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ABSTRACT VOLTERRA EQUATIONS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
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in partial fulfillment of the
requirements for the degree of
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in

The Department of Mathematics

by
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ABSTRACT

This dissertation is devoted to the study of the abstract Volterra equation

$$v(t) = A\int_0^t v(t-s)d\mu(s) + f(t) \qquad \text{for } t \geq 0, \tag{VE}$$

where A is a closed linear operator in a complex Banach space X, μ is a complex valued function of local bounded variation, and $f : [0,\infty) \to X$ is continuous and Laplace transformable. Laplace transform methods are used to characterize the existence and uniqueness of exponentially bounded solutions v for a given forcing term f, an operator A, and a given kernel μ. We extend the methods of a solution family (or a resolvent) for (VE) by studying integrated and analytic integrated solution operator families. These notions are employed to characterize those pairs (A,μ) for which (VE) has unique solutions for all sufficiently regular forcing terms f. Besides existence, uniqueness and wellposedness results for (VE), new results include Trotter-Kato type theorems for integrated solution operator families and a characterization of those pairs (A,μ) for which the integrated solution operator families are analytic in an open sector $\{\lambda \in \mathbb{C} \mid |\arg\lambda| < \alpha\}$ for some $\alpha \in (0, \frac{\pi}{2}]$.

INTRODUCTION

The theory of abstract Volterra equations has been developed due to its applications to problems in physics, engineering, and biology (see [G-L-S] or [Pr], for instance). The objective of this dissertation is to study the existence, uniqueness, continuous dependence, and analyticity of solutions to the abstract Volterra integral equation (or Volterra equation, for short)

$$v(t) = A \int_0^t v(t-s)d\mu(s) + f(t) \qquad \text{for } t \geq 0 \tag{VE}$$

by means of Laplace transform theory. Throughout, A is a closed linear operator with its domain $D(A)$ and range in a complex Banach space X, and $f \in C([0,\infty); X)$ is a forcing term. The function μ is complex valued and is assumed to be normalized and of bounded variation. A function $v \in C([0,\infty); X)$ with $\int_0^t v(t-s)d\mu(s) \in D(A)$ and which satisfies (VE) for every $t \geq 0$ is said to be a solution of the Volterra equation (VE). This work does not address the existence of solutions which are only local in time, i.e., those v which satisfy (VE) for $t \in [0,T]$ for some $T \geq 0$. In contrast to the abstract Cauchy problem $v'(t) = Av(t) + f(t)$, $v(0) = x$, where the existence of local solutions can be characterized (see [B-N]), the characterization of local solutions of (VE) remains unsolved.

In order to be able to apply Laplace transform methods to (VE) we will restrict our discussion to Laplace transformable forcing terms f, exponentially bounded kernels μ, and exponentially bounded solutions v. In this case, the Laplace transform converts

(VE) into the characteristic equation

$$(I - \widehat{d\mu}(\lambda)A)y(\lambda) = \widehat{f}(\lambda) \qquad \text{for } \lambda > \omega \tag{CE}$$

where $\widehat{d\mu}(\lambda) := \int_0^\infty e^{-\lambda t} d\mu(t)$, $\widehat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt$, $y(\lambda) = \widehat{v}(\lambda)$, and the number ω depends on the growth of functions v, μ, and f. The Laplace transform method simplifies the problem (VE) by eliminating the time variable in the characteristic equation (CE). The main body of this work concerns the interaction between the solution v of the Volterra equation (VE) and the solution y of the characteristic (or resolvent) equation (CE).

Chapter 1 describes notation, results from Laplace transform theory, and some elementary lemmas concerning the vector valued convolutions $t \mapsto \int_0^t f(t-s)d\mu(s)$. Chapter 2 characterizes the existence and uniqueness of exponentially bounded solutions of (VE) by the regularity properties of the solutions y of (CE) and the spectral properties of the operators $I - \widehat{d\mu}(\lambda)A$. It also characterizes the existence of exponentially bounded, continuous solutions of the Volterra equation

$$v(t) = \sum_{j=1}^{m} A_j \int_0^t v(t-s)d\mu_j(s) + f(t) \qquad \text{for } t \geq 0,$$

where $m \in \mathbb{N}$, A_j are closed linear operators in X, and the kernels μ_j are complex valued functions. Whereas Chapter 2 explores the solvability of (VE) for a given forcing term f, Chapter 3 investigates those pairs (A, μ) for which (VE) has unique solutions for *all* sufficiently regular f. In recent years, the method of an integrated semigroup with generator A has been applied successfully to the abstract Cauchy problem

$u'(t) = Au(t)$, $u(0) = x$, and the associated integral equation

$$u(t) = A\int_0^t u(s)ds + x = A\int_0^t u(t-s)ds + x.$$

The main idea there is to regularize the equation by integrating it $(n+1)$-times for a nonnegative integer n and to study the special Volterra equation

$$v(t) = A\int_0^t v(t-s)ds + \frac{t^n}{n!}x$$

instead, where v is the n-th antiderivative of u (see [A], [A-H-N], or [N]). Extending this method, we will show that the notion of an integrated solution operator family with generator (A, μ) is suitable for studying the wellposedness of the Volterra equation (VE). Also, Chapter 3 contains Trotter-Kato type approximations for Volterra equations, properties and a characterization of analytic integrated solution operator families, and an elementary example.

The starting point of this dissertation is the first chapter of the book *Evolutionary Integral Equations and Applications* by J. Prüss ([Pr]), which develops the elementary theory of strongly continuous solution families for (VE). We extend this theory in two directions. In Chapter 3 we study integrated solution operator families introduced by W. Arendt and H. Kellermann (see [A-K]) in 1987. Since one of the fundamental lemmas in that paper does not hold in the stated generality (Lemma 1.3, [A-K]), we reexamined their approach by assuming stronger regularity on the scalar function μ where it is required. The method of integrated solution operator families assumes the existence of $(I - \widehat{d\mu}(\lambda)A)^{-1}$ as bounded operators for $\mathrm{Re}\,\lambda > \omega$ for some constant

ω. As indicated in the example in Section 3.5, there are various cases where this assumption, which guarantees the solvability of (VE) for a large class of forcing terms f, is not satisfied. Thus, Chapter 2 studies (VE) without assuming the existence of $(I - \widehat{d\mu}(\lambda)A)^{-1}$ for any $\lambda \in \mathbb{C}$ and characterizes those forcing terms f for which there exist solutions of (VE). The results in Chapter 2 are modified after those in [N] and [A-H-N]. Some results on integrated solution operator families are modified after those on integrated semigroups in a preliminary version of a forthcoming monograph by W. Arendt, M, Hieber, and F. Neubrander (see [A-H-N]).

CHAPTER 1 PRELIMINARIES

In Section 1.1 we introduce notation, review some elementary facts from integration theory, and list some of the results from vector valued Laplace transform theory and some fundamental facts from functional analysis that are used throughout. In Section 1.2 we prove some elementary lemmas on vector valued convolutions $t \mapsto \int_0^t f(t-s)dg(s)$, $t \geq 0$, that will be used to discuss solutions of Volterra equations.

1.1 Notations and Some Results from Laplace Transform Theory

Let X be a complex Banach space. Let Ω be an (infinite or finite) interval in $[0,\infty)$. As usual, $C(\Omega;X)$ (respectively, $C^n(\Omega;X)$) denotes the space of all continuous (respectively, n-times continuously differentiable) functions from Ω to X. Let $[a,b] \subset \mathbb{R}$. A function $f : [a,b] \to X$ is of bounded variation if

$$\operatorname{var}(f;a,b) := \sup_{\pi} \sum_{j=1}^{n} \|f(t_j) - f(t_{j-1})\| < \infty$$

where $\pi = \{a = t_0 < \cdots < t_n = b\}$ is any partition of $[a,b]$. We denote by $BV([a,b];X)$ the space of all functions $f : [a,b] \to X$ of bounded variation and by $BV_{\mathrm{loc}}([0,\infty);X)$ the set of all functions $f : [0,\infty) \to X$ which are of bounded variation on $[0,b]$ for every $b \geq 0$. For $\epsilon \geq 0$ we define $BV_\epsilon([0,\infty);X)$ as the space of those functions $f \in BV_{\mathrm{loc}}([0,\infty);X)$ with $f(0) = 0$ and for which there exists a constant $M > 0$ such that $\operatorname{var}(f;0,t) \leq Me^{\epsilon t}$ for all $t \geq 0$.

Let s be a simple function on $[a,b]$, i.e., $s = \sum_{k=1}^{m} x_k \chi_{E_k}$ for $x_k \in X$ and characteristic functions χ_{E_k} of measurable sets E_k in $[a,b]$. The Bochner integral of s is defined as $\int_a^b s(t)dt = \sum_{k=1}^{m} x_k m(E_k)$. A function $f : [a,b] \to X$ is said to be Bochner integrable if there exists a sequence of simple functions s_n such that $\{s_n(t)\}_{n\in\mathbb{N}}$ converges to $f(t)$ for almost every $t \in [a,b]$ and $\lim_{n\to\infty} \int_a^b \|f(t) - s_n(t)\|dt = 0$. In this case the Bochner integral of the function f is defined as

$$\int_a^b f(t)dt = \lim_{n\to\infty} \int_a^b s_n(t)dt.$$

We denote by $L^1([a,b];X)$ the space of all Bochner integrable functions from $[a,b]$ to X and by $L^1_{\mathrm{loc}}([0,\infty);X)$ the space of all functions from $[0,\infty)$ to X which are Bochner integrable on $[0,b]$ for every $b \geq 0$. If a function $f : [a,b] \to X$ is weakly measurable, almost separably valued, and satisfies $\int_a^b \|f(t)\|dt < \infty$, then $f \in L^1([a,b];X)$ (for a proof, see Section 3.5, [H-P]). In many instances the weak measurability of a function $f : [a,b] \to X$ as well as the condition $\int_a^b \|f(t)\|dt < \infty$ can be easily verified. The following lemma is useful for checking if f is almost separably valued.

Lemma 1.1.1. Let Ω be a finite or infinite interval in $\mathbb{R}$. Let f be a function from Ω into X. Then the following two statements are equivalent.

(i) There exists a countable set $D_0 \subset f(\Omega)$ such that $f(\Omega) \subset \overline{D_0}$.

(ii) There exists a countable set $D \subset X$ such that $f(\Omega) \subset \overline{D}$.

Proof. The implication (i) $\Longrightarrow$ (ii) is obvious. We show that (ii) $\Longrightarrow$ (i). Suppose that D is a countable subset of X for which $f(\Omega) \subset \overline{D}$. Let D_1 be the set of the points x_m in D such that $B(x_m, \frac{1}{n}) \bigcap f(\Omega) \neq \emptyset$ for some $n \in \mathbb{N}$. It is clear that $D_1 \neq \emptyset$.

For every $m \in \mathbb{N}$ and for those $n \in \mathbb{N}$ for which $f(\Omega) \cap B(x_m, \frac{1}{n}) \neq \emptyset$, take a point $y_{m,n} \in f(\Omega) \cap B(x_m, \frac{1}{n})$. Let $D_0 = \{y_l \mid l \in \mathbb{N}\}$ be a denumeration of all those points $y_{m,n}$. We show that D_0 is a dense subset of $f(\Omega)$. Let $t \in \Omega$ and $k \in \mathbb{N}$. By the hypothesis, there exists an $x_m \in D_1$ such that $f(t) \in B(x_m, \frac{1}{2k})$ for some $m \in \mathbb{N}$. Since $B(x_m, \frac{1}{2k})$ contains a point $y_l \in D_0$, we obtain that $B(y_l, \frac{1}{k})$ contains $f(t)$. Thus, D_0 is a countable subset of $f(\Omega)$ for which $f(\Omega) \subset \overline{D_0}$. //

Lemma 1.1.2. $BV([0,b];X) \subset L^1([0,b];X)$.

Proof. Let $f \in BV([a,b];X)$. Let Q be the set of rational numbers in $[a,b]$. Let $O := Q \cup \bigcup_{n \in \mathbb{N}} E_n$ where $E_n := \{t \in [a,b] \mid \|f(t) - f(q)\| \geq \frac{1}{n}$ for all $q \in Q\}$. Notice that every set E_n is finite since $f \in BV([a,b];X)$. Suppose that $t \in O^c$. Then t is not an element of any E_n. Hence for every $n \in \mathbb{N}$, there exists a $q \in Q$ such that $\|f(t) - f(q)\| < \frac{1}{n}$. Thus, $f([a,b]) \subset \overline{O}$. Hence by Lemma 1.1.1, the function f is almost separably valued. Since every $f \in BV([0,b];X)$ is bounded and weakly measurable, it follows that $f \in L^1([a,b];X)$. //

A function f is of weak bounded variation (and hence weakly measurable) if and only if it is of bounded semivariation (namely, $\|\sum_{j=1}^{n} f(t_j) - f(s_j)\| \leq M$ for some constant $M > 0$ and any finite collection of nonoverlapping subintervals (s_j, t_j) of $[a,b]$, see [H-P] or [A-H-N]. A function of bounded semivariation is not necessarily almost separably valued. For example, the function $f : [0,1] \to L^\infty(\mathbb{R})$ defined by $f(t) := \chi_{[0,t]}$ is of bounded semivariation, but is not almost separably valued since $\|f(t) - f(s)\| = 1$ if $t \neq s$.

For functions $f : [a,b] \to X$ and $g : [a,b] \to \mathbb{C}$ (or for $f : [a,b] \to \mathbb{C}$ and $g : [a,b] \to X$) the Riemann-Stieltjes integral of f and g is defined as

$$\int_a^b f(s)dg(s) := \lim_{|\pi|\to 0} \sum_{j=1}^{n} f(\xi_j)[g(t_j) - g(t_{j-1})]$$

if the limit exists, where $\pi = \{a = t_0 < \cdots < t_n = b\}$ is any partition of $[a,b]$ with length $|\pi| = \max_j(t_j - t_{j-1})$ and $\xi_j \in [t_{j-1}, t_j]$. If one of f and g is continuous and the other is of bounded variation, then $\int_a^b f(s)dg(s)$ exists (for a proof, see the Appendix in [A-H-N] or Theorem 3.3.2 in [H-P]). If $\int_a^b f(s)dg(s)$ exists, then so does $\int_a^b g(s)df(s)$ and the integration by parts formula

$$\int_a^b f(s)dg(s) = f(b)g(b) - f(a)g(a) - \int_a^b g(s)df(s) \tag{1.1.1}$$

holds. If Bochner integrals and Riemann-Stieltjes integrals appear in a statement or an expression, the integral sign $\oint$ is used sometimes to emphasize Riemann-Stieltjes integrals. The Riemann integral of a function $f : [a,b] \to X$ is defined as the Riemann-Stieltjes integral $\oint_a^b f(t)dt$ if it exists. Note that if $f \in C([a,b]; X)$ and if $g : [a,b] \to \mathbb{C}$ is absolutely continuous, then the Riemann-Stieltjes integral $\oint f(t)dg(t)$ coincides with the Bochner integral $\int_a^b f(t)g'(t)dt$. In particular, if $f \in C([a,b]; X)$, then $\oint_a^b f(t)dt = \int_a^b f(t)dt$. Hence for $f \in C([a,b]; X)$ the integral $\int_a^b f(t)dt$ is unambiguous. If $g \in C([a,b]; \mathbb{C})$ and $F : [a,b] \to X$ is an antiderivative of a Bochner integrable function f, then $\oint_a^b g(t)dF(t) = \int_a^b g(t)f(t)dt$ (for a proof, see [A-H-N], for example).

Let $f : [0,\infty) \to X$ and $g : [0,\infty) \to \mathbb{C}$. If f and g are both locally Bochner integrable, then the convolution of f and g is defined as $(f*g)(t) := \int_0^t f(t-s)g(s)ds$, $t \geq 0$, as usual. If one of f and g is continuous and the other is of bounded variation, then the Stieltjes convolution of f and g is defined as $(f * dg)(t) := \int_0^t f(t-s)dg(s),\ t \geq 0$. It follows from (1.1.1) that

$$\int_0^t f(t-s)dg(s) = f(0)g(t) - f(t)g(0) + \int_0^t g(t-s)df(s), \quad t \geq 0. \tag{1.1.2}$$

For $n \in \mathbb{N}$, the n-th normalized antiderivative

$$t \mapsto \int_0^t \int_0^{s_{n-1}} \cdots \int_0^{s_1} f(s)ds \cdots ds_{n-2}ds_{n-1}, \qquad t \geq 0$$

of a function f is denoted by $f^{[n]}(t)$. Notice that

$$f^{[n]}(t) = \underbrace{1 * 1 * \cdots * 1}_{n-\text{times}} * f(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds \tag{1.1.3}$$

where 1 denotes the constant function $t \mapsto 1$. Some basic properties of the Stieltjes convolution will be proved in the next section.

For $f \in L^1_{\text{loc}}([0,\infty);X)$ and for $\lambda \in \mathbb{C}$ the Laplace integral of f at λ is defined as

$$\widehat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt = \lim_{T\to\infty} \int_0^T e^{-\lambda t} f(t)dt$$

if the limit exists. For the proofs of the following statements concerning the Laplace transform, see [A-H-N]. If $\widehat{f}(\lambda_0)$ exists for some $\lambda_0 \in \mathbb{C}$, then $\widehat{f}(\lambda)$ exists for all $\lambda \in \mathbb{C}$ with Re $\lambda >$ Re λ_0. Hence the abscissa of f is defined as

$$\text{abs}(f) := \inf\{\text{Re } \lambda \mid \widehat{f}(\lambda) \text{ exists}\}.$$

Clearly, $-\infty \leq \text{abs}(f) \leq \infty$. By $\mathbb{C}_\omega$ we denote the set of all complex numbers λ with Re $\lambda > \omega$. If $f \in L^1_{\text{loc}}([0,\infty);X)$ with $\text{abs}(f) < \infty$, then f is called Laplace transformable and the function $\widehat{f} : \mathbb{C}_{\text{abs}(f)} \to X$ is called the Laplace transform of f.

The function $\widehat{f}$ is analytic on $\mathbb{C}_{\mathrm{abs}(f)}$. We define the exponential growth bound of a function $f \in L^1_{\mathrm{loc}}([0,\infty);X)$ as

$$\omega(f) := \inf\{\omega \in \mathbb{R} \mid \sup_{t\geq\tau} \|\mathrm{e}^{-\omega t} f(t)\| < \infty \text{ for some } \tau \geq 0\}.$$

It is clear that $\mathrm{abs}(f) \leq \omega(f)$. In general $\omega(f)$ does not determine $\mathrm{abs}(f)$; for example, the function $t \mapsto \mathrm{e}^t \mathrm{e}^{\mathrm{e}^t} \cos \mathrm{e}^t$ $(t \geq 0)$ is Laplace transformable, but not exponentially bounded. However, a function is Laplace transformable if and only if its antiderivative is exponentially bounded. In particular, the following relation holds. If $\omega \geq 0$, then

$$\mathrm{abs}(f) \leq \omega \iff \omega(f^{[1]}) \leq \omega. \tag{1.1.4}$$

Thus, if $\mathrm{Re}\,\lambda > \max\{\mathrm{abs}(f),\ 0\}$, then $\widehat{f}(\lambda)$ and $\widehat{f^{[1]}}(\lambda)$ exist, and integrating by parts, we obtain

$$\lambda \widehat{f^{[1]}}(\lambda) = \widehat{f}(\lambda). \tag{1.1.5}$$

Let $a \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{C})$ with $\mathrm{abs}(|a|) \leq \omega$ for some $\omega \geq 0$ and let $\mu := a^{[1]}$. Then from (1.1.4), there exists a constant $M > 0$ such that $\sum_j |\mu(t_j) - \mu(t_{j-1})| \leq \sum_j \int_{t_{j-1}}^{t_j} |a(s)|ds = \int_0^t |a(s)|ds \leq M\mathrm{e}^{\omega t}$ for any partition $\{t_j\}$ of $[0,t]$. This shows that the antiderivatives of absolutely Laplace transformable functions are contained in $BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$. This will be used in Section 3.1.

For a function $f \in BV_{\mathrm{loc}}([0,\infty);X) \bigcup C([0,\infty);X)$ and for $\lambda \in \mathbb{C}$ the Laplace-Stieltjes integral of f at λ is defined as

$$\widehat{df}(\lambda) := \int_0^\infty \mathrm{e}^{-\lambda t} df(t) = \lim_{T\to\infty} \int_0^T \mathrm{e}^{-\lambda t} df(t)$$

if the limit exists. Sometimes it is convenient to use the notations $L(f) = (f)^\wedge$ and $L_s(f) = (\mathrm{d}f)^\wedge$ for $\widehat{f}$ and $\widehat{df}$, respectively. The abscissa of df is defined as

$$\mathrm{abs}(df) := \inf\{\mathrm{Re}\,\lambda \mid \widehat{df}(\lambda) \text{ exists}\}.$$

If $f \in BV_{\mathrm{loc}}([0,\infty);X) \bigcup C([0,\infty);X)$ with $\mathrm{abs}(df) < \infty$, then f is called Laplace-Stieltjes transformable and the function $\widehat{df} : \mathbb{C}_{\mathrm{abs}(df)} \to X$ is called the Laplace-Stieltjes transform of f. The function $\widehat{df}$ is analytic on $\mathbb{C}_{\mathrm{abs}(df)}$. If a function $f \in BV_{\mathrm{loc}}([0,\infty);X) \bigcup C([0,\infty);X)$ is exponentially bounded and if $f(0) = 0$, then for a nonnegative number $\omega \geq \omega(f)$, it follows from (1.1.1) that

$$\widehat{df}(\lambda) = \lambda \widehat{f}(\lambda) \tag{1.1.6}$$

for all $\lambda \in \mathbb{C}_\omega$. Let $\omega \in \mathbb{R}$. The Lipschitz continuous function space $Lip_\omega([0,\infty);X)$ is defined to be the space consisting of those functions $F : [0,\infty) \to X$ with $F(0) = 0$ and for which

$$\|F\|_{Lip_\omega} := \inf\{M \mid \|F(t+h) - F(t)\| \leq M \int_t^{t+h} e^{\omega r} dr \text{ for } t,h \geq 0\} < \infty.$$

It is clear that if $\omega \geq 0$ and $F \in Lip_\omega([0,\infty);X)$, then $\omega(F) \leq \omega$. If $f \in L^1_{\mathrm{loc}}([0,\infty);X)$ with $\omega(f) < \infty$, then for any number $\omega > \omega(f)$, $f^{[1]} \in Lip_\omega([0,\infty);X)$. The symbol $\mathbb{N}_0$ denotes the set of nonnegative integers. If $F \in Lip_\omega([0,\infty);X)$, then the k-th derivative of $\widehat{dF}$ is given by

$$\widehat{dF}^{(k)}(\lambda) = \int_0^\infty e^{-\lambda t}(-t)^k dF(t) \tag{1.1.7}$$

for every $\lambda \in \mathbb{C}_\omega$ and every $k \in \mathbb{N}_0$. Thus, every function $r : (\omega,\infty) \to X$ which has a Laplace-Stieltjes representation $r = \widehat{dF}$ for some $\omega \in \mathbb{R}$ and $F \in Lip_\omega([0,\infty);X)$ is contained in $C^\infty((\omega,\infty);X)$, and by (1.1.7), satisfies

$$\|(\lambda-\omega)^{(k+1)}\frac{r^{(k)}(\lambda)}{k!}\| \leq \|F\|_{Lip_\omega}$$

for all $\lambda > \omega$ and $k \in \mathbb{N}_0$. This shows that the Laplace-Stieltjes transform maps the space $Lip_\omega([0,\infty);X)$ into the Widder space $C_W^\infty((\omega,\infty);X)$, which is defined as the space consisting of all those functions $r \in C^\infty((\omega,\infty);X)$ for which

$$\|r\|_{W,\omega} := \sup_{k\in\mathbb{N}_0,\lambda>\omega} \|(\lambda-\omega)^{k+1}\frac{1}{k!}r^{(k)}(\lambda)\| < \infty.$$

The following is one of the key results in Laplace transform theory. For numerical functions, it was shown first by D. V. Widder in 1936. The generalization to Banach space valued functions was obtained by W. Arendt in 1987. The following formulation is taken from [N].

Theorem 1.1.3 (Widder's Theorem). The Laplace-Stieltjes transform is an isometric isomorphism from $Lip_\omega([0,\infty);X)$ onto $C_W^\infty((\omega,\infty);X)$.

It follows from Widder's Theorem that the Laplace transform is an injective operation on the Laplace transformable functions in $L^1_{\text{loc}}([0,\infty);X)$. To see this, let f be a function in $L^1_{\text{loc}}([0,\infty);X)$ with $\text{abs}(f) \leq \omega$ for some $\omega \geq 0$. Then the function $t \mapsto F(t) := \int_0^t f(s)ds$ is continuous on $[0,\infty)$ and $\omega(F) \leq \omega$. It follows that the function $H(t) := \int_0^t F(s)ds = f^{[2]}(t)$ for $t \geq 0$ is contained in $Lip_{\omega'}([0,\infty);X)$ for any $\omega' > \omega$, and

$$r(\lambda) := \int_0^\infty e^{-\lambda t}f(t)dt = \lambda\int_0^\infty e^{-\lambda t}dH(t)$$

for $\lambda > \omega'$. Thus, by Widder's Theorem, if $r \equiv 0$ on (ω', ∞), then $H \equiv 0$ on $[0, \infty)$. Thus, $f(t) = 0$ for almost all $t \geq 0$. This proves the following important corollary to Widder's Theorem.

Corollary 1.1.4 (Uniqueness Theorem). Let $f \in L^1_{\text{loc}}([0,\infty); X)$ with $\text{abs}(f) < \infty$. If there exists an $\omega \geq \text{abs}(f)$ such that $\widehat{f}(\lambda) = 0$ for all $\lambda > \omega$, then $f(t) = 0$ for almost all $t \geq 0$.

Stronger versions of the Uniqueness Theorem, which require only the condition that $\widehat{f}(\lambda_k) = 0$ for certain types of sequences $\{\lambda_k\}_{k \in \mathbb{N}}$ with $\text{Re}\, \lambda_k \longrightarrow \infty$ as $k \longrightarrow \infty$, can be found in [B-N].

Widder's Theorem and the following inversion theorem of the Laplace-Stieltjes transform (see [B-N]) will be crucial in characterizing solutions of Volterra equations in Chapter 2.

Theorem 1.1.5 (Phragmén-Doetsch Inversion Theorem). Let $F \in Lip_\omega([0,\infty); X)$ and define $r := \widehat{dF}$. Then for every $n \in \mathbb{N}$ with $n \geq \omega$ and every $t \geq 0$,

$$\|F(t) - \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{tnj} r(nj)\| \leq \frac{2}{n} \|F\|_{Lip_\omega}.$$

It follows from Widder' Theorem and Theorem 1.1.5 that the inverse Laplace-Stieltjes transform $L_s^{-1} : C_W^\infty((\omega,\infty); X) \to Lip_\omega([0,\infty); X)$ is given by

$$(L_s^{-1} r)(t) := F(t) = \lim_{n\to\infty} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{tnj} r(nj) \qquad \text{for } t \geq 0. \tag{1.1.8}$$

The statement (1.1.8) is called the Phragmén-Doetsch Inversion Formula.

The following theorem is due to B. Hennig and F. Neubrander [H-N]. It characterizes the pointwise convergence of a bounded sequence of functions in $Lip_\omega([0,\infty);X)$ in terms of their Laplace-Stieltjes transforms. It will be applied in proving Trotter-Kato type approximation theorems for integrated solution operator families in Section 3.3.

Theorem 1.1.6. Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence in $Lip_\omega([0,\infty);X)$ for which there exists a constant $M \geq 0$ such that $\|F_n\|_{Lip_\omega} \leq M$ for all $n \in \mathbb{N}$. Then the following are equivalent.

(i) There exist constants $a > \omega$ and $b > 0$ such that $\lim_{n\to\infty} \widehat{dF_n}(\lambda_k)$ exists for all $k \in \mathbb{N}_0$ where $\lambda_k := a + kb$.

(ii) There exists an $F \in Lip_\omega([0,\infty);X)$ such that $\|\widehat{dF}\|_{W,\omega} \leq M$ and $\{\widehat{dF_n}\}_{n\in\mathbb{N}}$ converges uniformly to $\widehat{dF}$ on every compact interval in (ω,∞).

(iii) $\lim_{n\to\infty} F_n(t)$ exists for every $t \geq 0$.

(iv) There exists an $F \in Lip_\omega([0,\infty);X)$ with $\|F\|_{Lip_\omega} \leq M$ such that $\{F_n\}_{n\in\mathbb{N}}$ converges uniformly to F on every compact interval in $[0,\infty)$.

If a function q has a Laplace or Laplace-Stieltjes representation, then q is analytic on a right half plane and it follows from Widder's Theorem that there exist constants M, $\omega > 0$ such that $\|\lambda q(\lambda)\| \leq M$ for all $\lambda \in \mathbb{C}_\omega$. A partial inverse of this statement is included in the following representation theorem (for a proof, see [B-N]), which will be applied in characterizing integrated solution operator families in Section 3.1.

Theorem 1.1.7. For $\omega \geq 0$ let $q : \mathbb{C}_\omega \to X$ be an analytic function for which there exists a constant M such that $\|\lambda q(\lambda)\| \leq M$ for all $\lambda \in \mathbb{C}_\omega$. Let $b > 0$. Then there exist an $f \in C([0,\infty);X)$ and a constant $C > 0$ such that $\|f(t)\| \le Ct^b e^{\omega t}$ for all $t \ge 0$ and $q(\lambda) = \lambda^b \widehat{f}(\lambda)$ for all $\lambda \in \mathbb{C}_\omega$.

Let $\omega \in \mathbb{R}$ and $0 < \theta \le \pi$. $\Sigma_{\omega,\theta}$ denotes the open sector $\{z \in \mathbb{C} \mid |\arg(z-\omega)| < \theta\}$. The following is a Laplace transform representation theorem of analytic functions on a sector $\Sigma_{\omega,\theta}$ for $0 < \theta \le \frac{\pi}{2}$. It will be applied in Section 3.4 where analytic solution operator families for the Volterra equation (VE) will be studied.

Theorem 1.1.8. Let $0 < \theta_0 \le \frac{\pi}{2}$, $\omega \in \mathbb{R}$. and let $q : (\omega,\infty) \to X$ be a function. Then the following are equivalent.

(i) There exists an analytic function $f : \Sigma_{0,\theta_0} \to X$ for which $q = \widehat{f}$ on (ω,∞), and

$$\sup_{z\in\Sigma_{0,\theta}} \|e^{-\omega z} f(z)\| < \infty \text{ for every } \theta \in (0,\theta_0).$$

(ii) The function q admits an analytic extension $\widetilde{q} : \Sigma_{\omega,\theta_0+\frac{\pi}{2}} \to X$ for which

$$\sup_{\lambda\in\Sigma_{\omega,\theta+\frac{\pi}{2}}} \|(\lambda-\omega)\widetilde{q}(\lambda)\| < \infty \text{ for every } \theta \in (0,\theta_0).$$

Moreover, if (i) holds, then for every $\theta \in (0,\theta_0)$, there exists a constant $C_\theta > 0$ such that

$$\|z^k f^{(k)}(z)\| \le C_\theta e^{\omega \operatorname{Re} z}(|\omega||z|+1)^k$$

for all $z \in \Sigma_\theta$.

Finally, we list three fundamental theorems from functional analysis which will be used in the following chapters. As usual, the set of all bounded linear operators from a Banach space X into a Banach space Y is denoted by $L(X,Y)$, and $L(X,X)$ is abbreviated by $L(X)$.

Theorem 1.1.9 (Uniform Boundedness Theorem). Let I be an index set and $T_\alpha \in L(X,Y)$ for all $\alpha \in I$. If for each $x \in X$ there exists a constant $M_x \geq 0$ such that $\sup_{\alpha\in I} \|T_\alpha(x)\| \leq M_x$, then $\{T_\alpha \mid \alpha \in I\}$ is uniformly bounded, i.e., there exists a constant $M \geq 0$ such that $\sup_{\alpha\in I} \|T_\alpha\| \leq M$.

The following is an extension of the Uniform Boundedness Theorem (see [Ly], [A-E-K], or [A-H-N]) which will be used in Section 3.2.

Theorem 1.1.10 (Uniform Exponential Boundedness Theorem). Let $F : [0,\infty) \longrightarrow L(X;Y)$ be a function such that $F(\cdot)x$ is exponentially bounded for each $x \in X$. Then there exist constants $M > 0$ and $\omega \in \mathbb{R}$ such that $\|F(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

Recall that a subset of X is called total if its linear span is dense in X.

Theorem 1.1.11 (Banach-Steinhaus Theorem). Let $\{T_n\}_{n\in\mathbb{N}}$ be a uniformly bounded sequence in $L(X,Y)$, i.e., $\sup_{n\in\mathbb{N}} \|T_n\| \leq M$ for some constant $M \geq 0$. If the sequence $\{T_n x\}$ converges for each x in a total subset of X, then there exists a $T \in L(X,Y)$ such that $\|T\| \leq M$ and $\{T_n x\}$ converges to Tx for every $x \in X$.

1.2 Vector Valued Convolutions

In this section we investigate regularity properties of vector valued Stieltjes convolutions $t \mapsto \int_0^t f(t-s)dg(s) = f * dg(t)$, and functions $t \mapsto \int_0^t S(t-s)f(s)ds$, where $\{S(t)\}_{t\geq 0}$ is a strongly continuous operator family. Also, we prove the multiplicativity of the Laplace transform of a Stieltjes convolution, i.e., $\widehat{f * dg} = \widehat{f} \cdot \widehat{dg}$, and establish the interchangeability in the order of iterated integrals.

Let X be a complex Banach space and let Ω be a finite or infinite interval in $[0, \infty)$. As usual, a function $f : \Omega \to X$ is said to be Lipschitz continuous if there exists a constant $M > 0$ such that $\|f(r) - f(s)\| \leq M|r - s|$ for all $r,\ s \in \Omega$. $Lip(\Omega; X)$ denotes the space consisting of all Lipschitz continuous functions f from Ω into X with norm $\|f\|_{Lip} := \inf\{M \mid \|f(r) - f(s)\| \leq M|r - s| \text{ for all } r,\ s \in \Omega\}$.

Proposition 1.2.1. Let one of E and F denote a Banach space X and the other $\mathbb{C}$. Then the following hold.

(a) If $f \in Lip([0,b]; E)$ and $g \in BV([0,b]; F)$, then $f * dg \in BV([0,b]; X)$.

(b) If $f \in C([0,b]; E)$ and $g \in BV([0,b]; F)$, then $f * dg \in L^1([0,b]; X)$.

(c) If $f \in C([0,b]; E)$ and $g \in BV([0,b]; F) \cap C([0,b]; F)$, then $f * dg \in C([0,b]; X)$.

(d) If $f \in L^1([0,b]; E)$ and $g \in Lip([0,b]; F)$, then $f * dg \in C([0,b]; X)$.

Proof. (a) Suppose that $f \in Lip([0,b]; X)$ and $g \in BV([0,b]; \mathbb{C})$. Let $M > 0$ be a constant such that $\|f(r) - f(s)\| \leq M|r - s|$ for all $r,\ s \in [0,b]$. Suppose that $g \in BV([0,b]; \mathbb{C})$. Let $0 = s_0 < \cdots < s_n = b$ be any partition of $[0,b]$. Then

$$\begin{aligned}
&\sum_{j=1}^{n} \|(f * dg)(s_j) - (f * dg)(s_{j-1})\| \\
\leq\ &\sum_{j=1}^{n} \| \int_{s_{j-1}}^{s_j} f(s_j - s)dg(s)\| + \| \int_0^{s_{j-1}} [f(s_j - s) - f(s_{j-1} - s)]dg(s)\| \\
\leq\ &\max_{0 \leq r \leq b} \|f(r)\| \sum_{j=1}^{n} \mathrm{var}(g; s_{j-1}, s_j) + M \sum_{j=1}^{n} (s_j - s_{j-1}) \mathrm{var}(g; 0, b) \\
=\ &[\ \max_{0 \leq r \leq b} \|f(r)\| + Mb\ ] \mathrm{var}(g; 0, b).
\end{aligned}$$

Hence $f * dg \in BV([0,b];X)$. The proof of the case that $f \in Lip([0,b];\mathbb{C})$ and $g \in BV([0,b];X)$ is the same.

(b) Suppose that $f \in C([0,b];X)$ and $g \in BV([0,b];\mathbb{C})$. Without loss of generality, we may assume that $f \not\equiv 0$ and that g is not constant. Recall that if a function $h : [0,b] \to X$ is weakly measurable, is almost separably valued, and satisfies $\int_0^b \|h(t)\|dt < \infty$, then $h \in L^1([0,b];X)$. Define $h := f * dg$ on $[0,b]$. First, we show that $h : [0,b] \to X$ is weakly measurable. Note that $C^1([0,b];X)$ is dense in $C([0,b];X)$ and that $C^1([0,b];X) \subset Lip([0,b];X)$. Hence for every $n \in \mathbb{N}$, there exists an $f_n \in C^1([0,b];X)$ such that $\max\limits_{0 \le t \le b} \|f(t) - f_n(t)\| \le \frac{1}{n}$ and thus

$$\| \int_0^t f(t-s)dg(s) - \int_0^t f_n(t-s)dg(s)\| \;\le\; \frac{1}{n}\text{var}(g;0,b)$$

for all $t \in [0,b]$. By (a), $f_n * dg$ is contained in $BV([0,b];X)$ and is hence weakly measurable on $[0,b]$ for every $n \in \mathbb{N}$. Thus, $f * dg$, the uniform limit of the sequence $\{f_n * dg\}_{n \in \mathbb{N}}$, is weakly measurable on $[0,b]$.

To show that $f * dg$ is almost separably valued on $[0,b]$, it will suffice by Lemma 1.1.1 to find a countable subset of X whose closure contains $(f * dg)([0,b])$. Let Q be the set of rational numbers in $[0,b]$ and let P be the set of all complex numbers with rational real and imaginary parts. Define $D := \{\sum\limits_{j=1}^{m} p_j f(q_j) \mid p_j \in P,\ q_j \in Q$, and $m \in \mathbb{N}\}$ which is countable in X. Let $t \in [0,b]$ and let ϵ be any positive number. Since $(f * dg)(t)$ exists, there exist a partition $0 = s_0 < s_1 < \cdots < s_m = t$ of $[0,t]$ and intermediate points $\xi_j \in [s_{j-1}, s_j]$ for $j \in \mathbb{N}$ with $1 \le j \le m$ for which $\| \int_0^t f(t-s)dg(s) - \sum\limits_{j=1}^{m} f(t-\xi_j)[g(s_j) - g(s_{j-1})]\| < \frac{\epsilon}{3}$. Since f is continuous, $\overline{f(Q)} = f([0,b])$. Therefore, for every $j \in \mathbb{N}$ with $1 \le j \le m$, we can find points $q_j \in Q$ and

$p_j \in P$ for which

$$\max_{1\le j\le m} \|f(t-\xi_j) - f(q_j)\| < \frac{\epsilon}{3\text{var}(g;0,b)}$$

and

$$\sum_{j=1}^{m} |g(s_j) - g(s_{j-1}) - p_j| \le \frac{\epsilon}{3\max\limits_{0\le t\le b}\|f(t)\|}.$$

Thus,

$$\begin{aligned}
&\|\int_0^t f(t-s)dg(s) - \sum_{j=1}^{m} f(q_j)p_j\| \\
\le\ &\|\int_0^t f(t-s)dg(s) - \sum_{j=1}^{m} f(t-\xi_j)[g(s_j) - g(s_{j-1})]\| \\
&+ \|\sum_{j=1}^{m} [f(t-\xi_j) - f(q_j)][g(s_j) - g(s_{j-1})]\| + \|\sum_{j=1}^{m} f(q_j)[g(s_j) - g(s_{j-1}) - p_j]\| \\
<\ &\epsilon.
\end{aligned}$$

This shows that $(f * dg)([0,b]) \subset \overline{D}$. Finally, the inequality $\int_0^b \|(f * dg)(t)\|dt < \infty$ follows from the inequality $\|\int_0^t f(t-s)dg(s)\| \le \max\limits_{0\le s\le b} \|f(s)\| \cdot \text{var}(g;0,b)$ for all $t \ge 0$. Therefore, $f * dg \in L^1([0,b];X)$. The case $f \in C([0,b];\mathbb{C})$ and $g \in BV([0,b];X)$ can be shown similarly by replacing the set P by a countable subset P_1 of X for which $g([0,b]) \subset \overline{P_1}$ (which exists by Lemma 1.1.2).

(c) Suppose that $f \in C([0,b];X)$ and $g \in BV([0,b];\mathbb{C}) \cap C([0,b];\mathbb{C})$. Let $\epsilon > 0$ be given. Since $f \in C([0,b];X)$ and since the variation of a continuous function of bounded variation is continuous, there exists $\delta = \delta(\epsilon) > 0$ such that $\|f(t) - f(s)\| < \epsilon$ and $\text{var}(g;s,t) < \epsilon$ for any $s < t \in [0,b]$ with $t - s < \delta$. Hence, for any $s < t \in [a,b]$ with $t - s < \delta$,

$$\begin{aligned}
\|(f * dg)(t) - (f * dg)(s)\| &\le \|\int_0^s [f(t-r) - f(s-r)]dg(r)\| + \|\int_s^t f(t-r)dg(r)\| \\
&\le \epsilon\ \text{var}(g;0,b) + \max_{0\le r\le b} \|f(r)\|\text{var}(g;s,t).
\end{aligned}$$

This shows that it follows that $f * dg \in C([a,b];X)$. The proof of the case that $f \in C([0,b];\mathbb{C})$ and $g \in BV([0,b];X) \cap C([0,b];X)$ is the same.

(d) Suppose that $f \in L^1([0,b];\mathbb{C})$ and $g \in Lip([0,b];X)$. We define the Riemann-Stieltjes integral $\int_0^t f(t-s)dg(s)$ of f and g as follows. For $t \in [0,b]$, define operators $K_t : C([0,b];\mathbb{C}) \to X$ by $K_t h := \int_0^t h(t-s)dg(s)$. Then $\|K_t h\| \leq \|g\|_{Lip}\|h\|_1$. Thus, K_t has a bounded extension to $L^1([0,b];\mathbb{C})$ and for every $t \geq 0$,

$$K_t f(t) = \int_0^t f(t-s)dg(s) := \lim_{n\to\infty} \int_0^t h_n(t-s)dg(s)$$

is well-defined, where $\{h_n\}_{n\in\mathbb{N}}$ is a sequence of continuous functions converging to f in $L^1([0,b];\mathbb{C})$. It follows from (c) that the function $t \mapsto \int_0^t h_n(t-s)dg(s),\ t \geq 0$ is continuous for every $n \in \mathbb{N}$. Moreover, $\|K_t f - K_t h_n\| \leq \|g\|_{Lip}\|f - h_n\|_1$. Thus, the function $t \mapsto K_t f$ is the uniform limit of the continuous functions. This shows that $f * dg \in C([0,b];X)$. The proof for the case that $f \in L^1([0,b];X)$ and $g \in Lip([0,b];\mathbb{C})$ is the same. //

Remark.

The Stieltjes convolution $f * dg$ of functions $f \in C([0,b];X)$ and $g \in BV([0,b];\mathbb{C})$ is not necessarily continuous. To see this, take $g \in BV([0,b];X)$ with $g(0) = 0$ and which is not continuous. Then $t \mapsto (1 * dg)(t) = g(t)$ is not continuous on $[0,b]$.

The Stieltjes convolution $f*dg$ of functions $f \in C([0,b];X)$ and $g \in BV([0,b];\mathbb{C})$ is not necessarily of bounded variation. To see this, consider the function $g(s) := \chi_{(0,\infty)}(s)$, and a function $f \in C([0,b];X)$ which is not of bounded variation. Then $f * dg = f$ is not of bounded variation on $[0,b]$.

Lemma 1.2.2. Let f be a continuous function on $[a,b] \times [c,d]$. Let g and h be functions of bounded variation on $[a,b]$ and $[c,d]$, respectively. Assume that one of the functions f, g, and h has values in a Banach space X and the other two in $\mathbb{C}$. Then

$$\int_a^b \int_c^d f(t,s)dh(t)dg(s) = \int_a^b \int_c^d f(t,s)dg(s)dh(t). \tag{1.2.1}$$

Proof. Let $f \in C([a,b] \times [c,d]; X)$, $g \in BV([a,b]; \mathbb{C})$, and $h \in BV([c,d]; \mathbb{C})$. Then $\int_a^b f(t,s)dh(t)$ and $\int_c^d f(t,s)dg(s)$ are continuous on $[c,d]$ and $[a,b]$, respectively. Hence the iterated integrals in (1.2.1) exist. If $X = \mathbb{C}$, then (1.2.1) follows from the mean value theorem for Riemann-Stieltjes integrals and the integral property $\int k d(\alpha + \beta) = \int k d\alpha + \int k d\beta$ (see Theorems 30.6 and 31.9, and Exercise 31.v in [B], for example). Let X be a complex Banach space and let $x^* \in X^*$. Then

$$\begin{aligned}
\langle \int_a^b \int_c^d f(t,s)dh(t)dg(s),\ x^* \rangle &= \int_a^b \int_c^d \langle f(t,s),\ x^* \rangle dh(t)dg(s) \\
&= \int_c^d \int_a^b \langle f(t,s),\ x^* \rangle dg(s)dh(t) \\
&= \langle \int_c^d \int_a^b f(t,s)dg(s)dh(t),\ x^* \rangle.
\end{aligned}$$

Since $x^* \in X^*$ is arbitrary, (1.2.1) follows. The proofs of the other cases are similar. //

Corollary 1.2.3. Let f be a continuous function on $[a,b] \times [a,b]$ and g be a function of bounded variation on $[a,b]$. Assume that one of f and g has values in a Banach space X and the other in $\mathbb{C}$. Then

$$\int_a^b \oint_a^b f(t,s)dg(s)dt = \oint_a^b \int_a^b f(t,s)dtdg(s). \tag{1.2.2}$$

Proof. Since both $t \mapsto \int_a^b f(t,s)dg(s)$ and $s \mapsto \int_a^b f(t,s)dt$ are contained in the space $C([a,b];X)$, $\oint_a^b \int_a^b f(t,s)dg(s)dt = \int_a^b \int_a^b f(t,s)dg(s)dt$ and $\oint_a^b f(t,s)dt = \int_a^b f(t,s)dt$ (see Section 1.1). Hence, by Lemma 1.2.2, the statement (1.2.2) follows. //

Lemma 1.2.4. Let $T \geq 0$ and $\lambda \in \mathbb{C}$. Let f be a continuous function and g be a function of bounded variation on $[a,b]$. Assume that one of f and g has values in a Banach space X and the other in $\mathbb{C}$. Then

$$\int_0^T \oint_0^t e^{-\lambda t} f(t-s)dg(s)dt = \oint_0^T \int_s^T e^{-\lambda t} f(t-s)dtdg(s). \qquad (1.2.3)$$

If f is Bochner integrable and g is Lipschitz continuous on $[0,T]$, then (1.2.3) remains valid.

Proof. By Proposition 1.2.1, the hypotheses on f and g imply that $f * dg$ is contained in $L^1([0,T];X)$ and that $s \mapsto \int_s^T e^{-\lambda t} f(t-s)dt$ is continuous on $[0,T]$. Hence the integrals in (1.2.3) exist. Let Ω be the region $\{(t,s) \in \mathbb{R}^2 \mid 0 \leq s \leq t \leq T\}$. Define $\tilde{f} : [0,T] \times [0,T] \to E$ by

$$\tilde{f}(t,s) = \begin{cases} e^{-\lambda t} f(t-s) & \text{if } (t,s) \in \Omega \\ e^{-\lambda t} f(0) & \text{otherwise.} \end{cases}$$

Then $\tilde{f}$ is continuous. By Corollary 1.2.3,

$$\int_0^T \oint_0^T \tilde{f}(t-s)dg(s)dt = \oint_0^T \int_0^T \tilde{f}(t-s)dtdg(s).$$

Since

$$\begin{aligned} \int_0^T \oint_0^T \tilde{f}(t-s)dg(s)dt &= \int_0^T \oint_0^t e^{-\lambda t} f(t-s)dg(s)dt + \int_0^T \oint_t^T e^{-\lambda t} f(0)dg(s)dt \\ &= \int_0^T \oint_0^t e^{-\lambda t} f(t-s)dg(s)dt + \int_0^T e^{-\lambda t}[g(T)-g(t)]dtf(0) \end{aligned}$$

and

$$\oint_0^T \int_0^T \tilde{f}(t-s)dtdg(s) = \oint_0^T \int_s^T e^{-\lambda t} f(t-s)dtdg(s) + \oint_0^T \int_0^s e^{-\lambda t} f(0)dtdg(s),$$

it suffices to show that

$$\int_0^T e^{-\lambda t}[g(T) - g(t)]dt = \oint_0^T \int_0^s e^{-\lambda t} dtdg(s). \tag{1.2.4}$$

When $\lambda = 0$,

$$\int_0^T [g(T) - g(t)]dt = Tg(T) - \int_0^T g(t)dt \qquad \text{and}$$

$$\oint_0^T \int_0^s dtdg(s) = Tg(T) - \int_0^T g(s)ds.$$

When $\lambda \neq 0$,

$$\int_0^T e^{-\lambda t}[g(T) - g(t)]dt = \frac{1 - e^{-\lambda T}}{\lambda} g(T) - \int_0^T e^{-\lambda t} g(t)dt \qquad \text{and}$$

$$\begin{aligned}
\oint_0^T \int_0^s e^{-\lambda t} dtdg(s) &= \oint_0^T \frac{1 - e^{-\lambda s}}{\lambda} dg(s) \\
&= \frac{1}{\lambda}\Big[g(T) - \int_0^T e^{-\lambda t} dg(s)\Big] \\
&= \frac{1}{\lambda}\Big[g(T) - e^{-\lambda t} g(T) - \int_0^T \lambda e^{-\lambda s} g(s)ds\Big] \\
&= \frac{1 - e^{-\lambda T}}{\lambda} g(T) - \int_0^T e^{-\lambda s} g(s)ds.
\end{aligned}$$

Thus, the statement (1.2.4) holds. //

As usual, we call an operator family $S = \{S(t)\}_{t \geq 0}$ in $L(X)$ strongly continuous if $S(\cdot)x : [0, \infty) \to X$ is continuous for every $x \in X$. The following will be applied in Chapter 3.

Lemma 1.2.5. Let S be a strongly continuous operator family. Let $f \in C([0,\infty);X)$ and let $\mu \in BV_{\text{loc}}([0,\infty);\mathbb{C})$ with $\mu(0)=0$. Then for every $t \geq 0$,

$$\int_0^t \oint_0^{t-s} S(t-s-r)f(s)d\mu(r)ds = \oint_0^t \int_0^{t-r} S(t-s-r)f(s)dsd\mu(r). \qquad (1.2.5)$$

Proof. For $t \geq 0$, let Ω be the region $\{(s,r) \in \mathbb{R}^2 \mid 0 \leq s \leq t,\ 0 \leq r \leq t-s\}$. Define $g : [0,t] \times [0,t] \to X$ by

$$g(s,r) = \begin{cases} S(t-s-r)f(s) & \text{if } (s,r) \in \Omega \\ S(0)f(s) & \text{otherwise.} \end{cases}$$

Then g is continuous on $[0,t] \times [0,t]$. For each $s \in [0,t]$,

$$\begin{aligned}\int_0^t g(s,r)d\mu(r) &= \int_0^{t-s} S(t-s-r)f(s)d\mu(r) + \int_{t-s}^t S(0)f(s)d\mu(r) \\ &= \int_0^{t-s} S(t-s-r)f(s)d\mu(r) + S(0)f(s)\mu(t) - S(0)f(s)\mu(t-s).\end{aligned}$$

The function $s \mapsto \int_0^t g(s,r)d\mu(r) - S(0)f(s)\mu(t)$ is continuous $[0,t]$. Since μ has at most countably many discontinuities, the function $s \mapsto S(0)f(s)\mu(t-s)$ is almost separably valued. It is clear that the function $s \mapsto S(0)f(s)\mu(t-s)$ is weakly measurable and norm-bounded on $[0,T]$. Thus,

$$s \mapsto \int_0^{t-s} S(t-s-r)f(s)d\mu(r) = \int_0^t g(s,r)d\mu(r) - S(0)f(s)\mu(t) + S(0)f(s)\mu(t-s)$$

is contained in $L^1([0,t];X)$. Hence the left hand side of (1.2.5) exists. For each $r \in [0,t]$,

$$\int_0^t g(s,r)ds = \int_0^{t-r} S(t-s-r)f(s)ds + \int_{t-r}^t S(0)f(s)ds.$$

Thus,

$$r \mapsto \int_0^{t-r} S(t-s-r)f(s)ds = \int_0^t g(s,r)ds - \int_{t-r}^t S(0)f(s)ds$$

is continuous in $[0,t]$. Hence the right hand side of (1.2.5) exists. Now, we show the equality in (1.2.5). By Corollary 1.2.3,

$$\int_0^t \oint_0^t g(s,r)d\mu(r)ds = \oint_0^t \int_0^t g(s,r)dsd\mu(r).$$

Calculating each side of this equation, we obtain that

$$\begin{aligned}
&\int_0^t \int_0^t g(s,r)d\mu(r)ds \\
&= \int_0^t \oint_0^{t-s} g(s,r)d\mu(r)ds + \int_0^t \oint_{t-s}^t S(0)f(s)d\mu(r)ds \\
&= \int_0^t \oint_0^{t-s} S(t-s-r)f(s)d\mu(r)ds + \int_0^t S(0)f(s)\left[\mu(t)-\mu(t-s)\right]ds,
\end{aligned}$$

and

$$\begin{aligned}
&\oint_0^t \int_0^t g(s,r)dsd\mu(r) \\
&= \oint_0^t \int_0^{t-r} g(s,r)dsd\mu(r) + \oint_0^t \int_{t-r}^t S(0)f(s)dsd\mu(r) \\
&= \oint_0^t \int_0^{t-r} g(s,r)dsd\mu(r) + \int_0^t S(0)f(s)ds\cdot\mu(t) - \int_0^t \mu(r)d\Big[\int_{t-r}^t S(0)f(s)ds\Big] \\
&= \oint_0^t \int_0^{t-r} g(s,r)dsd\mu(r) + \int_0^t S(0)f(s)ds\cdot\mu(t) - \int_0^t S(0)f(t-r)\mu(r)dr \\
&= \oint_0^t \int_0^{t-r} g(s,r)dsd\mu(r) + \int_0^t S(0)f(s)ds\cdot\mu(t) - \int_0^t S(0)f(s)\mu(t-s)ds \\
&= \oint_0^t \int_0^{t-r} S(t-s-r)f(s)dsd\mu(r) + \int_0^t S(0)f(s)\left[\mu(t)-\mu(t-s)\right]ds.
\end{aligned}$$

Therefore, the equality in (1.2.5) follows. //

Lemma 1.2.6. Suppose that $S:[0,\infty)\to L(X)$ is strongly continuous and that $f\in C([0,\infty);X)$. Then for $t\geq 0$,

$$\int_0^t \int_s^t S(s)f(r-s)drds = \int_0^t \int_0^r S(s)f(r-s)dsdr. \tag{1.2.6}$$

Proof. Let Ω be the region $\{(r,s) \in \mathbb{R} \mid 0 \leq s \leq r \leq t\}$. Define $g : [0,t] \times [0,t] \longrightarrow X$ by

$$g(r,s) = \begin{cases} S(s)f(r-s) & \text{if } (r,s) \in \Omega \\ S(s)f(0) & \text{otherwise.} \end{cases}$$

Since S is strongly continuous and since $f \in C([0,\infty); X)$, it follows that g is continuous on $[0,t] \times [0,t]$. For every $r \in [0,t]$,

$$\int_0^t g(r,s)ds = \int_0^r S(s)f(r-s)ds + \int_r^t S(s)f(0)ds.$$

Hence

$$r \mapsto \int_0^r S(s)f(r-s)ds = \int_0^t g(r,s)ds + \int_r^t S(s)f(0)ds$$

is continuous on $[0,t]$. For every $s \in [0,t]$,

$$\int_0^t g(r,s)ds = \int_0^s S(s)f(r-s)ds + \int_s^t S(s)f(0)ds.$$

Hence

$$s \mapsto \int_0^s S(s)f(r-s)ds = \int_0^t g(r,s)ds + \int_s^t S(s)f(0)ds$$

is continuous on $[0,t]$. Thus, the integrals in (1.2.6) exist. Now, we show that they are equal. By Lemma 1.2.2,

$$\int_0^t \int_0^t g(r,s)drds = \int_0^t \int_0^t g(r,s)dsdr.$$

Calculating each side of this equation, we obtain

$$\begin{aligned} \int_0^t \int_0^t g(r,s)drds &= \int_0^t \int_s^t S(s)f(r-s)drds + \int_0^t \int_0^s S(s)f(0)drds \\ &= \int_0^t \int_s^t S(s)f(r-s)drds + \int_0^t sS(s)f(0)ds, \end{aligned}$$

and

$$\int_0^t \int_0^t g(r,s)dsdr = \int_0^t \int_0^r S(s)f(r-s)dsdr + \int_0^t \int_r^t S(s)f(0)dsdr.$$

Thus, it suffices to show that $\int_0^t \int_r^t S(s)f(0)dsdr = \int_0^t sS(s)f(0)dsdr$. By the integration by parts formula (1.1.1),

$$\int_0^t \int_r^t S(s)f(0)dsdr = -\int_0^t r\Big[\int_r^t S(s)f(0)ds\Big] = \int_0^t r \cdot S(r)f(0)dr.$$

Therefore,

$$\int_0^t \int_s^t S(s)f(r-s)drds = \int_0^t \int_0^r S(s)f(r-s)dsdr. \quad //$$

The Laplace transform of a Stieltjes convolution has the following multiplicative property, which is essential for transforming the Volterra equation (VE) to the characteristic equation (CE).

Proposition 1.2.7. Suppose that $f \in C([0,\infty);X)$ with $\omega(f) < \infty$ and that $g \in BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$. Let $\omega \geq \max\{\omega(f),\epsilon\}$. Then $\mathrm{abs}(f*dg) \leq \omega$ and for $\lambda \in \mathbb{C}_\omega$,

$$\widehat{f*dg}(\lambda) = \widehat{f}(\lambda)\widehat{dg}(\lambda).$$

Proof. The proof is similar to the standard one for the Laplace transform (see [S], for example). It follows from Proposition 1.2.1 that $f*dg \in L^1_{\mathrm{loc}}([0,\infty);X)$. Let $T \geq 0$ and let $\lambda \in \mathbb{C}_\omega$. Applying Lemma 1.2.4, we obtain that

$$\int_0^T e^{-\lambda t}(f*dg)(t)dt$$
$$= \int_0^T \oint_0^t e^{-\lambda t} f(t-s)dg(s)dt = \oint_0^T \int_s^T e^{-\lambda t} f(t-s)dtdg(s)$$
$$= \oint_0^T \int_0^{T-s} e^{-\lambda(t+s)} f(t)dtdg(s) = \oint_0^T e^{-\lambda s} \int_0^{T-s} e^{-\lambda t} f(t)dtdg(s) =: \mathrm{I}_T.$$

Divide the region $\{(t,s) \in \mathbb{R}^2 \mid 0 \le s \le T,\ 0 \le t \le T-s\}$ of the integral I_T into three subregions as follows :

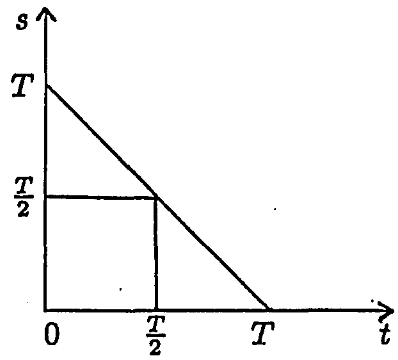

Figure 1.

Then

$$\begin{aligned} I_T &= \int_0^{\frac{T}{2}} e^{-\lambda s} dg(s) \int_0^{\frac{T}{2}} e^{-\lambda t} f(t)dt \\ &\quad + \int_{\frac{T}{2}}^{T} e^{-\lambda s} \int_0^{T-s} e^{-\lambda t} f(t)dtdg(s) + \int_0^{\frac{T}{2}} e^{-\lambda s} \int_{\frac{T}{2}}^{T-s} e^{-\lambda t} f(t)dtdg(s) \\ &=: \mathrm{I}(T) \quad + \quad \mathrm{II}(T) \quad + \quad \mathrm{III}(T) \qquad \text{(in order of the integrals).} \end{aligned}$$

Hence, it suffices to show that both $\mathrm{II}(T)$ and $\mathrm{III}(T)$ converge to 0 as $T \longrightarrow \infty$. We start with $\mathrm{II}(T)$. From the integration by parts formula (1.1.1) and by differentiating the integrand of the Riemann-Stieltjes integral in $\mathrm{II}(T)$,

$$\begin{aligned} \mathrm{II}(T) &= \int_{\frac{T}{2}}^{T} \Big[e^{-\lambda s} \int_0^{T-s} e^{-\lambda t} f(t)dt\Big] dg(s) \\ &= -e^{-\lambda \frac{T}{2}} g(\frac{T}{2}) \int_0^{\frac{T}{2}} e^{-\lambda t} f(t)dt - \int_{\frac{T}{2}}^{T} g(s)d\Big[e^{-\lambda s} \int_0^{T-s} e^{-\lambda t} f(t)dt\Big] \\ &= -e^{-\lambda \frac{T}{2}} g(\frac{T}{2}) \int_0^{\frac{T}{2}} e^{-\lambda t} f(t)dt \\ &\qquad + \int_{\frac{T}{2}}^{T} e^{-\lambda s} g(s) \Big[\lambda \int_0^{T-s} e^{-\lambda t} f(t)dt + e^{-\lambda(T-s)} f(T-s)\Big] ds \\ &=: \quad \mathrm{II}_1(T) \quad + \quad \mathrm{II}_2(T) \qquad \text{(in order of the integrals).} \end{aligned}$$

Since $\operatorname{Re}\lambda > \omega(g) \ge \omega$ and $\operatorname{Re}\lambda > \operatorname{abs}(f)$, we obtain that $\mathrm{II}_1(T) \longrightarrow 0$ as $T \longrightarrow \infty$.

The integral $\mathrm{II}_2(T)$ is estimated as follows :

$$\|\mathrm{II}_2(T)\|$$

$$\leq \Big[|\lambda| \sup_{0\leq u\leq \frac{T}{2}} \int_0^u \|e^{-\mathrm{Re}\,\lambda t} f(t)\| dt + \sup_{0\leq u\leq \frac{T}{2}} e^{-u\mathrm{Re}\,\lambda}\|f(u)\|\Big] \int_{\frac{T}{2}}^T e^{-\mathrm{Re}\,\lambda s}|g(s)|ds.$$

Since $\mathrm{Re}\,\lambda > \mathrm{abs}(f)$, it follows that $\sup_{T\geq 0}\sup_{0\leq u\leq \frac{T}{2}} \|\int_0^u e^{-\lambda t} f(t)dt\| < \infty$. Since $\mathrm{Re}\,\lambda > \omega(f)$, it follows that $\sup_{T\geq 0}\sup_{0\leq u\leq \frac{T}{2}} \|e^{-u\mathrm{Re}\,\lambda} f(u)\| < \infty$. These inequalities and the condition $\mathrm{Re}\,\lambda > \mathrm{abs}(g)$ imply that $\mathrm{II}_2(T) \longrightarrow 0$ as $T \longrightarrow \infty$. Thus, $\mathrm{II}(T) \longrightarrow 0$ as $T \longrightarrow \infty$. Applying the integration by parts formula (1.1.1) yields

$$\begin{aligned}
\mathrm{III}(T) &= \int_0^{\frac{T}{2}} \Big[e^{-\lambda s} \int_{\frac{T}{2}}^{T-s} e^{-\lambda t} f(t)dt\Big] dg(s) \\
&= -\int_0^{\frac{T}{2}} g(s) d\Big[e^{-\lambda s} \int_{\frac{T}{2}}^{T-s} e^{-\lambda t} f(t)dt\Big] \\
&= \int_0^{\frac{T}{2}} e^{-\lambda s} g(s)\Big[e^{-\lambda(T-s)} f(T-s) + \int_{\frac{T}{2}}^{T-s} e^{-\lambda t} f(t)dt\Big] ds.
\end{aligned}$$

It follows that

$$\|\mathrm{III}(T)\| \quad \leq \Big[\sup_{\frac{T}{2}\leq u\leq T} e^{-u\mathrm{Re}\,\lambda}\|f(t)\| + \sup_{\frac{T}{2}\leq u\leq T} \int_{\frac{T}{2}}^u e^{-\mathrm{Re}\,\lambda t}\|f(t)\|dt\Big] \int_0^{\frac{T}{2}} |e^{-\lambda s} g(s)|ds.$$

Since $\mathrm{Re}\,\lambda > \omega(f)$ and since $\mathrm{Re}\,\lambda > \mathrm{abs}(f)$, it follows that

$$\Big[\sup_{\frac{T}{2}\leq u\leq T} e^{-u\mathrm{Re}\,\lambda}\|f(t)\| + \sup_{\frac{T}{2}\leq u\leq T} \int_{\frac{T}{2}}^u e^{-\mathrm{Re}\,\lambda t}\|f(t)\|dt\Big] \longrightarrow 0$$

as $T \longrightarrow \infty$. This fact and the condition $\mathrm{Re}\,\lambda > \mathrm{abs}(g)$ imply that $\mathrm{III}(T) \longrightarrow 0$ as $T \longrightarrow \infty$. Consequently, I_T converges to $\widehat{dg}(\lambda)\widehat{f}(\lambda)$ as $T \longrightarrow \infty$ which implies that $\widehat{f * dg}(\lambda)$ exists and is equal to $\widehat{f}(\lambda)\widehat{dg}(\lambda)$ for every $\lambda \in \mathbb{C}_\omega$. //

Finally, we investigate the differentiability of Stieltjes convolutions.

Lemma 1.2.8. Suppose that f is a continuously differentiable function with $f(0) = 0$ and that g is a continuous function of local bounded variation on $[0, \infty)$. Assume that one of f and g has values in a Banach space X and the other in $\mathbb{C}$. Then $u := f * dg \in C^1([0, \infty); X)$, and for every $t \geq 0$,

$$u'(t) = \int_0^t f'(t-s)dg(s).$$

Proof. Let $H := f' * dg$. Then by Proposition 1.2.1, $H \in C([0, \infty); X)$. Hence it suffices to show that $u' = H$. Let $T \geq 0$. Applying Lemma 1.2.4 for $\lambda = 0$,

$$\begin{aligned} H^{[1]}(T) &= \int_0^T \int_0^t f'(t-s)dg(s)dt = \int_0^T \int_s^T f'(t-s)dtdg(s) \\ &= \int_0^T \int_0^{T-s} f'(t)dtdg(s) = \int_0^T f(T-s)dg(s) - \int_0^T f(0)dg(s) \\ &= \int_0^T f(T-s)dg(s) \\ &= u(T). \end{aligned}$$

Thus, $u'(t) = H(t)$ for every $t \geq 0$. //

Corollary 1.2.9. Let $n \in \mathbb{N}_0$. Suppose that f is an n-times continuously differentiable function with $f^{(k)}(0) = 0$ for $k \in \mathbb{N}$ with $0 \leq k \leq n-1$ and g is a continuous function of local bounded variation on $[0, \infty)$. Assume that one of f and g has values in a Banach space X and the other in $\mathbb{C}$. Then $u := f * dg \in C^n([0, \infty); X)$ and for every $t \geq 0$,

$$u^{(n)}(t) = \int_0^t f^{(n)}(t-s)dg(s).$$

Lemma 1.2.10. Let $f \in L^1([0,b];\mathbb{C})$ and $g \in Lip([0,b];X)$. Then the function $u := f * (dg^{[1]})$ is differentiable a.e. on $[a,b]$, and

$$u'(t) = \int_0^t f(t-s)dg(s) + f(t)g(0)$$

a.e. $t \in [0,b]$. Moreover, u is continuously differentiable if $g(0) = 0$.

Proof. By Proposition 1.2.1 (d), the function $t \mapsto \int_0^t f(t-s)dg(s)$ is well-defined and in $C([0,b];X)$. Let $0 \le T \le b$. By Lemma 1.2.4,

$$\int_0^T \int_0^t f(t-s)dg(s)dt = \int_0^T \int_s^T f(t-s)dtdg(s)$$
$$= \int_0^T \int_0^{T-s} f(t)dtdg(s) = \int_0^T f^{[1]}(T-s)dg(s)$$
$$= f^{[1]}(0)g(T) - f^{[1]}(T)g(0) - \int_0^T g(s)df^{[1]}(T-s)$$
$$= -f^{[1]}(T)g(0) + \int_0^T g(s)f(T-s)ds$$
$$= \int_0^T f(T-s)dg^{[1]}(s) - f^{[1]}(T)g(0)$$
$$= u(T) - f^{[1]}(T)g(0).$$

Therefore, $u'(t) = \int_0^t f(t-s)dg(s) + f(t)g(0)$ for almost all $t \in [0,b]$ and it follows from Proposition 1.2.1 (d) that u' is continuous if $g(0) = 0$. //

The following lemma will be used to prove a variation of constants formula for integrated solution operator families in Section 3.1.

Lemma 1.2.11. Suppose that $S : [0,\infty) \to L(X)$ is strongly continuous and that $f \in C^1([0,\infty);X)$. Then for $t \ge 0$,

$$\frac{d}{dt}\Big[\int_0^t S(t-s)f(s)ds\Big] = S(t)f(0) + \int_0^t S(t-s)f'(s)ds.$$

Proof. For $t \geq 0$,

$$\int_0^t S(t-s)f(s)ds = \int_0^t S(s)f(t-s)ds = \int_0^t S(s)\Big[f(0) + \int_0^{t-s} f'(r)dr\Big]ds$$
$$= \int_0^t S(s)f(0)ds + \int_0^t \int_0^{t-s} S(s)f'(r)drds.$$

Since S is strongly continuous, $s \mapsto S(s)f(0)$ is continuous on $[0,t]$. Hence it suffices to show that

$$\frac{d}{dt}\Big[\int_0^t \int_0^{t-s} S(s)f'(r)drds\Big] = \int_0^t S(t-s)f'(s)ds.$$

By change of variables and from Lemma 1.2.6,

$$\int_0^t \int_0^{t-s} S(s)f'(r)drds = \int_0^t \int_s^t S(s)f'(r-s)drds$$
$$= \int_0^t \int_0^r S(s)f'(r-s)dsdr = \int_0^t \int_0^r S(r-s)f'(s)dsdr.$$

From the proof of Lemma 1.2.6 we obtain that $r \mapsto \int_0^r S(r-s)f'(s)ds$ is continuous on $[0,t]$. Therefore,

$$\frac{d}{dt}\Big[\int_0^t \int_0^{t-s} S(s)f'(r)drds\Big] = \int_0^t S(t-s)f'(s)ds. \quad //$$

CHAPTER 2 EXISTENCE AND UNIQUENESS OF SOLUTIONS OF ABSTRACT VOLTERRA EQUATIONS

In this chapter we study the abstract Volterra equation

$$v(t) = A\int_0^t v(t-s)d\mu(s) + f(t) \qquad \text{for } t \geq 0 \tag{VE}$$

by applying Laplace transform theory. We assume that A is a closed linear operator with domain $D(A)$ and range in a complex Banach space X, that $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$, and that f is a Laplace transformable function in $C([0,\infty);X)$ (for notations, see Section 1.1).

A function $v \in C([0,\infty);X)$ is called a solution of (VE) if $\int_0^t v(t-s)d\mu(s) \in D(A)$ for every $t \geq 0$ and if it solves (VE).

A function $v \in C([0,\infty);X)$ is defined to be a weak solution of (VE) if

$$\langle v(t),\ y^*\rangle = \langle (v * d\mu)(t),\ z^*\rangle + \langle f(t),\ y^*\rangle$$

for all $t \geq 0$, $y^* \in D(A^*)$, and $z^* \in A^*y^*$. Note that A^* is multi-valued unless A is densely defined. However, $((\lambda I - A)^{-1})^* = (\overline{\lambda} I - A^*)^{-1}$ is single-valued and bounded (but not necessarily one-to-one) for every $\lambda \in \rho(A)$. For a definition and discussion of strong solutions of (VE), see Chapter 3. We first show that weak solutions of (VE) are solutions of (VE). The following is a slight generalization of a result from [Pr] (Proposition 1.4).

Theorem 2.1. Suppose that $\rho(A) \neq \emptyset$. Then every weak solution of (VE) is a solution of (VE).

Proof. Let $\lambda \in \rho(A)$. Suppose that $v \in C([0,\infty); X)$ is a weak solution of (VE). Let $t \geq 0$. Then

$$\langle v(t),\ y^* \rangle = \langle (v * d\mu)(t),\ z^* \rangle + \langle f(t),\ y^* \rangle$$

for every $y^* \in D(A^*)$ and every $z^* \in A^* y^*$. Hence

$$\langle (v * d\mu)(t),\ -z^* \rangle = \langle f(t) - v(t),\ y^* \rangle \quad \text{and thus,}$$

$$\langle (v * d\mu)(t),\ \overline{\lambda} y^* - z^* \rangle = \langle \lambda (v * d\mu)(t) + f(t) - v(t),\ y^* \rangle.$$

Let $x^* \in X^*$. Since $\overline{\lambda} \in \rho(A^*)$, there exists a $y_0^* \in D(A^*)$ such that $x^* \in (\overline{\lambda} I - A^*) y_0^*$. It follows that $x^* = \overline{\lambda} y_0^* - z_0^*$ for some $z_0^* \in A^* y_0^*$. Hence, $(\overline{\lambda} - A^*)^{-1} x^* = ((\lambda - A)^{-1})^* x^* = y_0^*$, and

$$\langle (v * d\mu)(t),\ x^* \rangle = \langle \lambda (v * d\mu)(t) + f(t) - v(t),\ ((\lambda I - A)^{-1})^* x^* \rangle.$$

Thus, $$\langle (v * d\mu)(t),\ x^* \rangle = \langle (\lambda I - A)^{-1} (\lambda (v * d\mu)(t) + f(t) - v(t)),\ x^* \rangle.$$

Since $x^* \in X$ is arbitrary, it follows that $(v * d\mu)(t) = (\lambda \mathrm{I} - \mathrm{A})^{-1} (\lambda (\mathrm{v} * \mathrm{d}\mu)(\mathrm{t}) + \mathrm{f}(\mathrm{t}) - \mathrm{v}(\mathrm{t}))$. Therefore, $\lambda (v * d\mu)(t) - A(v * d\mu)(t) = \lambda (v * d\mu)(t) + f(t) - v(t)$. From this equation we conclude that v is a solution of (VE). //

Exponentially bounded solutions of (VE) can be characterized in terms of the Laplace transforms of the functions involved in the equation (VE) and the characteristic equation

$$(I - \widehat{d\mu}(\lambda) A) y(\lambda) = \widehat{f}(\lambda) \qquad (\lambda > \omega) \tag{CE}$$

for some $\omega \geq 0$.

Theorem 2.2. Let $v \in C([0,\infty);X)$ with $\omega(v) < \infty$ and let ω be any number such that $\omega \geq \max\{\epsilon,\ \mathrm{abs}(f),\ \omega(v)\}$. Then the following are equivalent.

(i) v solves (VE).

(ii) $\widehat{d\mu}(\lambda)\widehat{v}(\lambda) \in D(A)$ and $(I - \widehat{d\mu}(\lambda)A)\widehat{v}(\lambda) = \widehat{f}(\lambda)$ if $\lambda \in \mathbb{C}_\omega$.

(iii) $\widehat{d\mu}(k)\widehat{v}(k) \in D(A)$ and $(I - \widehat{d\mu}(k)A)\widehat{v}(k) = \widehat{f}(k)$ if $\omega < k \in \mathbb{N}$.

Proof. To show (i) $\Longrightarrow$ (ii), suppose that v is an exponentially bounded solution of (VE). Since the Laplace transforms of the functions $t \mapsto v{*}d\mu(t)$ and $t \mapsto A(v{*}d\mu)(t) = v(t) - f(t)$ exist for $\lambda \in \mathbb{C}_\omega$, it follows from the closedness of A and Proposition 1.2.7 that $\widehat{d\mu}(\lambda)\widehat{v}(\lambda) \in D(A)$ and $A\widehat{d\mu}(\lambda)\widehat{v}(\lambda) = \widehat{v}(\lambda) - \widehat{f}(\lambda)$ for all $\lambda \in \mathbb{C}_\omega$. The implication (ii) $\Longrightarrow$ (iii) is obvious. We show the implication (iii) $\Longrightarrow$ (i) by means of the Phragmén-Doetsch Inversion Formula (1.1.8). Suppose that (iii) holds. Let $\omega' > \omega$. Observe that Proposition 1.2.7 and (1.1.4) imply that $v^{[2]}$, $f^{[2]}$, and $(v{*}d\mu)^{[2]} = v^{[2]}{*}d\mu$ are all contained in $Lip_{\omega'}([0,\infty);X)$. It follows from (1.1.6) and (1.1.5) that $\widehat{v}(\lambda) = \lambda\widehat{dv^{[2]}}(\lambda)$, $\widehat{f}(\lambda) = \lambda\widehat{df^{[2]}}(\lambda)$, and $\widehat{v}(\lambda)\widehat{d\mu}(\lambda) = (\widehat{v * d\mu})(\lambda) = \lambda(d(v * d\mu)^{[2]})^\wedge(\lambda)$ for $\lambda \in \mathbb{C}_\omega$. We obtain from these identities and from (iii) that

$$\widehat{dv^{[2]}}(k) - \widehat{df^{[2]}}(k) = A(d(v * d\mu)^{[2]})^\wedge(k) \tag{2.1}$$

for every $k \in \mathbb{N}$ with $k > \omega'$. Hence, by the inversion formula (1.1.8),

$$(v * d\mu)^{[2]}(t) = \lim_{n\to\infty} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{tnj}(d(v * d\mu)^{[2]})^\wedge(nj),$$

and

$$v^{[2]}(t) - f^{[2]}(t) = \lim_{n\to\infty} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{tnj} A(d(v * d\mu)^{[2]})^\wedge(nj).$$

Thus, by the closedness of A,

$$v^{[2]}(t) - f^{[2]}(t) = A(v * d\mu)^{[2]}(t)$$

for every $t \geq 0$. Since all the functions involved in this equation are twice differentiable and since A is closed, it follows that v solves (VE). //

Theorem 2.2 says that solving the equation (VE) can be simplified to solving the equation (CE) in which the time variable t is eliminated. The Laplace transform approach to (VE) consists of three steps. First, find a solution y of (CE) on (ω, ∞). Second, check whether or not the function y is a Laplace transform of a continuous exponentially bounded function. Finally, if y has a Laplace transform representation, then the inverse Laplace transform of y is an exponentially bounded solution of (VE). The main difficulty in the Laplace transform approach to (VE) is in the second step. In other words, it is hard to check if a function can be represented as a Laplace transform of a continuous, exponentially bounded function. On the other hand, the representation as a Laplace-Stieltjes transform of a Lip_ω-functions can be verified by Widder's Theorem. The following is a Hille-Yosida type characterization of the $Lip_\omega([0,\infty); X)$-solutions of (VE).

Corollary 2.3. Let $\omega \geq \max\{\epsilon, \text{ abs}(f)\}$ and $M > 0$ be a constant. Then the following are equivalent.

(i) There exists a solution $v \in Lip_\omega([0,\infty); X)$ of (VE) with $\|v\|_{Lip_\omega} \leq M$.

(ii) There exists a function $y \in C_W^\infty((\omega,\infty); X)$ for which $\|y\|_{W,\omega} \leq M$, $\widehat{d\mu}(\lambda)y(\lambda) \in D(A)$, and $(I - \widehat{d\mu}(\lambda)A)y(\lambda) = \lambda\hat{f}(\lambda)$ for every $\lambda \in \mathbb{C}_\omega$.

(iii) There exists a function $y \in C_W^\infty((\omega,\infty);X)$ for which $\|y\|_{W,\omega} \le M$, $\widehat{d\mu}(k)y(k) \in D(A)$, and $(I - \widehat{d\mu}(k)A)y(k) = k\hat{f}(k)$ for every $k \in \mathbb{N}$ with $k > \omega$.

Proof. It follows from Widder's Theorem (Theorem 1.1.3) that $F \in Lip_\omega([0,\infty);X)$ if and only if $\widehat{dF} : \lambda \mapsto \widehat{dF}(\lambda) = \lambda\hat{F}(\lambda)$ is contained in $C_W^\infty((\omega,\infty);X)$ and that $\|F\|_{Lip_\omega} \le M$ if and only if $\|\widehat{dF}\|_{W,\omega} \le M$. To show (i) $\Longrightarrow$ (ii), suppose that (i) holds. Then, by Theorem 2.2, $\lambda\widehat{d\mu}(\lambda)\hat{v}(\lambda) \in D(A)$ and $(I - \widehat{d\mu}(\lambda)A)\lambda\hat{v}(\lambda) = \lambda\hat{f}(\lambda)$ for all $\lambda \in \mathbb{C}_\omega$. Hence, setting $y(\lambda) := \lambda\hat{v}(\lambda) = \widehat{dv}(\lambda)$ for every $\lambda \in \mathbb{C}_\omega$, we obtain (ii). Obviously, (ii) implies (iii). To show (iii) $\Longrightarrow$ (i), suppose that (iii) holds. Then there exists a $v \in Lip_\omega([0,\infty);X)$ such that $y(\lambda) = \widehat{dv}(\lambda) = \lambda\hat{v}(\lambda)$ for every $\lambda \in \mathbb{C}_\omega$. Thus, $(I - \widehat{d\mu}(k)A)\hat{v}(k) = \hat{f}(k)$ for all $k \in \mathbb{N}$ with $k > \omega$. Hence statement (i) follows from Theorem 2.2. //

It follows from Theorem 2.2 that the exponentially bounded solutions of the Volterra equation (VE) are unique if and only if for any $\omega \ge 0$, the equation

$$(I - \widehat{d\mu}(\lambda)A)y(\lambda) = 0 \qquad (\lambda > \omega)$$

has no nonzero solution y which has a Laplace representation $y(\lambda) = \hat{v}(\lambda)$ for some $v \in L^1_{\text{loc}}([0,\infty);X)$. Another uniqueness theorem is given by the following generalization of the results in [H-P] and [L] for the abstract Cauchy problem. In the following theorem a condition on the range of $\widehat{d\mu}$ and the point spectrum $\sigma_p(A)$ of A implies that the equation (VE) has at most one exponentially bounded solution.

Theorem 2.4. Suppose that there exists a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ in $\mathbb{C}_\epsilon$ such that Re $\lambda_k \longrightarrow \infty$ as $k \longrightarrow \infty$, and for which either $\widehat{d\mu}(\lambda_k) = 0$ or $\widehat{d\mu}(\lambda_k)^{-1} \notin \sigma_p(A)$ for all $k \in \mathbb{N}$. Then (VE) has at most one exponentially bounded solution.

Proof. It suffices to show that $v \equiv 0$ is the only exponentially bounded solution to the equation

$$v(t) = A\int_0^t v(t-s)d\mu(s) \qquad \text{for } t \geq 0.$$

Suppose that $v \in C([0,\infty);X)$ is an exponentially bounded solution of this equation. Then it follows from Theorem 2.2 that

$$\widehat{v}(\lambda) = \widehat{d\mu}(\lambda)A\widehat{v}(\lambda)$$

for all $\lambda \in \mathbb{C}_\omega$, where $\omega = \max\{\epsilon,\ \omega(v)\}$. We claim that $\widehat{v} \equiv 0$ on $\mathbb{C}_\omega$. Let us assume not. Notice that $\widehat{v}(\lambda_k) = \widehat{d\mu}(\lambda_k)A\widehat{v}(\lambda_k)$ for sufficiently large $k \in \mathbb{N}$ since Re $\lambda_k \longrightarrow \infty$ as $k \longrightarrow \infty$. Since either $\widehat{d\mu}(\lambda_k) = 0$ or $\widehat{d\mu}(\lambda_k)^{-1} \notin \sigma_p(A)$, it follows that $\widehat{v}(\lambda_k) = 0$ for all sufficiently large k. Then since $\widehat{v} \not\equiv 0$ on $\mathbb{C}_\omega$, λ_k is a zero of order m of the analytic function $\widehat{v}$ for some $m \in \mathbb{N}$, i.e., $\widehat{v}^{(j)}(\lambda_k) = 0$ for $0 \leq j \leq m-1$ and $\widehat{v}^{(m)}(\lambda_k) \neq 0$. Hence, by the closedness of A,

$$\widehat{v}^{(m)}(\lambda_k) = A(\widehat{d\mu}\cdot\widehat{v})^{(m)}(\lambda_k) = A\sum_{j=0}^{m}\binom{m}{j}\widehat{d\mu}^{(j)}(\lambda_k)\cdot\widehat{v}^{(m-j)}(\lambda_k) = \widehat{d\mu}(\lambda_k)A\widehat{v}^{(m)}(\lambda_k).$$

Since $\widehat{v}^{(m)}(\lambda_k) \neq 0$, it follows that $\widehat{d\mu}(\lambda_k)^{-1} \in \sigma_p(A)$, which is a contradiction. Hence $\widehat{v} \equiv 0$ on $\mathbb{C}_\omega$. Thus, by the Uniqueness Theorem (Corollary 1.1.4) and from the continuity of v we conclude that $v \equiv 0$ on $[0,\infty)$. //

Next, for $2 \leq n \in \mathbb{N}$, we consider the Volterra equation

$$v(t) = \sum_{j=1}^{n} A_j \int_0^t v(t-s)d\mu_j(s) + f(t) \qquad \text{for } t \geq 0. \tag{2.2}$$

We assume that A_j are closed linear operators in a Banach space X, that μ_j are functions $BV_\epsilon([0,\infty);\mathbb{C})$ for some constant $\epsilon \geq 0$ and all $j \in \mathbb{N}$ with $1 \leq j \leq n$, and that f is a Laplace transformable function in $C([0,\infty);X)$. Exponentially bounded solutions of (2.2) can be characterized as in Theorm 2.1.

Theorem 2.5. Let v be an exponentially bounded function in $C([0,\infty);X)$. Suppose that $(v * d\mu_j)(t) \in D(A_j)$ for every $t \geq 0$ and $\text{abs}(A_j(v * d\mu_j)) < \infty$ for every $j \in \mathbb{N}$ with $1 \leq j \leq n-1$ for some integer $n \geq 2$. Let ω be any number such that $\omega \geq \max\{\epsilon,\ \text{abs}(f),\ \omega(v),\ \text{abs}(A_j(v * d\mu_j)) \text{ for } j \in \mathbb{N} \text{ with } 1 \leq j \leq n-1\}$. Then the following are equivalent.

(i) v solves (2.2).

(ii) If $\lambda \in \mathbb{C}_\omega$, then $\widehat{d\mu_j}(\lambda)\widehat{v}(\lambda) \in D(A_j)$ for every $j \in \mathbb{N}$ with $1 \leq j \leq n$ and

$$(I - \sum_{j=1}^{n} \widehat{d\mu_j}(\lambda)A_j)\widehat{v}(\lambda) = \widehat{f}(\lambda).$$

(iii) If $\omega < k \in \mathbb{N}$, then $\widehat{d\mu}(k)\widehat{v}(k) \in D(A_j)$ for every $j \in \mathbb{N}$ with $1 \leq j \leq n$ and

$$(I - \sum_{j=1}^{n} \widehat{d\mu_j}(k)A_j)\widehat{v}(k) = \widehat{f}(k).$$

Proof. The proof is similar to that of Theorem 2.2. To show (i) $\Longrightarrow$ (ii), suppose that v is an exponentially bounded solution to (2.2). Since v is a solution of (2.2), it follows from the hypothesis that $\text{abs}(A_n(v * d\mu_n)) \leq \omega$. Hence, Laplace transforming the equation (2.2) for $\lambda \in \mathbb{C}_\omega$, by Proposition 1.2.7 and the closedness of the operators

A_j, the statement (ii) follows. The implication (ii) $\Longrightarrow$ (iii) is obvious. We show the implication (iii) $\Longrightarrow$ (i) by means of the Phragmén-Doetsch Inversion Formula (1.1.8). Suppose that (iii) holds. By Proposition 1.2.7, $\text{abs}(v * d\mu_j) \leq \omega$ for all $j \in \mathbb{N}$ with $1 \leq j \leq n$. Let $\omega' > \omega$. Then we observe that Proposition 1.2.7 and (1.1.4) imply that $v^{[2]}$, $f^{[2]}$, and $(v * d\mu_j)^{[2]} = v^{[2]} * d\mu_j$ for $j \in \mathbb{N}$ with $1 \leq j \leq n$ are all contained in $Lip_{\omega'}([0,\infty);X)$. It follows from (1.1.8) and (1.1.5) that $\widehat{v}(\lambda) = \lambda\widehat{dv^{[2]}}(\lambda)$, that $\widehat{f}(\lambda) = \lambda\widehat{df^{[2]}}(\lambda)$, and that $\widehat{v}(\lambda)\widehat{d\mu}(\lambda) = \widehat{v * d\mu_j}(\lambda) = \lambda(d(v * d\mu_j)^{[2]})^\wedge(\lambda)$ for $j \in \mathbb{N}$ with $1 \leq j \leq n$ for every $\lambda \in \mathbb{C}_{\omega'}$. Hence, from (iii) and these identities,

$$\widehat{dv^{[2]}}(k) - \widehat{df^{[2]}}(k) = \sum_{j=1}^{n} A_j(d(v * d\mu_j)^{[2]})^\wedge(k)$$

for every $k \in \mathbb{N}$ with $k > \omega'$. As in Theorem 2.2 it follows from the inversion formula (1.1.8) and the closedness of the operators A_j that for every $t \geq 0$,

$$v^{[2]}(t) - f^{[2]}(t) = \sum_{j=1}^{n} A_j(v * d\mu_j)^{[2]}(t).$$

Since A_j are closed and all the functions involved in this equation are twice differentiable, the statement (i) follows. //

The $Lip_\omega([0,\infty);X)$-solutions of (2.2) can be characterized as follows. For closed linear operators A_j in a Banach space X for $j \in \mathbb{N}$, $1 \leq j \leq n$, $[\bigcap_{j=1}^{n} D(A_j)]$ denotes the Banach space $\bigcap_{j=1}^{n} D(A_j)$ equipped with the graph norm $\|x\|_G = \|x\| + \|A_1 x\| + \cdots + \|A_n x\|$ for $x \in \bigcap_{j=1}^{n} D(A_j)$.

Corollary 2.6. Let $\omega \geq \max\{\epsilon, \text{abs}(f)\}$ and $M > 0$. Then the following are equivalent.

(i) There exists a solution $v \in Lip_\omega([0,\infty);[\bigcap_{j=1}^{n-1} D(A_j)])$ of (2.2) with $\|v\|_{Lip_\omega} \le M$.

(ii) There exists a function $y \in C_W^\infty((\omega,\infty);[\bigcap_{j=1}^{n-1} D(A_j)])$ with $\|y\|_{W,\omega} \le M$ such that $\widehat{d\mu_n}(\lambda)y(\lambda) \in D(A_n)$ and $(I - \sum_{j=1}^{n} \widehat{d\mu_j}(\lambda)A_j)y(\lambda) = \lambda\hat{f}(\lambda)$ for all $\lambda \in \mathbb{C}_\omega$.

(iii) There exists a function $y \in C_W^\infty((\omega,\infty);[\bigcap_{j=1}^{n-1} D(A_j)])$ with $\|y\|_{W,\omega} \le M$ such that $\widehat{d\mu_n}(k)y(k) \in D(A_n)$ and $(I - \sum_{j=1}^{n} \widehat{d\mu_j}(k)A_j)y(k) = k\hat{f}(k)$ for all $k \in \mathbb{N}$ with $k > \omega$.

Proof. It follows from Widder's Theorem that $F \in Lip_\omega([0,\infty);[\bigcap_{j=1}^{n-1} D(A_j)])$ if and only if $\widehat{dF} : \lambda \mapsto \widehat{dF}(\lambda) = \lambda\hat{F}(\lambda)$ is contained in $C_W^\infty((\omega,\infty);[\bigcap_{j=1}^{n-1} D(A_j)])$ and that $\|F\|_{Lip_\omega} \le M$ if and only if $\|\widehat{dF}\|_{W,\omega} \le M$. To show (i) $\Longrightarrow$ (ii), suppose that (i) holds. Since $v \in Lip_\omega([0,\infty);[\bigcap_{j=1}^{n-1} D(A_j)])$, it holds that $\omega(v) \le \omega$ and $A_j v \in Lip_\omega([0,\infty);X)$ for $j \in \mathbb{N}$ with $1 \le j \le n-1$. Hence, by the closedness of A_j and by Proposition 1.2.7, we obtain $\text{abs}(A_j(v * d\mu_j)) = \text{abs}((A_j v) * d\mu_j) \le \omega$ for all $j \in \mathbb{N}$ with $1 \le j \le n-1$. Hence, from Theorem 2.5 and by the facts from Widder's Theorem mentioned at the beginning, it follows that $\widehat{d\mu_n}(\lambda)\hat{v}(\lambda) \in D(A_n)$ for all $\lambda \in \mathbb{C}_\omega$ and that $y := \hat{v} \in C_{W,\omega}^\infty([0,\infty);[\bigcap_{j=1}^{n-1} D(A_j)])$ with $\|y\|_{W,\omega} \le M$. Also, $(I - \sum_{j=1}^{n} \widehat{d\mu_j}(\lambda)A_j)y(\lambda) = \lambda\hat{f}(\lambda)$ for all $\lambda \in \mathbb{C}_\omega$. The implication (ii) $\Longrightarrow$ (iii) is obvious. To show (iii) $\Longrightarrow$ (i), suppose that (iii) holds. Then there exists $v \in Lip_\omega([0,\infty);[\bigcap_{j=1}^{n-1} D(A_j)])$ such that $y(\lambda) = \widehat{dv}(\lambda) = \lambda\hat{v}(\lambda)$ for every $\lambda \in \mathbb{C}_\omega$. Since $v \in Lip_\omega([0,\infty);[\bigcap_{j=1}^{n-1} D(A_j)])$ and since the operators A_j are closed, it follows from Proposition 1.2.7 that $\text{abs}(A_j(v*d\mu_j)) = \text{abs}((A_j v)*d\mu_j) \le \omega$ for all $j \in \mathbb{N}$ with $1 \le j \le n-1$. Hence, $\widehat{d\mu_j}(k)\hat{v}(k) \in D(A_j)$ for all $k \in \mathbb{N}$ with $k > \omega$ and for every $j \in \mathbb{N}$ with $1 \le j \le n$ and $(I - \sum_{j=1}^{n} \widehat{d\mu_j}(k)A_j)\hat{v}(k) = \hat{f}(k)$ for every $k \in \mathbb{N}$ with $k > \omega$. Thus, by Theorem 2.5, the statement (i) follows. //

CHAPTER 3 SOLUTION OPERATOR FAMILIES

This chapter is built on the discussion of Volterra equation

$$v(t) = A\int_0^t v(t-s)d\mu(s) + \frac{t^n}{n!}x \qquad \text{for } t \geq 0 \text{ and } x \in X, \tag{3.1}$$

where A is a closed linear operator in a complex Banach space X, $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$, and $n \in \mathbb{N}_0$. The equation (3.1) can be obtained by integrating the equation

$$u(t) = A\int_0^t u(t-s)d\mu(s) + x$$

n-times and setting $v(t) := u^{[n]}(t)$. For this reason we call (3.1) an integrated Volterra equation. Suppose that (3.1) has a unique exponentially bounded solution $v(\cdot) = v(\cdot, x)$ for every $x \in X$. For every $t \geq 0$, define a linear operator $S(t) : X \to X$ by $S(t)x := v(t,x)$. It follows from Theorem 2.2 that

$$(I - \widehat{d\mu}(\lambda)A)\widehat{v}(\lambda) = \frac{1}{\lambda^{n+1}}x$$

for all $\lambda > \omega$ for some number $\omega \geq \epsilon$. Assuming that $(I - \widehat{d\mu}(\lambda)A)^{-1}$ exists in $L(X)$ for $\lambda > \omega$, we obtain that

$$(I - \widehat{d\mu}(\lambda)A)^{-1}x = \lambda^{n+1}\int_0^\infty e^{-\lambda t}v(t)dt = \lambda^{n+1}\int_0^\infty e^{-\lambda t}S(t)xdt.$$

This observation motivates us to investigate those operators A for which there exist a constant $\omega \geq \epsilon$ and a strongly continuous, exponentially bounded family $\{S(t)\}_{t\geq 0}$ of operators in $L(X)$ for which $(I - \widehat{d\mu}(\lambda)A)^{-1} \in L(X)$ and

$$(I - \widehat{d\mu}(\lambda)A)^{-1}x = \lambda^{n+1}\int_0^\infty e^{-\lambda t}S(t)xdt = \lambda^{n+1}\widehat{S}(\lambda)x$$

for all $x \in X$ for all $\lambda > \omega$.

Section 3.1 deduces some properties of the operator family $\{S(t)\}_{t\geq 0}$, which is called the integrated solution operator family with generator (A, μ). We will show that the Volterra equation

$$v(t) = A \int_0^t v(t-s)d\mu(s) + f(t) \tag{VE}$$

has unique exponentially bounded solutions v for sufficiently regular functions f if (A, μ) generates an integrated solution operator family. Most results in Section 3.1 are generalization or modifications of results in [A-H-N] and [A-K]. Section 3.2 investigates the wellposedness of the equation (3.1) and its stronger version

$$v(t) = \int_0^t Av(t-s)d\mu(s) + \frac{t^n}{n!}x \qquad \text{for } t \geq 0 \text{ and } x \in D(A).$$

A function $v \in C([0, \infty); [D(A)])$ which satisfies the equation

$$v(t) = \int_0^t Av(t-s)d\mu(s) + f(t) \qquad \text{for } t \geq 0 \tag{3.2}$$

is called a strong solution of (VE). Section 3.3 shows Trotter-Kato type approximation theorems for stable sequences of integrated solution operator families. Section 3.4 shows some properties of analytic integrated solution operator family. Finally, Section 3.5 discusses an elementary example in terms of solution operator families.

3.1 Integrated Solution Operator Families

In this section we study integrated solution operator families associated with Volterra equations (VE). In a sense this is a generalization of the method of integrated semigroups, which has been used successfully to study the abstract Cauchy problem $u'(t) = Au(t)$, $u(0) = x$ (see for example, [A], [N], or [A-H-N]). We assume throughout that A is a closed linear operator with its domain $D(A)$ and range in a complex Banach space X and $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$.

Definition 3.1.1. Let $n \in \mathbb{N}_0$. Let $M > 0$ and $\omega \geq \epsilon$ be some constants. Suppose that $S : [0,\infty) \to L(X)$ is a strongly continuous mapping which satisfies the following.

(i) $\|S(t)\| \leq M\mathrm{e}^{\omega t}$ for all $t \geq 0$.

(iii) For every $\lambda > \omega$, $(I - \widehat{d\mu}(\lambda)A)^{-1}$ exists in $L(X)$ and

$$(I - \widehat{d\mu}(\lambda)A)^{-1}x = \lambda^{n+1}\widehat{S}(\lambda)x = \lambda^{n+1}\int_0^\infty \mathrm{e}^{-\lambda t}S(t)x\,dt \quad \text{for all } x \in X. \quad (3.1.1)$$

Then S is called the n-times integrated solution operator family (of exponential type $(M;\omega)$) with generator (A,μ). A 0-times integrated solution operator family is simply called a solution operator family.

Remark 3.1.2.

(i) If (3.1.1) holds for all $\lambda > \omega$, then it holds for all $\lambda \in \mathbb{C}_\omega$, i.e., if (A,μ) generates an integrated solution operator family, then $(I - \widehat{d\mu}(\lambda)A)^{-1}$ exists in $L(X)$ and (3.1.1) holds for every $\lambda \in \mathbb{C}_\omega$. This will be shown in Lemma 3.1.7.

(ii) It follows from (3.1.1) and the Uniqueness Theorem (Corollary 1.1.4) that for each $n \in \mathbb{N}_0$, every pair (A,μ) generates at most one n-times integrated solution operator family.

(iii) An n-times integrated solution operator family with generator (A,μ) where $\mu(s) = s$, is the n-times integrated semigroup generated by A.

(iv) If S is the n-times integrated solution operator family of exponential type $(M;\omega)$ with generator (A,μ), then $S^{[1]} \in Lip_\omega([0,\infty);L(X))$. From (1.1.5), $\lambda\widehat{S^{[1]}}(\lambda)x = \widehat{S}(\lambda)x$ for all $\lambda > \omega$ and $x \in X$. Hence $\{S^{[1]}(t)\}_{t\geq 0}$ is the $(n+1)$-times integrated, norm Lipschitz continuous solution operator family with generator (A,μ), and

$$(I-\widehat{d\mu}(\lambda)A)^{-1} = \lambda^{n+2}\int_0^\infty e^{-\lambda t}S^{[1]}(t)dt = \lambda^{n+1}\oint_0^\infty e^{-\lambda t}dS(t)$$

for all $\lambda > \omega$, where the Bochner and Riemann-Stieltjes integral is taken in the operator norm.

Lemma 3.1.3. Let S be an integrated solution operator family with generator (A,μ). Then $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$ for every $t \geq 0$ and every $x \in D(A)$.

Proof. Let S be the n-times integrated solution operator family of exponential type $(M;\omega)$ with generator (A,μ). Without loss of generality we may assume that $\widehat{d\mu} \not\equiv 0$ on (ω,∞). Let λ, $\nu > \omega$ with $\widehat{d\mu}(\nu) \neq 0$. Let $x \in D(A)$. Then $x = (I-\widehat{d\mu}(\nu)A)^{-1}z$ for some $z \in X$. Since $(I-\widehat{d\mu}(\lambda)A)^{-1}$ and $(I-\widehat{d\mu}(\nu)A)^{-1}$ are bounded and commute, and since A is closed,

$$\begin{aligned}\int_0^\infty e^{-\lambda t}S(t)xdt &= \int_0^\infty e^{-\lambda t}S(t)(I-\widehat{d\mu}(\nu)A)^{-1}zdt\\ &= \frac{(I-\widehat{d\mu}(\lambda)A)^{-1}}{\lambda^{n+1}}\left[(I-\widehat{d\mu}(\nu)A)^{-1}z\right]\\ &= (I-\widehat{d\mu}(\nu)A)^{-1}\left[\frac{(I-\widehat{d\mu}(\lambda)A)^{-1}}{\lambda^{n+1}}z\right]\\ &= (I-\widehat{d\mu}(\nu)A)^{-1}\int_0^\infty e^{-\lambda t}S(t)xdt\\ &= \int_0^\infty e^{-\lambda t}(I-\widehat{d\mu}(\nu)A)^{-1}S(t)zdt.\end{aligned}$$

Hence, by Corollary 1.1.4, $S(t)x = (I - \widehat{d\mu}(\nu)A)^{-1}S(t)(I - \widehat{d\mu}(\nu)A)x$ for almost all $t \geq 0$ and thus, $(I - \widehat{d\mu}(\nu)A)S(t)x = S(t)(I - \widehat{d\mu}(\nu)A)x$ for almost all $t \geq 0$. Since $\widehat{d\mu}(\nu) \neq 0$ and S is strongly continuous, we conclude that $AS(t)x = S(t)Ax$ for every $t \geq 0$ and every $x \in D(A)$. //

The following lemma shows that if (A, μ) generates an n-times integrated solution operator family S, then $t \mapsto S(t)x$ is a solution of (3.1) for every $x \in X$ and a solution of (3.2) for every $x \in D(A)$.

Lemma 3.1.4. Let S be an n-times integrated solution operator family with generator (A, μ) for some $n \in \mathbb{N}_0$. Then

$$S(t)x = \int_0^t S(t-s)Axd\mu(s) + \frac{t^n}{n!}x \qquad \text{for every } t \geq 0 \text{ and } x \in D(A), \tag{3.1.2}$$

$\int_0^t S(t-s)xd\mu(s) \in D(A)$, and

$$S(t)x = A\int_0^t S(t-s)xd\mu(s) + \frac{t^n}{n!}x \qquad \text{for every } t \geq 0 \text{ and } x \in X. \tag{3.1.3}$$

Proof. Let S be an n-times integrated solution operator family of exponential type $(M;\omega)$ with generator (A, μ). Let $\lambda > \omega$ and let $x \in D(A)$. We obtain from (3.1.1) and Proposion 1.2.7 that

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\frac{t^n}{n!}x\,dt &= \frac{1}{\lambda^{n+1}}x \\ &= \widehat{S}(\lambda)(I - \widehat{d\mu}(\lambda)A)x \\ &= \int_0^\infty e^{-\lambda t}\Big[S(t)x - \int_0^t S(t-s)Axd\mu(s)\Big]dt.\end{aligned}$$

The Uniqueness Theorem (Corollary 1.1.4) and the strong continuity of S yield the equation (3.1.2). It follows from Theorem 2.2 and (3.1.1) that the function $t \mapsto S(t)x$ satisfies (3.1.3) for every $t \geq 0$ and $x \in X$. //

The following is an immediate consequence of (3.1.3).

Corollary 3.1.5. Let S be an n-times integrated solution operator family. Then $S(0) = I$ if $n = 0$ and $S(0) = 0$ if $n \in \mathbb{N}$.

In fact, the properties of an integrated solution operator family derived in Lemma 3.1.3 and Lemma 3.1.4 characterize those exponentially bounded, strongly continuous operator families S in $L(X)$ which are integrated solution operator families.

Proposition 3.1.6. Let S be a strongly continuous operator family in $L(X)$ for which there exist constants $M > 0$ and $\omega \geq \epsilon$ such that $\|S(t)\| \leq Me^{\omega t}$. Let $n \in \mathbb{N}_0$. Then S is an n-times integrated solution operator family with generator (A, μ) if and only if S satisfies the following.

(i) $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$ for every $t \geq 0$ and $x \in D(A)$.

(ii) $S(t)x = A\int_0^t S(t-s)x\, d\mu(s) + \frac{t^n}{n!}x$ for every $t \geq 0$ and $x \in X$.

Proof. To show that (i) and (ii) are sufficient conditions for S to be an n-times integrated solution operator family, suppose that S satisfies (i) and (ii). For $x \in D(A)$ the functions $t \mapsto S(t)x$ and $t \mapsto AS(t) = S(t)Ax$ are continuous on $[0, \infty)$. Thus, by the closedness of A, we obtain that $A\int_0^t S(t-s)xd\mu(s) = \int_0^t S(t-s)Axd\mu(s)$.

Therefore,

$$S(t)x = \int_0^t S(t-s)Axd\mu(s) + \frac{t^n}{n!}x$$

for every $t \geq 0$ and $x \in D(A)$. For $\lambda > \omega$, take the Laplace transforms of this equation and the one in (ii). By Proposition 1.2.7, $\widehat{S}(\lambda)(I - \widehat{d\mu}(\lambda)A)x = \frac{1}{\lambda^{n+1}}x$ for all $x \in D(A)$. Moreover, by Theorem 2.2 it follows that $\widehat{d\mu}(\lambda)\widehat{S}(\lambda)x \in D(A)$ and $(I - \widehat{d\mu}(\lambda)A)\widehat{S}(\lambda)x = \frac{1}{\lambda^{n+1}}x$ for all $x \in X$. Hence $(I - \widehat{d\mu}(\lambda)A)^{-1}$ exist in $L(X)$ and $(I - \widehat{d\mu}(\lambda)A)^{-1}x = \lambda^{n+1}\widehat{S}(\lambda)x$ for all $\lambda > \omega$ and $x \in X$. Thus, S is the n-times integrated solution operator family with generator (A, μ). //

Lemma 3.1.7 Suppose that (A, μ) is a generator of an n-times integrated solution operator family S of exponential type $(M; \omega)$. Then

$$\frac{1}{\lambda^{n+1}}(I - \widehat{d\mu}(\lambda)A)^{-1}x = \widehat{S}(\lambda)x \qquad \text{for all } x \in X \tag{3.1.4}$$

holds for every $\lambda \in \mathbb{C}_\omega$.

Proof. By the definition of an n-times integrated solution operator family, (3.1.4) holds for every $\lambda > \omega$. It follows from elementary Laplace transform theory that $\widehat{d\mu}$ and $\widehat{S}$ exist on $\mathbb{C}_\omega$. Let $\lambda \in \mathbb{C}_\omega$. From the equation (3.1.3), the closedness of A, and Proposition 1.2.7, it follows that for every $x \in X$,

$$\begin{aligned}
\widehat{S}(\lambda)x - \frac{1}{\lambda^{n+1}}x &= \int_0^\infty e^{-\lambda t}S(t)xdt - \int_0^\infty e^{-\lambda t}\frac{t^n}{n!}xdt \\
&= A\int_0^\infty e^{-\lambda t}\int_0^t S(t-s)xd\mu(s)dt \\
&= \widehat{d\mu}(\lambda)A\widehat{S}(\lambda)x.
\end{aligned}$$

Similarly, from the equation (3.1.2) and Proposition 1.2.7, it follows that for every $x \in D(A)$,

$$\begin{aligned}\widehat{S}(\lambda)x - \frac{1}{\lambda^{n+1}}x &= \int_0^\infty e^{-\lambda t}S(t)xdt - \int_0^\infty e^{-\lambda t}\frac{t^n}{n!}xdt \\ &= \int_0^\infty e^{-\lambda t}\int_0^t S(t-s)Axd\mu(s)dt \\ &= \widehat{S}(\lambda)\widehat{d\mu}(\lambda)Ax.\end{aligned}$$

Hence we conclude that $(I - \widehat{d\mu}(\lambda)A)^{-1}$ exists in $L(X)$ and that (3.1.4) holds for all $\lambda \in \mathbb{C}_\omega$. //

Corollary 3.1.8. Let A be an unbounded, closed linear operator in a Banach space X and $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$. Assume that there exists a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ in $\mathbb{C}_\omega$ for which $\widehat{d\mu}(\lambda_k) = 0$ for all $k \in \mathbb{N}$ and Re $\lambda_k \longrightarrow \infty$ as $k \longrightarrow \infty$. Then (A,μ) does not generate an integrated solution operator family.

Proof. Assume that (A,μ) generates an n-times integrated solution operator family of exponential type $(M;\omega)$ for some constants $M > 0$, $\omega \geq \epsilon$, and $n \in \mathbb{N}_0$. Then by Lemma 3.1.7, the function $G(\lambda) := (I - \widehat{d\mu}(\lambda)A)^{-1}$ is analytic on $\mathbb{C}_\omega$. By assumption, there exists a $\lambda_k \in \mathbb{C}_\omega$ with $\widehat{d\mu}(\lambda_k) = 0$. Since the zeros of a nonzero Laplace-Stieltjes transform have no limit point, a small circle Γ around λ_k in $\mathbb{C}_\omega$ such that $\widehat{d\mu}$ has no zero in and on Γ can be chosen. Applying the Cauchy Integral Formula to the function G, it follows from the closedness of A that

$$\begin{aligned}Ax = AG(\lambda_k)x = A\frac{1}{2\pi i}\int_\Gamma \frac{G(\lambda)}{\lambda-\lambda_k}d\lambda &= \frac{1}{2\pi i}\int_\Gamma \frac{A(I-\widehat{d\mu}(\lambda)A)^{-1}x}{\lambda-\lambda_k}d\lambda \\ = \frac{1}{2\pi i}\int_\Gamma \frac{\widehat{d\mu}(\lambda)A(I-\widehat{d\mu}(\lambda)A)^{-1}x}{(\lambda-\lambda_k)\widehat{d\mu}(\lambda)}d\lambda &= \frac{1}{2\pi i}\int_\Gamma \frac{(I-\widehat{d\mu}(\lambda)A)^{-1}x - x}{(\lambda-\lambda_k)\widehat{d\mu}(\lambda)}d\lambda\end{aligned}$$

for all $x \in D(A)$. Thus, for some constant $M_1 > 0$, $\|Ax\| \leq M_1\|x\|$ for all $x \in D(A)$ which contradicts the unboundedness of A. //

If (A, μ) generates an n-times integrated solution operator family of exponential type $(M; \omega)$, then it follows from (3.1.4) that

$$\|(I - \widehat{d\mu}(\lambda)A)^{-1}\| \leq \frac{M|\lambda|^{n+1}}{\text{Re }\lambda - \omega} \qquad \text{for all } \lambda \in \mathbb{C}_\omega.$$

Observe that for any $\omega' > \omega$, the set $\{\frac{1}{\text{Re }\lambda-\omega} \mid \lambda \in \mathbb{C}_{\omega'}\}$ is bounded. From this we conclude that if (A, μ) generates an integrated solution operator family, then there exist constants $M > 0$, $\omega \geq \epsilon$, and $a \geq 0$ such that $(I - \widehat{d\mu}(\lambda)A)^{-1}$ exists in $L(X)$ and $\|(I - \widehat{d\mu}(\lambda)A)^{-1}\| \leq M|\lambda|^a$ for all $\lambda \in \mathbb{C}_\omega$. In fact, this polynomial boundedness of the operator $(I - \widehat{d\mu}(\lambda)A)^{-1}$ on a right half plane is a necessary and sufficient condition for a pair (A, μ) to generate an integrated solution operator family. The following is a modification of Theorem 3.2 in [A-K].

Theorem 3.1.9 A pair (A, μ) generates an integrated solution operator family if and only if there exist constants $M > 0$, $\omega \geq \epsilon$, and $a \geq 0$ such that $(I - \widehat{d\mu}(\lambda)A)^{-1} \in L(X)$ with $\|(I - \widehat{d\mu}(\lambda)A)^{-1}\| \leq M|\lambda|^a$ for all $\lambda \in \mathbb{C}_\omega$.

Proof. It is left to show that the conditions are sufficient for (A, μ) to generate an integrated solution operator family. Suppose that there exist constants $M > 0$, $\omega \geq \epsilon$, and $a \geq 0$ for which $(I - \widehat{d\mu}(\lambda)A)^{-1}$ exist in $L(X)$ and $\|(I - \widehat{d\mu}(\lambda)A)^{-1}\| \leq M|\lambda|^a$ for all $\lambda \in \mathbb{C}_\omega$. Then $\|\frac{\lambda(I-\widehat{d\mu}(\lambda)A)^{-1}}{\lambda^{a+1}}\| \leq M$ for all $\lambda \in \mathbb{C}_\omega$. Let n be an integer such that $n > a$ and let $b := n - a > 0$ for $\lambda \in \mathbb{C}_\omega$. Then $n \geq 1$, and by Theorem 1.1.7, there exists a function $S \in C([0,\infty); L(X))$ and a constant $C > 0$ such that $\|S(t)\| \leq Ct^b e^{\omega t}$ for all $t \geq 0$ and $\frac{(I-\widehat{d\mu}(\lambda)A)^{-1}}{\lambda^{a+1}} = \lambda^b \int_0^\infty e^{-\lambda t} S(t)dt$ for all $\lambda \in \mathbb{C}_\omega$. Hence, $\frac{(I-\widehat{d\mu}(\lambda)A)^{-1}}{\lambda^{n+1}} = \int_0^\infty e^{-\lambda t} S(t)dt$ for all $\lambda \in \mathbb{C}_\omega$. Therefore, (A,μ) generates an n-times integrated solution operator family. //

Analogously to integrated semigroups, a Hille-Yosida type characterization of an integrated solution operator family with generator (A,μ) is possible if A is a densely defined closed linear operator and μ is absolutely continuous in $BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$. The following is modified from and improves Theorem 2.2 in [A-K].

Theorem 3.1.10. Suppose that A is a densely defined, closed linear operator in a Banach space X and that μ is an absolutely continuous function in $BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$. Let $n \in \mathbb{N}_0$, $M > 0$, and $\omega \geq \epsilon$. Then the following are equivalent.

(i) The pair (A,μ) generates an n-times integrated solution operator family of exponential type $(M;\omega)$.

(ii) For every $\lambda > \omega$, $(I - \widehat{d\mu}(\lambda)A)^{-1} \in L(X)$ and the function $H : (\omega,\infty) \to L(X)$ defined by $H(\lambda) = \frac{1}{\lambda^{n+1}}(I - \widehat{d\mu}(\lambda)A)^{-1}$ is contained in $C^\infty((\omega,\infty); L(X))$ and satisfies the estimates

$$\left\|\frac{H^{(k)}(\lambda)}{k!}\right\| \leq \frac{M}{(\lambda-\omega)^{k+1}} \qquad \text{for all } k \in \mathbb{N}_0 \text{ and } \lambda > \omega. \tag{3.1.5}$$

Proof. To show (i) $\Longrightarrow$ (ii), suppose that S is the n-times integrated solution operator family of exponential type $(M;\omega)$ with generator (A,μ). Let $\lambda > \omega$, $x \in X$, and

$k \in \mathbb{N}_0$. From (3.1.1) and Remark 3.1.2 (iv), $\left[\frac{1}{\lambda^{n+1}}(I - \widehat{d\mu}(\lambda)A)^{-1}\right]x = \widehat{S}(\lambda)x = \left[\int_0^\infty e^{-\lambda t} dS^{[1]}(t)\right]x$ for all $x \in X$. By Widder's Theorem, $H \in C_W^\infty((\omega,\infty); L(X))$. This proves (3.1.5). To show (ii) $\Longrightarrow$ (i), suppose that (ii) holds. Then H is contained in $C_W^\infty((\omega,\infty); L(X))$. Hence, by Widder's Theorem, there exists $T \in Lip_\omega([0,\infty); L(X))$ such that $\widehat{dT} = H$ on (ω, ∞) and $\|T\|_{Lip_\omega} = \|H\|_{W,\omega} \leq M$. From (1.1.6),

$$\lambda^{n+2}\widehat{T}(\lambda) = (I - \widehat{d\mu}(\lambda)A)^{-1} \qquad \text{for } \lambda > \omega.$$

Hence, T is the $(n+1)$-times integrated solution operator family with generator (A,μ). We obtain from Lemma 3.1.4 that $T(0) = 0$ and

$$T(t)x = \int_0^t T(t-s)Ax\,d\mu(s) + \frac{t^{n+1}}{(n+1)!}x$$

for all $t \geq 0$ and all $x \in D(A)$. Let $g(t) := T(t)Ax$. Then

$$T(t)x = \int_0^t g(t-s)\mu'(s)ds + \frac{t^{n+1}}{(n+1)!}x = \int_0^t \mu'(t-s)dg^{[1]}(s) + \frac{t^{n+1}}{(n+1)!}x.$$

It follows from Lemma 1.2.10 that $t \mapsto T(t)x$ is continuously differentiable and that

$$\frac{dT(t)x}{dt} = \int_0^t \mu'(t-s)dg(s) + \frac{t^n}{n!}x = \int_0^t \mu'(t-s)dT(s)Ax + \frac{t^n}{n!}x$$

for all $x \in D(A)$. Next, we show that $T(\cdot)x$ is differentiable for all $x \in X$. Let $t \geq 0$. Since $T \in Lip_\omega([0,\infty); L(X))$, the difference quotients $D_h := \frac{T(t+h)-T(t)}{h}$ are uniformly bounded for h with $0 < |h| \leq 1$ and $t + h \geq 0$. Since $\lim_{h\to 0} D_h x$ exists for $x \in D(A)$, we obtain from the Banach-Steinhaus Theorem (Theorem 1.1.11) that there exist operators $S(t) \in L(X)$ such that $S(t)x = \lim_{h\to 0} D_h x = \lim_{h\to 0} \frac{T(t+h)x - T(t)x}{h}$ for all $x \in X$. Notice that S is of exponential type $(M;\omega)$. To prove the strong continuity of the operator family

S, let $x \in X$, $\epsilon > 0$, and $z \in D(A)$ with $\|x - z\| < \epsilon$. It follows from $\|S(t)x - S(t_0)x\| \leq \|S(t)\|\ \|x - z\| + \|S(t)z - S(t_0)z\| + \|S(t_0)\|\ \|x - z\|$, the continuity of $t \mapsto S(t)z$ for $z \in D(A)$, and the exponential boundedness of S that $t \mapsto S(t)x$ is continuous on $[0, \infty)$. Thus, the operator family T is strongly differentiable on $[0, \infty)$ and $\frac{dT(t)x}{dt} = S(t)x$ for all $t \geq 0$ and $x \in X$. Therefore,

$$(I - \widehat{d\mu}(\lambda)A)^{-1}x = \lambda^{n+2} \int_0^\infty e^{-\lambda t}T(t)x dt = \lambda^{n+1} \int_0^\infty e^{-\lambda t}S(t)x dt$$

for all $x \in X$ and $\lambda > \omega$. This shows that S is an n-times integrated solution operator family with generator (A, μ). //

For $n = 0$, Theorem 3.1.10 includes the Generation Theorem in [Pr] (Theorem 1.3) which was proved first by G. Da. Prato and M. Ianelli in 1980 ([Da P-I]). To see this let $a \in L^1_{\text{loc}}([0, \infty); \mathbb{C})$ such that $\int_0^\infty e^{-\omega t}|a(t)|dt < \infty$ (which is the condition on the kernel given in [Pr]). Then the function $t \mapsto \mu(t) := \int_0^t a(s)ds$ is absolutely continuous and by (1.1.4), $\mu \in BV_{\omega'}([0, \infty); \mathbb{C})$ for all $\omega' > \omega$.(see Section 1.1).

Theorem 3.1.11. Suppose that (A, μ) generates an n-times integrated solution operator family S for some $n \in \mathbb{N}_0$. Let f be a Laplace transformable function in $C([0, \infty); X)$. Define $w(t) := \int_0^t S(t - s)f(s)ds$ for every $t \geq 0$. If $w \in C^{n+1}([0, \infty); X)$, then $w^{(n+1)}$ is a solution of (VE). If $v \in C([0, \infty); X)$ is a solution of (VE), then $w \in C^{n+1}([0, \infty); X)$ and $v = w^{(n+1)}$.

Proof. Let $t \geq 0$ and $0 \leq s \leq t$. Since S is an n-times integrated solution operator family with generator (A,μ), it follows from (3.1.3) that

$$S(t-s)f(s) = A\int_0^{t-s} S(s-t-r)f(s)d\mu(r) + \frac{(t-s)^n}{n!}f(s).$$

Suppose that $t \mapsto w(t) := \int_0^t S(t-s)f(s)ds$ is a $C^{n+1}([0,\infty);X)$-function. Then it follows from (3.1.3), the closedness of A, and Lemma 1.2.5 that

$$\begin{aligned} w(t) &= \int_0^t \mathrm{S(t-s)f(s)ds} \\ &= A\int_0^t\int_0^{t-s} S(t-s-r)f(s)d\mu(r)ds + \int_0^t \frac{(t-s)^n}{n!}f(s)ds \\ &= A\int_0^t\int_0^{t-r} S(t-r-s)f(s)dsd\mu(r) + f^{[n+1]}(t) \\ &= A\int_0^t w(t-r)d\mu(r) + f^{[n+1]}(t). \end{aligned}$$

If $w \in C^{n+1}([0,\infty);\mathbb{C})$, then the closedness of A and Corollary 1.2.9 imply that

$$w^{(n+1)}(t) = A\int_0^t w^{(n+1)}(t-r)d\mu(r) + f(t).$$

For the converse, suppose that v is a solution of (VE). Since (A,μ) generates the n-times integrated solution operator family S, it follows from (3.1.3) that

$$\frac{(t-s)^n}{n!}v(s) = S(t-s)v(s) - A\int_0^{t-s} S(t-s-r)v(s)d\mu(r).$$

Hence

$$\begin{aligned} v^{[n+1]}(t) &= \int_0^t \frac{(t-s)^n}{n!}v(s)ds \\ &= \int_0^t S(t-s)v(s)ds - A\int_0^t\int_0^{t-s} S(t-s-r)v(s)d\mu(r)ds. \end{aligned} \tag{3.1.8}$$

By Lemma 1.2.5 and by change of variables,

$$\begin{aligned} A\int_0^t\int_0^{t-s} S(t-s-r)v(s)d\mu(r)ds &= A\int_0^t\int_0^{t-r} S(t-s-r)v(s)dsd\mu(r) \\ &= A\int_0^t\int_r^t S(t-s)v(s-r)dsd\mu(r). \end{aligned}$$

We claim that

$$\int_0^t \int_r^t S(t-s)v(s-r)ds d\mu(r) = \int_0^t S(t-s) \int_0^s v(s-r)d\mu(r)ds.$$

To prove this, let Ω be the region $\{(r,s) \in \mathbb{R}^2 \mid 0 \le r \le s \le t\}$. Define $g : [0,t]\times[0,t] \to X$ by

$$g(r,s) := \begin{cases} S(t-s)v(s-r) & \text{if } (r,s) \in \Omega \\ S(t-s)v(0) & \text{otherwise.} \end{cases}$$

Then g is continuous on $[0,t] \times [0,t]$. Hence, by Corollary 1.2.3, it follows that

$$\int_0^t \int_r^t S(t-s)v(s-r)ds d\mu(r) + \int_0^t \int_0^r S(t-s)v(0)ds d\mu(r)$$
$$= \int_0^t \int_0^s S(t-s)v(s-r)d\mu(r)ds + \int_0^t \int_s^t S(t-s)v(0)d\mu(r)ds.$$

It suffices to show that

$$\int_0^t \int_0^r S(t-s)v(0)ds d\mu(r) = \int_0^t \int_s^t S(t-s)v(0)d\mu(r)ds.$$

Since

$$\int_0^t \left[\int_0^r S(t-s)v(0)ds\right] d\mu(r)$$
$$= \int_0^t S(t-s)v(0)ds\mu(t) - \int_0^t \mu(r)d\left[\int_0^r S(t-s)v(0)ds\right]$$
$$= \int_0^t S(t-s)v(0)ds\mu(t) - \int_0^t \mu(r)S(t-r)v(0)dr,$$

and

$$\int_0^t \int_s^t S(t-s)v(0)d\mu(r)ds = \int_0^t S(t-s)v(0)\left[\mu(t) - \mu(s)\right]ds$$
$$= \int_0^t S(t-s)v(0)ds\mu(t) - \int_0^t S(t-s)v(0)\mu(s)ds,$$

the claim holds. Since $\int_0^t v(t-r)d\mu(r) \in D(A)$, and since $S(t)$ and A commute on the domain of A, it follows that

$$A\int_0^t \int_r^t S(t-s)v(s-r)ds d\mu(r) = A\int_0^t S(t-r) \int_0^s v(s-r)d\mu(r)ds$$
$$= \int_0^t S(t-r)A \int_0^s v(s-r)d\mu(r)ds$$
$$= \int_0^t S(t-s)v(s)ds - \int_0^t S(t-s)f(s)ds.$$

Hence it follows from (3.1.8) that

$$v^{[n+1]}(t) = \int_0^t S(t-s)f(s)ds = w(t).$$

Thus, $w \in C^{n+1}([0,\infty);X)$ with $w(0) = 0$ and $w^{(n+1)} = v$. //

Remark.

There is an essential difference between integrated solution operator families for Volterra equations (VE) and integrated semigroups for inhomogeneous abstract Cauchy problems. For an n-times integrated semigroup S with generator A, S satisfies the equation

$$S(t)x = \int_0^t S(t-s)Axds + \frac{t^n}{n!}x$$

for all $x \in D(A)$ and $t \geq 0$. Thus, the mapping $t \mapsto S(t)x$ for $t \geq 0$ is continuously differentiable for all $x \in D(A)$ and

$$S'(t)x = S(t)Ax + \frac{t^{n-1}}{(n-1)!}x.$$

If S is an n-times integrated solution operator family with generator (A,μ), S satisfies the equation

$$S(t)x = \int_0^t S(t-s)Axd\mu(s) + \frac{t^n}{n!}x$$

for all $x \in D(A)$ and $t \geq 0$. However, without any further regularity assumptions on μ, this does not yield the differentiability of $t \mapsto S(t)x$ for $x \in D(A)$.

Let $f \in C^{n+1}([0,\infty);X)$ and let $w(t) := \int_0^t S(t-s)f(s)ds$ for every $t \geq 0$. Then, by Lemma 1.2.11,

$$w'(t) = S(t)f(0) + \int_0^t S(t-s)f'(s)ds \qquad \text{for } t \geq 0.$$

Since in general there is no differentiability criteria for $t \mapsto S(t)f(0)$, we have to assume that $f^{(k)}(0) = 0$ for $0 \le k \le n$ in order to get that $w \in C^{n+1}([0,\infty); X)$ if $f \in C^{n+1}([0,\infty); X)$.

Corollary 3.1.12. Suppose that (A, μ) generates an n-times integrated solution operator family S for some $n \in \mathbb{N}_0$. Suppose that $f = g^{[n+1]}$ for some $g \in C([0,\infty); X)$. Then the function v defined as

$$v(t) := \int_0^t S(t-s)g(s)ds, \qquad \text{for } t \ge 0$$

is a solution of the Volterra equation $v(t) = A\int_0^t v(t-s)ds + f(t)$ for $t \ge 0$.

Proof. This corollary follows immediately from the previous remark and Theorem 3.1.11. However, since the proof of Theorem 3.1.10 is somewhat longwinded, we give an alternative direct proof. Let $t \ge 0$ and $0 \le s \le t$. Since S is an n-times integrated solution operator family generated by (A, μ), it follows from (3.1.3) that

$$S(t-s)g(s) = A\int_0^{t-s} S(t-s-r)g(s)d\mu(r) + \frac{(t-s)^n}{n!}g(s).$$

It follows from the closedness of A and Lemma 1.2.5 that

$$\begin{aligned} v(t) &= \int_0^t \left[A\int_0^{t-s} S(t-s-r)g(s)d\mu(r)ds + \frac{(t-s)^n}{n!}g(s)\right]ds \\ &= A\int_0^t\int_0^{t-s} S(t-s-r)g(s)d\mu(r)ds + \int_0^t \frac{(t-s)^n}{n!}g(s)ds \\ &= A\int_0^t\int_0^{t-r} S(t-r-s)g(s)dsd\mu(r) + g^{[n+1]}(t) \\ &= A\int_0^t v(t-r)d\mu(r) + f(t). \end{aligned}$$

Since S is strongly continuous and g is continuous, $v \in C([0,\infty);X)$. Therefore, v is an (exponentially bounded) solution of (VE). //

The following is a modification of Theorem 3.1.11 for strong solutions of (VE). The proof is omitted.

Corollary 3.1.13. Let (A,μ) be a generator of an n-times integrated solution operator family S for some $n \in \mathbb{N}_0$. Let $f \in C([0,\infty);X)$. Define $w(t) := \int_0^t S(t-s)f(s)ds$ for $t \geq 0$. Then (VE) has a strong solution $v \in C([0,\infty);[D(A)])$ if and only if $w \in C^{n+1}([0,\infty);[D(A)])$. When one of these equivalent conditions holds, $v = w^{(n+1)}$ is the unique strong solution of (VE).

3.2 The Wellposedness of Abstract Volterra Equations

Let A be a closed linear operator with its domain $D(A)$ and range in a complex Banach space X and let $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$. In Section 3.1 we showed that the Volterra equation

$$v(t) = A\int_0^t v(t-s)d\mu(s) + \frac{t^n}{n!}x \qquad \text{for } t \geq 0 \qquad (3.2.1)_n$$

has a solution $v : t \mapsto S(t)x$ for all $x \in X$ if (A,μ) generates an n-times integrated solution operator family S. In fact, $S(\cdot)x$ is the unique solution of the Volterra equation $(3.2.1)_n$ for all $x \in X$. The convolution notation $*$ is also used for the integral $\int_0^t S(t-s)f(s)ds =: (S*f)(t)$ for a strongly continuous mapping $S : [0,\infty) \to L(X)$ and $f \in C([0,\infty);X)$. The following result is similar to Proposition 2.1 in [O].

Proposition 3.2.1. If (A, μ) generates an integrated solution operator family, then (VE) has at most one solution.

Proof. Let S be the n-times integrated solution operator family generated by (A, μ) for some $n \in \mathbb{N}$. Let j_n denote the function $t \mapsto \frac{t^n}{n!}$ for $t \geq 0$. Suppose that $v(t) = A\int_0^t v(t-s)d\mu(s)$. Then

$$\begin{aligned} S * v(t) &= S * A(v * d\mu)(t) \\ &= \int_0^t S(r)A\int_0^{t-r} v(t-r-s)d\mu(s)dr \\ &= \int_0^t A\int_0^{t-r} S(r)v(t-r-s)d\mu(s)dr \\ &= A\int_0^t\int_0^{t-r} S(t-r-s)v(r)d\mu(s)dr \\ &= \int_0^t [S(t-r) - \frac{(t-r)^n}{n!}]v(r)dr \\ &= S * v(t) - j_n * v(t) \end{aligned}$$

for every $t \geq 0$. Thus, $j_n * v = 0$. Therefore, $v \equiv 0$. //

We will show next that $(3.2.1)_n$ has a unique, exponentially bounded solution for all $x \in X$ if and only if (A, μ) generates an n-times integrated solution operator family. To show this the following lemma is crucial.

Lemma 3.2.2. The following statements are equivalent.

(i) The Volterra equation $(3.2.1)_n$ has a unique, exponentially bounded solution for all $x \in X$.

(ii) The Volterra equation $(3.2.1)_n$ has a unique, exponentially bounded solution $v(\cdot) = v(\cdot, x)$ for all $x \in X$ and there exist constants $M > 0$, $\omega \geq \epsilon$ such that $\|v(t)\| \leq Me^{\omega t}\|x\|$ for all $x \in X$ and $t \geq 0$.

Proof. The implication (ii) $\Longrightarrow$ (i) is obvious. We show that (i) $\Longrightarrow$ (ii). Suppose that (i) holds. Considering $C([0,\infty);X)$ as the Fréchet space with the seminorms $p_T(f) = \sup_{0\le t\le T} \|f(t)\|$, $T \ge 0$, define a map $\phi : X \to C([0,\infty);X)$ by $x \mapsto v(\cdot,x)$. Then ϕ is linear since $v(\cdot,x)$ is a unique solution of $(3.2.1)_n$ for each $x \in X$. We show that ϕ is continuous. Suppose that a sequence $\{x_m\}_{m\in\mathbb{N}}$ converges to x in X, and the sequence $\{v(\cdot,x_m)\}_{m\in\mathbb{N}}$ converges to u in $C([0,\infty);X)$. Since the sequence $\{\int_0^t v(t-s,x_m)d\mu(s)\}_{m\in\mathbb{N}}$ converges to $\int_0^t u(t-s)d\mu(s)$ in X for every $t \ge 0$, we obtain from the closedness of A that $(u * d\mu)(t) \in D(A)$ and

$$u(t) = A\int_0^t u(t-s)d\mu(s) + \frac{t^n}{n!}x$$

for every $t \ge 0$ and $x \in X$. Hence ϕ is closed and everywhere defined, and therefore, continuous. For every $t \ge 0$ define $S(t) : x \mapsto v(t,x)$ on X. Clearly, the operators $S(t)$ are linear. Since $S(t)x = v(t,x) = \phi(x)(t)$, we obtain that $S(t) \in L(X)$. Since $t \mapsto S(t)x = v(t,x)$ is exponentially bounded for each $x \in X$, it follows from the Uniform Exponential Boundedness Theorem (Theorem 1.1.10) that there exist constants $M > 0$ and $\omega \ge \epsilon$ such that $\|S(t)x\| = \|v(t,x)\| \le Me^{\omega t}\|x\|$ for all $x \in X$ and $t \ge 0$. //

Theorem 3.2.3. Let A be a closed linear operator in a Banach space X and $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \ge 0$. The following are equivalent.

(i) The equation $(3.2.1)_n$ has unique, exponentially bounded solutions for all $x \in X$.

(ii) (A,μ) generates an n-times integrated solution operator family.

Proof. The implication (ii) $\Longrightarrow$ (i) was shown in Proposition 3.2.1. Suppose that (i) holds. By Lemma 3.2.2, there exists a strongly continuous operator family $\{S(t)\}_{t\ge 0} \subset$

$L(X)$ for which there exist constants $M > 0$, $\omega \geq 0$ such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and

$$S(t)x = A\int_0^t S(t-s)x d\mu(s) + \frac{t^n}{n!}x \qquad \text{for } t \geq 0$$

and for every $x \in X$. Let $x \in D(A)$ and $t \geq 0$. Define $v(t) := \int_0^t S(t-s)Axd\mu(s) + \frac{t^n}{n!}x$. Then $v(t) \in D(A)$ and

$$\begin{aligned} Av(t) &= A\int_0^t S(t-s)Axd\mu(s) + \frac{t^n}{n!}Ax \\ &= S(t)Ax - \frac{t^n}{n!}Ax + \frac{t^n}{n!}Ax \\ &= S(t)Ax. \end{aligned}$$

Thus,

$$v(t) = \int_0^t Av(t-s)d\mu(s) + \frac{t^n}{n!}x = A\int_0^t v(t-s)d\mu(s) + \frac{t^n}{n!}x.$$

Then, by the uniqueness of the solutions of $(3.2.1)_n$, $v(t) = S(t)x$. Thus, $AS(t)x = S(t)Ax$ for all $t \geq 0$ and $x \in D(A)$. Thus, by Proposition 3.1.6, S is an n-times integrated solution operator family generated by (A, μ). //

If (A, μ) generates an n-times integrated solution operator family S, then the equation

$$v(t) = \int_0^t Av(t-s)d\mu(s) + \frac{t^n}{n!}x \qquad \text{for } t \geq 0 \qquad (3.2.2)_n$$

also has an exponentially bounded solution $v : t \mapsto S(t)x$ in $C([0,\infty);[D(A)])$ for all $x \in D(A)$, i.e., the equation $(3.2.1)_n$ has exponentially bounded strong solutions for all $x \in D(A)$. Recall that $[D(A)]$ is defined as the Banach space $D(A)$ equipped with the graph norm $\|x\|_A = \|x\| + \|Ax\|$. The following lemma can be shown by modifying the proof of Lemma 3.2.2 by replacing the space X with $[D(A)]$.

Lemma 3.2.4. The following are equivalent.

(i) The equation $(3.2.2)_n$ has a unique, exponentially bounded solution in the space $C([0,\infty);[D(A)])$ for all $x \in D(A)$.

(ii) The equation $(3.2.2)_n$ has a unique, exponentially bounded solution $v(\cdot) = v(\cdot, x) \in C([0,\infty);[D(A)])$ for all $x \in D(A)$ and there exist constants $M > 0$, $\omega \geq \epsilon$ such that $\|v(t)\| \leq Me^{\omega t}\|x\|$ for all $x \in D(A)$ and $t \geq 0$.

By modifying the previous results on the equation $(3.2.1)_n$, it can be shown that $(3.2.2)_n$ has a unique, exponentially bounded solution in $C([0,\infty);[D(A)])$ for every $x \in D(A)$ if and only if (A,μ) generates an n-times integrated solution operator family.

Theorem 3.2.5. Let A be a closed linear operator in X and $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$. The following are equivalent.

(i) The equation $(3.2.2)_n$ has a unique, exponentially bounded solution in the space $C([0,\infty);[D(A)])$ for all $x \in D(A)$.

(ii) (A,μ) generates an n-times integrated solution operator family.

Proof. To show (ii) $\Longrightarrow$ (i), suppose that (A,μ) generates an n-times integrated solution operator family S of exponential type $(M;\omega)$. Then it follows from Lemma 3.1.3 and Lemma 3.1.4 that the function $t \mapsto S(t)x$ for $t \geq 0$ is an exponentially bounded solution in $C([0,\infty);[D(A)])$ of $(3.2.2)_n$ for all $x \in D(A)$. We show the uniqueness of $S(\cdot)x$ as an exponentially bounded solution of $(3.2.2)_n$ which is continuous in $\|\cdot\|_A$ for all $x \in D(A)$. Suppose that $v(t) = \int_0^t Av(t-s)d\mu(s)$ for every $t \geq 0$ for some exponentially bounded function $v \in C([0,\infty);[D(A)])$. It can be shown as in the proof of Proposition 3.2.1 that $v(t) = 0$ for all $t \geq 0$. We show (i) $\Longrightarrow$ (ii). This implication can be shown similarly to the proof of Theorem 3.2.3. Suppose that (i) holds. Then it follows by Lemma 3.2.4 and the closedness of A that there exists a strongly continuous, uniformly exponentially bounded operator family $\{S(t)\}_{t\geq 0} \subset L([D(A)])$ such that

$$S(t)x = A\int_0^t S(t-s)x d\mu(s) + \frac{t^n}{n!}x \qquad \text{for } t \geq 0$$

for every $x \in D(A)$. It can be shown by the same proof of Theorem 3.2.3 that S is an n-times integrated solution operator family with generator (A, μ). //

3.3 Approximations of Integrated Solution Operator Families

Let X be a Banach space and $\{S_m\}_{m\in\mathbb{N}}$ be a sequence of functions from $[0,\infty)$ to $L(X)$. If there exist constants $M > 0$ and $\omega \geq 0$ such that $\|S_m(t)\| \leq Me^{\omega t}$ for all $m \in \mathbb{N}$ and $t \geq 0$, then the sequence $\{S_m\}_{m\in\mathbb{N}}$ is said to be $(M;\omega)$-stable (or simply stable). Let $n \in \mathbb{N}_0$. In this section we prove Trotter-Kato type approximation theorems on the convergence of a stable sequence $\{S_m\}_{m\in\mathbb{N}}$ of n-times integrated solution operator families S_m with generators (A_m, μ_m) in terms of the convergence of the sequence $\{(I - \widehat{d\mu_m}(\lambda)A_m)^{-1}x\}_{m\in\mathbb{N}}$ for every $\lambda > \omega$ and $x \in X$.

Theorem 3.3.1. Let $n \in \mathbb{N}_0$, $\epsilon \geq 0$, $M > 0$, and $\omega \geq \epsilon$ be some constants. Let $\{S_m\}_{m\in\mathbb{N}}$ be an $(M;\omega)$-stable sequence of n-times integrated solution operator families S_m with generators (A_m, μ_m), where A_m are closed linear operators in X and $\mu_m \in$

$BV_\epsilon([0,\infty);\mathbb{C})$ for all $m \in \mathbb{N}$. Suppose that there exist A, a closed linear operator in X and a function $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ such that $(I-\widehat{d\mu}(\lambda)A)^{-1}$ exists in $L(X)$ for every $\lambda > \omega$ and $\lim_{m\to\infty}(I-\widehat{d\mu_m}(\lambda)A_m)^{-1}x = (I-\widehat{d\mu}(\lambda)A)^{-1}x$ for every $\lambda > \omega$ and $x \in X$. Then (A,μ) generates an $(n+1)$-times integrated solution operator family $T \in Lip_\omega([0,\infty);L(X))$ with $\|T\|_{Lip_\omega} \le M$. Moreover, for every $x \in X$, $\{S_m^{[1]}(t)x\}_{m\in\mathbb{N}}$ converges uniformly to $T(t)x$ on every compact interval in $[0,\infty)$. If, in addition, A is densely defined and μ is absolutely continuous on $[0,\infty)$, then (A,μ) generates an n-times integrated solution operator family of exponential type $(M;\omega)$.

Proof. Define $T_m(t)x := \int_0^t S_m(s)xds$ for every $m \in \mathbb{N}$, $t \ge 0$, and $x \in X$. Then the $(M;\omega)$-stability of $\{S_m\}_{m\in\mathbb{N}}$ implies that $T_m \in Lip_\omega([0,\infty);L(X))$ with $\|T_m\|_{Lip_\omega} \le M$ for all $m \in \mathbb{N}$. It follows from (1.1.6) that $\frac{1}{\lambda^{n+1}}(I-\widehat{d\mu_m}(\lambda)A_m)^{-1}x = \widehat{S_m}(\lambda)x = \widehat{dT_m}(\lambda)x = \lambda\widehat{T_m}(\lambda)x$ for every $\lambda > \omega$ and $x \in X$. Hence, T_m is the $(n+1)$-times integrated solution operator family of exponential type $(M;\omega)$ with generator (A_m,μ_m) and by the hypothesis, $\{\widehat{dT_m}(\lambda)x\}_{m\in\mathbb{N}}$ converges to $(I-\widehat{d\mu}(\lambda)A)^{-1}x$ for every $\lambda > \omega$ and $x \in X$. Since $\|T_m(\cdot)x\|_{Lip_\omega} \le M\|x\|$ for all $m \in \mathbb{N}$ and $x \in X$ and since the sequence $\{\widehat{dT_m}(\lambda)x\}_{m\in\mathbb{N}}$ is convergent for every $\lambda > \omega$, it follows from Theorem 1.1.6 that for every $x \in X$, there exists a $T_x \in Lip_\omega([0,\infty);X)$ with $\|T_x\|_{Lip_\omega} \le M\|x\|$ for which the sequence $\{T_m(\cdot)x\}_{m\in\mathbb{N}}$ converges uniformly to $T_x(\cdot)$ on every compact interval in $[0,\infty)$. Define $T(t)x := T_x(t)$ for every $t \ge 0$ and every $x \in X$. Then, by the uniqueness of a limit, $T(t) : X \to X$ is linear for every $t \ge 0$. Moreover, $T \in Lip_\omega([0,\infty);L(X))$ with $\|T\|_{Lip_\omega} \le M$. Since $\{T_m(t)x\}_{m\in\mathbb{N}}$ converges uniformly to $T(t)x$ on compact intervals in $[0,\infty)$, it follows from Theorem 1.1.6 that

$\{\widehat{dT_m}(\lambda)x\}_{m\in\mathbb{N}}$ converges uniformly to $\widehat{dT}(\lambda)x$ on compact intervals in (ω,∞). Hence, by the uniqueness of limits,

$$\frac{1}{\lambda^{n+2}}(I-\widehat{d\mu}(\lambda)A)^{-1}x=\frac{1}{\lambda}\widehat{dT}(\lambda)x=\widehat{T}(\lambda)x$$

for every $\lambda>\omega$ and $x\in X$. Thus, T is the $(n+1)$-times integrated solution operator family with generator (A,μ). Assuming that A is densely defined and μ absolutely continuous, the same reasoning as in the proof of Theorem 3.1.10 yields that $\frac{dT(t)x}{dt}=:$ $S(t)x$ exists for all $t\geq 0$ and $x\in X$, and that $S=\{S(t)\}_{t\geq 0}$ is an n-times integrated solution operator family generated by (A,μ). //

The previous theorem says that if $\{S_m\}_{m\in\mathbb{N}}$ is a stable sequence of n-times integrated solution operator families with generators (A_m,μ_m) where $\mu_m\in BV_\epsilon([0,\infty);\mathbb{C})$ for all $m\in\mathbb{N}$, and if A is densely defined closed linear operator in X and μ an absolutely continuous function in $BV_\epsilon([0,\infty);\mathbb{C})$, the strong convergence of the sequence $\{(I-\widehat{d\mu_m}(\lambda)A_m)^{-1}\}_{m\in\mathbb{N}}$ to $(I-\widehat{d\mu}(\lambda)A)^{-1}$ implies the existence of an n-times integrated solution operator family S with generator (A,μ) and the strong convergence of the sequence $\{S_m^{[1]}(\cdot)\}_{m\in\mathbb{N}}$ to $S^{[1]}(\cdot)$. In the following the uniform convergence of the sequence $\{S_m(\cdot)x\}_{m\in\mathbb{N}}$ to $S(\cdot)x$ on compact intervals in $[0,\infty)$ for all $x\in X$ will be shown with some additional assumptions on (A,μ) and (A_m,μ_m) for $m\in\mathbb{N}$.

Theorem 3.3.2. Let $n\in\mathbb{N}_0$, $\epsilon\geq 0$, $M>0$, and $\omega\geq\epsilon$ be some constants. Let $\{S_m\}_{m\in\mathbb{N}}$ be an $(M;\omega)$-stable sequence of n-times integrated solution operator families S_m with generators (A_m,μ_m) where A_m are densely defined, closed linear operators in a Banach space X and μ_m are absolutely continuous functions in $BV_\epsilon([0,\infty);\mathbb{C})$ for all $m \in \mathbb{N}$. Suppose that there exist a densely defined, closed linear operator A in X and an absolutely continuous function $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ such that $\widehat{d\mu} \not\equiv 0$ on (ω,∞), $(I-\widehat{d\mu}(\lambda)A)^{-1}$ exists in $L(X)$ for every $\lambda > \omega$, and $\lim_{m\to\infty}(I-\widehat{d\mu_m}(\lambda)A_m)^{-1}x = (I-\widehat{d\mu}(\lambda)A)^{-1}x$ for every $\lambda > \omega$ and $x \in X$. In addition, assume the following.

(i) $\bigcap_{m\in\mathbb{N}} D(A_m) \bigcap D(A)$ contains a dense subset D of X.

(ii) $\mu_m' \in Lip_\omega([0,\infty);\mathbb{C})$ for all $m \in \mathbb{N}$, $\|\mu_m'\|_{Lip_\omega} \leq M_1$ for some constant $M_1 > 0$ and all $m \in \mathbb{N}$, and $\{\mu_m'(t)\}_{m\in\mathbb{N}}$ converges to $\mu'(t)$ for every $t \geq 0$.

Then, (A,μ) generates an n-times integrated solution operator family S of exponential type $(M;\omega)$, and for every $x \in X$, $\{S_m(\cdot)x\}_{m\in\mathbb{N}}$ converges uniformly to $S(\cdot)x$ on compact intervals in $[0,\infty)$.

Proof. By Theorem 3.3.1, there exists an n-times integrated solution operator family S of exponential type $(M;\omega)$ with generator (A,μ). For the convergence, we first show that for every $y \in D$, the sequence $\{S_m(\cdot)y\}_{m\in\mathbb{N}}$ converges uniformly to $S(\cdot)y$ on compact intervals in $[0,\infty)$. Let $y \in D$. By Lemma 3.1.4,

$$S_m(t)y = \int_0^t S_m(t-s)A_m y\,d\mu_m(s) + \frac{t^n}{n!}y \tag{3.3.1}$$

and
$$S(t)y = \int_0^t S(t-s)Ay\,d\mu(s) + \frac{t^n}{n!}y \tag{3.3.2}$$

for every $t \geq 0$. Define $h_m(\lambda) := (I-\widehat{d\mu_m}(\lambda)A_m)^{-1}$ and $h(\lambda) := (I-\widehat{d\mu}(\lambda)A)^{-1}$ for every $m \in \mathbb{N}$ and $\lambda > \omega$. Then $\lim_{m\to\infty} h_m(\lambda)x = h(\lambda)x$ for every $\lambda > \omega$ and $x \in X$. Let $\lambda_0 > \omega$ such that $\widehat{d\mu}(\lambda_0) \neq 0$, Define $z := (I-\widehat{d\mu}(\lambda_0)A)y$. Then $y = h(\lambda_0)z$ and

$$\|S_m(t)y - S(t)y\| \leq \|S_m(t)\big(h(\lambda_0)z - h_m(\lambda_0)z\big)\| + \|S_m(t)h_m(\lambda_0)z - S(t)h(\lambda_0)z\|.$$

Since the sequence $\{S_m\}_{m\in\mathbb{N}}$ is stable, it suffices to estimate the second term in this expression. It follows from the assumption (ii) and Theorem 1.1.6 that $\{\widehat{d\mu}(\lambda_0)\}_{m\in\mathbb{N}}$ converges to $\widehat{d\mu}(\lambda_0)$, that $\{\mu'_m(\cdot)\}_{m\in\mathbb{N}}$ converges uniformly to $\mu'(\cdot)$ on compact intervals in $[0,\infty)$, and that $\mu' \in Lip_\omega([0,\infty);\mathbb{C})$. By (3.3.1) and (3.3.2),

$$|\widehat{d\mu}(\lambda_0)\widehat{d\mu_m}(\lambda_0)|\ \|S_m(t)h_m(\lambda_0)z - S(t)h(\lambda_0)z\|$$

$$= |\widehat{d\mu}(\lambda_0)\widehat{d\mu_m}(\lambda_0)|\ \|\int_0^t S_m(t-s)A_m h_m(\lambda_0)z d\mu_m(s)$$

$$- \int_0^t S(t-s)Ah(\lambda_0))z d\mu(s) + \frac{t^n}{n!}(h_m(\lambda_0) - h(\lambda_0))z\|$$

$$\leq \|\widehat{d\mu}(\lambda_0)\int_0^t S_m(t-s)[h_m(\lambda_0) - I]z d\mu_m(s) - \widehat{d\mu_m}(\lambda_0)\int_0^t S(t-s)[h(\lambda_0) - I]z d\mu(s)\|$$

$$+\frac{t^n}{n!}\|h_m(\lambda_0)z - h(\lambda_0)z.\|$$

Since $h_m(\lambda_0)z \longrightarrow h(\lambda_0)z$ as $m \longrightarrow \infty$, it suffices to estimate the first term in the last expression. This yields

$$\|\widehat{d\mu}(\lambda_0)\textstyle\int_0^t S_m(t-s)[h_m(\lambda_0)z - z]\mu'_m(s)ds -$$

$$\widehat{d\mu_m}(\lambda_0)\textstyle\int_0^t S(t-s)[h(\lambda_0)z - z]\mu'(s)ds\|$$

$$\leq\ |\widehat{d\mu}(\lambda_0) - \widehat{d\mu_m}(\lambda_0)|\ \|\textstyle\int_0^t S_m(t-s)[h_m(\lambda_0)z - z]\mu'_m(s)ds\|$$

$$+|\widehat{d\mu_m}(\lambda_0)|\ \|\textstyle\int_0^t S_m(t-s)[h_m(\lambda_0)z - z]\mu'_m(s)ds - \int_0^t S(t-s)[h(\lambda_0)z - z]\mu'(s)ds\|.$$

Since $\{\|\int_0^{\cdot} S_m(\cdot - s)[h_m(\lambda_0)z - z]\mu'_m(s)ds\|\ |\ m \in \mathbb{N}\}_{m\in\mathbb{N}}$ is uniformly bounded on compact intervals in $[0,\infty)$, it suffices to estimate the second term above.

$$\|\int_0^t S_m(t-s)[h_m(\lambda_0)z - z]\mu'_m(s)ds - \int_0^t S(t-s)[h(\lambda_0)z - z]\mu'(s)ds\|$$

$$\leq \|\int_0^t S_m(t-s)[h_m(\lambda_0)z - z]\mu'_m(s)ds\| + \|\int_0^t S_m(t-s)[h(\lambda_0)z - z][\mu'_m(s) - \mu'(s)]ds\|$$

$$+\|\textstyle\int_0^t \Big[S_m(t-s) - S(t-s)\Big][h(\lambda_0)z - z]\mu'(s)ds\|.$$

Since the sequence $\{S_m\}_{m\in\mathbb{N}}$ is stable and since $\mu'_m \in Lip_\omega([0,\infty);\mathbb{C})$ for all $m \in \mathbb{N}$, the first term converges uniformly to 0 as $m \longrightarrow \infty$ on compact intervals in $[0,\infty)$. Since $\{S_m\}_{m\in\mathbb{N}}$ is stable and $\{\mu'_m\}_{m\in\mathbb{N}}$ converges uniformly to μ on compact intervals in $[0,\infty)$, it suffices to estimate the third term above. By integration by parts,

$$\|\int_0^t \Big[S_m(t-s) - S(t-s)\Big](h(\lambda_0)z - z)\mu'(s)ds\|$$

$$\leq |\mu'(t)| \; \|\Big[S_m^{[1]}(t) - S^{[1]}(t)\Big](h(\lambda_0)z - z)\|$$

$$+\text{ess}\sup_{s\in[0,t]} |\mu''(s)| \int_0^t \|\Big[S_m^{[1]}(s) - S^{[1]}(s)\Big][h(\lambda_0)z - z]\|ds.$$

Since $\{S_m^{[1]}(\cdot)x\}_{m\in\mathbb{N}}$ converges uniformly to $T(\cdot)x = S^{[1]}(\cdot)x$ on compact intervals in $[0,\infty)$ for every $x \in X$ and since μ'' is essentially bounded on compact intervals in $[0,\infty)$, the last expression converges uniformly to 0 as $m \longrightarrow \infty$ on compact intervals in $[0,\infty)$. This shows that $\{S_m(\cdot)y\}_{m\in\mathbb{N}}$ converges uniformly to $S(\cdot)y$ on compact intervals in $[0,\infty)$ for every $y \in D$. Since $\overline{D} = X$ and $\{S_m\}_{m\in\mathbb{N}}$ is stable, it follows that $\{S_m(\cdot)x\}_{m\in\mathbb{N}}$ converges uniformly to $S(\cdot)x$ on compact intervals in $[0,\infty)$ for every $x \in X$. //

3.4 Analytic Integrated Solution Operator Families

In this section we will briefly discuss analytic integrated solution operator families for Volterra equations. These solution operator families are the analog of analytic integrated semigroups for the abstract Cauchy problem. In contrast to the characterization theorem (Theorem 3.1.10) of integrated solution operator families, the conditions on the function $H(\lambda) := (I - \widehat{d\mu}(\lambda)A)^{-1}$ on (ω,∞) which characterize analytic integrated solution operator families are much easier to be checked. We assume that A is a non-trivial, closed linear operator in a Banach space X and that $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$ with $\widehat{d\mu} \not\equiv 0$ on $\mathbb{C}_\epsilon$.

For $n = 0$ the following definition coincides with that of analytic resolvents in [Pr]. If $n = 0$, $\mu(t) = t$, and $\omega = 0$, then it reduces to the definition of a bounded, analytic, strongly continuous semigroup (see, for example, [G]).

Definition 3.4.1. Let $\theta_0 \in (0, \frac{\pi}{2}]$, $n \in \mathbb{N}_0$, $M > 0$, and $\omega \geq \epsilon$. Suppose that a function $S : \{0\} \cup \Sigma_{0,\theta_0} \to L(X)$ satisfies the following.

(i) The restriction $S|_{[0,\infty)}$ of S to $[0,\infty)$ is an n-times integrated solution operator family of exponential type $(M;\omega)$ with generator (A,μ).

(ii) S is analytic on the sector Σ_{0,θ_0}.

(iii) For every $\theta \in (0,\theta_0)$, there exists a constant $M_\theta \geq 0$ such that

$$\sup_{z\in\Sigma_{0,\theta}} \|e^{-\omega z}S(z)\| \leq M_\theta.$$

Then S is said to be an analytic n-times integrated solution operator family of analyticity type $(\omega;\theta_0)$ and with generator (A,μ).

Proposition 3.4.2. Let (A,μ) be the generator of an analytic n-times integrated solution operator family of analyticity type $(\omega;\theta_0)$. Then $\widehat{d\mu}$ admits an analytic continuation to the sector $\Sigma_{\omega,\theta_0+\frac{\pi}{2}}$, the function $H : (\omega,\infty) \to L(X)$ defined by $H(\lambda) = \frac{1}{\lambda^{n+1}}(I - \widehat{d\mu}(\lambda)A)^{-1}$ admits an analytic continuation to the sector $\Sigma_{\omega,\theta_0+\frac{\pi}{2}}$, and $\sup_{\lambda\in\Sigma_{\omega,\theta+\frac{\pi}{2}}} \|(\lambda-\omega)H(\lambda)\| < \infty$ for every $\theta \in (0,\theta_0)$.

Proof. By Theorem 3.1.7, the function H extends to the half plane $\mathbb{C}_\omega$ and $H(\lambda)x = \widehat{S}(\lambda)x$ for every $x \in X$ and $\lambda \in \mathbb{C}_\omega$. By Theorem 1.1.9, the function H also, admits an analytic continuation to the sector $\Sigma_{\omega,\theta_0+\frac{\pi}{2}}$ and $\sup_{\lambda\in\Sigma_{\omega,\theta+\frac{\pi}{2}}} \|(\lambda-\omega)H(\lambda)\| < \infty$ for every $\theta \in (0,\theta_0)$. Choose $x \in D(A)$ and $x^* \in X^*$ such that the function $\phi(\lambda) := \langle\lambda^{n+1}H(\lambda)x,\ x^*\rangle = \langle(I-\widehat{d\mu}(\lambda)A)^{-1}x,\ x^*\rangle = \int_0^\infty e^{-\lambda t}\langle S(t)x,\ x^*\rangle dt$ for $\lambda \in \Sigma_{\omega,\theta_0+\frac{\pi}{2}}$ on $\mathbb{C}_\omega$ is not a constant function. (Such x and x^* exist since A and μ are nontrivial.) Then ϕ is analytic and

$$\phi'(\lambda) = \widehat{d\mu}'(\lambda)\langle(I-\widehat{d\mu}(\lambda)A)^{-2}Ax,\ x^*\rangle = \widehat{d\mu}'(\lambda)\lambda^{2n+2}\langle H(\lambda)^2Ax, x^*\rangle$$

on $\Sigma_{\omega,\theta_0+\frac{\pi}{2}}$. The function ψ defined by $\psi(\lambda) := \langle(I-\widehat{d\mu}(\lambda)A)^{-2}Ax,\ x^*\rangle$ is not identically zero on $\Sigma_{\omega,\theta_0+\frac{\pi}{2}}$. Since otherwise, $\phi =$ a constant on $\mathbb{C}_\omega$, which is a contradiction. Also, ψ is analytic on $\Sigma_{\omega,\theta_0+\frac{\pi}{2}}$. Thus, $\widehat{d\mu}'(\lambda) = \frac{\phi'(\lambda)}{\psi(\lambda)}$ for $\lambda \in \mathbb{C}_\omega$ with $\psi(\lambda) \neq 0$. Hence $\widehat{d\mu}'$ extends analytically to $\Sigma_{\omega,\theta_0+\frac{\pi}{2}}$ and

$$\begin{aligned}\phi'(\lambda) &= \widehat{d\mu}'(\lambda)\langle(I-\widehat{d\mu}(\lambda)A)^{-2}Ax,\ x^*\rangle\\ &= \frac{\widehat{d\mu}'(\lambda)}{\widehat{d\mu}(\lambda)}\langle\widehat{d\mu}(\lambda)(I-\widehat{d\mu}(\lambda)A)^{-2}Ax,\ x^*\rangle\\ &= \frac{\widehat{d\mu}'(\lambda)}{\widehat{d\mu}(\lambda)}\langle(I-\widehat{d\mu}(\lambda)A)^{-2}x-(I-\widehat{d\mu}(\lambda)A)^{-1}x,\ x^*\rangle\\ &= \frac{\widehat{d\mu}'(\lambda)}{\widehat{d\mu}(\lambda)}g(\lambda),\end{aligned}$$

where the function $g(\lambda) := \langle(I-\widehat{d\mu}(\lambda)A)^{-2}x-(I-\widehat{d\mu}(\lambda)A)^{-1}x,\ x^*\rangle$ is analytic on $\Sigma_{\omega,\theta_0+\frac{\pi}{2}}$. Thus, $\widehat{d\mu}(\lambda) = \frac{\widehat{d\mu}'(\lambda)}{\phi'(\lambda)}g(\lambda)$ for $\lambda \in \mathbb{C}_\omega$ with $\phi'(\lambda) \neq 0$. Therefore, $\widehat{d\mu}$ extends analytically to $\Sigma_{\omega,\theta_0+\frac{\pi}{2}}$. //

Proposition 3.4.3. Let $n \in \mathbb{N}_0$, $\omega \geq \epsilon$, and $\theta_0 \in (0,\frac{\pi}{2}]$. Suppose that $(I-\widehat{d\mu}(\lambda)A)^{-1}$ exists in $L(X)$ for every $\lambda > \omega$ and that the function $H : (\omega,\infty) \to L(X)$ defined by

$H(\lambda) = \frac{1}{\lambda^{n+1}}(I - \widehat{d\mu}(\lambda)A)^{-1}$ admits an analytic extension H to the sector $\Sigma_{\omega,\theta_0+\frac{\pi}{2}}$ such that $\sup_{\lambda\in\Sigma_{\omega,\theta+\frac{\pi}{2}}} \|(\lambda-\omega)H(\lambda)\| < \infty$ for every $\theta \in (0,\theta_0)$. Then (A,μ) generates a norm continuous, $(n+1)$-times integrated solution operator family S which admits an extension S to $\{0\}\cup\Sigma_{0,\theta_0}$ such that S is analytic on Σ_{0,θ_0} and $\sup_{z\in\Sigma_{0,\theta}} \|e^{-\omega z}S'(z)\| < \infty$ for every $\theta \in (0,\theta_0)$.

Proof. It follows from Theorem 1.1.8 that there exists an analytic function $W : \Sigma_{0,\theta_0} \to L(X)$ such that $\sup_{z\in\Sigma_{0,\theta}} \|e^{-\omega z}W(z)\| < \infty$ for every $\theta \in (0,\theta_0)$ and

$$H(\lambda) = \frac{1}{\lambda^{n+1}}(I - \widehat{d\mu}(\lambda)A)^{-1} = \int_0^\infty e^{-\lambda t}W(t)dt$$

on (ω,∞). Defining $S(t) := \int_0^t W(s)ds$ for every $t \geq 0$, $H(\lambda) = \lambda\int_0^\infty e^{-\lambda t}S(t)dt$ for all $\lambda > \omega$. //

Theorem 3.4.4. Let A be a densely defined, closed linear operator in X and let μ be an absolutely continuous function in $BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$. Let $n \in \mathbb{N}_0$, $\omega \geq \epsilon$, and $\theta_0 \in (0,\frac{\pi}{2}]$. The following are equivalent.

(i) The pair (A,μ) generates an analytic, n-times integrated solution operator family S of analyticity type $(\omega;\theta_0)$.

(ii) $(I - \widehat{d\mu}(\lambda)A)^{-1}$ exist in $L(X)$ for all $\lambda > \omega$ and the function $H : (\omega,\infty) \to L(X)$ defined by $H(\lambda) = \frac{1}{\lambda^{n+1}}(I - \widehat{d\mu}(\lambda)A)^{-1}$ has an analytic continuation H to the sector $\Sigma_{\omega,\theta_0+\frac{\pi}{2}}$ such that $\sup_{\lambda\in\Sigma_{\omega,\theta+\frac{\pi}{2}}} \|(\lambda-\omega)H(\lambda)\| < \infty$ for every $\theta \in (0,\theta_0)$.

Proof. The implication (i) $\Longrightarrow$ (ii) was shown in Proposition 3.4.2. To show the implication (ii) $\Longrightarrow$ (i), suppose that (i) holds. It follows from Theorem 1.1.8 that there exists an analytic function $S : \Sigma_{0,\theta_0} \to L(X)$ such that $\sup_{z\in\Sigma_{0,\theta}} \|e^{-\omega z}S(z)\| < \infty$ for every $\theta \in (0,\theta_0)$ and

$$H(\lambda)x = \frac{1}{\lambda^{n+1}}(I - \widehat{d\mu}(\lambda)A)^{-1}x = \int_0^\infty e^{-\lambda t}S(t)x\,dt$$

for all $\lambda > \omega$ and $x \in X$. Since for some constant $M > 0$, $\|S(t)\| \le Me^{\omega t}$ for all $t > 0$, it follows that

$$\|\frac{1}{k!}H^{(k)}(\lambda)x\| = \|\frac{1}{k!}\int_0^\infty e^{-\lambda t}t^k S(t)x\,dt\| \le M\int_0^\infty e^{-(\lambda-\omega)t}\|x\| = M\frac{1}{(\lambda-\omega)^{k+1}}\|x\|$$

for every $k \in \mathbb{N}_0$, $\lambda > \omega$, and $x \in X$. Theorem 3.1.10 yields that (A,μ) generates an n-times integrated solution operator family S_1. Since

$$\frac{1}{\lambda^{n+1}}(I - \widehat{d\mu}(\lambda)A)^{-1}x = \int_0^\infty e^{-\lambda t}S(t)x\,dt = \int_0^\infty e^{-\lambda t}S_1(t)x\,dt$$

for all $x \in X$, the Uniqueness Theorem (Corollary 1.1.4) implies that $S(t) = S_1(t)$ for all $t > 0$. This shows (i). //

For $n = 0$, Theorem 3.4.4 coincides with the generation theorem of analytic resolvents in [Pr], and for $n = 0$ and $\mu(t) = t$, it is the generation theorem of analytic, strongly continuous semigroups. The following is an immediate consequence of Theorem 1.1.8. The estimate (3.4.1) improves Corollary 2.1 in [Pr].

Remark 3.4.5. Let S be an analytic, integrated solution operator family of analyticity type $(\omega;\theta_0)$ with generator (A,μ). Then, for every $\theta \in (0,\theta_0)$, there exists a constant $C_\theta > 0$ such that

$$\|z^k S^{(k)}(z)\| \le C_\theta e^{\omega \mathrm{Re}\, z}(|\omega||z| + 1)^k \tag{3.4.1}$$

for all $z \in \Sigma_\theta$.

3.5 An Example

In this section we demonstrate some results of Chapter 2 and 3 by discussing the delay problem

$$
\begin{aligned}
v'(t) &= Av(t) && \text{for } 0 \le t \le 1,\\
v'(t) &= Av(t) + Av(t-1) && \text{for } \quad t > 1,\\
v(0) &= x, && \qquad (3.5.1)
\end{aligned}
$$

where A is an unbounded, closed linear operator in a Banach space X and $x \in X$. Integrating (3.5.1), we obtain the equivalent equation

$$
v(t) = \begin{cases} \int_0^t Av(s)ds + x & \text{for } 0 \le t \le 1,\\ \int_0^1 Av(t-s)ds + 2\int_1^t Av(t-s)ds + x & \text{for } \quad t > 1. \end{cases}
$$

Thus, the problem (3.5.1) can be written as

$$
v(t) = \int_0^t Av(t-s)d\mu(s) + x \qquad \text{for } \ t \ge 0, \qquad (3.5.2)
$$

where

$$
\mu(t) = \begin{cases} t & \text{for } 0 \le t \le 1,\\ 2t & \text{for } \quad t > 1. \end{cases} \qquad (3.5.3)
$$

Clearly, $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ for all $\epsilon > 0$ and a simple calculation shows that $\widehat{d\mu}(\lambda) = \frac{1+e^{-\lambda}}{\lambda}$ for every $\lambda \in \mathbb{C}_0$. In the following the mild solutions of (3.5.2), i.e., the solutions of the equation

$$
v(t) = A\int_0^t v(t-s)d\mu(s) + x \qquad \text{for } \ t \ge 0 \qquad (3.5.4)
$$

with μ as in (3.5.3), will be studied.

We ask first for which unbounded operators A, the pair (A, μ) generates an analytic, integrated solution operator family. As in the proof of Corollary 3.1.8 it can be shown that if A is unbounded and (A, η) generates an analytic, integrated solution operator family of analyticity type $(\omega; \theta_0)$, then $\widehat{d\eta}(\lambda) \neq 0$ for all $\lambda \in \Sigma_{\omega,\theta_0+\frac{\pi}{2}}$. Since $\{\lambda \in \mathbb{C} \mid \widehat{d\mu}(\lambda) = 0\} = \{(2n+1)i \mid n \in \mathbb{N}\}$, no pair (A, μ) with A unbounded generates an analytic, integrated solution operator family.

Next, we ask for which operators A the pair (A, μ) generates an integrated solution operator family. Theorem 3.1.9 says that (A, μ) generates an integrated solution operator family if and only if there exist constants $\omega > 0$, $M \geq 0$, and $a \geq 0$ such that $(I - \widehat{d\mu}(\lambda)A)^{-1} \in L(X)$ and $\|(I - \widehat{d\mu}(\lambda)A)^{-1}\| \leq M|\lambda|^a$ for all $\lambda \in \mathbb{C}_\omega$. Let $\omega > 0$. Since

$$(I - \widehat{d\mu}(\lambda)A)^{-1} = \frac{1}{\widehat{d\mu}(\lambda)}(\frac{1}{\widehat{d\mu}(\lambda)} - A)^{-1}$$

for all $\lambda \in \mathbb{C}_\omega$, a necessary condition on A for (A, μ) to generate an integrated solution operator family is that $\frac{1}{\widehat{d\mu}(\lambda)} = \frac{\lambda}{1+e^{-\lambda}} \in \rho(A)$ for all $\lambda \in \mathbb{C}_\omega$. The function $\lambda \mapsto 1 + e^{-\lambda}$ maps the half plane $\mathbb{C}_\omega$ onto the open disk D with center $(1, 0)$ and radius $e^{-\omega}$, which is contained in a sector $\Sigma_{0,\theta}$ for some $\theta \in (0, \frac{\pi}{2}]$. The function $\lambda \mapsto \frac{1}{\lambda}$ maps D into the same sector $\Sigma_{0,\theta}$. Consequently, $\mathbb{C}_\omega$ is mapped into the sector $\Sigma_{0,\theta+\frac{\pi}{2}}$ by the function $\lambda \mapsto \frac{\lambda}{1+e^{-\lambda}}$. Thus, if there exist some constants $M \geq 0$, $\theta \in (0, \frac{\pi}{2}]$, and $b \geq -1$ for which A satisfies

$$\Sigma_{0,\theta+\frac{\pi}{2}} \subset \rho(A) \quad \text{and} \quad \|R(\lambda, A)\| \leq M|\lambda|^b \text{ for all } \lambda \in \Sigma_{0,\theta+\frac{\pi}{2}}, \tag{3.5.5}$$

then $(I - \widehat{d\mu}(\lambda)A)^{-1} \in L(X)$ and

$$\|(I - \widehat{d\mu}(\lambda)A)^{-1}\| = \frac{1}{|\widehat{d\mu}(\lambda)|}\|R(\frac{1}{\widehat{d\mu}(\lambda)}, A)\| \leq M\frac{1}{|\widehat{d\mu}(\lambda)|^{b+1}} \leq M_1|\lambda|^{b+1}$$

for a constant $M_1 > 0$ and all $\lambda \in \mathbb{C}_\omega$. Therefore, the condition (3.5.5) is sufficient for (A, μ) to generate an integrated solution operator family. Moreover, if the resolvent set of A does not contain a region $\Sigma_{\omega,\theta+\frac{\pi}{2}}$ for any $\omega \geq 0$ and $\theta \in (0, \frac{\pi}{2}]$, then (A, μ) does not generate an integrated solution operator family. In this case, the results of Chapter 2 can be applied. This will be done in the following.

Let A be the differential operator $\frac{d}{dr}$ in $X = C_0([0,\infty); \mathbb{C})$ with maximal domain. To solve (3.5.4) by means of Theorem 2.2, we solve first the characteristic equation of (3.5.4), i.e., the first order differential equation

$$(I - \frac{1+\mathrm{e}^{-\lambda}}{\lambda}\frac{d}{dr})y(\lambda)(r) = \frac{1}{\lambda}x(r) \qquad \text{for } r \geq 0 \tag{3.5.6}$$

for every $\lambda > \omega$ for $\omega \geq 0$. From the condition that $y(\lambda) \in C_0([0,\infty); \mathbb{C})$ for all $\lambda > \omega$, the solution

$$y(\lambda)(r) = \frac{1}{1+\mathrm{e}^{-\lambda}}x(r+\cdot)^\wedge(\frac{\lambda}{1+\mathrm{e}^{-\lambda}}) \qquad \text{for } r \geq 0$$

to (3.5.6) is obtained. Let $x(r) = \mathrm{e}^{-r}$ for every $r \geq 0$. Then

$$y(\lambda)(r) = \frac{1}{1+\mathrm{e}^{-\lambda}}\int_0^\infty \mathrm{e}^{-(\frac{\lambda}{1+\mathrm{e}^{-\lambda}})t}\mathrm{e}^{-(r+t)}dt = \frac{\mathrm{e}^{-r}}{\lambda+1+e^{-\lambda}} = \frac{x(r)}{\lambda(1-(-\frac{1+\mathrm{e}^{-\lambda}}{\lambda}))}.$$

Hence

$$y(\lambda) = x\sum_{k=0}^{\infty}(-1)^k\frac{1}{\lambda}(\frac{1+\mathrm{e}^{-\lambda}}{\lambda})^k$$

for $\lambda > \omega \geq 2$. Let

$$I_k(\lambda) := \frac{1}{\lambda}(\frac{1+\mathrm{e}^{-\lambda}}{\lambda})^k = \sum_{j=0}^{k}\binom{k}{j}\frac{1}{\lambda^k}\frac{\mathrm{e}^{-j\lambda}}{\lambda}$$

for every $k \in \mathbb{N}_0$ and $\lambda > 2$. Since the inverse Laplace transform of the function $\lambda \mapsto \frac{1}{\lambda^k}$ is $\frac{t^{k-1}}{(k-1)!}$ for $t \geq 0$ for every $k \in \mathbb{N}$ and since the inverse Laplace transform of the function $\lambda \mapsto \frac{e^{-j\lambda}}{\lambda}$ is $H_j = 1_{(j,\infty)}$, i.e., the unit step function at j, it follows that

$$L^{-1}I_k(t) = \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} H_j(s)ds = H_j^{[k]}(t) \qquad \text{for } t \geq 0.$$

Thus, the inverse Laplace transform of the function y, given by

$$L^{-1}y(t) = x\sum_{k=0}^{\infty}(-1)^k \sum_{j=0}^{k} \binom{k}{j} H_j^{[k]}(t) \qquad \text{for } t \geq 0, \tag{3.5.7}$$

is a mild, exponentially bounded solution of the equation (3.5.4) for $X = C_0([0,\infty);\mathbb{C})$, $A = \frac{d}{dr}$, and $x(r) = e^{-r}$.

Clearly, the problem (3.5.1) can be solved directly by the variation of constants formula (see also [Pr]). The problem (3.5.1) can be written as

$$v'(t) = Av(t) + f(t) \quad \text{for } t \geq 0, \qquad v(0) = x,$$

where

$$f(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq 1 \\ Av(t-1) & \text{for } \quad t > 1. \end{cases}$$

Suppose that A is a generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$. Then by the variation of constants formula,

$$v(t) = T(t)x + \int_0^t T(t-s)f(s)ds \qquad \text{for } t \geq 0$$

is a solution of (3.5.1). Thus, (3.5.1) has a mild solution $v(t) = T(t)x$ on $[0,1]$ for all $x \in X$. On $[0,2]$ the problem (3.5.1) has a mild solution for all $x \in D(A)$ which is given by

$$v(t) = \begin{cases} T(t)x & \text{for } 0 \le t \le 1 \\ T(t)x + \int_1^t T(t-s)Av(s-1)ds & \text{for } 1 < t \le 2 \end{cases}$$
$$= \begin{cases} T(t)x & \text{for } 0 \le t \le 1 \\ T(t)x + (t-1)T(t-1)Ax & \text{for } 1 < t \le 2. \end{cases}$$

In general, (3.5.1) has a mild solution v on $[0, n]$ for all $x \in D(A^{n-1})$ and it is given by

$$v(t) = \sum_{k=0}^{n-1} \frac{((t-k)_+)^k}{k!} T((t-k)_+)A^k x,$$

where $t_+ = \max\{t, 0\}$ for $t \in \mathbb{R}$.

Again, let $X = C_0([0,\infty);\mathbb{C})$, $A = \frac{d}{dr}$, and $x(r) = e^{-r}$. Then A is the generator of the shift semigroup $\{T(t)\}_{t\ge 0}$ defined by $T(t)g(s) = g(s+t)$ for $g \in X$. Let $x(r) = e^{-r}$. Then since $A^k x = (-1)^k x$ and $T((t-k)_+)A^k x = (-1)^k e^{-(t-k)_+} x$ for every $k \in \mathbb{N}_0$, the solution v to (3.5.1) with $X = C_0([0,\infty);\mathbb{C})$, $A = \frac{d}{dr}$, and $x(r) = e^{-r}$ is given as

$$v(t) = \sum_{k=0}^{\infty} \frac{((t-k)_+)^k}{k!} (-1)^k e^{-((t-k)_+)} x \qquad \text{for } t \ge 0. \tag{3.5.8}$$

By the uniqueness of Laplace transform, the solution (3.5.8) coincides with the solution (3.5.7).

REFERENCES

[A] W. Arendt, Vector-valued Laplace transforms and abstract Cauchy problems, Israel Journal of Mathematics 59, 327-351, 1987

[A-E-K] W. Arendt, O. El-Mennaoui, and V. Keyantuo, Local integrated semigroups : evolution with jumps of regularity, Journal of Mathematical Analysis and Applications 186, 572-592, 1994

[A-K] W. Arendt and H. Kellermann, Integrated solutions of Volterra integrodifferential equations and applications, In G. Da Prato and Iannelli, editors, Volterra integrodifferential Equations in Banach spaces and Applications, 21-51, Harlow, Essex, Longman Sci. Tech., 1989

[A-H-N] W. Arendt, M. Hieber, and F. Neubrander, Laplace Transforms and Evolution Equations, Monograph, in preparation

[B] R. G. Bartle, The Elements of Real Analysis, John Wiley & Sons, Inc., 1976

[B-N] B. Bäumer and F. Neubrander, Laplace transform methods for evolution equations, Conference del Seminario di Matematica dell' Università di Bari, to appear

[Da P-I] G. Da Prato and M. Iannelli, Linear integrodifferential equations in Banach spaces, Rend. Sem. Mat. Padova 62, 207-219, 1980

[G] J. Goldstein, Semigroups of Linear Operators and Applications, Oxford Math. Monographs, Clarendon Press, Oxford, 1985

[G-L-S] G. Gripenberg, S. -O. Londen, and O. Staffans, Volterra Integral and Functional Equations, Encyclopedia of Mathematics and Its Applications, Vol. 34, Cambridge University Press, 1990

[H-N] B. Hennig and F. Neubrander, On representations, inversions and approximations of Laplace transforms in Banach spaces, Applicable Analysis 49, 151-170, 1993

[H-P] E. Hille and R. S. Phillips, Functional Analysis and Semi-Groups, American Mathematical Society Colloquium Publications 31, 1957.

[L] Yu. I. Lyubich, The classical and local Laplace-transformation in an abstract Cauchy problem, Uspehi Mat. Nauk 21, 3-51, 1966

[N] F. Neubrander, The Lapace-Stieltjes transform in Banach spaces and abstract Cauchy problems, In Ph. Clément and G. Lumer, editors, Proceedings of the '3rd

International Workshop Conference on Evolution Equations, Control Theory, and Biomathematics', Han-sur-Lesse, Lecture Notes in Pure and Applied Math., vol 155, 417-431, Marcel Dekker, 1994

[O] H. Oka, Linear Volterra equations and integrated solution families, preprint, 1994

[Pa] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, 1983.

[Pr] J. Prüss, Evolutionary Integral Equations and Applications, Birkhäuser Verlag, 1993.

[S] M. G. Smith, Laplace Transform Theory, D. Van Nostrand Company LTD., London, 1966

[W-Z] R. L. Wheeden and A. Zygmund, Measure and Integral, Marcel Dekker, Inc., 1977

[W] D. V. Widder, The Laplace Transform, Princeton University Press, 1941.

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