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Multiplicities and Transforms of Ideals.

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UMI

MULTIPLICITIES AND TRANSFORMS OF IDEALS

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Abstract

Let (R, M_R) be a regular local ring of dimension 3 of the form $k [x, y, z]_{(x,y,z)}$, where k is an algebraically closed field and let I be an M_R-primary ideal that admits 3 generators. We prove that if I_1 is the proper transform of I to a quadratic transform (A, M_A) of (R, M_R) such that the analytic spread of I_1 is 3 and the generators of I_1 induced by those of I satisfy certain divisibility conditions, then the inequality of multiplicities

$$e_A (M (I_1)) < e_R (I)$$

is valid, where $M (I_1) \supsetneq I_1$ is an M_A-primary ideal associated to I_1 (the ideal I_1 may not be M_A-primary if dim (R) = 3) through an operation M that we define for ideals in a regular local ring .

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Chapter 1

Introduction

The theory of ideals in regular local noetherian rings has connections with many

geometric problems and has attracted the attention of many mathematicians. For instance,

consider the two dimensional case. Let R be a two dimensional regular local ring, with

maximal ideal M_R and residue field $k=R / M_R$ (e.g., take $R = k [x, y]_{(x,y)} = \{ f / g : f, g$

are polynomials in $k[x, y]$, $g(0, 0) \neq 0 \}$, where k is an algebraically closed field, for

instance $\mathbb{C}$). In this case the M_R-primary ideals of R, that is, those ideals I such that

$(M_R)^n \subseteq I$ for n large enough, are closely related to the singularities of the plane curves.

Zariski systematically studied these ideals (cf. [Z] , [ZS]), a study continued by other

authors (e.g., [Hu] , [R] in the 2-dimensional case, [J] , [L] in the d-dimensional one).

In all these studies, it is essential not to restrict one's attention to the basic ring R, but to

consider also "quadratic transforms" of R. Recall that given a point P of an algebraic

variety V (for simplicity, assumed non-singular), there is a canonical process to construct

a new variety V', and a morphism $\pi : V' \rightarrow V$, where $U = V \setminus \{ P \}$ is isomorphic to $\pi^{-1}(U)$

and $\pi^{-1}(P) = E$ is isomorphic to a projective space $(P)^{d-1}$, $d = \dim (V)$ (the points of E

correspond to lines that are tangent to V at P).V' is called the quadratic transform of V

with center P. A local ring of a point $Q \in E$ is called a quadratic transform of the local ring

of V at P. (This process can be formalized and applied to a local ring R, not necessarily

the local ring of an algebraic variety, this is reviewed in chapter 2). If I is an ideal of R

and A a quadratic transform of R, one may define the proper transform I_1 of I to R (the

ideal that I induces on A, when we disregard all "trivial factors"). In dimension 2, if I is

M_R-primary, its proper transform I_1 is again primary to the maximal ideal M_A of A, and

we may speak of the multiplicities $e_R(I)$ (of I in R) and $e_A(I_1)$ (of I_1 in A) (this is a

non-negative integer associated to the ideal, which in a sense measures its complexity, it

is defined if and only if I is M_R-primary (or the unit ideal, where the multiplicity will be

0)). It turns out that the inequality

$$(*)\, e_A(I_1) < e_R(I)$$

holds.

If $d = \dim(R) > 2$, if I_1 is an M_A-primary ideal, then (*) is still valid (cf. [J],

2.2). However, the condition "I_1 is M_A-primary" is not always fulfilled if $d > 2$. This

happens if the support of the proper transform is not finite. If I_1 is not M_A-primary, the

inequality (*) does not make sense, since $e_A(I_1)$ is not defined. Very little is known

about ideals I whose proper transform does not have finite support, i.e.such that for some

quadratic transform (A, M_A), the proper transform I_1 of I won't be M_A-primsry. But still

one could expect that somehow the ideal I improves when we take a proper transform.

More precisely, we try to assign to I a numerical invariant that drops when taking the

proper transform I_1. This invariant has to be such that it makes sense for a non primary

ideal since the proper transform I_1 to a quadratic transform A of R of an M_R-primary ideal

I may not be M_A-primary.

In order to define the aforementioned numerical invariant, we will introduce an

operation M on ideals J of a local ring R satisfying the following two conditions :

(i) $M (J) \supseteq J$ is an ideal, primary to the maximal ideal of R (hence the multiplicity

$e (M (J))$ of $M (J)$ is defined).

(ii) $M (J) = J$ if J is primary to the maximal ideal of the ambient ring of J.

Once the operation M is defined, we define the invariant $c (J)$ as:

$$c (J) := e (M (J)).$$

We are, thus, concerned with the relationship between $c (I)$ and $c (I_1)$. Namely, with the

relationship between the multiplicities of I ($M (I) = I$ since I is M_R-primary) and

$M (I_1)$. Our object of sudy becomes, then, the possible inequality

$$(\#) \; e_A(M (I_1)) < e_R (I).$$

In Chapter 2 we develop the background material that will be used later on. This

chapter consists of three sections. In the first one we give the basic results from ring theory

that constitute the basic "language" that will be used in the subsequent chapters ; in the

second one we introduce the concepts of integral dependence ond of reduction of ideals ;

in the third one we study the concept of blowing-up and related properties.

In Chapter 3 we define the operation M on ideals and present some properties and

concepts concerning this operation. In particular we study some properties relating the

operation M with the integral closure of an ideal and give some examples (and counter

examples).

In Chapter 4 we prove the main theorem of this work (cf. 4.3.12). This theorem

shows that the inequality (#) is valid for 3-dimensional regular local rings of the type

$R := k [x, y, z]_{(x,y,z)}$ (namely, polynomial rings in three indeterminates x, y, z over a

field k, say algebraically closed, localized at its maximal ideal (x, y, z)), provided I_1

satisfies some additional conditions (cf. 4.3.3). There is an introductory section in this chapter in which the main ideas of the proof of this result are presented. We believe that the proposed method of proof is interesting, in the sende that, suitably adjusted, it might give a more general theorem of the type of 4.3.12. In fact, most of the proof is valid under rather weak hypotheses, but to verify a crucial statement (cf. 4.3.10) we are forced to introduce the rather restrictive conditions 4.3.3. We hope that this could be done in greater generality, with other techniques. This is further discussed in Chapter 5 (cf.5.3)

In Chapter 5 we summarize what we did in chapters 3 and 4 and suggest directions for future work.

Chapter 2

Preliminary Results

2.1. Basic Results from Ring Theory.

We will let (R, M_R, k) denote a local ring with maximal ideal M_R and residue field k. We remark that, for us, "local ring" will mean "commutative noetherian ring with exactly one maximal ideal". We will, however, refer to R as of a noetherian local ring whenever we want to emphasize the fact that R is noetherian.

2.1.1. *Definition.* Let I be an ideal of R. The height of I and the dimension of I are defined, respectively, as:

$$\text{(a) } \text{ht}\,(\,I\,) = \inf\{\,\text{ht}\,(\,P\,) : I \subsetneq P \in \text{Spec}\,(\,R\,)\,\}$$

where ht (P) denotes the height of a prime ideal, that is, the supremum of lengths of chains of prime ideals $P_0 \subset P_1 \subset ... \subset P_n = P$, and

$$\text{(b) } \dim\,(\,I\,) = \text{Krull dim}\,(\,R\,/\,I\,).$$

The dimension dim (I) of I is also called the coheight of I, and denoted by cht (I) (cf. [M], pp.30, 31 or [Ku], 1.3, p. 40).

2.1.2. *Proposition.* The following inequality holds:

$$\text{ht}\,(\,I\,) + \text{cht}\,(\,I\,) \;\leq\; \dim R.$$

For a proof of 2.1.2 see [M], p. 31.

We review the Hilbert- Samuel function and the multiplicity of an ideal.

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2.1.3. *Definition*. Let (R, M_R, d) be a d-dimensional noetherian local ring; I $\subset$ R an ideal such that $\sqrt{I} = M_R$ (that is, an M_R-primary ideal). Then the Hilbert- Samuel function is defined as:

$$H_I (n) = \lambda (R / I^{n+1}) , \text{ n a natural number}$$

where λ ( ) denotes the length of an R-module (cf. [M], p. 97). When n $\gg$ 0, H_I (n) agrees with the values of a polynomial P_I (n) $\in$ Q [n] of degree d (cf. [M], p. 97), and

$$P_I (n) = (e / d!) n^d + (\text{ terms of lower order })$$

where e is a natural number (cf. [M], p. 107).

2.1.4. *Definition*. The integer e appearing in 2.1.3 is called the multiplicity of I in R and is denoted by e_R (I) or simply by e (I) if there is no possible confusion.

We now review the concept of regular sequence.

2.1.5. *Definition*. Let M be an R-module. An element $a \in$ R is said to be M-regular if a x = 0 with x $\in$ M implies that x = 0. A sequence a_1 , ... ,a_n of elements of R is an M-regular sequence if the following conditions hold:

(i) M $\neq$ (a_1 , ... ,a_n) M

(ii) a_1 is M-regular, a_2 is (M / a_1 M)-regular, ... , a_n is (M / (a_1,... ,a_{n-1}) M)-regular. If we regard R as an R-module, the definition of an R-regular sequence follows from 2.1.5. We will call it , then, a "regular sequence" if there is no possible confusion.

2.1.6 *Definition*. If x_1, ... , x_d generate an M_R-primary ideal, then {x_1, ... ,x_d} is said to be a system of parameters.

2.1.7 *Definition.* A local ring (R, M_R , d) is called regular if M_R is generated by d elements.

2.1.8. *Proposition.* If (R, M_R, d) is regular, then any minimal system of generators {x_1, ... , x_d } of M_R is a system of parameters and an R-regular sequence and any subsystem {x_{i1}, ... , x_{is}} of this system of parameters generates a prime ideal of R.

For a proof of 2.1.8 see [Ku], p. 168.

2.1.9. *Definition.* If (R, M_R) is a regular local ring, then any minimal system of generators of M_R is called a regular system of parameters.

2.1.10. *Definition.* Let M be a finitely generated module over a noetherian ring R. The number of elements of a maximal M-regular sequence in the ideal I $\subseteq$ R is called the I-depth of M, and denoted depth (I, M). If (R, M_R) is local then depth (M_R, M) is simply called the depth of M and denoted depth (M). In particular this defines the depth of R (depth (R)).

2.1.11. *Definition.* A noetherian ring R is called Cohen Macaulay (abbreviated to CM) if depth (R) = dim (R).

The following result can be found in [M], 17.11.

2.1.12. *Proposition.* If R is a CM ring and I = (x_1, ... , x_d), with (x_1, ... , x_d) forming an R-regular sequence, then

$$e_R (I) = \lambda (R / I).$$

We now introduce the concept of primary decomposition of an ideal.

2.1.13. *Definition*. Let I be a proper ideal of a ring R. A primary decomposition of I is an expression for I as an intersection of finitely many primary ideals of R. That is, an expression of the form

$$I = Q_1 \cap ... \cap Q_n, \text{ with } \sqrt{Q_i} = P_i \text{ prime, } i=1, ... , n.$$

A primary decomposition of I is said to be minimal when

(i) $P_1, ... P_n$ are n different ideals, and

(ii) $Q_j \not\supseteq \cap \{ Q_i : i \neq j, i=1, ..., n \}$ for all $j = 1, ... , n$.

The ideal is said to be decomposable when it has a primary decomposition.

2.1.14. *Remark.* In general not every proper ideal of a commutative ring R is decomposable (cf.[Sh], 4.30). However, there is a positive result in this direction which shows that every proper ideal in a noetherian ring possesses a primary decomposition. Moreover, any decomposable ideal has a minimal primary decomposition (cf. [Sh], 4.35) , in which case the set { $P_1, ... , P_n$ } is uniquely determined (cf. [Sh], 4.18). We can conclude this remark saying that every proper ideal of a noetherian ring possesses a minimal primary decomposition.

2.1.15. *Definition*. The set { $P_1, ... , P_n$ } of 2.1.14 is called the set of associated prime ideals of I and denoted by ass I. The members of ass I are called the associated primes of I, and are said to belong to I.

2.1.16. *Definition*. The minimal primes of I, that is, the minimal elements of ass I, are called the minimal or isolated primes of I. The remaining associated primes of I are called the embedded primes of I. Similarly, the components corresponding to the minimal primes

will be called the isolated components of I, and the ones corresponding to the embedded primes will be called the embedded components of I.

Now, in order to avoid confusion about terminology, we introduce the following definition.

2.1.17. *Definition.* Let M be a module over a noetherian ring R, P ∈ Spec (R). Then we say P is an associated prime of M if there exists m ∈ M with

$$\text{ann} (m) := (0 : m) = P.$$

The set of associated primes of M is denoted by Ass (M).

We have the following relationship between ass I and Ass (R / I).

2.1.18. *Proposition.* Let I be a proper ideal of a noetherian ring R. Then,

$$P \in \text{ass I if and only if } P \in \text{Ass} (R / I).$$

Proof. First of all, I has a primary decomposition because R is noetherian. We can, thus, form the finite set of associated primes of I, ass I. The result follows now from [Sh], 8.22.

We can, thus, indistinctively use the symbols ass I and Ass(R / I).

There is a relationship between the concepts of depth and associated primes: the equation depth (I, M) = 0 holds if and only if the ideal consists only of zero divisors of M. In particular, if (R, M_R) is a noetherian ring, depth (M) = 0 is equivalent to $M_R \in$ Ass (M). Therefore:

2.1.22. *Proposition.* If (R, M_R) is a local ring, I a proper ideal of R, then

$$\text{depth} (R / I) = 0 \text{ if and only if } M_R \in \text{ass I.}$$

2.2. Reductions and Integral Closure of an Ideal.

In this section we will introduce the concepts of a reduction of an ideal $I \subseteq R$, and of integral dependence of an ideal $I \subseteq R$ over an ideal $J \subseteq R$, where R is a noetherian ring. These concepts are equivalent in some sense (that of 2.2.9). We will study some properties related to these concepts. At the end of the section we show how to compute the "integral closure" of an ideal I generated by monomials in the ring $R = k [x, y, z]_{(x,y,z)}$.

2.2.1. *Definition*. Let I, J be ideals of R. Then J is called a reduction of I (or a reduction ideal of I) if

(i) $J \subseteq I$, and

(ii) $I^{n+1} = J I^n$ for some $n \in \mathbf{N}$.

The following is an important property of reduction ideals. For a proof see [M], 14.13.

2.2.2. *Proposition*. If I is an M_R-primary ideal and J a reduction ideal of I, then

(i) J is also M_R-primary, and

(ii) $e_R(I) = e_R(J)$.

The following proposition is a consequence of [M], 14.14.

2.2.3. *Proposition*. Let (R, M_R) be a d-dimensional noetherian local ring, with infinite residue field k; let I be an M_R-primary ideal. Then there is a reduction ideal J of I generated by a system of parameters $\{ x_1, \dots, x_d \}$.

2.2.4. *Proposition.* If R is a d-dimensional CM ring with infinite residue field k, I an M_R-primary ideal, then there is an R- regular sequence $x_1, \dots, x_d$ such that

$$e_R(I) = \lambda(R/(x_1, \dots, x_d)).$$

Proof. The generators of the ideal J of 2.2.3, being the generators of a primary ideal (cf. 2.2.2), form a regular sequence (cf [M],17.4) and by 2.2.2 and 2.1.12,

$$e_R(I) = e_R(J) = e_R(x_1, \dots, x_d) = \lambda(R/(x_1, \dots, x_d)).$$

2.2.5. *Definition.* A reduction ideal J of I is called a minimal reduction of I if it does not have any proper reductions.

The following theorem shows the existence of minimal reductions. This result can be found in [S], 1.7, p. 31.

2.2.6. *Theorem.* Let $J \subseteq I$ be a reduction. Then J contains a minimal reduction H of I. Moreover, if L is any ideal such that $H \subseteq L \subseteq I$, then any minimal set of generators of H can be extended to a minimal set of generators of L.

We now study the concept of integral dependence over an ideal I.

2.2.7. *Definition.* An element $x \in R$ is said to be integral over I if there are elements $a_1, \dots, a_n$ of R such that

$$x^n + a_1 x^{n-1} + \dots + a_n = 0$$

and $a_i \in I^i, i = 1, \dots, n$.

An ideal J is said ro be integral over I if $J \subseteq I$ and every element of J is integral over I. The integral closure of I is the ideal.

$$\overline{I} = \{ x \in R : x \text{ is integral over } I \}$$

and an ideal I for which $\bar{I} = I$ is called integrally closed or complete. Thus J is integral over I if $J \subseteq \bar{I}$.

The following propositions establish a connection between the concepts of a reduction of an ideal and of integral dependence over an ideal. Their proofs can be found in [HIO], 4.11 and 4.13.

2.2.8. *Proposition.* Let $I \subseteq J$ be two ideals of R. Then I is a reduction of J if and only if $J \subseteq \bar{I}$; i.e. if and only if J is integral over I.

2.2.9. *Proposition.* For any ideal I and any element $x \in R$, x is integral over I if and only if I is a reduction of $I + xR$.

2.2.10. *The Analytic Spread of an Ideal.* Given a local ring (R, M_R) with residue field k, I an ideal of R, the analytic spread of I is defined to be the integer s (I) given by:

$$(i) \quad s (I) = \dim G(I, R) \otimes_R k$$

(see 2.3.1 for a definition of G (I, R)). Or, more geometrically (cf. [HIO], 20.0), by:

$$(ii) \quad s (I) = \delta + 1$$

where $\delta = \dim \pi^{-1}(M_R)$; $\pi : Bl (I, R) \rightarrow Spec\ R$ the natural blowing-up morphism (cf. 2.3.3).

2.2.11. *Proposition.* The analytic spread of I satisfies the following :

$$ht (I) \leq s (I) \leq \dim R.$$

For a proof of 2.2.11 see [S], 2.3, p. 34.

If we let μ (J) denote the minimal number of generators of the ideal J, then the following proposition holds (cf. [HIO], 10.19).

2.1.12 *Proposition.* If k is infinite then,

$$s (I) = \mu (J)$$

for any minimal reduction ideal $J \subseteq I$.

Since, by 2.2.4, minimal reductions exist, 2.2.12 allows us to compute the analytic spread of an ideal using reductions of it.

There is actually a stronger result. The following proposition provides us with a criterion to decide whether a reduction ideal J of I is minimal. A proof of it can be found in [S], 2.2, p. 33.

2.2.13. *Proposition.* Let R have infinite residue field k. Let H be a reduction of I with minimal set of generators $a_1, \dots , a_r$. Then H is a minimal reduction of I if and only if

(i) $a_1, \dots , a_r$ are analytically independent in I. That is, if whenever F [$x_1, \dots , x_r$] $\in$ R [$x_1, \dots , x_r$] is a homogeneous form of degree t such that F [$a_1, \dots , a_r$] $\equiv 0$ mod I ^{t} M_R, then the coefficients of F are in M_R, and

(ii) $s (I) = r$.

2.2.14. *How to Compute the Integral Closure of an Ideal Generated by Monomials.* Let R denote the polynomial ring in three variables k[x, y, z] localized at the maximal ideal (x, y, z), that is R:= k[x, y, z]$_{(x,y,z)}$, and let k be infinite. Let I be an ideal of R generated by n monomials $x^{a(i)} y^{b(i)} z^{c(i)}$, i =1, ... , n.

In order to compute the integral closure $\bar{I}$ of I, consider the representative points of the monomials that generate I. That is, the points p (i) in N^3 whose coordinates are the exponents of x, y, and z for each monomial :

$$p (i) := (a (i), b (i), c (i)) \in N^3 , \quad i = 1, \dots , n.$$

We define the set $E(I) \subseteq \mathbb{R}^3$ to be

$$E(I) := (p(1) + E) \cup \ldots \cup (p(n) + E)$$

where $E := \{ (x_1, x_2, x_3) : x_i \geq 0 \} \subseteq \mathbb{R}^3$. Then $\bar{I}$ is also generated by monomials whose

representative points in $\mathbb{N}^3$ belong to the convex closure $N(I)$ of $E(I)$.

We can, thus, visualize the difference between the integral closure $\bar{I}$ and the ideal I

as given by those monomials whose representative points in $\mathbb{N}^3$ belong to $N(I)$ but not

to $E(I)$.

This result can be found in [T], section 1.2, p. 336.

2.2.15.*Example.* Let $I = (x^2, xy, z^2) \subseteq k[x, y, z]_{(x,y,z)}$. The representative points of I in

$\mathbb{N}^3$ are

$$p(1) = (2, 0, 0); \quad p(2) = (1,1,0); \quad p(3) = (0, 0, 2).$$

A computation shows that $(1, 0, 1) \in N(I) \cap \mathbb{N}^3$ and that this point is the only

one in $N(I) \cap \mathbb{N}^3$ that is not a linear combination of $p(1), p(2), p(3)$. The associated

monomial of this point is xz. Therefore

$$\bar{I} = (x^2, xy, z^2, xz)$$

2.3. Blowing-up.

2.3.1.*Definition.* Let a be an ideal of a noetherian ring R. The symbols B (a, R) and G $(a,$

R) wil denote the positively graded rings given by:

$$B(a, R) := a^0 \oplus \ldots \oplus a^n \oplus \ldots \ , \text{ and}$$

$$G(a, R) := a^0 / a^1 \oplus \ldots \oplus a^n / a^{n+1} \oplus \ldots$$

where $a^0 := R$.

2.3.2. *The Scheme* Bl (a, R). Given a positively graded ring $A = A_0 \oplus \ldots \oplus A_n \oplus \ldots$,

the set Proj A consists of all homogeneous prime ideals P which do not contain all of $A_+ :=$

$A_1 \oplus \ldots \oplus A_n \oplus \ldots$. It turns out that Proj A can be endowed with a topology and with a

sheaf of rings. Moreover, if we let $X := $ Proj A, and O_X its sheaf of rings, then (X, O_X)

is a scheme with

$$(i) \; O_{X, P} \approx A_{(P)}$$

where $P \in$ Proj A and $A_{(P)}$ is the subring of A_S consisting of elements of degree 0 in A_S,

where S denotes the set of homogeneous elements of $A \setminus P$. The ring A_S is endowed with

a natural grading: Let $(A_S)_n$ consist of all quotients $a / s \in A_S$, $a \in A$ homogeneous with

deg a - deg s = n. The ring $A_{(P)}$ consists of elements that are homogeneous and of degree

0 with respect to this grading. This is a local ring and is called the "homogeneous

localization" of A at P. Its maximal ideal consists of all quotients p / s, $p \in P$, $s \in S$ with

deg p = deg s.(Cf. [Ku], p. 77.)

The scheme Proj A ca n be covered by open affine schemes

$$(ii) \; (D_+ (f), O_X|_{D_+(f)}) \cong \text{Spec } A_{(f)}$$

where $f \in A_+$ is homogeneous and,

$$A_{(f)} = \{ \; a / f^n : a \in A \text{ homogeneous, deg } a = n \text{ deg } f \; \}.$$

We will denote the scheme Proj (B (a, R)) by the symbol Bl (a, R). Therefore, if

$a = (x_0, \ldots, x_n)$ then the scheme Bl (a, R) is covered by $D_+(x_0), \ldots, D_+(x_n)$ where

$$D_+(x) \cong \text{Spec } (B (a, R)_{(x)})$$

(cf. 2.3.2 (ii)), since B (a, R) is generated by $x_1, \ldots, x_n$ as an R-algebra.

The proofs of 2.3.2 (i), (ii) can be found in [H], 2.5, p. 76.

2.3.3. *The Blowing-up Morphism.* The blowing-up morphism with center a (or along the ideal a) is defined to be the morphism

$$\pi : \mathrm{Bl}\,(\,a\,,\,R\,) \to \mathrm{Spec}\,R$$

(cf. [HIO], 12.2 (c)).

2.3.4. *Universal Property of the Blowing-up.* Let $f : Z \to \mathrm{Sspec}\,R$ be a morphism succh that $a\,O_Z$ be an invertible sheaf of ideals over a scheme Z. That is, a locally free sheaf of ideals of rank 1 (cf. [H], p. 109). Then there exists a unique morphism $g : Z \to \mathrm{Bl}\,(\,a,\,R\,)$ such that $f = g\,\mathrm{o}\,\pi$ (cf. [H], 7.14, p. 164).

2.3.5. *The Exceptional Divisor E.* Let $X := \mathrm{Bl}\,(\,a,\,R\,)$. Then the sheaf of ideals $a\,O_X$ corresponds to a closed subscheme E of X via the surjection

$$B\,(\,a,\,R\,) \to G\,(\,a,\,R\,)$$

(since $G\,(\,a,\,R\,) \approx B(\,a,\,R\,)\,/\,a\,B\,(\,a,\,R\,)$) of graded rings. Taking Proj, this surjection induces a closed immersion

$$\mathrm{Proj}\,(\,G\,(\,a,\,R\,)\,) \to X.$$

We let $E := \mathrm{Proj}\,(\,G\,(\,a,\,R\,)\,)$, and call it the "exceptional divisor" of X.

The exceptional divisor E is actually an effective Cartier divisor. That is, one corresponding to an invertible sheaf of ideals.

In order to see that $a\,O_X$ is invertible, one considers the ideals $a\,O_{D+(x_i)}$ which correspond to $a\,B\,(\,a,\,R\,)_{(x_i)}$ (cf. 2.3.2). There is an R-algebra isomorphism

$$R\,[\,a\,/\,x\,] \approx B\,(\,a,\,R\,)_{(x)}$$

where $x \in a$ (cf. [HIO], 12.6). Therefore, one has to consider the ideals

$$a\,R\,[\,a\,/\,x_i\,] = (\,x_0,\,\ldots\,,\,x_n\,)\,R\,[\,a\,/\,x_i\,]$$

$i = 0, \ldots, n$. Now, since $x_j = (x_j / x_i) x_i$, we have that :

$$a \, R \, [\, a / x_i \,] = x_i \, R \, [\, a / x_i \,].$$

This is a principal ideal generated by a regular element (x_i is a unit in R_{x_i}, an over-ring of $R \, [\, a / x_i \,]$). This shows that $a \, O_X$ is an invertible sheaf of ideals.

2.3.6. *Proposition.* The blowing-up morphism π induces an isomorphism of schemes :

$$\pi \big|_{X \setminus E} : X \setminus E \to \operatorname{Spec} R \setminus V \, (\, a \,)$$

where $X := Bl \, (\, a, R \,)$.

For a proof of 2.3.6 see [H], 7.13 (b), p. 164.

2.3.7. *The Normalized Blowing-up.* If R is an integral domain, the normalized blowing - up of R along I, denoted $Bl \, (\, I, R \,)$ ', is the normalization of the integral scheme $Bl \, (\, I, R \,)$ (cf. [H], p. 91). If J is a reduction of I, then the normalized blowing-ups of J and I are canonically isomorphic ([T], p. 330). In particular,

$$Bl \, (\, I, R \,) \, ' \cong Bl \, (\, I^-, R \,) \, '$$

where I^- denotes the integal closure of I.

2.3.8. *The Proper Transform of an Ideal.* Let (R, M_R, d) be a regular local ring of dimension d, with infinite residue field k. Let $X := Bl \, (\, M_R, R \,)$, I an M_R-primary ideal. Let $v = \operatorname{ord} (\, M_R, I \,)$, that is, $I \subset (\, M_R \,)^v$, but, $I \not\subset (\, M_R \,)^{v+1}$. Then the global proper transform of I (relative to the blowing-up $\pi : X \to \operatorname{Spec} R$ centered at M_R) is defined to be the sheaf of ideals I' of O_X such that

$$(\, i \,) \quad I \, O_X = e^v I'$$

where e is the sheaf of ideals defining the exceptional divisor of X.

If U = Spec (R [M_R / x]) is an affine piece of X, then $I'|_U$ corresponds to the ideal I', where

$$(\text{ii}) \quad I' = I (R [M_R / x]) / (x)^\nu.$$

If x' $\in$ X, then the proper transform of I at x' is defined to be the ideal

$$(\text{iii}) \quad I'_{x'} \subset O_{X, x'} .$$

We remark that, for simplicity, we use the term "proper transform" understanding that the center of the blowing-up is M_R, although the proper transform can be defined relative to a blowing-up whose center might not necessarily be M_R.

2.3.9. *A Quadratic Transform of R.* Let (R, M_R, d) be as in 2.3.8. A regular local ring (A, M_A, d) is said to be a quadratic transform of R if

$$(\text{i}) \quad A = R [M_R / x]_m$$

where m denotes a maximal ideal of R [M_R / x] of height d and x $\in M_R \setminus (M_R)^2$ is an element of a minimal set of generators of M_R.

Given an M_R-primary ideal I $\subset$ R, the proper transform of I to the quadratic transform A is defined to be the ideal

$$(\text{i}) \quad I^A := I' A$$

where I' is the ideal of 2.3.8 (ii).

2.3.10. *Remark.* The ideal I^A corresponds to an ideal of the form $I'_{x'}$ as in 2.3.8 (iii), via an identification $O_{X, x'} \cong R [M_R / x]_\wp$, where x is a regular parameter, $\wp$ is a maximal ideal, x $\in \wp$ ([J], p. 5).

2.3.11. *Definition.* (*The support of an R-module*) . The support of an R-module M, denoted by Supp (M) , is defined to be the set

$$\text{Supp} (M) := \{ P \in \text{Spec } R : M_P \neq 0 \}.$$

The proof of the following proposition can be found in [Ku] , p. 79.

2.3.12. *Proposition.* If M is finitely generated, then

$$\text{Supp} (M) = V (\text{Ann} (M)) = \{ P \in \text{Spec } R : P \supset \text{Ann} (M) \}.$$

2.3 13. *Definition.* (*The Support of an Ideal*) . The support of an ideal I is defined as the support of the R-module R / I. That is,

$$\text{Supp} (I) := \text{Supp} (R / I) = \{ P \in \text{Spec } R : R_P \neq I_P \}$$

where $I_P = I R_P$.

2.3.14. *Proposition.* If I is an ideal of R, then

$$\text{Supp} (I) = V (I) .$$

Proof. By 2.3.12,

$$(i) \quad \text{Supp} (I) = \text{Supp} (R / I) = V (\text{Ann} (R / I)) .$$

Claim. Ann (R / I) = I.

Proof of claim. Let $r \in$ Ann (R / I) . Then r (1 + I) $\in$ I. Hence, r $\in$ I. Therefore, Ann (R / I) $\subseteq$ I, and, since the other inclusion also holds, the claim is proved.

Substituting in (i), 2.3.14 follows.

2.3.15. *Definition.* (*The Support of a Sheaf*) . Given a sheaf $\mathscr{F}$ over a scheme X, its support Supp $\mathscr{F}$ is defined as

$$\text{Supp} \mathscr{F} := \{ P \in X : \mathscr{F}_P \neq 0 \}.$$

2.3.16. *Definition. (The Support of a Sheaf of Ideals)*. The support of a sheaf of ideals

$J \subset O_X$, denoted by Supp J, is defined to be the support of the sheaf O_X / J, that is,

$$\text{Supp } J := \text{Supp } (O_X / J) ,$$

and J is said to be finitely supported if

$$O_{X,x} / J_x = 0 \quad (\text{ or, equivalently, if } O_{X,x} = J_x)$$

for all but finitely many $x \in X$.

If I is an M_R-primary ideal of height d, then Supp (I) = { M_R } due to 2.3.14. As for

the support of a proper transform I' , we have the following results.

2.3.17. *Proposition.* Let (R, M_R , d) be a regular local ring of dimension d, I an M_R-

primary ideal, I' the global proper transform of I. Then Supp $I' \subsetneq$ E, where E is the

exceptional divisor (2.3.5) . That is, Supp I' is a proper algebraic subset of the

exceptional divisor E of X = Bl (M_R , R) .

Proof. The exceptional divisor E corresponds to the invertible sheaf of ideals M_R $O_X \subsetneq$

O_X. If $M_R = (x_1, \dots, x_d)$, then, locally, this sheaf of ideals is a principal ideal generated

by a regular element x_i , and

$$M_R R [M_R / x_i] = (x_i) R [M_R / x_i]$$

for some i = 1, ... , d.

Let I_i be the ideal corresponding to $I'|_{U_i}$, where U_i = Spec (R [M_R / x_i]) . Then,

$$I_i = I (R [M_R / x_i]) / (x_i)^v.$$

So, in order to show that Supp $I' \subsetneq$ E, it suffices to show that if P $\in$ Supp (I_i) , then P $\neq$

(x_i) , and $x_i \in$ P.

To show that $P \neq (x_i)$, suppose that, on the contrary, $P = (x_i)$. Then, $(x_i) \supseteq I_i$, and, thus, $x_i \mid I_i$ which is not the case.

To show that $x_i \in P$, suppose that $x_i \notin P$. Since

$$(x_i)^v I_i = I R [M_R / x_i] \subset I_i ,$$

we have that

$$I \subset I R [M_R / x_i] \cap R \subset P \cap R =: P_0 .$$

But, since we are assuming $x_i \notin P$, $P \in U_i \subset X$, and $U_i \subset X \setminus E \cong \text{Spec } R \setminus \{ M_R \}$ (cf. 2.3.6). This means that $P_0 \neq M_R$, P_0 a prime ideal of R. But $I \subset P_0$ and, hence,

$$M_R = \sqrt{I} \subsetneq \sqrt{P_0} = P ,$$

which is a contradiction. Therefore, $x_i \in P$. This finishes the proof of 2.3.17.

2.3.18. *Corollary.* If $d = 2$, then Supp I' is finite.

Proof. In this case $E \cong \mathbf{P}^1$. So, by 2.3.17, Supp I' is a closed set in $\mathbf{P}^1$ (in the Zariski topology). Hence, Supp I' must be finite.

The following example shows that if dim $R \geq 3$, then I' may not be finitely supported.

2.3.19. *Example.* Let $R = k [x, y, z]_{(x,y,z)}$, with k an infinite field , and consider the M_R-primary ideal $I = (x^3, y^2, z)$. Then $R [M_R / x] = R [y / x , z / x]$, and if we let

$$x_1 := x , y_1 := y / x , z_1 := z / x ,$$

then $R [M_R / x] = R [y_1 , z_1]$, and

$$I R [M_R / x] = (x_1^3 , x_1^2 y_1^2 , x_1 z_1) = (x_1) (x_1^2 , x_1 y_1^2 , z_1) .$$

Therefore,

$$I' = (x_1^2 , x_1 y_1^2 , z_1) .$$

Then $I' \subset (x_1 , z_1) \subset (x_1 , y_1 - \alpha , z_1)$, $\alpha \in k$. Since k is infinite,

$$\text{Supp} (I') = V (I') = \{ P \in \text{Spec } R [y_1 , z_1] : P$$

Chapter 3

The Operation M

3.1.. Basic Definitions and Properties Related to the Operation M .

Let A be a regular local noetherian ring with maximal ideal M_A, and J an ideal of A.

Since A is noetherian, J has a minimal primary decomposition (2.1.14). That is, there is

an expression of the form

$$J = Q_1 \cap \ldots \cap Q_n, \quad \sqrt{Q_i} = P_i, \quad i = 1, \ldots, n$$

with $P_1, \ldots, P_n$ n different primes and $Q_j \not\supseteq \bigcap_{i \neq j} Q_i$ (2.1.13).

3.1.1. *Proposition.* For any proper ideal I of A

$\quad$ ht (I) = min { ht (P_i) : $P_i \in$ ass I } = min { ht (P) : P a minimal prime ideal of I}.

Proof. By definition, ht (I) = inf {ht (P) : I $\subsetneq$ P $\in$ Spec A } (2.1.1). But the minimal

elements are, precisely, the minimal members of ass I (cf. [Sh],4.24). Therefore, the

conclusion of 3.1.1 follows.

We will assume, for simplicity, throughout the rest of this section that the ring A is

3-dimensional and consider ideals J of A having height 2 or 3.

3.1.2. *Corollary.* If J is an ideal such that ht (J) $\geq$ 2, then J has at most one embedded

component, that is , ass J consists of isolated primes except for at most one embedded

prime (cf. 2.1.15).

Proof. It follows from 3.1.1 that, since ht (J) $\geq$ 2,

$$ht (P_i) = 2 \text{ or } 3, \quad i = 1, \ldots, n.$$

But, $P_i \subseteq M_A$ for all $i = 1, \ldots, n$. Therefore, $P_i = M_A$ or ht $(P_i) = 2$, in which case P_i is a minimal or isolated prime of J. Since all the primes P_i are different, the announced conclusion follows.

3.1.3. *Proposition.* An ideal J of A is M_A-primary if and only if ht $(J) = 3$.

Proof. Suppose ht $(J) = 2$. Then there is a $P \in$ ass J, ht $(P) = 2$ such that $\sqrt{J} \subseteq P$. Therefore $\sqrt{J} \neq M_A$, that is, J is not M_A-primary.

Now, suppose ht $(J) = 3$. Then 3.1.1 implies ht $(P_i) = 3$ for all $i = 1, \ldots, n$. So, since, by 3.1.2 there can be only one embedded component of J, $i = 1$ and, thus, $J = Q_1$ with $\sqrt{Q_1} = P_1 = M_A$. That is, J is M_A-primary. Proposition 3.1.3 is, thus, proved.

We remark here that in the present context, that is, when the ring A is 3-dimensional and ht $(J) \geq 2$, then :

(a) ht $(J) = 2 \Leftrightarrow J$ is not M_A-primary, and

(b) ht $(J) = 3 \Leftrightarrow J$ is M_A-primary.

3.1.4. *Proposition.* Let J be an ideal of A of height 2 and such that J has no embedded component. Then depth $(A / J) = 1$.

Proof. By 2.1.22, depth $(A / J) \neq 0$, and by [Ku], 3.9, p. 185, the inequality

$$\text{depth} (A / J) \leq \text{dim} (A / J)$$

holds. On the other hand, 2.1.2 implies that

$$\text{dim} (A / J) \leq 3 - 2 = 1.$$

Therefore $0 \neq$ depth $(A / J) \leq 1$. That is, depth $(A / J) = 1$.

We can conclude that there are two possibilities for an ideal $J \subset A$ having height 2.

3.1.5. *Proposition.* If ht (J) = 2, there are two possibilities for the primary decomposition of J :

(i)   If J has no embedded components (depth (A / J) = 1) :

$$J = Q_1 \cap \ldots \cap Q_n , \ \sqrt{Q_i} = P_i , \ i = 1, \ldots , n ;$$

P_i an isolated prime, ht (Q_i) = 2 for all i = 1, ... , n.

(ii) If J has an embedded component (depth (A / J) = 0) :

$$J = Q_1 \cap \ldots \cap Q_{n-1} \cap N , \ \sqrt{Q_i} = P_i , \ i = 1, \ldots , n ; \sqrt{N} = M_A ;$$

P_i an isolated prime, ht (Q_i) = 2, i =1, ... , n - 1 , and ht (N) = 3.

3.1.6. *Definition.* Let J be a proper ideal of A with ht (J) = 2 and minimal primary decomposition $J = Q_1 \cap \ldots \cap Q_n$. Then, the ideal J_0 is defined as :

$$J_0 := \bigcap\nolimits_{ht(Q_j)=2} Q_j .$$

That is :

(i) In case depth (A / J) = 1 (3.1.5 (i)),

$$J_0 = J$$

(ii) In case depth (A / J) = 0 (3.1.5 (ii),

$$J_0 = Q_1 \cap \ldots \cap Q_{n-1}$$

where $Q_n = N$, the embedded component.

3.1.7. *Remark.* (i)   In any of these two cases J_0 is defined in terms of the isolated (non-embedded) components of the minimal primary decomposition of J. Since these are uniquely determined ([Sh], 4.29), J_0 is well defined.

(ii)   We define J_0, in general, that is, for rings A of any dimension as :

$$J_0 := \bigcap \{ Q_j : Q_j \text{ is an isolated component } \}.$$

3.1.8. *Definition.* If J is M_A-primary, we will let $J_0 := A$.

We remark here that Definition 3.1.8 holds in general for any dimension. In our present context it says that $J_0 := A$ when ht (J) = 3 (cf. 3.1 (b)).

3.1.9. *Example.* Let $A := k [x, y, z]_{(x,y,z)}$, where k [x, y, z] is a polynomial ring in 3 indeterminates over an infinite field k, and let $J = (xy, y^2, z) A$. Then

$$J = (y, z) \cap (x, y^2, z) . So, J_0 = (y, z) , N = (x, y^2, z) .$$

The following proposition provides us with a characterization of the ideal J_0.

3.1.10. *Proposition.* Consider the scheme $X := $ Spec A, and let $U := X \setminus \{M_A\}$.

Let $Y := $ Spec (A / J) $\subset X$ as a subscheme, where J denotes an ideal of A of height 2, having an embedded component. If $Y_0' := Y \cap U$, then the scheme-theoretic closure Y_0 of Y_0' corresponds to the ideal J_0 . We can thus write $Y_0 = $ Spec (A / J_0).

Proof. Consider the closure Y_0 of Y_0' in X. By definition, Y_0 is a closed subscheme of X (say, defined by an ideal H of A) containing Y_0' and such that if Y_1 is another closed subscheme of X containing Y_0', then $Y_0 \subseteq Y_1$. Let W be the closed subscheme of X defined by J_0 . First, we claim that $Y_0 \subset W$. In order to see this let $N = (y_1, \dots , y_m)A$, where N is the embedded component of J; $D (y_i) = X \setminus V (y_i)$ (cf. [H], p. 70). To see that $Y_0' \subset W$, it suffices to show that $Y_0' |_{D(Y_i)} \subseteq W |_{D(Y_i)}$ for

i = 1, ... , m. In order to see this, consider the (restricted) associated sheaves of the ideals J_0 and J. Namely

$$(\dagger) \quad \tilde{J_0} |_{D(Y_i)} ; \tilde{J} |_{D(Y_i)} \quad i = 1, \dots , m.$$

These sheaves correspond to the ideals

$$(i) (J_0)_{Y_i} = (Q_1)_{Y_i} \cap \dots \cap (Q_{n-1})_{Y_i}$$

and

$$(\text{ ii }) \ J_{Y_i} = (Q_1)_{Y_i} \cap ... \cap (Q_{n-1})_{Y_i} \cap N_{Y_i}$$

respectively (cf. [H], 5.1 (c), p. 110) for i = 1, ... , m. But, since $y_i \in N$, $N_{Y_i} = A_{Y_i}$

and, hence, $J_{Y_i} = (J_0)_{Y_i}$. We have, thus that the sheaves (†), both correspond to the same

ideal. Therefore, they determine the same closed subscheme of X (cf. [H], 5.9, p. 116).

That is,

$$Y_0' \big|_{D(Y_i)} = W \big|_{D(Y_i)} \quad i = 1, ... , m.$$

Since W is closed in X, $Y_0 \subsetneq W$.

We can conclude that $H \supseteq J_0$. Next, we show that, indeed, $H = J_0$. First, we note

that, since $W \big|_U = Y_0 \big|_U$, it follows that H and J_0 have the same primary components of

height 2 : to see this, let $P \in$ ass H, P an isolated prime, then the corresponding component

(of height 2) is $HA_P \cap A$ (cf. [AM],4.8 and 4.9). Now, $P \in$ ass J_0 also. Therefore, the

corresponding component is given by $J_0 A_P \cap A$. It turns out that these components are

actually the same one, since there is a bijective correspondence between the prime ideals

of A_P and the prime ideals P' of A such that $P' \subseteq P$ (cf. [AM], 3.11 (iv)) under the natural

mapping $A \rightarrow A_P$ and, so, since they correspond to the same ideal $P \in A$, they must be

equal. Now, the fact that $W|_U = Y_0|_U$, implies that if P is an isolated prime of J_0, then it

is also an isolated prime of H. So H and J_0 have the same primary components of height

2. There are two possibilities for H: either (i) $H = Q_1 \cap ... \cap Q_{n-1} = J_0$ or (ii) H =

$Q_1 \cap ... \cap Q_{n-1} \cap N'$, N' being an embedded component . But (ii) is not possible

because $J_0 \subsetneq H$. This means that $H = J_0$, as desired.

We can now prove the following result.

3.1.11.*Proposition.* Let $I \subset J$ be two ideals of A such that $I = I_0 \cap N$, $J = J_0 \cap M$,

where $\sqrt{N} = \sqrt{M} = M_A$. Then, $I_0 \subsetneq J_0$.

Proof. Let $X := \text{Spec } A$ and $U := X \setminus \{M_A\}$ as in 3.1.10. If we let $Y := \text{Spec } (A / I)$,

$Z := \text{Spec } (A / J)$, then $Y|_U \supseteq Z|_U$, and

$$\text{cl} (Y|_U) \supseteq \text{cl} (Z|_U),$$

where cl ( ) denotes scheme-theoretic closure. These closed subschemes correspond to I_0

and J_0 by 3.1.10. Therefore,

$$\text{Spec} (A / I_0) \supseteq \text{Spec} (A / J_0).$$

That is, $I_0 \subsetneq J_0$ as claimed.

We will now define the ideal M (J), where J is an ideal of height 2 or 3. The next

one is the main definition of this chapter. This definition holds in general for a ring of any

dimension.

3.1.12. *Definition of the Ideal* M (J). Let J be an ideal of A. Then

$$M (J) = (J : J_0).$$

3.1.13. *Basic Properties of the Ideal* M (J).

(i) Since J_0 / J is an A-module, $M (J) = \text{Ann} (J_0 / J)$.

(ii) If J is M_A-primary, $M (J) = J$.

Proof. In this case $J_0 = A$ (3.1.8) and $M (J) = (J : A) = J$.

(iii) If dim $A = 3$ and ht (J) $= 2$ and J has no embedded component, then $M (J) = A$.

Proof. In this case, $J = J_0$ (3.1.6 (i)) and $M (J) = (J : J) = A$.

(iv) If dim $A = 3$ and ht (J) $= 2$ and J has an embedded component N, then $M (J) =$

$(N : J_0)$.

Proof. In this case, we can write $J = J_0 \cap N$ (3.1.6 (ii)). From this it follows that

$$M (J) = (J_0 \cap N : J_0) = (N : J_0).$$

The interesting case is that of 3.1.13 (iv).

3.1.14. *Proposition.* If J has an embedded component and ht (J) = 2, then M (J) is an M_A-primary ideal.

Proof. Let N be the embedded component of J. Therefore, if $r \in M_A$, there is an n $\in \mathbb{N}$ such that $r^n \in N$. Hence, $r^n J_0 \subsetneq N J_0 \subseteq N$. This says that $r^n \in (N : J_0) = M (J)$ (3.1.13 (iv)). That is, $\sqrt{M (J)} = M_A$, and the announced conclusion follows.

3.2. The Operation M and the Integral Closure of an Ideal.

In this section we let (A , M_A, d) be a regular local ring of dimensin d and J an ideal of A.

We will show that if $J = J^-$, i.e. if J is integrally closed, then $M (J) = M (J)^-$. First, we have the following proposition.

3.2.1. *Proposition.* Let I, J be any two ideals of A such that $I \subsetneq J$, and I is integrally closed $(I = I^-)$. Then $(I : J) = (I : J)^-$.

Proof. We will show that $(I : J)^- \subsetneq (I : J)$. Let $a \in (I : J)^-$, then a satisfies a relation

$$(i) \quad a^n + ... + b_1 a^{n-i} + ... + b_n = 0$$

where $b_i \in (I : J)^i$.

Claim. Let $f \in J$. Then $af \in I^-$.

Proof of the claim. Multiply (i) by f^n:

$$(ii) \quad (af)^n + ... + (b_i f^i)(af)^{n-i} + ... + b_n f^n = 0.$$

Since $b_i \in (I:J)^i$, b_i is a linear combination of elements $c_1 c_2 \dots c_i$, with $c_1, c_2, \dots, c_i \in$

$(I:J)$. Therefore, $b_i f^i \in I^i$ and (ii) can be expressed as:

$$(iii) \quad (af)^n + \dots + g_i (af)^{n-i} + \dots + g_n = 0$$

where $g_i \in I^i$. This shows that $af \in I^-$ and proves the claim.

Now, since $I = I^-$, if $f \in J$, then $af \in I$. Hence $aJ \subseteq I$. That is, $a \in (I:J)$. Therefore,

$(I:J)^- \subseteq (I:J)$ and the proof is complete.

3.2.2. Corollary. If J is an ideal of A such that J is complete (integrally closed) , then M

$(J) = M(J)^-$.

Proof. Let us assume that J is not M_A-primary. Then, if $J = J_0$, $M(J) = A$ and there is

nothing to prove. Otherwise, $J \subsetneq J_0$ and, by 3.2.1, $(J:J_0) = (J:J_0)^-$. That is, $M(J) =$

$M(J)^-$. If J is M_A-primary, $M(J) = J$ and , since $J = J^-$, $M(J) = M(J)^-$. Corollary

3.2.2 is , thus, proved.

We now show that there may be two ideals $J \subset I$ such that $M(J) \not\subset M(I)$.

3.2.3. Example. Let A be as in 3.1.9 . Then $J = (xy, y^2, z) \subset I = (x, y^2, z)$. However,

$$M(J) = ((xy, y^2, z) : (y, z)) = (x, y, z)$$

and

$$M(I) = ((x, y^2, z) : A) = (x, y^2, z).$$

Therefore, $M(J) \not\subset M(I)$.

Next we show that if we impose some conditions on the ideals $J \subset I$, the inclusion

$M(J) \subset M(I)$ holds.

3.2.4. Proposition. Let $I \subset J$ be two ideals of A satisfying the following conditions:

(i) J is integral over I ($I^- = J^-$);

(ii) J is complete ($J = J^-$);

(iii) J_0 is integral over I_0 ($I_0^- = J_0^-$).

Then $M (I) \subset M (J)$.

Proof. Let $a \in M (I)$ and $b \in J_0$.

Claim. ab is integral over J.

Proof of the claim. Since $b \in J_0$ and $I_0^- = J_0^-$, there is a relation

$$(i) \quad b^n + ... + t_i \, b^{n-i} + ... + t_n = 0$$

where $t_i \in (I_0)^i$. So, multiplying by a^n:

$$(ii) \quad (ab)^n + ... + (t_i a^i)(ab)^{n-i} + ... + t_n a^n = 0.$$

As in the proof of 3.2.1, $t_i a^i \in I^i$ for $i = 1, ... , n$. So,

$$(iii) \quad (ab)^n + ... + g_i \, (ab)^{n-i} + ... + g_n = 0$$

where $g_i \in I^i$. Since $I^i \subset J^i$, (iii) shows that ab is integral over J, proving the claim.

Now, since J is complete, $ab \in J$. This shows that $a \in (J : J_0) = M (J)$ and, hence, that $M (I) \subset M (J)$.

Next, we will see that $M (J)$ is not necessarily integral over $M (I)$, even if the ideals $I \subset J$ fulfill the conditions of 3.2.4.

3.2.5. *Example.* Let $A = k[x, y, z]_{(x,y,z)}$ and let $I \subset J$ be the ideals of A given by

$I = (x^2, xy, z^2)$ and $J = (x^2, xy, z^2, xz)$. Then :

(i) J is integral over I.

Proof. It suffices to show that xz is integral over I. But xz satisfies the relation

$$(xz)^2 - x^2 z^2 = 0.$$

Since $x^2 z^2 \in I^2$, xz is integral over I (2.2.7).

(ii) J is complete.

Proof. In 2.2.15 we saw thar $\bar{I}$=J.

(iii) J_0 is integral over I_0 .

Proof. We have that

$$I = (x, z^2) \cap (x^2, y, z^2) , J = (x, z^2) \cap (x^2, y, z).$$

Therefore, $I_0 = J_0 = (x, z^2)$.

One can now compute M (I) and M (J) :

$$M (I) = ((x^2, y, z^2) : (x, z^2)) = (x, y, z^2)$$

$$M (J) = ((x^2, y, z) : (x, z^2)) = (x, y, z).$$

So, $M (I) \subset M (J)$, as expected. We will see, however, that M (J) is not integral over M (I).

(iv) M (J) is not integral over M (I).

Proof. We compare the multiplicities e (M (I)) and e (M (J)).

To compute e (M (I)), note that A / M (I) has $\{1, z\}$ as basis. So,

$$e (M (I)) = \lambda (A / M (I)) = 2$$

(2.1.12).

Similarly one finds that,

$$e (M (J)) = 1$$

and, comparing multiplicities, e (M (J)) $\neq$ e (M (I)). This shows that M (I) is not a reduction of M (J) because the multiplicity of an ideal is preserved under reduction (2.2.2). This says that M (J) is not integral over M (I) (2.2.8).

Chapter 4

The Main Inequality

4.1. Introduction.

Let (R, M_R) be a 3-dimensional regular local ring having infinite residue field k, I an M_R-primary ideal and let (A, M_A) be a quadratic transform of (R, M_R), I_1 the proper transform of I with respect t o A. We define the ideal Γ:

$$\Gamma := M (I_1).$$

Assuming I_1 has height 2 and an embedded component, Γ is M_A-primary (3.1.14). Otherwise, $\Gamma = A$ (3.1.13 (iii)). In any case, it makes sense to consider the multiplicities $e_A (\Gamma)$ and $e_R (I)$. It is our hope that the inequality

$$(\#) \quad e_A(\Gamma) < e_R (I)$$

is valid. In this chapter we propose a strategy to show this inequality, which leads to an actual proof if certain additional assumptions on the generators of I_1 are made (cf. 4.3.3).

We will assume that I is an M_R-primary ideal generated by a system of parameters t_1, t_2, t_3 forming a regular sequence. It follows that the proper transform I_1 of I with respect to A is also generated by 3 elements. We will assume that, actually, the minimal number of generators of any minimal reduction of I_1 is 3, that is, that $s (I_1) = 3$.

Let $Z := Bl (A, I_1)$, $Z_0 := Spec\ A$ and consider the blowing -up morphism

$$\pi : Z \to Z_0$$

In 4.2 we shall see that the condition $s(I_1) = 3$ implies that if $o \in Z_0$ is the closed

point, then $\pi^{-1}(0) \subset Z$ is irreducible, so it has a generic point. So let Q be the generic

point of $\pi^{-1}(o)$ and let $B := O_{Z,Q}$. (Note that B is a 1-dimensional local ring.)

The idea to prove (#) is to verify the following two inequalities:

$$(i) \quad e_B(\Gamma^\sim B) < e_R(I)$$

$$(ii) \quad e_A(\Gamma^\sim) \le e_B(\Gamma^\sim B).$$

Clearly, (i) and (ii) together imply (#).

We now state a theorem of D. Katz that reduces the study of the multiplicity of an

ideal in a d-dimensional ring to that of the multiplicity of an ideal in a 1-dimensional ring.

For a proof see [K], 1.1.

4.1.1. *Theorem*. Let (R, M_R, d) be a d-dimensional CM local ring; $I = (a_1, \dots, a_d)$ an M_R-

primary ideal generated by a system of parameters. Let

$$\pi : Bl(R, I) \to Spec\, R$$

be the blowing-up morphism of R along I. Then $\pi^{-1}(0)$ is irreducible, where 0 is the

closed point of Spec R and if P is the generic point of $\pi^{-1}(0)$, then

$$e_R(I) = e_{O_{X,P}}(I\, O_{X,P})$$

where $X := Bl(R, I)$.

In section 4.2 we will prove that the inequality (i) holds. As for (ii), if I_1 is M_A-

primary, then $\Gamma^\sim = I_1$ (3.1.13 (ii)), thus $e_A(\Gamma^\sim) = e_A(I_1)$. On the other hand,

$e_B(I_1 B) = e_A(I_1)$ (4.1.1). Therefore (ii) holds in this case.

If I_1 is not M_A-primary, we do not know what happens in general. However in case

R is of the form $k[x, y, z]_{(x,y,z)}$, where x, y, z are indeterminates and k is an algebraically

closed field (and, so, A is of this form too) and if I_1 satisfies some additional conditions (those of 4.3.3), we can show that $e_A(\Gamma) = e_B (\Gamma B)$. That is, that (ii) holds. This is done in 4.3.

The strategy we propose is based on the proof of a theorem of B. Johnston (cf. [J], 2.2). This theorem states that in a regular local ring of dimension d having infinite residue field, the inequality $e_A (I_1) < e_R (I)$ holds whenever the ideal I_1 is an M_A-primary ideal. Our proof contains Johnston's case, and our presentation in this case is, however, more geometric.

After we have proved the main theorem of this chapter (4.3.12), we discuss the special case where the generators of I are monomials. This is done in section 4.3.

4.2. The inequality $e_B (\Gamma B) < e_R (I)$.

In order to show that the inequality $e_B (\Gamma B) < e_R (I)$ is valid, let us recall that if $\pi : Z \to Z_0$ is the blowing-up morphism, where $Z := Bl (A, I_1)$, $Z_0 := Spec A$, then we defined $B := O_{Z,Q}$, Q being the generic point of $\pi^{-1} (0)$, where 0 is the closed point of Z_0. In order to show that B is well defined, we need to show that $\pi^{-1} (0)$ is irreducible.

4.2.1. *Proposition.* If 0 is the closed point of Z_0 , then $\pi^{-1} (0)$ is irreducible.

Proof. We have that

$$\pi^{-1} (0) = Proj (Gr (A, I_1) \otimes_A k$$

where $\Lambda := A / I_1$, and k is the residue field of A. There is a surjective morphism of (A / I_1)-algebras

$$\psi : \Lambda [x_1, x_2, x_3] \to Gr (A, I_1) = \bigoplus_{m \geq 0} (I_1{}^m / I_1{}^{m+1})$$

given by

$$\psi(x_i) = u_i + (I_1)^2, \quad i = 1, 2, 3$$

where u_1, u_2, u_3 are the generators of I_1. Tensoring with k over Λ, we obtain a surjective morphism

$$k[x_1, x_2, x_3] \to Gr(A, I_1) \otimes_\Lambda k.$$

Taking Proj, we get a closed embedding

$$j : \pi^{-1}(0) \to (P_k)^2.$$

Since $s(I_1) = 3$,

$$\dim \text{Proj}(Gr(A, I_1) \otimes_\Lambda k) = \dim \pi^{-1}(0) = 3 - 1 = 2$$

(2.2.10 (ii)). Therefore, the map j is the identity map and $\pi^{-1}(0) \cong (P_k)^2$. This shows that $\pi^{-1}(0)$ is irreducible. Proposition 4.2.1 is, thus, proved.

Next, we want to show that the inequality $e_B(I^\sim B) < e_R(I)$ holds. This inequality can be expressed as

$$(\text{ i }) \quad e_B(I^\sim) < e_{B'}(IB'),$$

where $B' := O_{X,P}$, since $e_R(I) = e_{B'}(IB')$ (cf. 4.1.1).

Note that since $I_1 \subseteq I^\sim$ and, hence, $I_1 B \subseteq I^\sim B$, the inequality

$$e_B(I^\sim B) \le e_B(I_1 B)$$

holds provided the ideal $I_1 B$ is M_B-primary ([M], 14.4). So, in order to prove that the inequality (i) holds, it suffices to prove that the inequality

$$e_B(I_1 B) < e_{B'}(IB')$$

holds.

First, we will show that the ideal $I_1 B$ is M_B-primary and, second, that the latter inequality is valid.

4.2.2. *Proposition.* The ideal I_1B is M_B-primary.

Proof. First we note that $B = O_{Z,Q} \cong B(A, I_1)_{(Q)}$ (cf. [H],2.5 (a), p. 76). On the other hand, $B(A, I_1)_{(Q)} \cong A[I_1 / x]_P$, for some prime ideal P of $A[I_1 / x]$, $x \notin P$ (cf. [HIO], 12.7). Also, we know that $I_1 A[I_1 / x] = (x)A[I_1 / x]$, where x is a regular element (cf. [HIO], 12.8 or 2.3.5). Therefore,

$$I_1B = I_1 A[I_1 / x]_P = (x)A[I_1 / x]_P.$$

Since x is a non-zero divisor of B, we have that the set $\{x\}$ is independent in the sense of ([Ku], 4.13, p. 144) and, hence ht $(I_1B) = 1$ (cf. [Ku], 4.14, p. 145). Since dim(B) $= 1$, M_B is the only prime belonging to $I_1 B$. So, $I_1 B$ is M_B-primary.

4.2.3. *Proposition.* The inequality

$$e_B(I_1B) < e_{B'}(IB')$$

holds.

Proof. Consider the following solid diagram of schemes.

$$\varphi' : Z' \quad ---- \quad X'$$

$$\eta \downarrow \qquad \downarrow \eta'$$

$$\varphi : Z \quad ---- \quad X$$

$$\pi \downarrow \qquad \downarrow \pi'$$

$$\psi : Z_0 \quad \rightarrow \quad X_0$$

where $Z_0 := \text{Spec } A$, $X_0 := \text{Spec } R$, $Z := Bl(A, I_1)$, $X := Bl(R, I)$, Z', X' the respective normalizations; π, π' blowing-up morphisms. Since the given morphisms are birational, all the function fields can be identified with K (R), the fraction field of R.

The universal property of the blowing-up (2.3.4) implies the existence of the birational map $\varphi : Z \to X$, since $(\psi \circ \pi)^{-1} \, I \cdot O_Z = (\, x^r) \, I_1 \cdot O_Z$ is invertible in Z . The morphism φ induces the morphism $\varphi' : Z' \to X'$ and the diagram commutes.

Let $W := \pi^{-1} (0) \subseteq Z$, $W' := \pi'^{-1} (0') \subseteq X$. We write down the irreducible components of $\eta^{-1} (W) \subseteq Z'$ and $\eta'^{-1} (W') \subseteq X'$ as :

$$\eta^{-1} (W) = E_1 \cup ... \cup E_s \; ; \; \eta'^{-1} (W') = D_1 \cup ... \cup D_r$$

and let

$$W_j := O_{Z', Q_j} \, , \; j = 1, ... , s \; ; \; W_i' := O_{X', P_i} \, , \; i = 1, ... , r$$

where $Q_j \in E_j$, $P_i \in D_i$ are the generic points.

In order to show that the inequality $e_B (I_1 \, B) < e_{B'} (IB')$ holds, we apply the formulae :

(i) $e_B (I_1 \, B) = \sum_{1 \le j \le s} [k (Q_j) : k (Q)] \, e_{W_j} (I_1 \, W_j)$

(ii) $e_{B'} (IB') = \sum_{1 \le i \le r} [k (P_i) : k (P)] \, e_{W'_i} (I \, W_i)$

(cf. [ZS], Corollary 1, p. 299).

After these introductory remarks, now we prove 4.2.3 as follows :

(a) To show that there exists a surjective morphism $W \to W'$.

(b) To show that $s \le r$ and that

$$W_j = W_j' \; , \; j = 1, ... , s.$$

(c) To compare the terms of (i) with those of (ii) and to show that the inequality $e_B(I_1 \, B) < e_{B'} (IB')$ holds.

Proof of (a). We will show that the morphism $\psi : Z_0 \to X_0$ induces a finite surjective morphism $W \to W'$. To check this, we look at the standard affine pieces that cover X and Z.

The scheme X (resp. Z) is covered by affine open sets isomorphic to:

$$\text{Spec } R[I / t_j] , \quad j = 1, 2, 3 \text{ (resp. Spec } A[I_1 / u_j] , \quad j = 1, 2, 3).$$

Let us consider the affine sets corresponding to $j = 1$, in order to simplify the notation. The other cases can be treated in a similar way.

Consider the affine piece U of X (resp. V of Z) given by

$$U = \text{Spec } R[t_2 / t_1, t_3 / t_1] \text{ (resp. } V = \text{Spec } A[u_2 / u_1, u_3 / u_1]).$$

Then the morphism ψ induces a morphism $V \to U$, corresponding to the canonical homomorphism $R[I / t_1] \to A[I_1 / u_1]$, given by $t_i / t_1 \mapsto u_i / u_1$, $i = 2, 3$, since $t_i / t_1 = u_i / u_1$ in K (R), the fraction field of R.

The induced map of fibers can be identified with the map

$$k [x_2', x_3'] \to k [y_2', y_3'] ; x_i \mapsto y_i$$

where $x_i' = x_i / x_1, y_i' = y_i / y_1$, $i = 2, 3$. This map induces a surjective map

$$\text{Spec } k [y_2', y_3'] \to \text{Spec } k [x_2', x_3'].$$

Therefore, a finite surjective morphism

$$(\mathbf{P}_k)^2 \to (\mathbf{P}_k)^2$$

We saw that $W \cong (\mathbf{P}_k)^2$ (4.2.1). One can also prove in a similar way that $W' \cong (\mathbf{P}_k)^2$, provided s (I) = 3. But, indeed, this is the case since

$$\text{ht} (I) \leq s (I) \leq \dim (R)$$

(cf. 2.2.11) and, since I is M_R-primary, ht (I)=3 (3.1.3). Therefore, s (I) =3 and

W' $\cong$ (P_k)^2 and,thus, there is a finite surjective morphismW $\rightarrow$ W'. This completes the proof of (a).

Proof of (b). The local rings W_j , j =1, ... , s ; W_i' , i =1, ... , r are integrally closed, noetherian domains of dimension 1. Therefore, they are DVR's (cf. [M], 11.2).

The morphism φ' : Z ' $\rightarrow$ X' is finite. Therefore, given an irreducible component $E_{j'}$ of η^{-1} (W) $\subseteq$ Z ', there is an irreducible component $D_{i'}$ of η'^{-1} (W') such that φ' ($E_{j'}$) = $D_{i'}$. This means that $W_{j'}$ dominates $W''_{i'}$ (via the finite dominant morphism W $\rightarrow$ W' of (a)). Therefore $W_{j'}$ = $W'_{i'}$ due to the maximality of valuation rings ordered by domination (cf. [H], 6.1A, p. 40). Therefore s $\leq$ r and, after reordering, W_j = W'_j , j =1, ... , s. This proves (b).

Proof of (c). Finally, we compare the terms of (i) and (ii). Since s $\leq$ r, we only need to compare s terms. Also, since W_i= W'_i , i =1, ... , s , it suffices to show that

$$(iii) \quad [k (Q_i) : k (Q)] \leq [k (P_i) : k (P)] \quad i =1, ... , s$$

and

$$(iv) \quad e_{w_i} (I_1 W_i) < e_{w_i} (I W_i) \quad i =1, ... , s$$

hold.

Proof of (iii). k (Q_i) = W_i / max (W_i) = W'_i/max (W'_i) = k (P_i), and since k (P) $\subseteq$ k (Q),

$$k (P_i) = k (Q_i) \supseteq k (Q) \supseteq k (P)$$

and the inequality (iii) holds.

Proof of (iv). Let v ( ) denote the valuation (order) of the DVR W_j = W'_j. Then

$$v (J) = e_{w_j} (J W_j) = ord (J)$$

where J is any ideal, and if ord (J) = ζ, then $JW_j = (t^\zeta)$. Since I = $x^r I_1$ as sets,

$$e_{w_j}(IW_j) = v (I) = v (x^r I_1) = r\, v (x) + e_{w_j}(IW_j)$$

and $rv (x) > 0$ because each of these ideals has center in R [M_R / x] at an ideal $a \subsetneq$

R [M_R / x] containing x. Therefore, (iv) holds.

This shows that the terms of (i) are smaller than those of (ii). Therefore the

inequality $e_B (I_1 B) < e_{B'}(I B')$ holds. This proves 4.2.3.

4.2.4. *Corollary.* The inequality $e_B (\Gamma B) < e_R (I)$ holds.

Proof. The inequality $e_B (\Gamma B) < e_B (I_1 B)$ holds (cf. [M], 14.4 and 4.2.2). So, combining

this inequality with that of 4.2.3, we get that the inequality $e_B (\Gamma B) < e_{B'}(I B')$ is valid.

The desired conclusion follows since $e_{B'}(I B') = e_R (I)$ by 4.1.1.

4.3. The Inequality $e_A (\Gamma) < e_R (I)$.

We will assume throughout this section that the ring R is of the type

k [x, y, z] $_{(x, y, z)}$, a polynomial ring in three indeterminates x, y, z over an algebraically

closed field k, localized at the maximal ideal (x, y, z) $\subset$ k [x, y, z]. If A is the quadratic

transform of R, then A will be of the same type. So, let us write A = k [α, β, γ] $_{(\alpha, \beta, \gamma)}$,

where α, β, γ are three independent variables.

We will prove that the inequality

$$(\#)\; e_A (\Gamma) < e_R (I)$$

holds, where $\Gamma := M (I_1)$ and I_1 satisfies certain additional conditions (those of 4.3.3).

The precise statement is given in Theorem 4.3.12.

The proof of (#) depends heavily on the equality

$$(*)\; \lambda (A / \Gamma) = \lambda (B / \Gamma B)$$

where λ ( ) denotes the length of a module; and most of this section is devoted to proving this equality.

The proof of (*) is very technical and rather long, and it consists of everal parts. Here is an outline of several steps that will lead us to prove it and, hopefully, help the reader to follow it better.

(i) First, we start by giving a description of the ring B. We do this in 4.3.1.

(ii) Second, we show that λ (A / J) = λ (B / JB), where J is an M_A-primary ideal satisfying certain condition (cf. 4.3.2).

(iii) Finally, we show that if the generators of I_1 satisfy some conditions (cf. 4.3.3), then the ideal $\Gamma := M (I_1)$ will satisfy that condition of 4.3.2 and, hence, satisfy the equality (*). This constitutes the longest step and consists of several propositions.

Once (*) has been proven, we proceed to prove the main theorem of this chapter (4.3.12).

Let us now get started with the aforementioned steps.

First of all we will give a description of the ring B := $O_{Z,Q}$. This ring is well defined since we are assuming that s (I_1) = 3 (cf. 4.2.1). We recall that   Z := Bl (A, I_1), I_1 the proper transform of the M_R-primary ideal I to the quadratic transform A, and Q the generic point of π^{-1} (0) (4.2.1), where π : Bl (A, I_1) $\rightarrow$ Spec A and 0 is the closed point of Spec A.

We are assuming throughout this chapter that I is generated by 3 elements. So, it follows that I_1 is generated by 3 elements..

4.3.1. *Proposition.* Let I_1 be the proper transform of the M_R-primary ideal I to the quadratic transform A of R. Then if $A = k [\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)}$, and if $I_1 = (a, b, c)$, we have that the ring $B := O_{Z, Q}$ satisfies

$$B \approx k (\delta, \varepsilon) [\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)} / K \, k (\delta, \varepsilon) [\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon$ are independent variables and $K := \ker \Psi$, Ψ the surjective ring homomorphism given by

$$\Psi \colon k [\alpha, \beta, \gamma, \delta, \varepsilon] \to k [\alpha, \beta, \gamma] [a / c, b / c]$$

$$\alpha \mapsto \alpha, \beta \mapsto \beta, \gamma \mapsto \gamma, \delta \mapsto a / c, \varepsilon \mapsto b / c.$$

Proof. It is a consequence of 2.3.2 (i) and [HIO] , 12.7 that the ring $B := O_{Z, Q}$ satisfies the following isomorphism

$$(i) \quad O_{Z, Q} \approx A [a / c, b / c]_{(\alpha, \beta, \gamma)} = k [\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)} [a / c, b / c]_{(\alpha, \beta, \gamma)}$$

$$\approx k [\alpha, \beta, \gamma, a / c, b / c]_{(\alpha, \beta, \gamma)}.$$

On the other hand, Ψ induces a ring isomorphism

$$\Psi' \colon k [\alpha, \beta, \gamma] [a / c, b / c] \to k [\alpha, \beta, \gamma, \delta, \varepsilon] / K.$$

Let $(\alpha, \beta, \gamma)' := \Psi' (\alpha, \beta, \gamma)$. This is a prime ideal since

$$k [\alpha, \beta, \gamma] [a / c, b / c] / (\alpha, \beta, \gamma) \approx k [a / c, b / c]$$

is an integral domain. Now, Ψ' induces an isomorphism of local rings

$$(ii) \quad k [\alpha, \beta, \gamma] [a / c, b / c]_{(\alpha, \beta, \gamma)} \approx (k [\alpha, \beta, \gamma, \delta, \varepsilon] / K)_{(\alpha, \beta, \gamma)'}.$$

Let $S := k [\alpha, \beta, \gamma, \delta, \varepsilon] \setminus (\alpha, \beta, \gamma)$ and $S' := \{ s + K : s \in S \}$. Then, a direct calculation shows that

$$S' = (k [\alpha, \beta, \gamma, \delta, \varepsilon] / K) \setminus (\alpha, \beta, \gamma)'.$$

Therefore, (ii) can be expressed as

$$k [\alpha, \beta, \gamma] [a / c, b / c]_{(\alpha, \beta, \gamma)} \approx (k [\alpha, \beta, \gamma, \delta, \varepsilon] / K)_{s}.$$

But,

$$(k [\alpha, \beta, \gamma, \delta, \varepsilon] / K)_{s} \approx k [\alpha, \beta, \gamma, \delta, \varepsilon]_{s} / K k [\alpha, \beta, \gamma, \delta, \varepsilon]_{s}$$

(cf. [Sh], p. 99). That is,

(iii) $(k [\alpha, \beta, \gamma, \delta, \varepsilon] / K)_{s} \approx k [\alpha, \beta, \gamma, \delta, \varepsilon]_{(\alpha, \beta, \gamma)} / K k [\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)}.$

Since δ, ε are invertible elements in $k [\alpha, \beta, \gamma, \delta, \varepsilon]_{(\alpha, \beta, \gamma)}$,

(iv) $k [\alpha, \beta, \gamma, \delta, \varepsilon]_{(\alpha, \beta, \gamma)} = k (\delta, \varepsilon) [\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)}$

Finally, putting together (i), (ii), (iii) and (iv), one obtains the desired result. Namely, that

$$B \approx k (\delta, \varepsilon) [\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)} / K k (\delta, \varepsilon) [\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)}.$$

This completes the proof of 4.3.1.

Now that we have given a description of the ring B (step (i)), we proceed to complete step (ii).

4.3.2. *Proposition.* Let J be an M_A-primary ideal of A such that

$$K k (\delta, \varepsilon) [\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)} \subseteq J k (\delta, \varepsilon) [\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)}$$

where K is as in 4.3.1, and the right member of the inclusion denotes the extension of the ideal J under the inclusion morphism

$$\iota : k [\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)} \hookrightarrow k (\delta, \varepsilon) [\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)}.$$

Then, the equality $\lambda (A / J) = \lambda (B / JB)$ is valid.

Proof. Consider the exact sequence

$$0 \to J \to A \to A / J \to 0.$$

Tensoring with $L := k(\delta, \varepsilon)$ over k, one obtains the exact sequence

$$0 \to J \otimes_k L \to A \otimes_k L \to A/J \otimes_k L \to 0.$$

Since $J \subset A = k[\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)}$, $J = Jk[\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)}$. Therefore,

$$J \otimes_k L = Jk[\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)} \otimes_k k(\delta, \varepsilon),$$

and one can show that

$$Jk[\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)} \otimes_k k(\delta, \varepsilon) \approx Jk[\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)} k(\delta, \varepsilon)$$

where the right hand side of the isomorphism denotes the set of finite sums $\sum f_i m_i$, $f_i \in J$, $m_i \in L$. It turns out that this set is, simply, the ideal generated by the inclusion $\iota(J)$ of J in the ring $k(\delta, \varepsilon)[\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)}$. That is, $Jk(\delta, \varepsilon)[\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)}$, the extension of the ideal J. So,

$$J \otimes_k L \approx Jk(\delta, \varepsilon)[\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)}.$$

One can also check that :

$$A \otimes_k L \approx k(\delta, \varepsilon)[\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)}.$$

On the other hand, letting $d := \lambda(A/J)$, one has that

$$\lambda(A/J) = d \Leftrightarrow \dim_k(A/J) = d \Leftrightarrow A/J \approx k^d.$$

Therefore,

$$A/J \otimes_k L \approx k^d \otimes_k L \approx L^d.$$

One has, thus, the following exact sequence

$$0 \to Jk(\delta, \varepsilon)[\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)} \to k(\delta, \varepsilon)[\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)} \to L^d \to 0$$

and, so

$$L^d \approx L[\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)} / JL[\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)}.$$

Therefore,

$$\dim_L (L [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)} / J L [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)}) = d .$$

Claim. $L [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)} / J L [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)} \approx B / J B.$

Proof of the claim. Let us recall that

$$B \approx L [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)} / K L [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)}$$

where $L := k (\delta, \varepsilon)$ (cf. 4.3.1). Hence,

$$B / J B = (L [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)} / K L [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)}) / J B$$

$$\approx L [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)} / (K L [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)} + J L [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)})$$

$$\approx L [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)} / J L [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)}$$

since, by assumption, $K L [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)} \subseteq J L [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)}$. This proves the claim.

We conclude that $\dim_L (B / J B) = d$. Hence, $\lambda (B / J B) = d$. But $d := \lambda (A / J)$. Therefore, $\lambda (A / J) = \lambda (B / J B)$. The proposition is, thus, proved.

Next, we will restrict the discussion to those ideals I whose proper transform I_1 satisfy certain conditions and show that, for those ideals, our inequality (#) of 4.1 does hold.

4.3.3. *Conditions on* I_1. We will study the special case where the generators a, b, c of I_1 satisfy the following conditions.

(i) Two of the generators, say a and b, have a common factor d. So,

$$a = a_1 d , b = b_1 d ;$$

(ii) d and c do not have a common factor ;

(iii) a_1 , b_1 , c form an A-regular sequence ; and

(iv) one of the following is true :

a$_1$, d, c is an A-regular sequence or b$_1$, d, c is an A-regular sequence.

4.3.4. *Example.* Let R := k [x, y, z]$_{(x,y,z)}$, and let I = (x^3, y^2, z) $\subset$ R. This ideal is an

M$_R$-primary ideal. So, let I$_1$ be the proper transform of I at U = Spec (R [M$_R$ / x]), an

openaffine piece of X := Bl (M$_R$, R). Then,

$$I R [M_R / x] = x^v I_1$$

where v = ord (I , M$_R$), i.e. I $\subset$ (M$_R$)^v, but I $\not\subset$ (M$_R$)$^{v+1}$ (cf. 2.3.8 (ii)). Therefore, if

we let α = x, β = y / x, γ = z / x, then I$_1$ = (α^2, $\alpha \beta^2$, γ), an ideal of the ring

A := k [α, β, γ]$_{(\alpha,\beta,\gamma)}$. If we let a = α^2, b = $\alpha \beta^2$, c =γ, then the generators a, b, c of I$_1$

satisfy conditions 4.3.3. Here, the common factor of a and b is d = α. As for condition

4.3.3 (iv), it is the sequence b$_1$, d, c (namely, β^2, α, γ) the one that turns out to be a

regular one, whereas the sequence a$_1$, d, c (namely, α, α, γ) is not.

4.3.5. *Remarks.* (i) Conditions 4.3.3 (ii) and (iii) imply that d is the greatest common

divisor of a and b.

(ii) Condition 4.3.3 (iv) imposes a restriction on d. This condition implies that d cannot

have a common factor with both a and b.

4.3.6. *Proposition.* Let I$_1$ be such that can be generated by elements a , b , c, satisfying

conditions 4.3.3. Then the ideal (d, c) does not have an embedded component.

Proof. We first note that d, c form a regular sequence. In order to see this, we recall that,

since A is an UFD, both d and c are regular elements. So, in order to show that d, c is a

regular sequence, we will show that c is A / (d) - regular : let c' t' = 0 in A': = A / (d).

Then, c t = f d (in A). But, since d and c do not have a common factor, d must divide t.

That is, t' = 0 in A'. Now, since d, c form a regular sequence,

$$\text{depth} (A / (d, c)) = 3 - 2 = 1 \neq 0$$

(cf. [Ku], 3.4, p. 184). Therefore, the ideal (d, c) has no embedded component
(2.1. 22) as claimed.

The ideal I_1 can, thus, be expressed as

$$I_1 = (d, c) \cap (a_1, b_1, c)$$

where (a_1, b_1, c) is the embedded component of I_1.

4.3.7. *Proposition.* A necessary and sufficient condition for the sequence a_1, b, c to be
A- regular is that the sequence a_1, d, c is A- regular.

Proof. Necessity. Let b' = b_1' d' $\in$ A / (a_1, c) and let b' x' = 0, x' $\in$ A / (a_1, c). Then
b_1' d' x' = 0. Since a_1, b_1, c is regular, d' x' = 0 and since, by assumption, a_1, d, c is
regular , d x' = 0 implies that x' = 0 . Therefore, a_1, b, c is A- regular.

Sufficiency. We prove the counterpositive argument. So, let us suppose that a_1, d, c is not
regular. Then if d' $\in$ A / (a_1, c) , there exists x' $\in$ A / (a_1, c), x' $\neq$ 0 such that d' x' = 0
and, hence, such that b' x' = b_1' d' x' = 0. Therefore a_1, b, c is not A- regular.

4.3.8. *Remark.* A statement similar to 4.3.7 holds for the sequences b_1, a, c and
b_1, d, c respectively. We can, thus, replace 4.3.3 (iv) by the following condition.

(iv)' one of the following is true :

a_1, b, c is an A- regular sequence or b_1, a, c is an A- regular sequence.

In carrying out step (iii), namely, showing that the ideal $I^- := M (I_1)$ satisfy the
condition of 4.3.2 so that (*) holds (cf. 4.3.10), we will assume that the generators a,

b, c of the ideal I_1 satisfy conditions 4.3.3. A first result in this direction is the following one.

4.3.9. *Proposition.* Let Ψ be the ring homomorphism of 4.3.1 and let I_1 satisfy conditions 4.3.3. Then if $K = \ker \Psi$,

$$K = (c\,\delta - a, c\varepsilon - b, a_1\varepsilon - b_1\,\delta)\, k\, [\,\alpha, \beta, \gamma, \delta, \varepsilon\,]$$

where $I_1 = (\,a, b, c\,) = (\,a_1\, d, b_1\, d, c\,) \subset k\,[\,\alpha, \beta, \gamma\,]_{(\alpha, \beta, \gamma)}$.

Proof. Let us recall that the ring homomorphism Ψ is given by :

$$\Psi \colon k\,[\,\alpha, \beta, \gamma, \delta, \varepsilon\,] \to k\,[\,\alpha, \beta, \gamma\,]\,[\,a/c\,,\,b/c\,]$$

$$\alpha \mapsto \alpha\,,\, \beta \mapsto \beta\,,\, \gamma \mapsto \gamma\,,\, \delta \mapsto a/c\,,\, \varepsilon \mapsto b/c\,.$$

Let

$$\Psi_1 \colon k\,[\alpha, \beta, \gamma, \delta, \varepsilon\,] \to k\,[\,\alpha, \beta, \gamma\,]\,[\,\varepsilon\,]\,[\,a/c\,]\,;\, \delta \mapsto a/c$$

and

$$\Psi_2 \colon k\,[\,\alpha, \beta, \gamma\,]\,[\,\varepsilon\,]\,[\,a/c\,] \to k\,[\,\alpha, \beta, \gamma\,]\,[\,a/c\,]\,[\,b/c\,]\,;\, \varepsilon \mapsto b/c\,.$$

Then $\Psi = \Psi_2 \circ \Psi_1$. In order to compute K, we will compute $K_1 := \ker \Psi_1$ and $K_2 := \ker \Psi_2$.

The elements a and c do not have a common factor because d and c do not have a common factor and a_1, c form an A- regular sequence (cf. 4.3.3 (ii) and (iii)).
Therefore,

$$K_1 = (\,c\,\delta - a\,)\,k\,[\,\alpha, \beta, \gamma, \delta, \varepsilon\,]$$

(cf. [Ku], 5.10 b, p. 152).

As for K_2, we have the following lemma.

4.3.9.1. *Lemma.* $K_2 = (\,c\,\varepsilon - b, a_1\,\varepsilon - b_1\,(\,a/c\,)\,)\,k\,[\,\alpha, \beta, \gamma, \varepsilon\,]\,[\,a/c\,]$.

Proof of 4.3.9.1. The proof goes by induction on deg f, $f \in K_2$. Let $T := \varepsilon$ throughout the proof.

Let deg $f = 1$. Write $f = r\,T + s$. Then $r\,(b/c) - s = 0$. Therefore, $r\,b - c\,s = 0$ in $k[\alpha, \beta, \gamma][a/c]$.

Claim. $r = f_1\,a_1 + g_1\,c$ for suitable f_1, g_1 in $k[\alpha, \beta, \gamma][a/c]$.

Proof of the claim. By 4.3.3 (iv), one of the following conditions holds :

$$(i)\ a_1, b, c \text{ is regular} ;\ (ii)\ a, b_1, c \text{ is regular}$$

(cf. 4.3.8). So let us suppose that (i) holds. So, a_1, b, c is a regular sequence. Then if $b' \in k[\alpha, \beta, \gamma][a/c]/(c, a_1)$, b' is not a zero divisor and, since

$$k[\alpha, \beta, \gamma][a/c]/(c, a_1) \approx (k[\alpha, \beta, \gamma][T]/(c\,T - a))/(c, a_1)$$

$$\approx (k[\alpha, \beta, \gamma]/(c, a_1))[T],$$

b' is not a zero divisor in $(k[\alpha, \beta, \gamma]/(c, a_1))[T]$. Now, $r'\,b' = c'\,s' = 0$ in $k[\alpha, \beta, \gamma][a/c]/(c, a_1)$. Therefore, $r' = 0$. That is, $r \in (c, a_1)$ and the claim is proved.

Continuing with the proof of the lemma, we note that

$$c\,(r\,T - s) = r\,(c\,T - b)$$

since $c\,(r\,T - s) - r\,(c\,T - b) = -c\,s + r\,b = 0$ in $k[\alpha, \beta, \gamma][a/c]$. Therefore,

$$c\,(r\,T - s) = r\,(c\,T - b) = (f_1\,a_1 + g_1\,c)\,(c\,T - b)$$

$$= f_1\,a_1\,(c\,T - b) + g_1\,c\,(c\,T - b)$$

$$= f_1\,a_1\,c\,T - f_1\,a_1\,b + g_1\,c\,(c\,T - b)$$

$$= f_1\,(c\,(a_1\,T - b_1\,(a/c))) + g_1\,c\,(c\,T - b).$$

Therefore,

$$f = r\,T - s = f_1(\,a_1\,T - b_1\,(\,a\,/\,c\,)\,) + g_1(\,c\,T - b\,).$$

Since $T := \varepsilon$, f satisfies the conclusion of 4.3.9.1. This proves the case $\deg f = 1$.

Let $d = \deg f$. Let us suppose the lemma true for $\deg f < d$. Let $f \in K_2$, write

$$f = r\,T^d + g(\,T\,),\ \deg g \le d - 1\,.$$

Then,

$$f(\,b\,/\,c\,) = r\,(\,b^d\,/\,c^d\,) + g(\,b\,/\,c\,) = 0$$

and

$$c^d\ f(\,b\,/\,c\,) = r\,b^d + c\,g' = 0$$

where $g' \in k\,[\alpha,\,\beta,\,\gamma\,]$. Therefore, if we regard r', $(b^d)'$ as elements of $K\,[\alpha,\,\beta,\,\gamma\,]\,/\,(c)$,

then, since b, c is a regular sequence in $k\,[\alpha,\,\beta,\,\gamma\,]$, $(b^d)'$ is not a zero divisor and, hence,

$r'\,(b^d)' = 0$ implies that $r' = 0$. that is, $r = q\,c$, where $q \in k\,[\alpha,\,\beta,\,\gamma\,]$. So, $r - q\,c = 0$.

Let $h = f - q\,T^{d-1}(\,c\,T - b\,)$. Then

$$h(\,T\,) = r\,T^d + q\,c\,T^d + \text{lower degree terms.}$$

That is,

$$h(\,T\,) = (\,r - q\,c\,)\,T^d + \text{lower degree terms}$$

$$= \text{lower degree terms}$$

since $r - q\,c = 0$. So, $h \in K_2$ and $\deg h < d$. By the induction hypothesis on h :

$$h(\,T\,) = f_1(\,c\,T - b\,) + f_2(\,a_1\,T - b_1\,(\,a\,/\,c\,)\,).$$

Thus,

$$f(\,T\,) = (\,q\,T^{d-1} + f_1\,)(\,c\,T - b\,) + f_2(\,a_1\,T - b_1\,(\,a\,/\,c\,)\,).$$

Both $q\,T^{d-1} + f_1$ and f_2 belong to $k\,[\alpha, \beta, \gamma]\,[\,a\,/\,c\,]$. Therefore, f is of the requred form and the lemma is proved.

We conclude that, $K = (\,c\,\delta - a,\,c\,\varepsilon - b,\,a_1\,\varepsilon - b_1\,\delta\,)\,k\,[\,\alpha, \beta, \gamma, \delta, \varepsilon\,]$. Proposition 4.3.9 is, thus, proved.

We want to show that $\lambda\,(\,A\,/\,I^\sim\,) = \lambda\,(\,B\,/\,I^\sim B\,)$. To do that, we will use 4.3.2. So, we need to show that the ideal $I^\sim$ satisfies the condition of 4.3.2.

4.3.10. Proposition. $K\,k\,(\,\delta, \varepsilon\,)\,[\,\alpha, \beta, \gamma\,]_{(\alpha,\beta,\gamma)} \subseteq I^\sim\,k\,(\,\delta, \varepsilon\,)\,[\,\alpha, \beta, \gamma\,]_{(\alpha,\beta,\gamma)}$.

Proof. Let $H := k\,(\,\delta, \varepsilon\,)\,[\,\alpha, \beta, \gamma\,]_{(\alpha,\beta,\gamma)}$. We start by giving a description of the ring $I^\sim H$. $I^\sim H$ denotes the extension of the ideal $I^\sim$ of $A = k\,[\,\alpha, \beta, \gamma\,]_{(\alpha,\beta,\gamma)}$ under the inclusion homomorphism $\iota : A \hookrightarrow H$. Therefore, it is the ideal of H generated by $\iota\,(\,I^\sim\,)$. So, it consists of finite sums of the form $\sum x_i\,f_i$, where $x_i \in H$, $f_i \in I^\sim$.

On the other hand, since H is the localization of the ring $k\,[\,\alpha, \beta, \gamma, \delta, \varepsilon\,]$ at the ideal $(\,\alpha, \beta, \gamma\,)$, the ideal K consists of finite sums of the form $\sum y_i\,(g_i\,/\,1\,)$, where $y_i \in H$, $g_i\,/\,1 \in f\,(\,K\,)\,(\,g_i \in K\,)$, where f denotes the localization morphism

$$f : k\,[\,\alpha, \beta, \gamma, \delta, \varepsilon\,] \to H\,;\,f\,(\,g_i\,) = g_i\,/\,1$$

(recall that 1 is an element of the multiplicative set $S := k\,[\,\alpha, \beta, \gamma, \delta, \varepsilon\,]\setminus(\,\alpha, \beta, \gamma\,)$, and that $H = S^{-1}\,k\,[\,\alpha, \beta, \gamma, \delta, \varepsilon\,]$).

Now, since $g_i \in K = (\,c\,\delta - a,\,c\,\varepsilon - b,\,a_1\,\varepsilon - b_1\,\delta\,)$ (cf, 4.3.8), $y_i\,(\,g_i\,/\,1\,)$ can be expressed as a linear combination of a_1, b_1, c with coefficients in H. Therefore, the finite sum $\sum y_i\,g_i$ can also be expressed as a linear combination of these elements and with coefficients in H. On the other hand, the ideal $(\,a_1, b_1, c\,)$ of A is the embedded component of the ideal I_1 (recall that $(\,I_1\,)_0 = (\,d, c\,)$ (cf. 4.3.5) and that

$I_1 = (d, c) \cap (a_1, b_1, c))$. Therefore, $(a_1, b_1, c) \subset ((a_1, b_1, c) : (d, c)) = \Gamma$. We have,

thus, shown that any element of K H is of the form $\sum x_i f_i$, $x_i \in$ H, $f_i \in (a_1, b_1, c) \subsetneq \Gamma$.

This means that K H $\subseteq \Gamma$ H, which is what we wanted to prove.

The following proposition follows now from 4.3.2 and 4.3.10.

4.3.11. *Proposition.* $\lambda (A / \Gamma) = \lambda (B / \Gamma B)$.

Once we have proved 4.3 (*), we proceed to prove that our main inequality 4.3

(#) is valid. The next one is the main theorem of this chapter.

4.3.12. *Theorem.* Let I be an M_R-primary ideal, where $R = k [x, y, z]_{(x,y,z)}$, k an

algebraically closed, such that I is generated by 3 elements and let I_1 be the proper

transform of I to the quadratic transform $A = k [\alpha, \beta, \gamma]_{(\alpha, \beta, \gamma)}$. Let I_1 be such that

$s (I_1) = 3$, that is, such that the minimal number of generators of any minimal reduction

of I_1 is 3, and such that it admits generators a, b, c of I_1 satisfying conditions 4.3.3. Then,

the inequality

$$e_A (\Gamma) < e_R (I)$$

holds, where $\Gamma := M (I_1)$.

Proof. Consider the ideal ΓB. Since B is 1-dimensional, $s (\Gamma B) = 1$ (cf. 2.2.11). B is

a CM ring because B is a 1-dimensional integral domain (B is a quotient of a ring with a

prime ideal (4.3.1)). So, depth (B) = dim (B).

The ideal ΓB has a reduction that is generated by one element f (2.2.6). So, let

(f) B $\subseteq \Gamma B$ be a minimal reduction. Then

$$(i) \quad \lambda (B / \Gamma B) \leq \lambda (B / (f) B).$$

On the other hand,

$$(\text{ ii }) \; e_A(\Gamma^-) \leq \lambda(A/\Gamma^-)$$

(cf. [M], 14.10). But, $\lambda(A/\Gamma^-) = \lambda(B/\Gamma^-B)$ (4.3.11). Therefore, it follows from (i) and (ii) that

$$e_A(\Gamma^-) \leq \lambda(B/(f)B).$$

But, since B is CM, $\lambda(B/(f)B) = e_B((f)B)$ (2.1.12). Hence, the inequality

$$e_A(\Gamma^-) \leq e_B((f)B)$$

is valid. But $e_B((f)B) = e_B(\Gamma^-B)$ (2.2.2). So, the inequality

$$e_A(\Gamma^-) < e_B(\Gamma^-B)$$

holds. This shows that the inequality (ii) of 4.1 holds. Also, in 4.2 we showed that the inequality (i) holds. That is, that the inequality

$$e_B(\Gamma^-B) < e_R(I)$$

holds. Combining the last two inequalities, we arrive at the announced conclusion. Namely, that the inequality

$$e_A(\Gamma^-) < e_R(I)$$

is valid.

4.4. The Case of Certain Monomial Ideals.

In this section we let $R := k[x, y, z]_{(x,y,z)}$, k an infinite field; $X := Bl(R, M_R)$, and will consider monomial ideals I that are M_R-primary and generated by 3 elements. These ideals must be of the form $I = (x^N, y^M, z^L)$, where N, M, L $\in$ N.

We will show directly that the inequality (#) of 4.1 holds for this type of ideals. We will also show that these ideals satify conditions 4.3.3.

Take the affine piece $U = \text{Spec} (R [M_R / x])$ of X, and consider the ideal I' given by

$$I' := I (R [M_R / x]) / (x)^v$$

where $v = \text{ord} (I, M_R)$, that is, $I \subseteq (M_R)^v$, but, $I \not\subseteq (M_R)^{v+1}$. This ideal correspomds to $I' |_U$, where I' is the global proper transform of I with respect to X (cf. 2.3.8 (ii)).

In order to study the ideal $I (R [M_R / x])$, let

$$(i) \quad \alpha := x, \quad \beta := y / x, \quad \gamma := z / x .$$

Therefore, $I (R [M_R / x]) = (\alpha^N, \alpha^M \beta^M, \alpha^L \gamma^L)$.

The ring $A := k [\alpha, \beta, \gamma]_{(x,y,z)}$ is a quadratic transform of the ring R (cf. 2.3.9), and the proper transform I_1 of I to A is given by $I_1 = I' A$ (cf. 2.3.9 (ii)).

We want to show in this section that the inequality

$$(\#) \quad e_A (I^-) < e_R (I)$$

is valid. It is not difficult to show that the only significant case is that where the exponents N, M, L satisfy the double inequality $N > M > L$. In any other case we get either (1) $I_1 = A$, in which case the multiplicity drops when we take the proper transform of I; or (2) $I_1 = (I_1)_0$, in which case $I^- := M (I_1) = A$, in which case (#) holds; or (3) I_1 is an M_A- primary ideal, in which case $I_1 = I^-$ and (#) holds (cf. [J], 2.2). We will, thus, assume that $N > M > L$.

4.4.1. *Theorem.* Let $I = (x^N, y^M, z^L) \subset R$ be an ideal, where $N > M > L$. Then the proper transform I_1 of I to the quadratic transform $A = k [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)}$, where α, β, γ are as in (i) above, is given by

$$I_1 = (\alpha^{N-L}, \alpha^{M-L} \beta^M, \gamma^L) A$$

and the inequality

$$e_A(\Gamma) < e_R(I)$$

is valid, where $\Gamma := M(I_1)$.

Proof. We have that

$$I(R[M_R/x]) = (\alpha^N, \alpha^M \beta^M, \alpha^L \gamma^L) = (\alpha^L)(\alpha^{N-L}, \alpha^{M-L}\beta^M, \gamma^L) = (\alpha^L) I'$$

where $\alpha = x$, $\beta = y/x$, $\gamma = z/x$. Therefore, since $I_1 = I' A$, our first conclusion follows.

In order to show that our inequality holds, note that we can express I_1 as:

$$I_1 = (\alpha^{M-L}, \gamma^L) \cap (\alpha^{N-M}, \beta^M, \gamma^L) = (I_1)_0 \cap N$$

where $(I_1)_0 = (\alpha^{M-L}, \gamma^L)$ is the non-embedded part of the primary decomposition of I_1,

and $N = (\alpha^{N-M}, \beta^M, \gamma^L)$ is the embedded component of I_1. Therefore, we have that

$N \subsetneq \Gamma$ and, hence,

$$(i) \quad e_A(\Gamma) \le e_A(N)$$

(cf. [M] , 14.4). On the other hand,

$$(ii) \quad e_A(N) = \lambda(A/N) = (N-M)ML$$

and

$$(iii) \quad e_R(I) = \lambda(R/I) = NML.$$

Combining (i), (ii) and (iii), we get that the inequality

$$e_A(\Gamma) < e_R(I)$$

does hold, proving 4.4.1.

We will now show that the generators of I_1 satisfy conditions 4.3.3, assuming the condition $N > M > L$ is fulfilled.

4.4.2. *Proposition.* Let I and I_1 be as in 4.4.1, then the generators of I_1 satisfy conditions 4.3.3.

Proof. In order to see that the generators of I_1 satisfy the conditions of 4.3.3, let $a := \alpha^{N-L}$, $b := \alpha^{M-L} \beta^M$, $c := \gamma^L$ and note that $d := \alpha^{M-L}$ is a common factor of a and b. Since α and γ are independent variables in A, c and d do not have a common factor. So, conditions (i) and (ii) of 4.3.3 hold.

To see that (iii) does hold, we note that $a_1 := \alpha^{N-M}$, $b_1 := \beta^M$, $c_1 := \gamma^L$ form an A- regular sequence.

Finally, to see that (iv) holds, we note that the sequence b_1, d, c is A- regular and 4.4.2 is, thus, proved.

Note that we could have given an alternative proof that the inequality (#) holds using Theorem 4.3.12 and 4.4.2, provided we could show that s (I_1) = 3. We will show that this happens if I_1 is integrally closed.

4.4.3. *Proposition.* If $I_1 = I_1^-$, then s (I_1) = 3.

Proof. Since $I_1 = I_1^-$ has an embedded component, depth (A / I_1^-) = 0 (2.1.22). It follows that s (I_1) = s (I_1^-) = 3 (cf. [N], 1.3).

4.4.4. *Remark.* According to 4.4.3, one could show that s (I_1) = 3 by showing that I_1 is complete. In our present case I_1 is a monomial ideal, that is, an ideal generated by monomials, and there is a method to obtain the completion (integral closure) of ideals of this type (cf. 2.2.14). It may not be true in general that I_1 is complete, say if N, M, L are large so that the convex closure N (I_1) of E (I_1) admits more integral points than

E (I_1) does. In example 4.3.4 one can easily see, using this method, that $I_1 = (\alpha^2, \alpha \beta^2, \gamma)$ is complete and, hence, that s (I_1) = 3.

Chapter 5

Summary and Conclusions

In this dissertation we study the problem of associating a numerical invariant to an ideal in a regular noetherian local ring, which will get strictly smaller when we take the proper transform of such an ideal to a quadratic transform of the ring. The importance of such result in proofs involving mathematical induction should be clear. Here, we provide a partial solution. Our approach is to attach to an ideal I of a regular local ring (R, M_R) the multiplicity $e_R(M (I))$, where M (I) is a certain M_R-primary ideal that we associate to I. In Chapter 3 we introduce the operation M (I) and investigate some of its basic properties. In Chapter 4 we study the behavior of the multiplicity of M (I) under quadratic transformations. (In Chapter 2 we collected some preliminary results that are necessary in the following chapters .) Next we present a more detailed summary of Chapter 3 (cf. 5.1) and Chapter 4 (cf. 5.2). We conclude with a section (5.3) where open problems, and possible future work are discussed.

5.1. Summary of Chapter 3.

In this chapter we introduced an operation M on ideals J of R, where R is a regular local ring. This operation assigns to J an ideal M (J) such that $J \subseteq M (J)$ and M (J) is M_R-prrimary or M (J) = R and, if J is M_R-primary, then M (J) = J. In any case, the multiplicity of M (J) is well defined.

In section 3.2, we study the behavior of the operation M under reductions of ideals. One result in this direction is that if I is integrally closed, then M (I) is also integrally

closed (3.2.2). In 3.2.4, we give sufficient conditions ontwo ideals $I \subsetneq J$, where I is a reduction of J, to insure the inclusion $M (I) \subseteq M (J)$.We do not necessarily get an integral extension as is shown inexample 3.2.5, where we give two ideals $I \subsetneq J$ satisfying the conditions of 3.2.4 and, hence, the inclusion $M (I) \subseteq M (J)$, but $M (J)$ not being integral over $M (I)$.

5.2. Summary of Chapter 4.

In this chapter we are concerned with the inequality

$$(\#) \quad e_A(\Gamma) < e_R(I)$$

where $\Gamma := M (I_1)$, I_1 as usual, the proper transform of the M_R-primary ideal I to a quadratic transform A of R, R being a 3-dimensional regular local (noetherian) ring.

In order to study the possible inequality (#), we first impose some conditions on both I and I_1. We first assume that I is generated by 3 elements forming a regular sequence. We also assume that I_1 does not have any minimal reduction that is generated by less than 3 elements. That is to say, that the analytic spread of I_1 is $s (I_1) = 3$ (2.2.12).

The strategy we propose is to consider the following two inequalities :

$$(i) \quad e_B(\Gamma B) < e_R(I)$$

$$(ii) \quad e_A(\Gamma) \leq e_B(\Gamma B)$$

where $B := O_{Z,Q}$, Q being the generic point of $\pi^{-1} (0)$, where

$$\pi : Z \to Z_0$$

is the blowing-up morpphism for $Z := Bl (A, I_1)$, $Z_0 := Spec A$, and 0 is the closed point of Z_0. To show that B is well defined we proved that $\pi^{-1}(0)$ is irreducible under the assumption that $s (I_1) = 3$ (cf. 4.2.1).

Obviously, the two innequalities (i) and (ii) would imply the inequality (#). In section 4.3 we achieved to show that (i) holds in a general setting (cf. 4.2.4). To prove that (ii) holds, it suffices to verify that the inequality

$$\lambda (A / \Gamma) \leq \lambda (B / \Gamma B)$$

holds, where λ ( ) denotes the length of a module, since $e_A (\Gamma) \leq \lambda (A / \Gamma)$ (cf. [M], 14.10). We have not succeded to verify this in general. But if $R = k [x, y, z]_{(x,y,z)}$, with k an algebraically closed field, one can give an explicit description of the ring B. Indeed, any quadratic transform A of R will be of the form $A = k [\alpha,\beta, \gamma]_{(\alpha,\beta,\gamma)}$ and B will be a quotient of a ring of the form $H = k (\delta, \varepsilon) [\alpha, \beta, \gamma]_{(\alpha,\beta,\gamma)}$ (where α, β, γ , δ, ε denote indeterminates) by an ideal J. If one could verify that $J \subseteq \Gamma H$, one would obtain, indeed, an *equality* $\lambda (A / \Gamma) = \lambda (B / \Gamma B)$. We can verify that this inclusion of ideals does hold, provided we impose some additional hypotheses on I_1 ; these are rather technical and appear in 4.3.3 . That is, if R $[x, y, z]_{(x,y,z)}$ and the additional hypotheses (4.3.3) do hold, then the inequality (#) is valid.

In the last section of the chapter we discuss the case of M_R-primary ideals that are generated by 3 monomials and show that for this type of ideals the inequality (#) holds. We also show that that these ideals satisfy the hypotheses of 4.3.3. In some special cases we can show that the analytic spread of the proper transform I_1 of an ideal of this type is 3. That is, whenever we can show that I_1 is a complete ideal (cf. [N], 1.3) following the procedure given in 2.2.14.

5.3. Suggestions for Future Work.

It would be natural to try to eliminate the (probably superfluous) hypotheses that we use in chapter 4. One could try, for instance, the following .

(1) To eliminate the condition s (I_1) = 3. The other possibility is that s (I_1) = 2. Probably a special argument could show that (#) is valid under the latter condition.

(2) To prove directly that λ (A / Γ) $\leq \lambda$ (B / Γ B). Probably even the *equality* holds. If R = k [x, y, z] $_{(x,y,z)}$, our proof reduces to showing an inclusion of certain ideals. This inclusion could be true in general. In our work we verify it with some direct calculations that obligue us to make some additional hypotheses (cf. 4.3.3), but maybe this can be verified, in general, in a different way.

The condition s (I_1) = 3 deserves some attention. It is known that if s (I_1) < 3, then I_1^- does not have an embedded component (cf. [N], 1.3). One can ask whether I_1 has an embedded component. In general, it is not true that if s (J) < 3, then J does not have an embedded component. The following example, that was communicated by W. Heinzer, shows the latter claim.

5.3.1. *Example.* Consider the ideal

$$J = (\, x^4, x^3\, y, x^2\, y^2\, z, x\, y^3, y^4 \,) \, R$$

of the ring R = k [x, y, z] $_{(x,y,z)}$. One can check that (J : $x^2\, y^2$) = (x, y, z) R = M_R, showing that $M_R \in$ ass J, that is, that J has an embedded component. Now let

$$P = (\, x, y \,) \, R.$$

Then s (P) = 2 . In fact, it is easy to check that $J^2 = P^8$. Now, using the fact that for an

ideal I, $s(I) = s(I^n)$, for $n \geq 1$, we have that

$$s(J) = s(J^2) = s(P^8) = s(P) = 2.$$

Still one can ask whether $s(I_1) = 3$ whenever I_1 has an embedded prime. The previous example illustrates the fact that, were this the case, it would not be the consequence of a general fact.

The search of examples (or counterexamples) using computers (say, using the system MACAULAY or the European CoCoA) is another possibility for future work. In particular, one could use examples to test the inequality (#) or the inclusion $J \subseteq \Gamma H$ mentioned in 5.2.

Another invariant that could give a variant of the formula (#) is the following one. If I is an M_R-primary ideal in aregular local ring (R, M_R) having infinite residue field k and (A, M_A) is a quadratic transform of R, let

$$I^* := M(I_1^-)$$

where I_1 is the proper transform of I to A. One may conjecture that

$$(*) \quad e_A(I^*) < e_R(I).$$

Indeed, if (#) were valid, we could verify (*) (assuming $\dim(R) = 3$), provided we could show the following :

5.3.2. *Conjecture.* There is a minimal reduction J of I such that $J = (a, b, c)$ and such that if $J_1 = (a_1, b_1, c_1)$ is the proper transform of J to A, then $M(I_1^-) \supseteq M(J_1)$.

Note that since I is an M_R-primary ideal its height is 3 (cf. 3.1.3), then $s(I) = 3$ (cf. 2.2.11), and if J is a minimal reduction of I, then $\mu(J) = s(I) = 3$ (cf. 2.1.12) where $\mu(J)$ denotes the number of a minimal set of generators of J.

Also note that, since I_1^- is complete and integral over J_1 (since I is integral over J),the inclusion $M(I_1^-) \supseteq M(J_1)$ of 5.3.2 will hold if $(I_1^-)_0$ is integral over $(J_1)_0$, according to 3.2.4.

Indeed, if 5.3.2 is true, we take a reduction $J = (a, b, c)$ of I as in 5.3.2. Then

$$e_R(I) = e_R(J) > e_A(M(J_1)) \geq e_A(M(I_1^-)) = e_A(I^*)$$

where the first equality holds by 2.2.2 ; the first inequality holds because we are assuming that (#) is valid ; and the second inequality holds by 5.3.2 and [M], 14.4

If one can prove 5.3.2, one could try to prove (*) directly using the technique of Chapter 4. As we explain next, there are certain advantages.

First, we replace I by $J = (a, b, c)$. If we let J_1 denote the proper transform of J with respect to A, then the inequality (*) easily follows if $s(J_1) < 3$. In fact, then $s(J_1^-) = s(J_1)$ (the analytic spread depends on the normalized blowing-up (cf. 2.3.7)). Now, since (as is well known) from the fact that J is a reduction of I it follows that J_1 is a reduction of I_1, we conclude that

$$s(J_1^-) = s(I_1^-) = 2.$$

But, since $s(I_1^-) < 3$, it follows that I_1^- has no embedded component (cf. [N], 1.3), hence $M(I_1^-) = A$ and hence $e_A(I^*) = e_A(A) = 0$; so clearly (*) holds here. So, the non trivial case is $s(J_1) = 3$, i. e. in this case we do not need this assertion. The strategy of Chapter 4 applies with little change . If in adition J_1 satisfies the conditions of 4.3.3, then, as in Chapter 4, we get the inequality (since we have some freedom to choose the reduction $J = (a, b, c)$ of I, it is enough to show that the hypotheses of 4.3.3 hold for one suitable choice). It is, thus, interesting to try to prove 5.3.2.

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Vita

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Date of Examination:

July 5, 1995   _____