

Louisiana State University

LSU Scholarly Repository

LSU Historical Dissertations and Theses

Graduate School

1995

On Some Problems in the Algebraic Theory of Quadratic Forms.

Hamza Y. Ahmad

Louisiana State University and Agricultural & Mechanical College

Follow this and additional works at: https://repository.lsu.edu/gradschool_disstheses

Recommended Citation

Ahmad, Hamza Y., "On Some Problems in the Algebraic Theory of Quadratic Forms." (1995). *LSU Historical Dissertations and Theses*. 5991.

https://repository.lsu.edu/gradschool_disstheses/5991

This Dissertation is brought to you for free and open access by the Graduate School at LSU Scholarly Repository. It has been accepted for inclusion in LSU Historical Dissertations and Theses by an authorized administrator of LSU Scholarly Repository. For more information, please contact gradetd@lsu.edu.

INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI

**A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
313/761-4700 800/521-0600**

ON SOME PROBLEMS IN THE ALGEBRAIC THEORY OF
QUADRATIC FORMS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Hamza Y. Ahmad
B.S., Kuwait University, 1990
M.S., Louisiana State University, 1992
August, 1995

UMI Number: 9609065

UMI Microform 9609065

Copyright 1996, by UMI Company. All rights reserved.

**This microform edition is protected against unauthorized
copying under Title 17, United States Code.**

UMI

**300 North Zeeb Road
Ann Arbor, MI 48103**

Acknowledgments

The author wishes to thank his advisor, Prof. Jack Ohm, for help and encouragement during the preparation of this dissertation. Also thanks to Prof. Jurgen Hurrelbrink for many discussions.

The author also wishes to thank his family and friends, for their support. Special thanks go to Prof. Mowaffaq Hajja from Al-Yarmouk University for his continuous support and encouragement.

Table of contents

| | |
|--|----|
| Acknowledgements | ii |
| Abstract | iv |
| Chapter | |
| 1. Introduction | 1 |
| 1.1 Terminology, notations and basic facts | 1 |
| 1.2 Function fields | 4 |
| 2. Witt kernels of inseparable extensions | 7 |
| 2.1 Inseparable quadratic extensions | 8 |
| 2.2 Excellence of quadratic extensions | 15 |
| 2.3 Bi-quadratic extensions | 21 |
| 3. Function fields of quadratic forms and similarity | 26 |
| 3.1 Pfister neighbors and special Pfister neighbors | 27 |
| 3.2 Function fields of special Pfister neighbors | 35 |
| 3.3 The Zariski cancellation problem | 39 |
| 3.4 Similarity and function fields | 40 |
| 4. The Pfister-Leep conjecture | 49 |
| 4.1 A System of forms | 50 |
| 4.2 The main result | 54 |
| References | 59 |
| Appendixes: Letters of permission | 61 |
| Vita | 63 |

Abstract

This work consists of results on three questions in the algebraic theory of forms.

The first question deals with characterizing the Witt kernel (i.e. the anisotropic non-singular quadratic forms over that become hyperbolic) over a given field extension. For separable quadratic and bi-quadratic extension this is well known (for example see [B1, 4.2 and 4.3], [B2, p. 121], [L, p. 200], [ELW, 2.12]). In chapter 2, we provide answers to this question for inseparable quadratic and bi-quadratic extensions. We provide theorem 2.1.5, which in particular answers question 4.4 in [B2]. From this result we prove the excellence property for inseparable quadratic extension, which is in turn used to find the Witt kernels of inseparable bi-quadratic extensions.

In the third chapter we study the relation between similarity of quadratic forms and isomorphism and place equivalence of their function fields. In sections 3.1 and 3.2, we show that the function fields of special Pfister neighbors of the same Pfister form are isomorphic. Also we show that any Pfister neighbor of codimension ≤ 4 is special; in particular this implies place equivalence implies birational equivalence in this case. Together with the main result of [H3], this gives an affirmative answer of the quadratic Zariski problem in dimension 3. (see §3.3). In §3.4 we provide few results on the problem of descent of similarity over field extensions and some examples where similarity is determined by their generic splitting tower.

In the last chapter we provide a positive answer for the following conjecture of Pfister-Leep in the special case $d =$ the characteristic of the field k

CONJECTURE. For a fixed d , if k is a C_0^d -field, then k is a p -field for some prime $p \neq d$.

Chapter 1

Introduction

1.1 Terminology, Notations and Basic Facts

Our standard references on quadratic forms are Lam's book [L], Scharlau's book [Sch] and Baeza's lecture notes [B2].

Let k be a field. The set of non-zero elements of k will be denoted by k^* .

A *quadratic k -form* q (or simply a form) is a map from a finite dimensional k -vector space V to k satisfying: (i) For every $a \in k$ and $\mathbf{x} \in V$, $q(a\mathbf{x}) = a^2q(\mathbf{x})$, and (ii) $B_q(\mathbf{x}, \mathbf{y}) := q(\mathbf{x} + \mathbf{y}) - q(\mathbf{x}) - q(\mathbf{y})$ is a bilinear map. (V, q) is called a quadratic space. For simplicity we write q for (V, q) . The dimension of q , $\dim q := \dim V$. After fixing a basis of V , we may consider the map q as a homogeneous polynomial of degree 2 in n coordinate indeterminates, where $n = \dim V$. We will sometimes work interchangeably with both the quadratic map and the polynomial and make no distinction between them.

Two quadratic spaces (V, q_1) , (V, q_2) are *isometric*, denoted by $q_1 \cong q_2$, if there exists a linear isomorphism $L : V \rightarrow V$ such that for every $\mathbf{x} \in V$, $q_1(\mathbf{x}) = q_2(L\mathbf{x})$ (This equivalent to saying that the polynomial q_1 can be obtained from the polynomial q_2 by an invertible linear change of variables). The forms q_1 and q_2 are called *similar*, denoted by $q_1 \approx q_2$, if there exists $a \in k^*$ such that $q_1 \cong aq_2$. A form q is called *non-singular* if the subspace $V^\perp := \{\mathbf{x} \in V \mid B_q(\mathbf{x}, \mathbf{y}) = 0 \text{ for all } \mathbf{y} \in V\} = 0$, or equivalently, if the matrix $(B_q(\mathbf{e}_i, \mathbf{e}_j))$ with respect to a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of V is nonsingular.

Two vectors \mathbf{x} and \mathbf{y} are called *orthogonal* if $B_q(\mathbf{x}, \mathbf{y}) = 0$. The form q is said to *represent* an element $a \in k$, written $q \sim a$, if there exists a vector

$\mathbf{x} \in V$ such that $q(\mathbf{x}) = a$. The set of elements non-trivially represented by q over k is denoted by $D_k(q)$; the subscript will sometimes be omitted. The form q is called *isotropic* if it represents 0 non-trivially; otherwise it is called *anisotropic*. A subspace W of V is said to be *totally isotropic* if $q|_W = 0$. If (V_1, q_1) and (V_2, q_2) are quadratic spaces, then their *orthogonal sum* (V, q) is defined by: $V = V_1 \oplus V_2$ and $q(\mathbf{x}_1 \oplus \mathbf{x}_2) := q_1(\mathbf{x}_1) + q_2(\mathbf{x}_2)$. In this case we write $q = q_1 \perp q_2$.

The non-singular isotropic two dimensional is called a *hyperbolic plane*, and will be denoted by H . With a suitable choice of basis, the polynomial corresponding to H is XY . A form is called *hyperbolic* if it is isometric to $r \times [0, 0] := [0, 0] \perp \dots \perp [0, 0]$ (r summands). Any non-singular form q can be decomposed into $q \cong rH \perp q_1$ with q_1 anisotropic. The positive integer r is uniquely determined by q and so is q_1 , up to isometry. The form q_1 is called the *anisotropic part* or the *kernel* of q , and r is called the *Witt index* of q . Two nonsingular forms q_1 and q_2 are said to be *Witt equivalent* if they have isometric kernel forms. This is an equivalence relation and the equivalence classes form an abelian group called the *Witt group* of k and denoted by $W(k)$. (Actually $W(k)$ is a ring with the multiplication being the tensor product of forms, but it will not be needed in this work.)

Let q be a k -form and L/k be a field extension. Then q is also an L -form. We write q_L or $q \otimes L$ for the L -form q .

1.1.1 Forms over fields of characteristic $\neq 2$. Over such fields, any quadratic form is isometric to a diagonal form $\langle a_1, \dots, a_n \rangle := a_1 X_1^2 + \dots + a_n X_n^2$. (This corresponds to choosing a basis $\{e_1, \dots, e_n\}$ of pairwise orthogonal vectors with $q(e_i) = a_i$.) The form q is non-singular if the product $a_1 \cdots a_n \neq 0$. The class of the element $\det(q) := a_1 \cdots a_n$ in $k^*/(k^*)^2$ is called

the *determinant* of q . The determinant of a quadratic form q does not change when q is replaced by another form isometric to q . The hyperbolic plane H is isometric to $\langle 1, -1 \rangle$.

1.1.2 Forms over fields of characteristic 2. For nonsingular two dimensional quadratic space (P, q) over a field of characteristic 2 there exists a basis \mathbf{e}, \mathbf{f} of P such that $B_q(\mathbf{e}, \mathbf{f}) = 1$. With respect to such a basis, q is denoted by $[a, b]$, where $q(\mathbf{e}) = a$, and $q(\mathbf{f}) = b$. Arf showed in [1, Satz 2] that any nonsingular quadratic space (P, q) is the orthogonal sum of such nonsingular two dimensional subspaces:

$$(P, q) = P_1 \perp \dots \perp P_n.$$

In particular, the dimension of a nonsingular quadratic space is even. For $i = 1, \dots, n$, let $\mathbf{e}_i, \mathbf{f}_i$ be a basis of P_i such that $B_q(\mathbf{e}_i, \mathbf{f}_i) = 1$, and let $a_i = q(\mathbf{e}_i)$ and $b_i = q(\mathbf{f}_i)$. Then $\mathbf{e}_1, \mathbf{f}_1, \dots, \mathbf{e}_n, \mathbf{f}_n$ is a basis of P with respect to which $q = [a_1, b_1] \perp \dots \perp [a_n, b_n]$. The *Arf invariant* of a non-singular quadratic form is defined to be the class of the element $a_1 b_1 + \dots + a_n b_n$ in k modulo the (additive) subgroup $\wp P(k) := \{c^2 - c \mid c \in k\}$. The following known result (see [Sch, p. 341, Lemma 4.4.(i)]) will be used frequently in chapter 2 without reference.

PROPOSITION *Let q_1 and q_2 be non-singular forms of dimension 2. Then q_1 and q_2 are isometric if and only if q_1 and q_2 represent a common element and have the same Arf invariant.*

1.1.3 Pfister forms. Let k be a field of characteristic $\neq 2$. The 0-fold Pfister form is $\langle 1 \rangle$, the 2-fold Pfister form is $\langle 1, a \rangle \perp d \langle 1, a \rangle$, and an n -fold Pfister form is defined to be $Q \perp dQ$ where $d \in k^*$ and Q is an $(n - 1)$ -fold Pfister form. The following important properties of Pfister forms will be used frequently in chapter 2.

For a Pfister form P we have :

- (i) P is non-singular form of dimension 2^n , and for $n > 1$, P has determinant 1.
- (ii) If P is isotropic, then P is hyperbolic.
- (iii) If P represents $a \in k^*$, then $P \cong aP$.

It is known that a four dimensional form is similar to a Pfister form if and only if it has determinant 1.

For fields of characteristic 2, the 0-fold Pfister form is $[1, a]$ where $a \in k^*$, and an n -fold Pfister form is defined to be $Q \perp dQ$ where $d \in k^*$ and Q is an $(n - 1)$ -fold. In characteristic 2, a four dimensional form is similar to a Pfister form if and only if it has trivial Arf invariant. The properties (ii) and (iii) above still hold for Pfister forms in characteristic 2.

1.2 Function Fields

Here we recall some of the basic definitions and results about function fields of quadratic forms that will be used frequently in chapter 3.

Let k be a field of characteristic $\neq 2$. A *function field* is a finitely generated field extension K/k . We use \dim for the transcendence degree of a function field. A pure transcendental function field is also said to be *rational*, and we consider the case that $K = k$ to be a rational extension of $\dim 0$. A *generic zero* of an irreducible polynomial $f(X)$ in n indeterminates (over k) is an n -tuple (x) of elements (from Ω) such that $f(x) = 0$ and $\dim k(x)/k = n - 1$. Every irreducible polynomial has a generic zero, which determines a field extension $k(x)/k$ up to k -isomorphism; and the *function field* of $f(X)$ is defined to be this extension $k(x)/k$ (or, more precisely, any element of the isomorphism class of this extension). irreducible quadratic form $Q(X_0, X_1, \dots, X_n)$ is the function field of the *dehomogenized* polynomial $Q^{af} := Q(1, X_1, \dots, X_n)$ (which is also

irreducible) and is denoted $k(Q)$. The adjective ‘affine’ will henceforth be dropped, it being understood that by function field of a form we mean its *affine* function field. (Note: The function field of a form is independent of which indeterminate is used to dehomogenize the form.) A function field of a *nonsingular* quadratic form is also called a *quadratic function field* for short. Two irreducible forms Q and Q' are said to be *birationally equivalent* if $k(Q) \cong k(Q')$.

Any quadratic form of dimension > 2 is irreducible, hence it has a function field. A two dimensional form is irreducible if and only if it is anisotropic. Also, any two non-singular isotropic form of the same dimension have the same function field.

PROPOSITION 1.2.1 (cf. e.g. [KI, p. 72, Proposition 3.8]) *An irreducible nonsingular form Q is isotropic iff $k(Q)/k$ is rational.*

The next theorem is essentially an application of the Cassels-Pfister subform theorem (cf. [L, p. 262, Theorem 2.8]).

THEOREM 1.2.2 *If P is an anisotropic Pfister form and Q an irreducible nonsingular form, then P becomes isotropic over $k(Q)$ (if and) only if Q is similar to a subform of P .*

COROLLARY 1.2.3 *If an irreducible nonsingular form Q is birationally equivalent to a subform of an anisotropic Pfister form P , then Q is similar to a subform of P . In particular birationally equivalent irreducible Pfister forms are isometric.*

We will need the following recent and powerful result (see [H4, theorem 1], also [HR2]).

THEOREM 1.2.4 *Let P and ψ be k -forms with P anisotropic and $\dim P \leq 2^n < \dim \psi$ for some $n \geq 0$. Then P stays anisotropic over $k(\psi)$.*

Following [K1], a field K/k is called a *generic zero field* for a non-singular k -form q if q is isotropic over K and for any L/k with q_L isotropic, there exists a k -place $\gamma : K \rightarrow L \cup \{\infty\}$. The function field of q is a generic zero field for q . A *generic splitting tower* k_0, k_1, \dots, k_h of a non-singular quadratic k -form q (assumed irreducible if $\dim = 2$) can be constructed (inductively) as follows: set $k_0 = k$ and $q_0 = (q)_{an}$, the anisotropic part of q . If $\dim q_0 \leq 0$, stop with $k_0 = k$. Otherwise choose k_1 a generic zero field of q_0 . We repeat this by choosing k_i a generic zero field of q_i , the anisotropic part of q over k_{i-1} ; and letting q_i be the kernel of $q \otimes k_i$. We stop at h (called the *height* of q) is the smallest integer such that $\dim q_h \leq 1$. The form q_{h-1} is a Pfister form, and is called the *leading form* of q . The degree of q , written $\deg q$, is n where 2^n is the dimension of the leading form. One generic splitting tower can be constructed by taking k_i , $i > 0$, to be the function field of q_{i-1} over k_{i-1} .

Chapter 2

Witt kernels of inseparable extensions

An important question in the algebraic theory of quadratic forms is to determine the behavior of quadratic forms over field extensions. More precisely, if K/k is a field extension, then the inclusion $i : k \rightarrow K$ induces a group homomorphism $i^* : W(k) \rightarrow W(K)$ of the Witt groups given by $\varphi \mapsto \varphi \otimes K$ for any $\varphi \in W(k)$. The question is to determine the kernel of i^* , that is, to determine the anisotropic k -forms that become hyperbolic over K . This kernel is called the Witt kernel of the extension K/k . When K/k is an algebraic extension of odd degree, Springer [Sp] showed that i^* is a monomorphism. If k has characteristic $\neq 2$ and K/k is a quadratic extension, it is a well known

Theorem. ([4, p. 200, Lemma 3.1 and Theorem 3.2]) If φ is an anisotropic quadratic k -form that becomes isotropic over $K = k(\sqrt{d})$, then there exists $c \in k$ such that $c(X^2 - dY^2)$ is a subform of φ ; and if φ becomes hyperbolic over K , then there is a k -form q such that $\varphi \cong q \perp -dq$.

Similar results hold when k is of characteristic 2 and K is a separable quadratic extension: If $K = k[X]/(X^2 + X + b)$ and φ is a nonsingular anisotropic quadratic k -form which becomes isotropic over K , then there exists $c \in k$ such that $c(X^2 + XY + bY^2)$ is a subform of φ (see [B2, p. 121, Theorem 4.2]).

In this chapter we determine the Witt kernel for purely inseparable quadratic extensions (§2.1) and for bi-quadratic extensions over fields of characteristic 2 (§2.3). In particular, the first section answers in the affirmative a question of Baeza [B1, 4.4].

2.1 Inseparable quadratic extensions

Throughout this section, let k be a fixed field of characteristic two and $K = k(\sqrt{d})$ be an inseparable quadratic extension of k . Here we prove (see 2.1.8) that if a non-singular anisotropic k -form φ becomes hyperbolic over K , then φ is *isometric* to $q \perp dq$ for some k -form q . This answers a question of Baeza [B1, 4.4] who showed the corresponding weaker assertion with isometry replaced by Witt equivalence. Actually we prove a sharper result (see 2.1.6). The results in this section have been published as [A].

PROPOSITION 2.1.1 (cf. [A, 2.1]) *Let (V, q) be a nonsingular quadratic space and S be a totally isotropic subspace of V . If the Witt index of q is greater than $\dim(S)$, then S is contained in a totally isotropic subspace of dimension $1 + \dim(S)$.*

PROOF. The space V contains a totally isotropic subspace T of dimension equal to the Witt index of q ; for if $r \times [0, 0]$ is a subform of q , we can pick an isotropic vector \mathbf{e}_i in each $[0, 0]$ and $\mathbf{e}_1, \dots, \mathbf{e}_r$ span a totally isotropic subspace. Let $R = S \cap T$. Then $S = R \oplus S_0$ and $T = R \oplus T_0$, where S_0 and T_0 are subspaces of S and T respectively. Since by hypothesis $\dim(T) > \dim(S)$, $\dim(T_0) - \dim(S_0) = \dim(T) - \dim(S) > 0$; and since (V, q) is nonsingular, $\dim(S_0^\perp) = \dim(V) - \dim(S_0)$. Note that

$$\begin{aligned} \dim(T_0 \cap S_0^\perp) &= \dim(T_0) + \dim(S_0^\perp) - \dim(T_0 + S_0^\perp) \\ &\geq \dim(T_0) + \dim(V) - \dim(S_0) - \dim(V) > 0 \end{aligned}$$

Thus, there exists a non-zero vector $\mathbf{v} \in T_0 \cap S_0^\perp$. Since $\mathbf{v} \in T (= R \oplus T_0)$ and T is totally isotropic, $\mathbf{v} \in R^\perp$. Hence $\mathbf{v} \in (R \oplus S_0)^\perp = S^\perp$. Also $\mathbf{v} \notin S$; otherwise, $\mathbf{v} \in S \cap T = R$, hence $\mathbf{v} \in R \cap T_0 = 0$, a contradiction. Then the subspace generated by S and \mathbf{v} . is totally isotropic, because S is totally

isotropic, \mathbf{v} is isotropic, and $\mathbf{v} \in S^\perp$. Since $\mathbf{v} \notin R = S \cap T$, this subspace contains S properly. \square

The following lemma is a slight variation of [B1, Lemma 3.1].

LEMMA 2.1.2 (cf. [A, 2.2]) *Let (V, q) be a nonsingular quadratic space and $\mathbf{x}_1, \dots, \mathbf{x}_s \in V$ be linearly independent vectors such that $B_q(\mathbf{x}_i, \mathbf{x}_j) = 0$ for $1 \leq i, j \leq s$. Let $q(\mathbf{x}_i) = a_i$ for $i = 1, \dots, s$. Then there exist $b_1, \dots, b_s \in k$ such that $[a_1, b_1] \perp \dots \perp [a_s, b_s]$ is a subform of q .*

PROOF. Let $n = \dim(V)$. Since q is nonsingular and $\mathbf{x}_1, \dots, \mathbf{x}_s$ are linearly independent, $\dim(\{\mathbf{x}_1, \dots, \mathbf{x}_s\}^\perp) = n - s$ and $\dim(\{\mathbf{x}_2, \dots, \mathbf{x}_s\}^\perp) = n - s + 1$ (if $s = 1$, put $\{\mathbf{x}_2, \dots, \mathbf{x}_s\}^\perp = V$). Thus there exists $\mathbf{y}_1 \in \{\mathbf{x}_2, \dots, \mathbf{x}_s\}^\perp$ such that $B_q(\mathbf{x}_1, \mathbf{y}_1) = 1$. Let W be the subspace generated by $\mathbf{x}_1, \mathbf{y}_1$. Then $W = [a_1, b_1]$, where $b_1 = q(\mathbf{y}_1)$. In particular W is nonsingular, and hence $V = W \perp W^\perp = [a_1, b_1] \perp W^\perp$ (by [B2, p. 10, Proposition 3.2]). If $s = 1$ then we are done. If $s > 1$, then the statement follows by induction on s since $\mathbf{x}_2, \dots, \mathbf{x}_s \in W^\perp$. \square

We will need the following criterion for representing elements of k over $K = k(\sqrt{d})$.

PROPOSITION 2.1.3 (cf. [A, 2.3]) *Let q be a nonsingular k -form and let $a \in k$ which is not represented by q over k . Then q represents a over K (if and) only if either*

- (i) *there exist c and e in k such that $[a(c^2 + d), e]$ is a subform of q , or*
- (ii) *there exist $b, e, f \in k$ such that $[b, e] \perp [a + bd, f]$ is a subform of q .*

PROOF. Suppose q represents a over K . Let \mathbf{v} be a K -vector such that $q(\mathbf{v}) = a$. Since q does not represent a , $\mathbf{v} = \mathbf{v}_1 + \sqrt{d}\mathbf{v}_2$ where \mathbf{v}_1 and \mathbf{v}_2 are k -vectors and $\mathbf{v}_2 \neq 0$. Since $a \in k$ and $\sqrt{d} \notin k$, $a = q(\mathbf{v}) = q(\mathbf{v}_1) + dq(\mathbf{v}_2) +$

$\sqrt{d}B_q(\mathbf{v}_1, \mathbf{v}_2)$, which implies that

$$\begin{aligned} 0 &= B_q(\mathbf{v}_1, \mathbf{v}_2), \text{ and} \\ a &= q(\mathbf{v}_1) + dq(\mathbf{v}_2). \end{aligned}$$

If $\mathbf{v}_1 = c\mathbf{v}_2$ for some $c \in k$, then $a = (c^2 + d)q(\mathbf{v}_2)$. In this case q represents $a(c^2 + d)$ over k ; hence by Lemma 2.1.2 (with $s = 1$), there exists $e \in k$ such that $[a(c^2 + d), e]$ is a subform of q . On the other hand, if \mathbf{v}_1 and \mathbf{v}_2 are independent, let $b = q(\mathbf{v}_2)$ and apply 2.1.2 to $\mathbf{v}_1, \mathbf{v}_2$ to conclude that there exist $e, f \in k$ such that $[b, e] \perp [a + bd, f]$ is a subform of q . \square

Since the form $[0, e]$ is isotropic, the case of 2.1.3 with $a = 0$ is

COROLLARY 2.1.4 (cf. [A, 2.4]) *Let q be a nonsingular anisotropic k -form. Then q becomes isotropic over K (if and) only if there exist $b, e, f \in k$ such that $[b, e] \perp [bd, f]$ is a subform of q .*

(This corollary also appears as a part of the proof of [B1, Lemma 4.3].)

THEOREM 2.1.5 (cf. [A, 2.5]) *Let (V, q) be a nonsingular anisotropic quadratic k -form and let $K = k(\sqrt{d})$. If the Witt index of q over K equals $r \geq 1$, then either*

- (i) *there exist $a, b \in k$ such that $[a, b] \perp d[a, b]$ is a subform of q , or*
- (ii) *there exist $a_i, b_i, c_i \in k$ ($i = 1, \dots, r$) such that $([a_1, b_1] \perp [da_1, c_1]) \perp \dots \perp ([a_r, b_r] \perp [da_r, c_r])$ is a subform of q .*

PROOF. The form q becomes isotropic over K because $r \geq 1$. Thus, by 2.1.4 there exist $a_1, b_1, c_1 \in k$ such that $[a_1, b_1] \perp [da_1, c_1]$ is a subform of q . Let m be the largest positive integer such that there exist a quadratic space (V', q') and $a_i, b_i, c_i \in k$, $1 \leq i \leq m$, such that

$$\begin{aligned} (V, q) &\cong ([a_1, b_1] \perp [da_1, c_1]) \perp \dots \perp ([a_m, b_m] \perp [da_m, c_m]) \perp (V', q') \\ &\cong L_1 \perp \dots \perp L_m \perp (V', q') \end{aligned}$$

where $L_i = [a_i, b_i] \perp [da_i, c_i]$ for $i = 1, \dots, m$. If $m = r$, then we have case (ii) and we are done. So assume that $m < r$. Let $\mathbf{e}_i, \mathbf{f}_i$ be the basis associated to $[a_i, b_i]$ and $\mathbf{g}_i, \mathbf{h}_i$ be the basis associated to $[da_i, c_i]$.

The K -vectors

$$\mathbf{v}_1 = \sqrt{d}\mathbf{e}_1 + \mathbf{g}_1, \mathbf{v}_2 = \sqrt{d}\mathbf{e}_2 + \mathbf{g}_2, \dots, \mathbf{v}_m = \sqrt{d}\mathbf{e}_m + \mathbf{g}_m$$

are linearly independent over K , and pairwise orthogonal because $\mathbf{v}_i \in K \otimes_k L_i$, for $i = 1, \dots, m$. Moreover, they are isotropic since $q(\mathbf{v}_i) = q(\sqrt{d}\mathbf{e}_i + \mathbf{g}_i) = dq(\mathbf{e}_i) + q(\mathbf{g}_i) = da_i + da_i = 0$. Thus they generate over K a totally isotropic subspace of dimension m .

Since the Witt index of q over K is $r > m$, $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is contained in a totally isotropic K -subspace S of dimension $> m$ by 2.1.1. Now, choose $\mathbf{v}_0 \in S$ to be linearly independent of $\mathbf{v}_1, \dots, \mathbf{v}_m$ over K and let

$$\mathbf{v} = \mathbf{v}_0 - B_q(\mathbf{v}_0, \mathbf{h}_1)\mathbf{v}_1 - \dots - B_q(\mathbf{v}_0, \mathbf{h}_m)\mathbf{v}_m.$$

Since $\mathbf{v}_i, \mathbf{h}_i \in K \otimes_k L_i$ for $i = 1, \dots, m$,

$$\begin{aligned} B_q(\mathbf{v}, \mathbf{h}_i) &= B_q(\mathbf{v}_0, \mathbf{h}_i) + B_q(\mathbf{v}_0, \mathbf{h}_i)B_q(\mathbf{v}_i, \mathbf{h}_i) \\ &= B_q(\mathbf{v}_0, \mathbf{h}_i) + B_q(\mathbf{v}_0, \mathbf{h}_i)B_q(\sqrt{d}\mathbf{e}_i + \mathbf{g}_i, \mathbf{h}_i) \\ &= B_q(\mathbf{v}_0, \mathbf{h}_i) + B_q(\mathbf{v}_0, \mathbf{h}_i)(\sqrt{d}0 + 1) \\ &= 0. \end{aligned}$$

Moreover, since $\mathbf{v}_0, \dots, \mathbf{v}_m \in S$, they are isotropic and pairwise orthogonal, hence \mathbf{v} is isotropic. Also, \mathbf{v} is K -linearly independent of $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ because \mathbf{v}_0 is.

Write

$$\mathbf{v} = (\alpha_1\mathbf{e}_1 + \beta_1\mathbf{f}_1 + \gamma_1\mathbf{g}_1 + \delta_1\mathbf{h}_1) + \dots + (\alpha_m\mathbf{e}_m + \beta_m\mathbf{f}_m + \gamma_m\mathbf{g}_m + \delta_m\mathbf{h}_m) + \mathbf{u}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i \in K$ ($i = 1, \dots, m$) and $\mathbf{u} \in V'_K$. Since $0 = B_q(\mathbf{v}, \mathbf{h}_i) = \gamma_i$ and $0 = B_q(\mathbf{v}, \mathbf{v}_i) = B_q(\mathbf{v}, \sqrt{d}\mathbf{e}_i + \mathbf{g}_i) = \sqrt{d}B_q(\mathbf{v}, \mathbf{e}_i) + B_q(\mathbf{v}, \mathbf{g}_i) = \sqrt{d}\beta_i + \delta_i$, ($i = 1, \dots, m$), we have

$$(*) \quad \mathbf{v} = [\alpha_1\mathbf{e}_1 + \beta_1(\mathbf{f}_1 + \sqrt{d}\mathbf{h}_1)] + \dots + [\alpha_m\mathbf{e}_m + \beta_m(\mathbf{f}_m + \sqrt{d}\mathbf{h}_m)] + \mathbf{u}_1 + \sqrt{d}\mathbf{u}_2.$$

where $\mathbf{u} = \mathbf{u}_1 + \sqrt{d}\mathbf{u}_2$ and $\mathbf{u}_1, \mathbf{u}_2 \in V'$.

Suppose first $\beta_1 = 1$. Let \mathbf{x} and \mathbf{y} be the k -vectors such that $\mathbf{v} = \mathbf{x} + \sqrt{d}\mathbf{y}$. Since $\beta_1 = 1$, we can solve (*) for \mathbf{f}_1 and \mathbf{h}_1 to conclude that

$$\begin{aligned} & k\text{-span}(\{\mathbf{e}_1, \mathbf{g}_1, \mathbf{f}_1, \mathbf{h}_1\}) + L_2 + \dots + L_m + V' \\ &= k\text{-span}(\{\mathbf{e}_1, \mathbf{g}_1, \mathbf{x}, \mathbf{y}\}) + L_2 + \dots + L_m + V'. \end{aligned}$$

Thus $\mathbf{e}_1, \mathbf{g}_1, \mathbf{x}, \mathbf{y}$ are linearly independent over k . Since $1 = B_q(\mathbf{e}_1, \mathbf{v}) = B_q(\mathbf{e}_1, \mathbf{x} + \sqrt{d}\mathbf{y}) = B_q(\mathbf{e}_1, \mathbf{x}) + \sqrt{d}B_q(\mathbf{e}_1, \mathbf{y})$, we have

$$1 = B_q(\mathbf{e}_1, \mathbf{x}) \text{ and } 0 = B_q(\mathbf{e}_1, \mathbf{y}).$$

Similarly, because $\sqrt{d} = B_q(\mathbf{g}_1, \mathbf{v})$, we have

$$1 = B_q(\mathbf{g}_1, \mathbf{y}) \text{ and } 0 = B_q(\mathbf{g}_1, \mathbf{x}).$$

Since by definition $q(\mathbf{e}_1) = a_1$ and $q(\mathbf{g}_1) = a_1d$,

$$q(\mathbf{g}_1) = dq(\mathbf{e}_1).$$

Also \mathbf{x}, \mathbf{y} are orthogonal and $dq(\mathbf{x}) = q(d\mathbf{y})$ because $0 = q(\mathbf{v}) = q(\mathbf{x} + \sqrt{d}\mathbf{y}) = q(\mathbf{x}) + dq(\mathbf{y}) + \sqrt{d}B_q(\mathbf{x}, \mathbf{y})$. Let W be the space generated by $\mathbf{e}_1, \mathbf{x}, \mathbf{g}_1, \mathbf{y}$. Then with respect to the basis $\{\mathbf{e}_1, \mathbf{x}, \mathbf{g}_1, d\mathbf{y}\}$, we can write $q|_W = [a, b] \perp d[a, b]$, where $a = q(\mathbf{e}_1)$ and $b = q(\mathbf{x})$. In particular, W is a nonsingular subspace of V , hence $V = W \perp W^\perp$ (by [B2, p. 10, Proposition 3.2]), i.e. $[a, b] \perp d[a, b]$ is a subform of q .

If $\beta_1 \neq 0$, then replace \mathbf{v} by $(1/\beta_1)\mathbf{v}$, and similarly if $\beta_i \neq 0$ for some $i = 1, \dots, m$. Hence we may assume that for $i = 1, \dots, m$, $\beta_i = 0$. That is, $\mathbf{v} = \alpha_1\mathbf{e}_1 + \dots + \alpha_m\mathbf{e}_m + \mathbf{u}$. Let $\alpha_i = t_i + \sqrt{d}s_i$ where $t_i, s_i \in k$ for $i = 1, \dots, m$

and let $p = a_1 t_1^2 + a_1 ds_1^2 + \cdots + a_m t_m^2 + a_m ds_m^2$. Note that $p = q'(\mathbf{u})$ since

$$\begin{aligned} 0 &= q(\mathbf{v}) = q(\alpha_1 \mathbf{e}_1 + \cdots + \alpha_m \mathbf{e}_m) + q'(\mathbf{u}) \\ &= a_1 \alpha_1^2 + \cdots + a_m \alpha_m^2 + q'(\mathbf{u}) \\ &= p + q'(\mathbf{u}) \end{aligned}$$

Since the subform $([a_1, b_1] \perp [da_1, c_1]) \perp \cdots \perp ([a_m, b_m] \perp [da_m, c_m])$ represents every element the shape $a_1 z_1^2 + a_1 dw_1^2 + \cdots + a_m z_m^2 + a_m dw_m^2$ (with $z_i, w_i \in k$ for $i = 1, \dots, m$), it represents p and $p(c^2 + d)$ for any $c \in k$. Hence, because q is anisotropic, q' cannot represent p over k and $[p(c^2 + d), c']$ cannot be a subform of q' for any $c, c' \in k$. Also \mathbf{u} cannot be 0, otherwise since \mathbf{v} is a non-zero isotropic vector, $\alpha_j \neq 0$ for some $j \leq m$, and

$$\begin{aligned} 0 &= q(\mathbf{v}) = q(\alpha_1 \mathbf{e}_1 + \cdots + \alpha_m \mathbf{e}_m) \\ &= a_1 \alpha_1^2 + \cdots + a_m \alpha_m^2 \\ &= a_1 t_1^2 + a_1 ds_1^2 + \cdots + a_m t_m^2 + a_m ds_m^2; \end{aligned}$$

hence $([a_1, b_1] \perp [da_1, c_1]) \perp \cdots \perp ([a_m, b_m] \perp [da_m, c_m])$ is isotropic over k , a contradiction. Thus, q' represents p over K . Therefore by proposition 2.1.3, there exist $a_{m+1}, b_{m+1}, c_{m+1} \in k$ such that $[a_{m+1}, b_{m+1}] \perp [p + a_{m+1}d, c_{m+1}]$ is a subform of q' . Hence,

$$\begin{aligned} q &\cong ([a_1, b_1] \perp [da_1, c_1]) \perp \cdots \perp ([a_m, b_m] \perp [da_m, c_m]) \\ &\quad \perp ([a_{m+1}, b_{m+1}] \perp [p + a_{m+1}d, c_{m+1}]) \perp q'' \end{aligned}$$

for some k -form q'' . Let \mathbf{e}, \mathbf{f} be the basis associated to $[a_{m+1}, b_{m+1}]$ and \mathbf{g}, \mathbf{h} be the basis associated to $[p + a_{m+1}d, c_{m+1}]$. Applying Lemma 2.1.2 to the vectors $\mathbf{e}_1, \mathbf{g}_1, \dots, \mathbf{e}_m, \mathbf{g}_m, \mathbf{e}$, and $\mathbf{g}' := \mathbf{g} + t_1 \mathbf{e}_1 + s_1 \mathbf{g}_1 + \cdots + t_m \mathbf{e}_m + s_m \mathbf{g}_m$, we

get: $([a_1, b'_1] \perp [da_1, c'_1]) \perp \dots \perp ([a_{m+1}, b'_{m+1}] \perp [da_{m+1}, c'_{m+1}])$ is a subform of q , where $b'_1, c'_1, \dots, b'_{m+1}, c'_{m+1} \in k$. This contradicts the choice of m . This concludes the proof of the theorem. \square

COROLLARY 2.1.6 (cf. [A, 2.6]) *Let q be a nonsingular anisotropic k -form. If the Witt index of q over K is greater than $(1/4) \dim(q)$, then there exist $a, b \in k$ such that $[a, b] \perp d[a, b]$ is a subform of q .*

PROOF. Let r be the Witt index of q over K . Since $r > (1/4) \dim(q)$, q cannot have a subform of the shape $([a_1, b_1] \perp [da_1, c_1]) \perp \dots \perp ([a_r, b_r] \perp [da_r, c_r])$ because this form has dimension $4r > \dim q$. Thus theorem 2.1.5 implies that there exist $a, b \in k$ such that $[a, b] \perp d[a, b]$ is a subform of q . \square

Corollary 2.1.6 gives a lower bound on the Witt index of q over K to guarantee the splitting off of a subform of the shape $[a, b] \perp d[a, b]$. We failed to decide whether the conclusion of 2.1.6 holds under a weaker hypothesis. The best one can hope for is

QUESTION 2.1.7 If $q \cong_K H \perp H \perp q'$, where H is a hyperbolic plane, does it follow that there exist $a, b \in k$ such that $[a, b] \perp d[a, b]$ is a subform of q ?

In view of Theorem 2.1.5, it is enough to answer the following:

QUESTION 2.1.7' If $q \cong ([a_1, b_1] \perp [da_1, c_1]) \perp \dots \perp ([a_r, b_r] \perp [da_r, c_r])$ with $r \geq 2$, does it follow that there exist $a, b \in k$ such that $[a, b] \perp d[a, b]$ is a subform of q ?

COROLLARY 2.1.8 (cf. [A, 2.8]) *Let q be a nonsingular anisotropic k -form. If q becomes hyperbolic over K , then there exists a k -form q' such that $q \cong q' \perp dq'$.*

PROOF. By 2.1.6, there exist a k -form q_0 and $a, b \in k$ such that $q \cong [a, b] \perp d[a, b] \perp q_0$. Over K , q and $[a, b] \perp d[a, b]$ are hyperbolic. Hence

by the cancellation theorem, q_0 is hyperbolic over K . By induction we have $q_0 \cong q'_0 \perp dq'_0$ for some k -form q'_0 . Thus $q \cong ([a, b] \perp q'_0) \perp d([a, b] \perp q'_0)$. \square

REMARK 2.1.9 (cf. [A, 2.9]) A nonsingular anisotropic k -form q of dimension 6 cannot become hyperbolic over K . For if the Witt index of q over K is 3, then by 2.1.6, $q \cong [a, b] \perp d[a, b] \perp [e, f]$, for some $a, b, e, f \in k$. Over K , q and $[a, b] \perp d[a, b]$ are hyperbolic, and thus by the cancellation theorem (cf. [B2, p. 82, Corollary 4.3]), $[e, f]$ would be isotropic over K . By 2.1.4 this is impossible. In the same way one can prove: if the dimension of q is not divisible by 4, then q cannot become hyperbolic over K .

If q is a nonsingular anisotropic k -form of dimension 6 and q has Witt index 2 over K , then it is of the shape $[a, b] \perp d[a, b] \perp [e, f]$ for some $a, b, e, f \in k$. More generally by using 2.1.6 inductively as in the proof of 2.1.8, we obtain :

COROLLARY 2.1.10 (cf. [A, 2.10]) *Let q be a nonsingular anisotropic k -form of dimension $4m + 2$. If the Witt index of q over K is $2m$ then there exist $e, f \in k$ and a k -form q' such that $q \cong q' \perp dq' \perp [e, f]$.*

2.2 Excellence of quadratic extensions

In this section we develop the tools needed to determine the Witt kernel of bi-quadratic extensions. We start with

DEFINITION Let L/k be a field extension. An L -form φ is said to be defined over k if there exists a k -form γ such that $\varphi \cong \gamma_L$. The extension L/k is called excellent if the anisotropic part over L of any non-singular k -form is defined over k .

Some examples of excellent extensions are the algebraic extensions of odd degree and the purely transcendental extensions. It is also known that separable quadratic extension are excellent. This an immediate consequence of

the following proposition which follows by repeatedly applying [B2, p. 121, Theorem 4.2] (and the cancellation theorem).

PROPOSITION 2.2.1 ([B2, p. 121]) *Let k be a field of characteristic 2 and let $K = k(\beta)/k$ where $\beta \notin k$ and $\beta^2 - \beta = b \in k$. If a non-singular anisotropic k -form q has Witt index s over K , then the $q \cong c_1[1, b] \perp \dots \perp c_s[1, b] \perp q_0$ for some $c_1, \dots, c_s \in k$ and a k -form q_0 . In particular, if q becomes hyperbolic over K , then $q \cong c_1[1, b] \perp \dots \perp c_r[1, b]$ for some $c_1, \dots, c_r \in k$*

In 2.2.3 below we show that inseparable quadratic extensions are also excellent

REMARK 2.2.2 Let $K = k(\sqrt{d})$.

(i) Since $[ad, c] \cong d[a, c']$ for some $c' \in k$, theorem 2.1.5 implies that any non-singular anisotropic k -form q can be written as

$$q \cong q_1 \perp (q_0 \perp dq_0) \perp ([a_1, b_1] \perp d[a_1, c_1]) \perp \dots \perp ([a_r, b_r] \perp d[a_r, c_r]),$$

where q_0 and q_1 are k -forms and $a_i, b_i, c_i \in k$, ($i = 1, \dots, r$) such that the Witt index of q over $K = k(\sqrt{d})$ equals $\dim(q_0) + r$. In particular, q_1 remains anisotropic over K , and $[a_i, b_i] \perp d[a_i, c_i]$ is not hyperbolic over K .

(ii) Let $a, b, c \in k$. Then, over K , we have

$$[a, b] \perp d[a, c] \cong [a, b] \perp [a, c] \cong [0, b] \perp [a, b + c] \cong H \perp [a, b + c]$$

The first isometry follows because $d \in K^2$. To see the second isometry, take $\mathbf{u}_1, \mathbf{v}_1, \mathbf{u}_2,$

\mathbf{v}_2 to be a basis associated with the form $q \cong [a, b] \perp [a, c]$ and note that

$$q(\mathbf{u}_1 + \mathbf{u}_2) = q(\mathbf{u}_1) + q(\mathbf{u}_2) = a + a = 0,$$

$$B_q(\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}_1) = B_q(\mathbf{u}_1, \mathbf{v}_1) + B_q(\mathbf{u}_2, \mathbf{v}_1) = 1 + 0 = 1,$$

$$q(\mathbf{v}_2 + \mathbf{v}_1) = q(\mathbf{v}_1) + q(\mathbf{v}_2) = b + c,$$

$$B_q(\mathbf{u}_2, \mathbf{v}_1 + \mathbf{v}_2) = B_q(\mathbf{u}_2, \mathbf{v}_1) + B_q(\mathbf{u}_2, \mathbf{v}_2) = 0 + 1 = 1, \text{ and}$$

$$\begin{aligned}
0 &= B_q(\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_2) = B_q(\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}_1 + \mathbf{v}_2) \\
&= B_q(\mathbf{v}_1, \mathbf{u}_2) = B_q(\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2)
\end{aligned}$$

So, rewriting the form q with respect to the new basis $\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}_1, \mathbf{u}_2, \mathbf{v}_2 + \mathbf{v}_1$ gives $q \cong [0, b] \perp [a, b + c]$.

THEOREM 2.2.3 *Let q be a non-singular anisotropic k -form. Then the anisotropic part of q over $K = k(\sqrt{d})$ is defined over k ; i.e. K/k is excellent.*

PROOF. We may assume that q is anisotropic over k (otherwise we take the anisotropic part of q instead.) Write

$$q \cong q_1 \perp (q_0 \perp dq_0) \perp ([a_1, b_1] \perp d[a_1, c_1]) \perp \dots \perp ([a_r, b_r] \perp d[a_r, c_r]),$$

as in remark 2.2.2(i). Over K , $q_0 \perp dq_0$ is hyperbolic, q_1 is anisotropic, and by 2.2.2(ii), $[a_i, b_i] \perp d[a_i, c_i] \cong H \perp [a_i, b_i + c_i]$. Therefore

$$q_K \cong (\dim q_0 + r)H \perp q_1 \perp [a_1, b_1 + c_1] \perp \dots \perp [a_r, b_r + c_r].$$

Since the Witt index of q_K equals $\dim q_0 + r$ (see 2.2.2(i)), the anisotropic part of q over K is $q_1 \perp [a_1, b_1 + c_1] \perp \dots \perp [a_r, b_r + c_r]$, which is defined over k . \square

COROLLARY 2.2.4 *Let $K = k(\sqrt{d})$ be a quadratic extension over k . Let σ and δ be non-singular k -forms and let γ be a non-singular K -form. If $\sigma_K \cong \delta_K \perp \gamma$, then γ is defined over k .*

PROOF. It is enough to show that the anisotropic part of γ is defined over k . Since $\sigma_K \cong \delta_K \perp \gamma$, we have $[\gamma] = [\sigma_K \perp -\delta_K]$ in the Witt ring of K . So the anisotropic part of γ is isometric to that of $(\sigma \perp -\delta)_K$ which, by the previous theorem, is defined over k . \square

To determine the Witt kernel for bi-quadratic extensions, one needs a "characteristic 2" analogue of [ELW, Proposition 2.11.(a)]. For separable quadratic extensions we have

THEOREM 2.2.5' *Let k be a field of characteristic 2. Let K/k be an excellent extension of k . Let $q \cong e_1[1, b] \perp \dots \perp e_r[1, b]$ where $b \in k^*$ and $e_1, \dots, e_r \in K$. If q is defined over k , then there exist $c_1, \dots, c_r \in k$ such that $q \cong c_1[1, b] \perp \dots \perp c_r[1, b]$.*

The proof is identical to that of [ELW, Proposition 2.11.(a)]; hence omitted.

For the case of inseparable quadratic extensions we have

THEOREM 2.2.5 *Let K/k be an excellent extension of k , and let $d \in k^*$. Let γ be a non-singular K -form such that the form $\gamma \perp d\gamma$ is defined over k . Then there exists a k -form δ such that $\gamma \perp d\gamma \cong (\delta \perp d\delta)_K$.*

The proof will be broken into two lemmas.

LEMMA 2.2.6 *Let $a, b, c, d \in k$ such that the form $\delta \cong [a, b] \perp d[a, b]$ represents c . Then $\delta \cong [c, b'] \perp d[c, b']$ for some $b' \in k$.*

PROOF. Let $\alpha \cong [1, ab] \perp d[1, ab]$. Since $[a, b] \cong a[1, ab]$, we have $\delta \cong a\alpha$. Since δ represents c , we have α represents ac . But α is a Pfister form; therefore $\alpha \cong aca$. Thus we have

$$\begin{aligned} \delta &\cong a\alpha \cong a^2c\alpha \cong ca \\ &\cong c[1, ab] \perp dc[1, ab] \end{aligned}$$

Since $c[1, ab]$ represents c we have $c[1, ab] \cong [c, b']$ for some $b' \in k$. □

LEMMA 2.2.7 *Let K be a field extension of k , and let $d \in k^*$. Let γ be a non-singular K -form such that the form $\gamma \perp d\gamma$ represents an element of k . Then there exist $a \in k$, $b \in K$, and a K -form γ_1 such that*

$$\gamma \perp d\gamma \cong ([a, b] \perp d[a, b]) \perp (\gamma_1 \perp d\gamma_1)$$

PROOF. Let (V, γ) be a non-singular K -quadratic space. If d is a square in K , then $\gamma \perp d\gamma$ is a hyperbolic form of dimension $2(\dim \gamma)$ which is divisible

by 4. Hence in this case we take $a = b = 0$ and γ_1 to be the hyperbolic form of dimension $(\dim \gamma - 1)$.

If γ is isotropic, then $\gamma \cong [0, 0] \perp \gamma_1$ and we then let $a = b = 0$. So, we may assume that γ is anisotropic. Suppose $\gamma \perp d\gamma$ represents an element $c \in k^*$. So, there exist $\mathbf{v}_1, \mathbf{v}_2 \in V$ not both zero such that

$$c = \gamma(\mathbf{v}_1) + d\gamma(\mathbf{v}_2)$$

If \mathbf{v}_1 (respectively, \mathbf{v}_2) is the zero vector, then $\gamma(\mathbf{v}_1)$ (respectively, $\gamma(\mathbf{v}_2)$) equals c (respectively, $\frac{c}{d}$). So, γ represents an element a of k^* where $a = c$ or $a = \frac{c}{d}$. Therefore $\gamma \cong [a, b] \perp \gamma_1$ for some $b \in K$ and a K -form γ_1 , and the conclusion follows. Therefore we may assume that \mathbf{v}_1 and \mathbf{v}_2 are both non-zero.

Assume first that \mathbf{v}_1 and \mathbf{v}_2 are not orthogonal or \mathbf{v}_1 and \mathbf{v}_2 are linearly dependent. Since γ is non-singular, in either case the vectors $\mathbf{v}_1, \mathbf{v}_2$ are contained in a non-singular two dimensional subspace V_0 of V . Then

$$(A) \quad (V, \gamma) \cong (V_0, \gamma|_{V_0}) \perp \gamma_1$$

for some K -form γ_1 . Let $f = \gamma(\mathbf{v}_1)$ and $g = \gamma(\mathbf{v}_2)$. Then by 2.1.2, we have

$$(B) \quad \begin{cases} (V_0, \gamma|_{V_0}) \cong [f, f'], & \text{and} \\ (V_0, \gamma|_{V_0}) \cong [g, g'], \end{cases}$$

for some $f', g' \in K$. Since $f + d_2g = \gamma(\mathbf{v}_1) + d\gamma(\mathbf{v}_2) = c$, the form $[f, f'] \perp d[g, g']$ represents $c \in k$. Thus by 2.2.6, there exists $b \in K$ such that

$$(C) \quad [f, f'] \perp d[g, g'] \cong [c, b] \perp d[c, b]$$

From the equations (A), (B) and (C) we have

$$\gamma \perp d\gamma \cong ([f, f'] \perp \gamma_1) \perp d([g, g'] \perp \gamma_1)$$

$$\begin{aligned}
&\cong ([f, f'] \perp d[g, g']) \perp (\gamma_1 \perp d\gamma_1) \\
&\cong ([c, b] \perp d[c, b]) \perp (\gamma_1 \perp d\gamma_1)
\end{aligned}$$

as desired.

Now assume that \mathbf{v}_1 and \mathbf{v}_2 are orthogonal and linearly independent. Then by 2.1.2,

$$(D) \quad \gamma \cong [e_1, f_1] \perp [e_2, f_2] \perp \gamma_0$$

where $e_i = \gamma(\mathbf{v}_i)$, $i = 1, 2$, $f_1, f_2 \in K$ and γ_0 is a K -form. Since $e_1 + de_2 = \gamma(\mathbf{v}_1) + d\gamma(\mathbf{v}_2) = c$, the form $[e_1, f_1] \perp d[e_2, f_2]$ represents $c \in k$. Thus by 2.1.2, there exists $b, r, s \in K$ such that

$$(E) \quad [e_1, f_1] \perp d[e_2, f_2] \cong [c, b] \perp [r, s]$$

From the equations (D) and (E) we have

$$\begin{aligned}
\gamma \perp d\gamma &\cong ([e_1, f_1] \perp [e_2, f_2] \perp \gamma_0) \perp d([e_1, f_1] \perp [e_2, f_2] \perp \gamma_0) \\
&\cong ([e_1, f_1] \perp d[e_2, f_2] \perp \gamma_0) \perp d([e_1, f_1] \perp d[e_2, f_2] \perp \gamma_0) \\
&\cong ([c, b] \perp [r, s] \perp \gamma_0) \perp d([c, b] \perp [r, s] \perp \gamma_0) \\
&\cong ([c, b] \perp d[c, b]) \perp (([r, s] \perp \gamma_0) \perp d([r, s] \perp \gamma_0)) \\
&\cong ([c, b] \perp d[c, b]) \perp (\gamma_1 \perp d\gamma_1),
\end{aligned}$$

where $\gamma_1 := ([r, s] \perp \gamma_0)$. This completes the proof of the lemma. \square

REMARK 2.2.8 (i) The forms $[a, b]$, $a[1, ab]$ and $a[1, a^2b^2]$ are isometric because they are two dimensional forms representing a common element a and have the same Arf invariant (see the introduction or [Sch, Lemma 4.4.(i), p. 341]).

(ii) Let $K = k(\sqrt{d})$, $a \in k$ and $b \in K$. By (i), the form $[a, b]$ is isometric to $a[1, a^2b^2]$; hence is defined over k because $a^2b^2 \in k$. So in the conclusion of the previous lemma we may assume that both a and b to be in k .

PROOF (of Theorem 2.2.5). Since $\gamma \perp d\gamma$ is defined over k , it represents an element of k . So, by the previous lemma and remark

$$\gamma \perp d\gamma \cong ([a, b] \perp d[a, b]) \perp (\gamma_1 \perp d\gamma_1)$$

where $a, b \in k$ and γ_1 is a K -form. If $\dim(\gamma) = 2$, then we are done. If $\dim(\gamma) > 2$, then 2.2.4 implies that $\gamma_1 \perp d_2\gamma_1$ is defined over k . We may then use induction to find a k -form δ_1 such that $\gamma_1 \perp d\gamma_1 \cong (\delta_1 \perp d\delta_1)_K$; and therefore

$$\begin{aligned} \gamma \perp d\gamma &\cong ([a, b] \perp d[a, b]) \perp (\delta_1 \perp d\delta_1)_K \\ &\cong (([a, b] \perp \delta_1)) \perp d([a, b] \perp \delta_1)_K \end{aligned}$$

and we set $\delta = [a, b] \perp \delta_1$. □

2.3 Bi-quadratic extensions

We are now ready to describe the Witt kernel of bi-quadratic extensions (i.e., degree 4 extensions which are the composite of two quadratic extension). We start with the inseparable case first. One distinguishes between two types of inseparable bi-quadratic extensions: the purely inseparable case where $L = k(\sqrt{d_1}, \sqrt{d_2})$ with non-square elements $d_1, d_2 \in k$; and the case L/k contains an intermediate separable extension. In the latter case $L = k(\beta, \sqrt{d})$ for some non-square element $d \in k$ and $\beta \notin k$ such that $\beta^2 - \beta = b \in k$.

THEOREM 2.3.1 *Let L/k be an inseparable bi-quadratic extension over k . Let q be an anisotropic non-singular k -form such that q is hyperbolic over L .*
(i) If $L = k(\sqrt{d_1}, \sqrt{d_2})$ with $d_1, d_2 \in k$, then q is Witt equivalent to a form of the shape

$$(q_1 \perp d_1 q_1) \perp (q_2 \perp d_2 q_2)$$

for some k -forms q_1 and q_2 .

(ii) If $L = k(\beta, \sqrt{d})$ where $d \in k - k^2$, $\beta \notin k$ and $\beta^2 - \beta = b \in k$, then q is Witt equivalent to a form of the shape

$$(c_1[1, b] \perp \dots \perp c_r[1, b]) \perp (q_0 \perp dq_0)$$

for some $c_i \in k$ ($i = 1, \dots, r$) and a k -form q_0 .

PROOF. For (i), let $K = k(\sqrt{d_1})$. If q is hyperbolic over K , the theorem follows immediately from 2.1.8. So, assume q_K is not hyperbolic. Let φ denote the anisotropic part of q over K . By 2.2.3, φ is defined over k . Since q is hyperbolic over $L = K(\sqrt{d_2})$, φ_L is hyperbolic; hence there exists a K -form q_2 such that $\varphi \cong q_2 \perp d_2q_2$. By 2.2.5, we may assume that q_2 is a k -form. Consider the k -form $\alpha := q \perp -(q_2 \perp d_2q_2)$. Over K , the form α is hyperbolic because (in $W(K)$) $[\alpha_K] = [q \perp -(q_2 \perp d_2q_2)_K] = [\varphi \perp -\varphi] = 0$. So, by 2.1.8, α is Witt equivalent (over k) to $q_1 \perp d_1q_1$ for some k -form q_1 . Therefore in the Witt ring of k we have

$$[q \perp -(q_2 \perp d_2q_2)] = [q_1 \perp d_1q_1];$$

or equivalently,

$$[q] = [(q_2 \perp d_2q_2) \perp (q_1 \perp d_1q_1)]$$

as desired.

For (ii), we let $K = k(\beta)$. If q_K is hyperbolic, then we are done by 2.2.1. So, assume that q_K is not hyperbolic and let φ be its anisotropic part. As in part (i), it follows that $\varphi \cong q_0 \perp d_2q_0$ for some k -form q_0 and the k -form $\alpha := q \perp q_0 \perp d_2q_0$ is hyperbolic over K . Now, by 2.2.1, α is Witt equivalent (over k) to $c_1[1, b] \perp \dots \perp c_r[1, b]$ for some $c_i \in k$ ($i = 1, \dots, r$). Therefore q is Witt equivalent to $(c_1[1, b] \perp \dots \perp c_r[1, b]) \perp (q_0 \perp dq_0)$ \square

For separable bi-quadratic extensions one can use an argument similar to that in the proof of 2.3.1 (or similar to [ELW, Proposition 2.12], using 2.2.5') to get

THEOREM 2.3.2 *Let $L = k(\alpha, \beta)$ be a (separable) bi-quadratic extension over k with $\alpha^2 - \alpha = a \in k$ and $\beta^2 - \beta = b \in k$. Let q be an anisotropic non-singular k -form. If q is hyperbolic over L , then q is Witt equivalent to a form of the shape*

$$(e_1[1, a] \perp \dots \perp e_r[1, a]) \perp (f_1[1, b] \perp \dots \perp f_s[1, b])$$

for some $e_i, f_j \in k$ ($i = 1, \dots, r; j = 1, \dots, s$).

We conclude the chapter by an example which shows that the Witt equivalence in the conclusions of theorems 2.3.1 and 2.3.2 above cannot be improved to isometry.

EXAMPLE . Let k_0 be a fixed field of characteristic two. Let r, s, t, u be algebraically independent elements over k_0 and set

$$q \cong [1, r] \perp t[1, s] \perp u[1, r + s].$$

Let α, β (in the algebraic closure of k) be such that $\alpha^2 - \alpha = r$ and $\beta^2 - \beta = r + s$.

Then

(i) The form q is anisotropic over k because r, s, t, u are algebraically independent elements over k_0 (see [L, ex. 1, p. 273]).

(ii) Over the fields $K_1 = k(\sqrt{t})$, $K_2 = k(\sqrt{u})$, $K_3 = k(\alpha)$ and $K_4 = k(\beta)$, the form q is isotropic and have Witt index 1. We see this as follows: First over K_1 , $t \in K_1^2$ and $[1, r] \perp t[1, s] \cong [1, r] \perp [1, s] \cong H \perp [1, r + s]$ (cf. see remark 2.2.2(ii)). Therefore

$$(1) \quad q_{K_1} \cong H \perp [1, r + s] \perp u[1, r + s].$$

Since $r+s$ and u are algebraically independent over $k_0(\sqrt{t})$, the form $[1, r+s] \perp u[1, r+s]$ is anisotropic over K_1 , and therefore q_{K_1} has Witt index 1. Similarly, we can show that q_{K_2} also has Witt index 1.

Now over $K_3 = k(\alpha)$, the form $[1, r]$ is isotropic and $[1, r+s] \cong_{K_3} [1, s]$ (because they represent 1 and have the same Arf invariant over K_3). Therefore, over K_3 ,

$$(2) \quad q_{K_3} \cong H \perp t[1, r+s] \perp u[1, r+s].$$

and $t[1, s] \perp u[1, s]$ is anisotropic over K_3 because s, t and u are algebraically independent over $k_0(\alpha)$. Therefore, q_{K_3} has Witt index 1. Likewise, q_{K_4} has Witt index 1 too.

(iii) The form q is hyperbolic over the fields $L_1 = k(\sqrt{t}, \sqrt{u})$, $L_2 = k(\sqrt{t}, \beta)$ and $L_3 = k(\alpha, \beta)$:

Note that $u \in L_1^2$ and therefore the form $[1, r+s] \perp u[1, r+s] \cong [1, r+s] \perp [1, r+s] \cong 2H$. Since $K_1 \subset L_1$, we have from equation (1) above that

$$q_{L_1} \cong H \perp [1, r+s] \perp u[1, r+s] \cong 3H;$$

that is, q_{L_1} is hyperbolic.

Since β belongs to L_2 and L_3 , $[1, r+s] \cong H$ over L_2 and L_3 because $\beta^2 + \beta + (r+s) = 0$. Therefore the form $[1, r+s] \perp u[1, r+s]$ (respectively, $t[1, r+s] \perp u[1, r+s]$) is hyperbolic over L_2 (respectively, L_3). Therefore equation (1) (respectively, equation (2)) implies that q_{L_2} (respectively, q_{L_3}) is hyperbolic.

(iv) Theorems 2.3.1 and 2.3.2 imply that over k the form q is Witt equivalent to forms of the shape

$$(a) \quad (q_1 \perp tq_1) \perp (q_2 \perp uq_2).$$

$$(b) \quad (q_1 \perp tq_1) \perp (c_1[1, r+s] \perp \dots \perp c_n[1, r+s]).$$

$$(c) \quad (b_1[1, r] \perp \dots \perp b_m[1, r]) \perp (c_1[1, r+s] \perp \dots \perp c_n[1, r+s]).$$

where $b_j, c_i \in k$ and q_1 and q_2 are non-singular k -forms. This Witt equivalence cannot be improved to isometry. For if q is isometric to (a), (b) or (c), then by comparing dimensions we have either $\dim q_1 \geq 2$, $\dim q_2 \geq 2$, $m \geq 2$ or $n \geq 2$. This respectively imply that the Witt index over K_1 , K_2 , K_3 or K_4 is ≥ 2 ; contradicting part (ii). \square

Chapter 3

Function fields of quadratic forms and similarity

Throughout this chapter k will denote a field of characteristic $\neq 2$. The term form will mean a non-singular quadratic form. For an irreducible form q , $k(q)$ will denote the function field of q . (see the introduction chapter for definitions.)

Let Q_1 and Q_2 be two non-singular k -forms of the same dimension. In this chapter we will study the relation between the following equivalences of Q_1 and Q_2 :

- (1) The forms Q_1 and Q_2 are similar over k .
- (2) The forms Q_1 and Q_2 are birationally equivalent over k ; i.e. the fields $k(Q_1)$ and $k(Q_2)$ are k -isomorphic.
- (3) Q_1 becomes isotropic over $k(Q_2)$ and Q_2 becomes isotropic over $k(Q_1)$.

(Since the function field is a generic zero field, the condition (3) is equivalent to the existence of a k -place of $k(Q_2)$ in $k(Q_1)$ and a k -place of $k(Q_1)$ in $k(Q_2)$.) Clearly (1) \Rightarrow (2) \Rightarrow (3). The question is to decide when the converse of these implications holds. The question when birational equivalence of quadratic forms implies similarity was first investigated by A. Wadsworth (see [W]). He proved that (3) \Rightarrow (1) for forms (of the same dimension) of dimension ≤ 4 . Also Wadsworth ([W]) and Knebusch ([K1, theorem 5.8]) independently noted that the same holds if Q_1 is a Pfister neighbor of codimension ≤ 1 . Knebusch (cf. [K1, p. 72–73]) also pointed out that in general (2) \Rightarrow (1) does not hold. He gave an example of two non-similar six dimensional Pfister neighbors which define the same function field. we will give an example of two non-similar birationally equivalent 5-dimensional forms. In view of [W], this is an

example of minimal dimension. We also show that (see 3.2.2) (3) \Rightarrow (1) in the class of *special Pfister neighbors*. In particular, we show (3) \Rightarrow (1) if Q_1 is a Pfister neighbor of codimension ≤ 4 . In §3.3 we point out the relation of (3) \Rightarrow (2) to the Zariski cancellation problem. The contents of §3.1-3.3 are a joint work with Prof. Jack Ohm and will appear as [AO] in the Journal of Algebra.

In section 3.4 we discuss the question of descent of similarity over field extensions; and we give some classes of forms for which the similarity class is determined by the generic splitting tower.

3.1 Pfister neighbors and special Pfister neighbors

DEFINITION The k -forms Q and P are called *neighbors* if Q is isotropic over $k(P)$ and P is isotropic over $k(Q)$.

REMARK 3.1.1 Let P and Q be neighbors.

(i) Since Q is isotropic over $k(P)$, the function field $k(P)(Q)$ is a pure transcendental extension of $k(P)$ (recall 1.2.1 in the introduction). Likewise, $k(P)(Q)$ is a pure transcendental extension of $k(Q)$ too.

(ii) Let L/k be a field extension. If P is isotropic over L , then $L(P)$ is a pure transcendental over L . Since $k(P) \subset L(P)$ and Q is isotropic over $k(P)$, Q is isotropic over $L(P)$, which implies that Q is also isotropic over L . Therefore P and Q are neighbors (if and) only if they have the same isotropy fields.

(iii) (see [K1, p. 73]) Suppose further that $\dim P \geq \dim Q$. Then $k(Q)$ embeds into $k(P)(Q)$ which, by part(i), is a pure transcendental extension of $k(P)$. Therefore by [O1], there exists a k -embedding of $k(Q)$ into $k(P)$.

Recall that the 0-fold Pfister form is the form $\langle 1 \rangle$; and an n -fold Pfister form is a form $Q \perp dQ$, where Q is an $(n - 1)$ -fold Pfister form. A neighbor of an n -fold Pfister form with $n > 1$ is called a *Pfister neighbor*.

Let P be an n -fold Pfister form and Q a Pfister neighbor of P . By definition, P is isotropic, hence hyperbolic, over the function field of Q . Therefore by the Cassels-Pfister subform theorem it follows that Q is similar to a subform of P . Also since Q becomes isotropic over $k(P)$, it follows from the main theorem in [H4] (see theorem 1.2.4 in the introduction) that $2 \dim Q > \dim P$. Conversely, let Q' be a form similar to a subform of the Pfister form P and $2 \dim Q' > \dim P$, i.e. $P \cong aQ' \perp R$ for some $a \in k^*$ and a form R of dimension $< \dim Q$. Clearly P is isotropic over $k(Q)$. Now over $K(P)$, P is isotropic and therefore hyperbolic. Hence in the Witt ring of $k(P)$, $[Q'] = [-R]$. Since $\dim Q' > \dim R$, we have Q' is isotropic over $k(P)$.

So, we conclude that a form Q is a neighbors of a Pfister form P in the sense defined above) if and only if Q is similar to a subform of P and $2 \dim Q > \dim P$; i.e. our definition of a Pfister neighbor coincides with that given originally by Knebusch in [K2, p.2, Definition 7.4].

By [K1 p. 73 and K2 pp. 2-3], a Pfister neighbor Q is the neighbor of a unique (upto isometry) Pfister form Q^+ . Also the form Q^- such that $aQ^+ \cong Q \perp Q^-$, for some $a \in k^*$, is also uniquely determined (up to isometry) by Q . We will continue to use the notation Q^+ for the Pfister form associated to Q , and Q^- for the complement of Q in Q^+ . The dimension of Q^- , is called *the codimension of the Pfister neighbor Q* .

DEFINITION 3.1.2 (cf. [AO, 1.1]) Let P is a Pfister form of dimension ≥ 2 , $c \in k^*$ and P_1 is a non-zero subform of P . We shall call the such a triple (P, c, P_1) , a *Pfister triple*; the form $P \perp cP_1$, *the form defined by the triple*; the form $P \perp \langle c \rangle$, *the base form of the triple*; and the form $P \perp cP$, *the associated Pfister form of the triple*. A form Q which is similar to a form defined by a Pfister triple will be called a *special Pfister neighbor*.

We emphasize that the special Pfister neighbors are Pfister neighbors with the associated Pfister being $Q^+ \cong P \perp cP$, the Pfister form defined by the triple.

If Q is any Pfister neighbor, then by 3.1.1 (iii), the function field of Q embeds into the function field of Q^+ . For special Pfister neighbors, one can give the following refinement which is essentially due to Knebusch (cf. [K1, pp. 72–73], also see cf. [AO, 1.2 and 1.3]).

THEOREM 3.1.3 *Let P be a Pfister form of dimension ≥ 2 , $c \in k^*$ and P_1 a non-zero subform of P . Then*

- (i) *The function field of $P \perp cP_1$ is (k -isomorphic to) a pure transcendental extension of the function field of the base form $P \perp \langle c \rangle$.*
- (ii) *The function field of the Pfister form $P \perp cP$ is a pure transcendental extension of the function field of $P \perp cP_1$.*

PROOF. Let X be a set of coordinate indeterminates for P and x be generic zero for the polynomial $P(X) + c$, and let Y be a set of coordinate indeterminates for (the dehomogenized polynomial) P_1^{af} and y be a corresponding set of elements algebraically independent over $k(x)$. (We take $Y = \emptyset$ if $\dim P_1 = 1$.)

Since P is a Pfister form and $P \sim_{k(y)} P_1^{af}(y)$, we have $P \cong_{k(y)} P_1^{af}(y)P$. This means there exists a nonsingular $k(y)$ -matrix M such that if $X' := XM$, then $P(X') = P_1^{af}(y)P(X)$. Therefore, if $x' = xM$, then

$$P(x') + cP_1^{af}(y) = P_1^{af}(y)[P(x) + c] = 0.$$

Then $k(x, y) = k(x', y)$; and since $\dim k(x, y)/k = \dim(P \perp cP_1) - 2$, it follows that $k(x', y)$ is the function field of $P \perp cP_1$. But $k(x)$ is the function field of $P \perp \langle c \rangle$, so we proved (i).

Now by part (i), $P \perp cP$ and $P \perp cP_1$ have function fields which are pure transcendental over the function field of $P \perp \langle c \rangle$. So, (ii) follows. \square

In particular, by taking $P = \langle 1, a \rangle$ in 1.3, in which case $P \perp c \langle 1 \rangle = \langle 1, a, c \rangle$ and $P \perp cP = \langle 1, a, c, ac \rangle$, we have the very special case:

COROLLARY 3.1.4 (cf. [AO, 1.4]) *A simple transcendental extension of a function field of a nonsingular conic is a quadratic function field.*

Recall (cf. e.g. [O2, p. 27]) that a field extension K/k is called *ruled* if there exists an intermediate field L ($k \subseteq L \subset K$) such that K is simple transcendental over L . The function fields K/k defined by Pfister triples (P, c, P_1) with $\dim P_1 > 1$ are ‘quadratically ruled’, in the sense that L/k may additionally be chosen to be a quadratic function field. While the Pfister triple construction produces examples of ruled quadratic function fields, the problem of giving a complete characterization of such extensions remains open. For example, the following generalization of 3.1.3 gives further examples of quadratically ruled quadratic function fields:

PROPOSITION *Let P be a Pfister form, P_1 a non-zero subform of P and $c, b_1, \dots, b_m \in k^*$, then the function field of $b_1P \perp \dots \perp b_mP \perp cP_1$ is pure transcendental over the function field of $b_1P \perp \dots \perp b_mP \perp \langle c \rangle$.*

NOTATION. If P and Q are forms, we shall write $Q \preceq P$ if Q is similar to a subform of P , and $Q \prec P$ if Q is similar to a proper subform of P , i.e. if $Q \preceq P$ and Q is not similar to P .

The next theorem gives a necessary and sufficient condition for an anisotropic Pfister neighbor to be special.

THEOREM 3.1.5 (cf. [AO, 2.2]) *If Q is an anisotropic Pfister neighbor with complement Q^- and associated Pfister form Q^+ , then Q is special iff there exists a Pfister form P_0 such that $Q^- \preceq P_0 \prec Q^+$. (In particular, anisotropic Pfister neighbors of codimension ≤ 3 are special.)*

PROOF. If $Q^- = 0$, then $Q = Q^+$; hence in this case Q is special (Pfister) and $Q^- \preceq \langle 1 \rangle \prec Q^+$. Therefore we may assume $Q^- \neq 0$, or equivalently, that there exists $c \in k^*$ such that $Q^- \sim c$. If $aQ^+ \cong Q \perp Q^-$, then $Q^+ \sim ac$ and $Q^+ \cong acQ^+ \cong cQ \perp cQ^-$, and $cQ^- \sim 1$. Thus, by replacing Q with the similar Pfister neighbor cQ , we may assume $Q^+ \cong Q \perp Q^-$ and $Q^- \sim 1$.

If: By 1.2.2 the hypothesis implies that Q^- is a subform of P_0 and P_0 is a subform of Q^+ . Since P_0 is a proper subform of Q^+ , by [L, p. 293, Exercise 8] there exists a Pfister form R of dimension ≥ 2 such that $Q^+ \cong P_0 \otimes R$. But then, writing $R = \langle 1, d \rangle \otimes R_1$ with R_1 Pfister, we have

$$Q^+ \cong P_0 \otimes (\langle 1, d \rangle \otimes R_1) \cong S \perp dS,$$

where $S := P_0 \otimes R_1$ is Pfister. But Q^- is a subform of P_0 and P_0 is a subform of S , so there exists a form Q' such that $S \cong Q^- \perp Q'$. By cancellation of Q^- in the expression

$$Q^+ \cong Q \perp Q^- \cong Q^- \perp Q' \perp dS,$$

we have $Q \cong Q' \perp dS$. Therefore Q is similar to $dQ \cong dQ' \perp S$, hence dQ is defined by the Pfister triple (S, d, Q') . (Note that $Q' \neq 0$ since $\dim S = (1/2) \dim P > \dim Q^-$.)

Only if: We are given a Pfister triple (P_0, d, T) and an element $b \in k^*$ such that $bQ \cong P_0 \perp dT$. Since Q is a neighbor of $P_0 \perp dP_0$, and by the uniqueness of the Pfister form associated to a Pfister neighbor, we have $Q^+ \cong P_0 \perp dP_0$. By definition of Q^+ , there exists $a \in k^*$ such that $aQ^+ \cong Q \perp Q^- \cong b(P_0 \perp dT) \perp Q^-$, or $Q^+ \cong ab(P_0 \perp dT) \perp aQ^-$. Since $Q^+ \sim ab$, $Q^+ \cong abQ^+ \cong P_0 \perp dT \perp bQ^-$. Cancelling P_0 from

$$P_0 \perp dP_0 \cong Q^+ \cong P_0 \perp dT \perp bQ^-,$$

we get $dP_0 \cong dT \perp bQ^-$. Therefore $P_0 \cong T \perp (db)Q^-$, so $Q^- \preceq P_0 \prec Q^+$. \square

Before stating the next theorem, we shall give a lemma needed for its proof.

LEMMA 3.1.6 (cf. [AO, 2.3]) *Let $R \preceq P_0 \preceq P$ be forms such that P_0 and P are Pfister and R represents 1. If there exists $c \in k^*$ such that $R \perp \langle c \rangle \preceq P$ and $R \perp \langle c \rangle \not\preceq P_0$, then there exists $b \in k^*$ such that*

$$R \perp \langle c \rangle \preceq P_0 \perp \langle b \rangle \preceq P_0 \perp bP_0 \preceq P.$$

PROOF. Since R and P_0 represent 1, by 1.2.2 we can write $P_0 \cong R \perp R'$ and $P \cong P_0 \perp P'_0$. Then

$$\begin{aligned} R \perp \langle c \rangle \perp (\dots) \cong P &\cong P_0 \perp P'_0 \\ &\cong (R \perp R') \perp P'_0. \end{aligned}$$

By Witt cancellation of R , this implies $R' \perp P'_0 \sim c$. Hence there exist $a, b \in k$ such that $c = a + b$, where $R' \sim a$ and $P'_0 \sim b$. Moreover, $b \neq 0$ since $R \perp \langle c \rangle \not\preceq P_0$. Thus, $R \perp \langle c \rangle \preceq R \perp R' \perp \langle b \rangle \cong P_0 \perp \langle b \rangle \preceq P_0 \perp bP_0$, and by the following lemma, $P_0 \perp bP_0 \preceq P$. Finally, by [L, p. 280, Theorem 1.9] the three conditions P_0 and P are Pfister, $P \cong P_0 \perp P'_0$, and $P'_0 \sim b \in k^*$ imply $P_0 \perp bP_0$ is a subform of P . \square

THEOREM 3.1.7 (cf. [AO, 2.4]) *Let Q' be a form of dimension $\leq n$, and let P be a Pfister form of dimension $\geq 2^n$ ($n \geq 1$). If $Q' \preceq P$, then there exists a Pfister form P_0 such that $Q' \preceq P_0 \prec P$.*

PROOF. If $Q' = 0$, we take $P_0 = \langle 1 \rangle$; so we may assume $Q' \neq 0$. By replacing Q' with a similar form, we may further assume $Q' \sim 1$. Choose a form R which is maximal with respect to the properties: $R \sim 1$, $R \preceq Q'$, and there exists a Pfister form P_R of dimension $\leq 2^{\dim R - 1}$ such that $R \preceq P_R \prec P$. (Note that $R = \langle 1 \rangle = P_R$ is one such form.)

Claim: R is similar to Q' . If not, then there exists $c \in k^*$ such that $R \perp \langle c \rangle \preceq Q'$. Moreover, $R \perp \langle c \rangle \not\preceq P_R$ by the maximality of R . Therefore by 2.1.6, there exists $b \in k^*$ such that

$$R \perp \langle c \rangle \preceq P_R \perp bP_R \preceq P,$$

and

$$\dim(P_R \perp bP_R) = 2 \dim P_R \leq 2^{\dim R} < 2^{\dim Q'} \leq \dim P.$$

Let $R' = R \perp \langle c \rangle$ and $P_{R'} = P_R \perp bP_R$. Then

$$\dim P_{R'} \leq 2^{\dim R} = 2^{\dim R' - 1} < \dim P,$$

so $R' \preceq Q'$, $\dim P_{R'} \leq 2^{\dim R' - 1}$, and $R' \preceq P_{R'} \prec P$, which contradicts the maximality of R . \square

COROLLARY 3.1.8 (cf. [AO, 2.5]) *If Q is a Pfister neighbor of codimension $\leq n$ of an anisotropic Pfister form P of dimension $\geq 2^n$ ($n \geq 2$), then Q is special. (In particular, if $\dim P = 4$ or 8 , then all neighbors of P are special; and if $\dim P > 8$, then every subform of P of codimension ≤ 4 is special.)*

PROOF. By 3.1.7 there exists a Pfister form P_0 such that $Q^- \preceq P_0 \prec P$, and then 3.1.5 applies. \square

Knebusch [KII, p. 3, (7.8)] calls a Pfister neighbor Q *excellent* if either $\text{codim } Q \leq 2$ or its successive complements $Q^-, (Q^-)^-, \dots$ of dimension > 2 are again Pfister neighbors. These forms have been studied in [KII] and, in a generalized setting, in [HR1].

COROLLARY 3.1.9 (cf. [AO, 2.7]) *If Q is an anisotropic Pfister neighbor whose complement Q^- is also a Pfister neighbor, then Q is special. In particular, anisotropic excellent Pfister neighbors are special.*

PROOF. It suffices by 3.1.5 to see that $(Q^-)^+ \prec Q^+$, hence by 1.2.2 to see that Q^+ is isotropic over $k((Q^-)^+)$. Since Q^- is a Pfister neighbor, there exists an embedding of $k(Q^-)$ in $k((Q^-)^+)$ by 3.1.1(iii). But Q^+ is isotropic over $k(Q^-)$ since Q^- is a subform of Q^+ , and therefore a fortiori Q^+ is isotropic over $k((Q^-)^+)$.

For the second assertion, if Q is an anisotropic excellent Pfister neighbor and $\text{codim } Q \leq 2$, then Q is special by 3.1.8, while if $\text{codim } Q > 2$, then its complement Q^- is a Pfister neighbor. \square

We note below in a final remark for this section that the collection of special Pfister neighbors is strictly larger than the collection of excellent Pfister neighbors.

EXAMPLE (cf. [AO, 2.8]) of a non-special Pfister neighbor.

We have seen that every anisotropic Pfister neighbor of codimension ≤ 4 is special. We now give an example of an 11-dimensional Pfister neighbor of codimension 5 which is not special. Let a, b, c, d be algebraically independent elements over a field k_0 (of characteristic $\neq 2$); let $k = k_0(a, b, c, d)$; and let Q' be the k -form $\langle 1, a, b, c, d \rangle$.

CLAIM: The 5-dimensional 'generic' form $Q' = \langle 1, a, b, c, d \rangle$ is not a Pfister neighbor.

Suppose we have established the Claim. Then Q' is a subform of the 16-dimensional Pfister form $P = \langle 1, a \rangle \otimes \langle 1, b \rangle \otimes \langle 1, c \rangle \otimes \langle 1, d \rangle$, and P is anisotropic by [L, p. 273, Exercise 1]. Let $P \cong Q \perp Q'$. Then Q is a Pfister neighbor of P . By the Claim Q' is not similar to a subform of an 8-dimensional Pfister form, so Q cannot be special by 3.1.5.

PROOF OF THE CLAIM. If Q' is a Pfister neighbor, then by the lemma below Q' represents its determinant $abcd$, or equivalently, $Q' \perp \langle -abcd \rangle \cong \langle 1, a, b, c \rangle \perp$

$d\langle 1, -abc \rangle$ is isotropic. But this is not the case by [L, p. 273, Exercise 1]. \square

LEMMA (cf. [KII, p. 10]) *An anisotropic 5-dimensional form Q is a Pfister neighbor iff Q represents its determinant.*

PROOF. \Rightarrow : A 5-dimensional anisotropic Pfister neighbor Q is special by 3.1.8. This means there exist $a, d \in k^*$ and a 4-dimensional Pfister form P such that $Q \cong a(P \perp \langle d \rangle)$, which implies $Q \sim ad = \det Q$.

\Leftarrow : If $Q \sim d := \det Q$, then $Q \cong P \perp \langle d \rangle$ for some 4-dimensional anisotropic form P of determinant 1. If $P \sim a$, then aP is Pfister; so $aQ \cong aP \perp \langle ad \rangle$ implies Q is a (special) Pfister neighbor. \square

REMARK The above example also yields an example of a special Pfister neighbor which is not excellent: Take $Q^+ = Q \perp Q'$, with Q, Q' as above. Then $Q^+ \perp Q$ is a special Pfister neighbor of $Q^+ \perp Q^+ = Q^+ \perp Q \perp Q'$, but the complement Q' of $Q^+ \perp Q$ is not a Pfister neighbor; so $Q^+ \perp Q$ is not excellent.

3.2 Function fields of special Pfister neighbors

We have seen in the previous section (cf. 3.1.3(ii)) that if Q_1 and Q_2 are two special Pfister neighbors of the (same) Pfister form P , then $k(P)$ is a pure transcendental extension of $k(Q_1)$ and $k(Q_2)$. The main result of this section is to show that if also $\dim Q_1 = \dim Q_2$, then $k(Q_1)$ and $k(Q_2)$ are k -isomorphic. We start with the following special case.

LEMMA 3.2.1 (cf. [AO, 1.5.1]) (Transposition Lemma) *Let P be a Pfister form; let $b, c \in k^*$; and let $Q = (P \perp bP) \perp \langle c \rangle$ and $Q' = (P \perp cP) \perp \langle b \rangle$. Then*

- (1) Q and Q' define the same function field; and
- (2) if $\dim P \geq 2$, then Q and Q' are similar (if and) only if $bP \cong cP$.

PROOF. (1) If P is isotropic, then the function fields of Q and Q' are rational (1.2.1) and of the same dimension, hence isomorphic; so we may assume P is anisotropic. Let (x, y) be a generic zero for the polynomial $P(X) + cP(Y) + b$. Since $P \sim_{k(y)} P(y)$, the Pfister property of P yields $P \cong_{k(y)} P(y)P$. Therefore there exists a tuple (x') of elements such that $P(x) = P(y)P(x')$ and $k(y, x) = k(y, x')$. Then

$$\begin{aligned} P(y)(P(x') + c) + b &= 0, \text{ hence} \\ P(x') + c + (1/P(y))b &= 0. \end{aligned}$$

Let $y'_i = y_i/P(y)$. Since $P(y') = 1/P(y)$, it follows that $k(y) = k(y')$ and $P(x') + c + P(y')b = 0$. But $k(x, y) = k(x', y) = k(x', y')$, so then (x', y') must be a generic zero for $P(X') + bP(Y') + c$. Thus, $k(Q) = k(x, y) = k(x', y') = k(Q')$.

(2) Suppose Q is similar to Q' . Then bQ is similar to cQ' ; and since bQ and cQ' are of odd dimension and have the same determinant bc , this implies $bQ \cong cQ'$. Therefore $bP \perp P \perp \langle bc \rangle \cong cP \perp P \perp \langle bc \rangle$, and hence by Witt cancellation $bP \cong cP$. \square

THEOREM 3.2.2 (cf. [AO, 1.6]) *Let (P_1, a_1, P'_1) and (P_2, a_2, P'_2) be Pfister triples which define forms $P_1 \perp a_1P'_1$ and $P_2 \perp a_2P'_2$ of the same dimension.*

Then the following are equivalent:

- (1) $k(P_1 \perp \langle a_1 \rangle) \cong k(P_2 \perp \langle a_2 \rangle)$,
- (2) $k(P_1 \perp a_1P'_1) \cong k(P_2 \perp a_2P'_2)$,
- (3) $k(P_1 \perp a_1P_1) \cong k(P_2 \perp a_2P_2)$,
- (4) $P_1 \perp a_1P_1 \cong P_2 \perp a_2P_2$.

PROOF. (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4): By 3.1.3 $k(P_i \perp a_iP'_i)$ is a pure transcendental extension of $k(P_i \perp \langle a_i \rangle)$ and $k(P_i \perp a_iP_i)$ is a pure transcendental extension of $k(P_i \perp a_iP'_i)$, so (1) implies (2) implies (3); and (3) implies (4) by 1.2.3.

(4) \Rightarrow (1): The proof of this will proceed through a sequence of lemmas.

The first lemma is due Arason (cf. [Ars, p. 454, Lemma 1.7]). The proof provided here is simpler.

LEMMA 3.2.3 (cf. [AO, 1.6.1]) (Exchange Lemma) *Let P_i be a Pfister form and $a_i \in k^*$ ($i = 1, 2$). If $P_1 \perp a_1 P_1 \cong P_2 \perp a_2 P_2$, then there exists $c \in k^*$ such that for $i = 1, 2$, $P_i \perp \langle a_i \rangle$ is similar to $P_i \perp \langle c \rangle$.*

PROOF. Since $P_i \perp a_i P_i \sim a_j, j \neq i$, and $P_i \perp a_i P_i$ is Pfister,

$$P_i \perp a_i P_i \cong a_j (P_i \perp a_i P_i) \cong a_j P_i \perp a_j a_i P_i.$$

Putting these isomorphisms together with the isomorphism $P_1 \perp a_1 P_1 \cong P_2 \perp a_2 P_2$ given by the hypothesis, we have

$$a_2 P_1 \perp a_2 a_1 P_1 \cong a_1 P_2 \perp a_1 a_2 P_2,$$

or, in the Witt ring,

$$a_1 a_2 ([P_1] - [P_2]) = [a_1 P_2] - [a_2 P_1].$$

But $[P_1] - [P_2] = [\langle 1, -1 \rangle \perp (\dots)]$, which, since the left and right sides of the above equality involve forms of the same dimension, implies that $a_1 P_2 \perp (-a_2 P_1)$ is isotropic. Thus, there exist $b_1, b_2 \in k^*$ such that $P_i \sim b_i$ and $a_2 b_1 = a_1 b_2$, or $\langle a_1 b_1 \rangle \cong \langle a_2 b_2 \rangle$. We assert that $c = a_1 b_1$ has the desired properties. Since P_i is Pfister and $P_i \sim b_i$, $b_i P_i \cong P_i$. Therefore $P_i \perp \langle a_i \rangle \cong b_i P_i \perp \langle a_i \rangle \cong b_i (P_i \perp \langle a_i b_i \rangle)$. \square

LEMMA 3.2.4 (cf. [AO, 1.6.2]) *Let P_i be Pfister and $a_i, b_i \in k^*$ ($i = 1, 2$). If $(P_1 \perp a_1 P_1) \perp \langle b_1 \rangle$ and $(P_2 \perp a_2 P_2) \perp \langle b_2 \rangle$ have the same associated Pfister form, then there exist $c, d \in k^*$ such that for $i = 1, 2$, $(P_i \perp a_i P_i) \perp \langle b_i \rangle$ is birationally equivalent to $(P_i \perp c P_i) \perp \langle d \rangle$.*

PROOF. Note first that two forms which are defined by Pfister triples and which are birationally equivalent have the same associated Pfister form, by

3.1.3 and 1.2.3. Therefore we can successively apply the Exchange Lemma, the Transposition Lemma, and the Exchange Lemma again to conclude that there exist $c, d \in k^*$ such that for $i = 1, 2$, $(P_i \perp a_i P_i) \perp \langle b_i \rangle$ is birationally equivalent to $(P_i \perp a_i P_i) \perp \langle c \rangle$ is birationally equivalent to $(P_i \perp c P_i) \perp \langle a_i \rangle$ is birationally equivalent to $(P_i \perp c P_i) \perp \langle d \rangle$. \square

LEMMA 3.2.5 (cf. [AO, 1.6.3]) *Let Q_i be a Pfister form of dimension 2^n ($n \geq 1$) and $b_i \in k^*$ ($i = 1, 2$), and suppose Q is a common Pfister subform of Q_1 and Q_2 of dimension $< 2^n$. If $Q_1 \perp \langle b_1 \rangle$ and $Q_2 \perp \langle b_2 \rangle$ have the same associated Pfister form, then there exist Pfister forms P_1, P_2 of dimension 2^{n-1} and elements $c, d \in k^*$ such that Q is a common subform of P_1 and P_2 and for $i = 1, 2$, $(P_i \perp c P_i) \perp \langle d \rangle$ is birationally equivalent to $Q_i \perp \langle b_i \rangle$. (Hence $Q \perp c Q$ is a common Pfister subform of $P_1 \perp c P_1$ and $P_2 \perp c P_2$.)*

PROOF. Since Q is a proper Pfister subform of the Pfister form Q_i , there exists a Pfister subform P_i of Q_i and an element $a_i \in k^*$ such that Q is a subform of P_i and $Q_i \cong P_i \perp a_i P_i$ (cf. [L, p. 293, Exercise 8]). The assertion now follows from the preceding lemma. \square

We are now ready to finish the proof of 3.2.2. Let \mathcal{S}_i ($i = 1, 2$) be the set of all forms birationally equivalent to $P_i \perp \langle a_i \rangle$ and of the shape $Q \perp \langle b \rangle$ with Q Pfister and $b \in k^*$; and suppose we have chosen elements $Q_1 \perp \langle b_1 \rangle$ from \mathcal{S}_1 and $Q_2 \perp \langle b_2 \rangle$ from \mathcal{S}_2 such that Q_1 and Q_2 have a common Pfister subform Q of maximal possible dimension among such choices. By 3.2.4 Q cannot be a *proper* subform of, say, Q_1 ; so $Q \cong Q_1 \cong Q_2$.

In view of the fact that the associated Pfister form is preserved under birational equivalence (3.1.3 and 1.2.3), it suffices to note that $Q \perp \langle b_1 \rangle$ and $Q \perp \langle b_2 \rangle$ are similar, hence birational, by the Exchange Lemma. \square

3.3 The Zariski cancellation problem

We noticed in 3.1.2(i) that if Q_1 and Q_2 are neighbors, the field $k(Q_1)(Q_2)$ is pure transcendental over both $k(Q_1)$ and $k(Q_2)$. If $\dim Q_1 = \dim Q_2$, should $k(Q_1)$ and $k(Q_2)$ be k -isomorphic? Note that by 3.1.2(iii), $k(Q_1)$ and $k(Q_2)$ can be embedded in one another. The situation is identical to that of the problem of birational cancellation which can be stated as follow: (See [O2] and [O3] for a detailed exposition.)

THE ZARISKI CANCELLATION PROBLEM (ZCP) Suppose K/k and K'/k are quadratic function fields and there exists a finite set of elements (t) algebraically independent over K and a finite set of elements (t') algebraically independent over K' such that $K(t) \cong K'(t')$. Does it follow that $K \cong K'$?

ZCP has an affirmative answer if $\dim K/k \leq 1$; but the answer is *no* in general by a (difficult) counter-example (see [BCSS]) which is a function field of a cubic surface.

In our set-up, the fields K and K' are function fields of quadratic forms. One may ask ZCP has an affirmative answer with this extra hypothesis. This will be referred to as the **QUADRATIC ZARISKI CANCELLATION** problem, or simply the quadratic ZCP.

COROLLARY 3.3.1 (cf. [AO, 2.6]) *If Q is an anisotropic Pfister neighbor of codimension ≤ 4 , then the quadratic function fields which are overfields of $k(Q)$ are exactly the pure transcendental extensions K of $k(Q)$ such that $k(Q) \subseteq K \subseteq k(Q^+)$.*

PROOF. By 3.1.8 Q is special, and therefore the pure transcendental extensions between $k(Q)$ and $k(Q^+)$ are all function fields of special Pfister neighbors of Q^+ , by 3.1.3. Suppose then that R is a nonsingular form such that $k(Q) \subseteq k(R)$. Since Q^+ becomes isotropic over $k(Q)$ and a fortiori over $k(R)$,

R is similar to a subform of Q^+ by 1.2.2. But then R is a Pfister neighbor of Q^+ of codimension ≤ 4 , hence special by 3.1.8, and by 3.1.3 $k(R)$ is isomorphic to a pure transcendental extension of $k(Q)$. \square

COROLLARY 3.3.2 (cf. [AO, 2.6.1]) *The quadratic Zariski problem has an affirmative answer if K is the function field of a Pfister neighbor of codimension ≤ 4 .*

PROOF. The hypothesis of the ZCP implies K embeds in K' (cf. [O1], or 3.1.2 (iii)) and $\dim K = \dim K'$, so K' is an algebraic extension of K . Corollary 3.3.1 implies that K' has to equal K in this case. \square

The quadratic ZCP is known to have an affirmative answer if $\dim K/k \leq 3$ or if K is a function field of (cf. [O4] for details). The case that $\dim K/k = 3$, i.e. that $K = k(P)$ for P a form of dimension 5, was pointed out to us by Detlev Hoffmann: If P is not a Pfister neighbor, then this case follows from the Main Theorem of [H3], while if P is a Pfister neighbor, it follows from two of our main results, 3.2.2 and 3.1.8.

3.4 Similarity and function fields

As a consequence of theorem 3.2.2, one can construct examples of birationally equivalent forms that are non-similar. We explicitly produce two five dimensional non-similar form that are birationally equivalent. In view of Wadsworth's results in [W], our example has minimal possible dimension. Also, from the last paragraph of the previous section, Pfister neighbors are the only source of such examples in dimension 5.

EXAMPLE 3.4.1 (cf. [AO, 1.5]) Fix a field k_0 and let a, b, c be algebraically independent over k_0 , and let $k = k_0(a, b, c)$. Let

$$Q \cong \langle 1, a, b, ab, c \rangle \text{ and } Q' \cong \langle 1, a, c, ac, b \rangle$$

By [L, p. 273, Exercise 1] $P = \langle 1, a \rangle$ is anisotropic over $k_0(a)$, and then also Q and Q' are anisotropic over k . Clearly Q and Q' are special Pfister neighbors of the Pfister form $\langle 1, a, b, ab \rangle \perp c \langle 1, a, b, ab \rangle$, hence they are birationally equivalent by 3.2.2. Moreover, by (2) of the transposition lemma 3.2.1, Q cannot be similar to Q' ; for otherwise bP represents c (over k), which is easily seen to contradict to the assumption that a, b, c are algebraically independent over k_0 .

Now we turn our attention to the problem of descent of similarity over field extensions:

QUESTION. Let L/k be a field extension and let P and Q be anisotropic k -forms. Under what conditions (imposed on either the extension or the forms) the similarity (isometry) of P and Q over L implies the similarity of P and Q over k .

The question is widely open, and our results are rather elementary and limited. Of course one is particularly interested in the case where the field L is a function field of a quadratic form.

Note the the descent of isometry is known if L/k is either a pure transcendental or an algebraic extension of odd degree; for in these cases $P \perp -Q$ is hyperbolic over L if and only if it is hyperbolic over k . (cf. [L, p. 255 lemma 1.1] for rational extensions, and Springer's theorem [L, p. 198] for the odd degree extension.) In the next two results we show one actually gets similarity descent over rational extensions and odd degree extensions.

PROPOSITION 3.4.2 *Let L/k be a purely transcendental extension and P and Q be anisotropic k -forms. Then P is similar to Q over L (if and) only if P is similar to Q over k .*

PROOF. We only need to show this for simple transcendental extensions. Let $L = k(X)$ where X is transcendental over k . Assume first that k is infinite. By hypothesis, there exist $f \in k(X)$ and an invertible matrix $M = (f_{ij})$ over $k(X)$ such that

$$P(Y) = f \cdot Q(MY) \quad (*)$$

Let $d(X) = \det(M) \in L^*$. Since k is infinite, we can choose $x \in k$ such that $d(x), f(x)$ and $f_{ij}(x)$ are defined, and $f(x), d(x)$ are non-zero. Let $M' = (f_{ij}(x))$ and $b = f(x)$. Then $b \in k^*$ and M' is an invertible k -matrix. Equation (*) gives

$$P(Y) = b \cdot Q(M'Y),$$

hence P and Q are similar over k .

Now assume that k is finite. In this case the anisotropic k -forms have dimension ≤ 2 . (Actually, over finite fields any homogeneous polynomial in more than 2 variables has a zero. cf. [G].) If $\dim P = 1$, then there nothing to prove. If $2 = \dim P = \dim Q$, then $P = a \langle 1, d_1 \rangle$ and $Q = b \langle 1, d_2 \rangle$. Since $P \approx Q$ over L , P and Q must have the same determinant, i.e. $\langle d_1 \rangle = \langle d_2 \rangle$ over L . Since k is algebraically closed in L , we have $\langle d_1 \rangle = \langle d_2 \rangle$ over k and we are done □

The descent of similarity over algebraic extension of odd degree follow from the following theorem which is a generalization of [L, p. 208, Scharlau's Norm principle] (for odd degree extensions).

PROPOSITION 3.4.3 *Let L/k be an algebraic extension of odd degree, and P and Q be anisotropic k -forms. If $P \cong aQ$ for $a \in L^*$, then $P \cong N_{L/k}(a)Q$ over k .*

PROOF. Let $L_0 = k(a)$, $m := [L : L_0]$ and $n := [L_0 : k]$, m and n are odd. By Springer's theorem $P \cong aQ$ over L implies $aQ \cong P$ over L_0 . By applying to

last isometry the transfer s^* map induced by the linear functional $s : L_0 \rightarrow k$ defined by $s(1) = 1$, and $s(a) = \dots = s(a^{n-1}) = 0$, we get (cf. [L, p. 195-196 theorems 1.6 and 1.7]):

$$P \cong N_{L_0/k}(a)Q \text{ over } k.$$

Note that $N_{L/k}(a) = N_{L_0/k}(N_{L/L_0}(a)) = N_{L_0/k}(a^n) = (N_{L_0/k}(a))^n$. Since n is odd, we have, over k ,

$$N_{L/k}(a)Q \cong (N_{L_0/k}(a))^n Q \cong (N_{L_0/k}(a))Q \cong P$$

as desired. □

THEOREM 3.4.4 *Let ψ be a k -form. Let Q and P be anisotropic forms such that Q and P represent a common element over k and $\dim P \leq 2^n < \dim \psi$ for some $n \geq 0$. Then Q is a subform of P over $k(\psi)$ (if and) only if Q is a subform of P over k .*

PROOF. We induct on $\dim Q$. If $\dim Q = 1$, then the hypothesis Q and P represent a common element over k implies that Q is a subform of P .

So assume that $\dim Q > 1$ and let $a \in D_k(Q) \cap D_k(P)$. Write $Q = \langle a \rangle \perp Q_1$ and $P = \langle a \rangle \perp P_1$. By hypothesis $\dim Q \leq \dim P \leq 2^n < \dim \psi$. So theorem 1.2.4, implies that Q and P stay anisotropic over $k(\psi)$. Since $Q_{k(\psi)}$ is a subform of $P_{k(\psi)}$, we can write

$$P \cong_{k(\psi)} \langle a \rangle \perp P_1 \cong_{k(\psi)} \langle a \rangle \perp Q_1 \perp R$$

for some $k(\psi)$ -form R . By cancelling $\langle a \rangle$, we have that Q_1 is a subform of P_1 over $k(\psi)$. This implies, since Q_1 is a k -form, that there exists an element $a_1 \in k^*$ represented over k by Q_1 that is also represented over $k(\psi)$ by P_1 . Therefore the k -form $P_1 \perp \langle -a_1 \rangle$ is isotropic over $k(\psi)$. But $\dim(P_1 \perp \langle -a_1 \rangle) = \dim P \leq 2^n < \dim \psi$. By 1.2.4 implies that $P_1 \perp \langle -a_1 \rangle$ is isotropic over k . Since P_1

is anisotropic, this implies that P_1 represents a_1 over k . In other words we concluded that Q_1 and P_1 represent a common element over k . Thus by the inductive hypothesis applied to Q_1 and P_1 , we have Q_1 is a subform of P_1 and therefore $Q = \langle a \rangle \perp Q_1$ is a subform of $P = \langle a \rangle \perp P_1$. \square

REMARK 3.4.5 In the previous proposition, the hypothesis Q and P represent a common element over k can be relaxed if the $\dim P < 2^n (< \dim \psi)$. For then the inductive step can be modified as follows: Let $Q = \langle a \rangle$. Since Q is a subform of P over $k(\psi)$, the k -form $P \perp \langle -a \rangle$ is isotropic over $k(\psi)$ and has dimension $= \dim P + 1 \leq 2^n < \dim \psi$. Therefore by 1.2.4, $P \perp \langle -a \rangle$ is isotropic over k ; i.e. $Q = \langle a \rangle$ is a subform of P .

When $\dim Q = \dim P$, 3.4.4 and this remark give

COROLLARY 3.4.6 *Let ψ be a k -form. Let Q and P be anisotropic forms such that $\dim P < 2^n < \dim \psi$ for some $n \geq 0$. Then $Q \cong P$ are isometric over $k(\psi)$ (if and) only if $Q \cong P$ over k .*

That is we have isometry descent in this case. If further we assume that $\dim P$ is odd, one can get a similarity descent result :

COROLLARY 3.4.7 *Let ψ be a k -form. Let Q and P be anisotropic forms such that $\dim P$ is odd and $\leq 2^n < \dim \psi$ for some $n \geq 0$. Then $Q \approx P$ are isometric over $k(\psi)$ (if and) only if $Q \approx P$ over k .*

PROOF. Since $\dim P$ is odd we can find a k -form P' which is similar to P such that $\det P' = \det Q$. Over $k(\psi)$, $Q \approx P'$. By comparing determinants (remember $\dim P$ is odd), we conclude that P' and Q are actually isometric over $k(\psi)$; hence isometric over k by 3.4.6. \square

COROLLARY 3.4.8 *Let Q and ψ be k -forms. Let P be an anisotropic n -fold Pfister k -form and $2^n < \dim \psi$. If Q is similar to a subform of P over $k(\psi)$,*

then Q is similar to a subform of P over k . In particular, if P' is another Pfister k -form and $P \cong P'$ over $k(\psi)$, then $P \cong P'$ over k .

PROOF. Let $a \in D_k(Q)$. Then the form $Q_1 := aQ$ represents 1. Over $k(\psi)$, Q_1 is similar to a subform of the Pfister form P , hence by 1.2.2 Q_1 is a subform of P over $k(\psi)$. Since both P and Q_1 represent 1, theorem 3.4.4 implies that Q_1 is a subform of P over k . \square

Recall that Q is called a Pfister neighbor if it is similar to a subform of a Pfister form P such that $2 \dim Q \geq \dim P$. If Q is a Pfister neighbor of dimension $2^n + m < 2^{(n+1)}$ then the Witt index Q over (its function field) $k(Q)$ is known to be m (see for example [H4, cor. 2]). In the following proposition we record a (partial) converse of this.

PROPOSITION 3.4.9 *Let P be an n -fold Pfister form. Let Q_0 be a non-zero k -form of dimension $\leq \dim P$ such that the form $Q := P \perp Q_0$ is anisotropic. Then Q is a Pfister neighbor iff the Witt index Q over $k(Q) = \dim Q_0$*

REMARK Note that the Witt index Q over $k(Q)$ is always $\leq \dim Q_0$, for otherwise P will become isotropic over $k(Q)$ contradicting 1.2.4.

PROOF. Assume that the Witt index Q over $k(Q)$ equals $\dim Q_0$. Since $\dim P = 2^n$ and $\dim Q > \dim P \geq \dim Q_0$, theorem 1.2.4 implies that Q_0 and P stay anisotropic over $k(Q)$. Therefore $\iota_W(Q_{k(Q)}) = \dim Q_0$ implies that $-Q_0$ is a subform of P over $k(Q)$. By 3.4.8, Q_0 is similar to a subform of P over k , i.e., there exists $a \in k$ such that aQ_0 is a subform of P over k . Hence $Q = P \perp Q_0$ is a neighbor of the Pfister form $P \perp aP$ \square

REMARK Note that in the proof of 3.4.9 we only needed that Witt index Q over a field L equals $\dim Q_0$ where L/k is a function field of a quadratic form of dimension greater than $\dim P$.

We now return to the relation between birational equivalence and function fields. In our example 3.4.1 (and also in Knebúsch's example [K1, p. 73]) of birational but non-similar forms, we notice that the anisotropic parts of these forms over their (common) function fields are not similar. This leads to the following (open) question which is a special case of the similarity descent question:

Q1. Let P and Q be nonsingular birationally equivalent quadratic k -forms such that P is similar to Q over $k(P)$ ($= k(Q)$). Should P and Q be similar over k ?

This question is equivalent to the following question proposed by professor Ulf Rehmann.

Q2. Let P and Q be nonsingular anisotropic quadratic k -forms of the same dimension which define the same generic splitting tower over k . Should P and Q be similar over k ?

REMARK Let P and Q be anisotropic forms that define the same generic splitting tower. Let $k_0 = k, k_1, \dots, k_h$ be a common splitting tower and $P_0 = P, P_1, \dots, P_{h-1}$ (respectively, $Q_0 = Q, Q_1, \dots, Q_{h-1}$) be the corresponding anisotropic kernels of P (respectively, Q). Then P_i is isotropic over $k_i(Q_i)$ and conversely. This fact will be used frequently in what follows.

Our next theorem will answer in the affirmative Q1 for forms of odd dimension. We will need the following lemma which is a special case of a result of Fitzgerald [F, theorem 1.6].

LEMMA 3.4.10 *Let P and Q be anisotropic k -forms. If $Q \cong P$ over $k(P)$, then either*

- (1) Q and P are similar over k , or
- (2) $P \perp Q$ is similar to a Pfister form.

PROOF. Let $\varphi = P \perp -Q$. Then $2 \dim P = \dim \varphi > \dim \varphi - 2^{\deg(\varphi)}$. Also φ is hyperbolic over $k(P)$ because $Q \cong P$ over $k(P)$. Fitzgerald's theorem [F, theorem 1.6], φ is either hyperbolic or an anisotropic Pfister form; hence (1) and (2) follow respectively. \square

Notice that case (2) can happen only if $\dim P$ is a 2-power. So the lemma provides an isometry descent result in the case $\dim P$ not a 2-power and $L = k(p)$.

THEOREM 3.4.11 *Let P and Q be non-singular quadratic k -forms such that $\dim P$ is odd. If P is similar to Q over $k(P)$, then P and Q are similar over k .*

Notice we do not need the hypothesis P and Q are birationally equivalent.

PROOF. Since P has odd dimension, we can replace P by a similar form such that we may assume that Q and P have the same determinant. Over $k(P)$, P and Q are similar forms with the same determinant and odd dimension, they must be isometric over $k(P)$. Since $\dim P$ is not a 2-power, the previous lemma implies that P and Q must be similar over k , as desired. \square

We now provide few classes of forms where the question Q2 above has a positive answer.

PROPOSITION 3.4.12 *Let P and Q be anisotropic k -forms and $P \cong P_0 \perp \langle a \rangle$ for some Pfister form P' and $a \in k$. Then P and Q are similar if and only if P and Q define the same generic splitting tower.*

PROOF. Suppose that P and Q define the same generic splitting tower. P is clearly a Pfister neighbor of the Pfister form $P_0 \perp aP_0$, hence Q is also a Pfister neighbor of this Pfister form too. In particular, the anisotropic kernels P_1 and Q_1 of P and Q over $k(P)$, respectively, are defined over k . Clearly, $P_1 \cong -aP_0$ over $k(P)$ which is a codimension 1 Pfister neighbor. But P_1 and Q_1 have the

same generic zero fields, hence they must be similar (over $k(P)$). By 3.4.8, P_1 and Q_2 are both codimension 1 neighbors of P_0 over k , hence are similar. Therefore P and Q are neighbors of $P_0 \perp aP_0$ with similar complements, hence are similar. \square

We have seen that the 5-dimensional Pfister neighbors are special, hence are similar to a form of shape $P \perp \langle a \rangle$, where P is a Pfister form. The preceding proposition implies the similarity class of such a form is determined by its generic splitting tower. The results in [W] and the main theorem in [H4], imply that the same hold for forms of dimension ≤ 4 and the 5-dimensional none Pfister neighbor forms. So we have

COROLLARY 3.4.13 *The similarity class of any form of dimension ≤ 5 is determined by its generic splitting tower.*

COROLLARY 3.4.14 *Let P be an anisotropic Pfister neighbor of codimension ≤ 5 . Let Q be an anisotropic form such that P and Q define the same generic splitting tower over k . Then P and Q are similar.*

PROOF. Let π be the Pfister form associated to P . The hypothesis imply that Q is also a neighbor of π . Let P_0 and Q_0 be the complements of P and Q , respectively, in P_0 . By the previous corollary we have $P_0 \sim Q_0$ over $k(P)$. Note that P is isotropic over $k(P_0)$, because π is. Therefore, $k(P)(P_0)$ is rational over $k(P_0)$. Likewise $k(P)(Q_0)$ is rational over $k(Q_0)$. By the ZCP (which holds here since $\dim P_0 \leq 5$), P_0 and Q_0 are birationally equivalent over k . In the case P_0 is not a 5-dimensional neighbor, this implies that P_0 and Q_0 are similar. This in turn shows that their complements P and Q in π are similar. Now let P_0 and Q_0 be 5-dimensional neighbors. Then we can repeat the same argument above with P_0 and Q_0 replaced by their complements to conclude that P_0 and Q_0 are similar; this implies that $P \sim Q$ are similar. \square

Chapter 4

The Pfister-Leep Conjecture

In analogy to algebraically closed fields, a field k is called a C_0^d -field if every system of r forms of degree d over k in n variables where $n > r$ has a common non-trivial zero over k . For a prime p , the field k is called a p -field if $[L : k]$ is a power of p for every finite extension L/k .

In [P1], Pfister proof the following

THEOREM ([P1, Theorem 2]): *If k is a p -field, then for any d not divisible by p , k is a C_0^d -field.*

See also [P2, Theorem 2]. A special case is

COROLLARY ([P1, Corollary 1]): *If k is a p -field for some prime $p \neq 2$, then k is a C_0^2 -field*

Pfister conjectured that the converse of this corollary is true.

PFISTER'S CONJECTURE ([P1, Conjecture 3]): *If k is a C_0^2 -field, then k is a p -field for some prime $p \neq 2$.*

In [L2, Theorems 5.4, 5.5], Leep proved this conjecture for fields of characteristic 0 or 2 and gave the following generalized version of Pfister's conjecture to higher degree forms (see [L2, 1.4]):

THE CONJECTURE OF PFISTER-LEEP. For a fixed d , if k is a C_0^d -field, then k is a p -field for some prime $p \neq d$.

In this chapter we show that the Pfister-Leep conjecture is true if d is a power of the characteristic of the field k . Note that if k is a $C_0^{q^i}$ -field, then k is also a C_0^q -field (because if $\{F_1, \dots, F_r\}$ is a system of forms of degree q ,

then $\{F_1^i, \dots, F_r^i\}$ is an equivalent system of forms of degree q^i). Therefore we need only to consider the case when d is equal to the characteristic of k .

4.1 A system of forms

Let k be a fixed field and let $d > 1$ be a fixed integer. In this section, we define a system of forms of degree d that will be used in the proof of the special case of the conjecture. We take our variables to be Z, X_1, X_2, \dots

Define $f : \{2, 3, \dots\} \rightarrow \{1, 2, \dots\}$ and $g : \{2, 3, \dots\} \rightarrow \{1, d, d^2, \dots\}$ as follows:

For $n > 0$, let

$$n = a_0 + a_1d + \dots + a_r d^r$$

be the d -adic expansion of n , where $a_t \in \{0, 1, \dots, d-1\}, 0 \leq t \leq r$, and $a_r \neq 0$. Set $g(n) = d^{r+1}$; and

$$f(n) = \begin{cases} d^{r-1} & \text{if } n = d^r; \\ d^r & \text{if } n = a_r d^r \text{ and } a_r > 1; \\ a_0 d + a_1 d^2 + \dots + a_{r-1} d^r & \text{if } n \neq a_r d^r. \end{cases}$$

Now, define the form ϕ_n of degree d as follows:

$$\phi_n = \begin{cases} X_n Z^{d-1} - X_{f(n)}^d, & \text{if } n = d^r; \\ X_n^d - X_{g(n)}^{a_r} Z^{d-a_r}, & \text{if } n = a_r d^r \text{ and } a_r > 1; \\ X_n^d - X_{f(n)} X_{g(n)}^{a_r} Z^{d-a_r-1}, & \text{otherwise.} \end{cases}$$

REMARK 4.1.1

(i) Note that $n < d^{r+1} = g(n)$, hence $f(n) < d^{r+1}$. In particular, The form ϕ_n does not involve the variables $X_t, t > d^{r+1}$

(ii) Also $n \leq d^m$, implies $r \leq m$, with equality only if $n = d^m$. Hence $n < d^m$ implies $f(n) < d^m$ and $g(n) \leq d^m$.

(iii) If $n = a_r d^r$, then $dn = a_r g(n)$. If $n \neq a_r d^r$, then $g(n) > n$ and $dn = f(n) + a_r g(n)$.

(iv) If $n \neq a_r d^r$ and $a_r = d-1$, then $n - f(n) = (d-1)(g(n) - n) > 0$.

To give the reader a feeling of how the forms ϕ_n look, we list these forms for the case $d = 3$ and $1 < n \leq 9$.

$$\begin{aligned} \phi_2 &= X_2^3 - X_3^2 Z, & \phi_3 &= X_3 Z^2 - X_1^3, \\ \phi_4 &= X_4^3 - X_3 X_9 Z, & \phi_5 &= X_5^3 - X_6 X_9 Z, & \phi_6 &= X_6^3 - X_9^2 Z, \\ \phi_7 &= X_7^3 - X_3 X_9^2, & \phi_8 &= X_8^3 - X_6 X_9^2, & \phi_9 &= X_9 Z^2 - X_3^3. \end{aligned}$$

LEMMA 4.1.2 *Let m be an integer ≥ 1 , and let z, x_1, x_2, \dots be elements from a field. If $z = 0$ and the forms $\phi_2, \dots, \phi_{d^m}$ defined above vanish on (z, x_1, x_2, \dots) , then $x_n = 0$ for $n < d^m$.*

PROOF. Suppose that $1 \leq n < d^m$, and let $n = a_0 d^0 + a_1 d + \dots + a_r d^r$ be the d -adic expansion of n , $a_r \neq 0$. We use induction on n to show that under the given hypotheses, $x_n = 0$.

If $n=1$, then $d \leq d^m$. Hence $\phi_d := X_d Z^{d-1} - X_{f(d)}^d = X_d Z^{d-1} - X_1^d$ vanishes on $(0, x_1, x_2, \dots)$, which implies that $x_1 = 0$.

Now assume $n > 1$. There will be three cases according to how ϕ_n is defined.

Case I. $n = d^r$. Since $n < d^m$, $dn \leq d^m$; hence by hypothesis $\phi_{dn} := X_{dn} Z^{d-1} - X_{f(dn)}^d$ vanishes on $(0, x_1, \dots)$. This implies $x_{f(dn)} = 0$. But $n = d^r$ implies $f(dn) = f(d^{r+1}) = d^r = n$, so $x_n = 0$.

Case II. $n = a_r d^r$, $a_r \neq 1$. Then our hypothesis that $\phi_n = X_n^d - X_{g(n)}^{a_r} Z^{d-a_r}$ vanishes on $(0, x_1, \dots)$ again implies $x_n = 0$.

Case III. $n \neq a_r d^r$. Suppose first that $a_r < d - 1$. Then our hypothesis that $\phi_n = X_n^d - X_{f(n)} X_{g(n)}^{a_r} Z^{d-a_r-1}$ vanishes on $(0, x_1, \dots)$ implies $x_n = 0$, because $d - a_r - 1 > 0$. Now assume $a_r = d - 1$. By remark 4.1.1.(iv), $f(n) < n < d^m$; hence $x_{f(n)} = 0$ by induction hypothesis. Therefore, $\phi_n = X_n^d - X_{f(n)} X_{g(n)}^{d-1}$ vanishes on $(0, x_1, \dots)$ again implies $x_n = 0$ as desired. \square

LEMMA 4.1.3 Let m be an integer ≥ 1 , and let z, x_1, x_2, \dots be elements from a field. If $z = 1$ and the forms $\phi_2, \dots, \phi_{d^m}$ defined above vanish on (z, x_1, x_2, \dots) , then

$$x_n = \epsilon_n x_1^n, \text{ for } n \leq d^m$$

where ϵ_n is a d -power root of unity.

REMARK 4.1.4 From the proof of this lemma we shall see that

(i) $\epsilon_n = 1$ if $n = d^r$. In particular, for any n , $\epsilon_{g(n)} = 1$ since $g(n)$ is a d -power.

(ii) ϵ_n is a d -th root of unity if $n = ad^r$, $1 < a < d$.

(iii) $\epsilon_n = \epsilon \epsilon_{f(n)}$ where ϵ is a d -th root of unity, if $n \neq ad^r$, $1 \leq a < d$.

PROOF OF 4.1.3 Again, let $n = a_0 d^0 + a_1 d + \dots + a_r d^r$ be the d -adic expansion of n , $a_r \neq 0$, and $n \leq d^m$. If $n = 1$, then $x_1 = 1 \cdot x^1$, so we may assume $n > 1$.

Case I. $n = d^r$. We induct on r . If $r = 1$, then $\phi_d := X_n Z^{d-1} - X_1^d$ vanishes on $(1, x_1, x_2, \dots)$ implies $x_d = x_1^d$. If $r > 1$, then similarly, $x_{d^r} = x_{d^{r-1}}^d$. But by induction hypothesis, $x_{d^{r-1}} = x_1^{d^{r-1}}$, so $x_{d^r} = x_1^{d^r}$.

Case II. $n = a_r d^r$, $1 < a_r < d$. Since $n \leq d^m$, $\phi_n = X_n^d - X_{g(n)}^{a_r} Z^{d-a_r}$ vanishes on $(1, x_1, x_2, \dots)$, which implies that $x_n^d = x_{g(n)}^{a_r}$. Since $d^r < n \leq d^m$, $r+1 \leq m$. Therefore, $g(n) = d^{r+1} \leq d^m$, hence by case I we have $x_{g(n)} = x_1^{g(n)}$. Hence $x_n^d = x_{g(n)}^{a_r} = x_1^{a_r g(n)}$. Therefore, $x_n = \epsilon_n x_1^n$, Where ϵ_n is a d -th root of unity.

Case III. $n \neq a_r d^r$. Again, since $g(n) = d^{r+1} \leq d^m$, by case I, $x_{g(n)} = x_1^{g(n)}$. Thus, $\phi_n := X_n^d - X_{f(n)} X_{g(n)}^{a_r} Z^{d-a_r-1}$ vanishes on $(1, x_1, x_2, \dots)$ implies

$$x_n^d = x_{f(n)} x_1^{a_r g(n)}.$$

So, it is enough to show that $x_{f(n)} = \epsilon_{f(n)} x_1^{f(n)}$, where $\epsilon_{f(n)}$ is a d -power root of unity; for then we have

$$\begin{aligned} x_n^d &= \epsilon_{f(n)} x_1^{f(n)+a_r g(n)} \\ &= \epsilon_{f(n)} x_1^{dn}, \quad \text{by remark 4.1.1.(ii),} \end{aligned}$$

and therefore,

$$x_n = \epsilon \epsilon_{f(n)}^{1/d} x_1^n.$$

where ϵ is a d -th root of unity, and we may take $\epsilon_n = \epsilon \epsilon_{f(n)}^{1/d}$, a d -power root of unity.

Claim : If n is an integer > 1 and there exists m such that $n \leq d^m$ and $\phi_2, \phi_3, \dots, \phi_{d^m}$ vanish on $(1, x_1, x_2, \dots)$, then $x_{f(n)} = \epsilon_{f(n)} x_1^{f(n)}$, where $\epsilon_{f(n)}$ is a d -power root of unity.

Proof of the claim. We may assume $f(n) > 1$, for otherwise $f(n) = 1$ implies $n = d$ and we are done by case *I*. Note that since $n \leq d^m$, $f(n) < d^m$ by remark 4.1.1.(ii). To show that $x_{f(n)} = \epsilon_{f(n)} x_1^{f(n)}$, we induct on the "length" of n ; i.e. on the quantity $r - t$, where

$$n = a_t d^t + \dots + a_r d^r \quad \text{with } a_t a_r \neq 0.$$

If $r - t = 0$, then $f(n) = d^{r-1}$ or d^r , hence by case *I* applied to $f(n)$ we have $x_{f(n)} = \epsilon_{f(n)} x_1^{f(n)}$, where $\epsilon_{f(n)}$ is a d -th root of unity.

So assume that $r - t > 0$. Then

$$f(n) = a_t d^{t+1} + \dots + a_s d^{s+1} \quad \text{where } a_s \neq 0, \text{ and } s < r.$$

The length of $f(n)$ equals $s - t < r - t$. Therefore, by induction hypothesis applied to $f(n)$,

$$x_{f(f(n))} = \epsilon_{f(f(n))} x_1^{f(f(n))}.$$

Also, since $g(f(n))$ is a d -power, $x_{g(f(n))} = x_1^{g(f(n))}$ by case *I*.

If $f(n) = a_t d^{t+1}$, then again by cases *I* and *II*, we get $x_{f(n)} = \epsilon_{f(n)} x_1^{f(n)}$, where $\epsilon_{f(n)}$ is a d -th root of unity. So, assume $f(n) \neq a_s d^{s+1}$. Then $\phi_{f(n)} := X_{f(n)}^d - X_{f(f(n))} X_{g(f(n))}^{a_s} Z^{d-a_s-1}$ vanishes on $(1, x_1, x_2, \dots)$ implies that

$$\begin{aligned} x_{f(n)}^d &= x_{f(f(n))} x_{g(f(n))}^{a_s} \\ &= \epsilon_{f(f(n))} x_1^{f(f(n))+a_s g(f(n))} \\ &= \epsilon_{f(f(n))} x_1^{df(n)}, \quad \text{by remark 4.1.1.(iii).} \end{aligned}$$

Hence,

$$x_{f(n)} = \epsilon \epsilon_{f(f(n))}^{1/d} x_1^{f(n)},$$

where ϵ is a d -th root of unity. We set $\epsilon_{f(n)} = \epsilon \epsilon_{f(f(n))}^{1/d}$, a d -power root of unity.

□

4.2 The main result

In this section we will prove

THEOREM 4.2.1 *Let k be a field of characteristic d . Given a polynomial h over k of degree d^m , ($m \geq 1$) in one variable, there exists a system S of $r (= d^m - 1)$ forms of degree d in $r + 1$ variables such that h has a zero in k if and only if the system S has a common non-trivial k -zero.*

As a corollary we have

COROLLARY 4.2.2 *Let k be a field of characteristic d . If k is a C_0^d -field, then*

- (i) *every polynomial of d -power degree has a zero in k .*
- (ii) *k is a p -field for some prime p not dividing d .*

PROOF. Since k is a C_0^d -field, the system S in the theorem has a non-trivial k -zero. Therefore, the polynomial h has a zero in k ; hence (i) follow. Now (ii) follows from (i) and the following proposition which was proved by Leep for the case $d = 2$; the proof of the general case is identical.

PROPOSITION ([L1, prop. 4.4]) *A field k is a p -field for some prime number p not dividing d if and only if every polynomial of d -power degree has a zero in k .*

Before starting the proof of 4.2.1, we need the following: Define the functions

$$i : \{d, d+1, d+2, \dots\} \rightarrow \{1, 2, \dots\},$$

$$j : \{d, d+1, d+2, \dots\} \rightarrow \{1, d, d^2, \dots\}$$

as follows:

For any integer $n \geq d$, write the d -adic expansion of n :

$$n = a_0 d^0 + a_1 d + \dots + a_r d^r,$$

where $a_r \neq 0$. Set $j(n) = d^r$; and

$$i(n) = \begin{cases} a_r d^{r-1}, & \text{if } n = a_r d^r; \\ a_0 + a_1 d + \dots + a_{r-1} d^{r-1}, & \text{if } n \neq a_r d^r. \end{cases}$$

Now, for $n \geq 0$, define the monomials Y_n (of degree d) as follows:

$$Y_n = \begin{cases} X_1^n Z^{d-n} & \text{if } 0 \leq n < d; \\ X_{i(n)}^d & \text{if } d \leq n = a_r d^r; \\ X_{i(n)} X_{j(n)}^{a_r} Z^{d-a_r-1}, & \text{if } d < n \neq a_r d^r. \end{cases}$$

Suppose we are given a polynomial

$$h = X^{d^m} + c_{d^m-1} X^{d^m-1} + \dots + c_1 X + c_0,$$

with coefficients c_i from k . Define ϕ_h (a form of degree d) to be

$$\phi_h = Y_{d^m} + c_{d^m-1} Y_{d^m-1} + \dots + c_1 Y_1 + c_0 Y_0.$$

REMARK 4.2.3

(i) Note that $i(n) < d^r = j(n)$. Also, if $d \leq n < d^m$, then $r < m$, hence $i(n) < d^{m-1}$ and $j(n) \leq d^{m-1}$. In particular, for h of degree d^m , the form ϕ_h involves only the variables Z, X_1, \dots, X_{d^m-1} .

(ii) Let $n \geq d$. If $n = a_r d^r$, then $di(n) = n$; and if $n \neq a_r d^r$, then $n = i(n) + a_r j(n)$.

(iii) If h has degree d , then ϕ_h is nothing but the homogenization of h .

PROOF OF 4.2.1

Let k be a field of characteristic d . Throughout the proof, for $n > 0$, let $n = a_0 + \dots + a_r d^r$, $a_r \neq 0$, be the d -adic expansion of n . For any elements $z, x_1, x_2, \dots, x_{d^m-1}$ of k , and for $n = 0, \dots, d^m$ let

$$y_n = \begin{cases} x_1^n z^{d-n} & \text{if } 0 \leq n < d; \\ x_{i(n)}^d & \text{if } d \leq n = a_r d^r; \\ x_{i(n)} x_{j(n)}^{a_r} z^{d-a_r-1}, & \text{if } d < n \neq a_r d^r. \end{cases}$$

Take S to be the system consisting of the d^{m-1} forms $\phi_h, \phi_2, \dots, \phi_{d^m-1}$. These forms have degree d , and by 4.1.1.(i),(ii) and 4.2.3.(i), the system involves the $d^{m-1} + 1$ variables Z, X_1, \dots, X_{d^m-1} .

Claim: The system S has a non-trivial k -zero if and only if the polynomial h has a k zero.

If $m = 1$, then, as noted in 4.2.3(iii), $S = \{\phi_h\}$ is just the homogenization of h , and therefore the claim is proved in this case. So we may assume $m > 1$.

First, assume that the system $\phi_h, \phi_2, \dots, \phi_{d^m-1}$ has a non-trivial common zero $(z, x_1, x_2, \dots, x_{d^m-1})$ over k . Then z cannot be zero. Otherwise, if $z = 0$, then by Lemma 4.1.2,

$$x_n = 0 \text{ for } 1 \leq n < d^{m-1}.$$

By 4.2.3.(i), $d \leq n < d^m$ implies $i(n) < d^{m-1}$. Hence $x_{i(n)} = 0$ for $d \leq n < d^m$, which implies that $y_n = 0$ for $0 \leq n < d^m$. Therefore ϕ_h vanishes on $(z, x_1, x_2, \dots, x_{d^m-1})$ implies $0 = y_{d^m} = x_{i(d^m)}^d$. But $i(d^m) = d^{m-1}$, so $x_{d^m-1} = 0$. Therefore, $z = 0$ leads to the trivial solution, a contradiction.

So, we may assume that $z = 1$. Claim: x_1 is a zero of h . Note that since the characteristic of $k = d$, all the d -power roots of unity are equal to 1. Therefore lemma 4.1.3 implies that

$$x_n = x_1^n \text{ for } 1 \leq n \leq d^{m-1}.$$

Therefore, for $0 \leq n \leq d^m$,

$$\begin{aligned} y_n &= \begin{cases} x_1^n & \text{if } 0 \leq n < d; \\ x_1^{di(n)} & \text{if } d \leq n = a_r d^r; \\ x_1^{i(n)+a_r j(n)} & \text{if } 0 < n \neq a_r d^r. \end{cases} \\ &= x_1^n, \quad (\text{by 4.2.3.(ii)}). \end{aligned}$$

Now, ϕ_h vanishes on $(1, x_1, \dots, x_{d^{m-1}})$ implies

$$\begin{aligned} 0 &= y_{d^m} + c_{d^m-1} y_{d^m-1} + \dots + c_0 y_0 \\ &= x_1^{d^m} + c_{d^m-1} x_1^{d^m-1} + \dots + c_0 \\ &= h(x_1). \end{aligned}$$

Hence x_1 is a zero of h .

Conversely, Assume that there exists $\alpha \in k$ such that $h(\alpha) = 0$. Put $z = 1$ and $x_n = \alpha^n$ for $n \geq 1$. We verify that $(z, x_1, \dots, x_{d^{m-1}})$ is a common zero of the forms $\phi_h, \phi_1, \dots, \phi_{d^{m-1}}$. As above, we have by 4.2.3.(ii),

$$\begin{aligned} y_n &= \begin{cases} \alpha^n & \text{if } 0 \leq n < d; \\ \alpha^{di(n)} & \text{if } d \leq n = a_r d^r; \\ \alpha^{i(n)+a_r j(n)}, & \text{if } d < n \neq a_r d^r. \end{cases} \\ &= \alpha^n; \end{aligned}$$

and therefore

$$\begin{aligned} 0 &= h(\alpha) = \alpha^{d^m} + c_{d^m-1} \alpha^{d^m-1} + \dots + c_0 \\ &= y_{d^m} + c_{d^m-1} y_{d^m-1} + \dots + c_0 y_0; \end{aligned}$$

Hence, ϕ_h vanishes on $(z, x_1, \dots, x_{d^{m-1}})$.

To verify that ϕ_n vanishes on $(z, x_1, \dots, x_{d^m-1})$ for $1 < n \leq d^{m-1}$, first assume that $n = d^r$. Then $f(n) = d^{r-1}$, and therefore $x_n z^{d-1} - x_{f(n)}^d = \alpha^n - \alpha^{df(n)} = \alpha^{d^r} - \alpha^{d(d^{r-1})} = 0$. Hence, ϕ_n vanishes on $(z, x_1, \dots, x_{d^m-1})$ in this case. Now assume that $n = a_r d^r$, $a_r \neq 1$. Then $g(n) = d^{r+1}$, and we have $x_n^d - x_{g(n)}^{a_r} z^{d-a_r} = \alpha^{dn} - \alpha^{a_r g(n)} = \alpha^{a_r d^{r+1}} - \alpha^{a_r d^{r+1}} = 0$, hence, ϕ_n vanishes on $(z, x_1, \dots, x_{d^m-1})$ in this case too. Finally, assume that $n \neq a_r d^r$. By 4.1.1.(iii), $dn = f(n) + a_r g(n)$; hence, $x_n^d - x_{f(n)} x_{g(n)}^{a_r} z^{d-a_r-1} = \alpha^{dn} - \alpha^{f(n)+a_r g(n)} = \alpha^{dn} - \alpha^{dn} = 0$, so ϕ_n vanishes on $(z, x_1, \dots, x_{d^m-1})$. This completes the proof of the theorem. \square

References

- [A] Ahmad, H. : On quadratic forms over inseparable quadratic extensions. Arch. Math. 63, 23–29 (1994).
- [AO] Ahmad, H. and Ohm, J. : Function fields of Pfister neighbors. To appear in J. of Algebra.
- [Ar] Arf, C. : Untersuchungen über quadratische Formen in Körpern der Charakteristik 2. J. reine angew Math. 183, 148–167 (1941).
- [Ars] Arason, J. Kr. : Cohomologische invarianten quadratischer Formen. J. of Algebra 36, 448–491
- [B1] Baeza, R. : Ein Teilformensatz für quadratische Formen in Charakteristik 2. Math. Z. 135, 175–184 (1974).
- [B2] Baeza, R. : Quadratic Forms over Semilocal Rings. (Lect. Notes Math. Vol. 655). Springer-Verlag, Berlin Heidelberg New York, 1978.
- [BCSS] Beauville, A. ; Colliot-Thélène, J. -L. ; Sansuc, J. -J. and Sir Swinnerton-Dyer, P. : Variétés stables rationnelles non rationnelles. Annals of Math. 121, 283–318
- [ELW] Elman, R. ; Lam, T. Y. and Wadsworth, A. : Amenable fields and Pfister extension. Conf. Quad. Forms, 1976, (G. Orezech, ed.), Queen's papers on pure and appl. math. 46 (1977), Queen's University, Kingston, Ont. , 445–492.
- [F] Fitzgerald, R. : Function fields of quadratic forms. Math. Z. 178, 63–76 (1981).
- [G] Greenberg, M. : Lectures on Forms in Many Variables. New York 1968.
- [H1] Hoffmann, D. : Function fields of quadratic forms. Doctoral Dissertation, University of California, Berkeley, California 1992.
- [H2] Hoffmann, D. : On 6-dimensional quadratic forms isotropic over the function field of a quadric. Comm. in Alg. 22(6), 1999–2014 (1994).
- [H3] Hoffmann, D. : Isotropy of 5-dimensional quadratic forms over the function field of a quadric. To appear in Proceedings of Symposia in Pure Mathematics.
- [H4] Hoffmann, D. : Isotropy of quadratic forms over the function field of a quadric. To appear in Math. Z.
- [HR1] Hurrelbrink, J. and Rehmann, U. : Splitting patterns of excellent quadratic forms. J. reine angew. Math. 444, 183–192 (1993).

- [HR2] Hurrelbrink, J. and Rehmann, U. : Splitting patterns of quadratic forms. Preprint (1994).
- [K1] Knebusch, M. : Generic splitting of quadratic forms I. Proc. London Math. Soc. 33, 65–93 (1976).
- [K2] Knebusch, M. : Generic splitting of quadratic forms II. Proc. London Math. Soc. 34, 1–31 (1977).
- [L] Lam, T. Y. : The Algebraic Theory of Quadratic Forms. Benjamin, Reading, Mass. 1973
- [Le1] Leep, D. : Systems of quadratic forms. J. reine angew. Math. 350, 109–116 (1984).
- [Le2] Leep, D. : Pfister's conjecture on quadratic C_0 -fields. J. reine angew. Math. 404, 209–220 (1990).
- [O1] Ohm, J. : On subfields of rational function fields. Arch. Math 42, 136–138
- [O2] Ohm, J. : On ruled fields. Sem. de theorie des nombres, Bordeaux 1, Serie 2, vol. 1, 27–50
- [O3] Ohm, J. : Function fields of conics, a theorem of Amitsur-MacRae, and a problem of Zariski. Algebraic geometry and its applications (C. Bajaj, ed.), 333–363,
- [O4] Ohm, J. : The Zariski problem for function fields of quadratic forms. To appear in Proc. Am. Math. Soc.
- [P1] Pfister, A. : Systems of quadratic forms. Bull. Soc. Math. France, Mémoire 59, 115–123 (1979).
- [P2] Pfister, A. : A new proof of the homogeneous nullstellensatz for p -fields and applications to topology. Recent Advances in Real Algebraic Geometry and Quadratic Forms, 1991-1992 (B. Jacob, T. Y. Lam and R. Robson ed.), Contemporary math. 155, 221–230 (1994).
- [Sch] Scharlau, W. : Quadratic and Hermetian Forms. Springer-Verlag, Berlin, 1985.
- [Sp] Springer, T. A. : Sur les formes quadratiques d'indices zero, C. R. Acad. Sci. 234, 1517–1519 (1952).
- [W] Wadsworth, A. : Similarity of quadratic forms and isomorphism of their function fields. Trans. Amer. Math. Soc. 208, 352–358

Appendixes: Letters of permission



LOUISIANA STATE UNIVERSITY

AGRICULTURAL AND MECHANICAL COLLEGE

Department of Mathematics

Hamza Y. Ahmad • 504/388-1225
E-mail: ahmad@math.lsu.edu

7 July 1995

Academic Press, Journal of Algebra
Copy-right Clearance
Orlando, FL 32887-4900

Dear Sir/Madam,

I have submitted the article "Function fields of Pfister neighbours" by Hamza Ahmad and Jack Ohm, to be published in the Journal of Algebra and has been accepted. The acceptance was around March this year. Parts of this article will be included as a part my Ph. D. dissertation at Louisiana State University, Baton Rouge, Louisiana. Before I could submit my dissertation, I am required to obtain a written permission from your company to include this article in my dissertation.

I would appreciate it if you provide me with such permission for both inclusion in my dissertation and for micro-filming the dissertation by the school library later. Since the deadline for submitting the dissertation is by July 14. I would very much appreciate it if you process my request quickly. Please fax copies of the permission to me (Hamza Ahmad) at

(504) 388 - 4278 Math Dept., LSU

My address is: Dept. of Mathematics, LSU, Baton Rouge, LA 70803. ... Thank you

Sincerely,

Hamza Ahmad

July 10, 1995

PERMISSION GRANTED, provided that 1) complete credit is given to the source, including the Academic Press copyright notice; 2) the material to be used has appeared in our publication without credit or acknowledgement to another source and 3) if commercial publication should result, you must contact Academic Press again.

We realize that University Microfilms must have permission to sell copies of your thesis, and we agree to this. However, we must point out that we are not giving permission for separate sale of your article.

Cheryl A. Under
Permissions Department
ACADEMIC PRESS, INC.
Orlando, FL 32867

Birkhäuser



Unsere Referenz:
Our Reference:

rh
Ihre Referenz:
Your Reference:

Basel
July 12, 1995

FAX: 001/504 388 4275

Mr. Hamza Ahmad
Department of Mathematics
Louisiana State University
Louisiana
USA

Re: Reprint permission from our journal
ARCHIV DER MATHEMATIK

Dear Mr. Ahmad,

We refer to your fax message to Springer Verlag in New York. Your request has been forwarded to us, as the above journal is published by Birkhäuser Basel.

We herewith grant you permission to reproduce parts of your article published in Archiv der Mathematik (Vol. 63 pp. 23-29), for your dissertation. Permission is granted, provided that full reference to the original source of publication is made (i.e. name, vol. and issue no. of journal, authors and title of article, name and place of publisher).

Sincerely yours,
BIRKHÄUSER VERLAG AG

R. Hersberger

Regina Hersberger
Assistant
Rights & Licences

Birkhäuser
Verlag AG

Klosterberg 23
P.O. Box 133
CH-4010 Basel
Switzerland

Telefon
061/271 74 00

Telefax
061/271 76 66

Buchhaltung/
Auslieferung

Printed: 191
CB-4106 Thewiss
Telefon 061/721 77 40
Telefax 061/721 79 50
Telex 963 475



Vita

Hamza Ahmad was born in Kuwait on December 29, 1967. He graduated from high school in June 1986. He attended Kuwait University where he received a Bachelor of Science in theoretical physics in January 1990. In August 1990, he began his graduate study at Louisiana State University, and is presently a candidate for the doctoral degree in mathematics at LSU.

DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Hamza Y. Ahmad
Major Field: Mathematics
Title of Dissertation: On Some Problems in the Algebraic Theory of Quadratic Forms

Approved:



Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:











Date of Examination:

May 2, 1995
