)$ $(2'\ 0'\ 6\ 2\ 0\ 8\ 7\ 3)$

$(2\ 1\ 2'\ 3\ 4)$ $(0\ 1\ 0'\ 5\ 8)$ $(0\ 2\ 6\ 0'\ 2'\ 9)$ $(3\ 4\ 6\ 5\ 8\ 9)$

Figure 2.11. A consistent ordering of O_2.

in Figure 2.3. We obtain O_1 by adding $1'$, $2'$, and $7'$ in series with 1, 2, and 7, respectively, and adding $4'$ and $4''$ in series with 4. Figure 2.10 gives a consistent ordering of the circuits of O_1.

Example 2.2.9. Let K_5 be labelled as in Figure 2.4. We obtain a regular, non-graphic matroid O_2 from $M^*(K_5)$ by adding elements $0'$ and $2'$ in series with 0 and 2, respectively. Figure 2.11 gives a consistent ordering of O_2.

2.3. A Characterization of Non-Binary Orderable Matroids

In this section, we prove Theorem 2.1.2. We begin by finding the orderable series extensions of uniform matroids and their consistent orderings. These results allow us to characterize the non-binary orderable matroids that are 3-connected, from which we obtain the full characterization using the canonical tree decomposition of Cunningham and Edmonds [6].

A uniform matroid is binary if and only if it is graphic. Thus, the binary uniform matroids are certainly orderable, as are those whose rank is at most two. Proposition 2.2.3

implies this list is complete.

Corollary 2.3.1. *A uniform matroid is orderable if and only if it is binary or has rank at most two.*

The next two results deduce the structure of a consistent ordering of a series extension of a non-binary uniform matroid, and show that such an ordering can be used to consistently order the underlying uniform matroid. For a non-coloop element e of a matroid M, we denote the series class of M containing e by S_e or sometimes by $S_e(M)$.

Let M be a matroid with a consistent ordering. Suppose X and Y are disjoint subsets of a circuit C of M. We say X and Y are *adjacent* if there is an adjacent pair of elements x and y, where x belongs to X and y belongs to Y. Let $\mathcal{B}$ be the union of a set of blocks that belong to a common circuit of M. If there is a listing $B_1, B_2, \ldots, B_k$ of the blocks in $\mathcal{B}$ such that B_i and B_{i+1} are adjacent for all i in $[k-1]$, then $\mathcal{B}$ is a *section*. Finally, let S be a series class of M. If a block of M is contained in S and is maximal with this property, then it is called an *S-block*.

Lemma 2.3.2. *Let M be an orderable series extension of a non-binary uniform matroid $U_{r,n}$ and fix a consistent ordering of M. Let C be a circuit of M, and let x and y be elements of C from distinct series classes of M.*

(i) If a section K in C is adjacent to a pair of S_x-blocks, then K must contain an S_y-block.

(ii) Every series class S of M has the same number of S-blocks.

Proof. For (i), suppose to the contrary that there is a section K in C that contains no S_y-block and is adjacent to a pair of distinct S_x-blocks. As M is non-binary, $2 \leq r \leq n-2$

and there is a circuit D_x of M that contains K and S_x but avoids S_y. Let $D_y = (D_x - S_x) \cup S_y$. Observe that, since M is a series extension of $U_{r,n}$, the set D_y is a circuit. The consistency of D_y with C implies that K is not adjacent to S_y-blocks in D_y, but the consistency of D_y with D_x gives that K can only be adjacent to S_y-blocks in D_y, a contradiction.

We now deduce (ii) from (i). Let S be a series class of $E(M)$ for which the number of S-blocks is as large as possible. We may assume this number exceeds one. In a circuit C containing S, let K be a minimal section that is adjacent to a pair of distinct S-blocks. Note that the number of such minimal sections in C equals the number of S-blocks. Let S' be a series class of M contained in C that is distinct from S. Part (i) implies there is an S'-block in K and, as K contains no S-blocks, (i) further implies that there is exactly one S'-block in K. Thus there are the same number of S'-blocks as S-blocks. Part (ii) now follows. □

Proposition 2.3.3. *Let $U_{r,n}$ be a non-binary uniform matroid. If a series extension of $U_{r,n}$ is orderable, then so is $U_{r,n}$.*

Proof. Let M be an orderable series extension of $U_{r,n}$ and fix a consistent ordering of M. By Lemma 2.3.2(ii), there is an integer $k \geq 1$ such that every series class of M is divided into exactly k blocks. If $k = 1$, the result follows immediately, so assume $k \geq 2$.

Let $[n]$ be the ground set of $U_{r,n}$. Consider the circuit C of M that contains the set $\{1, 2, \ldots, r+1\}$. Label the S_1-blocks in C as $B_1, B_2, \ldots, B_k$, such that B_i and B_{i+1} abut a section K_i that does not contain S_1-blocks, as in Figure 2.12.

Applying Lemma 2.3.2(i), we see that each section K_i contains exactly one S_j-block

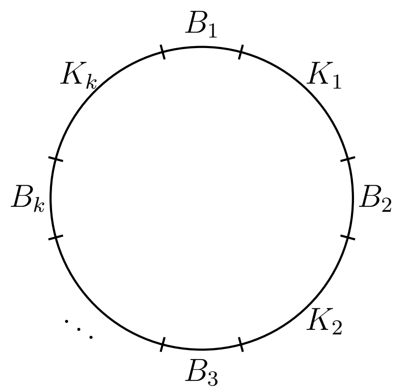

Figure 2.12. The circuit C in the proof of Proposition 2.3.3.

for all j in $\{2, 3, \ldots, r+1\}$. Thus, $B_i \cup K_i$ defines a permutation of $\{1, 2, \ldots, r+1\}$ that begins with 1. We show this permutation is the same for all i.

Without loss of generality, suppose the block in K_1 adjacent to B_1 is an S_2-block. If the block in K_2 adjacent to B_2 is an S_j-block with $j \neq 2$, then the S_j-blocks in K_1 and K_2 abut a section that contains no S_2-block, contradicting Lemma 2.3.2(i). Thus the block in K_2 adjacent to B_2 is an S_2-block. Repeating this argument gives that $B_1 \cup K_1$ and $B_2 \cup K_2$ define the same permutation on $\{1, 2, \ldots, r+1\}$. It follows that $B_i \cup K_i$ defines the same permutation on $\{1, 2, \ldots, r+1\}$ for all i in $[n]$. Thus $B_i \cup K_i \cup B_{i+1}$ defines the same reversible cyclic ordering on $\{1, 2, \ldots, r+1\}$ for all i in $[n]$; it is this reversible cyclic ordering that we extract from C and use to order the circuit $\{1, 2, \ldots, r+1\}$ in $U_{r,n}$.

In this way, every circuit of $U_{r,n}$ is ordered using the corresponding circuit of M. Since the ordering of M is consistent, so too is the ordering it gives to $U_{r,n}$. □

Theorem 2.3.4. *Let $U_{r,n}$ be a non-binary uniform matroid of rank at least three. If M is a matroid with a series minor isomorphic to $U_{r,n}$, then M is not orderable.*

Proof. By [11, Proposition 5.4.2], we may write $U_{r,n} = M \backslash X / Y$ where each element of Y

is in series with an element of $M\backslash X$ not in Y. By Corollary 2.3.1, the matroid $U_{r,n}$ is not orderable. Therefore, by Proposition 2.3.3, neither is its series extension $M\backslash X$. Thus, M is not orderable. □

Recall that, in a balanced series extension N of a matroid M without coloops, each element of M is replaced by k elements in series for some positive integer k.

Lemma 2.3.5. *Let N be a balanced series extension of $U_{2,n}$ for some $n \geq 4$. Then N is orderable.*

Proof. Let $[n]$ be the ground set of $U_{2,n}$. For each x in $E(U_{2,n})$, let $S_x = \{x_0, x_1, \ldots, x_{k-1}\}$ be the series class of N that corresponds to x. Subscript arithmetic will be done modulo k. Let $\mathcal{C}_1$ be the set of circuits of N containing S_1. For the circuit $C = S_1 \cup S_x \cup S_y$ in $\mathcal{C}_1$ with $x < y$, give C the ordering

$$(y_0\ 1_0\ x_0\ y_1\ 1_1\ x_1\ \ldots\ y_{k-1}\ 1_{k-1}\ x_{k-1}).$$

To see that the circuits in $\mathcal{C}_1$ are consistent, suppose e and f belong to common circuits in $\mathcal{C}_1$. Then at least one of e and f, say e, is in S_1. If f is in S_1, then e and f are never adjacent. Otherwise, f is in S_z for some $z > 1$, so $e = 1_s$ and $f = z_t$ for some s and t. If $s = t$, then e and f are always adjacent; if $s \neq t$, then e and f are never adjacent.

Now let $\mathcal{C}_2$ be the set of circuits of N not containing S_1. For the circuit $D = S_x \cup S_y \cup S_z$ in $\mathcal{C}_2$ with $1 < x < y < z$, give D the ordering

$$(z_1\ x_0\ y_1\ z_2\ x_1\ y_2\ \ldots\ z_0\ x_{k-1}\ y_0).$$

Note that x_i is always adjacent to y_{i+1} and z_{i+1}. Further, the blocks $z_i x_{i-1} y_i$ are ordered so that y_i is always adjacent to z_{i+1}. Thus, the circuits in $\mathcal{C}_2$ are consistent with those in

C_1.

Finally, the circuits in $\mathcal{C}_2$ are consistent. Suppose instead that there are circuits C and D in $\mathcal{C}_2$ and elements x_s and y_t in $C \cap D$ so that x_s and y_t are adjacent in C but not in D. Assume $x < y$. From C, we have that $t = s + 1$, but, from D, we have that $t \neq s + 1$, a contradiction. □

The following proposition specializes some of the results about uniform matroids to $U_{2,n}$ with $n \geq 4$. These rank-two uniform matroids will serve as the foundation from which all non-binary orderable matroids are built.

Proposition 2.3.6. *Let M be an orderable series extension of $U_{2,n}$ for some $n \geq 4$, and fix a consistent ordering of M. Then*

(i) for all series classes S of M, every S-block of the ordering consists of a single element; and

(ii) M is a balanced series extension of $U_{2,n}$.

Proof. Statement (ii) follows from combining (i) with Lemma 2.3.2(ii), so it suffices to show (i). Let $E(U_{2,n}) = [n]$. Suppose, to the contrary, that M has an S_1-block B of size at least two.

Applying Lemma 2.3.2(i), we have that B is adjacent to both an S_2-block and an S_3-block in the circuit of M containing $\{1, 2, 3\}$. Let 1_2 be the element of B adjacent to the S_2-block and let 1_3 be the element of B adjacent to the S_3-block, where 1_2 and 1_3 are necessarily distinct. In the circuit of M containing $\{1, 2, 4\}$, Lemma 2.3.2(i) now gives that B is adjacent to both an S_2-block and an S_4-block. Consistency dictates that 1_2 is again adjacent to the S_2-block. Therefore 1_3 is now adjacent to the S_4-block.

Now consider the circuit of M containing $\{1, 3, 4\}$. Consistency with the two aforementioned circuits requires that 1_3 be adjacent to both an S_3-block and an S_4-block. As $|B| \geq 2$, this is a contradiction. □

The next theorem identifies all orderable matroids that are 3-connected and non-binary.

Theorem 2.3.7. *If M is a 3-connected non-binary orderable matroid, then $M \cong U_{2,n}$ for some $n \geq 4$.*

The next two results will be used in the proof of this theorem.

Proposition 2.3.8. *If M is an orderable matroid, then M has no minor isomorphic to $U_{3,5}$.*

Proof. Assume instead that $M \backslash X/Y \cong U_{3,5}$, with X coindependent and Y independent. Then $M^*/X \backslash Y \cong U_{2,5}$ where M^*/X has rank two. Thus, after deleting a set Z of loops from M^*/X, we obtain a parallel extension of $U_{2,n}$ for some $n \geq 5$. This makes $M \backslash (X \cup Z)$ an orderable series extension of $U_{n-2,n}$, contradicting Theorem 2.3.4. □

Proposition 2.3.9. *If M is an orderable matroid, then M has no minor isomorphic to $\mathcal{W}^3$.*

The proof of this proposition will rely on the next lemma and its corollary. This second pair of results will use the following modification of the (e, f, g)-property. A matroid M has the *series (e, f, g)-property* if

(i) M has a circuit containing $\{e, f, g\}$;

(ii) $S_f(M)$ is distinct from both $S_e(M)$ and $S_g(M)$; and

(iii) M has a circuit D containing f but not $\{e, g\}$ and, for each d in D, there is a circuit of M containing $\{e, f, g, d\}$.

Note that e and g may be equal in this definition.

Lemma 2.3.10. *Suppose that M has the series (e, f, g)-property and that N is a series extension of M. Then, in a consistent ordering of N, if $S_e(N) \neq S_g(N)$, then no $S_f(N)$-block is adjacent to both an $S_e(N)$-block and an $S_g(N)$-block; and, if $S_e(N) = S_g(N)$, then no $S_f(N)$-block is adjacent to two $S_e(N)$-blocks.*

Proof. Let D be the circuit of M whose existence is guaranteed by condition (iii). Let D' be the circuit of N corresponding to D, and let B_f be an $S_f(N)$-block. Notice D must have an element d not in $\{e, f, g\}$, so $D' - (S_e(N) \cup S_f(N) \cup S_g(N))$ is nonempty. If $S_e(N) = S_g(N)$ and B_f is adjacent to two $S_e(N)$-blocks, then e is not in D, so B_f is not adjacent to any elements of $D' - B_f$, a contradiction. Now suppose $S_e(N) \neq S_g(N)$ and, without loss of generality, suppose e is in D but g is not. If B_f is adjacent to an $S_e(N)$-block and an $S_g(N)$-block, then all of the elements in $D' - B_f$ adjacent to B_f are in $S_e(N)$. This contradicts the fact that B_f is adjacent to an $S_e(N)$-block and an $S_g(N)$-block in a common circuit. □

Corollary 2.3.11. *Let C be a circuit of a matroid M. Suppose that C contains an element c so that M has the series (e, c, g)-property for every choice of e and g in $C - c$. Then no series extension of M is orderable.*

Proof of Proposition 2.3.9. Assume instead that $M \backslash X / Y \cong \mathcal{W}^3$, with X coindependent and Y independent. Let L be the set of loops of M^*/X, and let N denote $M^*/(X \cup L)$. Note that N is a loopless rank-3 extension of $\mathcal{W}^3$, so $\mathrm{si}(N)$ is 3-connected. Further, N is a

Number n of elements	Restrictions of O_7 with n elements
0	$U_{0,0}$
1	$U_{1,1}$
2	$U_{2,2}$
3	$U_{2,3}$, $U_{3,3}$
4	$U_{2,4}$, $U_{3,4}$, $U_{2,3} \oplus U_{1,1}$
5	$U_{2,4} \oplus U_{1,1}$, $P(U_{2,3}, U_{2,3})$, $U_{2,4} \oplus_2 U_{2,3}$
6	$P(U_{2,4}, U_{2,3})$, $M(K_4)$, $\mathcal{W}^3$
7	O_7

Figure 2.13. Choices for K, the complement of si(N) in $PG(2,3)$.

parallel extension of si(N), which makes N^* an orderable series extension of co(N^*).

2.3.11.1. si(N) *is ternary.*

To see this, first note that, as N^* is orderable, it has no $U_{3,5}$-minor by Proposition 2.3.8. Thus, si(N) has no $U_{2,5}$-minor. As si(N) is 3-connected and its rank and corank each exceed two, [11, Proposition 12.2.15] gives that si(N) has no $U_{3,5}$-minor. The rank of F_7^* exceeds three, so si(N) also has no F_7^*-minor.

Finally, suppose si(N) has an F_7-minor. Then si$(N)|Z \cong F_7$ for some set Z. As F_7 has no $\mathcal{W}^3$-minor, si(N) has an element e not in Z. Then si$(N)/e$ has a $U_{2,5}$-restriction, a contradiction. We conclude that si(N) has no F_7-minor. Thus 2.3.11.1 holds.

By 2.3.11.1, si(N) has the form $PG(2,3) - K$, where K is a restriction of O_7, the complement of $\mathcal{W}^3$ in $PG(2,3)$. The matroid O_7 is obtained from $M(K_4)$ by adding a point freely to an existing 3-point line; the fifteen restrictions of O_7 are given in Figure 2.13. In the remainder of the proof, we eliminate each possibility for K.

If $K = U_{0,0}$, then si$(N) = PG(2,3)$. Let si(N) be labelled as in Figure 2.14. Suppose N^* has a consistent ordering, and let B_x, B_y, and B_z be S_x-, S_y-, and S_z-blocks in a common circuit C of N^*, where x, y, and z are elements of $E(\text{si}(N))$. Assume also that

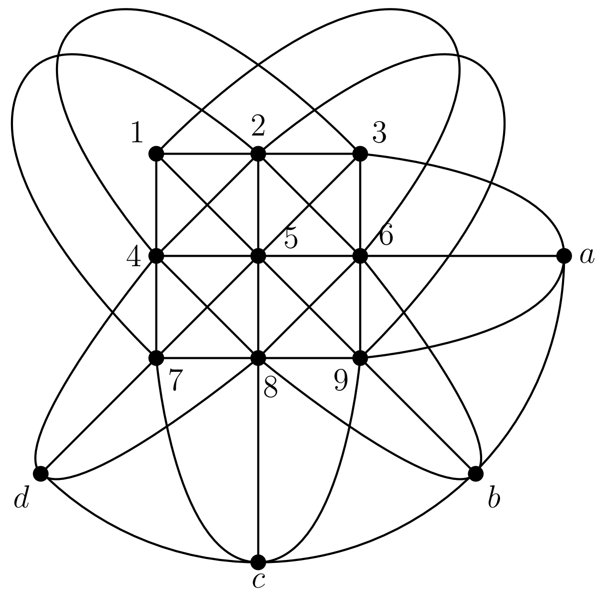

Figure 2.14. $PG(2,3)$ in the proof of Proposition 2.3.9.

B_y is adjacent to B_x and B_z. Then, by Lemma 2.3.10, $\mathrm{co}(N^*)$ does not have the series (x, y, z)-property. We show next that

2.3.11.2. *x, y, and z are collinear in* $\mathrm{si}(N)$*, and $x \neq z$.*

Suppose x, y, and z are not collinear in $\mathrm{si}(N)$. Then one easily finds circuits of $\mathrm{co}(N^*)$ that verify the series (x, y, z)-property in $\mathrm{co}(N^*)$, a contradiction. Similarly, when $x = z$ there are circuits of $\mathrm{co}(N^*)$ that verify the series (x, y, z)-property in $\mathrm{co}(N^*)$, a contradiction. Thus, 2.3.11.2 holds.

By symmetry, we may assume that C is the circuit $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of $\mathrm{co}(N^*)$; let C' be the corresponding circuit of N^*. Consider an S_1-block B in C'. The block B is adjacent to an S_e- and S_f-block for some e and f in $C - 1$. By 2.3.11.2, the elements 1, e, and f are collinear in $\mathrm{si}(N)$; without loss of generality, say $e = 2$ and $f = 3$. Let B_3 be the S_3-block adjacent to B. By repeatedly applying 2.3.11.2, we have that B_3 is adjacent to

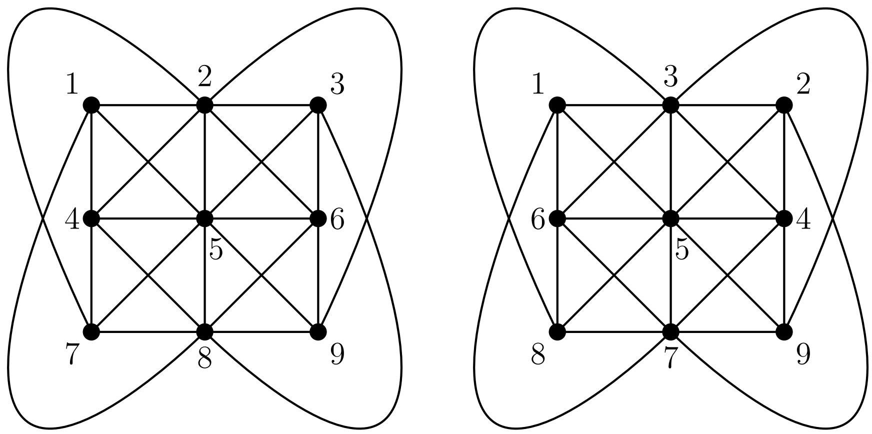

Figure 2.15. Two geometric representations of the matroid $AG(2,3)$.

an S_2-block B_2, the block B_2 is adjacent to another S_1-block B_1, the block B_1 is adjacent to another S_3-block, and so on. It follows that C' has a proper subset X of elements not adjacent to any element of $C' - X$, a contradiction.

If $K = U_{2,4}$, then $\text{si}(N) = AG(2,3)$. Figure 2.15 gives two labelled copies of $\text{si}(N)$ in order to illustrate some of the symmetries of this matroid. Using Corollary 2.3.11, let c be the element 1 in the circuit $C = \{1,2,3,4,5,6\}$ of $\text{co}(N^*)$. When $e = g = 2$, the circuits C and $\{1,2,3,7,8,9\}$ certify that $\text{co}(N^*)$ has the series $(2,1,2)$-property with $D = \{1,3,4,6,7,9\}$. Since $\text{co}(N^*)$ has a doubly transitive automorphism group, it follows that $\text{co}(N^*)$ has the series $(e,1,e)$-property for each e in C. When $\{e,g\} = \{2,3\}$, the circuits C and $\{1,2,3,7,8,9\}$ certify that $\text{co}(N^*)$ has the series $(2,1,3)$-property with $D = \{1,3,4,6,7,9\}$. The circuits C, $\{1,2,4,5,7,8\}$, and $\{1,2,4,6,8,9\}$ certify that $\text{co}(N^*)$ has the series $(2,1,4)$-property with $D = \{1,3,4,6,7,9\}$. A symmetric set of circuits certifies that $\text{co}(N^*)$ has the series $(e,1,g)$-property for each independent set $\{e,1,g\}$ contained in C. By Corollary 2.3.11, N^* is not orderable.

If $K = U_{2,4} \oplus U_{1,1}$, then $\mathrm{si}(N) = AG(2,3)\backslash 9$ with $AG(2,3)$ labelled as in Figure 2.15. Using Corollary 2.3.11 again, let c be the element 1 in the circuit $C = \{1,2,3,7,8\}$ of $\mathrm{co}(N^*)$. When $e = g = 2$, the circuits C and $\{1,2,4,6,8\}$ certify that $\mathrm{co}(N^*)$ has the series $(2,1,2)$-property with $D = \{1,3,4,6,7\}$. A symmetric set of circuits certifies that $\mathrm{co}(N^*)$ has the series $(e,1,e)$-property for each e in $C-1$.

When $\{e,g\} = \{2,3\}$, the circuits C and $\{1,2,4,6,8\}$ certify that $\mathrm{co}(N^*)$ has the series $(2,1,3)$-property with $D = \{1,3,4,6,7\}$. From Figure 2.15, we see that the cases $\{e,g\} = \{2,7\}$ and $\{e,g\} = \{3,8\}$ are symmetric; and the circuits C and $\{1,2,4,5,7,8\}$ certify that $\mathrm{co}(N^*)$ has the series $(2,1,7)$-property with $D = \{1,3,4,5,8\}$. The cases $\{e,g\} = \{2,8\}$ and $\{e,g\} = \{3,7\}$ are also symmetric, and the circuits C and $\{1,2,4,6,8\}$ certify that $\mathrm{co}(N^*)$ has the series $(2,1,8)$-property with $D = \{1,3,4,6,7\}$. Finally, the circuits $\{1,2,4,5,7,8\}$ and $\{1,3,5,6,7,8\}$ certify that $\mathrm{co}(N^*)$ has the series $(7,1,8)$-property with $D = \{1,2,3,4,5,6\}$. Corollary 2.3.11 now implies N^* is not orderable.

The next five cases make frequent use of Proposition 2.3.6(ii). The strategy is to contract strategic parallel classes of N to get parallel extensions of $U_{2,4}$. These parallel extensions are dual to orderable series extensions of $U_{2,4}$, and Proposition 2.3.6(ii) implies that the parallel classes of such a parallel extension have the same size. For each case, we view $\mathrm{si}(N)$ as a restriction of the labelled copy of $PG(2,3)$ in Figure 2.14. For each element e in $E(N)$, let p_e be the size of the parallel class of N containing e.

If $K = U_{1,1}$, then $\mathrm{si}(N) = PG(2,3)\backslash d$. The following equations are obtained by applying Proposition 2.3.6(ii) in the minors $N/\mathrm{cl}(\{a\})$, $N/\mathrm{cl}(\{b\})$, and $N/\mathrm{cl}(\{c\})$, respectively:

$$p_b + p_c = p_1 + p_2 + p_3 = p_4 + p_5 + p_6 = p_7 + p_8 + p_9;$$

$$p_a + p_c = p_1 + p_5 + p_9 = p_2 + p_6 + p_7 = p_3 + p_4 + p_8;$$

$$p_a + p_b = p_3 + p_6 + p_9 = p_2 + p_5 + p_8 = p_1 + p_4 + p_7.$$

Combining these equations, we obtain

$$3(p_a + p_b) + 3(p_a + p_c) + 3(p_b + p_c) = 3(p_1 + p_2 + \cdots + p_9),$$

which implies

$$2(p_a + p_b + p_c) = p_1 + p_2 + \cdots + p_9,$$

and therefore

$$3(p_a + p_b + p_c) = |E(N)|.$$

We conclude that exactly one-third of the elements of $E(N)$ lie on the line $\{a, b, c\}$. By symmetry, the same is true of the lines $\{1, 6, 8\}$, $\{3, 5, 7\}$, and $\{2, 4, 9\}$, so now four disjoint lines each account for one-third of the elements in N, a contradiction.

If $K = U_{2,3}$, then $\operatorname{si}(N) = PG(2, 3)\backslash\{b, c, d\}$. The following equations are obtained by applying Proposition 2.3.6(ii) in the minors $N/\operatorname{cl}(\{1\})$, $N/\operatorname{cl}(\{2\})$, and $N/\operatorname{cl}(\{3\})$, respectively:

$$p_2 + p_3 + p_a = p_4 + p_7 = p_6 + p_8 = p_5 + p_9 = \frac{1}{4}\big(|E(N)| - p_1\big); \tag{2.1}$$

$$p_1 + p_3 + p_a = p_4 + p_9 = p_5 + p_8 = p_6 + p_7 = \frac{1}{4}\big(|E(N)| - p_2\big); \tag{2.2}$$

$$p_1 + p_2 + p_a = p_5 + p_7 = p_6 + p_9 = p_4 + p_8 = \frac{1}{4}(|E(N)| - p_3).$$

Solving equations (2.1) and (2.2) for $|E(N)|$, we see that

$$p_1 + 4p_2 + 4p_3 = 4p_1 + p_2 + 4p_3,$$

so $p_1 = p_2$. Through additional substitutions, it follows that $p_i = p_j$ for each $i, j \neq a$. But now $p_a = 0$, a contradiction.

If $K = P(U_{2,3}, U_{2,4})$, then $\mathrm{si}(N) = PG(2,3)\backslash\{7, 8, 9, a, b, d\} \cong P_7$. From the minors $N/\operatorname{cl}(\{1\})$ and $N/\operatorname{cl}(\{3\})$ and Proposition 2.3.6(ii), we get the equations

$$p_2 + p_3 = p_4 + p_c = p_5 = p_6,$$

$$p_1 + p_2 = p_6 + p_c = p_4 = p_5.$$

It follows that $p_c = 0$, a contradiction.

If $K = \mathcal{W}^3$, then $\mathrm{si}(N) = PG(2,3)\backslash\{6, 8, 9, b, c, d\} \cong O_7$. From the minors $N/\operatorname{cl}(\{7\})$ and $N/\operatorname{cl}(\{5\})$ we get the equations

$$p_1 + p_4 = p_3 + p_5 = p_2 = p_a,$$

$$p_3 + p_7 = p_4 + p_a = p_1 = p_2.$$

It follows that $p_4 = 0$, a contradiction.

If $K = O_7$, then $\mathrm{si}(N) = PG(2,3)\backslash 6, 8, 9, a, b, c, d \cong \mathcal{W}^3$. From the minors $N/\operatorname{cl}(\{2\})$ and $N/\operatorname{cl}(\{4\})$ we get the equations

$$p_1 + p_3 = p_4 = p_5 = p_7,$$

$$p_1 + p_7 = p_2 = p_3 = p_5,$$

so $p_1 = 0$, a contradiction.

For the next six cases, we continue to view N as a parallel extension of a restriction of $PG(2,3)$, with $PG(2,3)$ labelled as in Figure 2.14. However, we now represent the deletion of an element e from $PG(2,3)$ by setting p_e to be 0. Each of these cases is eliminated using the following assertion.

2.3.11.3. *Let N be a restriction of $PG(2,3)$ such that*

(i) $p_c = p_d = 0$*;*

(ii) $p_x \neq 0$ *for each x in $\{a, b, 1\}$;*

(iii) $\mathrm{si}(N/\mathrm{cl}(\{x\})) \cong U_{2,4}$ *for each x in $\{a, b, 1\}$; and*

(iv) p_2 *and* p_3 *are not both zero.*

Then N^* is not orderable.

To see this, we first use the minors $N/\mathrm{cl}(\{a\})$ and $N/\mathrm{cl}(\{b\})$ to establish the equations

$$p_b = p_1 + p_2 + p_3 = p_4 + p_5 + p_6 = p_7 + p_8 + p_9,$$

$$p_a = p_2 + p_6 + p_7 = p_1 + p_5 + p_9 = p_3 + p_4 + p_8,$$

from which we obtain

$$3p_a = p_1 + p_2 + \cdots + p_9 = 3p_b,$$

so $p_a = p_b$, and $|E(N)| = 5p_a$. Now, $N/\mathrm{cl}(\{1\})$ gives that

$$|E(N)| - p_1 = 4(p_2 + p_3 + p_a),$$

and substituting $5p_a$ for $|E(N)|$ produces

$$p_a = p_1 + 4(p_2 + p_3).$$

Finally, since $p_a = p_1 + p_2 + p_3$, we deduce that $p_2 + p_3 = 0$, a contradiction. Thus 2.3.11.3 holds.

The six options for K eliminated by 2.3.11.3 are the matroids $U_{2,2}$, $U_{3,3}$, $U_{3,4}$, $U_{2,3} \oplus U_{1,1}$, $P(U_{2,3}, U_{2,3})$, and $U_{2,3} \oplus_2 U_{2,4}$. It is straightforward to check that, for each K in

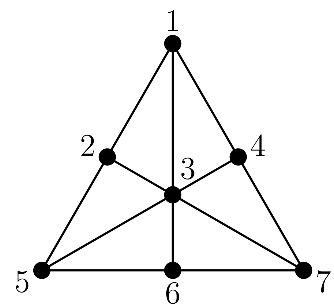

Figure 2.16. F_7^- in the proof of Proposition 2.3.9.

this list, we may set classes of $PG(2,3)$ equal to zero in such a way that the zeroed classes form a restriction isomorphic to K, and the conditions of 2.3.11.3 hold. For example, $U_{2,3} \oplus_2 U_{2,4}$ is produced when p_5, p_7, p_9, p_c, and p_d are the zeroed classes.

In the final case, $K = M(K_4)$ and $\text{si}(N) = F_7^-$. Label F_7^- as in Figure 2.16, and, for each e in $[7]$, let $S_e = S_e(N^*)$. Suppose N^* has a consistent ordering, and let B be an S_1-block in the ordering. In N^*, there is a circuit corresponding to each circuit $\{1,2,3,5,7\}$, $\{1,3,4,5,7\}$, and $\{1,3,5,6,7\}$ of $\text{co}(N^*)$; let $\mathcal{X}$ be the collection of these circuits of N^*. Similarly, let $\mathcal{Y}$ be the collection of circuits of N^* corresponding to the circuits $\{1,2,3,4\}$, $\{1,4,5,6\}$, and $\{1,2,6,7\}$ of $\text{co}(N^*)$.

Suppose B is adjacent to an S_e-block for some e in $\{2,4,6\}$. Then the consistency of the circuits in $\mathcal{X}$ implies that B is adjacent to an S_e-block for every e in $\{2,4,6\}$. The circuits in $\mathcal{Y}$ now imply that B is adjacent to an S_2-, S_4-, and S_6-block. Further, B is not adjacent to an S_e-block for any e in $\{3,5,7\}$. It follows that, in the circuit of N^* corresponding to $\{1,2,3,5,7\}$, the block B must be adjacent to a pair of S_2-blocks, contradicting the fact that B is adjacent to both an S_2-block and an S_4-block in the circuit of N^* corresponding to $\{1,2,3,4\}$.

We now know that, for each e in $\{2,4,6\}$, the block B is not adjacent to an

S_e-block. The circuits in $\mathcal{Y}$ now imply that, in the circuit of N^* corresponding to $\{1,2,3,5,7\}$, the block B is adjacent to an S_e-block for every e in $\{3,5,7\}$. This contradiction implies N^* is not orderable. $\square$

The next proposition is a result of Oxley [10] (see also [11, Corollary 12.2.18]). We will use it to prove Theorem 2.3.7.

Proposition 2.3.12. *A 3-connected non-binary matroid whose rank and corank exceed two has a minor isomorphic to one of $\mathcal{W}^3$, P_6, Q_6, and $U_{3,6}$.*

Proof of Theorem 2.3.7. Assume that the theorem fails for M. Then $r(M) \geq 3$. As P_6, Q_6, and $U_{3,6}$ each have $U_{3,5}$ as a minor, Proposition 2.3.12 and Propositions 2.3.8 and 2.3.9 now imply that $r^*(M) \leq 2$, so $r^*(M) = 2$. As M is 3-connected, it follows that $M \cong U_{n-2,n}$ for some $n \geq 5$. Hence M has a $U_{3,5}$-minor, a contradiction. $\square$

Section 1.1 gives an introduction to matroid-labelled trees and canonical tree decompositions. Let T be a tree decomposition of a matroid M, and let N and p be a vertex label and edge label of T, respectively. For the remainder of this section, we define $M_{p,N}$ and $M'_{p,N}$ to be the matroids such that $M = M_{p,N} \oplus_2 M'_{p,N}$ with basepoint p, where $E(M_{p,N})$ contains the subset of $E(M)$ corresponding to the component of $T \backslash p$ containing N. Notice that if the vertex labels M_1 and M_2 lie in different components of $T \backslash p$, then $M_{p,M_1} = M'_{p,M_2}$.

In the next four lemmas, M is assumed to be a connected, orderable, non-binary matroid whose canonical tree decomposition is T.

Lemma 2.3.13. *Suppose that T has a vertex label U that is isomorphic to $U_{2,n}$ for some $n \geq 4$. Then, for all $e, f \in E(U)$,* *(i)* e *is an edge label of* T*, unless* M *is a parallel extension of* $U_{2,n}$*;*

(ii) all circuits of $M'_{e,U}$ *containing* e *have the same size; and*

(iii) the circuits of $M'_{e,U}$ *containing* e *have the same size as the circuits of* $M'_{f,U}$ *containing* f*.*

Proof. We may assume that M is not a parallel extension of $U_{2,n}$ otherwise (i) holds. For each element y of $E(U)$ that labels an edge of T, let C_y be a circuit of $M'_{y,U}$ that contains y. As M is not a parallel extension of $U_{2,n}$, we may assume that $|C_x| \geq 3$ for some element x. Let M'' be the matroid that is obtained from U by attaching each C_y via 2-sum. This matroid is a restriction of M having $C_x - x$ as a non-trivial series class. Moreover, M'' is a series extension of $U_{2,n}$ and it is orderable. Thus, by Proposition 2.3.6(ii), M'' is a balanced series extension of $U_{2,n}$. Hence (i) holds. Furthermore, $|C_x| = |C_y| \geq 3$ for all y in $E(U) - \{x\}$. Parts (ii) and (iii) now follow without difficulty. □

The next lemma generalizes Lemma 2.3.13(ii) to arbitrary edges of T.

Lemma 2.3.14. *Suppose that* T *has a vertex label* U *that is isomorphic to* $U_{2,n}$ *for some* $n \geq 4$*, and suppose* e *is an edge label of* T*. Then the circuits of* $M'_{e,U}$ *that contain* e *all have the same size.*

Proof. Let N be the endpoint of e in the same component of $T\backslash e$ as U. If $U = N$, then the assertion holds by Lemma 2.3.13(ii), so assume otherwise. Let f be the label of the edge incident with U that lies on the path connecting U to N in T. Next, let T' be the subtree of $T\backslash\{e, f\}$ containing N, and let M' be the matroid with tree decomposition T'.

Fix a circuit C of M' that contains e and f. Observe that, for each circuit D of $M'_{e,N}$ that contains e, there is a circuit $(D - e) \cup (C - e)$ of $M'_{f,U}$ that contains f. By

Lemma 2.3.13(ii), the quantity $|(D-e)\cup(C-e)|$ is the same for each choice of D, so every such circuit D has the same size. □

Lemma 2.3.15. *The tree T has exactly one 3-connected non-binary vertex label, and this label is isomorphic to $U_{2,n}$ for some $n \geq 4$.*

Proof. As M is non-binary, it has at least one 3-connected non-binary vertex label N. For each element y of $E(N)$ that labels an edge of T, let C_y be a circuit of $M'_{y,N}$ that contains y. Let M'' be the matroid that is obtained from N by attaching each C_y via 2-sum. Then M'' is a restriction of M. Thus M'' is an orderable series extension of N. By Propositions 2.3.8, 2.3.9, and 2.3.12, $N \cong U_{2,n}$ for some $n \geq 4$. Now suppose T has a pair of 3-connected non-binary vertex labels $N_1 \cong U_{2,n_1}$ and $N_2 \cong U_{2,n_2}$ with $n_1, n_2 \geq 4$. Let e_1 and e_2 be the edge labels of T incident with N_1 and N_2 that lie on the path connecting N_1 and N_2 in T.

By Lemma 2.3.13(ii), the circuits of M'_{e_1,N_1} containing e_1 all have size k and the circuits of M'_{e_2,N_2} containing e_2 all have size ℓ, where k and ℓ are integers exceeding one. Let $\{e_1, x, y\}$ be a circuit of N_1. By Lemma 2.3.13(i), x and y are also edge labels of T; let C_x be a circuit of M'_{x,N_1} containing x, and C_y be a circuit of M'_{y,N_1} containing y. Then $k = |C_x| = |C_y|$ by Lemma 2.3.13(iii). Now there is a circuit of M'_{e_2,N_2} containing e_2 that also contains $C_x - x$ and $C_y - y$. Thus, $\ell \geq 2(k-1)+1$. A symmetric argument gives that $k \geq 2(\ell-1)+1$, and substitution yields that $k \leq 1$, a contradiction. □

The next lemma rules out 3-connected binary vertex labels that are not circuits or cocircuits. It uses the following result of Seymour [15].

Proposition 2.3.16. *Let M be a 3-connected binary matroid with at least four elements.*

If $e \in E(M)$, then M has an $M(K_4)$-minor using e.

Lemma 2.3.17. *No vertex of T is labelled by a 3-connected binary matroid with at least four elements.*

Proof. Suppose B is such a vertex label of T, let U be the unique vertex label with $U \cong U_{2,n}$ and $n \geq 4$ given by Lemma 2.3.15, and say $E(U) = \{e_1, e_2, \ldots, e_n\}$. Let $p \in E(B)$ and $e_1 \in E(U)$ be the labels of the edges incident with B and U, respectively, that lie on the path connecting B to U in T. By Proposition 2.3.16, B has a minor isomorphic to $M(K_4)$ that uses p.

This minor can be written in the form $B/I\backslash I^*$, where I is independent in B and I^* is coindependent in B. This makes B/I a rank-three binary matroid with $M(K_4)$ as a restriction, so after deleting the loops from B/I, we obtain a parallel extension of either $M(K_4)$ or F_7. Dually, after deleting the coloops from $B\backslash I^*$, we obtain a series extension of $M(K_4)$ or F_7^*. Thus B has a restriction N_1 using p that is a series extension of $M(K_4)$ or F_7^*.

Suppose q is an edge label of T that is used in N_1, and choose a circuit C_q of $M'_{q,B}$ that contains q. Form the matroid N_2 from N_1 by replacing q with $C_q - q$ in $E(N_1)$ for each q in $E(N_1) - p$ that is an edge label of T. Then N_2 is a series extension of $M(K_4)$ or F_7^* that appears as a restriction of $M_{p,B}$. Now, for each i in $\{2, 3\}$, let C_{e_i} be a circuit of $M'_{e_i,U}$ that contains e_i. Then $M'_{p,B}$ has a circuit C_p that contains p and both $C_{e_2} - e_2$ and $C_{e_3} - e_3$. Form the matroid N from N_2 by taking the 2-sum of N_2 and C_p across the basepoint p. Then N is a restriction of M that is a series extension of $M(K_4)$ or F_7^*. For each element x of $M(K_4)$ or F_7^*, let S_x be $S_x(N)$. By Lemma 2.3.14, every circuit of N_2

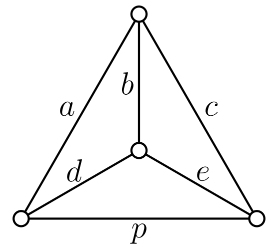

Figure 2.17. K_4 in the proof of Lemma 2.3.17.

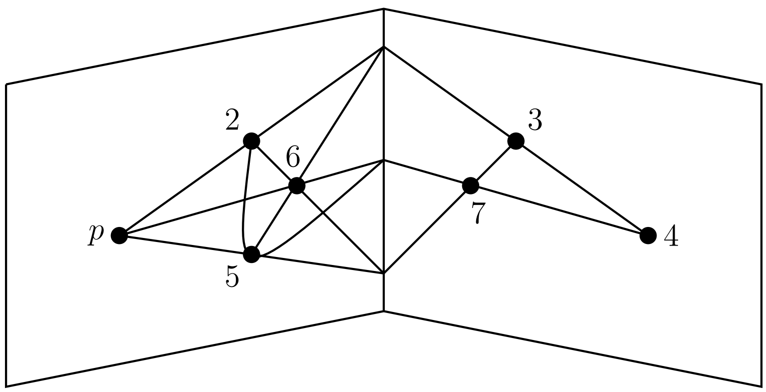

Figure 2.18. F_7^* in the proof of Lemma 2.3.17.

that contains p has the same size.

Suppose first that N is a series extension of $M(K_4)$ with K_4 labelled as in Figure 2.17. Thus, every circuit of N that contains S_p has the same size. Since all circuits of N containing S_p have the same size, $|S_d| + |S_e| = |S_a| + |S_b| + |S_e|$, so

$$|S_d| = |S_a| + |S_b|. \tag{2.3}$$

Similarly, $|S_a| + |S_c| = |S_d| + |S_b| + |S_c|$, so

$$|S_a| = |S_d| + |S_b|. \tag{2.4}$$

Equations (2.3) and (2.4) imply that $|S_b| = 0$, a contradiction.

Now suppose that N is a series extension of F_7^* with F_7^* labelled as in Figure 2.18. Since the circuits of N containing S_p must have the same size,

$$|S_2| + |S_5| = |S_4| + |S_7|,$$

$$|S_2| + |S_6| = |S_3| + |S_7|,$$

and

$$|S_5| + |S_6| = |S_3| + |S_4|.$$

Together, these equations imply that

$$|S_2| = |S_7|. \tag{2.5}$$

Fix a consistent ordering of M. This induces a consistent ordering of N. Consider the circuit $C = S_p \cup S_2 \cup S_3 \cup S_4$ of N. Notice that M has, as a restriction, a series extension U' of $U_{2,n}$ whose ground set contains C. Specifically, $C = S_{e_1} \cup S_{e_2} \cup S_{e_3}$, where S_{e_i} is $S_{e_i}(U')$.

Let t be an arbitrary member of the series class S_2 of N. In U', the element t belongs to the class S_{e_1}, so $\{t\}$ is an S_{e_1}-block in the ordering of C by Proposition 2.3.6(i). Lemma 2.3.2(i) implies that t is adjacent to some element $x \in S_{e_2}$ and some $y \in S_{e_3}$; notice that, in N, the elements x and y both belong to S_p. Thus, every element of S_2 is adjacent to a pair of elements from S_p in C. In particular, t is not adjacent to any element of S_2 or of S_3. Now observe that t is adjacent to this same pair $\{x, y\}$ in the circuit $S_p \cup S_2 \cup S_5 \cup S_6$ of N, so t is also not adjacent to any element of S_6. It follows that t is adjacent to a pair of elements from S_7 in the circuit $S_2 \cup S_3 \cup S_6 \cup S_7$ of N. Therefore $|S_2| < |S_7|$, contradicting (2.5). □

Proposition 2.3.18. *Let M'' be obtained from M by parallel-path addition. Then M is orderable if and only if M'' is orderable.*

Proof. In forming M'' from M, let P' be added in parallel to P. As M'' has M as a restriction, M is orderable if M'' is. Conversely, fix a consistent ordering of M and let C'' be a circuit of M''. If C'' does not meet P', give C'' the same ordering in M'' that it has in M. Otherwise, C'' contains P' and either $C'' = P \cup P'$, or there is a circuit C of M such that $C = (C'' - P') \cup P$. In the the latter case, give C'' the same ordering in M'' that C has in M by replacing every element $p \in P$ by the corresponding element $p' \in P'$.

If $C'' = P \cup P''$, take a circuit D of M containing P. Let $B_1, B_2, \ldots, B_k$ be the P-blocks of D, numbered sequentially as they appear in a traversal of the ordering of D in M. For each i in $[k]$, let $B'_i = \{p' : p \in B_i\}$. Now, order C'' as $B_1, B'_1, B_2, B'_2, \ldots, B_k, B'_k$. It is straightforward to check that this gives a consistent ordering of M''. □

We are now ready to prove the main result of the chapter, which was given as Theorem 2.1.2 in the introduction and is restated here for convenience.

Theorem 2.3.19. *Let M be a connected non-binary matroid. Then M is orderable if and only if it can be obtained from a balanced series extension of $U_{2,n}$ for some $n \geq 4$ by a sequence of parallel-path additions.*

Proof. By Lemmas 2.3.5 and 2.3.18, a matroid obtained from $U_{2,n}$ by the given operations is certainly orderable, so it remains to show the converse.

We may assume that M is simple, as adding an element in parallel is a parallel-path addition of size one. If $M \cong U_{2,n}$, the result holds, so assume otherwise. Let T be the canonical tree decomposition of M. Lemmas 2.3.15 and 2.3.17 imply that there is a single vertex label U of T for which $U \cong U_{2,n}$ and $n \geq 4$, and every vertex of $T - U$ is labelled by a circuit or a cocircuit. By Lemma 2.3.13(i), each e in $E(U)$ labels an edge of T. Let T'_e be the component of $T \backslash e$ that does not have U as a vertex. As M is simple, the leaves of T are labelled by circuits. Therefore, if every T'_e has only one vertex, then M is a series extension of $U_{2,n}$, and the result holds by Proposition 2.3.6(ii). We show that, if this is not the case, then each T'_e can be reduced to a single vertex labelled by a circuit via a sequence of deletions that can be undone by parallel-path additions.

Suppose T'_e has at least two vertices. Since only one vertex of T'_e is adjacent to U, not all vertices of T'_e are leaves of T. We now observe that

2.3.19.1. *T'_e has a vertex v that*

(i) is adjacent to a leaf of T; and

(ii) has exactly one neighbor that is not a leaf of T

If L is the set of leaves of T, such a vertex v can be found as a leaf of $T - L$. Since the leaves of T are labelled by circuits and T is canonical, v is labelled by a cocircuit C^*. Lemma 2.3.14 now implies that the circuits that label the leaves of T adjacent to C^* all have the same size, and every element of C^* must be used as a basepoint labelling an edge of T.

We can delete all but one of the leaves, C say, of T that are adjacent to C^*, along with the corresponding basepoints in C^*, since the circuit that labels each deleted leaf can be added via a parallel-path addition. As C^* is now a pair of parallel elements, we can delete the leaf labelled C and relabel v with C. At this point, v is a leaf, and is either adjacent to U, in which case the work on this subtree is complete, or v is adjacent to another vertex of T'_e labelled by a circuit C'. In the latter case, keep T canonical by contracting the edge of T between v and C' and labelling the resulting vertex with the circuit that is the 2-sum of C and C'.

Provided the modification of T'_e continues to have at least two vertices, condition 2.3.19.1 continues to hold, and the process described in the previous paragraph can be repeated. Thus, we may assume T'_e consists of a single vertex labelled by a circuit. By applying this pruning process on the other subtrees attached to U, the tree T is reduced to the decomposition tree of a balanced series extension of $U_{2,n}$. Thus, M can be obtained from a balanced series extension of $U_{2,n}$ by a sequence of parallel-path additions. □

2.4. Theta-Orderability

Recall that theta-orderability of a matroid requires a consistent ordering of the matroid with respect to the theta-graphs of that matroid. Each of the elementary properties of orderability given in Proposition 2.2.1 also holds for theta-orderability. Their straightforward proofs are omitted.

Proposition 2.4.1. *Let M be a matroid.*

(i) If M is theta-orderable, then $M\backslash e$ is theta-orderable for all e in $E(M)$.

(ii) If $r(M) \leq 2$, then M is theta-orderable.

(iii) M is theta-orderable if and only if the connected components of M are theta-orderable.

(iv) M is theta-orderable if and only if $\mathrm{si}(M)$ is theta-orderable.

Next we prove Theorem 2.1.6, a characterization of binary theta-orderable matroids.

Proof of Theorem 2.1.6. It is clear that a graphic matroid is theta-orderable. Moreover, Wagner [20] proved that a matroid is graphic if and only if it has no set of incompatible arcs. Now suppose that M has a circuit C and a set $\{A_1, A_2, A_3\}$ of incompatible arcs of C. It remains to show that M is not theta-orderable. Our proof of this is a straightforward modification of Wagner's proof that no graphic matroid has a set of incompatible arcs [20, Lemma 2]. Assume that M is theta-orderable. Because each of A_1, A_2, and A_3 is an arc, for each i in $\{1,2,3\}$, there is a theta-graph of M in which A_i is a theta-arc. As M is theta-orderable, A_i is a block in a consistent ordering of M. As $\{A_1, A_2, A_3\}$ is an incompatible set, there are distinct elements e_1, e_2, and e_3 of C such that $e \in A_1 \cap A_2 \cap A_3$ and $e_i \in A_i - (A_j \cup A_k)$ for all $\{i,j,k\} = \{1,2,3\}$. For each h in $\{2,3\}$, the set $A_1 \cup A_h$ is a block in C in which e appears between e_1 and e_h. Then e does not appear between e_2 and e_3 in $A_2 \cup A_3$, a contradiction. □

To prove Theorem 2.1.7, we will establish the following equivalent version of it.

Theorem 2.4.2. *A simple connected non-binary matroid is theta-orderable if and only if it is a balanced series extension of* $U_{2,n}$ *for some* $n \geq 4$.

Proof. First, for $n \geq 4$, the matroid $U_{2,n}$ and its series extensions have no theta-graphs. Therefore, consistent orderings of these matroids are also theta-orderings.

Conversely, suppose M is a simple connected non-binary orderable matroid. By Theorem 2.3.19, for some $n \geq 4$, we can obtain M from a balanced series extension B of $U_{2,n}$ by a sequence of parallel-path additions. It now suffices to show that the sequence of parallel-path additions is empty.

Suppose to the contrary that P' is a set added in parallel to a subset P of a series class S of B. Note $|P| \geq 2$ since M is simple. Now, by Proposition 2.3.6, each S-block in a consistent ordering of B contains a single element. As B is a restriction of M, this implies that the elements of P are not a block in a consistent ordering of M. Since M has a theta-graph with P and P' as theta-arcs, this is a contradiction. □

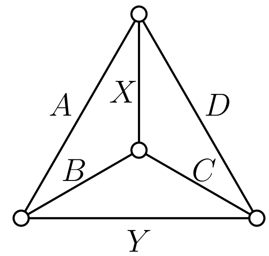

Figure 2.19. K_4 in the proof of Proposition 2.5.3.

2.5. Characterizing 3-Connected Orderable Binary Matroids

This section proves the following partial result towards Conjecture 2.1.4. Theorem 2.1.5 is an immediate consequence of this result.

Theorem 2.5.1. *A* 4*-connected binary orderable matroid with no series minor isomorphic to* F_7^* *is graphic.*

Our proof will require the next three results, the first of which is due to Seymour [16]. Two elements are *opposite* in $M(K_4)$ if they form a matching in the K_4.

Theorem 2.5.2. *Let* M *be a* 4*-connected binary matroid and let* e *and* f *be elements of* M*. Suppose there is no* $M(K_4)$*-minor of* M *in which* e *and* f *are opposite elements. Then there is a graph* G *with* $M = M(G)$ *or* $M^*(G)$*, and* e *and* f *are adjacent edges in* G*.*

Proposition 2.5.3. *In a consistent ordering of a series extension* M *of* $M(K_4)$*, if two elements correspond to opposite elements in the* $M(K_4)$*, then they are not adjacent.*

Proof. Let A, B, C, D, X, and Y be the series classes of M, labelled as in Figure 2.19. Take elements x in X and y in Y, and suppose x and y are adjacent in the given consistent ordering of M.

In the circuit $A \cup X \cup C \cup Y$, we have that y is adjacent to at most one member of C. Therefore, in $B \cup C \cup Y$, there must be an element, b_y, of B that is adjacent to y. Similarly, in $A \cup X \cup B$, there must be an element, b_x, of B adjacent to x. Now, in $B \cup X \cup D \cup Y$, we have the block $b_x x y b_y$, so no member of D is adjacent to y. By symmetry, no member of A is adjacent to y. Since y is adjacent to at most one element in Y, it follows that there is no second element of $A \cup D \cup Y$ adjacent to y, a contradiction. □

Lemma 2.5.4. *Suppose M is a binary matroid with no series minor isomorphic to F_7^*. If e and f are opposite elements in an $M(K_4)$-minor of M, then e and f are not adjacent in any consistent ordering of M.*

Proof. Assume that M has a consistent ordering in which e and f are adjacent. Let N be an $M(K_4)$-minor of M in which e and f are opposite elements, and write $N = M\backslash X/Y$ with X coindependent and Y independent. Then $N^* = M^*/X\backslash Y$, where $r(M^*/X) = r(N^*) = 3$. Since M^*/X is binary and $N^* \cong M(K_4)$, if L is the set of loops of M^*/X, then $M^*/X\backslash L$ is a parallel extension of either $M(K_4)$ or F_7. It follows that $M\backslash(X \cup L)$ is a series extension of $M(K_4)$ or F_7^*. By assumption, M has no series minor isomorphic to F_7^*, so $M\backslash(X \cup L)$ is a series extension of $M(K_4)$. However, e and f are adjacent in the consistent ordering of $M\backslash(X \cup L)$ inherited from M and correspond to opposite elements in N, a contradiction by Proposition 2.5.3. □

We now prove the main result of this section.

Proof of Theorem 2.5.1. Let M be a 4-connected binary orderable matroid that does not have F_7^* as a series minor. Take a consistent ordering of M and assume M is not graphic. Suppose M is cographic, letting $M = M^*(G)$ for some graph G. Take an edge e of G with endpoints u and v. Let $(x_1\ x_2\ \cdots\ x_n\ e)$ be the ordering on the edges meeting u, and let $(e\ y_1\ y_2\ \cdots\ y_m)$ be the ordering on the edges meeting v. Then we may assume the ordering on the bond that is the symmetric difference of these two vertex bonds is $(x_1\ x_2\ \cdots\ x_n\ y_1\ y_2\ \cdots\ y_m)$, so x_n and y_1 are adjacent. Combining Lemma 2.5.4 and Theorem 2.5.2, we now have that x_n and y_1 share an endpoint in G. Hence, $\{e, x_n, y_1\}$ is a triangle in G, a contradiction as M is 4-connected.

We may now assume that M is not cographic. Let e and f be adjacent elements of M. By Theorem 2.5.2, e and f appear as opposite elements in some $M(K_4)$-minor of M. Lemma 2.5.4 now gives a contradiction. $\square$

Chapter 3. On the Cogirth of Binary Matroids

3.1. Introduction

For an arbitrary graph G without isolated vertices, the well-known fact that the degree sum of G is twice the number of edges of G implies that

$$\frac{|E(G)|}{\delta(G)} \geq \frac{1}{2}|V(G)|,$$

where $\delta(G)$ is the minimum degree of G. In a matroid M of nonzero rank, the *cogirth*, $g^*(M)$, of M is the size of a smallest cocircuit of M. As $U_{r,n}$ shows, $\frac{|E(M)|}{g^*(M)}$ can be arbitrarily close to 1 even for simple matroids, although it is bounded below by $\frac{1}{2}(r(M)+1)$ when M is graphic.

In this chapter, we show that, when M is binary,

$$\frac{|E(M)|}{g^*(M)} \geq 2$$

unless M simplifies to a projective geometry. We also characterize the matroids that achieve equality in this bound. Both of these results are special cases of results for matroids representable over arbitrary finite fields. The next two results are the main results of the chapter.

Theorem 3.1.1. *For $r \geq 1$, let M be a rank-r matroid representable over $GF(q)$ whose simplification is not P_r. Then*

$$\frac{|E(M)|}{g^*(M)} \geq \frac{q}{q-1}.$$

Moreover, equality holds if and only if M is loopless and, for a fixed embedding of $\operatorname{si}(M)$ *in* P_r,

The material in this chapter was published in *Advances in Applied Mathematics*, Volume 147, Cameron Crenshaw and James Oxley, "On the cogirth of binary matroids," Copyright Elsevier (2023).

(i) the complement of $\mathrm{si}(M)$ *in* P_r *is isomorphic to* P_k *for some* k *with* $1 \leq k < r$*; and*

(ii) if P *is a copy of* P_{k+1} *in* P_r *containing the complement of* $\mathrm{si}(M)$*, then the parallel classes of the elements in* $E(M) \cap E(P)$ *all have the same size; and*

(iii) $|E(N)| \geq (q-1)|E(M) - E(N)|$ *for every restriction* N *of* M *that simplifies to* A_r*.*

This theorem excludes the matroids M for which $\mathrm{si}(M) \cong PG(r-1, q)$. These excluded matroids are covered by the next result.

Proposition 3.1.2. *For* $r \geq 1$*, let* M *be a matroid that simplifies to* $PG(r-1,q)$*. Then*

$$\frac{|E(M)|}{g^*(M)} \geq \frac{q^r - 1}{q^{r-1}(q-1)}.$$

Moreover, equality holds if and only if M *is loopless and all its parallel classes have the same size.*

Condition (i) in Theorem 3.1.1 says that M simplifies to a *Bose-Burton geometry* [5], that is, a matroid that is obtained from $PG(r-1,q)$ by deleting some $PG(k-1,q)$ where $1 \leq k < r$. In each of our results, the bound on $\frac{|E(M)|}{g^*(M)}$ is relatively easy to obtain. The core of each proof involves characterizing when equality holds in the bound. The proofs appear in Section 3.3.

Our results have implications for linear codes. A *linear code* of *length* n and *rank* k is a k-dimensional subspace of the n-dimensional vector space over $GF(q)$. Such a code, C, is also known as a *q-ary code* (see, for example, [8]). The minimum distance d of C is the minimum number of coordinates in which two vectors in C differ or, equivalently, the minimum number of non-zero coordinates in a non-zero vector in C. The *relative distance* of C is d/n. A *generator matrix* for C is a $k \times n$ matrix A over $GF(q)$ such that C equals the row space of A. Let M be $M[A]$, the vector matroid of A. Then the cocircuits of M

coincide with the minimal non-empty supports of the vectors in C. Thus the cogirth of M is the minimum distance of C, and $g^*(M)/|E(M)|$ is the relative distance, d/n, of C.

When $\frac{d}{n} > 1 - \frac{1}{q}$, the Plotkin bound [13] for q-ary codes [1, 4, 7] asserts that $|C| \leq \frac{qd}{qd-(q-1)n}$. Moreover, when $q = 2$ and $\frac{d}{n} = \frac{1}{2}$, Plotkin showed that $|C| \leq 4d$. Theorem 3.1.1 describes the matroids that do not simplify to $PG(r-1, q)$ for which the relative distance of the corresponding linear code equals $1 - \frac{1}{q}$.

3.2. Preliminaries

In a matroid M of rank at least one, a loop contributes to $|E(M)|$ but not to $g^*(M)$. Since our concern here is on bounding $\frac{|E(M)|}{g^*(M)}$ below, we shall focus on matroids without loops. It will be convenient here to deal with the parallel classes in such a matroid M by assigning, to each element of $\mathrm{si}(M)$, a *weight* $w(e)$ that is equal to the cardinality of the parallel class of M that contains e. Thus we deal with simple matroids with associated weight functions that take a positive-integer value on each element. For a set X in such a matroid N, we write $w(X)$ for $\sum_{x\in X} w(x)$ and write $w(N)$ for $w(E(N))$. The weight function of $N\backslash Y$ is the restriction of the weight function of N to $E(N) - Y$. When Y is contracted from N, we replace each parallel class P by a single element e_P whose weight in the contraction is $w_N(P)$. We will call this weighted simple matroid the *weighted contraction* of Y and denote it by N/Y, even though the underlying matroid is actually $\mathrm{si}(N/Y)$. The cogirth of a weighted matroid is the minimum weight of a cocircuit.

3.3. The Proofs

We begin with a lemma that serves as the base case for both of the inductive arguments that prove the inequalities in the main results.

Lemma 3.3.1. *Let M be a simple, rank-2 matroid, and let w be a weight function on M. Then*

$$\frac{w(M)}{g^*(M)} \geq \frac{|E(M)|}{|E(M)|-1},$$

with equality if and only if w is constant.

Proof. Let $w_1 \leq w_2 \leq \cdots \leq w_n$ be the weights of the elements of M. Because the cocircuits of M coincide with the complements of the parallel classes in the rank-2 matroid M, we deduce that $g^*(M) = w_1 + w_2 + \cdots + w_{n-1}$. Thus, the desired inequality is equivalent to

$$(n-1)(w_1 + w_2 + \cdots + w_n) \geq n(w_1 + w_2 + \cdots + w_{n-1}).$$

Subtracting $(n-1)(w_1 + w_2 + \cdots + w_{n-1})$ from each side, we obtain

$$(n-1)w_n \geq w_1 + w_2 + \cdots + w_{n-1}, \tag{3.1}$$

which is true since $w_n \geq w_i$ for all i. Note that equality holds in (3.1) if and only if $w_i = w_n$ for all i. □

The following is the main result of the chapter. It is equivalent to Theorem 3.1.1 and is stated here in terms of weights.

Theorem 3.3.2. *Let M be a simple, rank-r matroid representable over $GF(q)$, and let w be a weight function on M. Suppose $M \not\cong P_r$. Then*

$$\frac{w(M)}{g^*(M)} \geq \frac{q}{q-1}.$$

Moreover, equality holds if and only if, for a fixed embedding of M in P_r,

(i) the complement of M is isomorphic to P_k, with $1 \leq k < r$; and

(ii) *if P is a copy of P_{k+1} containing the complement of M in P_r, then w is constant on P; and*

(iii) $w(N) \geq (q-1)w(E(M) - E(N))$ *for every A_r-restriction N of M.*

Proof. We begin by proving the displayed inequality by induction on r. Lemma 3.3.1 gives the result when $r = 2$ since $|E(M)| < |E(P_2)| = q + 1$. Suppose $r \geq 3$. If there is an e in $E(M)$ with $M/e \ncong P_{r-1}$, then, by induction,

$$(q-1)w(M) > (q-1)w(M/e) \geq qg^*(M/e) \geq qg^*(M).$$

Thus we may assume that $M/e \cong P_{r-1}$ for all e in $E(M)$. Take a line of P_r that meets both $E(M)$ and $E(P_r) - E(M)$. Let X be the set of elements of M on this line and e be a maximum-weight element of X. Let $Y = X - e$. Note that $|Y| \leq q - 1$, so

$$w(Y) \leq w(e)(q-1). \tag{3.2}$$

Observe that $M\backslash Y/e$ has rank $r-1$ but is not isomorphic to P_{r-1} so, by the induction assumption,

$$(q-1)w(M\backslash Y/e) \geq qg^*(M\backslash Y/e).$$

Now

$$w(M\backslash Y/e) = w(M) - w(e) - w(Y),$$

and

$$g^*(M\backslash Y/e) \geq g^*(M) - w(Y).$$

Thus

$$(q-1)w(M) \geq qg^*(M) + w(e)(q-1) - w(Y),$$

so, by (3.2),

$$(q-1)w(M) \geq qg^*(M)$$

as desired.

Next we characterize when equality is achieved in the last bound. Let M^c be the complement of the fixed embedding of M in P_r. When $M^c \cong P_k$ for $1 \leq k < r$, a hyperplane of P_r either contains this P_k or meets it in a P_{k-1}. Thus a cocircuit of M is isomorphic to either A_r or $A_r - A_k$. We call these *type-I* and *type-II cocircuits*, respectively, noting that there are no type-I cocircuits when $k = r-1$.

3.3.2.1. *Suppose M satisfies (i) and (ii). If C^* is a type-II cocircuit of M, then*

$$w(C^*) = \frac{q-1}{q}w(M).$$

As M satisfies (i), $M^c \cong P_k$. Since C^* is a type-II cocircuit, there is a restriction A of P_r isomorphic to A_r such that A meets M^c and $C^* = E(M) \cap E(A)$. Let H be the hyperplane of P_r that is the complement of A.

Now, P_r consists of $\frac{q^{r-k}-1}{q-1}$ copies of P_{k+1} containing M^c, and the pairwise intersection of these copies is M^c. Thus M is the disjoint union of $\frac{q^{r-k}-1}{q-1}$ copies of A_{k+1}. By (ii), the elements in each A_{k+1} have the same weight. To complete the proof of 3.3.2.1, we show that C^* contains exactly $\frac{q-1}{q}$ of the elements of each A_{k+1}.

Consider the complementary A_{k+1} to M^c in a fixed P_{k+1}. Note that P_{k+1} consists of $q+1$ copies of P_k, including M^c, that contain $H \cap E(M^c)$, which is isomorphic to P_{k-1}. Therefore, this A_{k+1} is the disjoint union of q copies of A_k. Now H meets P_{k+1} at a P_k distinct from M^c. Thus A meets P_{k+1} in a set that is the union of q disjoint copies of A_k, one of which is in M^c. This implies that $C^* \cap A_{k+1}$ is the disjoint union of $q-1$ copies of A_k, and 3.3.2.1 follows.

Now assume that $\frac{w(M)}{g^*(M)} = \frac{q}{q-1}$. Then equality holds in (3.2) so $|Y| = q-1$ and $w(y) = w(e)$ for all $y \in Y$. The former implies that every line of P_r that meets both M and M^c contains exactly q points of M. This means that every line that contains two points of M^c lies entirely in M^c. Thus M^c is a flat of P_r, proving (i).

As $w(y) = w(e)$ for all $y \in Y$, it follows that w is constant on each line of P_r that meets both M and M^c. Since a P_k contained in a P_{k+1} meets every line of the P_{k+1}, (ii) is satisfied. It now follows from 3.3.2.1 that $\frac{w(M)}{g^*(M)} = \frac{q}{q-1}$ if and only if M satisfies (i) and (ii), and the type-I cocircuits of M have weight at least $\frac{q-1}{q}w(M)$. It is straightforward to check that this third condition is equivalent to (iii), so the theorem holds. □

The reader may find condition (iii) of Theorem 3.3.2 unsatisfying, and the next proposition offers a potential replacement, (iii)′. The example that follows Proposition 3.3.3 shows that conditions (i), (ii), and (iii)′ do not guarantee $\frac{w(M)}{g^*(M)} = \frac{q}{q-1}$ for a matroid M meeting the hypotheses of Theorem 3.3.2. In addition, the example illustrates the potential difficulty of finding a satisfactory replacement for (iii).

Proposition 3.3.3. *Let M be a simple, rank-r matroid representable over $GF(q)$, and let w be a weight function on M. Suppose that $M \ncong P_r$ and that $\frac{w(M)}{g^*(M)} = \frac{q}{q-1}$. Then*

(iii)′ $q^{r-1}w(e) \leq w(M)$ for all e in $E(M)$.

Proof. By Theorem 3.3.2, $M^c = P_k$. If $k = r-1$, then Theorem 3.3.2(ii) implies that (iii)′ holds with equality. Thus we may assume that $k < r-1$. Extend the weight function of M to P_r by assigning each element of M^c a weight of one. Then contract M^c from P_r to form a weighted matroid $M' \cong P_{r-k}$. Fix an element e in $E(M)$, and let e' be the image of e in M'. Note that $w(M') = w(M)$ and $w(e') = q^k w(e)$ under this transformation. Moreover, the type-I cocircuits of M correspond to the cocircuits of M', so the weight of each cocircuit of M' equals the weight of the corresponding type-I cocircuit of M.

Let C^* be a cocircuit of M' that avoids e' and let H be the complementary hyperplane to C^* in M'. Since $\frac{w(M)}{g^*(M)} = \frac{q}{q-1}$, it follows that

$$\frac{q}{q-1} w(C^*) \geq w(M'),$$

and subtracting $w(C^*)$ from each side produces

$$\frac{1}{q-1} w(C^*) \geq w(H). \tag{3.3}$$

Note that (3.3) holds for an arbitrary cocircuit of M' avoiding e', so we have such an inequality for every such cocircuit. Moreover, C^* and H partition $E(M')$ so, for a fixed $f \in E(M' \backslash e')$, its weight contributes to exactly one side of each inequality. Now, there are $\frac{q^{r-k-1}-1}{q-1}$ total inequalities as this is the number t of hyperplanes of M' containing e'. Similarly, $w(f)$ contributes to the right-hand side of exactly $\frac{q^{r-k-2}-1}{q-1}$ of these inequalities as this is the number s of hyperplanes of M' containing both e' and f. Hence $w(f)$ contributes to the left-hand side of $t - s$ of these inequalities. Summing these inequalities gives

$$\frac{t-s}{q-1} w(M' \backslash e') \geq s w(M' \backslash e') + t w(e'),$$

and this simplifies to

$$w(M') \geq q^{r-k-1} w(e').$$

Finally, we substitute $w(M)$ for $w(M')$ and $q^k w(e)$ for $w(e')$ to obtain

$$w(M) \geq q^{r-1} w(e)$$

as desired. □

Example 3.3.4. Let $q = 2$ and let $M = P_4 - p$ for some $p \in E(P_4)$. Then $M \cong P_4 - P_1$. Take a hyperplane H of P_4 containing p, and note that H has $|P_3|$ elements. Then $H - p$ is a hyperplane of M and the corresponding cocircuit C^* is type-I and has $|A_4|$ elements.

Assign the weight 2 to each element of $H - p$ and the weight 1 to each element of C^*. Then

$$w(M) = 2(|P_3| - 1) + 1 \cdot |A_4| = 2(6) + 8 = 20.$$

Observe that conditions (i) and (ii) of Theorem 3.3.2 hold and, since, for all e in $E(M)$,

$$q^{r-1} w(e) \leq 2^3(2) < 20 = w(M),$$

so does (iii)$'$. However, $w(C^*) = 8$, so the equation

$$\frac{w(M)}{g^*(M)} = \frac{q}{q-1} \tag{3.4}$$

fails.

Now, in P_4, take a line in $C^* \cup p$ and another in H that each meet in $\{p\}$. Swap the weights 1 and 2 on the elements of M on these lines. Note that (i), (ii), and (iii)$'$ continue to hold, and $w(M)$ is unchanged. However, it is straightforward to check that the weights of the type-I cocircuits of M are at least 10, so (3.4) holds by Theorem 3.3.2. Thus, characterizing the matroids for which equality holds in Theorem 3.3.2 requires not only restricting the weights themselves, but also controlling their distribution.

Finally, we prove a proposition equivalent to Proposition 3.1.2 stated here in terms of weights.

Proposition 3.3.5. *Let M be a matroid isomorphic to $PG(r-1, q)$ and w be a weight function on $E(M)$. Then*

$$\frac{w(M)}{g^*(M)} \geq \frac{q^r - 1}{q^{r-1}(q-1)}.$$

Moreover, equality holds if and only if w is constant.

Proof. We prove the inequality by induction on r. It is trivial when $r = 1$ and is true for $r = 2$ by Lemma 3.3.1, so suppose $r \geq 3$. Let C^* be a cocircuit of M of weight $g^*(M)$ and let H be the complementary hyperplane to C^* in M. Choose Z as a maximum-weight hyperplane of H. Then, letting Y be the complement of Z in H, we have

$$\frac{w(M)}{g^*(M)} = \frac{w(C^*) + w(Y) + w(Z)}{w(C^*)} = 1 + \frac{w(Y) + w(Z)}{w(C^*)}. \tag{3.5}$$

Observe that the weighted contraction of Z from M is isomorphic to P_2 so, by the inequality for $r = 2$, we get that

$$\frac{w(M/Z)}{g^*(M/Z)} \geq \frac{q+1}{q}.$$

Now, since C^* is also a minimum-weight cocircuit of M/Z, we rewrite this inequality as

$$\frac{w(Y) + w(C^*)}{w(C^*)} \geq \frac{q+1}{q}. \tag{3.6}$$

It follows that $qw(Y) \geq w(C^*)$. Substituting into (3.5), we obtain

$$\frac{w(M)}{g^*(M)} \geq 1 + \frac{1}{q} \cdot \frac{w(Y) + w(Z)}{w(Y)}. \tag{3.7}$$

Finally, the hyperplane H is isomorphic to P_{r-1}, and our choice of Z makes Y a minimum-weight cocircuit of H. Thus, by induction,

$$\frac{w(Y) + w(Z)}{w(Y)} \geq \frac{q^{r-1} - 1}{q^{r-2}(q-1)}. \tag{3.8}$$

Substituting (3.8) into (3.7) gives the desired inequality.

One easily checks that, when w is constant,

$$\frac{w(M)}{g^*(M)} = \frac{q^r - 1}{q^{r-1}(q-1)}. \tag{3.9}$$

We now use induction on r to prove that the elements of M have the same weight when (3.9) holds. When $r = 1$, this is trivial, and Lemma 3.3.1 handles the rank-2 case.

Suppose $r \geq 3$. Since (3.9) holds, equality holds in (3.6) and (3.8). It follows from the latter using the induction assumption that the weight function w on $E(M)$ is constant on the hyperplane H of M. From the former, we deduce that, in M/Z, every point has equal weight. It follows that every hyperplane H' of M containing Z has the same weight. Hence the cocircuit $E(M) - H'$ has the same weight as C^*. Replacing H by H', we deduce that w is constant on the elements of H'. Letting H' range over all of the hyperplanes of M containing Z, we deduce that w is constant on $E(M)$. $\square$

Chapter 4. A Binary-Matroid Analogue of a Graph-Connectivity Theorem of Jamison and Mulder.

4.1. Introduction

Jamison and Mulder [9] defined a graph G to be Θ_3-*closed* if, whenever distinct vertices x and y of G are joined by three internally disjoint paths, x and y are adjacent. For disjoint graphs G_1 and G_2, a 1-*sum* of G_1 and G_2 is a graph that is obtained by identifying a vertex of G_1 with a vertex of G_2. Following Jamison and Mulder, we define a 2-*sum* of G_1 and G_2 to be a graph that is obtained by identifying an edge of G_1 with an edge of G_2. Note that, in contrast to some other definitions of this operation, we retain the identified edge as an edge of the resulting graph. The main result of Jamison and Mulder's paper is the following.

Theorem 4.1.1. *A connected graph G is Θ_3-closed if and only if G can be built via 1-sums and 2-sums from cycles and complete graphs.*

This chapter generalizes Theorem 4.1.1 to binary matroids; all matroids considered here are binary. Let T be a theta-graph of M with theta-arcs A_1, A_2, and A_3. If M has an element e such that, for every i, either $A_i \cup e$ is a circuit of M, or $A_i = \{e\}$, then e *completes* T in M, and T is said to be *complete.* A matroid M is *matroid* Θ_3-*closed* if every theta-graph of M is complete. The next theorem is the main result of this chapter.

Theorem 4.1.2. *A matroid M is matroid Θ_3-closed if and only if M can be built via direct sums and parallel connections from circuits, cycle matroids of complete graphs, and projective geometries.*

Suppose M is isomorphic to the cycle matroid of a graph G. Two vertices in G that are joined by three internally disjoint paths are adjacent via an edge e exactly when the corresponding theta-graph of M is completed by e. In other words, G is Θ_3-closed if and only if M is matroid Θ_3-closed. This allows us to refer to M as Θ_3-*closed* without ambiguity. We will denote the class of Θ_3-closed matroids by Θ_3.

Section 4.2 introduces supporting results. The 3-connected matroids that are matroid Θ_3-closed are characterized in Section 4.3, and the proof of Theorem 4.1.2 appears in Section 4.4.

4.2. Preliminaries

Our first proposition collects some essential properties of Θ_3-closed matroids. These properties will be used frequently and often implicitly.

Proposition 4.2.1. *If $M \in \Theta_3$, then*

(i) $\mathrm{si}(M) \in \Theta_3$*;*

(ii) $M|F \in \Theta_3$ *for every flat F of M; and*

(iii) $M/e \in \Theta_3$ *for every $e \in E(M)$.*

Proof. Parts (i) and (ii) are straightforward. For part (iii), let T be a theta-graph of M/e. Then $[(M/e)|T]^*$ is obtained from $U_{2,3}$ by adding elements in parallel to the existing elements. Since M is binary, it follows that $M^*/(E(M)-(T\cup e))$ is obtained from $[(M/e)|T]^*$ by adding e as a coloop or by adding e in parallel to one of the existing elements. As $M \in \Theta_3$, it follows that $M/e \in \Theta_3$. □

The following is an immediate consequence of Proposition 4.2.1.

Corollary 4.2.2. *If $M \in \Theta_3$ and N is a parallel minor of M, then $N \in \Theta_3$.*

From, for example, [11, Exercise 8.3.3], if $M = M_1 \oplus_2 M_2$, then M_1 and M_2 are parallel minors of M. The next result now follows from Corollary 4.2.2.

Corollary 4.2.3. *If $M \oplus_2 N$ is in Θ_3, then M and N are in Θ_3.*

We conclude this section with a result about constructing larger matroids in Θ_3 from smaller ones. Recall that, for sets X and Y in a matroid M, the *local connectivity* between X and Y, denoted $\sqcap(X, Y)$, is defined by $\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y)$. We will use the following result about local connectivity from, for example, [11, Lemma 8.2.3].

Lemma 4.2.4. *Let X_1, X_2, Y_1, and Y_2 be subsets of the ground set of a matroid M. If $X_1 \supseteq Y_1$ and $X_2 \supseteq Y_2$, then $\sqcap(X_1, X_2) \geq \sqcap(Y_1, Y_2)$.*

Proposition 4.2.5. *For matroids M and N, the parallel connection $P(M, N)$ is in Θ_3 if and only if $M \in \Theta_3$ and $N \in \Theta_3$.*

Proof. Let p be the basepoint of the parallel connection. When p is a loop or a coloop of M, the matroid $P(M, N)$ is $M \oplus (N/p)$ or $(M \backslash p) \oplus N$, respectively. In these cases, it follows using Proposition 4.2.1 that the result holds. Thus we may assume that p is neither a loop nor a coloop of M or N. Suppose $P(M, N) \in \Theta_3$. Let B_M be a basis for M containing p. Extend B_M to a basis B for $P(M, N)$. Then M is obtained from $P(M, N)$ by contracting the elements of $B - B_M$ and then contracting all of the resulting loops. The remaining elements of $E(N) - p$ are now parallel to p. We deduce that M, and similarly N, is a parallel minor of $P(M, N)$. Hence, by Corollary 4.2.2, M and N are in Θ_3.

Conversely, let T be a theta-graph of $P(M, N)$ with theta-arcs A_1, A_2, and A_3. Then we may assume that $|A_i| \geq 2$ for each i, otherwise T is complete. Suppose $p \in A_1$. Then $A_1 \cup A_2$ is a circuit containing p, so it is contained in $E(M)$ or $E(N)$ depending on which of these sets contains $A_1 - p$. It follows that the same set contains A_2 and, likewise A_3, so T is complete. Hence we may assume that $p \notin T$.

Suppose that each of the theta-arcs of T meets both $E(M\backslash p)$ and $E(N\backslash p)$. Let $T_M = E(T) \cap E(M)$, and similarly for T_N. Note that T_M and T_N are independent, and $T_M \cup T_N = E(T)$, so

$$\begin{aligned} \sqcap(T_M, T_N) &= r(T_M) + r(T_N) - r(T_M \cup T_N) \\ &= |T_M| + |T_N| - (|T_M| + |T_N| - 2) \\ &= 2. \end{aligned}$$

However, $\sqcap(E(M), E(N)) = 1$, contradicting Lemma 4.2.4.

Next, suppose that each of A_1 and A_2 meets both $E(M\backslash p)$ and $E(N\backslash p)$. Then, from above, we may assume that $A_3 \subseteq E(M\backslash p)$. The circuits $A_1 \cup A_3$ and $A_2 \cup A_3$ have the form $(C_1 - p) \cup (D_1 - p)$ and $(C_2 - p) \cup (D_2 - p)$, respectively, for circuits C_1 and C_2 of M containing p, and circuits D_1 and D_2 of N containing p. Because $A_3 \subseteq E(M)$ and $A_1 \cap A_2 = \emptyset$, it follows that $D_1 - p$ and $D_2 - p$ are disjoint. However, since M is binary, $D_1 \triangle D_2$ contains a circuit of $P(M, N)$ that is properly contained in the circuit $A_1 \cup A_2$, a contradiction.

Now, suppose that A_1 meets both $E(M\backslash p)$ and $E(N\backslash p)$. Then, from above, each of the remaining theta-arcs of T lies in $E(M\backslash p)$ or $E(N\backslash p)$. We may assume that $A_2 \subseteq E(M\backslash p)$. Suppose $A_3 \subseteq E(N\backslash p)$. Then the circuits $A_1 \cup A_2$ and $A_3 \cup A_2$ have the form $(C_1 - p) \cup (D_1 - p)$ and $(C_3 - p) \cup (D_3 - p)$, respectively, for circuits C_1 and C_3 of M containing p, and circuits D_1 and D_3 of N containing p. Now, since $A_2 \subseteq E(M\backslash p)$ and A_1 meets $E(M\backslash p)$, the set $C_1 - p$ properly contains A_2. Further, as A_3 does not meet $E(M\backslash p)$, we have that $A_2 = C_3 - p$. This means $A_2 \cup p$ is the circuit C_3, but $A_2 \cup p$ is properly contained in C_1, a contradiction.

We conclude that $A_3 \subseteq E(M\backslash p)$. Form T' from T by replacing the portion of A_1 in $E(N\backslash p)$ by p. Observe that T' is isomorphic to a series minor of T, so T' is a theta-graph. Moreover, T' is a theta-graph of M, so it is completed in M by an element f. Now, since T and T' share a theta-arc, f also completes T in $P(M, N)$.

We are left to consider the case when each theta-arc of T is contained in either $E(M)$ or $E(N)$. If all three theta-arcs belong to $E(M)$, say, then T is complete in the corresponding matroid, and so is complete in $P(M, N)$. Otherwise, p completes T. $\square$

4.3. The 3-Connected Θ_3-Closed Matroids

The proof of Theorem 4.1.2 will use the canonical tree decomposition of Cunningham and Edmonds [6] and, in support of that approach, this section proves the following 3-connected form of Theorem 4.1.2.

Theorem 4.3.1. *Let M be a simple 3-connected Θ_3-closed matroid. Then M is a projective geometry or the cycle matroid of a complete graph.*

The proof of this theorem relies on the next two propositions.

Proposition 4.3.2. *If M is a simple matroid in Θ_3 and M has a spanning $M(K_{r+1})$-restriction, then $M \cong M(K_{r+1})$ or P_r.*

Proof. We shall view M as being the restriction of P_r to the set X. Recall that the number of nonzero entries of a vector is its *weight*, and that the *distance* between two vectors is the number of coordinates upon which they disagree. Because M has an $M(K_{r+1})$-restriction, we may assume that X contains the set Z of vectors of weight one or two. We may assume that $Z \neq X$. Then M has an element e of weight at least three. We shall establish that $M \cong P_r$ by proving the following three assertions.

(i) M has an element of weight three;

(ii) if the matroid M has every element of weight $k-1$ and an element of weight k, for some k exceeding two, then M has every element of weight k; and

(iii) if M has every element of weight k, where $3 \leq k < r$, then M has an element of weight $k+1$.

Let e_i denote the weight-1 element whose nonzero entry is in the ith position. To show (i), we may assume e has weight $k \geq 4$. Say $e = e_1 + e_2 + \cdots + e_k$. Let $Y = \{e, e_1, e_2, e_4, e_5, \ldots, e_k, e_1 + e_3, e_2 + e_3\}$. Then $M|Y$ is a theta-graph having theta-arcs $\{e, e_4, e_5, \ldots, e_k\}$, $\{e_1, e_2 + e_3\}$, and $\{e_2, e_1 + e_3\}$. This theta-graph forces $e_1 + e_2 + e_3$ to be an element of M, so (i) holds.

To prove (ii), we may assume $k < r$. Suppose g is an element of weight k not in M, and let f be an element of weight k in M with minimum distance from g. Let s label a row where f is 1 and g is 0, and let t label a row where g is 1 and f is 0. Next, as $k \geq 3$, there are two additional rows, u and v, distinct from s where f is 1. Now, the set $\{f, e_u, e_v, e_s + e_t\}$ is independent, so the theta-arcs $\{f, e_s + e_t\}$, $\{e_u, f + e_u + e_s + e_t\}$, and $\{e_v, f + e_v + e_s + e_t\}$ form a theta-graph in M. This theta-graph implies that $f + e_s + e_t$ belongs to M. However, $f + e_s + e_t$ has weight k and is a smaller distance from g than f, a contradiction. Thus (ii) holds.

Finally, let f be an element of weight $k+1$ for some k with $3 \leq k < r$. By symmetry, we may assume that the set of rows in which f is nonzero contains $\{1, 2, 3\}$. Then $\{f, e_1, e_2, e_3\}$ is independent in M, and the sets $\{e_1, f+e_1\}$, $\{e_2, f+e_2\}$, and $\{e_3, f+e_3\}$ are the theta-arcs of a theta-graph in M. This theta-graph shows that f belongs to M. Thus (iii) holds. Hence the proposition holds as well. $\square$

The second proposition that we use to prove Theorem 4.3.1 will follow from the following three results.

Lemma 4.3.3. *Let M be a simple rank-r matroid in Θ_3. Suppose that*

(i) $r \geq 4$;

(ii) $E(M)$ *has a subset* P *such that* $M|P \cong P_{r-1}$; *and*

(iii) $E(M) - P$ *contains an independent set of size at least three.*

Then $M \cong P_r$.

Proof. View M as a restriction of P_r, and let $\{e, f, g\}$ be an independent set in $E(M) - P$. Let p be a point in $E(P_r) - P$ that is not in $\{e, f, g\}$. Observe that, for each x in $\{e, f, g\}$, the third point on the line in P_r containing $\{x, p\}$ is in P. Thus there are three lines of M that meet at p. Provided p is not coplanar with $\{e, f, g\}$, these lines define a theta-graph in M that is completed by p, so p is in M. It remains to show that the point q of $E(P_r) - (P \cup \{e, f, g\})$ that is coplanar with $\{e, f, g\}$ belongs to M. But one easily checks that $P_r \backslash q$ is not in Θ_3 when $r \geq 4$. Thus $M \cong P_r$. □

Corollary 4.3.4. *Let M be a simple rank-r matroid in Θ_3 with $r \geq 3$. If M has a basis B and an element b in B so that, for each $\{x, y\} \subseteq B - b$, the set $\{b, x, y\}$ spans an F_7-restriction of M, then $M \cong P_r$.*

Proof. Let $B = \{b_1, b_2, \ldots, b_r\}$ with $b = b_1$. If $r = 3$, then the result is immediate, so suppose $r \geq 4$. By induction, $M|\operatorname{cl}(B - b_r)$ is isomorphic to P_{r-1}. Since $M|\operatorname{cl}(\{b_1, b_2, b_r\}) \cong F_7$, we see that this restriction contains an independent set of three elements that avoids $\operatorname{cl}(B - b_r)$. Lemma 4.3.3 now implies that $M \cong P_r$. □

The next result was proved by McNulty and Wu [12, Lemma 2.10].

Lemma 4.3.5. *Let M be a 3-connected binary matroid with at least four elements. Then, for any two distinct elements e and f of M, there is a connected hyperplane containing e and avoiding f.*

For a simple binary matroid M, we now define the smallest Θ_3-closed matroid whose ground set contains $E(M)$. Let $M_0 = M$ and $r(M) = r$. Suppose $M_0, M_1, \ldots, M_k$ have been defined. The simple binary matroid M_{k+1} is obtained from M_k by ensuring that, whenever T is an incomplete theta-graph of M_k, the element x that completes T is in $E(M_{k+1})$. Since each M_i is a restriction of P_r, there is a j for which $M_{j+1} = M_j$. When this first occurs, we call M_j the Θ_3-*closure* of M. Evidently this is well defined. By associating M with its ground set, the Θ_3-closure is a closure operator (but not necessarily a matroid closure operator) on the set of subsets of the ground set of any projective geometry containing M.

Proposition 4.3.6. *Let M be a simple 3-connected matroid in Θ_3, and let k be an integer exceeding two. If M has a simple minor N whose Θ_3-closure is P_k, then M is a projective geometry.*

Proof. Take subsets X and Y of $E(M)$ such that $M/X\backslash Y = N$ with X independent and Y coindependent in M. The matroid M/X is in Θ_3 and has N as a spanning restriction. Therefore M/X has P_k as a restriction, so P_k is a minor of M. From here, the proof is by induction on the rank, r, of M.

If $r = k$, the result is immediate, so assume $r > k$. By the Splitter Theorem [14], P_k can be obtained from M by a sequence of single-element contractions and deletions, all while staying 3-connected. Let e be the first element that is contracted in this sequence.

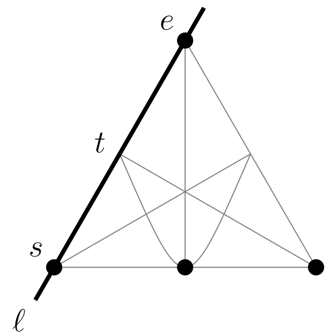

Figure 4.1. In the proof of 4.3.6.1, each of $M|\pi_2$ and $M|\pi_3$ has this form.

Note that $\mathrm{si}(M/e)$ is a 3-connected member of Θ_3 that has P_k as a minor. By induction, $\mathrm{si}(M/e) \cong P_{r-1}$. Fix an embedding of M in P_r. We may assume that $M \ncong P_r$. Then some line ℓ of P_r through e is not contained in $E(M)$. For each subset Z of $E(P_r)$, let $\mathrm{cl}_P(Z)$ be its closure in P_r. Since $\mathrm{si}(M/e) \cong P_{r-1}$, there is an element s of $E(M)$ that is in $\ell - \{e\}$. Let t be the point of P_r in $\ell - \{e\}$ that is not in $E(M)$.

4.3.6.1. *Let F be a rank-4 flat of M containing ℓ. Then $M|F$ is isomorphic to one of $P(F_7, U_{2,3})$ or $F_7 \oplus U_{1,1}$ where the F_7-restriction of $M|F$ contains s but avoids e.*

To see this, first note that, by Proposition 4.2.1(ii), $M|F$ is in Θ_3. Recall that each line of $\mathrm{cl}_P(F)$ through e contains another point of F. Then there are three planes, π_1, π_2, and π_3, of $M|F$ containing ℓ. Let $\mathcal{P}$ be this set of planes. Therefore, each plane in $\mathcal{P}$ has at least one pair of points that are not on ℓ such that these two points are collinear with either s or t. Call such a pair of points a *target pair.*

Suppose π_1 has a target pair collinear with t. Note that if two distinct planes in $\mathcal{P}$ each have a target pair collinear with t, then these pairs, along with $\{e, s\}$, are the theta-arcs of an incomplete theta-graph in $M|F$, an impossibility. Consequently, neither π_2 nor π_3 has a target pair collinear with t, so both have the form in Figure 4.1. The restriction

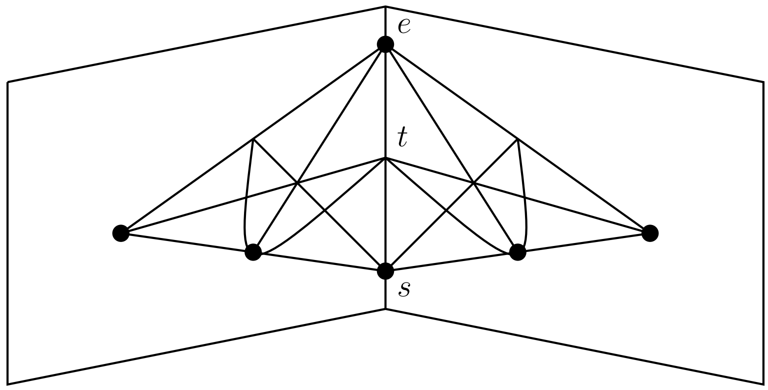

Figure 4.2. The matroid $M|(\pi_2 \cup \pi_3)$ in the proof of 4.3.6.1.

of M to $\pi_2 \cup \pi_3$ is given in Figure 4.2.

The plane π_1 adds a pair of points collinear with t to $M|(\pi_2 \cup \pi_3)$, and one readily checks that the addition of this pair gives a restriction of $M|F$ that is isomorphic to $F_7 \oplus_2 M(K_3)$. A theta-graph of $M|F$ now gives that $M|F$ has a restriction isomorphic to $P(F_7, U_{2,3})$. It follows that $M|F \cong P(F_7, U_{2,3})$ otherwise, by Lemma 4.3.3, $M|F \cong P_4$ and we obtain the contradiction that $t \in F$.

We may now suppose that no π_i has a target pair collinear with t. It follows that each target pair in each π_i is collinear with s, and that e is the only element of F outside of the target pairs. Observe that the target pairs must be coplanar as, otherwise, we can find an incomplete theta-graph in $M|F$. Thus $M|F \cong P(F_7, U_{1,1})$. Noting that e is not in the F_7-restriction of $M|F$, we conclude that 4.3.6.1 holds.

Since M is 3-connected, it follows from 4.3.6.1 that $r \geq 5$. By Lemma 4.3.5, there is a connected hyperplane H of M containing s and avoiding e. Let $s = b_1$ and let $\{b_1, b_2, \ldots, b_{r-1}\}$ be a basis B of $M|H$. For distinct elements i and j of $\{2, 3, \ldots, r-1\}$, let $F_{i,j}$ denote the rank-4 flat $M|\operatorname{cl}(\{e, s, b_i, b_j\})$ of M. By 4.3.6.1, $M|F_{i,j}$ is isomorphic to $F_7 \oplus U_{1,1}$ or $P(F_7, U_{2,3})$. Let X be the subset of $F_{i,j}$ such that $M|X \cong F_7$, and recall that

$e \notin X$ and $s \in X$. The hyperplane H either contains X or meets X along one of the lines $\mathrm{cl}(\{s, b_i\})$ or $\mathrm{cl}(\{s, b_j\})$. We deduce the following.

4.3.6.2. *For each pair $\{i, j\}$ in $\{2, 3, \ldots, r-1\}$, if the element $s + b_i$ is not in $E(M)$, then $s + b_j$ is.*

Suppose $s + b_2$ is not in $E(M)$. By 4.3.6.2, the element $s + b_i$ belongs to $E(M)$ for every i in $\{3, 4, \ldots, r-1\}$. Consequently, for each pair $\{i, j\}$ in $\{3, 4, \ldots, r-1\}$, the hyperplane H contains the copy of F_7 in $M|F_{i,j}$, and this F_7 is spanned by $\{s, b_i, b_j\}$. Corollary 4.3.4 now implies that $M|\,\mathrm{cl}(B - b_2) \cong P_{r-2}$.

Now, since $M|H$ is connected, H contains an element f that is not in $\mathrm{cl}(B-b_2) \cup b_2$. The line $\mathrm{cl}(\{b_2, f\})$ meets the projective geometry $\mathrm{cl}(B - b_2)$, so $b_2 + f$ is also in M. Now consider $Y = \mathrm{cl}(\{e, s, b_2, b_2 + f\})$. The intersection $H \cap Y$ contains the line $\{b_2, f, b_2 + f\}$ and also the element s. By applying 4.3.6.1 to $M|Y$, we see that $H \cap Y$ is a Fano plane containing s and b_2. Since $s + b_2$ is not in $E(M)$, we have a contradiction.

We conclude that $s + b_i$ is in $E(M)$ for every i in $\{2, 3, \ldots, r-1\}$. The flat $M|F_{i,j}$ now meets H in a Fano plane for every pair $\{i, j\}$ in $\{2, 3, \ldots, r-1\}$ so, by Corollary 4.3.4, $M|H$ is a projective geometry. Finally, as M is 3-connected, there is an independent set of three elements in $E(M)$ avoiding H. Hence, by Lemma 4.3.3, $M \cong P_r$. □

We are now ready to prove Theorem 4.3.1.

Proof of Theorem 4.3.1. Let r be the rank of M. If M is graphic, then Theorem 4.1.1 gives that $M \cong M(K_{r+1})$, so we may assume that M is not graphic. Thus M has a minor N isomorphic to F_7, F_7^*, $M^*(K_{3,3})$, or $M^*(K_5)$. By Proposition 4.3.6, it now suffices to show that the Θ_3-closure, $\Theta(N)$, of N is a projective geometry.

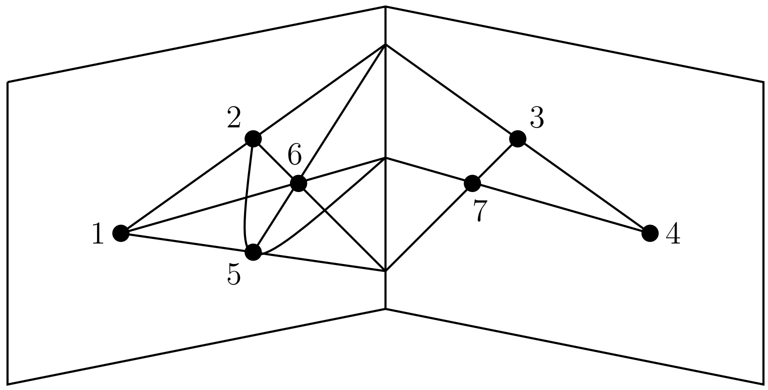

Figure 4.3. F_7^* in the proof of Theorem 4.3.1.

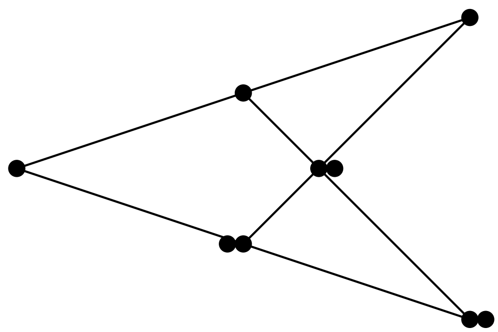

Figure 4.4. A quotient of $M^*(K_{3,3})$.

This is immediate when $N \cong F_7$, so suppose N is isomorphic to F_7^*, labelled as in Figure 4.3. The theta-graphs of N imply that, in $\Theta(N)$, the plane containing $\{1, 2, 5, 6\}$ is isomorphic to F_7. Proposition 4.3.6 now implies that M is isomorphic to P_r.

Next, suppose $N \cong M^*(K_{3,3})$. The complement of N in P_4 is $U_{2,3} \oplus U_{2,3}$; let x be an element of this complement. The elementary quotient of N obtained by extending N by x and then contracting x is shown in Figure 4.4. The three pairwise-skew 2-element circuits of this quotient correspond to three lines in the extension of N by x where the union of these lines has rank four. Thus N contains a theta-graph that is completed by x. It follows that x and, symmetrically, every point in the complement of N in P_4 belongs to $\Theta(N)$. Thus $\Theta(N) \cong P_4$. Hence M is a projective geometry.

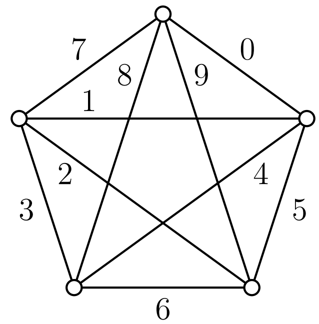

Figure 4.5. K_5 in the proof of Theorem 4.3.1.

$$
\begin{array}{cccccc|cccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \\
\end{array}
$$
$$
\left[\begin{array}{cccccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}\right]
$$

Figure 4.6. A binary representation of $M^*(K_5)$.

Now suppose $N \cong M^*(K_5)$, where K_5 is labelled as in Figure 4.5. Figure 4.6 gives a corresponding binary representation of N. The dual of a theta-graph with theta-arcs of size at least two is a triangle with no trivial parallel classes. Therefore, we can detect theta-graph restrictions of N by contracting elements of N^* to produce such a triangle. For example, $N^*/5, 8$ is the dual of a theta-graph in N with theta-arcs $\{1, 2\}$, $\{3, 7\}$, and $\{0, 4, 6, 9\}$. This theta-graph is completed by the element $[1\ 1\ 0\ 0\ 0\ 0]^T$, so this element belongs to $\Theta(N)$. The following is a list of duals of theta-graphs of N and the corresponding elements of $\Theta(N)$ that they produce using this reasoning.

- $N^*/5, 8$ gives $[1\ 1\ 0\ 0\ 0\ 0]^T \in \Theta(N)$.
- $N^*/4, 9$ gives $[1\ 0\ 1\ 0\ 0\ 0]^T \in \Theta(N)$.
- $N^*/3, 9$ gives $[1\ 0\ 0\ 1\ 0\ 0]^T \in \Theta(N)$.

- $N^*/2,8$ gives $[1\ 0\ 0\ 0\ 1\ 0]^T \in \Theta(N)$.
- $N^*/0,6$ gives $[0\ 1\ 1\ 0\ 0\ 0]^T \in \Theta(N)$.
- $N^*/1,8$ gives $[0\ 1\ 0\ 0\ 1\ 0]^T \in \Theta(N)$.
- $N^*/0,3$ gives $[0\ 1\ 0\ 0\ 0\ 1]^T \in \Theta(N)$.
- $N^*/1,9$ gives $[0\ 0\ 1\ 1\ 0\ 0]^T \in \Theta(N)$.
- $N^*/0,2$ gives $[0\ 0\ 1\ 0\ 0\ 1]^T \in \Theta(N)$.
- $N^*/6,7$ gives $[0\ 0\ 0\ 1\ 1\ 0]^T \in \Theta(N)$.
- $N^*/5,7$ gives $[0\ 0\ 0\ 1\ 0\ 1]^T \in \Theta(N)$.
- $N^*/4,7$ gives $[0\ 0\ 0\ 0\ 1\ 1]^T \in \Theta(N)$.

It is now straightforward to find theta-graphs in $\Theta(N)$ that are completed by the elements $[1\ 0\ 0\ 0\ 0\ 1]^T$, $[0\ 1\ 0\ 1\ 0\ 0]^T$, and $[0\ 0\ 1\ 0\ 1\ 0]^T$, so $\Theta(N)$ contains every vector of weight 1 or 2. Thus $\Theta(N)$ properly contains $M(K_7)$, so, by Proposition 4.3.2, $\Theta(N) \cong P_6$. □

4.4. The Main Result

This section proves Theorem 4.1.2. For a review of canonical tree decompositions, see Section 1.1.

Proof of Theorem 4.1.2. Since circuits, cycle matroids of complete graphs, and projective geometries are in Θ_3, by Proposition 4.2.5, every matroid that can be built from such matroids by a sequence of parallel connections is in Θ_3.

To prove the converse, we begin by noting that loops can be added via direct sums and that parallel elements can be added via parallel connections of circuits, so we may assume M is simple. Let T be the canonical tree decomposition of M. The proof is by induction on $|V(T)|$.

If $|V(T)| = 1$, then M is 3-connected and the result holds by Theorem 4.3.1. Now assume T has at least two vertices, and let N be a matroid labelling a leaf v of T. Since M is simple, N is not a cocircuit. We may now write $M = N \oplus M_1$, where, by Corollary 4.2.3, N and M_1 are in Θ_3. Thus N is a circuit, the cycle matroid of a complete graph of rank at least three, or a projective geometry of rank at least three. Moreover, by induction, M_1 is a parallel connection of circuits, cycle matroids of complete graphs, and projective geometries.

Let N_1 be the label of the neighbor of v in T, and suppose N_1 is not a cocircuit. In this case, each of N and N_1 is a circuit, the cycle matroid of a complete graph of rank at least three, or a projective geometry of rank at least three, but they are not both circuits. Therefore, if p is the basepoint of the 2-sum $N \oplus_2 N_1$, there are circuits in N and N_1 that form a theta-graph that is completed by p, a contradiction. Thus N_1 is a cocircuit.

Now let k be the degree of N_1 in T. Evidently N_1 has at least k elements, but, since M is simple, N_1 has at most $k+1$ elements. If $k = 2$, then N_1 has three elements as N_1 labels a vertex of T. Otherwise $k \geq 3$, so there are circuits in M that form a theta-graph that is completed by an element of N_1. Hence N_1 has $k+1$ elements. It now follows that M is the parallel connection of circuits, cycle matroids of complete graphs, and projective geometries. □

Appendix. Copyright Information

Chapter 2

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Ordering Circuits of Matroids

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PDF

DOI: https://doi.org/10.37236/11117

Published
2022-11-18

Abstract

The cycles of a graph give a natural cyclic ordering to their edge-sets, and these

Issue
Volume 29, Issue 4 (2022)

Article Number
P4.31

Chapter 3

On the cogirth of binary matroids

Author: Cameron Crenshaw,James Oxley

Publication: Advances in Applied Mathematics

Publisher: Elsevier

Date: June 2023

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Bibliography

[1] Berlekamp, E. R., *Algebraic Coding Theory*, McGraw-Hill, New York, 1968.

[2] Bixby, R. E., A strengthened form of Tutte's characterization of regular matroids, *J. Combin. Theory Ser. B* **20** (1976), 216–221.

[3] Bixby, R. E., Kuratowski's and Wagner's theorems for matroids, *J. Combin. Theory Ser. B* **22** (1977), 31–53.

[4] Blake, I. F., and Mullin, R. C., An Introduction to Algebraic and Combinatorial Coding Theory, Academic Press, New York, 1976.

[5] Bose, R. C. and Burton, R. C., A characterization of flat spaces in a finite geometry and the uniqueness of the Hamming and MacDonald codes, *J. Combinatorial Theory* **1** (1966), 96–104.

[6] Cunningham, W. H., A combinatorial decomposition theory, Ph.D. thesis, University of Waterloo, 1973.

[7] Guruswami, V., Introduction to Coding Theory, Notes 4, Elementary Bounds on Codes, January, 2010. Retrieved May 23, 2022 from `http://www.cs.cmu.edu/~venkatg/teaching/codingtheory/notes/notes4.pdf`.

[8] Hill, R., A First Course in Coding Theory, Oxford University Press, New York, 1986.

[9] Jamison, R. E. and Mulder, H. M., Graphs constructible from cycles and complete graphs, *Discrete Math.* **340** (2017), 2301–2306.

[10] Oxley, J. G., On nonbinary 3-connected matroids, *Trans. Amer. Math. Soc.* **300** (1987), 663–679.

[11] Oxley, J., *Matroid Theory*, 2nd ed., Oxford University Press, New York, 2011.

[12] McNulty, J. and Wu, H., Connected hyperplanes in binary matroids, *J. Combin. Theory Ser. B* **79** (2000), 87–97.

[13] Plotkin, M., Binary codes with specified minimum distance, *IRE Trans.* **IT-6** 1960, 445–450.

[14] Seymour, P. D., Decomposition of regular matroids, *J. Combin. Theory Ser. B* **28** (1980), 305–359.

[15] Seymour, P. D., Minors of 3-connected matroids, *European J. Combin.* **6** (1985), 375–382.

[16] Seymour, P. D., Adjacency in binary matroids, *European J. Combin.* **7** (1986), 171–176.

[17] Tutte, W. T., Matroids and graphs, *Trans. Amer. Math. Soc.* **90** (1959), 527–552.

[18] Tutte, W. T., Selected Papers of W. T. Tutte, Vol. II, McCarthy, D., Stanton R. G. (eds.), Charles Babbage Research Centre, Winnipeg, 1979.

[19] Vardy, A., The intractability of computing the minimum distance of a code, *IEEE Trans. Inform. Theory* **43** (1997), 1757–1766.

[20] Wagner, D. K., A circuit characterization of graphic matroids, *J. Combin. Theory Ser. B* **118** (2016), 284–290.

[21] Whitney, H., 2-isomorphic graphs, *Amer. J. Math.* **55** (1933), 245–254.

Vita

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