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Analytic Continuation of Toeplitz Operators and Commuting Families of C*-Algebras

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ANALYTIC CONTINUATION OF TOEPLITZ OPERATORS AND COMMUTING FAMILIES OF C^*-ALGEBRAS

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This thesis is dedicated to my mother and father,

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and

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Abstract

In this thesis we consider the Toeplitz operators on the weighted Bergman spaces and their analytic continuation. We proved the commutativity of the C^*-algebras generated by the analytic continuation of Toeplitz operators with special class of symbols that are invariant under suitable subgroups of $SU(n, 1)$, and we showed that commutative C^*-algebras with symbols invariant under compact subgroups of $SU(n, 1)$ are completely characterized in terms of restriction to multiplicity free representations. Moreover, we extended the restriction principal to the analytic continuation case for suitable maximal abelian subgroups of the universal covering group $\widetilde{SU(n, 1)}$, and we obtained the generalized Segal-Bargmann transform, where we used it describe the spectral representation of Toeplitz operators. In particular, we proved that Toeplitz operators are unitarily equivalent to a convolution operator, and we used the Fourier transform to describe their spectra.

Chapter 1. Introduction

Toeplitz operators were first studied by the mathematician Otto Toeplitz. Originally, Toeplitz operators were defined by infinite matrices of the form

$$A = \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \\ \cdots & a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ \cdots & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ \cdots & a_2 & a_1 & a_0 & a_{-1} & \cdots \\ \cdots & a_3 & a_2 & a_1 & a_0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Figure 1.1. Toeplitz Matrix

Where the entries on the diagonals are fixed. The matrix A corresponds to the sequence $(a_n)_{-\infty}^{\infty}$, this matrix can be viewed as an operator acting on the Hilbert space ℓ^2. This raises the problem of characterizing the sequences $(a_n)_{-\infty}^{\infty}$ such that the corresponding Toeplitz matrix is the matrix of a bounded linear operator on ℓ^2 [1]. The characterization was given by Otto Toeplitz in the special case where $a_{-n} = \overline{a_n}$.

Theorem 1.0.1 (O. Toeplitz) *The Toeplitz matrix corresponding to a two-sided sequence $(a_n)_{-\infty}^{\infty}$ is the matrix of a bounded operator on ℓ^2 if and only if there exists $f \in L^\infty(\partial D, d\sigma)$ such that $a_n = \int_{\partial D} f(z)\,\overline{z^n}\,d\sigma(z)$.*

Let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$, and consider the weighted Bergman space

$$A^2_\lambda(\mathbb{B}^n) = \{f \in L^2(\mathbb{B}^n, d\mu_\lambda) : f \; is\, holomorphic\, on\, \mathbb{B}^n\}$$

where

$$d\mu_\lambda(z) = \frac{\Gamma(\lambda)}{\pi^n \Gamma(\lambda - n)}(1 - |z|^2)^{\lambda-n-1} dz$$

The Toeplitz operator on the weighted Bergman space $A^2_\lambda(\mathbb{B}^n)$, with symbol $\phi \in L^\infty(\mathbb{B}^n)$

and weight $\lambda > n$ is defined by

$$T_\phi^{(\lambda)} : A_\lambda^2(\mathbb{B}^n) \to A_\lambda^2(\mathbb{B}^n)$$

$$T_\phi^{(\lambda)} = P_\lambda(\phi f)$$

where $P_\lambda : L^2(\mathbb{B}^n, d\mu_\lambda) \to A_\lambda^2(\mathbb{B}^n)$ is the orthogonal projection

$$P_\lambda f(x) = \int_{\mathbb{B}^n} f(y) K_\lambda(x,y) d\mu_\lambda(y)$$

and K_λ is the kernel, which is given by

$$K_\lambda(x,y) = (1- < x, y >)^{-\lambda}$$

This definition of the Toeplitz operator works when $\lambda > n$. So, a natural question would be, is it possible to extend the definition of the weighted Bergman spaces, and Toeplitz operators to include the weights λ that are less than n? It turns out that the weighted Bergmann spaces does not reduce to zero when we consider $\lambda > 0$, this is called the continuous part of the Wallach set [18]. In [3], the authors constructed Hilbert spaces of holomorphic functions with reproducing kernel

$$K_\lambda(z,w) = (1 - z.\bar{w})^{-\lambda}$$

with weights $\lambda > 0$, and they proved these spaces coincide with the weighted Bergman spaces when we restrict the values of λ to $\lambda > n$.

In [7] the authors generalized the work of Chailuek and Hall, and proved the existence of analytic continuation of weighted Bergman spaces to include all weights α in $\mathbb{R} - V$ where V a discrete set. These spaces are Sobolev spaces of holomorphic functions of certain order. Moreover, they proved the existence and uniqueness of the analytic continuation of the Toeplitz operators.

One interesting problem is to find families of commutative C^*-algebras generated by Toeplitz operators, and the spectral decomposition of the Toeplitz operators. For example, it was shown that the C^*-algebra generated by Toeplitz operators is commutative if and only if there is a pencil of hyperbolic geodesics such that the symbols of the Toeplitz operators are constant on the cycles of this pencil [15]. Another approach to this problem was presented in [5] where the authors used tools from representation theory to construct commuting families of C^*-algebras. In particular, they have applied the results in [4], to show that Toeplitz operators with symbols that are invariant under maximal abelian subgroups of $SU(n, 1)$ generates commutative C^*-algebras. Moreover, they applied the restriction principal to show that Toeplitz operators on the weighted Bergman space can be realized as a convolution operator against a certain function.

In this thesis we will construct commuting families of C^*-algebras generated by the analytic continuation of the Toeplitz operators, and we will show that the restriction principal can be used to find the spectral decomposition of the analytic continuation of the Toeplitz operators. In other words, our main goal is extend the results in [5] to the analytic continuation case. This thesis is arranged as follow: Chapter 2 reviews the basics on bounded symmetric domains, weighted Bergman spaces and representation theory. Chapter 3 describes the work in [2] on the analytic continuation of Toeplitz operators. In chapter 4 we construct commutative families of C^*-algebras generated by the analytic continuation of Toeplitz operators. Further, we show that the if H is a compact subgroup of $SU(n, 1)$, then the C^*-algebra generated by Toeplitz operators with $H-$invariant symbols is dense in the algebra of bounded operators on the weighted Bergman space. Furthermore, we show that the weighted Bergman space admits a decomposition into a direct sum of irreducible $U(n)-$modules and $\mathbb{T}^n-$modules. Chapter 5 discusses the spectral decomposition of the Toeplitz operators. First, we show that the restriction principal can be applied in the analytic continuation case, this allows us to write the Toeplitz operator as a convolution operator, then we use the Fourier transform to describe the spectrum of the Toeplitz operators. This only works when we have symbols invariant under a maximal abelian subgroup of $SU(n, 1)$. We will use a different approach to deal with the case when we have symbols invariant under the group $U(n)$, we will show that the analytic continuation of Toeplitz operator with $U(n)-$invariant symbols is unitarily equivalent to a multiplication operator and we will give explicit description of the eigenvalues of the Toeplitz operators.

Chapter 2. Preliminaries

2.1. Bounded Symmetric Domains

The definitions and examples in this preliminary section were taken from [18]. Bounded symmetric domains are generalization of the open unit disk. The unit disk in the complex plane can be realized as the upper half plane using the Cayley transform, similarly bounded symmetric domains can be realized as symmetric tubular domains, where the symmetric tubular domains are generalization of the upper half plane. Let V be an n-dimensional vector space. An open unit ball $\Omega \subset V$ is called symmetric if the group of automorphisms $\mathrm{Aut}(\Omega)$ acts transitively on Ω.

Jordan triple product is a map $V \times V \times V \to V$, $(z, a, w) \to \{za^*w\}$ which is symmetric, bilinear in z and w, and conjugate-linear in a. The Bergman endomorphism associated with $(a, b) \in Z \times Z$ is defined by

$$B(a, b)z := z - 2\{ab^*z\} + \{a\{bz^*b\}^*a\}$$

The map $B(z, \xi)$ is invertible when (z, ξ) belongs to the unit ball Ω. The quasi-inverse of z with respect to ξ is defined as

$$z^\xi := B(z, \xi)^{-1}(z - \{z\xi^*z\})$$

which can be uniqely written as

$$z^\xi = \frac{p(z, \xi)}{\Delta(z, \xi)}$$

where $p(z, \xi)$ is a sesqui-polynomial having no common factor with $\Delta(z, \xi)$, and the map

$$\Delta(z, \xi) : Z \times Z \to \mathbb{C}$$

is a sesqui-polynomial with $\Delta(0,0) = 1$, and Δ is called the Jordan triple determinant associated with Ω.

Example 2.1.1 *Let $\mathbb{B} = \{z \in \mathbb{C} : |z|< 1\}$ be the open unit disk in $\mathbb{C}$ with associated Jordan triple product $\{za^*w\} = z\bar{a}w$. The Bergman endomorphism is given by*

$$B(a,b)z = (1 - a\bar{b})^2 z$$

and the quasi-inverse of z with respect to ξ is

$$z^\xi = \frac{z}{1 - z\bar{\xi}}$$

Therefore, the Jordan triple determinant $\Delta(z,\xi)$ is given by

$$\Delta(z,\xi) = 1 - z\bar{\xi}$$

Example 2.1.2 *Let $Z = M_{n\times n}(\mathbb{C})$ be the space of $n \times n$ matrices over $\mathbb{C}$. Consider the unit ball*

$$\Omega = \{z \in M_{n\times n}(\mathbb{C}) : I_n - zz^* > 0\}$$

the group of automorphism $Aut(\Omega)$ can be realized as $SU_n(\mathbb{C})$ which acts transitively on Ω by the fractional linear transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z := (az + b)(cz + d)^{-1}$$

The associated Jordan triple product on $M_{n\times n}(\mathbb{C})$ is given by

$$\{za^*w\} = \frac{1}{2}(za^*w + wa^*z)$$

the Bergman endomorphism is $B(a,b)z = (I - ab^*)z(I - ba^*)$, and the quasi-inverse of z with respect to ξ is $z^\xi = (I - z\xi^*)^{-1}z$. Therefore, the Jordan triple determinant is given by

$$\Delta(z,\xi) = Det(I - z\xi^*)$$

2.1.1. Joint Peirce Decomposition

Let Z be an irreducible Jordan triple with an associated irreducible bounded symmetric domain Ω. An element $c \in Z$ is called tripotent if $\{cc^*c\} = c$. Let $a, b \in Z$, if $\{aa^*b\} = 0$ then a,b are called orthogonal. A non zero tripotent b is called primitive if it can't be written as a sum of two orthogonal tripotents. A frame of Z is a maximal set of orthogonal primitive tripotents. The cardinality of a frame is called the rank of Z. Let $e_1, ..., e_r$ be a frame of Z. Set $Z_{ij} := \{z \in Z : \{e_k e_k^* z\} = \frac{\delta_{ik} + \delta_{jk}}{2} z \ \ where\ 1 \leq k \leq r\}$, then we have the joint Peirce decomposition

$$Z = \sum_{0 \leq i \leq j \leq r}^{\oplus} Z_{ij}$$

Define the numbers $a := dim(Z_{ij})$ for $1 \leq i < j \leq r$, and $b := dim(Z_{0j})$ for $1 \leq j \leq r$, the numbers a,b are called the characteristic multipliers of Z. The dimension of Z is given by

$$n = r + \frac{a}{2} r(r-1) + rb$$

and the genus of Z is

$$p := 2 + r(a-1) + b$$

Example 2.1.3 *Let Z be the space of $m \times m$ matrices over the complex numbers $\mathbb{C}$. A frame of Z is given by he matrices e_i where the (i,i)-th entry is 1, and zero elsewhere. The characteristic multipliers of Z are $a = 2, b = 0$, and the genus of Z is $p = 2 + m$.*

2.2. Weighted Bergman Space

Let Z be an irreducible Jordan triple of dimension n and rank r with an associated symmetric domain Ω. The volume of Ω is

$$\pi^{-n}\int_{\Omega} dv(z) = \prod_{k=1}^{r} \frac{\Gamma(1+(k-1)\frac{a}{2}}{\Gamma(b+2+(r+k-2)\frac{a}{2})}$$

$$= \frac{\Gamma_{\Lambda}(p-\frac{n}{r})}{\Gamma_{\Lambda}(p)}$$

where

$$\Gamma_{\Lambda}(y) := (2\pi)^{(n-r)/2}.\prod_{j=1}^{r}\Gamma(y-\frac{a}{2}(j-1))$$

for $y > \frac{a}{2}(r-1)$.

The space $L^2(\Omega, d\mu_{\Omega})$ is equipped with the inner product

$$(f,g)_{\Omega} := \int_{\Omega} \overline{f(z)}g(z)d\mu_{\Omega}$$

where $d\mu_{\Omega}$ is the probability measure

$$d\mu_{\Omega} := \pi^{-n}\frac{\Gamma_{\Lambda}(p)}{\Gamma_{\Lambda}(p-\frac{n}{r})}dv(z)$$

Definition 2.2.1 *The Bergman space on Ω is the subspace of $L^2(\Omega)$ consisting of holomorphic function*

$$H^2(\Omega) := \{f \in L^2(\Omega) : f \text{ is holomorphic}\}$$

The Bergman space is a closed subspace of $L^2(\Omega)$, and the orthogonal projection

$$P_{\Omega} : L^2(\omega) \to H^2(\omega)$$

is called the Bergman projection. Since the Bergman space is a closed subspace of $L^2(\Omega)$, then $H^2(\Omega)$ is also a Hilbert space. Moreover, it's a reproducing kernel Hilbert space with kernel

$$K(z,w) = \Delta(z,w)^{-p}$$

where $\Delta(z,w)$ is the Jordan triple determinant, and p is the genus. The kernel is invariant under $Aut(\Omega)$ in the sense that

$$K(z,w) = Det\, g'(z)\,.K(g(z),g(w))\,.\,\overline{Det\, g'(w)}$$

for all $g \in Aut(\Omega)$. This allows us to define a measure on Ω that is invariant under the action of $Aut(\Omega)$

$$d\mu(z) = \Delta(z,z)^{-p} dv(z)$$

The function $\Delta(z,z)^{\lambda-p}$ is integrable for all $\lambda > p-1$, and

$$\int_\Omega \Delta(z,z)^{\lambda-p}.dv(z) = \pi^n \frac{\Gamma_\Lambda(\lambda - \frac{n}{r})}{\Gamma_\Lambda(\lambda)}$$

Therefore,

$$d\mu_\lambda(z) := \pi^{-n} \frac{\Gamma_\Lambda(\lambda)}{\Gamma_\Lambda(\lambda - \frac{n}{r})} \Delta(z,z)^{\lambda-p} dv(z)$$

defines a probability measure on Ω.

Define the Hilbert space $L^2_\lambda(\Omega) := L^2(\Omega, d\mu_\lambda)$, the weighted Bergman space $H^2_\lambda(\Omega)$ of weight $\lambda > p-1$ is defined as the subspace of holomorphic functions in $L^2_\lambda(\Omega)$. The space $H^2_\lambda(\Omega)$ is a reproducing kernel Hilbert space with kernel

$$K_\lambda(z,w) = \Delta(z,w)^{-\lambda}$$

Example 2.2.1 *Consider the unit disk Ω in $\mathbb{C}$. We have n=r=1, b=0, a=2, and p=2. The Jordan triple determinant is $\Delta(z,w) = 1 - z\overline{w}$. For $\lambda > 1$ The kernel of the weighted Bergman space is*

$$K_\lambda(z,w) = (1 - z\overline{w})^{-\lambda}$$

where $\lambda = 2$ corresponds to the unweighted Bergman space, and $\lambda = 1$ corresponds to Hardy space on the unit disk with kernel (Szego kernel)

$$K(z,w) = (1 - z\overline{w})^{-1}$$

2.2.1. Sobolev Spaces

Let s be a real number. The Sobolev space $W^s(\mathbb{R}^n)$ is the space of generalized functions u whose Fourier transform $\widetilde{u}(\xi)$ is locally integrable, and such that

$$\|u\|_s^2 = \int_{\mathbb{R}^n} |\widetilde{u}(\xi)|^2 (1 + |\xi|^2)^{s/2} d\xi < \infty. \tag{2.2.1}$$

The Fourier transform of the space W^s will be denoted by $\widetilde{W^s}$. The spaces W^s, and $\widetilde{W^s}$ are Hilbert spaces with norm

$$< u, v > = \int \widetilde{u}(\xi)\overline{\widetilde{v}(\xi)}(1 + |\xi|^2)^{s/2} d\xi. \tag{2.2.2}$$

If we take s=0, then space $\widetilde{W^s}$ is the space $L^2(\mathbb{R}^n)$. By Plancherel's theorem we have $W^0 = F^{-1}\widetilde{W^0} = L^2(\mathbb{R}^n)$. If $s = m$ where m is positive integer, then the space $W^m(\mathbb{R}^n)$ consists of square integrable functions $u(x)$ whose generalized derivatives $D^k u(x)$ are also square integrable for $1 \leq |k| \leq m$. In this case the norm (2.2.1) is equivalent to the norm

$$\|u\|^2 = \sum_{|k| \leq m} \int |D^k u(x)|^2 dx = \sum_{|k| \leq m} \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} |\xi^k \tilde{u}(\xi)|^2 d\xi \tag{2.2.3}$$

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. For $s > 0$ the Sobolev space $W^s(\Omega)$ is defined by

$$W^s(\Omega) := \{f : \Omega \to \mathbb{R} \; where \, f \, has \, an \, extension \, Lf \, onto \, \mathbb{R}^n \, such \, that \, f \in W^s(\mathbb{R}^n)\}$$

The norm on $W^s(\Omega)$ is defined by $\|u\|_s = inf\|Lf\|_s$, where the infimum is taken over all the extensions Lf. The space $W^{-s}(\Omega)$ is defined as the dual space of $W^s(\Omega)$ [8].

2.3. Representation Theory

Representation theory studies algebraic structures by representing their elements as linear transformations over a vector space. This field of mathematics emerged after an observation of Dedekind, he noticed that if we have a finite group and we replaced each entry in the multiplication table then the determinant of the resulting matrix factors into a product of irreducible polynomials each one occurs with multiplicity equal to it's of degree [9].

A unitary representation of a group G is a group homomorphism π into the group of unitary operators on a Hilbert space $\mathcal{H}_\pi$ such that the map $g \mapsto \pi(g)x$ is continuous for every $x \in \mathcal{H}_\pi$. Unitary representations usually arises when the group G acts on a locally compact Hausdorff space Ω. If G acts on Ω, then G acts on the functions defined on Ω by

$$\pi(g)f(x) = f(g^{-1}x)$$

and this action can be modified to obtain a unitary representation of the group G. For example, if the space Ω admits a $G-$invariant Radon measure μ, then the representation

$$\pi(g)f(x) = f(g^{-1}x)$$

is a unitary representation on $L^2(\Omega, d\mu)$. If we consider the more general case where Ω admits a Radon measure μ such that

$$d\mu(gx) = \varphi(g, x)d\mu(x) \tag{2.3.1}$$

where φ is a positive continuous function that satisfy the cocycle relation

$$\varphi(xy, p) = \varphi(x, yp).\varphi(y, p)$$

Then the representation

$$\pi(g)f(x) = \varphi(g, g^{-1}x)^{-1/2} f(g^{-1}x)$$

is a unitary representation on $L^2(\Omega, d\mu)$. A Radon measure that satisfy 2.3.1 is called a strongly quasi-invariant measure [10].

We can consider representations of the group G on $L^2(G)$ by allowing G to acts on itself. If G acts on itself by left translation then, we get the left regular representation $\pi(g)f(x) = f(g^{-1}x)$, and the right regular representation is defined by $\pi(g)f(x) = f(xg)$. The right regular representation is usually used when we have an object G whose elements need not have inverses.

Let π be a representation of a group G on a Hilbert space $\mathcal{H}_\pi$, a subspace M of $\mathcal{H}_\pi$ is called invariant subspace for π if $\pi(g)M \subset M$. The representation π is called irreducible if it doesn't have non-trivial invariant subspaces. Let $u \in \mathcal{H}_\pi$, the closed linear span $\mathcal{M}$ of $\{\pi(g)u : g \in G\}$ in $\mathcal{H}_\pi$ is called the cyclic subspace generated by u. If the $\mathcal{M} = \mathcal{H}_\pi$, then u is called cyclic vector for π. The representation π is called cyclic representation if it has cyclic vector [10]. Zorn's lemma can be used to show that every unitary representation can be decomposed into a direct sum of cyclic representations.

Theorem 2.3.1 (Schur's Lemma) *Let π be a unitary representation of a group G, and $\mathcal{C}(\pi)$ be the space of bounded operators on $\mathcal{H}_\pi$ that intertwine with π. The representation π is irreducible if and only if $\mathcal{C}(\pi)$ consists only of scalar multiples of the identity.*

Let π be a unitary representation of a group G on a separable Hilbert space $\mathcal{H}$. Let $R(\pi)$ be the smallest weakly closed algebra of bounded linear operators that contains $\pi(g)$ for every $g \in G$. The representation π is called primary if the center of $R(\pi)$ consists only of scalar operators. If every primary representation of a group G decomposes as a direct sum of the same irreducible representation, then the group G is said to be of Type I [11].

2.3.1. Induced Representations

Suppose we have a locally compact group G, and a closed subgroup $H \subset G$. If we have a unitary representations of H, then we can use it to construct a unitary representation for the group G, the new representation can be considered as an extension of the representation of H. The process of constructing new representations of G out of representations of it's closed subgroup is very important tool for providing irreducible representation. The induced representations are the unitary representation of G that are coming from the action of the group G on sections of homogeneous vector bundles on G/H [10].

Let $q : G \to G/H$ be the quotient map, and assume σ is a unitary representation of the subgroup H on the Hilbert space $\mathcal{H}_\sigma$. Let $C(G, \mathcal{H}_\sigma)$ be the space of all continuous functions from G to $\mathcal{H}_\sigma$. Define the space

$$\mathcal{F}_0 := \{f \in C(G, \mathcal{H}_\sigma) : q(\operatorname{supp} f)\, is\, compact\, and\, f(xh) = \sigma(h^{-1})f(x)\, for\, x \in G, h \in H\}$$

Proposition 2.3.1 *[10] If $\alpha : G \to \mathcal{H}_\sigma$ is continuous with compact support then the* *function*

$$f_\alpha(x) = \int_H \sigma(\nu)\alpha(x\nu)d\nu$$

belongs to the space $\mathcal{F}_0$ *and is left uniformly continuous on* G*. Moreover, every element of* $\mathcal{F}_0$ *is of the form* f_α *for some* $\alpha \in C_c(G, \mathcal{H}_\sigma)$*.*

Let $< u, v >_\sigma$ *denote the inner product on the Hilbert space* $\mathcal{H}_\sigma$*. The group* G *acts on the space* $\mathcal{F}_0$ *by left translation* $f \mapsto L_x f$*. Assume there is an invariant measure* μ *on* G/H *under the left translation action. Since the representation* σ *is a unitary representation of the subgroup* H*, then for* $f(x)$ *and* $g(x)$ *in* $\mathcal{F}_0$*, the inner product* $< f(x), g(x) >_\sigma$ *depends on the coset* $q(x)$ *of* x*. So the inner product defines an integrable function on* $C_c(G/H)$*, this make it possible to define an inner product on* $\mathcal{F}_0$

$$< f, g >= \int_{G/H} < f(x), g(x) >_\sigma \, d\mu(xH)$$

Let $\mathcal{F}$ *be the Hilbert space completion of* $\mathcal{F}_0$*, then the left translation operators* L_x *extends to a unitary operators on* $\mathcal{F}$*, so we have a continuous map* $x \mapsto L_x f$ *from* G *to* $\mathcal{F}$ *for each* $f \in \mathcal{F}_0$*. So, they define a unitary representation of* G*. This representation is denoted by* $\mathrm{Ind}_H^G(\sigma)$*, and is called the representation induced by* σ *[10]. The construction of the induced representation relies on the assumption that* G/H *admits an invariant measure. There are two ways to construct the induced representation when* G/H *does not have an invariant measure. The first method is to replace the measure* μ *with a quasi-invariant measure on* $\mathcal{F}$*, and change the action of the group* G *to make it unitary. The second method requires modifying the space* $\mathcal{F}_0$ *and the advantage in this case is that the induced representation does not depend on the quasi-invariant measure.*

Geometrically, the space $\mathcal{F}_0$ *can be viewed as the space of compactly supported con-*

tinuous sections of a vector bundle over G/H, and the induced representations $\mathrm{Ind}_H^G(\sigma)$ as the representations of G on sections of homogeneous Hermitian vector bundles over G/H, for further details we refer to [10].

Chapter 3. Analytic Continuation of Toeplitz Operators

3.1. Introduction

The weighted Bergman space of weight $\alpha > -1$ on the unit ball $\mathbb{B}^n$ is defined as

$$A^2_\alpha(\mathbb{B}^n) = \{f \in L^2(\mathbb{B}^n, d\mu_\alpha) : f \textit{ is holomorphic on } \mathbb{B}^n\}$$

where

$$d\mu_\alpha(z) = \frac{\Gamma(\alpha+n+1)}{\pi^n \Gamma(\alpha+1)}(1-|z|^2)^\alpha dz$$

A holomorphic function $f(z) = \sum_j f_j z^j$ on the unit ball $\mathbb{B}^n$ belongs to the weighted Bergman space $A^2_\alpha(\mathbb{B}^n)$ if and only if

$$\|f\|^2_\alpha := \sum_j |f_j|^2 \frac{j!\,\Gamma(\alpha+n+1)}{\Gamma(|j|+\alpha+n+1)} < \infty$$

the factor $\dfrac{\Gamma(\alpha+n+1)}{\Gamma(|j|+\alpha+n+1)}$ remains positive even when $\alpha > -n-1$. This was the motivation to extend the space A^2_α to include the weights $\alpha > -n-1$, by setting

$$A^2_\alpha(\mathbb{B}^n) := \{f \textit{ holomorphic on } \mathbb{B}^n : \|f\|_\alpha < \infty\}$$

We will see later that this definition can be extended further to include all real numbers except possibly a discrete set, where the discrete set arises from the poles of the Gamma function.

3.2. Toeplitz Operators and Pseudo-differential Operators

Pseudo-differential operators were used [7] to show that weighted Bergman spaces can be realized as holomorphic Sobolev spaces of certain order, this new realization made it possible to define weighted Bergman spaces $H^2_\lambda(\Omega)$ almost everywhere on $\mathbb{R}$. In this section we will describe how Toeplitz operators were defined using pseudo-differential operators, and how this gives us a new realization of Bergman spaces on bounded strictly pseudoconvex domains Ω.

Definition 3.2.1 *[13] Suppose $\Omega \subset \mathbb{C}^n$ is open, $u \in C^2(\Omega)$, and $z \in \Omega$. The Levi form of u at z is the quadratic form*

$$w \mapsto Lev(u)(z; w) := \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}} w_j \overline{w_k}$$

Definition 3.2.2 *[13] A bounded domain Ω is strictly pseudoconvex if there exists a neighbourhood V of $bd(\Omega)$ and a function $r \in C^2(V)$ with the following properties:*

1. $\Omega \cap V = \{z \in V : r(v) < 0\}$
2. $dr(z) \neq 0$ *for every* $z \in V$
3. $Lev(r)(y, w) > 0$ *for every* $y \in bd(\Omega)$ *and every nonzero* w *in the complex tangent space to the boundary at* w

The Hardy space $H^2(\partial\Omega)$ consists of $L^2(\partial\Omega)$ functions that have holomorphic Poisson extension in Ω. The Poisson extension operator $\mathbf{K}$ is defined as

$$\mathbf{K} : C^\infty(\partial\Omega) \to C^\infty(\overline{\Omega}), \;\; \Delta\mathbf{K}u = 0 \;\; on \;\; \Omega, \; and \; \mathbf{K}u|_{\partial\Omega} = u$$

Another useful operator is the operator that restricts the values to the boundary of Ω

$$\gamma : C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$$

$$\gamma u := u|_{\partial\Omega}$$

The Poisson extension operator acts as a bounded operator from the Sobolev space $W^s(\partial\Omega)$ onto the subspace of harmonic function $W^{s+1/2}_{harm}(\Omega) \subset W^{s+1/2}(\Omega)$, and from $W^s_{hol}(\partial\Omega)$ onto $W^{s+1/2}_{hol}(\Omega)$, where $W^s_{hol}(\Omega)$ is the subspace of holomorphic functions in $Ws(\Omega)$. The operator γ acts as a bounded operator from $W^s_{harm}(\Omega)$ onto $W^{s-1/2}(\partial\Omega)$, and from $W^s_{hol}(\Omega)$ onto $W^{s-1/2}_{hol}(\partial\Omega)$. Therefore, the space $W^s_{hol}(\Omega)$ is isomorphic to $W^{s-1/2}_{hol}(\partial\Omega)$, and $W^s_{harm}(\Omega)$ is isomorphic to $W^{s-1/2}(\partial\Omega)$.

Let $m \in (-\infty, \infty)$. The Hörmander class of symbols of order m is denoted by S^m and it is defined as the set of all functions $\sigma(x,\xi)$ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for any two multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$|(D^\alpha_x D^\beta_\xi \sigma)(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}$$

the pseudo-differential operator T_σ associated to a symbol σ is defined by

$$(T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix.\xi} \sigma(x,\xi)\hat{\varphi}(\xi).dx$$

Definition 3.2.3 *[2] Let A be a pseudo-differential operator of order m, and symbol p. If the total symbol $p(x,\xi)$ has an asymptotic expansion $p(x,\xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x,\xi)$, then the operator A is called classical pseudo-differential operator. Where the asymptotic expansion $p(x,\xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x,\xi)$ means $p(x,\xi) - \sum_{j=0}^{k-1} p_{m-j}(x,\xi)$ belongs to the Hörmander class $S^{Re\, m-k}$ for each k.*

Let Ψ^m denote the space of all ΨDO of order m. If $A \in \Psi^m$, then A maps $W^s_{hol}(\partial\Omega)$ into $W^{s-Re\, m}_{hol}(\partial\Omega)$ for any $s \in \mathbb{R}$. Let $\mathcal{H}_A$ be the completion of $C^\infty(\partial\Omega)$ with respect to the norm $\|u\|^2_A := < Au, u >_{\partial\Omega}$. Then $\mathcal{H}_A = W^{\frac{m}{2}}(\partial\Omega)$ as sets with equivalent norms [2].

Let A be a pseudo-differential operator of order m, the generalized Toeplitz operator $T_A : W^{Re\, m}_{hol}(\partial\Omega) \to H^2(\partial\Omega)$ is defined as $T_A = \Pi A$ where

$$\Pi : L^2(\partial\Omega) \to H^2(\partial\Omega)$$

is the orthogonal projection.

If the order m is non zero real number number, then A^r is also a pseudo-differential operator of order mr. Let $\mathcal{H}_{T_A}$ be the completion of $C^\infty_{hol}(\partial\Omega)$ with respect to the norm

$$\|x\|^2_{T_A} :=< T_A x, x >_{\partial\Omega}$$

then $\mathcal{H}_{T_A} = W^{m/2}_{hol}(\partial\Omega)$ with equivalent norms . The Poisson extension operator $\mathbf{K}$ acts from $W^s_{hol}(\partial\Omega)$ into $W^{s+1/2}_{hol}(\Omega)$. Therefore, $\mathbf{K}W^{m/2}_{hol}(\partial\Omega) = W^{(m+1)/2}_{hol}(\Omega)$.

Let ρ be a function on the bounded strictly pseudoconvex domain Ω such that: $\rho \in C^\infty(\overline{\Omega})$, $\rho > 0$ on Ω, and $\rho = 0, \nabla\rho \neq 0$ on $\partial\Omega$. For $\alpha > -1$ define the space

$$A^2_{\alpha,\rho} := \{f \in L^2(\Omega, c_{\alpha,\rho}\rho^\alpha dz) : f \textit{ is holomorphic on } \Omega\}$$

where $c_{\alpha,\rho} = (\int_\Omega \rho^\alpha dz)^{-1}$. We will show that this reproducing kernel Hilbert space coincide with a certain Sobolev space. Let $\alpha > -1$, and consider $f = \mathbf{K}u \in C^\infty_{hol}(\overline{\Omega})$.

$$\begin{aligned} \|f\|^2 = \int_\Omega |f|^2 \rho^\alpha dz &=< \rho^\alpha f, f >_\Omega \\ &=< \rho^\alpha \mathbf{K}u, \mathbf{K}u >_\Omega =< \mathbf{K}^*\rho^\alpha\mathbf{K}u, u >_{\partial\Omega} \\ &=< T_{\mathbf{K}^*\rho^\alpha\mathbf{K}}u, u > \end{aligned}$$

The operator $\mathbf{K}^*\rho^\alpha\mathbf{K}$ is a pseudo-differential operator of order $-(\alpha+1)$. Therefore,

$$A^2_{\alpha,\rho} = \mathbf{K}W^{-(\alpha+1)/2}_{hol}(\partial\Omega) = W^{-\alpha/2}_{hol}(\Omega)$$

this equality says the space $A^2_{\alpha,\rho}$ does not depend on ρ.

Theorem 3.2.1 *[7]*

For $\alpha > -1$ the function

$$C(\alpha) := \int_\Omega \rho^\alpha(z) dz$$

extends to a holomorphic function on $\mathbb{C} - \{-1, -2, -3, ...\}$.

For $\alpha \in \mathbb{R} - \{-1, -2, -3, ...\}$, define the space

$$A^2_{\alpha,\#} := W^{-\alpha/2}_{hol}(\Omega)$$

with norm

$$\|f\|^2_{\alpha\# m\rho} := \sum_{|v|\leq n} \|\partial^v f\|^2_{\alpha+2m,\rho}$$

where m is nonnegative integer with $m > -\dfrac{\alpha+1}{2}$. The function f is in the space $A^2_{\alpha,\#}$ if and only if $\partial^v f \in A^2_{\alpha+2m,\rho}$ for all multi-indices v with $|v| \leq m$.An equivalent norm on $A^2_{\alpha,\#}$ is

$$\|f\|^2_{\alpha\flat\, m\rho} := \sum_{j=0}^{m} \left\| D^j f \right\|^2_{\alpha+2m,\rho}$$

where

$$D := \sum_{j=1}^{n} (\overline{\partial_j}\rho)\partial_j$$

For $\alpha > -1$, and $\phi \in L^\infty(\Omega)$ the Toeplitz operator $T^{(\alpha,\rho)}_\phi$ is defined by

$$T^{(\alpha,\rho)}_\phi : A^2_{\alpha,\rho} \to A^2_{\alpha,\rho}$$

$$T^{(\alpha,\rho)}_\phi f = P_\alpha(\phi f)$$

where $P_\alpha : L^2_{\alpha,\rho} \to A^2_{\alpha,\rho}$ is the orthogonal projection. The analytic continuation of $T^{(\alpha,\rho)}_\phi$ is defined as follows.

Definition 3.2.4 ([7]) *Let ρ be a positively signed defining function for Ω, $\phi \in L^\infty(\Omega)$, $\alpha_0 \in \mathbb{R}$, and m non negative integer such that $m > -\dfrac{\alpha_0+1}{2}$. Assume for each $f \in C^\infty_{hol}(\overline{\Omega})$, and $z \in \Omega$, the integral*

$$F^\circ_{\alpha,\phi,f}(z) := T^{(\alpha,\rho)}_\phi f(z)$$

can be analytically continued from $\alpha > -1$ as a holomorphic function in α to some neighborhood of α_0, the function $F^{\circ}_{\alpha,\phi,f}$ belongs to $A^2_{\alpha_0\#}$, and the operator $T : f \longmapsto F^{\circ}_{\alpha,\phi,f}$ extends to a bounded operator on $A^2_{\alpha_0\#}$. Then we say the Toeplitz operator $T_\phi^{(\circ,\alpha_0,\rho)}$ exists, and we set $T_\phi^{(\circ,\alpha_0,\rho)} := T$.

Let $Z_\rho := \{\alpha \in \mathbb{C} - \{-1,-2,-3,...\} : C_\rho(\alpha) = 0\}$. For $k = 0,1,2,...$ let $BC^k(\Omega)$ be the subspace of functions in $C^k(\Omega)$ with derivatives less than or equal to k are bounded.

$$BC^k(\mathbb{B}^n) := \{f \in C^k(\Omega) : \partial^\nu \overline{\partial^\mu} f \in L^\infty(\Omega) \ for \ every \ |\nu|+|\mu| \leq k\}$$

The following theorem shows that the existence of the analytic continuation of Toeplitz operators is guaranteed when the symbol ϕ is in $BC^k(\Omega)$.

Theorem 3.2.2 ([7]) *Let $k \geq 0$ be an integer. If $\phi \in BC^k(\Omega)$, then $T_\phi^{(\alpha,\rho)}$ exists for all $\alpha \in (-k-1,\infty) - (Z_\rho \cup \{-2,-3,-4,...\})$.*

Theorem 3.2.3 ([7]) *If $\phi \in L^\infty(\Omega)$ has a compact support on Ω, then $T_\phi^{(\alpha,\rho)}$ exists and unique for all $\alpha \in \mathbb{R} - Z_\rho$*

Chapter 4. C^*-algebra Generated by Toeplitz Operators

In this chapter we will construct commuting families of C^*-algebras. Representation theory can be used to find commuting families of C^*-algebras generated by Toeplitz operators, it has been shown that certain multiplicity free representations corresponds to commuting families of Toeplitz operators with symbols admitting an invariant property. In particular, if $H \subset G$ is a compact subgroup then $\pi_\alpha|_H$ is multiplicity free if and only if the family of Toeplitz operators over $A^2_\alpha(\mathbb{B}^n)$ with H-invariant symbols is commutative [4, Theorem 6.4].

4.1. Reproducing Kernel of $A^2_{\alpha,\#}(\mathbb{B}^n)$

The reproducing kernel of the weighted Bergman space $A^2_\alpha(\mathbb{B}^n)$ when $\alpha > -1$ is given by

$$K_\alpha(z,w) = (1 - z.\bar{w})^{-(\alpha+n+1)}$$

The authors [3] showed that the same formula gives the reproducing kernel when the we consider the weight $\alpha > -n-1$. I will use the symbol $K_{\alpha,\#}$ for the kernel of $A^2_{\alpha,\#}(\mathbb{B}^n)$, and K_α for the kernel of the space $A^2_\alpha(\mathbb{B}^n)$.

We will follow [3] in defining a norm on $A^2_{\alpha,\#}(\mathbb{B}^n)$. In particular, define the operator

$$N := \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}$$

Then $Nz^r = |r|z^r$ for all multi-indices r. Choose a non-negative integer m such that $m > -\dfrac{\alpha+1}{2}$. A function f is in $A^2_{\alpha,\#}(\mathbb{B}^n)$ if and only if $\partial^v f \in A^2_{\alpha+2m}(\mathbb{B}^n)$ for all multi-indices v with $|v| \leq m$ [2]. Therefore, a function f is in $A^2_{\alpha,\#}(\mathbb{B}^n)$ if and only if $N^k f \in A^2_{\alpha+2m}(\mathbb{B}^n)$ for all k with $0 \leq k \leq m$.

Define the operators

$$A := (I + \frac{N}{\alpha+n+1+m})...(I + \frac{N}{\alpha+n+1+2m-1})$$

$$B := (I + \frac{N}{\alpha+n+1})...(I + \frac{N}{\alpha+n+1+m-1})$$

The inner product on $A^2_{\alpha,\#}(\mathbb{B}^n)$ is defined by

$$< f, g >_{\alpha,\#} := < Af, Bg >_{\alpha+2m}$$

where the inner product $< Af, Bg >_{\alpha+2m}$ is taken over the usual weighted Bergman space $A^2_{\alpha+2m}(\mathbb{B}^n)$ [3]. The functions $\{z^r\}$ forms an orthogonal basis for the space $A^2_{\alpha,\#}(\mathbb{B}^n)$, and direct calculations show that

$$< z^r, z^s >_{\alpha,\#} = \ \delta_{r,s}\frac{r!\ \Gamma(\alpha+n+1)}{\Gamma(\alpha+n+1+|r|)} \tag{4.1.1}$$

where $\delta_{r,s} = 1$ when $r = s$, and zero otherwise.

The kernel of the weighted Bergman space $A^2_{\alpha+2m}(\mathbb{B}^n)$, is given by

$$K_{\alpha+2m}(z, w) = (1 - z.\overline{w})^{-(\alpha+n+1+2m)}$$

Let $f \in A^2_{\alpha,\#}(\mathbb{B}^n)$, we have

$$< f, (AB)^{-1}K_{\alpha+2m} >_{\alpha,\#}=< Af, B(AB)^{-1}K_{\alpha+2m} >_{\alpha+2m}=< Af, A^{-1}K_{\alpha+2m} >_{\alpha+2m}$$

$$=< f, AA^{-1}K_{\alpha+2m} >_{\alpha+2m}=< f, K_{\alpha+2m} >_{\alpha+2m}= f(z)$$

Therefore, the kernel of $A^2_{\alpha,\#}(\mathbb{B}^n)$ is

$$K_{\alpha,\#}(z, w) = \overline{(AB)^{-1}K_{\alpha+2m}})(w) = (1 - z.\bar{w})^{-\alpha-n-1}$$

where the last equality holds because for any non-zero s we have

$$(I + \frac{N}{s})(1 - \bar{z}.w)^{-s} = (1 - \bar{z}.w)^{-(s+1)}$$

This proves the following proposition.

Proposition 4.1.1 *[3] For $\alpha > -n-1$, the reproducing kernel of the space $A^2_{\alpha,\#}(\mathbb{B}^n)$ is given by*

$$K_{\alpha,\#}(z,w) = (1 - z.\bar{w})^{-\alpha-n-1}$$

4.2. Toeplitz Operators on $\mathbb{B}^n$

Let $I_{n,1}$ be the diagonal matrix $I_{n,1} := diag(1,1,...,-1)$. The map

$$<,>_{n,1}\colon \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}, \quad < z, w >_{n,1} := w^* I_{n,1} z$$

is a Hermitian form. The group $U(n,1)$ is the subgroup of $Gl(n+1,\mathbb{C})$ that preserves the Hermitian form

$$< Az, Bw >_{n,1} = < z, w >_{n,1}$$

for all matrices $A, B \in Gl(n+1,\mathbb{C})$. The group $SU(n,1)$ is the subgroup of $U(n,1)$ consisting of matrices with determinant one. The group $SU(n,1)$ acts transitively on the unit ball $\mathbb{B}^n$ by the fractional transform

$$\begin{pmatrix} a & v \\ w^t & d \end{pmatrix} .z := (az+v)(w^t z + d)^{-1}$$

where $a \in M_n(\mathbb{C})$, $d \in \mathbb{C}$, and $v, w \in \mathbb{C}^n$.

Toeplitz operators on $A^2_\alpha(\mathbb{B}^n)$ are given by

$$T^{(\alpha)}_\varphi : A^2_\alpha(\mathbb{B}^n) \to A^2_\alpha(\mathbb{B}^n)$$

$$T^{(\alpha)}_\varphi(f) = P_\alpha(f\,\varphi)$$

where P_α is the Bergman projection, and $\varphi \in L^\infty(\mathbb{B}^n)$.

The proof of the following lemma can be found in [4].

Lemma 4.2.1 *Let $\mathcal{L}(A^2_\alpha(\mathbb{B}^n))$ be the space of bounded linear operators on $A^2_\alpha(\mathbb{B}^n)$. For every $\alpha > -1$, the map*

$$L^\infty \to \mathcal{L}(A^2_\alpha(\mathbb{B}^n))$$

$$\varphi \longmapsto T^{(\alpha)}_\varphi$$

is injective.

Let $\pi : H \to \mathcal{H}$ be a unitary representation of a Lie group H. we will use the notation $End_H(\mathcal{H})$ as in [4] to denote the space of bounded operators that intertwine with the representation π. Multiplicity free representations are important in constructing commuting families of Toeplitz operators. If the Lie group H is of type I domain, then the representation π is multiplicity free if and only if $End_H(\mathcal{H})$ is commutative [12].

Let $\alpha > -1$, the representation

$$\pi_\alpha(g)(f)(z) := j_\alpha(g^{-1}, z) f(g^{-1}z)$$

is the holomorphic discrete series representation of $SU(n,1)$ on $A^2_\alpha(\mathbb{B}^n)$. Where

$$j_\alpha : SU(n,1) \times \mathbb{B}^n \to \mathbb{C}$$

$$j_\alpha\left(\begin{pmatrix} a & v \\ w^t & d \end{pmatrix}, z\right) = (w^t z + d)^{-\alpha-n-1}$$

We lift the map j_α to $\widetilde{SU(n,1)} \times \mathbb{B}^n$ to make it well defined for non integer values of α. Where $\widetilde{SU(n,1)}$ is the universal covering of $SU(n,1)$.

For $\varphi \in L^\infty(\mathbb{B}^n)$ and $h \in \widetilde{SU(n,1)}$, let $\varphi_h(z) := \varphi(h^{-1}z)$. It was shown in [4] that for $\alpha > -1$ we have

$$\pi_\alpha(g) \circ T_\varphi^{(\alpha)} = T_{\varphi_g}^{(\alpha)} \circ \pi_\alpha(g)$$

Let H be a subgroup of $\widetilde{SU(n,1)}$, we say φ is H-invariant if $\varphi_h(z) = \varphi(z)$ for every $h \in H$, and for a.e $z \in \mathbb{B}^n$. The following result says that, for a compact subgroup H, the space of Toeplitz operators with H-invariant symbols is dense in the space of operators that intertwine with the representation π.

Theorem 4.2.1 ([4]) *Let H be a compact subgroup of G. The space $\mathcal{T}^{(\alpha)}(L^\infty(\mathbb{B}^n)^H)$ of Toeplitz operators on $A^2_\alpha(\mathbb{B}^n)$ with H- invariant symbols is dense in $End_{\widetilde{H}}(A^2_\alpha(\mathbb{B}^n))$ of $\widetilde{H}$-intertwining operators on $A^2_\alpha(\mathbb{B}^n)$ under the strong operator topology.*

The following result characterize the commutativity of Toeplitz operators in terms of multiplicity free representations.

Theorem 4.2.2 ([4]) *Let $\widetilde{H}$ be a closed subgroup of $\widetilde{SU(n,1)}$. And $L^\infty(\mathbb{B}^n)^{\widetilde{H}}$ be the set of $\widetilde{H}$- invariant symbols.*

I- If for $\alpha > -1$ the algebra of bounded $\widetilde{H}$-intertwining operators for $\pi_\alpha|_{\widetilde{H}}$ is commutative, then the C^*-algebra generated by Toeplitz operators with $\widetilde{H}$- invariant symbols is commutative.

II- If $\widetilde{H}$ is compact, then the C^*-algebra generated by Toeplitz operators with $\widetilde{H}$-invariant symbols is commutative if and only if $\pi_\alpha|_{\widetilde{H}}$ is multiplicity free.

4.3. Representation of $SU(n,1)$ on $A^2_{\alpha,\#}(\mathbb{B}^n)$

In this section we will construct commuting families of C^*-algebras, these families are generated by the analytic continuation of Toeplitz operators. In particular, we will extend [5, Theorem 2] to include the case $\alpha > -n-1$. Moreover, we will prove a nice density theorem for the C^*-algebra generated by the analytic continuation of Toeplitz operators with symbols satisfying an invariant property.

Let $G = SU(n,1)$, and H be a subgroup of G. For $\alpha > -1$, we consider the representation of $\widetilde{SU(n,1)}$ on $A^2_\alpha(\mathbb{B}^n)$, defined by

$$\pi_\alpha(g)f(z) = j_\alpha(g^{-1}, z)f(g^{-1}.z)$$

The map j_α satisfy the cocycle relation

$$j_\alpha(gh, z) = j_\alpha(g, h.z)j_\alpha(h, z).$$

On the space $A^2_{\alpha,\#}(\mathbb{B}^n)$, the representation

$$\pi_\alpha(g)f(z) = j_\alpha(g^{-1}, z)f(g^{-1}.z)$$

is unitary. We only need to see the action on the kernel to show the unitarity of the representation. We will use the notation $K^w_{\alpha,\#}(z) := K_{\alpha,\#}(z, w)$.

The transformation rule of the Bergman kernel [18] says that; If $\Psi : \mathbb{B}^n \to \mathbb{B}^n$ is bi-holomorphic mapping, then

$$K(z, w) = \mathrm{Det}\Psi(z)' K(\Psi(z), \Psi(w))\overline{\mathrm{Det}\Psi(w)'}$$

where $\Psi(z)'$ is the complex derivative.

When we consider the action of the group $SU(n, 1)$ on $\mathbb{B}^n$, the transformation rule read as

$$j_\alpha(g, z)K_{\alpha,\#}(gz, gw)\overline{j_\alpha(g, w)} = K_{\alpha,\#}(z, w)$$

If we apply the representation π_α on $K^w_{\alpha,\#}(z)$, then we get

$$\pi_\alpha(g)K^w_{\alpha,\#}(z) = j_\alpha(g^{-1}, z)K^w_{\alpha,\#}(g^{-1}z) \tag{4.3.1}$$

and we have

$$
\begin{aligned}
j_\alpha(g,z)K^w_{\alpha,\#}(gz) =& j_\alpha(g,z)K_{\alpha,\#}(gz,w)\\
=& j_\alpha(g,z)\,\overline{j_\alpha(g,g^{-1}w)}\,K_{\alpha,\#}(gz,g(g^{-1}w))\,\overline{j_\alpha(g,g^{-1}w)^{-1}}\\
=& \overline{j_\alpha(g,g^{-1}w)^{-1}}\,K_{\alpha,\#}(z,g^{-1}w)\\
=& \overline{j_\alpha(g^{-1},w)}\,K^{g^{-1}w}_{\alpha,\#}(z)
\end{aligned}
$$

where $j_\alpha(g,g^{-1}w)^{-1} = j_\alpha(g^{-1},w)$ follows from the cocycle relation of j_α. Therefore, the representation in equation 4.3.1 becomes

$$\pi_\alpha(g)K^w_{\alpha,\#}(z) = j_\alpha(g^{-1},z)K^w_{\alpha,\#}(g^{-1}z) = \overline{j_\alpha(g,w)}K^{gw}_{\alpha,\#}(z)$$

The reproducing kernel has the property $K_{\alpha,\#}(z,w) =< K^w_{\alpha,\#}, K^z_{\alpha,\#} >$. Therefore,

$$
\begin{aligned}
< \pi_\alpha\, K^w_{\alpha,\#}, \pi_\alpha\, K^z_{\alpha,\#} >=& < \overline{j_\alpha(g,w)}K^{gw}_{\alpha,\#}, \overline{j_\alpha(g,z)}K^{gz}_{\alpha,\#} >\\
=& \overline{j_\alpha(g{\cdot}w)} < K^{gw}_{\alpha,\#},\, K^{gz}_{\alpha,\#} > j_\alpha(g,z)\\
=& j_\alpha(g,z)\,\overline{j_\alpha(g,w)}K_{\alpha,\#}(gz,gw)\\
=& < K^w_{\alpha,\#}, K^z_{\alpha,\#} >
\end{aligned}
$$

this shows that the representation π_α is unitary.

When working in the space $A^2_{\alpha,\#}(\mathbb{B}^n)$ we choose a non-negative integer m such that $m > -\dfrac{\alpha+1}{2}$. The space $A^2_{\alpha,\#}(\mathbb{B}^n)$ is independent of the integer m when $\alpha > -n-1$, because the inner product on the monomials

$$< z^r, z^s >_{\alpha,\#} = \delta_{r,s}\frac{r!\;\Gamma(\alpha+n+1)}{\Gamma(\alpha+n+1+|r|)}$$

and the kernel

$$K_{\alpha,\#}(z,w) = (1 - z.\bar{w})^{-\alpha-n-1}$$

are both independent of m. The problem when considering the case $\alpha < -n-1$ is that the space $A^2_{\alpha,\#}(\mathbb{B}^n)$ depends on the integer m, and we lose the unitarity of the representation π_α. So, we will focus on the analytic continuation case where $\alpha > -n-1$. Moreover, if $\alpha_0 > -n-1$, and m be a non negative integer such that $m > -\dfrac{\alpha_0+1}{2}$. The analytic continuation of the Toeplitz operator $T_\varphi^{(\alpha,\rho)}$ was denoted by $T_\varphi^{(\circ,\alpha_0,\rho)}$, in this section we will omit ρ because we can take $\rho(z) = (1-|z|^2)$. In particular, $T_\varphi^{(\circ,\alpha_0)}$ will represent the analytic continuation of $T_\varphi^{(\alpha)}$.

We have seen in Chapter 3, if $\varphi \in BC^n(\mathbb{B}^n)$, then $T_\varphi^{(\circ,\alpha)}$ exists for all $\alpha \in (-n-1,\infty) - (Z_\rho \cup \{-2,-3,-4,...\})$. From now on, when we write $\alpha > -n-1$, we will also mean that we are excluding the points in $Z_\rho \cup \{-2,-3,-4,...\}$.

Proposition 4.3.1 *Let $\mathcal{L}(A^2_{\alpha,\#}(\mathbb{B}^n))$ be the space of bounded linear operators on the weighted Bergman space $A^2_{\alpha,\#}(\mathbb{B}^n)$. For every $\alpha > -n-1$, the map*

$$BC^n(\mathbb{B}^n) \to \mathcal{L}(A^2_{\alpha,\#}(\mathbb{B}^n))$$

$$\varphi \longmapsto T_\varphi^{(\circ,\alpha)}$$

is injective.

Proof: For $\alpha > -n-1$, take $\varphi_1, \varphi_2 \in BC^n(\mathbb{B}^n)$ such that $T_{\varphi_1}^{(\circ,\alpha)} = T_{\varphi_2}^{(\circ,\alpha)}$. The restriction of α on $(-1,\infty)$ implies $T_{\varphi_1}^{(\alpha)} = T_{\varphi_2}^{(\alpha)}$, and $\varphi_1 = \varphi_2$ follows from Lemma 4.2.1.

□

The connection between representation theory and the commutativity of Toeplitz operators was presented in [4]. The following proposition is the cornerstone in finding commuting families of Toeplitz operators using representation theory. In particular, we need the Toeplitz operators to commute with the representation π_α, and this can be done by restricting the representation π_α to suitable subgroup of $\widetilde{SU(n,1)}$.

Let $\varphi \in BC^n(\mathbb{B}^n)$ and $h \in \widetilde{SU(n,1)}$, define $\varphi_h(z) := \varphi(h^{-1}z)$. The proof of the following proposition follows from [4] in the case where $\alpha > -1$, and the uniqueness of the analytic continuation for the case $\alpha > -n-1$.

Proposition 4.3.2 *For every Toeplitz operator $T_\varphi^{(\circ,\alpha)}$ we have*

$$\pi_\alpha(g) \circ T_\varphi^{(\circ,\alpha)} = T_{\varphi_g}^{(\circ,\alpha)} \circ \pi_\alpha(g)$$

Corollary 4.3.1 *Let $\widetilde{H}$ be a subgroup of $\widetilde{SU(n,1)}$. The symbol φ is $\widetilde{H}$-invariant if and only if $T_\varphi^{(\circ,\alpha)}$ intertwine with $\pi_\alpha|_{\widetilde{H}}$.*

Multiplicity free representations corresponds to commuting families of Toeplitz operators, we will show this also remains true when we consider the analytic continuation case. Let H be a Lie group of type I, and $\pi : H \to \mathcal{L}(\mathcal{H})$ be a unitary representation of H on a Hilbert space $\mathcal{H}$. The representation π admits a unique direct integral decomposition (up to unitary equivalence)

$$\pi \simeq \int_{\widehat{H}}^{\oplus} m_\pi(\tau)\tau d\mu(\tau)$$

where $\widehat{H}$ is the set of equivalence classes of irreducible unitary representation of H, μ is a Borel measure on $\widehat{H}$, and $m_\pi : \widehat{H} \to \mathbb{N} \cup \{\infty\}$ is the multiplicity function [12]. A unitary representation π is called multiplicity free if $m_\pi(\tau) \leq 1$ for almost all $\tau \in \widehat{H}$.

Let $End_H(\mathcal{H})$ be the set of operators that intertwine with the representation π, the ring $End_H(\mathcal{H})$ is commutative if and only if the representation π is multiplicity free [12]. Let $\widetilde{H}$ be a subgroup of $\widetilde{SU(n,1)}$, and $BC^n(\mathbb{B}^n)^{\widetilde{H}}$ be the set of symbols in $BC^n(\mathbb{B}^n)$ that are $\widetilde{H}$-invariant. Also, we will use $\mathcal{T}^{(\circ,\alpha)}(BC^n(\mathbb{B}^n)^{\widetilde{H}})$ to denote the C^*-algebra generated by Toeplitz operators on the Hilbert space $A^2_{\alpha,\#}(\mathbb{B}^n)$.

Lemma 4.3.1 ([7]) *Let Ω be a domain in $\mathbb{C}^n$ and $F(x,y)$ be analytic on $\Omega \times \bar{\Omega}$ such that $F(x,\bar{x}) = 0$ on Ω. Then F is identically zero on $\Omega \times \bar{\Omega}$.*

The following lemma appeared in the proof of Theorem 2 in [7]. We will need it to show that the space of generalized Toeplitz operators with $H-$invariant symbols is dense in $End_{\widetilde{H}}(A^2_{\alpha,\#}(\mathbb{B}^n))$.

Lemma 4.3.2 *Let W be a finite dimensional subspace of $A^2_{\alpha,\#}(\mathbb{B}^n)$ of dimension qr. Assume W has basis $\{f_1, ..., f_q, g_1, ..., g_r\}$. If $u \in \mathbb{C}^{q\times r}$ is such that*

$$\sum_j \sum_i < T_\varphi^{(\alpha)} f_i, g_j > \overline{u_{ij}} = 0$$

for every $\varphi \in BC^n(\mathbb{B}^n)$, then $u = 0$.

Proof: Suppose we have $u \in \mathbb{C}^{q\times r}$ such that

$$\sum_j \sum_i < T_\varphi^{(\alpha)} f_i, g_j > \overline{u_{ij}} = 0$$

for every φ. Then

$$\int_\Omega \varphi(z) \sum_j \sum_i \overline{u_{ij}} f_i(z) \overline{g_j(z)}.d\mu(z) = 0$$

for all $\varphi \in BC^n(\mathbb{B}^n)$. Therefore, we have

$$\sum_j \sum_i \overline{u_{ij}} f_i(z) \overline{g_j(z)} = 0 \tag{4.3.2}$$

almost everywhere on Ω. Since the left hand side of equation 4.3.2 is continuous, then the equation holds everywhere on Ω. Define the function

$$F(x,y) = \sum_j \sum_i \overline{u_{ij}} f_i(x) \overline{g_j(\overline{y})}$$

then $F(z,\bar{z}) = 0$ on Ω. Therefore, the function F is zero on $\Omega \times \overline{\Omega}$. Since the functions $f_i's$ and $g_i's$ are linearly independent, then $u_{ij} = 0$ for every i,j.

□

The C^*- algebra $\mathcal{T}^{(\alpha)}(BC^n(\mathbb{B}^n)^H)$, Toeplitz operators with H- invariant symbols, is dense in $End_{\widetilde{H}}(A^2_{\alpha,\#}(\mathbb{B}^n))$ under the strong operator topology, the following proposition shows that this density result still holds for the analytic continuation of Toeplitz operators with H- invariant symbols, the proof is close to the proof of [7, Theorem 2]. If $F(\alpha, x)$ is a function that depends on α, then we will use the notation $An_{\alpha_0}(F)$ to denote the analytic continuation of the function F to a neighborhood of α_0.

Proposition 4.3.3 *Let H be a compact subgroup of $SU(n,1)$. For every $\alpha > -n-1$, the space $\mathcal{T}^{(\circ,\alpha)}(BC^n(\mathbb{B}^n)^{\widetilde{H}})$ is dense in $End_{\widetilde{H}}(A^2_{\alpha,\#}(\mathbb{B}^n))$ the space of $\widetilde{H}$-intertwining operators on $A^2_{\alpha,\#}(\mathbb{B}^n)$ under the strong operator topology.*

Proof: Since H is compact, then $A^2_{\alpha,\#}(\mathbb{B}^n)$ decomposes into an orthogonal finite dimensional $\widetilde{H}-$invariant subspaces. Let $\alpha_0 > -n-1$ and W be a $\widetilde{H}-$invariant finite dimensional subspace of $A^2_{\alpha_0,\#}(\mathbb{B}^n)$ with basis $f_1, ..., f_q, g_1, ..., g_r$. Let T be a bounded operator on $A^2_{\alpha_0,\#}(\mathbb{B}^n)$, we will show that there exists $\varphi \in BC^n(\mathbb{B}^n)$ such that

$$< Tf_i, g_j > = < T^{(\alpha_0)}_{\varphi} f_i, g_j >$$

Define the operator

$$R : BC^n(\mathbb{B}^n) \to \mathbb{C}^{q\times r}$$

$$(R\varphi)_{ij} := < T^{(\alpha_0)}_{\varphi} f_i, g_j >$$

which can be written as

$$(R\varphi)_{ij} = An_{\alpha_0}(< T^{(\alpha)}_{\varphi} f_i, g_j >)$$

Suppose $u \in \mathbb{C}^{q\times r}$ is orthogonal to the image of the operator R, that is

$$\sum_j \sum_i < T_\varphi^{(\alpha_0)} f_i, g_j > \overline{u_{ij}} = 0$$

for every $\varphi \in BC^n(\mathbb{B}^n)$. So, we have

$$An_{\alpha_0}(\sum_j \sum_i < T_\varphi^{(\alpha)} f_i, g_j > \overline{u_{ij}}) = 0$$

which implies

$$\sum_j \sum_i < T_\varphi^{(\alpha)} f_i, g_j > \overline{u_{ij}} = 0$$

Therefore, by Lemma 4.3.2 we have $u_{ij} = 0$. Which means that the image of R is $\mathbb{C}^{q\times r}$.

Since W was an arbitrary finite dimensional $\widetilde{H}-$invariant subspace, then $\mathcal{T}^{(\circ,\alpha)}(BC^n(\mathbb{B}^n)^{\widetilde{H}})$ is dense in $End_{\widetilde{H}}(A^2_{\alpha,\#}(\mathbb{B}^n))$ in the weak operator topology. Moreover, the space $\mathcal{T}^{(\circ,\alpha)}(BC^n(\mathbb{B}^n)^{\widetilde{H}})$ is a convex subspace. Therefore, the space $\mathcal{T}^{(\circ,\alpha)}(BC^n(\mathbb{B}^n)^{\widetilde{H}})$ is dense in $End_{\widetilde{H}}(A^2_{\alpha,\#}(\mathbb{B}^n))$ in the strong operator topology.

□

One of our main goals is to establish a connection between the commutativity of a subspace of generalized Toeplitz operators and multiplicity free representations. First, we need the following Lemma. The proof can be found in [4, Lemma 6.3].

Lemma 4.3.3 *Suppose that V is a linear subspace of the space $\mathcal{L}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. If V consists of operators that commutes, then the closure of V in the strong operator topology also consists of operators that commutes.*

Theorem 4.3.1 *Let $\alpha > -n-1$.*

1) If $End_{\widetilde{H}}(A^2_{\alpha,\#}(\mathbb{B}^n))$ is commutative, then $\mathcal{T}^{(\alpha)}(BC^n(\mathbb{B}^n)^{\widetilde{H}})$ is a commutative C^*-algebra.

2) If $\widetilde{H}$ is compact, then $\mathcal{T}^{(\alpha)}(BC^n(\mathbb{B}^n)^{\widetilde{H}})$ is a commutative if and only if $\pi_\alpha|_{\widetilde{H}}$ is multiplicity free.

Proof:

(1) Using proposition 4.3.2 we have every Toeplitz operator with $\widetilde{H}$-invariant symbol commute with the representation π. Therefore $\mathcal{T}^{(\alpha)}(BC^n(\mathbb{B}^n)^{\widetilde{H}}) \subset End_{\widetilde{H}}(A^2_{\alpha,\#}(\mathbb{B}^n))$, and the commutativity of $End_{\widetilde{H}}(A^2_{\alpha,\#}(\mathbb{B}^n))$ implies $\mathcal{T}^{(\alpha)}(BC^n(\mathbb{B}^n)^{\widetilde{H}})$ is also commutative.

(2) If the C^*-algebra $\mathcal{T}^{(\alpha)}(BC^n(\mathbb{B}^n)^{\widetilde{H}})$ is commutative, then by proposition 4.3.3 and lemma 4.3.3, we have the space $End_{\widetilde{H}}(A^2_{\alpha,\#}(\mathbb{B}^n))$ is commutative. Therefore, the representation $\pi_\alpha|_{\widetilde{H}}$ is multiplicity free. Conversely, if the representation $\pi_\alpha|_{\widetilde{H}}$ is multiplicity free, then $End_{\widetilde{H}}(A^2_{\alpha,\#}(\mathbb{B}^n))$ is commutative, and the commutativity of $\mathcal{T}^{(\alpha)}(BC^n(\mathbb{B}^n)^{\widetilde{H}})$ follows from (1).

$\square$

4.4. Representation of $U(n)$ on $A^2_{\alpha,\#}(\mathbb{B}^n)$

We will need this section when we discuss the spectrum of Toeplitz operators with symbols invariant under $U(n)$ and $\mathbb{T}^n$. This will be done in chapter 5.

The group $U(n)$ acts on the unit ball $\mathbb{B}^n$ by

$$U(n) \times \mathbb{B}^n \to \mathbb{B}^n$$

$$(g, z) \mapsto g.z$$

this action induces an action on the space $A^2_{\alpha,\#}(\mathbb{B}^n)$. In particular, we have a unitary representation

$$\pi_\alpha : U(n) \times A^2_{\alpha,\#}(\mathbb{B}^n) \to A^2_{\alpha,\#}(\mathbb{B}^n)$$

$$\pi_\alpha(g)f(z) = f(g^{-1}z)$$

Let $\varphi \in BC^n(\mathbb{B}^n)$ and $h \in U(n)$, define $\varphi_h(z) := \varphi(hz)$. The following proposition characterizes the Toeplitz operators that intertwine with the representation π_α. The proof follows from proposition 4.3.2 by only restricting to the subgroup $U(n)$.

Proposition 4.4.1 *If $\alpha > -n-1$, then we have*

$$\pi_\alpha(h) \circ T_\varphi^{(\circ,\alpha)} = T_{\varphi_h}^{(\circ,\alpha)} \circ \pi_\alpha(h)$$

for all $\varphi \in BC^n(\mathbb{B}^n)$, and $h \in U(n)$.

Corollary 4.4.1 *Let H be a closed subgroup of $U(n)$. If $\varphi \in BC^n(\mathbb{B}^n)$ is $H-$ invariant, then*

$$\pi_\alpha(h) \circ T_\varphi^{(\circ,\alpha)} = T_\varphi^{(\circ,\alpha)} \circ \pi_\alpha(h)$$

for all $h \in U(n)$.

Let $\mathbb{T}^n$ be the subgroup of $U(n)$ consisting of diagonal matrices. The subgroup $\mathbb{T}^n$ acts on $\mathbb{B}^n$ by $(t,z) \mapsto tz$, and the representation on $A^2_{\alpha,\#}(\mathbb{B}^n)$ is also given by

$$\pi_\alpha : \mathbb{T}^n \times A^2_{\alpha,\#}(\mathbb{B}^n) \to A^2_{\alpha,\#}(\mathbb{B}^n)$$

$$\pi_\alpha(g)f(z) = f(g^{-1}z)$$

which is a unitary representation. The algebra $\mathcal{P}(\mathbb{C}^n)$ of polynomial functions on $\mathbb{C}^n$ is dense in $A^2_{\alpha,\#}(\mathbb{B}^n)$. Moreover, $\mathcal{P}(\mathbb{C}^n)$ is $U(n)-$invariant, which allows us to decompose the space $A^2_{\alpha,\#}(\mathbb{B}^n)$ into a direct sum of Hilbert spaces.

The following result was proved in [14], the proof for the analytic continuation case , where $\alpha > -n-1$, is the same because $\mathcal{P}(\mathbb{C}^n)$ is $\mathbb{T}^n-$invariant, and dense in $A^2_{\alpha,\#}(\mathbb{B}^n)$. It shows that the space $A^2_{\alpha,\#}(\mathbb{B}^n)$ decompose into a direct sum of irreducible $\mathbb{T}^n-$modules.

Proposition 4.4.2 *The algebra $\mathcal{P}(\mathbb{C}^n)$ decomposes into an irreducible $\mathbb{T}^n-$modules*

$$\mathcal{P}(\mathbb{C}^n) = \sum_{m \in \mathbb{N}^n} \mathbb{C}z^m$$

where $\mathbb{C}z^a \not\approx \mathbb{C}z^b$ as $\mathbb{T}^n-$modules and $\mathbb{C}z^a \perp \mathbb{C}z^b$ whenever $a \neq b$. Moreover, for every $\alpha > -n-1$ we have

$$A^2_{\alpha,\#}(\mathbb{B}^n) = \bigoplus_{m \in \mathbb{N}^n} \mathbb{C}z^m$$

as orthogonal Hilbert spaces.

When $\alpha > -1$, the space $A^2_{\alpha,\#}(\mathbb{B}^n)$ also admits a decomposition into a direct sum of irreducible $U(n)-$modules, this is a well known result and the proof can be found in [14]. The decomposition also holds for the case $\alpha > -n-1$, because the algebra $\mathcal{P}(\mathbb{C}^n)$ is $U(n)-$invariant and dense in $A^2_{\alpha,\#}(\mathbb{B}^n)$.

Proposition 4.4.3 *Let $\mathcal{P}^k(\mathbb{C}^n)$ be the space of homogeneous polynomials of degree k, the space $\mathcal{P}^k(\mathbb{C}^n)$ is an irreducible $U(n)-$module, and the decomposition of $\mathcal{P}(\mathbb{C}^n)$ is given by*

$$\mathcal{P}(\mathbb{C}^n) = \sum_{k \in \mathbb{N}} \mathcal{P}^k(\mathbb{C}^n)$$

Moreover, if $\alpha > -n-1$, then

$$A^2_{\alpha,\#}(\mathbb{B}^n) = \bigoplus_{k \in \mathbb{N}} \mathcal{P}^k(\mathbb{C}^n)$$

as orthogonal Hilbert spaces.

Chapter 5. Spectral Representation of Toeplitz Operators

The main goal of this chapter is to diagonalize Toeplitz operators with symbols invariant under a subgroup of the group of all biholomorphisms of $\mathbb{B}^n$. We will use the notation $A^2_\lambda(\mathbb{B}^n)$ to denote the weighted Bergman space, and in this case $\lambda = \alpha + n + 1$. The kernel of the reproducing Hilbert space $A^2_\lambda(\mathbb{B}^n)$ is given by

$$K_\lambda(z, w) = (1 - < z, w >)^{-\lambda}$$

Also, we will use the notation $T^{(\lambda)}_\varphi$ to denote the Toeplitz operator for the case $\lambda > n$, and for the analytic continuation case when $\lambda > 0$. In particular, when we write $\lambda > 0$, we mean $\lambda \in \mathbb{R}^+ - Z_\rho$ where

$$Z_\rho := \{\lambda \in \mathbb{R} - \{n, n-1, n-2, ...\} : C_\rho(\lambda) = 0\}$$

and

$$C(\lambda) := \int_{\mathbb{B}^n} (1 - |z|^2)^{\lambda - n - 1} dz$$

these values of λ ensures the existence of the Toeplitz operators.

On the unit ball $\mathbb{B}^n$, the Segal-Bargman transform was used to derive a formula for the spectrum of the Toeplitz operators, which was given as an integral formula in terms of the symbols [15, 16]. We will extend the results in [5] to the case when the weight $\lambda > 0$. In particular, we will consider the class of symbols in $BC^n(\mathbb{B}^n)$ that are invariant under maximal abelian subgroups of $SU(n, 1)$, and we will show that Toeplitz operators can be realized as a convolution operator. Moreover, we will use the Fourier transform to find the the spectral representation of the Toeplitz operators.

The group $U(n)$ is not a maximal abelian subgroup of $SU(n, 1)$, so we will adapt another approach to describe the spectrum of Toeplitz operators with $U(n)-$invariant symbols. In particular, we will show that the work in [14] still holds in the analytic continuation case.

5.1. The Restriction Principle

Let V be a vector space and J an $\mathbb{R}$-linear endomorphism such that $J^2 = -I$, then J is called a complex structure on V. If the a vector space V has a complex structure J, then we can define the complex multiplication

$$(a + ib).v = av + bJv$$

where $a, b \in \mathbb{R}$. In other words, equipping a vector space with a complex structure allows us to view it as a complex vector space. An almost complex structure J on a smooth manifold M is a differentiable vector bundle isomorphism

$$J : T(M) \to T(M)$$

such that

$$J_x : T_x(M) \to T_x(M)$$

is a complex structure on $T_x(M)$, that is $J_x^2 = -I$ for every x in the manifold M. A real submanifold S of M is totally real if

$$T_xS \cap J_x(T_xS) = \{0\} \quad for \ \ any \ \ x \in S$$

Let $M \subset \mathbb{B}^n$ be a submanifold. If $f \mapsto f|_M$ is injective on the space of holomorphic functions, then M is called **restriction injective** [5]. Let H be a closed subgroup of $G := SU(n, 1)$, and $\widetilde{H}$ be the inverse image of H in the universal covering of G. Assume that the submanifold $M := H.\, z_0 = \widetilde{H}.\, z_0$ is restriction injective, then there exists a measure on M that is invariant under the submanifolds H and $\widetilde{H}$ [5].

Proposition 5.1.1 *For $\lambda > 0$, the map $\chi_\lambda : \tilde{H}_{z_0} \to \mathbb{C}$, defined by*

$$\chi_\lambda(h) = j_\lambda(h, z_0)^{-1}$$

is a unitary character satisfying

$$j_\lambda(hk, z_0) = j_\lambda(h, z_0)\chi_\lambda(k)^{-1}$$

for all $h \in \tilde{H}$, $k \in \tilde{H}_{z_0}$.

Proof: The case for $\lambda > n$ was proved in [5], and the case when $\lambda > 0$ is almost the same, in fact, both of the them follows from the cocycle relation. Assume $\lambda > 0$, and k_1, $k_2 \in \tilde{H}_{z_0}$, then we have

$$\chi_\lambda(k_1 k_2) = j_\lambda(k_1 k_2, z_0)^{-1} = j_\lambda(k_1, k_2.z_0)^{-1} j_\lambda(k_2, z_0)^{-1}$$

since $k_2 \in \tilde{H}_{z_0}$ then $k_2.z_0 = z_0$. Therefore

$$\chi_\lambda(k_1 k_2) = j_\lambda(k_1, z_0)^{-1} j_\lambda(k_2, z_0)^{-1} = \chi_\lambda(k_1)\chi_\lambda(k_2)$$

so the map χ_λ is a multiplicative character. Moreover, for $h \in \tilde{H}$ and $k \in \tilde{H}_{z_0}$

$$j_\lambda(hk, z_0) = j_\lambda(h, k.z_0) j_\lambda(k, z_0) = j_\lambda(h, z_0)\chi_\lambda(k)^{-1}$$

□

For $\lambda > n$ the character χ_λ is a representation of $\tilde{H}_{z_0}$, so we can consider the induced representation $\mathrm{Ind}_{\tilde{H}_{z_0}}^{\tilde{H}} \chi_\lambda$, we will denote the induced representation by τ_λ. The Hilbert space for the induced representation is denoted by $L^2_{\chi_\lambda}(M, d\mu)$ which consists of the functions $f : \tilde{H} \to \mathbb{C}$ that are square integrable with respect to the invariant measure

μ and they satisfy

$$f(kh) = \chi_\lambda(h)^{-1} f(k) \tag{5.1.1}$$

for all $k \in \tilde{H}$ and $h \in \tilde{H}_{z_0}$. Moreover, $\widetilde{H}$ acts by

$$h.f(k) = f(h^{-1}k)$$

which is a unitary representation, and the space M is identified with $\tilde{H}/\tilde{H}_{z_0} \cong H/H_{z_0}$.

There are five conjugacy class of maximal abelian subgroups of $SU(n,1)$. In this section, when we write a subgroup H of $SU(n,1)$ we mean a maximal abelian subgroup that is either quasi-elliptic, quasi-parabolic or quasi-hyperbolic. We will define a differential operator N for each one of them. We will consider these subgroups in more detail in **section 5.2**.

The quasi-elliptic abelian subgroup corresponds to the maximal compact torus in $SU(n,1)$, and it is given by

$$E(n) = \left\{ k_{t,a} = \begin{pmatrix} at_1 & & & & \\ & at_2 & & & \\ & & \ddots & & \\ & & & at_n & \\ & & & & a \end{pmatrix} \middle| \begin{smallmatrix} a,t_1,\ldots,t_n \in \mathbb{T} \\ \mathrm{Det}(k_{t,a})=1 \end{smallmatrix} \right\}$$

For this subgroup we define the operator $N_{E(n)} := 0$ to be the zero operator.

The Quasi-parabolic subgroup can be written as a subgroup of $SU(n,1)$ as:

$$P(n) = \left\{ p_{t,y,a} = \begin{pmatrix} at_1 & & & & & \\ & at_2 & & & & \\ & & \ddots & & & \\ & & & at_{n-1} & & \\ & & & & a(1+i\frac{y}{2}) & a(i\frac{y}{2}) \\ & & & & a(-i\frac{y}{2}) & a(1-i\frac{y}{2}) \end{pmatrix} \middle| \begin{smallmatrix} a,t_1,\ldots,t_{n-1} \in \mathbb{T},\, y \in \mathbb{R} \\ a^{n+1} t_1 \ldots t_{n-1} = 1 \end{smallmatrix} \right\}$$

For this subgroup we define the operator $N_{P(n)} := y\dfrac{\partial}{\partial y}$

The quasi-hyperbolic can be written as a subgroup of $SU(n,1)$ as:

$$H(n) = \left\{ h_{t,s,a} = \begin{pmatrix} at_1 & & & & & \\ & at_2 & & & & \\ & & \ddots & & & \\ & & & at_{n-1} & & \\ & & & & a\cosh s & a\sinh s \\ & & & & a\sinh s & a\cosh s \end{pmatrix} \middle| \begin{smallmatrix} a,t_1,\dots,t_{n-1}\in\mathbb{T},\, s\in\mathbb{R} \\ a^{n+1}t_1\dots t_{n-1}=1 \end{smallmatrix} \right\}$$

For this subgroup we define the operator $N_{H(n)} := \dfrac{\sinh\, s}{\cosh\, s}\dfrac{\partial}{\partial\, s}$

Suppose $n \geq \lambda > 0$, let m be the smallest non-negative integer such that $\lambda + 2m > n$. Let H ba a maximal abelian subgroup of $SU(n,1)$, define the operators:

$$\begin{aligned} A_\lambda &:= (I + \frac{N_H}{\lambda + 2m - 1})...(I + \frac{N_H}{\lambda + m}) \\ B_\lambda &:= (I + \frac{N_H}{\lambda + m - 1})...(I + \frac{N_H}{\lambda}) \end{aligned} \tag{5.1.2}$$

We define the space $\mathcal{V}_\lambda$ to be the space of analytic functions $f \ : \ \widetilde{H} \ \to \ \mathbb{C}$ such that $A_\lambda B_\lambda\, f \in L^2_{\chi_{\lambda+2m}}(M, d\mu)$ and satisfy

$$f(hh_1) = \chi_\lambda(h_1)^{-1} f(h)$$

for all $h \in \tilde{H}$, and $h_1 \in \tilde{H}_{z_0}$. Moreover, the space $\mathcal{V}_\lambda$ is equipped with the inner product

$$< f, g >_\lambda := < A_\lambda B_\lambda\, f, A_\lambda B_\lambda\, g >_{L^2_{\chi_{\lambda+2m}}}$$

If $\lambda > n$, then we take $A_\lambda = B_\lambda = I$.

Define the map $D_\lambda : \widetilde{H} \to \mathbb{C}$ by $D_\lambda(h) = j_\lambda(h, z_0)$, the cocycle relation of j_λ implies

$$D_\lambda(hh_1) = \chi_\lambda(h_1)^{-1} D_\lambda(h)$$

If we apply the operators $A_\lambda B_\lambda$ on D_λ, then we have

$$A_\lambda B_\lambda\, D_\lambda(h) = c D_{\lambda+2m}(h) \tag{5.1.3}$$

in the quasi-elliptic and quasi-parabolic case, where c is a constant that depends on h. Therefore, D_λ belongs to the space $\mathcal{V}_\lambda$. In fact, equation 5.1.3 does not hold for the quasi-hyperbolic case, but we still have $D_\lambda \in \mathcal{V}_\lambda$ even in this case, we will prove this in section 5.2.

Now, define the restriction operator $R_\lambda : A^2_\lambda(\mathbb{B}^n) \to \mathcal{V}_\lambda$ by

$$R_\lambda(f)(h) = D_\lambda(h) f|_M(h.z_0)$$

and let $\mathcal{S}_\lambda$ be the closure of $R_\lambda(A^2_\lambda(\mathbb{B}^n))$ in $\mathcal{V}_\lambda$. The point evaluation maps are continuous in $A^2_\lambda(\mathbb{B}^n)$, it follows that R_λ is closed. Therefore, the operator $R^*_\lambda : \mathcal{S}_\lambda \to A^2_\lambda(\mathbb{B}^n)$ is well-defined and

$$\begin{aligned} R^*_\lambda f(z) &= < R^*_\lambda f, K_z >_\lambda = < f, R_\lambda K_z >_{\mathcal{S}_\lambda} \\ &= \int_{\widetilde{H}/\widetilde{H}_{z_0}} A_\lambda B_\lambda f(h) \, \overline{A_\lambda B_\lambda \Big(D_\lambda(h) K(h.z_0, z) \Big)} \, d\mu(h) \end{aligned}$$

Using integration by parts, the operator R^*_λ can be written as

$$R^*_\lambda f(z) = \int_{\widetilde{H}/\widetilde{H}_{z_0}} f(h) \; Q \, \overline{A_\lambda B_\lambda \Big(D_\lambda(h) K(h.z_0, z) \Big)} \, d\mu(h)$$

where Q is a polynomial differential operator acting on h.

The following proposition shows that the restriction operator R_λ intertwine the representations $(\pi_\lambda|_{\widetilde{H}}, A^2_\lambda(\mathbb{B}^n))$ and $(\tau_\lambda, \mathcal{S}_\lambda)$ where τ_λ is the left regular representation on $\mathcal{S}_\lambda$.

Proposition 5.1.2 *If $f \in A^2_\lambda(\mathbb{B}^n)$, and $h, h_1 \in \widetilde{H}$, then* $R_\lambda(\pi_\lambda(h_1) f)(h) = (\tau_\lambda(h_1) R_\lambda f)(h)$

Proof:

$$\begin{aligned}
R_\lambda(\pi_\lambda(h_1)f)(h) =& D_\lambda(h)(\pi_\lambda(h_1)f)|_M(h.z_0) \\
=& D_\lambda(h)j_\lambda(h_1^{-1}, h.z_0)f(h_1^{-1}h.z_0) \\
=& j_\lambda(h, z_0)j_\lambda(h_1^{-1}, h.z_0)f(h_1^{-1}h.z_0) \\
=& j_\lambda(h_1^{-1}h, z_0)f(h_1^{-1}h.z_0) \\
=& \tau_\lambda(h_1)(j_\lambda(h, z_0)f(h.z_0)) \\
=& (\tau_\lambda(h_1)R_\lambda f)(h)
\end{aligned}$$

We will need the following lemma in the next section. Following [5], for $h, k \in \widetilde{H}$ we define the operator

$$R_\lambda(h,k) = D_\lambda(h)\, Q\overline{A_\lambda B_\lambda(D_\lambda(k)K_\lambda(k.z_0, h.z_0))}$$

Lemma 5.1.1 *Let $f \in \mathcal{S}_\lambda$, then*

$$R_\lambda R_\lambda^*(f)(h) = \int_M f(k)R_\lambda(h,k)\, d\mu(k)$$

Proof:

$$R_\lambda^*(f)(z) = \int_{\widetilde{H}/\widetilde{H}_{z_0}} f(h)\, Q\, \overline{A_\lambda B_\lambda\Big(D_\lambda(h)K(h.z_0, z)\Big)}\, d\mu(h)$$

Therefore, $R_\lambda R_\lambda^*(f)$ becomes

$$\begin{aligned}
R_\lambda R_\lambda^*(f)(h) =& R_\lambda\Big(\int_{\widetilde{H}/\widetilde{H}_{z0}} f(k)\, Q\, \overline{A_\lambda B_\lambda(D_\lambda(k)K_\lambda(k.z_0, z))}.d\mu(k)\Big) \\
=& D_\lambda(h) \int_{\widetilde{H}/\widetilde{H}_{z0}} f(k)\, Q\, \overline{A_\lambda B_\lambda(D_\lambda(k)K_\lambda(k.z_0, h.z_0))}.d\mu(k) \\
=& \int_M f(k)R_\lambda(h,k)\, d\mu(k)
\end{aligned}$$

$\square$

Since the operator $R_\lambda R_\lambda^*$ is a positive operator, then we can consider $\sqrt{R_\lambda R_\lambda^*}$. The polar decomposition says that there exists a unitary isomorphism $U_\lambda : \mathcal{S}_\lambda \to A^2_\lambda(\mathbb{B}^n)$ such that

$$R_\lambda^* = U_\lambda \sqrt{R_\lambda R_\lambda^*}$$

the operator U_λ is called the Segal-Bargmann transform. Since the operator R_λ intertwine the representations $\pi_\lambda|_{\widetilde{H}}$, and τ_λ, then U_λ is a unitary $\widetilde{H}$-isomorphism. The above discussion can be summarized in the following theorem, which shows that [5, Theorem 1] holds for $\lambda > 0$.

Theorem 5.1.1 *The Segal-Bargmann transform* $U_\lambda : \left(\mathcal{S}_\lambda, \tau_\lambda\right) \to \left(A^2_\lambda(\mathbb{B}^n), \pi_\lambda|_{\widetilde{H}}\right)$ *is a unitary* $\widetilde{H}-$*isomorphism.*

If $\widetilde{H}$ is Type I subgroup, then the representation $\pi_\lambda|_{\widetilde{H}}$ admits a unique direct integral decomposition (up to a unitary equivalence)

$$\pi_\lambda|_{\widetilde{H}} \simeq \int_{\widehat{\widetilde{H}}}^{\oplus} m_{\pi_\lambda}(\sigma)\sigma d\mu_\lambda(\sigma)$$

where $\widehat{\widetilde{H}}$ is the set of equivalence classes of irreducible unitary representation of $\widetilde{H}$, μ_λ is a Borel measure on $\widehat{\widetilde{H}}$, and $m_{\pi_\lambda} : \widehat{\widetilde{H}} \to \mathbb{N} \cup \{\infty\}$ is the multiplicity function. Moreover, multiplicity free representations corresponds to commutative C^*-algebras, in particular, Theorem 4.3.1 says that If $\widetilde{H}$ is compact, then $\mathcal{T}^{(\lambda)}(BC^n(\mathbb{B}^n)^{\widetilde{H}})$ is commutative if and only if $\pi_\lambda|_{\widetilde{H}}$ is multiplicity free. Additionally, if an operator commutes with the representation π_λ, then the operator also has a direct integral decomposition. Furthermore, Corollary 4.3.1 says that Toeplitz operator $T^{(\lambda)}_\varphi$ is an intertwining operator for π_λ if and only if the symbol φ is $\widetilde{H}-$invariant. Since the Segal-Bargmann transform U_λ commute with π_λ then the diagonalization of the Toeplitz operator is

$$U_\lambda^* T_\varphi^{(\lambda)} U_\lambda = \int_{\widehat{\widehat{H}}}^{\oplus} m_{\varphi,\lambda}(\sigma) id_{\mathcal{H}_\sigma} d\mu_\lambda(\sigma)$$

where $m_{\varphi,\lambda} : \widehat{\widehat{H}} \to \mathbb{C}$. The set $(m_{\varphi,\lambda}(\sigma))_\sigma$ is the spectrum of the Toeplitz operator $T_\varphi^{(\lambda)}$.

5.2. Maximal Abelian Subgroups of $SU(n,1)$

Our main goal in this section is to describe the spectral decomposition of Toeplitz operators with symbols that are invariant under the maximal abelian subgroups of $SU(n,1)$. In particular, we will apply the restriction principle to write the Toeplitz operators as a convolution operator on certain sections of line bundles. In fact, we will show that the work in [5] can be generalized to the analytic continuation case.

There are five conjugacy classes of maximal abelian subgroups of $SU(n,1)$. It is easier to describe their action on an unbounded realization of the unit ball $\mathbb{B}^n$. Let D_n be the unbounded domain defined by

$$D_n = \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : Im(z_n) - |z'|^2 > 0\}$$

The Cayley transform defines a biholomorphism $\mathbb{B}^n \to D_n$, which is given by

$$z \mapsto \frac{i}{1+z_n}(z', 1-z_n)$$

with inverse

$$z \mapsto \frac{1}{1-iz_n}(-2iz', 1+iz_n)$$

Moreover, if we define

$$C := \begin{pmatrix} i & & & & \\ & \ddots & & & \\ & & i & & \\ & & & -i & i \\ & & & 1 & 1 \end{pmatrix}$$

then the group $CSU(n,1)C^{-1}$ realizes the group of biholomorphisms of D_n. Each maximal abelian subgroup is conjugate to one of the following representative [17, 5] :

Quasi-elliptic: This maximal abelian subgroup corresponds to the maximal compact torus in $SU(n,1)$, it is given by the $\mathbb{T}^n-$action on the unit ball

$$t.z = (t_1 z_1, ..., t_n z_n)$$

where $t \in \mathbb{T}^n$ and $z \in \mathbb{B}^n$.

Quasi-parabolic: This group is isomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}$, and it acts on D_n by

$$(t,y).(z^{'}, z_n) = (tz^{'}, z_n + y)$$

Quasi-hyperbolic: This group is isomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}^{+}$ and acts on D_n by

$$(t,r).(z^{'}.z_n) = (rtz^{'}, r^2 z_n)$$

Nilpotent: This group is isomorphic to $\mathbb{R}^{n-1} \times \mathbb{R}$ and acts on D_n by

$$(b,s).(z^{'}, z_n) = (z^{'} + b, z_n + 2i < z^{'}, b > +s + i|b|^2)$$

Quasi-nilpotent: This group is isomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}^{+}$ where $1 \leq k \leq n-2$, this group acts on D_n by

$$(t,b,s).(z^{'}, z^{''}, z_n) = (tz^{'}, z^{''} + b, z_n + 2i < z^{''}, b > +s + i|b|^2)$$

We will restrict our work to the first three classes of subgroups. In particular, when we say a maximal abelian subgroup, we mean it is either Quasi-elliptic, Quasi-parabolic, or Quasi-hyperbolic.

Let H be a maximal abelian subgroup of $SU(n,1)$, and $\widetilde{H}$ be the subgroup of $\widetilde{SU(n,1)}$ that covers H. The character $\chi_\lambda : H_{z_0} \to \mathbb{C}$ can be lifted to $\widetilde{H}$. This can be done by constructing a holomorphic embedding $\widetilde{H} \hookrightarrow H/H_{z_0} \times \mathbb{R}$, and χ_λ extends to $\widetilde{H}$ by defining

$$\chi_\lambda((h,x)) := e^{2\pi i \lambda x}$$

for all $h \in H/H_{z_0}$ and $x \in \mathbb{R}$. This construction will be clear when we examine each case of the maximal abelian subgroups of $SU(n,1)$ in the following subsections.

Let f be in $\mathcal{S}_\lambda$, and define $\widetilde{f} := f\chi_\lambda$, we will show that the operator $R_\lambda R_\lambda^*$ can be written as a convolution operator. In particular, we will show that

$$R_\lambda R_\lambda^* f = \chi_{-\lambda}.(\widetilde{f} * \phi_H)$$

where $\phi_H \in L^1(H/H_{z_0})$. Therefore, the space $\mathcal{S}_\lambda$ can be realized as

$$\mathcal{S}_\lambda = \{f \in \mathcal{V}_\lambda : (For\ \psi \in \widehat{H/H_{z_0}}\ such\ that\ \widehat{\phi_H}(\psi) = 0)\, \mathcal{F}_{H/H_{z_0}} \widetilde{f}(\psi) = 0\}$$

where $\mathcal{F}_{H/H_{z_0}}$ is the Fourier transform on H/H_{z_0}.

5.2.1. Quasi-elliptic

Following [5], the quasi-elliptic abelian subgroup corresponds to the maximal compact torus in $SU(n,1)$, and it is given by

$$E(n) = \left\{ k_{t,a} = \begin{pmatrix} at_1 & & & & \\ & at_2 & & & \\ & & \ddots & & \\ & & & at_n & \\ & & & & a \end{pmatrix} \middle| \begin{smallmatrix} a,t_1,\ldots,t_n \in \mathbb{T} \\ \mathrm{Det}(k_{t,a})=1 \end{smallmatrix} \right\}$$

The subgroup in the universal covering of $SU(n,1)$ that corresponds to $E(n)$ can be identified with

$$\widetilde{E(n)} = \{(t_1, t_2, ..., t_n, x) \in \mathbb{T}^n \times \mathbb{R} \,:\, e^{2\pi i(n+1)x} t_1 t_2 ... t_n = 1\}$$

and the product on $\widetilde{E(n)}$ is then given by

$$(t, x_1).(s, x_2) = (ts, x_1 + x_2)$$

Fix $z_0 = ((2n)^{-1/2}, ..., (2n)^{-1/2}) \in \mathbb{B}^n$. The action of $E(n)$ on z_0 is given by

$$k_{t,a}.z_0 = \frac{1}{\sqrt{2n}}(t_1, ..., t_n)$$

and the stabilizer of z_0 are given by

$$E(n)_{z_0} = \{k_{t,a} : t_i's = 1\} = \{k_{t,a} : a^{n+1} = 1\} \simeq \mathbb{Z}_n$$

$$\widetilde{E(n)_{z_0}} = \{(1, 1, ..., 1, \frac{m}{n+1}) : m \in \mathbb{Z}\} \simeq \mathbb{Z}$$

Let $q = (t, x) \in \widetilde{E(n)}$, then $D_\lambda(q) = e^{-2\pi i\lambda x}$. The differential operators A_λ and B_λ (5.1.2) acts on D_λ as an identity map. In particular, $A_\lambda B_\lambda(D_\lambda(q)) = D_\lambda(q)$. Therefore, $D_\lambda \in \mathcal{V}_\lambda$ for every $\lambda > 0$.

Let f be in $\mathcal{S}_\lambda$, by Lemma 5.1.1 we have that the operator $R_\lambda R_\lambda^*$ is given by the integral

$$R_\lambda R_\lambda^*(f)(h) = \int_M A_\lambda B_\lambda(f(k))R_\lambda(h, k)\, d\mu(k)$$

where

$$R_\lambda(h, k) = D_\lambda(h)\overline{A_\lambda B_\lambda(D_\lambda(k)K_\lambda(k.z_0, h.z_0))}$$

Let $h = (t, x)$ and $k = (s, y)$. Since the differential operators A_λ and B_λ acts as the identity map, then the operator $R_\lambda(h, k)$ can be written as

$$\begin{aligned}
R_\lambda(h, k) &= D_\lambda(h)\overline{A_\lambda B_\lambda(D_\lambda(k)K_\lambda(k.z_0, h.z_0))} \\
&= D_\lambda(h)\overline{D_\lambda(k)K_\lambda(k.z_0, h.z_0)} \\
&= D_\lambda(h)\overline{D_\lambda(k)}K_\lambda(h.z_0, k.z_0) \\
&= e^{-2\pi i\lambda(x-y)}(1- < h.z_0, k.z_0 >)^{-\lambda} \\
&= e^{-2\pi i\lambda(x-y)}(1 - \frac{1}{2n}\sum_{i=1}^{n} t_i(s_i)^{-1})^{-\lambda}
\end{aligned}$$

Furthermore, if $f \in \mathcal{S}_\lambda$, then the operator $R_\lambda R_\lambda^*$ becomes

$$R_\lambda R_\lambda^* f(t,x) = \int_{\mathbb{R}\times\mathbb{T}^n} f(s,y) e^{-2\pi i(x-y)} (1 - \frac{1}{2n}\sum_{i=1}^{n} t_i(s_i)^{-1})^{-\lambda} ds$$

$$= e^{-2\pi i(x-y)} (\widetilde{f} * \phi_{E(n)})(t)$$

where $\widetilde{f}(t) = e^{2\pi i\lambda x} f((t,x).z_0)$ for every $(t,x) \in \widetilde{E(n)}$, and the map $\phi_{E(n)}$ is given by

$$\phi_{E(n)}(t) = (1 - \frac{1}{2n}\sum_{i=1}^{n} t_i)^{-\lambda}$$

where $t \in \mathbb{T}^n$. The formulas for $R_\lambda R_\lambda^*$ and $R_\lambda(h,k)$ agrees exactly with work in [5] in the quasi-elliptic case, and the Fourier transform of $\phi_{E(n)}$ is given by

$$\widehat{\phi_{E(n)}}(\beta) = \begin{cases} (2n)^{-|\beta|} \dfrac{\Gamma(\lambda+|\beta|)}{\Gamma(\lambda)} \dfrac{1}{\beta_1! \ldots \beta_n!} & ;\ \beta_i' s \geq 0 \\ \\ 0 & ;\ otherwise \end{cases}$$

where $\beta = (\beta_1, ..., \beta_n) \in \mathbb{Z}^{n-1}$.

5.2.2. Quasi-parabolic

The quasi-parabolic subgroup is isomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}$, and it acts on D_n, the unbounded realization of the unit ball, by

$$(t,y).(z', z_n) = (tz', z_n + y)$$

where $t \in \mathbb{T}^{n-1}$, $y \in \mathbb{R}$ and $(z', z_n) \in D_n$.

The Quasi-parabolic can be written as a subgroup of $SU(n,1)$ as:

$$P(n) = \left\{ p_{t,y,a} = \begin{pmatrix} at_1 & & & & & \\ & at_2 & & & & \\ & & \ddots & & & \\ & & & at_{n-1} & & \\ & & & & a(1+i\frac{y}{2}) & a(i\frac{y}{2}) \\ & & & & a(-i\frac{y}{2}) & a(1-i\frac{y}{2}) \end{pmatrix} \middle| \begin{smallmatrix} a,t_1,\ldots,t_{n-1} \in \mathbb{T},\, y \in \mathbb{R} \\ a^{n+1} t_1 \ldots t_{n-1} = 1 \end{smallmatrix} \right\}$$

The group $P(n)$ acts on the unit ball $\mathbb{B}^n$ by

$$p_{t,y,a}.(z^{'}, z_n) = \Big(\frac{2}{-iyz_n + 2 - iy} tz^{'}, \frac{(2+iy)z_n + iy}{-iyz_n + 2 - iy}\Big)$$

Let $\widetilde{P(n)}$ be the subgroup of $\widetilde{SU(n,1)}$ that covers $P(n)$. This group can be identified with

$$\widetilde{P(n)} = \{(t, y, x) : t \in \mathbb{T}^{n-1}, x, y \in \mathbb{R} \quad and \quad e^{2\pi i(n+1)x} t_1 ... t_{n-1} = 1\}$$

with product

$$(t, y, x).(r, w, m) = (tr, y + w, x + m)$$

and the projection map is given by

$$\widetilde{P(n)} \to P(n)$$

$$(t, y, x) \mapsto p_{t,y,e^{2\pi ix}}$$

Let $z_0 = \Big(\frac{1}{\sqrt{2(n-1)}}, ..., \frac{1}{\sqrt{2(n-1)}}, 0\Big)$ be in $\mathbb{B}^n$. For each $q = (t, y, x) \in \widetilde{P(n)}$ we have

$$D_\lambda(q) = 2^\lambda e^{-2\pi i\lambda x}(2 - iy)^{-\lambda}$$

The operator $N = y\frac{\partial}{\partial y}$, and direct calculations shows that

$$A_\lambda B_\lambda(D_\lambda(q)) = 2^{\lambda+2m} e^{-2\pi i\lambda x}(2 - iy)^{-(\lambda+2m)}$$

Also, we have

$$\begin{aligned} |A_\lambda B_\lambda(D_\lambda(q))|^2 &= 2^{2(\lambda+2m)}|2 - iy|^{-2(\lambda+2m)} \\ &= \Big(1 + \frac{|y|^2}{4}\Big)^{-(\lambda+2m)} \end{aligned}$$

Therefore, $D_\lambda \in \mathcal{S}_\lambda$ for all $\lambda > \frac{1}{2} - 2m$.

Let $h = (t, y, x)$ and $k = (t^{'}, y^{'}, x^{'})$. The operator $R_\lambda(h, k)$ can be written as

$$
\begin{aligned}
R_\lambda(h,k) &= D_\lambda(h) Q\, \overline{A_\lambda B_\lambda(D_\lambda(k) K_\lambda(k.z_0, h.z_0))} \\
&= e^{-2\pi i(x-x^{'})}\, \overline{Q A_\lambda B_\lambda \Big(1 - \frac{i}{2}(y' - y) - \frac{1}{2(n-1)} \sum_{i=1}^{n-1} t_i' t_i^{-1}\Big)^{-\lambda}} \\
&= e^{-2\pi i(x-x^{'})} Q\, A_\lambda B_\lambda \Big((1 - \frac{i}{2}(y - y^{'}) - \frac{1}{2(n-1)} \sum_{i=1}^{n-1} t_i (t_i^{'})^{-1})^{-\lambda}\Big)
\end{aligned}
$$

where Q, A_λ and B_λ acts on $y^{'}$. Moreover, the operator Q has the form

$$
Q = c_1 + c_2 y \frac{\partial}{\partial y} + c_3 y^2 \frac{\partial^2}{\partial y^2} + ... + c_{2m} y^{2m} \frac{\partial^{2m}}{\partial y^{2m}}
$$

where $c_i^{'} s$ are constants .

If $f \in \mathcal{S}_\lambda$, then the operator $R_\lambda R_\lambda^*$ becomes

$$
\begin{aligned}
R_\lambda R_\lambda^* f(t,y,x) = \int\limits_{\mathbb{R}\times\mathbb{T}^{n-1}} & f(t^{'}, y^{'}, z^{'}) e^{-2\pi i (x - x^{'})}\ Q\, A_\lambda B_\lambda \Big((1 - \frac{i}{2}(y - y^{'}) \\
& - \frac{1}{2(n-1)} \sum_{i=1}^{n-1} t_i (t_i^{'})^{-1})^{-\lambda}\Big) dt^{'} dy^{'} \\
= {} & e^{-2\pi i \lambda x} (\widetilde{f} * \phi_{P(n)})(t, y)
\end{aligned}
$$

where $\widetilde{f}(t,y) = e^{2\pi i \lambda x} f(t,y,x)$ and the function $\phi_{P(n)}$ is given by

$$
\phi_{P(n)}(t,y) = Q A_\lambda B_\lambda \Big((1 - \frac{i}{2} y - \frac{1}{2(n-1)} \sum_{i=1}^{n-1} t_i)^{-\lambda}\Big)
$$

The function $\phi_{P(n)}$ is in $L^1(P(n))$ for all $\lambda > 0$. To prove this, first notice that

$$
AB\Big((1 - \frac{i}{2} y - \frac{1}{2(n-1)} \sum_{i=1}^{n-1} t_i)^{-\lambda}\Big) = \Big(1 - \frac{1}{2(n-1)} \sum_{i=1}^{n-1} t_i\Big)^{2m} \Big(1 - \frac{i}{2} y - \frac{1}{2(n-1)} \sum_{i=1}^{n-1} t_i\Big)^{-(\lambda+2m)}
$$

since $|\frac{1}{2(n-1)} \sum_{i=1}^{n-1} t_i| \leq 1$, then by following an argument similar to [5, p. 212] we get

$$
|\Big(1 - \frac{i}{2} y - \frac{1}{2(n-1)} \sum_{i=1}^{n-1} t_i\Big)^{-(\lambda+2m)}| \leq |1 + (|y| - 1)^2|^{-(\lambda+2m)/2} \tag{5.2.1}
$$

Since the operator Q is a polynomial differential operator, then it is enough to consider $y^r \frac{\partial^r}{\partial y^r}$. Equation 5.2.1 implies, $y^r \Big(1 - \frac{i}{2}y - \frac{1}{2(n-1)} \sum_{i=1}^{n-1} t_i\Big)^{-(\lambda+2m+r)}$ is in $L^1(P(n))$ for all $0 \leq r \leq 2m$. Therefore, $\phi_{P(n)}$ is in $L^1(P(n))$ for all $\lambda > 1 - 2m$.

5.2.3. Quasi-hyperbolic

The quasi-hyperbolic subgroup is isomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}^+$ and it acts on the unbounded domain D_n by

$$(t, r).(z^{'}, z_n) = (rtz^{'}, r^2 z_n)$$

As a subgroup of $SU(n, 1)$ it can be written as [5]

$$H(n) = \left\{ h_{t,s,a} = \begin{pmatrix} at_1 & & & & & \\ & at_2 & & & & \\ & & \ddots & & & \\ & & & at_{n-1} & & \\ & & & & a\cosh s & a\sinh s \\ & & & & a\sinh s & a\cosh s \end{pmatrix} \middle| \begin{smallmatrix} a, t_1, \ldots, t_{n-1} \in \mathbb{T},\, s \in \mathbb{R} \\ a^{n+1} t_1 \ldots t_{n-1} = 1 \end{smallmatrix} \right\}$$

and it acts on the unit ball $\mathbb{B}^n$ by

$$h_{t,s,a}.(z^{'}, z_n) = \Big(\frac{tz^{'}}{z_n \sinh s + \cosh s}, \frac{z_n \cosh s + \sinh s}{z_n \sinh s + \cosh s}\Big)$$

The group $H(1)$ is simply connected. So we have $H(1) \simeq \widetilde{H(1)} \simeq \mathbb{R}$. When $n > 1$, then $\widetilde{H(n)}$ can be identified with

$$\widetilde{H(n)} = \{(t, s, x) : t \in \mathbb{T}^{n-1},\, s, x \in \mathbb{R} \, : e^{2\pi i(n+1)x} t_1 t_2 \ldots t_{n-1} = 1\}$$

with

$$(t, s, x).(t^{'}, s^{'}, x^{'}) = (tt^{'}, s + s^{'}, x + x^{'})$$

Let $z_0 = (\frac{1}{\sqrt{2(n-1)}}, \ldots, \frac{1}{\sqrt{2(n-1)}}, 0)$, then we have

$$h_{t,s,a}.z_0 = (\frac{t}{\cosh s} z_0^{'}, \tanh s)$$

so, we have $H(n).z_0 \simeq H(n)/H(n)_{z_0} \simeq \mathbb{T}^{n-1} \times \mathbb{R}$.

Let $q = (t, s, x) \in \widetilde{H(n)}$, then we have $D_\lambda(q) = e^{-2\pi i\lambda x}(\cosh\, s)^{-\lambda}$. The operator N is defined by

$$N = \frac{\sinh\, s}{\cosh\, s}\frac{\partial}{\partial\, s}$$

Direct calculations show that

$$(I + \frac{N}{r})(\cosh\, s)^{-l} = (r-l)r^{-1}(\cosh\, s)^{-l} + lr^{-1}(\cosh\, s)^{-(l+2)} \tag{5.2.2}$$

and for all $l > 0$, we have

$$\int_{\mathbb{R}} (\cosh\, s)^{-l}\, ds\, < \infty$$

This with equation 5.2.2 implies

$$\int_{\mathbb{R}} A_\lambda B_\lambda((\cosh\, s)^{-\lambda})^2\, ds\, < \infty$$

Therefore, D_λ is in $\mathcal{S}_\lambda$ for all $\lambda > 0$.

Let $h = (t, s, x)$ and $k = (t', s', x')$. The operator $R_\lambda(h, k)$ can be written as

$$\begin{aligned} R_\lambda(h,k) &= D_\lambda(h) Q\, \overline{A_\lambda B_\lambda(D_\lambda(k) K_\lambda(k.z_0, h.z_0))} \\ &= e^{-2\pi i(x-x')}\, Q \overline{A_\lambda B_\lambda\Big(\cosh(s'-s) - \frac{1}{2(n-1)}\sum_{i=1}^{n-1} t_i' t_i^{-1}\Big)^{-\lambda}} \\ &= e^{-2\pi i(x-x')} Q\, A_\lambda B_\lambda\Big((\cosh(s-s') - \frac{1}{2(n-1)}\sum_{i=1}^{n-1} t_i (t_i')^{-1})^{-\lambda}\Big) \end{aligned}$$

where Q, A_λ and B_λ acts on s'. Moreover, the operator Q has the form

$$Q = a_1(s) + a_2(s)\tanh(s)\frac{\partial}{\partial s} + a_3 \tanh^2(s)\frac{\partial^2}{\partial s^2} + ... + a_{2m}(s)\tanh^{2m}(s)\frac{\partial^{2m}}{\partial s^{2m}}$$

and $a_i(s)$ are functions

$$a_i(s) = (\sum_j c_j \frac{\partial^j}{\partial\, s^j}) \tanh\, s = \sum_j c_j \tanh^{(j)}(s)$$

where $c_i's$ are constants.

If $f \in \mathcal{S}_\lambda$, then the operator $R_\lambda R_\lambda^*$ becomes

$$R_\lambda R_\lambda^* f(t,y,x) = \int\limits_{\mathbb{R}\times\mathbb{T}^{n-1}} f(t^{'},y^{'},z^{'})e^{-2\pi i(x-x^{'})}\, Q\, A_\lambda B_\lambda\Big((\cosh(s-s^{'}) - \frac{1}{2(n-1)}\sum_{i=1}^{n-1} t_i(t_i^{'})^{-1})^{-\lambda}\Big)dt^{'}ds^{'}$$

$$= e^{-2\pi i\lambda x}(\widetilde{f} * \phi_{H(n)})(t,s)$$

where $\widetilde{f}(t,s) = e^{2\pi i\lambda x} f(t,s,x)$ and the function $\phi_{P(n)}$ is given by

$$\phi_{H(n)}(t,s) = QA_\lambda B_\lambda\Big((\cosh\ s - \frac{1}{2(n-1)}\sum_{i=1}^{n-1} t_i)^{-\lambda}\Big)$$

Remark 5.2.1 *Direct calculations shows that*

$$A_\lambda B_\lambda((\cosh s)^{-\lambda}) = \sum_{j=1}^{2m} c_j(\cosh s)^{-(\lambda+2j)}$$

where $c_j's$ are constants. Also, we have

$$A_\lambda B_\lambda\Big((\cosh s\ - \frac{1}{2(n-1)}\sum_{i=1}^{n-1} t_i)^{-\lambda}\Big) = \sum_{j=1}^{2m}\gamma_j(s)(\cosh s - \frac{1}{2(n-1)}\sum_{i=1}^{n-1} t_i)^{-(\lambda+2j)}$$

where $\gamma_j(s)$ are functions that are constant multiples of $\operatorname{sech} s$ *and* $\tanh s$*. Now consider*

$$\frac{\partial}{\partial s}(\gamma_j(s)(\cosh s - \frac{1}{2(n-1)}\sum_{i=1}^{n-1} t_i)^{-(\lambda+2j)}) = \gamma_j^{'}(s)(\cosh s - \frac{1}{2(n-1)}\sum_{i=1}^{n-1} t_i)^{-(\lambda+2j)} + \gamma_j(s)(-\lambda-2j)\sinh s\ (\cosh s - \frac{1}{2(n-1)}\sum_{i=1}^{n-1} t_i)^{-(\lambda+2j+1)}$$

It is readily seen that

$$\frac{\partial^k}{\partial s^k}\Big(\gamma_j(s)(\cosh s - \frac{1}{2(n-1)}\sum_{i=1}^{n-1} t_i)^{-(\lambda+2j)}\Big) = \sum_{r=0}^{k} a_r(s)(\cosh s - \frac{1}{2(n-1)}\sum_{i=1}^{n-1} t_i)^{-(\lambda+2j+r)}$$

where $a_r(s)$ is a linear combination of $\gamma_j(s)^{(i_r)}$ and powers of $\sinh^{q_r}$ and $\cosh^{w_r}$, where the powers q_r, w_r are strictly less than $\lambda + 2j + r$. Since

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1} t_i \le \frac{1}{2}$$

it follows that

$$|\cosh s - \frac{1}{2(n-1)}\sum_{i=1}^{n-1} t_i|^{-\lambda} \le |\cosh s - \frac{1}{2}|^{-\lambda}$$

Moreover, $|\tanh\, s|$ and $|\operatorname{sech} s|$ are bounded above by 1, and

$$\int_{\mathbb{R}} (\sinh s)^q (\cosh s - \frac{1}{2})^{-v}\, ds < \infty$$

$$\int_{\mathbb{R}} (\cosh s)^w (\cosh s - \frac{1}{2})^{-v}\, ds < \infty$$

for all $q, w < v$ where $v > 1$. Therefore, we have $\phi_{H(n)} \in L^1(H(n).z_0)$ for all $\lambda > 0$.

5.3. $T_\varphi^{(\lambda)}$ with Symbols Invariant under Maximal Abelian Subgroups

In this section we will show that the analytic continuation of Toeplitz operators can be written as a convolution operator on sections of line bundle. It was shown in [5, Theorem 5.1] that for $\lambda > n$, the operator

$$R_\lambda T_\varphi^{(\lambda)} R_\lambda^* : \mathcal{S}_\lambda \to \mathcal{S}_\lambda$$

can be written as

$$R_\lambda T_\varphi^{(\lambda)} R_\lambda^* f = f * \nu_\varphi$$

we will show that we can extend it to cover the analytic continuation case. First, we will prove the following Lemma, which shows how the representation π_λ acts on the kernel of $A^2_\lambda(\mathbb{B}^n)$.

Lemma 5.3.1 *Let $g \in G$, and $z \in \mathbb{B}^n$, we have*

$$\pi_\lambda(g)K_z = \overline{j_\lambda(g,z)}K_{g.z}$$

proof: Let $f \in A^2_\lambda(\mathbb{B}^n)$, then we have

$$\begin{aligned}
< f, \pi_\lambda(g)K_z >= &< \pi_\lambda(g)^{-1}f, K_z > \\
= &< \pi_\lambda(g^{-1})f, K_z > \\
= &\pi_\lambda(g^{-1})f(z) = j_\lambda(g,z)f(gz) \\
= &< f, \overline{j_\lambda(g,z)}, K_{g.z} >
\end{aligned}$$

$\square$

Theorem 5.3.1 *Let $\lambda_0 > 0$ and H be a maximal abelian subgroup of $SU(n,1)$ such that $H.z_0$ is restriction injective. Assume $\varphi \in BC^n(\mathbb{B}^n)^H$ is $H-$invariant, then*

$$R_{\lambda_0}T^{(\lambda_0)}_\varphi R^*_{\lambda_0}f = A_{\lambda_0}B_{\lambda_0}(f) * \nu^{\lambda_0}_\varphi$$

where $\nu^{\lambda_0}_\varphi : H/H_{z_0} \to \mathbb{C}$ is given by

$$\nu^{\lambda_0}_\varphi(h.z_0) := An_{\lambda_0}\Big(\overline{A_{\lambda_0}B_{\lambda_0}(< \overline{j_\lambda(h,z_0)}K_{h.z_0,\lambda}, \;\; \varphi K_{z_0,\lambda} >_\lambda)}\Big)$$

and An_{λ_0} denotes the operation of taking the analytic continuation.

Proof: Let $h \in \widetilde{H}$, we have

$$\begin{aligned}
R_{\lambda_0} T_\varphi^{(\lambda_0)} R_{\lambda_0}^* f(h) &= D_{\lambda_0}(h) T_\varphi^{(\lambda_0)} R_{\lambda_0}^* f(h.z_0) \\
&= D_{\lambda_0}(h) < T_\varphi^{(\lambda_0)} R_{\lambda_0}^* f, K_{h.z_0,\lambda_0} >_{\lambda_0} \\
&= D_{\lambda_0}(h) An_{\lambda_0}\Big(< T_\varphi^{(\lambda)} R_{\lambda_0}^* f, K_{h.z_0,\lambda_0} >_{A_\lambda^2} \Big) \\
&= D_{\lambda_0}(h) An_{\lambda_0}\Big(< \varphi R_{\lambda_0}^* f, K_{h.z_0,\lambda_0} >_{A_\lambda^2} \Big) \\
&= D_{\lambda_0}(h) An_{\lambda_0}\Big(\int_{\mathbb{B}^n} \varphi(z) R_{\lambda_0}^* f(z) \overline{K_{h.z_0,\lambda_0}(z)}\, d\mu_\lambda(z) \Big) \\
&= D_{\lambda_0}(h) An_{\lambda_0}\Big(\int_{\mathbb{B}^n} \varphi(z) \int_{H/H_{z_0}} A_{\lambda_0} B_{\lambda_0}(f(k)) \overline{A_{\lambda_0} B_{\lambda_0}(D_{\lambda_0}(k) K_{z,\lambda_0}(k.z_0))}\, dk \\
&\qquad \overline{K_{h.z_0,\lambda_0}(z)}\, d\mu_\lambda(z) \Big)
\end{aligned}$$

So, the operator $R_{\lambda_0} T_\varphi^{(\lambda_0)} R_{\lambda_0}^* f(h)$ can be written as

$$An_{\lambda_0}\Big(\int\limits_{\mathbb{B}^n} \varphi(z) \Big[\int\limits_{H/H_{z_0}} A_{\lambda_0} B_{\lambda_0}(f(k)) \overline{A_{\lambda_0} B_{\lambda_0}(D_{\lambda_0}(k) K_{z,\lambda_0}(k.z_0) \overline{D_{\lambda_0}(h)} K_{h.z_0,\lambda_0}(z))dk} \Big]\, dz \Big)$$

and by interchanging the order of integration we get

$$An_{\lambda_0} \int\limits_{H/H_{z_0}} A_{\lambda_0} B_{\lambda_0}(f(k)) \overline{A_{\lambda_0} B_{\lambda_0}(< \overline{j_\lambda(h,z_0)} K_{h.z_0,\lambda},\ \varphi \overline{j_\lambda(k,z_0)} K_{k.z_0,\lambda} >_\lambda)} d\mu(k)$$

For $k \in \widetilde{H}$, define $W_k := < \overline{j_\lambda(h,z_0)} K_{h.z_0,\lambda},\ \varphi \overline{j_\lambda(k,z_0)} K_{k.z_0,\lambda} >_\lambda$. Then W_k can be written as

$$\begin{aligned}
W_k &= < \overline{j_\lambda(h,z_0)} \pi(k^{-1}) K_{h.z_0,\lambda},\ \overline{j_\lambda(k,z_0)} \pi(k^{-1})(\varphi K_{k.z_0,\lambda}) >_\lambda \\
&= < \overline{j_\lambda(h,z_0) j_\lambda(k^{-1},h.z_0)} K_{k^{-1}h.z_0,\lambda},\ \varphi \overline{j_\lambda(k,z_0) j_\lambda(k^{-1},k.z_0)} K_{z_0,\lambda} >_\lambda \\
&= < \overline{j_\lambda(k^{-1}h,z_0)} K_{k^{-1}h.z_0,\lambda},\ \varphi K_{z_0,\lambda} >_\lambda
\end{aligned}$$

Therefore,

$$R_{\lambda_0}T_{\varphi}^{(\lambda_0)}R_{\lambda_0}^{*}f(h) = \int\limits_{H/H_{z_0}} A_{\lambda_0}B_{\lambda_0}(f(k))\, An_{\lambda_0}\Big(\overline{A_{\lambda_0}B_{\lambda_0}(<\overline{j_\lambda(k^{-1}h,z_0)}K_{k^{-1}h.z_0,\lambda},\varphi K_{z_0,\lambda}>_\lambda)}\Big)$$

$\square$

Integration by parts implies the following corollary.

Corollary 5.3.1 *The operator $R_{\lambda_0}T_{\varphi}^{(\lambda_0)}R_{\lambda_0}^{*}$ can be written as*

$$R_{\lambda_0}T_{\varphi}^{(\lambda_0)}R_{\lambda_0}^{*}f(h) = f * Q\,\nu_{\varphi}^{\lambda_0}$$

where Q is a polynomial differential operator.

For $f \in \mathcal{S}_\lambda$, we can write

$$\begin{aligned} U_\lambda^{*}T_{\varphi}^{(\lambda)}U_\lambda f &= (\sqrt{R_\lambda R_\lambda^{*}})^{-1}\, R_\lambda T_{\varphi}^{(\lambda)}R_\lambda^{*}(\sqrt{R_\lambda R_\lambda^{*}})^{-1} \\ &= (R_\lambda R_\lambda^{*})^{-1}\, R_\lambda T_{\varphi}^{(\lambda)}R_\lambda^{*} \end{aligned}$$

This with Corollary 5.3.1 implies the following proposition, which gives the spectral representation of the Toeplitz operators.

Theorem 5.3.2 *Let $\lambda > 0$, H be a maximal abelian subgroup of $SU(n,1)$, and let $A = \{\psi \in \widehat{H/H_{z_0}} : \widehat{\phi_H^{\lambda}}(\psi) \neq 0\}$. If $\varphi \in BC^n(\mathbb{B}^n)^H$ is $H-$invariant. Then*

$$\mathscr{F}U_\lambda^{-1}T_{\varphi}^{(\lambda)}U_\lambda\mathscr{F}^{-1}\omega(\psi) = \frac{\widehat{Q\nu_{\varphi}^{\lambda}}(\psi)}{\widehat{\phi_H^{\lambda}}(\psi)}\omega(\psi)$$

for all $\psi \in A$, and $\omega \in L^2(\widehat{H/H_{z_0}})$ such that $supp\,\omega \subseteq supp\,\widehat{\phi_H^{\lambda}}$. Where $\mathscr{F}f(\psi) = \mathscr{F}_{H/H_{z_0}}\widetilde{f}(\psi)$

5.4. $T_{\varphi}^{(\lambda)}$ with Radial and Separately Radial Symbols

In this section we will consider the problem of finding the spectrum of the Toeplitz operators for a special class of symbols. In particular, we will consider the class of symbols that is invariant under the group $U(n)$, we can't apply the restriction principal in this case because $U(n)$ is not a maximal abelian subgroup of $SU(n,1)$. Our approach will follow the work in [14]. First, we recall that if π is a representation of a group G on a Hilbert space $\mathcal{H}$, and H is a closed subgroup of a group G, then $End_H(\mathcal{H})$, the algebra of bounded operators intertwining with with the representation π is commutative if and only if π is multiplicity free. In particular, $End_{\mathbb{T}^n}(A^2_\lambda(\mathbb{B}^n))$, and $End_{U(n)}(A^2_\lambda(\mathbb{B}^n))$ are commutative. This follows from **Proposition** 4.4.2, and 4.4.3.

The functions $\{z^r\}$ forms an orthogonal basis for the space $A^2_\lambda(\mathbb{B}^n)$, and direct calculations show that

$$\|z^r\|^2_\lambda = \frac{r!\ \Gamma(\lambda)}{\Gamma(\lambda+|r|)}$$

In this section we will consider an orthonormal basis $\{e_r(z)\}_{r\in\mathbb{N}^n}$ for $A^2_\lambda(\mathbb{B}^n)$ where

$$e_r(z) := \sqrt{\frac{\Gamma(\lambda+|r|)}{r!\ \Gamma(\lambda)}}z^r$$

Also, we will need the following two well-known results. We will present them without a proof, the proof can be found in [14].

Proposition 5.4.1 ([14]) *Let H be a lie group and π be a unitary representation on a Hilbert space $\mathcal{H}$. Suppose $\mathcal{H}$ contains a dense subspace V that can be decomposed as*

$$V = \sum_{i\in I} V_i$$

where the subspaces V_i are mutually orthogonal, closed in $\mathcal{H}$ and irreducible $H-$invariant modules. Then the following are equivalent

1) $V_{i_1} \not\approx V_{i_2}$ as $H-$modules for every $i_1 \neq i_2$

2) $End_H(\mathcal{H})$ is commutative.

For a suitable index J, the algebra $End_H(\mathcal{H})$ can be realized as $\ell^\infty(J)$. This was described in [14] as follows; Choose $(e_l)_{l\in L}$ an orthonormal basis of $\mathcal{H}$, write the index set L as

$$L = \bigcup_{j\in J} L_j$$

such that, for every $j \in J$ the set $(e_l)_{l\in L_j}$ is an orthonormal basis for V_j. Define the operator $R : \mathcal{H} \to \ell^2(L)$ by

$$R(x) := (< x, e_l >)_{l\in L}$$

the adjoint operator

$$R^*(y) = \sum_{l\in L} y_l e_l$$

This gives an injective homomorphism

$$\Psi : End_H(\mathcal{H}) \to B(\ell^2(L))$$

$$T \mapsto RTR^*$$

In fact, the map Ψ is gives an isomoprhism between $End_H(\mathcal{H})$ and $\ell^\infty(J)$.

Proposition 5.4.2 ([14]) *Suppose the assumption of 5.4.1 are satisfied. Every operator $T \in End_H(\mathcal{H})$ is unitarily equivalent to RTR^*, which is the multiplication operator on $\ell^2(L)$ given by the function*

$$\gamma_T : L \to \mathbb{C}$$

$$\gamma_T(l) =< T(e_l), e_l >$$

Furthermore, the function γ_T is constant on L_j for every $j \in J$, so this induces a function

$\widehat{\gamma}_T : J \to \mathbb{C}$ that belongs to $\ell^\infty(J)$ given by

$$\widehat{\gamma}_T(j) = \gamma_T(l)$$

for every $l \in L_j$. In particular, the map Ψ realizes an isomorphism between the algebra $End_H(\mathcal{H})$ and $\ell^\infty(J)$ given by

$$T \mapsto \widehat{\gamma}_T$$

As a consequence of these two propositions, we have the following result which generalizes [14, Theorem 3.2].

Theorem 5.4.1 *Let $\lambda > 0$. Every operator $T \in End_{\mathbb{T}^n}(A^2_\lambda(\mathbb{B}^n))$ is unitarily equivalent to RTR^* where*

$$R : A^2_\lambda(\mathbb{B}^n) \to \ell^2(\mathbb{N}^n)$$

$$f \mapsto (< f, e_m >_\lambda)_{m \in \mathbb{N}^n}$$

Moreover, the operator RTR^* is the multiplication operator given by

$$\gamma_T : \mathbb{N}^n \to \mathbb{C}$$

$$\gamma_T(m) =< T(e_m), e_m >_\lambda$$

Proposition 5.4.2 implies the following isomorphism of algebras. In fact, realizing $End_{\mathbb{T}^n}(A^2_\lambda(\mathbb{B}^n))$ as $\ell^\infty(\mathbb{N}^n)$ gives another way to see the commutativity of the algebra $End_{\mathbb{T}^n}(A^2_\lambda(\mathbb{B}^n))$.

Corollary 5.4.1 *The map $T \mapsto \gamma_T$ is an isomorphism between $End_{\mathbb{T}^n}(A^2_\lambda(\mathbb{B}^n))$ and $\ell^\infty(\mathbb{N}^n)$.*

Theorem 5.4.2 *Let $\lambda > 0$. Every operator $T \in End_{U(n)}(A^2_\lambda(\mathbb{B}^n))$ is unitarily equivalent to RTR^* where*

$$R : A^2_\lambda(\mathbb{B}^n) \to \ell^2(\mathbb{N}^n)$$

$$f \mapsto (< f, e_m >_\lambda)_{m\in\mathbb{N}^n}$$

Moreover, the operator RTR^* is the multiplication operator given by

$$\gamma_T : \mathbb{N}^n \to \mathbb{C}$$

$$\gamma_T(m) =< T(e_m), e_m >_\lambda$$

Furthermore, for every $k \in \mathbb{N}$ choose $u_k \in \mathcal{P}^k(\mathbb{C}^n)$, and consider the function

$$\widehat{\gamma}_T : \mathbb{N} \to \mathbb{C}$$

$$\widehat{\gamma}_T(k) =< T(u_k), u_k >_\lambda$$

Then

$$\widehat{\gamma}_T(|m|) = \gamma_T(m)$$

for every $m \in \mathbb{N}^n$.

Proof: Let $T \in End_{U(n)}(A^2_\lambda(\mathbb{B}^n))$, Proposition 5.4.2 shows that T is unitarily equivalent to RTR^*. Moreover, for every $k \in \mathbb{N}$, we have $\gamma_T(m_1) = \gamma_T(m_2)$ for every m_1 and m_2 such that where $e_{m1}, e_{m2} \in \mathcal{P}^k(\mathbb{C}^n)$, and this happens exactly when $k = |m_1|= |m_2|$. Schur's lemma implies that the operator T acts as multiplication by a scalar. So, we have

$$< T(u_k), u_k >_\lambda=< T(e_m), e_m >_\lambda$$

whenever $k = |m|$. Therefore, $\widehat{\gamma}_T(|m|) = \gamma_T(m)$.

$\square$

This theorem together with **Proposition** 5.4.2 proves the following corollary.

Corollary 5.4.2 *The map* $T \mapsto \widehat{\gamma}_T$ *is an isomorphism between* $End_{U(n)}(A^2_\lambda(\mathbb{B}^n))$ *and* $\ell^\infty(\mathbb{N})$.

5.4.1. $T^{(\lambda)}_\varphi$ with $\mathbb{T}^n-$ Invariant Symbols

In this subsection we will consider symbols $\varphi \in BC^n(\mathbb{B}^n)$ that are $\mathbb{T}^n-$ invariant, we will use the notation $BC^n(\mathbb{B}^n)^{\mathbb{T}^n}$ to denote this class of symbols. If a symbol is $\mathbb{T}^n-$ invariant then it is called separately radial symbols. In fact, a symbol φ is $\mathbb{T}^n-$invariant if and only if $\varphi(z_1, ..., z_n) = \varphi(|z_1|, ..., |z_n|)$, for almost all $z \in \mathbb{B}^n$.

Remark 5.4.1 *The torus is a maximal abelian subgroup of* $SU(n, 1)$, *and the restriction principal can be applied in this case. In fact, it is equivalent to the Quasi-elliptic case that was discussed before.*

Let $\mathcal{T}^{(\lambda)}(BC^n(\mathbb{B}^n)^{\mathbb{T}^n})$ be the C^*- algebra generated by Toeplitz operators $T^{(\lambda)}_\varphi$ with $\varphi \in BC^n(\mathbb{B}^n)$. Since the Toeplitz operators with $\mathbb{T}^n$- invariant symbols intertwine the representation π_λ then $\mathcal{T}^{(\lambda)}(BC^n(\mathbb{B}^n)^{\mathbb{T}^n}) \subset End_{\mathbb{T}^n}(A^2_\lambda(\mathbb{B}^n))$, and the commutativity of $End_{\mathbb{T}^n}(A^2_\lambda(\mathbb{B}^n))$ proves the following result.

Proposition 5.4.3 *Let* $\lambda > 0$. *The* C^*- *algebra generated by Toeplitz operators with symbols in* $BC^n(\mathbb{B}^n)^{\mathbb{T}^n}$ *is commutative.*

The following result shows that the Toeplitz operators with symbols in $BC^n(\mathbb{B}^n)^{\mathbb{T}^n}$ are diagonizable.

Proposition 5.4.4 *Assume* $\lambda_0 > 0$ *and* $\varphi \in BC^n(\mathbb{B}^n)^{\mathbb{T}^n}$. *The Toeplitz operator* $T^{(\lambda_0)}_\varphi$ *is* *unitarliy equivalent to the operator $RT_\varphi^{(\lambda_0)}R^*$, where*

$$R : A^2_{\lambda_0}(\mathbb{B}^n) \to \ell^2(\mathbb{N}^n)$$

$$R(f) = (< f, e_m >_{\lambda_0})_{m\in\mathbb{N}^n}$$

Moreover, $RT_\varphi^{(\lambda_0)}R^* = \gamma_{\varphi,\lambda_0} I$ where the eigenvalues $\gamma_{\varphi,\lambda_0}$ is given by

$$\begin{aligned}\gamma_{\varphi,\lambda_0}(m) = &< T_\varphi^{(\lambda_0)}(e_m), e_m >_{\lambda_0} \\ =&\frac{2^n\Gamma(\lambda_0+|m|)}{m!\,\Gamma(\lambda_0-n)} An_{\lambda_0}\int_{D_n}\varphi(r)|r|^{2m}(1-|r|^2)^{\lambda-n-1}\prod_{j=1}^n r_j\,dr\end{aligned}$$

where $D_n := \{r \in \mathbb{R}^n_+ : r_1 + ... + r_n < 1\}$.

Proof: Proposition 5.4.1 shows that Toeplitz operators $T_\varphi^{(\lambda_0)}$ with $\mathbb{T}^n-$invariant symbols are unitarly equivalent to $RT_\varphi^{(\lambda_0)}R^*$. Moreover, the eigenvalues $\gamma_{\varphi,\lambda_0}$ can be calculated using the polar coordinates. In particular, we write $z_j = r_je^{i\theta_j}$ where $r_j \in [0, 1)$. The eigenvalues $\gamma_{\varphi,\lambda_0}(m)$ can be written as

$$\begin{aligned}\gamma_{\varphi,\lambda_0}(m) =&< T_\varphi^{(\lambda_0)}(e_m), e_m >_{\lambda_0} \\ &= An_{\lambda_0}\Big(\int_{\mathbb{B}^n}\varphi(z)|e_m|^2\,d\mu_\lambda(z)\Big) \\ &= An_{\lambda_0}\Big(\frac{\Gamma(\lambda+|m|)}{m!\,\Gamma(\lambda)}\int_{\mathbb{B}^n}\varphi(z)\,|z|^{2m}(1-|z|^2)^{\lambda-n-1}\,dv(z)\Big) \\ &= \frac{\Gamma(\lambda_0+|m|)}{m!\,\Gamma(\lambda_0)}\frac{\Gamma(\lambda_0)}{\pi^n\Gamma(\lambda_0-n)}An_{\lambda_0}\Big(\int_{\mathbb{B}^n}\varphi(|z_1|,...,|z_n|)\,|z|^{2m}(1-|z|^2)^{\lambda-n-1}\,dz\Big) \\ &= \frac{\Gamma(\lambda_0+|m|)}{m!\,\pi^n\Gamma(\lambda_0-n)}An_{\lambda_0}\int_0^{2\pi}...\int_0^{2\pi}\int_{D_n}\varphi(r_1,...,r_n)\,|r|^{2m}(1-|r|^2)^{\lambda-n-1}\prod_{j=1}^n r_j\,dr\,d\theta \\ &= \frac{2^n\Gamma(\lambda_0+|m|)}{m!\,\Gamma(\lambda_0-n)}An_{\lambda_0}\int_{D_n}\varphi(r)\,|r|^{2m}(1-|r|^2)^{\lambda-n-1}\prod_{j=1}^n r_j\;dr\end{aligned}$$

where $\varphi(z_1, ..., z_n) = \varphi(|z_1|, ..., |z_n|)$ because φ is $\mathbb{T}^n-$ invariant.

□

5.4.2. $T_\varphi^{(\lambda)}$ with $U(n)-$ Invariant Symbols

As in the previous section, we will use $BC^n(\mathbb{B}^n)^{U(n)}$ to denote the class of symbols in $BC^n(\mathbb{B}^n)$ that are $U(n)-$ invariant. A symbol φ is $U(n)-$ invariant if and only if there exists a function $\mathbf{a}$ defined on $[0,1]$ such that $\varphi(z) = \mathbf{a}(|z|)$ for almost all $z \in \mathbb{B}^n$. Consider the C^*- algebra $\mathcal{T}^{(\lambda)}(BC^n(\mathbb{B}^n)^{U(n)})$ that is generated by Toeplitz operators with symbols in $BC^n(\mathbb{B}^n)^{U(n)}$. This algebra is contained in $End_{U(n)}(A^2_\lambda(\mathbb{B}^n))$, the algebra of bounded operates that intertwine with the representation π_α, since the algebra $End_{U(n)}(A^2_\lambda(\mathbb{B}^n))$ is commutative then $\mathcal{T}^{(\lambda)}(BC^n(\mathbb{B}^n)^{U(n)})$ is commutative. This proves the first part following result, and the second part follows from Theorem 5.4.2.

Proposition 5.4.5 *Let $\lambda > 0$. The C^*- algebra generated by Toeplitz operators $T_\varphi^{(\lambda)}$ with $\varphi \in BC^n(\mathbb{B}^n)^{U(n)}$ is commutative. Moreover, every Toeplitz operator $T_\varphi^{(\lambda)}$ with $\varphi \in BC^n(\mathbb{B}^n)^{U(n)}$ is unitarly equivalent to the multiplication operator $RT_\varphi^{(\lambda)}R^*$, where*

$$R : A^2_\lambda(\mathbb{B}^n) \to \ell^2(\mathbb{N}^n)$$

$$R(f) = (< f, e_m >_\lambda)_{m\in\mathbb{N}^n}$$

and $RT_\varphi^{(\lambda)}R^* = \gamma_{\varphi,\lambda} I$.

The eigenvalues $\gamma_{\varphi,\lambda}$ can be computed more explicitly, this can be done by the help of the following lemma.

Lemma 5.4.1 ([19]) *Let dv be the normalized volume measure on $\mathbb{B}^n$, and $d\sigma$ be the normalized surface measure. The measures v and σ are related by*

$$\int_{\mathbb{B}^n} f(z)\, dv(z) = 2n \int_0^1 r^{2n-1}\, dr \int_{\mathbb{S}_n} f(r\zeta)\, d\sigma(\zeta)$$

Corollary 5.4.3 *Let $\lambda_0 > 0$. The Toeplitz operator $T_\varphi^{(\lambda_0)}$ is unitarly equivalent to*

$RT_{\varphi}^{(\lambda_0)}R^*$ *where*

$$
\begin{aligned}
RT_{\varphi}^{(\lambda_0)}R^* =& \gamma_{\varphi,\lambda_0} I \\
&= \frac{4n\Gamma(\lambda_0+|m|)}{m!\,(n-1)!\,\Gamma(\lambda_0-n)} An_{\lambda_0} \int_0^1 r^{2n+2m-1}\boldsymbol{a}(r)(1-r^2)^{\lambda-n-1}dr
\end{aligned}
$$

Proof:

$$
\begin{aligned}
\gamma_{\varphi,\lambda_0}(m) &= < T_{\varphi}^{(\lambda_0)}(e_m), e_m >_{\lambda_0} \\
&= An_{\lambda_0}\Big(\int_{\mathbb{B}^n} \varphi(z)|e_m|^2\, d\mu_\lambda(z)\Big) \\
&= An_{\lambda_0}\Big(\frac{\Gamma(\lambda+|m|)}{m!\,\Gamma(\lambda)} \int_{\mathbb{B}^n} \varphi(z)\,|z|^{2m}(1-|z|^2)^{\lambda-n-1}\,dv(z)\Big) \\
&= \frac{\Gamma(\lambda_0+|m|)}{m!\,\Gamma(\lambda_0)} \frac{\Gamma(\lambda_0)}{\pi^n\Gamma(\lambda_0-n)}\Big(An_{\lambda_0}\int_{\mathbb{B}^n} \boldsymbol{a}(|z|)\,|z|^{2m}(1-|z|^2)^{\lambda-n-1}\,dz\Big) \\
&= \frac{2n\Gamma(\lambda_0+|m|)}{m!\,\pi^n\Gamma(\lambda_0-n)}\,An_{\lambda_0}\Big(\int_0^1 r^{2n-1}dr\int_{\mathbb{S}_n} \boldsymbol{a}(|r\zeta|)\,|r\zeta|^{2m}(1-|r\zeta|^2)^{\lambda-n-1}\,d\sigma(\zeta)\Big) \\
&= \frac{2n\Gamma(\lambda_0+|m|)}{m!\,\pi^n\Gamma(\lambda_0-n)}\,An_{\lambda_0}\Big(\int_0^1 r^{2n-1}dr\int_{\mathbb{S}_n} \boldsymbol{a}(r)\,r^{2m}(1-r^2)^{\lambda-n-1}\,d\sigma(\zeta)\Big) \\
&= \frac{2n\Gamma(\lambda_0+|m|)}{m!\,\pi^n\Gamma(\lambda_0-n)}\,\frac{2\pi^n}{(n-1)!}An_{\alpha_0}\Big(\int_0^1 r^{2n+2m-1}\boldsymbol{a}(r)\,(1-r^2)^{\lambda-n-1}dr\Big) \\
&= \frac{4n\Gamma(\lambda_0+|m|)}{m!\,(n-1)!\,\Gamma(\lambda_0-n)}\,An_{\lambda_0}\int_0^1 r^{2n+2m-1}\boldsymbol{a}(r)\,(1-r^2)^{\lambda-n-1}dr
\end{aligned}
$$

□

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Vita

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