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Congruences Between Coefficients of a Class of Eta-Quotients and their Applications to Combinatorics

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CONGRUENCES BETWEEN COEFFICIENTS OF A CLASS OF ETA-QUOTIENTS AND THEIR APPLICATIONS TO COMBINATORICS

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Contents

List of Tables

Abstract

Ramanujan in 1920s discovered remarkable congruence properties of the partition function $p(n)$. Later, Watson and Atkin proved these congruences using the theory of modular forms. Atkin, Gordon, and Hughes extended these works to k-colored partition functions. In 2010, Folsom-Kent-Ono and Boylan-Webb proved the congruences of $p(n)$ by studying a ℓ-adic module associated with a certain sequence of modular functions which are related to $p(n)$.

Primary goal of this thesis is to generalize the work of Atkin, Gordon, Hughes, Folsom-Kent-Ono, and Boylan-Webb about the partition function to a larger class of partition functions. For this purpose we study a closely related two parameters family of related functions $p_{[1^c\ell^d]}(n)$ for arbitrary integers c, d. We can define it in the following way:

$$\sum_{n=0}^{\infty} p_{[1^c\ell^d]}(n)q^n := \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^c(1-q^{\ell n})^d}.$$

In this dissertation we prove an infinite family of congruences for the function $p_{[1^c\ell^d]}(n)$ for $\ell = 5, 7, 11, 13,$ and 17. Then we use it to find congruences for ℓ-regular partitions, ℓ-core partitions, ℓ-colored generalized Frobenius partitions.

Next, we study the ℓ-adic module structures related to $p_{[1^c\ell^d]}(n)$. Then we prove an upper bound for the rank of a ℓ-adic module associated with the partition function $p_{[1^c\ell^d]}(n)$ and use that to discuss ℓ-adic properties of $p_{[1^c\ell^d]}(n)$.

Chapter 1. Introduction

Famous German mathematician Martin Eichler once said, "There are five elementary arithmetical operations: addition, subtraction, multiplication, division, and, modular forms". This is because many naturally arising sequences of integers appear as coefficients of modular forms and proving arithmetic identities among integers in those sequences would be significantly harder without modular forms. For example, Conway's famous Moonshine conjecture shows that the least dimension of the space needed to express the symmetries of the largest sporadic simple group is associated with the Fourier coefficients of the well-known elliptic j-function which generates the field of modular functions on $SL_2(\mathbb{Z})$. As another example, in the quantum theory of black holes in string theory, the physical problem of counting the dimensions of certain eigenspaces has led to the study of Fourier coefficients of certain meromorphic modular forms [11].

Fourier coefficients of modular forms are also being studied to understand combinatorial objects such as integer partitions, and their variants. In recent years, many deep results involving integer partitions are proved using the theory of modular forms.

Let n be a positive integer. An (integer) partition of n is a non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_r \geq 1$ that sum to n. Let $p(n)$ be the number of partitions of n. By convention, we take $p(0) = 1$ and $p(n) = 0$ for negative n. For example, if $n = 4$, we have

$$4, \qquad 3+1, \qquad 2+2, \qquad 2+1+1, \qquad 1+1+1+1.$$

Therefore, $p(4) = 5$.

In 1800s Euler found the following generating function for integer partitions which connects the partitions and modular forms as follows.

$$\sum_{n=0}^{\infty} p(n)q^n = (1 + q^1 + q^{1+1} + \cdots)(1 + q^2 + q^{2+2} + \cdots)\cdots = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = \frac{q^{1/24}}{\eta(z)},$$

where $q = e^{2\pi i z}$, z is in the complex upper half plane denoted by $\mathbb{H}$, and

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$$

is a modular form.

In 1920s, Ramanujan studied the integer partitions and discovered remarkable congruence properties in [36, 37, 38]. They were improved later by Watson [41], and Atkin [2].

Theorem 1.0.1 (Ramanujan, Watson, Atkin)**.** *For all positive integers j and n, we have*

$$p(5^j n + \delta_{5,j}) \equiv 0 \pmod{5^j},$$

$$p(7^j n + \delta_{7,j}) \equiv 0 \pmod{7^{[\frac{j+2}{2}]}},$$

$$p(11^j n + \delta_{11,j}) \equiv 0 \pmod{11^j},$$

where $24\delta_{\ell,j} \equiv 1 \pmod{\ell^j}$ for $\ell \in \{5, 7, 11\}$ and $0 \leq \delta_{\ell,j} < \ell^j$.

Definition 1.0.2. *Let ℓ be a prime number. If for all positive integers n and a positive integer m such that $0 \leq m < \ell$, we have*

$$p(\ell n + m) \equiv 0 \pmod{\ell},$$

then we say that $p(n)$ has a Ramanujan congruence at prime ℓ.

A surprising fact is that, Ramanujan congruences only exist for $p(n)$ at these 3 primes. This was another Ramanujan's conjecture and was proved in 2003 by Ahlgren and Boylan in [4] using the theory of modular forms modulo ℓ.

These fascinating congruence properties not only hold for the partition function itself, but also are expected to hold for generalized partitions. In this dissertation we study a two-parameter family of partition generating functions $p_{[1^c\ell^d]}(n)$ defined in the following way:

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^c(1-q^{\ell n})^d} =: \sum_{n=0}^{\infty} p_{[1^c\ell^d]}(n)q^n.$$

This partition function also has been studied in recent years, for example see Chan and Toh [14], and Wang [43].

Generalizing results for $p(n)$ to other partition functions has been a main research topic since 1960s. The first attempt was to study the k-colored partitions ($p_{-k}(n)$) which are the coefficients of the k^{th} power of the generating function of the partition function

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^k} =: \sum_{n=0}^{\infty} p_{-k}(n)q^n.$$

In 1960s Atkin in [1] proved congruences for $p_{-k}(n)$ for small primes using modular equations. Notice that we can obtain the generating function for k-colored partitions by setting $c = k, d = 0$ in the generating function of the two parameter family of partition generating functions $p_{[1^c\ell^d]}(n)$.

The primary goal of this dissertation is to generalize two methods of proving congruences for the partition function $p(n)$ to the two-variable generalized partition function $p_{[1^c\ell^d]}(n)$. From these results, we expect to give a unified proof for some of the recent results about partition congruences and obtain new results for some partition functions. Finally, we use these congruences to prove congruences for more complex partitions like k-colored generalized Frobenius partitions.

The first method is the traditional method used by Watson [41] for genus zero modular curves. In [34], we generalized the proof given by Atkin in [1] where he proved the Ramanujan congruences for $p_{-k}(n)$ modulo small prime powers. His proof uses modular equations for prime numbers which we define in Section 2.2.4. Our generalization enables us to prove Ramanujan congruences for a large class of partition functions including the k-colored partitions for negative values of k. We also generalized the work of Gordon in [18] and Hughes in [23] of proving congruences for $p_{-k}(n)$ modulo powers of 11 and 17 respectively.

Theorem 1.0.3 (Petta Mestrige, S.,[34]). *For $\ell = 5, 7, 11, 13, 17$, for any integers c, d, for any positive integer r, and for any non negative integer m, we have*

$$p_{[1^c \ell^d]}(\ell^r m + n_{r,\ell}) \equiv 0 \pmod{\ell^{A_r}}, \tag{1.0.1}$$

where $24n_{r,\ell} \equiv (c + \ell d) \pmod{\ell^r}$, and A_r is an explicitly calculable non-negative integer defined in equation (4.3.1)*, and it depends on the integers c, d, the positive integer r, and the prime ℓ.*

There are some values of $c, d, r,$ and ℓ such that $A_r = 0$. In these cases, the statement given in Theorem 1.0.3 is vacuously true, and we further discussed this case in Corollary 1.0.6. Moreover, we obtain the following corollary, this is similar to Gordon's Theorem 1.1 in [18].

Corollary 1.0.4. *For $\ell = 5, 7,$ and 13 and for any positive integer r, we have*

$$p_{[1^c \ell^d]}(n) \equiv 0 \pmod{\ell^{\frac{1}{2}\alpha_\ell r + \epsilon}},$$

where $|c + \ell d| \neq 0, 24n \equiv (c + \ell d) \pmod{\ell^r}$, and $\epsilon = \epsilon(c, d) = \mathcal{O}(\log |c + \ell d|)$. Here, $\alpha_\ell = \alpha_\ell(c, d)$ depends on the residue of $c + \ell d \pmod{24}$, and for $c + \ell d > 0$, the values of α_ℓ are stated in Table 1.1.

For $c + \ell d < 0$, the entries in the last column need to be changed to 1 for $\ell = 5, 7$. If $c = -\ell d$, the statement of Corollary (1.0.4) *still holds with $\epsilon = \epsilon(c, d) = \mathcal{O}(1)$.*

Remark 1.0.5. *This is the same shape as Gordon's result (Theorem 2, [18]) for k-color partitions with k replaced by $c + \ell d$.*

Table 1.1. Values of $\alpha_\ell(c, d)$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$\ell = 5$	2	1	1	1	2	2	1	1	1	1	1	0	0	0	1	1	0	0	0	1	1	0	0	0
$\ell = 7$	1	1	1	2	1	1	1	0	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0
$\ell = 13$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Even though Ramanujan's congruences and in general congruences among coefficients of modular forms were studied for so many years, little attention has given to incongruences between them. Proving incongruences between modular forms has been an interesting problem in recent years. Anderson in [6] proved a criterion to determine congruences and incongruences for a large class of modular forms including eta quotients. In [20], Garthwaite and Jameson developed the above mentioned criterion and proved incongruences between modular forms and partitions. Here we use the expansions of the generating modular functions for $p_{[1^c\ell^d]}(n)$ in terms of the explicit basis of modular forms to determine incongruences.

Corollary 1.0.6. *Let A_r be the non-negative integer defined in equation* (4.3.1)*, and n_r be the integer mentioned in Theorem 1.0.3. If $A_r = 0$ then for $\ell = 5, 7,$ and 13, there is some integer m such that*

$$p_{[1^c\ell^d]}(\ell^r m + n_r) \not\equiv 0 \pmod{\ell}. \tag{1.0.2}$$

This result holds for $r = 1$ when $\ell = 11$ and 17.

Remark 1.0.7. *For $\ell = 11$, we expect the incongruences should hold for all positive integers r since A_r is the best possible bound as shown in [18], but we do not prove it in this thesis.*

Theorem 1.0.3 can be applied to prove several recent results about partitions. For example, let $b_\ell(n)$ be the ℓ-regular partition function and it counts the number of partitions of a natural number n with the condition that the parts are not divisible by ℓ. We provide more details about $b_\ell(n)$ in Section 3.2. In 2017 and 2018, Wang proved the following Ramanujan congruences for $b_\ell(n)$.

Corollary 1.0.8 (Wang, [43, 44]). *For any positive integer k and for $m > 0$,*

$$b_5\left(5^{2k}m + \frac{5^{2k}-1}{6}\right) \equiv 0 \pmod{5^k},$$
$$b_7\left(7^{2k-1}m + \frac{3\cdot 7^{2k-1}-1}{4}\right) \equiv 0 \pmod{7^k}.$$

In 1992, using the theory of modular forms modulo ℓ, Kimming and Olsson [29] proved several exceptional congruences for $p_{-k}(n)$ and we state them in Theorem 3.1.4. We can employ Theorem 1.0.3 to prove most of these congruences.

Corollary 1.0.9. *For all positive integers m, we have*

$$p_{-3}(11m+7) \equiv 0 \pmod{11},$$

$$p_{-5}(11m+8) \equiv 0 \pmod{11},$$

$$p_{-7}(11m+9) \equiv 0 \pmod{11},$$

$$p_{-3}(17m+15) \equiv 0 \pmod{17}.$$

Next we look at a more modern way of proving partition congruences that was introduced by Folsom, Ono and Kent in [17].

There is a sequence of modular functions ($L_\ell(c,d,b;z)$, defined in (4.2.3)) that relate to $p(n)$. We study the ℓ-adic module structure associated with these forms. Then, the partition congruences can be obtained by calculating a bound for the rank of an ℓ-adic module associated with these forms. More explicitly, Folsom-Kent-Ono proved that using the theory of modular forms modulo ℓ developed by Serre, the sequence $L_\ell(c,d,b;z)$ lies in a $\mathbb{Z}/\ell^m\mathbb{Z}$ module with rank $\leq \lfloor\frac{\ell-1}{12}\rfloor - \lfloor\frac{\ell^2-1}{24\ell}\rfloor$. This gives a conceptual proof for the fact that all Ramanujan congruences are the congruences that stated in theorem 1.0.1.

The partition function $p(n)$ and its associated ℓ-adic module structures have been studied further by Boylan and Webb in [12]. Their work has been extended by Belmont, Lee, Musat and Trebat-Leder in [13] to include k-colored partitions and Andrews spt-function. In this dissertation, we extend the work of Boylan and Web further, and obtain results about ℓ-adic module structures associated with the partition function $p_{[1^c\ell^d]}(n)$.

Definition 1.0.10. *For $b, c, d \geq 0$, we set*

$$P_\ell(c,d,b;z) := \sum_{n=0}^{\infty} p_{[1^c\ell^d]}\left(\frac{\ell^b n + \ell d + c}{24}\right) q^{n/24}.$$

Following [12] and [13], for $c, d \geq 0$, we define

$$\begin{aligned} \Lambda_\ell^{odd}(c,d,b,m) &:= \textit{Span}_{\mathbb{Z}/\ell^m\mathbb{Z}}\{L_\ell(c,d,\beta;z) \pmod{\ell^m} : \beta \geq b, \text{ and } b, \beta \text{ are odd}\} \\ \Lambda_\ell^{even}(c,d,b,m) &:= \textit{Span}_{\mathbb{Z}/\ell^m\mathbb{Z}}\{L_\ell(c,d,\beta;z) \pmod{\ell^m} : \beta \geq b, \text{ and } b, \beta \text{ are even}\} \end{aligned} \tag{1.0.3}$$

Let $\triangle_\ell(c,d,b,m)$ denote the $\mathbb{Z}/\ell^m\mathbb{Z}$-module that works for most integers c and d. We define this module using the $\mathbb{Z}/\ell^m\mathbb{Z}$-modules defined in (1.0.3).

$$\triangle_\ell^{\mathsf{odd}}(c,d,b,m) = \begin{cases} \Lambda_\ell^{odd}(c,d,b,m) & \text{if } c \geq 0, d \geq 0, \\ \Lambda_\ell^{\mathsf{odd}}(c-2, 2\ell+d, b, m) & \text{if } c \geq 2, d < 0, 2\ell + d \geq 0 \\ \Lambda_\ell^{\mathsf{odd}}(\ell, \ell+d-1, b, m) & \text{if } c = 1, d < 0, \ell + d \geq 1, \\ \Lambda_\ell^{\mathsf{odd}}(2\ell+c-2, 2\ell+d-2, b, m) & \text{if } c < 0, d < 0, \ell + c, \ell + d \geq 2. \end{cases}$$

Here we consider only four cases. There are other possible cases to consider. For example, $d = 1$, and $c < 0$. This case is similar to the case $c = 1$, and $d < 0$ by the symmetry of the operator $T_\ell(c,d)$ which we define in Section 5.1.

We will define the quantity $v(c,d)$ in (5.1.6). This gives a lower bound for the prime ℓ such that the lemmas and theorems are valid.

Theorem 1.0.11. *Let S_k denotes the cusp forms on $SL_2(\mathbb{Z})$ of weight k, let $\ell \geq v(c,d)$ be prime and $m > 0$. Then there is an integer $b_\ell(c,d,m)$ that satisfies,*

1. The nested sequence of $\mathbb{Z}/\ell^m\mathbb{Z}$-modules

$$\triangle_\ell^{odd}(c,d,1,m) \supseteq \triangle_\ell^{odd}(c,d,3,m) \cdots \supseteq \triangle_\ell^{odd}(c,d,2b+1,m) \supseteq \cdots$$

is constant for all b with $2b+1 \geq b_\ell(c,d,m)$. Moreover if one denotes the stabilized $\mathbb{Z}/\ell^m\mathbb{Z}$-module by $\Omega_\ell^{odd}(c,d,m)$, then its rank $r_\ell(c,d)$ is at most $R_\ell(c,d)$, where

$$R_\ell(c,d) := \begin{cases} \dim\left(S_{(\lfloor \frac{e}{2} \rfloor+1)(\ell-1)}\right) - \left\lfloor \frac{e(\ell^2-1)}{24\ell} \right\rfloor & \text{if } c,d \geq 0, \\ \dim\left(S_{(\lfloor \frac{e}{2} \rfloor+2)(\ell-1)}\right) - \left\lfloor \frac{e(\ell^2-1)}{24\ell} \right\rfloor & \text{if } c=1, d<0, \text{ and } \ell+d \geq 1, \\ & \text{or if } c \geq 2, d<0, \text{ and } 2\ell+d \geq 0, \\ \dim\left(S_{(\lfloor \frac{e}{2} \rfloor+3)(\ell-1)}\right) - \left\lfloor \frac{e(\ell^2-1)}{24\ell} \right\rfloor & \text{if } c<0, d<0, \text{ and } 2\ell+c, 2\ell+d \geq 2. \end{cases}$$

and

$$e = \max\{c,d\} \qquad \text{if } c,d \geq 0, \text{ but not both zero},$$

$$e \in \begin{cases} \{c-2, 2\ell+d\} & \text{if } c \geq 2, d<0, \text{ and } 2\ell+d \geq 0, \\ \{\ell, \ell+d-1\} & \text{if } c=1, d<0, \text{ and } \ell+d \geq 1, \\ \{2\ell+d-2, 2\ell+c-2\} & \text{if } c<0, d<0, \text{ and } 2\ell+c, 2\ell+d \geq 2, \end{cases}$$

then we choose e between the two values such that the quantity $R_\ell(c,d)$ is the minimum.

2. The nested sequence of even $\mathbb{Z}/\ell^m\mathbb{Z}$-modules $\{\triangle_\ell^{even}(c,d,b,m) : b \geq b_\ell(c,d,m)\}$ is constant for all b with $2b \geq b_\ell(c,d,m)$. If we denote the stable module by $\Omega_\ell^{even}(c,d,m)$, then its rank is at most $R_\ell(c,d)$.

This theorem provides a conceptual explanation for the existence of congruences for a large class of partitions. Moreover, we see that these partitions are in a space with small dimension modulo ℓ^m. Hence the partition functions satisfy a nice congruence properties among each other.

For example, when $c = 2$ and $d = 8$, we have the rank of the stabilized module is at most $\dim(S_{5(\ell-1)}) - \lfloor\frac{8(\ell^2-1)}{24\ell}\rfloor$ when $\ell > 7$. We see that the right hand of the inequality is zero when $\ell = 11$. Therefore, from Theorem 1.0.11 we see that Ramanujan congruences occur when $\ell = 11$.

If we take $c = 1$ and $d = -1$, we get ℓ-regular partitions $b_\ell(n)$. Theorem 1.0.11 says that the stabilized module has rank at most $\dim\left(S_{(\ell-1)(\frac{\ell+3}{2})}\right) - \lfloor\frac{\ell(\ell^2-1)}{24\ell}\rfloor$. Hence we have Ramanujan congruences for $\ell = 5, 7,$ and 11. These results are justified by the Wang's congruences stated in Theorem 1.0.8.

As the final example, we take $c = 4$ and $d = -3$. As we will shown in Section 6.2, when $\ell = 7$, the generating function of the partition function $p_{[1^4\ell^{-3}]}(n)$ arises as a part of the generating function of another partition function called 7-colored Frobenius partition function. We apply Theorem 1.0.11 to prove striking congruences. The theorem says that the rank of the stabilized module is bounded by $\dim(S_{\lfloor\frac{2\ell+1}{2}\rfloor(\ell-1)}) - \lfloor\frac{(2\ell-1)(\ell^2-1)}{24\ell}\rfloor$. Therefore, the congruences must exist for $\ell = 5, 7, 11,$ and 23. In fact calculating $p_{[1^4\ell^{-3}]}(n)$, we see that, for all $m \geq 1$, we have

$$p_{[1^4 23^{-3}]}(23^2 m + 372) \equiv 0 \pmod{23}.$$

In Chapter 2, we give a brief introduction to modular forms, congruence subgroups, and the operators on modular forms. We also introduce several modular functions, modular equations, and the filtrations of modular forms. Moreover, we state the basic results that we need later when we prove our results in Chapters 4 and 5.

In Chapter 3, we introduce the partition functions that we are working on in this dissertation. We also state several recent congruence relations of these partitions. We will prove some of these relations later in Chapter 6 using Theorem 1.0.3 and Theorem 1.0.11.

In Chapter 4, first we prove Theorem 1.0.3 for primes $\ell = 5, 7$, and 13. For that, we obtain several inequalities that give us a lower bound for the ℓ-adic orders of the Fourier coefficients of certain modular functions. Then we construct modular functions such that they generate the partitions $p_{[1^c\ell^d]}(n)$. In the later part of the Chapter, we prove Theorem 1.0.3 for primes 11 and 17.

In Chapter 5, we prove Theorem 1.0.11. We first obtain Lemmas about the action of the operators on forms modulo prime powers. Then we construct an injective function to determine an upper bound for the rank of the stabilized module.

Finally, in Chapter 6, we apply Theorem 1.0.3 and Corollary 1.0.4 to several partition functions and prove congruences and incongruences. Next, we apply Theorem 1.0.11 and calculate upper bounds of the ranks of certain ℓ-adic modules related to some partition functions.

Chapter 2. Preliminaries

In this chapter, we briefly discuss the basic definitions, properties and theorems about modular forms. Detailed explanations can be found in [15], [25], and [39].

2.1. Basic definitions of modular forms

Let $\mathbb{G}$ be a finite index subgroup of $SL_2(\mathbb{Z})$ and k be an integer. A modular form f of weight k is a function on the extended complex upper-half plane $\mathbb{H}^* := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ satisfying the following three conditions:

1. f is a holomorphic function on $\mathbb{H}$.

2. f is holomorphic at all elements of $\mathbb{P}^1(\mathbb{Q})$ and z in $\mathbb{H}$.

3. f satisfies the following transformation property,

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \quad \text{for any matrix } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{G}.$$

Furthermore, if f vanishes at all points in $\mathbb{P}^1(\mathbb{Q})$, f is called a cusp form. In the case of $k = 0$, we allow f to have poles and we say that f is a modular function on $\mathbb{G}$.

Remarks:

1. For a given weight k, the set of modular forms forms a finite dimensional vector space, denoted by $M_k(\mathbb{G})$. We use $S_k(\mathbb{G})$ to denote the space of cusp forms on $\mathbb{G}$. By replacing the holomorphic requirement with meromorphic in condition (2), we get weakly holomorphic modular forms and we denote this set by $M_k^!(\mathbb{G})$.

2. On $SL_2(\mathbb{Z})$, every modular form or function f satisfies $f(z) = f(z+1)$, and hence f has a Fourier expansion $\sum_{n=m}^{\infty} a_n q^n$, where $q = e^{2\pi i z}$, and m is some finite integer. For examples, the well-known weight -k Eisenstein series, the weight-12 delta function, and the

elliptic j-function have the following q-expansions.

$$E_k(z) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\frac{n^{k-1}q^n}{1-q^n} = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n,$$

$$\Delta(z) := q\prod_{n=1}^{\infty}(1-q^n)^{24} = q - 24q^2 + 252q^3 + 1472q^4 - 4830q^5 + \cdots,$$

$$j(z) := \frac{E_4^3(z)}{\Delta(z)} = q^{-1} + 744 + 196884q + 21493760q^2 \ldots,$$

where B_k is the k-th Bernoulli number and $\sigma_k(n)$ is the k-th power divisor function of n.

Let $M_k := M_k(SL_2(\mathbb{Z}))$ and $S_k := S_k(SL_2(\mathbb{Z}))$.

Theorem 2.1.1 (Theorem 1.23, [32]). *If $k \geq 4$ is even, then M_k is generated by the monomials of the form*

$$E_4(z)^a E_6(z)^b,$$

where $4a + 6b = k$ and $a, b \geq 0$.

Proposition 2.1.2 (Proposition 1.25, [32]). *If $k \geq 4$ is even, then*

$$\dim_{\mathbb{C}}(M_k) = \dim_{\mathbb{C}}(S_k) + 1,$$

$$\dim(M_k) = \begin{cases} \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \pmod{12}, \\ \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2 \pmod{12}. \end{cases}$$

There are some available tools to construct modular forms or functions explicitly on $SL_2(\mathbb{Z})$ and its finite index subgroups. One of them is the Dedekind eta function

$$\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = \Delta(z)^{\frac{1}{24}}.$$

This is a weight $1/2$ modular form of $SL_2(\mathbb{Z})$ under the following transformation property.

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Then, we have

$$\eta\left(\frac{az+b}{cz+d}\right) = \nu(\gamma)(cz+d)^{1/2}\eta(z),$$

where $\nu(\gamma)$ is given by

$$\nu(\gamma) = \begin{cases} \left(\frac{d}{|c|}\right) exp\left(\frac{\pi i}{12}(a+d-3)c - bd(c^2-1)\right) & \text{if } c \not| 2, \\ \\ \left(\frac{c}{|d|}\right) exp\left(\frac{\pi i}{12}(a-2d)c - bd(c^2-1) + 3d - 3\right) \epsilon(c,d) & \text{if } c|2, \end{cases}$$

where $\left(\frac{c}{d}\right)$ is Kronecker-Legendre symbol and $\epsilon(c,d) = -1$ when $c \leq 0$ and $d < 0$ and $\epsilon(c,d) = 1$ otherwise.

2.2. Congruence subgroups, modular curves, and modular forms on congruence subgroups

Let N be a positive integer. The principal congruence subgroup of level N for $SL_2(\mathbb{Z})$ is defined by

$$\Gamma(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod N \right\}$$

Definition 2.2.1. *A subgroup Γ of $SL_2(\mathbb{Z})$ is a congruence subgroup of level N, if $\Gamma(N) \subset \Gamma$ for some $N \in \mathbb{Z}^+$.*

Other important congruence subgroups:

$$\begin{aligned} \Gamma_0(N) :=& \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod N \right\} \\ \Gamma_1(N) :=& \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod N \right\} \end{aligned} \tag{2.2.1}$$

We can calculate the index of $\Gamma(N)$ in $SL_2(\mathbb{Z})$ using the fact that $\Gamma(N)$ is the kernel of the natural homomorphism $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})$. Hence we see that the index $[SL_2(\mathbb{Z}) : \Gamma(N)]$ is finite for all N and if ℓ is a prime, we have

$$[SL_2(\mathbb{Z}) : \Gamma(N)] = N^3 \prod_{\ell | N} \left(1 - \frac{1}{\ell^2}\right).$$

Similarly, we can show that

$$[SL_2(\mathbb{Z}) : \Gamma_1(N)] = N^2 \prod_{\ell | N} \left(1 - \frac{1}{\ell^2}\right),$$

$$[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{\ell | N} \left(1 + \frac{1}{\ell}\right).$$

2.2.1. Modular curves

The group $SL_2(\mathbb{Z})$ and its subgroups act on the extended complex upper half plane $\mathbb{H}^*$ by linear fractional transformation.

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az+b}{cz+d}$$

Let $\pi : \mathbb{H} \to Y(\Gamma)$ be the natural surjection defined by $\pi(z) = \Gamma z$.

Definition 2.2.2 (Definition 2.2.1, [15])**.** *Let Γ be a subgroup of $SL_2(\mathbb{Z})$. For each point $z \in \mathbb{H}$, let Γ_z be the isotropy subgroup of z, i.e., the z-fixing subgroup of Γ,*

$$\Gamma_z = \{\gamma \in \Gamma | \gamma(z) = z\}.$$

A point $z \in \mathbb{H}$ is an elliptic point for Γ if Γ_z is non-trivial as a group of transformations, that is, if the containment $\{\pm I\}\Gamma_z \supset \{\pm I\}$ of matrix groups is proper. The corresponding point $\pi(z) \in Y(\Gamma)$ is also called an elliptic point.

The cups of Γ are the Γ-equivalent classes of $\mathbb{Q} \cup \{\infty\}$. In the case of $SL_2(\mathbb{Z})$, all rational numbers are equivalent to ∞ since every rational number can be written as $\frac{a}{c}$ with $\gcd(a, c) = 1$ and there exists a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ which sends ∞ to a/c if $c \neq 0$. Hence $SL_2(\mathbb{Z})$ has only one cusp ∞. Let ℓ be a prime. On $\Gamma_0(\ell)$, all rational numbers are equivalent to 0 or ∞. Hence it has two cusps 0, and ∞.

For any subgroup Γ of $SL_2(\mathbb{Z})$, we denote $Y(\Gamma)$ the quotient space of orbits under Γ,

$$Y(\Gamma) := \Gamma\backslash\mathbb{H} = \{\Gamma z | z \in \mathbb{H}\}.$$

The curve $Y(\Gamma)$ is a non-compact Riemann surface, and it can be compactified by adjoining cusps to it. We denote $X(\Gamma)$ the compactification of $Y(\Gamma)$,

$$X(\Gamma) := \Gamma\backslash(\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}).$$

For the congruence subgroups $\Gamma_0(N), \Gamma_1(N)$, and $\Gamma(N)$, we denote the compactified modular curves by $X_0(N), X_1(N)$, and $X(N)$.

Theorem 2.2.3 (Theorem $3.1.1$,[15]). *Let Γ be a finite index subgroup of $SL_2(\mathbb{Z})$. Let ϵ_2 and ϵ_3 denote the number of Γ-inequivalent elliptic points of order 2 and 3 in $X(\Gamma)$, let ϵ_∞ the number of Γ-inequivalent cusps of $X(\Gamma)$. Then the genus of $X(\Gamma)$ is*

$$g = 1 + \frac{d}{12} - \frac{\epsilon_2}{4} - \frac{\epsilon_3}{3} - \frac{\epsilon_\infty}{2}.$$

where

$$d = \begin{cases} [SL_2(\mathbb{Z}) : \Gamma]/2 & \text{if } -I \notin \Gamma, \\ [SL_2(\mathbb{Z}) : \Gamma] & \text{otherwise.} \end{cases}$$

For example, the genus of $X_0(5)$ is 0. This follows from the facts that $\Gamma_0(5)$ has 2 cusps and 2 elliptic points which are of order 2, and the index of $\Gamma_0(5)$ of $\Gamma_0(1)$ is 6. Hence we see that the genus of $X_0(5)$ is zero. It can be proved similarly that the genera of $X_0(7)$ and $X_0(13)$ are 0 and the genera of $X_0(11)$ and $X_0(17)$ are 1.

If the genus of a modular curve is zero, the modular function field is generated by a single element and the generator has one simple pole and one simple zero. From this fact we can see that all the modular functions on $SL_2(\mathbb{Z})$ are generated by the j-function. If the modular curve has genus 1, it is an elliptic curve and the field of modular functions is generated by two elements.

2.2.2. Dedekind eta-products and quotients as modular forms

Definition 2.2.4. *We say $\chi : \mathbb{Z} :\to \mathbb{C}$ a Dirichlet Character modulo $m \geq 1$, if it satisfies the following conditions for all $a, b \in \mathbb{Z}$.*

$$\chi(ab) = \chi(a)\chi(b),$$

$$\chi(a) \begin{cases} = 0, & \textit{if } gcd(a,m) > 1, \\ \\ \neq 0, & \textit{if } gcd(a,m) = 1, \end{cases}$$

$$\chi(a+m) = \chi(a).$$

Definition 2.2.5 (Definition 1.15, [32])**.** *If χ is a Dirichlet character modulo N, then we say that a form $f(z) \in M_k(\Gamma_1(N))$ (resp. $S_k(\Gamma_1(N))$) has Nebentypus character χ if*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for all $z \in \mathbb{H}$ and all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. The space of such modular forms (resp. cusp forms) is denoted by $M_k(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N), \chi)$).

If $\chi = \chi_0$ is trivial, then we denote $M_k(\Gamma_0(N), \chi_0)$ (resp. $S_k(\Gamma_0(N), \chi_0)$) by $M_k(\Gamma_0(N))$ (resp. $S_k(\Gamma_0(N))$).

The following proposition is useful to identify Dedekind eta products and quotients as modular forms on congruence subgroups on $\Gamma_0(N)$ and $\Gamma_1(N)$.

Proposition 2.2.6 (Proposition 2.27, [30])**.** *Suppose that there is some $N \geq 1$ such that $f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z)$, with the additional properties that $k := \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$ and*

$$\sum_{\delta|N} \delta \cdot r_\delta \equiv \sum_{\delta|N} \frac{N}{\delta} \cdot r_\delta \equiv 0 \pmod{24}.$$

If the expression

$$\frac{N}{24} \sum_{\delta|N} \frac{(c,\delta)^2 \cdot r_\delta}{(c, \frac{N}{c}) \cdot c\delta}$$

is non-negative (resp. vanishes) for each divisor $c|N$ then,

$$f(z) \in M_k\left(\Gamma_0(N), \chi\right) \quad (\textit{resp} \;\; f(z) \in S_k\left(\Gamma_0(N), \chi\right)).$$

The character here is $\chi(d) := (\frac{(-1)^k s}{d})$, where $s := \prod_{\delta|N} \delta^{r_\delta}$.

The following theorem can be used to determine the order of vanishing of an eta-quotient at cusps.

Theorem 2.2.7 (Theorem 1.65, [32])**.** *Let c, d and N be positive integers with $d|N$ and $\gcd(c, d) = 1$. If $f(z)$ is an eta-quotient satisfying the conditions of Proposition 2.2.6 for $N \geq 1$, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24} \sum_{\delta|N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, \frac{N}{d}) d\delta}.$$

A theorem of Sturm provides a criterion for deciding when two modular forms with integer coefficients congruent modulo a prime number. Let $m \geq 1$ be an integer, and let $f(z) = \sum_{n=0}^{\infty} a(n) q^n$ be a modular form with integer coefficients. Then we define

$$\mathrm{ord}_m(f(z)) = \min\{n : a(n) \not\equiv 0 \pmod{\ell}\}$$

If no such n exists, then we say that $\mathrm{ord}_m(f(z)) := \infty$.

Theorem 2.2.8 (Sturm, Theorem 2.3, [42])**.** *Suppose that N is a positive integer, ℓ is prime, and $f(z), g(z) \in M_k(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$. If*

$$\mathrm{ord}_\ell(f(z) - g(z)) > \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma_0(N)],$$

then we have $f(z) \equiv g(z) \pmod{\ell}$.

2.2.3. Several modular functions

Then we study several modular functions on the congruence subgroup $\Gamma_0(N)$ where N is a prime or a prime square. These functions play a special role in our work.

The function $g_\ell(z)$:

For a prime ℓ, we define $g_{\ell,r}(z)$ on $\mathbb{H}$ by

$$g_{\ell,r}(z) := \left\{\frac{\eta(\ell z)}{\eta(z)}\right\}^r \tag{2.2.2}$$

Theorem 2.2.9 (Theorem 1, chapter 7, [25]). *If $\ell(>3)$ is a prime and r is an integer such that $r(\ell-1) \equiv 0 \pmod{24}$, then $g_{\ell,r}(z)$ is a modular function on $\Gamma_0(\ell)$.*

The function $\phi_\ell(z)$:

Let $\ell > 3$ be a prime. Let $\phi_\ell(z)$ be defined in $\mathbb{H}$ by

$$\phi_\ell(z) := \frac{\eta(\ell^2 z)}{\eta(z)}. \tag{2.2.3}$$

Theorem 2.2.10 (Theorem 2, Chapter 7, [25]). *If ℓ is a prime greater than 3 then $\phi_\ell(z)$ is a modular function on $\Gamma_0(\ell^2)$.*

The function $S_{\ell,r}(z)$:

Let $\ell > 3$ be a prime number and r be any integer. We define $S_{\ell,r}(z)$ by

$$S_{\ell,r}(z) := \sum_{k=0}^{\ell-1} \phi_\ell^r\left(\frac{z}{-\ell k z+1}\right). \tag{2.2.4}$$

Theorem 2.2.11 (Theorem 03, [25]). *$S_{\ell,r}(z)$ is a modular function with respect to $\Gamma_0(\ell)$ and is holomorphic for all $z \in \mathbb{H}$. If $r > 0$, $S_{\ell,r}(z)$ has at most a pole at 0 and is holomorphic at ∞. If $r < 0$, $S_{\ell,r}(z)$ is holomorphic at 0 and has at most a pole at ∞.*

The following theorem describes the Fourier expansion of $S_{\ell,r}(z)$.

Theorem 2.2.12 (Theorem 4, [25]). *Let the integer $p_r(n), n \geq 0$, be defined by means of,*

$$\prod_{m=1}^{\infty}(1-q^m)^r =: \sum_{n=0}^{\infty} p_r(n) q^n, \tag{2.2.5}$$

and for a prime $\ell > 3$, we put $v = \frac{\ell^2-1}{24}$. Then if r is even

$$S_{\ell,r}(z) = -\ell^{-r/2} e^{(\pi i r/4)(1-\ell)} + \phi_\ell^r(z) + \ell^{1-r/2} e^{(\pi i r/4)(1-\ell)} \prod_{n=1}^{\infty}(1-x^n)^{-r} \sum_{\substack{n=0 \\ n \equiv rv \pmod{\ell}}}^{\infty} p_r(n) q^n. \tag{2.2.6}$$

If r is odd,

$$S_{\ell,r}(z) = \phi_\ell^r(z) + \ell^{(1-r)/2} e^{(\pi i r(1-\ell)^2/8} e^{(\pi i r/4)(1-\ell)} \prod_{n=1}^{\infty}(1-x^n)^{-r} \times \sum_{n=0}^{\infty} \left(\frac{rv-n}{\ell}\right) p_r(n) q^n. \tag{2.2.7}$$

2.2.4. Modular equations

Let $\ell \geq 5$ be a prime number and $g_\ell(z)$ be the modular form $g_{\ell,r}(z)$ where r is the least positive integer satisfying the condition $r(\ell-1) \equiv 0 \pmod{24}$.

By the *modular equation* for prime ℓ, we refer to an algebraic equation connecting two modular forms $g_\ell(z)$ and $\phi_\ell(\frac{z}{\ell})$.

Theorem 2.2.13. *For $\ell = 5, 7, 13$ these equations are,*

$(\ell = 5)$

$$\phi_5^5(z) = g_5(5z)\left(5^2\phi_5^4(z) + 5^2\phi_5^3(z) + 5 \cdot 3\phi_5^2(z) + 5\phi_5(z) + 1\right). \tag{2.2.8}$$

$(\ell = 7)$

$$\phi_7^7(z) = g_7^2(7z)\{343\phi_7^6(z) + 343\phi_7^5(z) + 147\phi_7^4(z) + 49\phi_7^3(z) + 21\phi_7^2(z) + 7\phi_7(z) + 1\} + g_7(7z)\{7\phi_7^4(z) + 35\phi_7^5(z) + 49\phi_7^6(z)\}. \tag{2.2.9}$$

$(\ell = 13)$

$$\phi_{13}^{13}(z) + \sum_{r=1}^{13} \sum_{p=\lfloor (r+2)/2 \rfloor}^{7} m_{r,p} g_{13}^p(13z) \phi_{13}^{13-r}(z) = 0, \tag{2.2.10}$$

where,

$$
(m_{r,p}) = \begin{bmatrix}
-11{\cdot}13 & -36{\cdot}13^2 & -38{\cdot}13^3 & -20{\cdot}13^4 & -6{\cdot}13^5 & -13^6 & -13^6 \\
 & 204{\cdot}13 & 346{\cdot}13^2 & 222{\cdot}13^3 & 74{\cdot}13^4 & 13^6 & 13^6 \\
 & -36{\cdot}13 & -126{\cdot}13^2 & -102{\cdot}13^3 & -38{\cdot}13^4 & -7{\cdot}13^5 & -7{\cdot}13^5 \\
 & & 346{\cdot}13 & 422{\cdot}13^2 & 184{\cdot}13^3 & 37{\cdot}13^4 & 3{\cdot}13^5 \\
 & & -38{\cdot}13 & -102{\cdot}13^2 & -56{\cdot}13^3 & -13^5 & -15{\cdot}13^4 \\
 & & & 222{\cdot}13 & 184{\cdot}13^2 & 51{\cdot}13^3 & 5{\cdot}13^4 \\
 & & & -20{\cdot}13 & -38{\cdot}13^2 & -13^4 & -19{\cdot}13^3 \\
 & & & & 74{\cdot}13 & 37{\cdot}13^2 & 5{\cdot}13^3 \\
 & & & & -6{\cdot}13 & -7{\cdot}13^2 & -15{\cdot}13^2 \\
 & & & & & 13^2 & 3{\cdot}13^2 \\
 & & & & & -13 & -7{\cdot}13 \\
 & & & & & & 13 \\
 & & & & & & -1
\end{bmatrix}
$$

.

Here we explain how to obtain the equation for the prime 5, and the other modular equations can be obtained similarly.

Proof. First notice that to find an equation between the two functions $\phi_5(z)$ and $g_5(5z)$, it is enough to consider the order of vanishings at cusps. This is because the zeros or poles of these functions occur at only cusps since they have infinite product expansions. Here $\phi_5(z) = \frac{\eta(25z)}{\eta(z)}$ and $g_5(5z) = \left(\frac{\eta(25z)}{\eta(5z)}\right)^6$ are modular functions on $X_0(25)$. The modular curve $X_0(25)$ has 6 cusps $0, \infty, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}$, and $\frac{4}{5}$. Using Theorem 2.2.7, we calculate the order of vanishing of these modular functions at cusps and they are stated in Table 2.1.

Table 2.1. The order of vanishing at cusps

	0	∞	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$
$\phi_5(z)$	-1	1	0	0	0	0
$g_5(5z)$	-1	5	-1	-1	-1	-1

We first consider $\frac{1}{g_5(5z)}$. This function has only a pole of order 5 at ∞ and holomorphic at every point on the upper half plane. Now by subtracting relevant negative powers of $\phi_5(z)$ and comparing the Fourier expansions, we remove the pole at ∞. The resulting function,

$$\frac{1}{g_5(5z)} - \frac{1}{\phi_5^5(z)} - \frac{5}{\phi_5^4(z)} - \frac{15}{\phi_5^3(z)} - \frac{5^2}{\phi_5^2(z)} - \frac{5^2}{\phi_5(z)}$$

does not have any poles on the modular curve $X_0(25)$. Hence by Liouville's theorem, it is a constant. Then comparing Fourier expansions we see that the constant is zero. $\square$

Remark 2.2.14. *Gordon in [18] found a modular equation for* $\ell = 11$ *and Hughes in [23] found a modular equation for* $\ell = 17$. *However these equations are complicated and we do not state them in this dissertation.*

2.3. Operators on modular forms

In this section, we introduce some operators acting on the space of modular forms. Let $f(z) \in M_k^! (\Gamma_0(N))$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{Q})$. The slash operator is defined as

$$(f(z)\Big|_k \gamma)(z) = (\det(\gamma))^{\frac{k}{2}} (cz+d)^{-k} f(\gamma z).$$

We now introduce Hecke operators for modular forms. These operators play a significant role establishing the congruences between the Fourier coefficients of modular forms and functions.

Definition 2.3.1 (Definition 2.1, [32]). *If* ℓ *is a prime number and* $f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_k (\Gamma_0(N), \chi)$, *then the action of the Hecke operator* $T_{\ell,k,\chi}$ *on* $f(z)$ *is defined by*

$$f(z)|T_{\ell,k,\chi} := \sum_{n=0}^{\infty} \left(a(\ell n) + \chi(\ell) \ell^{k-1} a(n/\ell) \right) q^n.$$

If $\ell \nmid n$, *then we agree that* $a(n/\ell) = 0$. *More generally, if* m *is a positive integer, then the action of* $T_{m,k,\chi}$ *is defined by*

$$f(z)|T_{m,k,\chi} := \sum_{n=0}^{\infty} \left(\sum_{d|gcd(m,n)} \chi(d) d^{k-1} a(mn/d^2) \right) q^n.$$

Note that $\chi(n) = 0$ *if* $\gcd(N, n) \neq 1$.

If χ is the trivial character modulo N, then we write $T_{\ell,k} = T_{\ell,k,\chi}$.

Proposition 2.3.2 (Proposition 2.3, [32]). *Suppose that*

$$f(z) \in M_k(\Gamma_0(N), \chi).$$

If $m \geq 2$, then $f(z)|T_{m,k,\chi} \in M_k(\Gamma_0(N), \chi)$. Moreover, $f(z)|T_{m,k,\chi}$ is a cusp form if $f(z)$ is a cusp form.

There are two other operators acting on the space of modular forms that are important for us.

Definition 2.3.3. *For a Laurent series $f(z) = \sum_{n \geq N} a(n)q^n$, we define the U_ℓ and V_ℓ operators by*

$$U_\ell \left(\sum_{n \geq N}^{\infty} a(n)q^n \right) := \sum_{\ell n \geq N}^{\infty} a(\ell n)q^n, \tag{2.3.1}$$

and

$$V_\ell \left(\sum_{n \geq N}^{\infty} a(n)q^n \right) := \sum_{n \geq N}^{\infty} a(n)q^{\ell n}.$$

Let $g(z) = \sum_{n \geq N} b(n)q^n$ be another Laurent series. The following simple property will play a key role in Section 4.2 when constructing the sequence of modular forms that is related to the generating function of the partition $p_{[1^c \ell^d]}(n)$.

$$U_\ell \left(f(z)g(\ell z) \right) = g(z) U_\ell \left(f(z) \right). \tag{2.3.2}$$

Proposition 2.3.4 ([3], Lemma 7)**.** *If $f(z)$ is a modular function for $\Gamma_0(N)$, if $\ell^2 | N$, then $U_\ell\Big(f(z)\Big)$ is a modular function for $\Gamma_0(N/\ell)$.*

Lemma 2.3.5 (Lemma 2.1, [12])**.** *Let $f(z) \in M_k^!(\Gamma_0(\ell^j))$ and $k \in \mathbb{Z}$. Then we have*

$$1.\ \ U_\ell\Big(f(z)\Big) \in \begin{cases} M_k^!(\Gamma_0(\ell)) & \text{if} \quad j \in \{0,1\}, \\ M_k^!\Big(\Gamma_0(\ell^{j-1})\Big) & \text{otherwise} \end{cases} \tag{2.3.3}$$

$$2.\ \ V_\ell\Big(f(z)\Big) \in M_k^!\Big(\Gamma_0(\ell^{j+1})\Big)$$

U_ℓ operator is an extremely useful tool to detecting congruences of partition functions.

For example, consider the Fourier expansion of $\phi_5(z)$ at ∞.

$$\phi_5(z) = q\prod_{n=1}^{\infty}\left(\frac{1-q^{25n}}{1-q^n}\right) = q + q^2 + 2q^3 + 3q^4 + 5q^5 + 7q^6 + 11q^7 + 15q^8 + 22q^9 + 30q^{10} + \cdots$$

Now using (2.3.2) and U_5 operator, we see that

$$U_5\left(\phi_5(z)\right) = \prod_{n=1}^{\infty}(1-q^{5n})U_5\left(\sum_{n=0}^{\infty}p(n)q^{n+1}\right) = \prod_{n=1}^{\infty}(1-q^{5n})\sum_{n=1}^{\infty}p(5n-1)q^n.$$

Now applying the U_5 operator to the Fourier expansion of $\phi_5(z)$ at ∞, we have that

$$\prod_{n=1}^{\infty}(1-q^{5n})\sum_{n=1}^{\infty}p(5n-1)q^n = 5q + 30q^2 + \cdots .$$

Even though this not a proof for the famous Ramanujan congruence modulo 5, but it gives a way to check for congruences computationally.

Let $f(z) \in M_k^!(\Gamma_0(\ell))$. We can define the U_ℓ operator in the following way.

$$U_\ell\Big(f(z)\Big) = \ell^{k/2-1}\sum_{j=0}^{\ell-1} f(z)|_k\left(\begin{smallmatrix}1 & j\\ 0 & \ell\end{smallmatrix}\right) \tag{2.3.4}$$

The expansion given in (2.3.4) is useful when calculating the Fourier expansion at the cusp 0 of $\Gamma_0(\ell)$.

Definition 2.3.6. *Let $\gamma_1, \gamma_2, \cdots, \gamma_{\ell+1}$ be a set of coset representatives of $\Gamma_0(\ell)\backslash SL_2(\mathbb{Z})$. If $f(z) \in M_k^!(\Gamma_0(\ell))$, then we define the trace operator of f by*

$$\mathrm{Tr}(f) := \sum_{j=1}^{\ell+1} f|_k\gamma_j.$$

The trace $\mathrm{Tr}(f)$ is independent of the choice of coset representatives and it is a modular form of weight k on $SL_2(\mathbb{Z})$.

Lemma 2.3.7 (Lemma 7, [39])**.** *For $f(z) \in M_k^!(\Gamma_0(\ell))$, we have*

$$\mathrm{Tr}(f) = f + \ell^{1-\frac{k}{2}}\left(f|_k W(\ell)\right)|U(\ell), \tag{2.3.5}$$

where $W(\ell) := \left(\begin{smallmatrix}0 & 1\\ -\ell & 0\end{smallmatrix}\right)$.

The Atkin-Lehner operator $W(\ell)$ stabilizes $\Gamma_0(\ell)$. Hence if $f(z) \in M_k^!(\Gamma_0(\ell))$ then $f(z)|_k W(\ell) \in M_k^!(\Gamma_0(\ell))$. Therefore, we substitute $f(z)|_k W(\ell)$ for f in equation (2.3.5) and we obtain

$$\ell^{\frac{k}{2}-1} \operatorname{Tr}\Big(f(z)|_k W(\ell)\Big) = \ell^{\frac{k}{2}-1} f|_k W(\ell) + f|U(\ell). \tag{2.3.6}$$

For a rational integer a, let $\pi_\ell(a)$ to be the ℓ-adic order of a. If

$$f(z) := \sum_{n \geq n_0} a(n) q^n,$$

we define,

$$\pi_\ell\left(f(z)\right) := \min_{n \geq n_0} \left\{\pi_\ell(a(n)\right\}. \tag{2.3.7}$$

The following version of the statement in equation (2.3.6) is useful when finding forms on $SL_2(\mathbb{Z})$ such that they congruent to forms on $\Gamma_0(\ell)$ modulo certain powers of ℓ.

Lemma 2.3.8. *Let $f(z) \in M_k^!(\Gamma_0(\ell))$. Then $f(z)|U(\ell)$ is congruent modulo ℓ^j to a form on $SL_2(\mathbb{Z})$ with the same weight for any positive j if*

$$\pi_\ell\Big(\ell^{\frac{k}{2}-1} f(z)|_k W(\ell)\Big) \geq j.$$

Now we define another modular form. We use this form in Chapter 5 to prove Theorem 1.0.11. Let

$$h_\ell(z) = h(z) := E_{\ell-1}(z) - \ell^{\ell-1} E_{\ell-1}(z)|V(\ell) \tag{2.3.8}$$

From Lemma 2.3.5, we see that $h(z) \in M_{\ell-1}(\Gamma_0(\ell))$. We need the following two lemmas of Serre [39].

Lemma 2.3.9 (Lemma 8, [39])**.** *Let $a \geq 4$ be an integer and assume that $\ell - 1|a$, then we have*

$$h(z) \equiv 1 \pmod{\ell}, \quad h(z)|_a W(\ell) \equiv 0 \pmod{\ell^{1+a/2}} \tag{2.3.9}$$

Lemma 2.3.10 (Lemma 9, [39]). *Let f be a modular form of weight k with rational coefficients on $SL_2(\mathbb{Z})$. Then for any integer $m \geq 0$, we have*

$$\pi_\ell\Bigg(\operatorname{Tr}\left(f(z)h(z)^{\ell^m}\right) - f(z)\Bigg) \geq \inf\left(m+1+\pi_\ell(f(z)), \ell^m + 1 + \pi_\ell(f|_k W(\ell)) - \frac{k}{2}\right).$$

2.4. Modular forms modulo ℓ

The modern approach of studying the sequence $(L_\ell(c,d,b;z))$ stated in (4.2.3) is using the theory of modular forms modulo a prime ℓ which was developed by Serre [39] in 1970s. In fact this has been used by Boylan and Ahlgren in [4] to prove that there are no Ramanujan congruences for primes other than $5, 7$ and 11.

2.4.1. Structure of modular forms modulo ℓ

Theorem 2.4.1 (Clausen-von Staudt). *Let B_k be the k^{th} Bernouilli number. Then for all $k \in \mathbb{Z}$ and for a prime ℓ, we have*

$$B_k + \sum_{\substack{\ell \text{ prime} \\ \ell-1|k}} \frac{1}{\ell} \in \mathbb{Z}.$$

In particular, $v_\ell(B_k) = -1$ if $\ell - 1|k$ and $v_\ell(B_k) \geq 0$ otherwise.

Theorem 2.4.2 (Kummer congruences). *If $k, k' \in \mathbb{N}$ are even and with $k \equiv k' \not\equiv 0 \pmod{\ell - 1}$ then*

$$\frac{B_k}{k} \equiv \frac{B_{k'}}{k'} \pmod{\ell}.$$

More generally, if $k, k' \in \mathbb{N}$ are even, not divisible $\ell - 1$, and $k \equiv k' \pmod{\ell^a(\ell-1)}$ then

$$(1-\ell^{k-1})\frac{B_k}{k} \equiv (1-\ell^{k'-1})\frac{B_{k'}}{k'} \pmod{\ell^{a+1}}$$

Recall

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

Clausen-von Staudt's theorem says $v_\ell(\frac{k}{B_k}) \geq a+1$ when $k \equiv 0 \pmod{\ell^a(\ell-1)}$. Hence we have $E_{\ell-1}(z) \equiv 1 \pmod{\ell}$ and $E_{\ell^a(\ell-1)}(z) \equiv 1 \pmod{\ell^{a+1}}$.

As in Theorem 2.1.1, a modular form $f(z)$ of weight k can be written uniquely as

$$f(z) = \sum_{m,n=0}^{\infty} \alpha_{m,n} E_4(z)^m E_6(z)^n,$$

where $\alpha_{m,n} \in \mathbb{Q}$ such that $4m+6n=k$. Now let $f(z)$ has the following Fourier expansion.

$$f(z) := \sum_{r=0}^{\infty} a(r) q^r.$$

Then comparing coefficients, we have $a(r) \in \mathbb{Q}$ and $a(r)$ is ℓ-integral for all r if and only if $\alpha_{m,n}$ is ℓ-integral for all m,n. Now assume $f(z)$ is a modular form with ℓ-integral coefficients. Then consider the formal power series

$$\overline{f(z)} = \sum_{r=0}^{\infty} \overline{a(r)} q^r$$

where $\overline{a(r)}$ is the reduction modulo ℓ. If $f(z)$ and $g(z)$ are modular forms such that $f(z) \equiv g(z) \pmod{\ell}$ then we have the equality $\overline{f} = \overline{g}$.

From this way we obtain a subalgebra (over $\mathbb{F}_\ell$) of $\mathbb{F}_\ell[[q]]$ and called the algebra of modular forms modulo ℓ and it is denoted by $M(\ell)$.

Let $A(X,Y)$ to be the polynomial

$$E_{\ell-1}(z) = A(E_4(z), E_6(z)).$$

The polynomial A has rational, ℓ-integral coefficients. Reducing the coefficients modulo ℓ, we have $\overline{A(X,Y)} \in \mathbb{F}_\ell[X,Y]$ and from [40], we have

$$M(\ell) = \mathbb{F}_\ell[\overline{E_4(z)}, \overline{E_6(z)}] \equiv \mathbb{F}_\ell[X,Y]/(\overline{A(x,y)} - 1).$$

For any even integer $k \geq 4$, we denote by $M_k(\mathbb{F}_\ell)$ the subspace of $M(\ell)$ consisting of reductions of modular forms of weight k. Then we have

$$M_k(\mathbb{F}_\ell) \subseteq M_{k+\ell-1}(\mathbb{F}_\ell) \subseteq M_{k+2(\ell-1)}(\mathbb{F}_\ell) \subseteq \cdots .$$

Now for $s \in \mathbb{Z}/(\ell-1)\mathbb{Z}$ we set,

$$M^s(\mathbb{F}_\ell) = \bigcup_{k \equiv s \pmod{\ell-1}} M_k(\mathbb{F}_\ell).$$

Then we have

$$M(\ell) = \bigoplus_{\alpha \in \mathbb{Z}/(\ell-1)\mathbb{Z}} M^\alpha(\mathbb{F}_\ell).$$

2.4.2. Filtrations of modular forms

Let $f(z)$ and $g(z)$ are modular forms of weight k_1 and k_2 respectively and that they have ℓ-integral coefficients in their q-expansions. Furthermore, we assume that $f(z) \equiv g(z) \not\equiv 0 \pmod{\ell}$. Then both $\overline{f(z)}$ and $\overline{g(z)}$ belong to $M^s(\mathbb{F}_\ell)$ for some $s \in \mathbb{Z}/(\ell-1)\mathbb{Z}$. Hence we have $k_1 \equiv k_2 \pmod{\ell-1}$.

Let ℓ be a prime, then we denote $\mathbb{Z}_{(\ell)}$ be the localization of $\mathbb{Z}$ at prime ℓ.

Let $k \geq 4$ be even, and let $f(z) \in M_k \cap \mathbb{Z}_{(\ell)}[[q]]$ with $f(z) \not\equiv 0 \pmod{\ell}$. The filtration of $f(z)$ is defined by

$$\omega_\ell(f) := \inf\{k' : f \equiv g \pmod{\ell}, g \in M_{k'} \cap \mathbb{Z}_{(\ell)}[[q]]\}.$$

If $f(z) \equiv 0 \pmod{\ell}$, we set $\omega_\ell(f) = -\infty$. The following Lemma of Serre describes the basic properties of filtrations of modular forms.

Lemma 2.4.3 (Lemma 2, [39]). *Let $\ell \geq 5$ be a prime, and $f(z) \in M_k \cap \mathbb{Z}_{(\ell)}[[q]]$. Then we have*

1. $\omega_\ell\left(f(z)|U_\ell\right) \leq \ell + \dfrac{\omega_\ell(f(z)) - 1}{\ell}$,
2. *If* $\omega_\ell(f(z)) = \ell - 1$, *then* $\omega_\ell\left(f(z)|U_\ell\right) = \ell - 1$,
3. *if* $\omega_\ell(f(z)) = a(\ell - 1)$, *for some* $a \geq 2$, *then* $\omega_\ell\left(f(z)|U(\ell)\right) \leq (a-1)(\ell - 1)$,
4. *If* $\omega_\ell\left(f(z)\right) = \ell - 1$, *then* $\omega_\ell\left(\Delta(z)^{\frac{\ell^2-1}{24}} f(z)|U(\ell)\right) \in \{0, \ell - 1\}$.

A theorem of Serre provides a key fact of congruences among modular forms.

Theorem 2.4.4 (Theorem 1, [39]). *Let $f(z)$ and $g(z)$ be two modular forms with rational coefficients, of respective weights k_1 and k_2. Assume also that $f(z) \neq 0$ and that:*

$$\pi_\ell\left(f(z) - g(z)\right) \geq \pi_\ell(f(z)) + m, \quad \text{for some integer } m \geq 0,$$

$f(z)(\not\equiv 0 \pmod{\ell}) \in M_{k_1} \cap \mathbb{Z}_{(\ell)}[[q]]$ *and* $g(z) \in M_{k_2} \cap \mathbb{Z}_{(\ell)}[[q]]$ *with* $f(z) \equiv g(z) \pmod{\ell^m}$. *Then we have* $k_1 \equiv k_2 \pmod{\ell^{m-1}(\ell - 1)}$.

Chapter 3. Several Partition Functions and Their Congruences

3.1. The k-colored partition function

The k-colored partition functions (also known as multi-partitions or multi-colored partitions) is denoted by $p_{-k}(n)$ and it counts the partitions of n into k colors. For example $p_{-2}(3)$, counts partitions of 3 into 2 colors and there are 10 such partitions.

$$3_1,\ \ 2_1{+}1_1,\ \ 1_1{+}1_1{+}1_1,\ \ 3_2,\ \ 2_2{+}1_2,\ \ 1_2{+}1_2{+}1_2,\ \ 2_1{+}1_2,\ \ 2_2{+}1_1,\ \ 1_2{+}1_1{+}1_1,\ \ 1_2{+}1_2{+}1_1.$$

Notice here that, if a partition consists with same parts, then we need to number the colors and arrange the parts in a decreasing order with respect to the colors. The following function is the generating function of $p_{-k}(n)$.

$$\sum_{n=0}^{\infty} p_{-k}(n)q^n := \prod_{m=1}^{\infty} \frac{1}{(1-q^n)^k} = \frac{q^{\frac{k}{24}}}{\eta^k(z)}. \tag{3.1.1}$$

This partition function has been studied by many researchers in the past half century. In 1960's Atkin studied this function using modular equations and proved congruence relations modulo primes $\ell = 2, 3, 5, 7, 13$ in [1].

Theorem 3.1.1 (Theorem 1, [18]). *Let $k > 0$ and ℓ be one of the primes $2, 3, 5, 7,$ or 13. Then if $24m \equiv k \pmod{\ell^r}$ we have $p_{-k}(m) \equiv 0 \pmod{\ell^{\alpha r/2+\epsilon}}$ where $\epsilon = \epsilon(k) = \mathcal{O}(log(k))$ and $\alpha = \alpha(k, \ell)$ depending on ℓ and the residue of k modulo 24 according to Table 3.1 for $k > 0$.*

Table 3.1. Values of $\alpha(k, \ell)$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2	-	-	-	-	-	-	-	3	-	-	-	-	-	-	-	3	-	-	-	-	-	-	-	3
3	-	-	3	-	-	2	-	-	3	-	-	2	-	-	1	-	-	2	-	-	1	-	-	0
5	2	1	1	1	2	2	1	1	1	1	1	0	0	0	1	1	0	0	0	1	1	0	0	0
7	1	1	1	2	1	1	1	0	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0
13	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

In 1981, Basil Gordon in [18] extended Atkin's result to prime 11 using a linear basis for modular functions on $\Gamma_0(11)$ constructed by A.O.L. Atkin.

Theorem 3.1.2 (Gordon, 1981). *If $24n \equiv k \pmod{11^r}$, then $p_{-k}(n) \equiv 0 \pmod{11^{\frac{r \cdot \alpha_{11}}{2}+\epsilon}}$ where $\epsilon = \epsilon(k) = \mathcal{O}\Big(\log|k|\Big)$, if $k \geq 0, \alpha_{11}(k)$ depends on the residue of $k \pmod{120}$ and they are listed in Table 3.2.*

Table 3.2. Values of $\alpha_{11}(k)$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	2	1	2	1	1	1	2	2	1	1	2	2	1	2	1	0	0	1	1	0	0	1	1	0
24	1	1	1	1	2	2	1	1	2	2	1	0	0	0	0	1	1	0	0	1	1	1	0	0
48	1	1	2	2	1	1	1	0	1	0	1	0	0	1	1	0	0	1	0	1	0	1	0	0
72	2	1	1	1	2	1	2	1	2	1	2	2	1	1	1	2	1	2	1	2	1	1	1	0
96	0	0	1	0	1	0	1	0	1	1	0	0	0	1	0	1	0	1	0	1	1	0	0	0

Here the entry is $\alpha_{11}(24i + j)$ where rows are labeled as $24i$ and columns are labeled as j. When $k < 0$, the last column is changed to $2, 2, 2, 0, 2$.

This result has been extended to prime 17 by Kim Hughes in 1991 in [23] using a linear basis for modular functions on $\Gamma_0(17)$.

Theorem 3.1.3 (Hughes, 1991). *Let k be a nonzero integer. If $24n \equiv k \pmod{17^r}$, then $p_{-k}(n) \equiv 0 \pmod{17^{\frac{r\cdot\alpha_{17}}{2}+\epsilon}}$, where $\epsilon = \epsilon(k) = \mathcal{O}(\log(k))$ and $\alpha_{17} = \alpha_{17}(k)$ depends on the residue of $k \pmod{96}$. The dependence of α_{17} on k is given in Table 3.3 for $k > 0$.*

Table 3.3. Values of α_{17}

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
24	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
48	1	1	1	1	1	2	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0
72	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Here the entry is $\alpha_{17}(24i + j)$ where rows are labeled as $24i$ and columns are labeled j. When $k < 0$, the last column is changed to $0, 2, 0, 0$.

In [33, 34] we have extended all of these results to two-parameter partition function that we called $p_{[1^c\ell^d]}(n)$.

The k-colored partitions also satisfy more general ℓ-adic congruences. In [29], Kimming and Olsson studied these properties using the theory of modular forms modulo ℓ and proved the following result.

Theorem 3.1.4 (Theorem 4, [29]). *For $n \in \mathbb{N}$, we have*

$$\begin{aligned} p_{-3}(11n+7) &\equiv 0 \pmod{11}, \\ p_{-5}(11n+8) &\equiv 0 \pmod{11}, \\ p_{-7}(11n+9) &\equiv 0 \pmod{11}, \\ p_{-3}(17n+15) &\equiv 0 \pmod{17}, \\ p_{-9}(17n+11) &\equiv 0 \pmod{17}, \\ p_{-13}(17n+14) &\equiv 0 \pmod{17}, \\ p_{-7}(19n+9) &\equiv 0 \pmod{19}, \\ p_{-9}(19n+17) &\equiv 0 \pmod{19}, \\ p_{-13}(19n+14) &\equiv 0 \pmod{19}, \end{aligned} \tag{3.1.2}$$

In 2016, Dawsey and Wagner, in [31] proved the following result using the theory of newforms.

Theorem 3.1.5 (Theorem 1-Theorem 3, [31]). *For a prime ℓ and positive integers h and t, let $k + h = \ell t$, and let $\delta_{k,\ell}$ be the integer such that $24\delta_{k,\ell} \equiv k \pmod{\ell}$. Then we have the following Ramanujan-type congruence*

$$p_{-k}(\ell n + \delta_{k,\ell}) \equiv 0 \pmod{\ell}$$

if any of the following holds:

(1). We have $h \in \{4, 8, 14\}$ and $\ell \equiv 2 \pmod{3}$.

(2). We have $h \in \{6, 10\}$ and $\ell \equiv 3 \pmod{4}$.

(3). We have $h = 26$ and $\ell \equiv 11 \pmod{12}$.

3.2. The ℓ-regular partition function

The ℓ-regular partition function is denoted by $b_\ell(n)$ and it counts the number of partitions of a natural number n with the condition that parts are not divisible by ℓ. For example, there are 4 3-regular partitions.

$$4, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

The generating function for these partitions is given by the following product function.

$$\sum_{n=0}^{\infty} b_\ell(n) q^n := \prod_{m=1}^{\infty} \frac{(1-q^{\ell n})}{(1-q^n)} = q^{\frac{1-\ell}{24}} \frac{\eta(\ell z)}{\eta(z)}.$$

Congruences properties of ℓ-regular partitions have been studied by several researchers in recent years. The following theorems describe several recent results about $b_\ell(n)$.

Theorem 3.2.1 (Liuquan Wang, [44])**.** *For any positive integer k and for $n > 0$,*

$$b_5\left(5^{2k} n + \frac{5^{2k}-1}{6}\right) \equiv 0 \pmod{5^k}.$$

Theorem 3.2.2 (Liuquan Wang, [45])**.** *For any positive integer k and for $n > 0$,*

$$b_7\left(7^{2k} n + \frac{7^{2k}-1}{4}\right) \equiv 0 \pmod{7^k}.$$

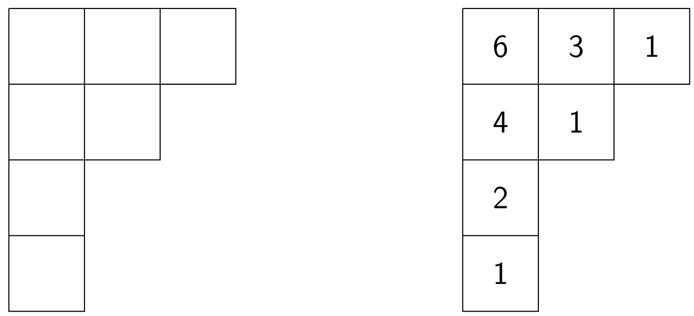
Webb in [42] used Sturm's criterion (Theorem 2.2.8) to prove the following result which was previously conjectured by Callin.

Theorem 3.2.3. *For all integers n and r with $n \geq 0$ and $r \geq 2$, we have*

$$b_{13}\left(3^r n + \frac{5 \cdot 3^{r-1} - 1}{2}\right) \equiv 0 \pmod{3}.$$

3.3. The t-core partition function

t-cores are partitions of a number n with hook numbers of the entries of the Ferrers diagram not divisible by ℓ and denoted by $a_t(n)$. For example, the Ferrers diagram of the partition $3+2+1+1$ and Ferrers diagram with their hook numbers are shown in the following diagrams.



t-cores are also played a special role in a variety of areas. Especially when t is a prime ℓ, they characterize ℓ modular irreducible representation of the symmetric group S_n. They also have been used to define a crank function (t-core rank) that can be used to prove the famous Ramanujan congruences.

$a_t(n)$ has the following generating function:

$$\sum_{n=0}^{\infty} a_t(n)q^n := \prod_{m=1}^{\infty} \frac{(1-q^{tm})^t}{(1-q^m)}$$

t-cores satisfy the famous Ramanujan congruences when $t = 5, 7, 11$ due to the bijection discovered by Frank Garvan, Dennis Stanton, and Donshu Kim in [21]. As a corollary of our main theorem we prove these congruences in chapter 6.

In 2002, Matthew Boylan [13] proved congruences for 2^t-core partitions by studying the action of the Hecke operator $T_{\ell,k}$ when ℓ is an odd prime, on $S_{4^{t+1}-4}$ with integer coefficients modulo 2.

Theorem 3.3.1 (Boylan, [8]). *For any positive integer $t \geq 1$, if $p_1, \cdots, p_{(4^t-1)/3}$ are distinct* *odd primes, then*

$$a_{2^t}\left(\frac{p_1\cdots p_{\frac{(4^t-1)}{3}}N-\frac{4^t-1}{3}}{8}\right)\equiv 0 \pmod 2,$$

for every N *with* $\gcd(N,\prod p_i)=1$.

3.4. The k-colored generalized Frobenius partition function

k-colored Generalized Frobenius partition function $c\phi_k(n)$ is introduced by G.E. Andrews [5] in 1984. They are two-array partitions with non negative entries with monotonically decreasing order on each row,

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_d \\ \\ b_1 & b_2 & \cdots & b_d \end{pmatrix}$$

and satisfy the following conditions

(i). $n=d+\sum_{i=1}^{d}a_i+\sum_{i=1}^{d}b_i$,

(ii). Parts can appear at most k times,

(iii). Parts are colored, colors are numbered from 1 to k and ordered in decreasing order.

Ex: $c\phi_2(2)=9$; $\begin{pmatrix}1_1\\0_1\end{pmatrix}$, $\begin{pmatrix}1_1\\0_2\end{pmatrix}$, $\begin{pmatrix}1_2\\0_1\end{pmatrix}$, $\begin{pmatrix}1_2\\0_2\end{pmatrix}$, $\begin{pmatrix}0_1\\1_1\end{pmatrix}$, $\begin{pmatrix}0_2\\1_1\end{pmatrix}$, $\begin{pmatrix}0_1\\1_2\end{pmatrix}$, $\begin{pmatrix}0_2\\1_2\end{pmatrix}$, $\begin{pmatrix}0_2 & 0_1\\0_2 & 0_1\end{pmatrix}$.

The generating function for $c\phi_k(n)$ is denoted by $C\Phi_k(q)$ and it is given by the following function.

$$C\Phi_k(q)=\sum_{n=0}^{\infty}c\phi_k(n)q^n=\prod_{m=0}^{\infty}\frac{1}{(1-q^m)^k}\sum_{m_1,m_2,\cdots m_{k-1}\in\mathbb{Z}}q^{Q(m_1,m_2,\cdots m_{k-1})}$$

$$\text{where } Q(m_1,m_2,\cdots m_{k-1})=\sum_{i=1}^{k-1}m_i^2+\sum_{1\le i<j\le k-1}m_im_j.$$

Now we state several congruence properties of $c\phi_k(n)$. Luis Kolitsch in 1989 in [27] found an identity between $p(n)$ and $c\phi_5(n)$ and an identity between $p(n)$ and $c\phi_7(n)$. Then he used them to prove congruences for $c\phi_5(n)$ and $c\phi_7(n)$.

Theorem 3.4.1 (Kolitsch, [27]). *For r a positive integer, let δ_r and λ_r be the reciprocals of 24 modulo 5^r and 7^r respectively. Then for all $n \geq 0$,*

$$\begin{aligned} c\phi_5\left(5^{r-1}n + \frac{\delta_r + 1}{5}\right) &\equiv 0 \pmod{5^{r+1}}, \\ c\phi_7\left(7^{r-1}n + \frac{\lambda_r + 2}{7}\right) &\equiv 0 \pmod{7^{\lfloor \frac{r+4}{2} \rfloor}}. \end{aligned} \tag{3.4.1}$$

Frank Garvan and James Sellers [22] in 2013 proved a result that can be used to prove congruences for $c\phi_k(n)$ for a large number of colors from known congruences modulo primes.

Theorem 3.4.2 (Garvan–Sellers, [22]). *Let ℓ be prime and let r be an integer such that $0 < r < \ell$. If*

$$c\phi_k(\ell n + r) \equiv 0 \pmod{\ell},$$

for all $n \geq 0$, then

$$c\phi_{\ell N+k}(\ell n + r) \equiv 0 \pmod{\ell},$$

for all $N \geq 0$ and $n \geq 0$.

In 2012, Peter Paule and Silviu Radu [35] used modular forms to prove the following result which was previously conjectured by James Sellers in 1994.

Theorem 3.4.3 (Radu-Paule, [35]). *For each $r \geq 0$, let λ_r be the smallest positive integers such that $12\lambda_r \equiv 1 \pmod{5^r}$. Then for all $n \geq 0$ we have,*

$$c\phi_2(5^r n + \lambda_r) \equiv 0 \pmod{5^r}. \tag{3.4.2}$$

In 2018, H.H. Chan, Y. Yang, and L. Wang further investigated modular properties of the generating function $C\Phi_k(q)$. Let

$$\mathfrak{A}_k(q) := \prod_{n=1}^{\infty}(1-q^m)^k C\Phi_k(q) = \sum_{m_1, m_2, \cdots m_{k-1} \in \mathbb{Z}} q^{Q(m_1, m_2, \cdots m_{k-1})}.$$

Theorem 3.4.4 (H.H. Chan, L. Wang, Y. Yang, 2018)**.** *If $k = 2r + 1$ is odd,*

$$\mathfrak{A}_k(q) \in M_{(k-1)/2}\left(\Gamma_0(k), \frac{(-1)^r \cdot k}{\cdot}\right)$$

Using this result they found different representations for $C\Phi_k(q)$ for various k values.

Theorem 3.4.5 (Chan, Wang, Yang [16])**.** *Let $C\Phi_k(q)$ be the generating function of the k-colored generalized Frobenius partitions. We have*

$$C\Phi_5(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^{5n}} + 25q \prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)^6}, \tag{3.4.3}$$

$$C\Phi_7(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^{7n}} + 49 \cdot q \prod_{n=1}^{\infty} \frac{(1-q^{7n})^3}{(1-q^n)^4} + 343 \cdot q^2 \prod_{n=1}^{\infty} \frac{(1-q^{7n})^7}{(1-q^n)^8}, \tag{3.4.4}$$

$$C\Phi_{11}(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^{11n}} + 11 \sum_{j=1}^{\infty} p(11j-5)q^j. \tag{3.4.5}$$

$$C\Phi_{13}(q) = \frac{1}{(q^{13};q^{13})_\infty} + 13 \sum_{n=0}^{\infty} p(13n-7)q^n + 26 \cdot q \prod_{n=1}^{\infty} \frac{(1-q^{13n})}{(1-q^n)^2}. \tag{3.4.6}$$

They have also found expressions for $C\Phi_k(q)$ with $k = 4, 6, 8, 10, 12, 15$, and 16.

Chapter 4. Ramanujan Congruences Modulo Powers of Primes $5, 7,$ and 13

4.1. Important inequalities

Theorem 4.1.1 (Newton's Formula, Theorem 9, Chapter 7, [25])**.** *Let*

$$f(x) = x^q - p_1 x^{q-1} + p_2 x^{q-2} - \cdots + (-1)^q p_q,$$

where $p_i \in \mathbb{Z}$ with roots $\phi_1, \cdots, \phi_q$. For a positive integer h, we put, $S_h = \sum_{i=1}^{q} \phi_i^h$. Then if $1 \leq h \leq q$,

$$S_h - p_1 S_{h-1} + p_2 S_{h-2} - \cdots + (-1)^{h-1} p_{h-1} S_1 + (-1)^h p_h h = 0.$$

if $h > q$,

$$S_h - p_1 S_{h-1} + p_2 S_{h-2} - \cdots + (-1)^q p_q S_{h-q} = 0.$$

As described in [25], using the notation of Newton's formula, we let $S_{r,\ell}(z)$ be the sum of the r^{th} power of the roots of the modular equation for prime ℓ.

$$S_{r,\ell} = \sum_{k=0}^{\ell-1} \phi_\ell^r(\zeta_\ell^k q^{\frac{1}{\ell}}). \tag{4.1.1}$$

Let $h(z)$ be a modular function on $\Gamma_0(\ell)$. As shown in Lemma 2, Chapter 8 of [25], we have

$$U_\ell(h(z)) = \frac{1}{\ell} \sum_{k=0}^{\ell-1} h\left(\zeta_\ell^k q^{\frac{1}{\ell}}\right). \tag{4.1.2}$$

Now combining equations (4.1.1) and (4.1.2),

$$S_{r,\ell}(z) = \ell \cdot U_\ell\left(\phi_\ell^r(z)\right). \tag{4.1.3}$$

We use the modular equations to find Laurent series expansions for $S_{r,\ell}(z)$ in $g_\ell(z)$. Lemmas 4.1.2-4.1.4 describe the ℓ-adic orders of the coefficients in those expansions. For convenience, we use $a_{r,p}$ to denote $a_{r,p}^{(\ell)}$ which are the coefficients of $g_\ell(z)$ in the series expansion of $S_{r,\ell}(z)$.

Lemma 4.1.2. *Let r be a non zero integer. Then $S_{r,5}(z)$ is a Laurent polynomial in $g_5(z)$ of the form $S_{r,5}(z) = \sum_{p=-\infty}^{\infty} a_{r,p} g_5(z)^p$, where $a_{r,p}$ is an integer divisible by 5 and satisfying*

$$\pi_5(a_{r,p}) \geq \left\lfloor \frac{5p - r + 1}{2} \right\rfloor, \tag{4.1.4}$$

for $r > 0$, $a_{r,p} \neq 0$ for $\lfloor \frac{r+4}{5} \rfloor \leq p \leq r$ and for $r < 0$, $a_{r,p} = 0$ for $\lfloor \frac{r+4}{5} \rfloor > p$.

Proof. For $r > 0$ this inequality can be found in [25]. We prove this inequality for $r < 0$ here. To calculate $S_{r,5}(z)$ when $r < 0$, we modify the modular equation for prime 5 by dividing the both sides of (2.2.8) by $\phi_5^5(z)$ and $g_5(5z)$.

$$\left(\phi_5^{-1}(z)\right)^5 + 5\left(\phi_5^{-1}(z)\right)^4 + 5 \cdot 3\left(\phi_5^{-1}(z)\right)^3 + 5^2\left(\phi_5^{-1}(z)\right)^2 + 5^2\left(\phi_5^{-1}(z)\right) - \left(g_5^{-1}(5z)\right) = 0.$$

Then using Newton's formula, we have

$$S_{-1,5}(z) = -5,$$

$$S_{-2,5}(z) = -5,$$

$$S_{-3,5}(z) = 25,$$

$$S_{-4,5}(z) = -25,$$

$$S_{-5,5}(z) = 5g_5^{-1}(z),$$

$$S_{r,5}(z){=}-5S_{r+1,5}(z) - 15S_{r+2,5}(z) - 25S_{r+3,5}(z) - 25S_{r+4,5}(z) + g^{-1}(z)S_{r+5,5}(z).$$

First of all using induction with the help of the recurrence relation satisfied by $S_{r,5}(z)$ we see that for $r < 0$, $a_{r,p} = 0$ for $\lfloor \frac{r+4}{5} \rfloor > p$.

Now notice that above claim is true for $r = -1 \cdots -5$. Then we assume the claim holds for any $r < -5$. Again from the recurrence relation, we have

$$a_{r,p} = -5a_{r+1,p} - 15a_{r+2,p} - 25a_{r+3,p} - 25a_{r+4,p} + a_{r+5,p+1} \tag{4.1.5}$$

By the assumption, the 5-adic valuation of the terms in the right hand side of (4.1.5) satisfies the inequality (2.2.8). Hence, we have

$$\begin{aligned}\pi_5\left(a_{r,p}\right) \geq & \min\{\pi_5(a_{r+1,p})+1, \pi_5(a_{r+2,p})+1, \pi_5(a_{r+3,p})+2,\\ & \pi_5(a_{r+4,p})+2, \pi_5(a_{r+5,p+1})\}\\ = & \left\lfloor \frac{5p-r+1}{2} \right\rfloor.\end{aligned}$$

□

Lemma 4.1.3. *Let r be a non zero integer. Then, $S_{r,7}(z)$ is a Laurent polynomial in $g_7(z)$ of the form $S_{r,7}(z)=\sum_{p=-\infty}^{\infty} a_{r,p} g_7^p(z)$, where $a_{r,p}$ is an integer divisible by 7 and satisfying*

$$\pi_7(a_{r,p}) \geq \left\lfloor \frac{7p-2r+3}{4} \right\rfloor, \tag{4.1.6}$$

for $r>0$, $a_{r,p} \neq 0$ for $\lfloor \frac{2r+6}{7} \rfloor \leq p \leq 2r$. For $r<0$, $a_{r,p}=0$ for $\lfloor \frac{2r+6}{7} \rfloor > p$.

Proof. For $r>0$, the proof of the inequality can be found in [25]. We here prove it when $r<0$. To calculate $S_{r,7}(z)$, when $r<0$, we modify the modular equation by dividing (2.2.9) by $\phi_7^7(z)$ and $g_7^2(7z)$. Then using Newton's formula, we have

$$S_{-1,7}(z) = -7,$$

$$S_{-2,7}(z) = 7,$$

$$S_{-3,7}(z) = -7^2,$$

$$S_{-4,7}(z) = -7^2 - 2^2 \cdot 7 g_7^{-1}(z),$$

$$S_{-5,7}(z) = 7^3 + 10 \cdot 7 g_7^{-1}(z),$$

$$S_{-6,7}(z) = 7^3,$$

$$S_{-7,7}(z) = 7^3 g_7^{-1}(z) + 7 g_7^{-2}(z).$$

For $r \leq -8$,

$$\begin{aligned} S_{r,7}(z) =& -7S_{r+1,7}(z) - 3\cdot 7S_{r+2,7}(z) - 7^2 S_{r+3}(z) \\ & - (3\cdot 7^2 + 7g_7^{-1}(z))S_{r+4,7}(z) - (7^3 + 5\cdot 7)S_{r+5}(z) \\ & - (7^3 + 7^2 g_7^{-1}(z))S_{r+6,7}(z) + g_7^{-2}(z)S_{r+7,7}(z). \end{aligned} \tag{4.1.7}$$

Similar to the proof of Lemma 4.1.2, we can use the recurrence relation satisfied by $S_{r,7}(z)$ to see that for $r < 0$, $a_{r,p} = 0$ for $\lfloor \frac{2r+6}{7} \rfloor > p$.

Now notice that (4.1.6) is true for $r = -1 \cdots -7$. Then we assume the claim holds for any $r < -7$. Again from the recurrence relation (4.1.7), we have

$$\begin{aligned} a_{r,p} =& -7a_{r+1,p} - 3\cdot 7a_{r+2,p} - 7^2 a_{r+3,p} - 3\cdot 7^2 a_{r+4,p} - 7a_{r+4,p+1} \\ & - 7^3 a_{r+5,p} - 5\cdot 7a_{r+5,p+1} - 7^3 a_{r+6,p} - 7^2 a_{r+6,p+1} + a_{r+7,p+2}. \end{aligned} \tag{4.1.8}$$

Now by the assumption, the terms in the right hand side of (4.1.8) satisfy inequality (4.1.6). Hence, we have

$$\begin{aligned} \pi_7(a_{r,p}) \geq \min\Big\{ & \pi_7(a_{r+1,p}) + 1, \pi_7(a_{r+2,p}) + 1, \pi_7(a_{r+3,p}) + 2, \pi_7(a_{r+4,p}) + 2, \\ & \pi_7(a_{r+4,p+1}) + 1, \pi_7(a_{r+5,p}) + 3, \pi_7(a_{r+5,p+1}) + 1, \pi_7(a_{r+6,p}) + 3, \\ & \pi_7(a_{r+6,p+1}) + 2, \pi_7(a_{r+7,p+2}) \Big\} \\ = & \left\lfloor \frac{7p - 2r + 3}{4} \right\rfloor. \end{aligned}$$

□

Lemma 4.1.4. *Let r be a non zero integer. Then $S_{r,13}(z)$ is a Laurent polynomial in $g_{13}(z)$ of the form $S_{r,13}(z) = \sum_{p=-\infty}^{\infty} a_{r,p} g_{13}^p(z)$, where $a_{r,p}$ is an integer divisible by 13, and satisfying*

$$\pi_{13}(a_{r,p}) \geq \left\lfloor \frac{13p - 7r + 13}{14} \right\rfloor, \tag{4.1.9}$$

for $r > 0$, $a_{r,p} \neq 0$ for $\lfloor \frac{7r+12}{13} \rfloor \leq p \leq 7r$. For $r < 0$, $a_{r,p} \neq 0$ for $\lfloor \frac{7r+12}{13} \rfloor \leq p \leq 0$.

Proof. For $r > 0$, this is proved in [7]. For $r < 0$, we modify (2.2.10) by dividing it by $\phi_{13}^{13}(z)$ and $g_{13}^{7}(z)$. Then we derive $S_{r,13}(z)$ for negative r using Theorem 4.1.1.

$$S_{r,13}(z) := \sum_{\lfloor \frac{7r+12}{13} \rfloor \leq p \leq 0} a_{r,p} g_{13}^{p}(z). \tag{4.1.10}$$

Here $\{a_{r,p}\}_{r<0,p\leq 0}$ given in the matrix below,

$$\begin{bmatrix}
\cdots & & & & & & & 13 \\
\cdots & & & & & & -2{\cdot}13 & -13 \\
\cdots & & & & & & & 13^2 \\
\cdots & & & & & 26 & 12{\cdot}13^3 & 467{\cdot}13^3 \\
\cdots & & & & & -260 & 2018{\cdot}13^2 & 467{\cdot}13^3 \\
\cdots & & & & 130 & -20{\cdot}13^4 & 36{\cdot}13^4 & 2807{\cdot}13^3 \\
\cdots & & & & 98{\cdot}13 & -26336{\cdot}13^2 & -12{\cdot}13^5 & 935{\cdot}13^4 \\
\cdots & & & -70{\cdot}13 & 84{\cdot}13^4 & -684{\cdot}13^4 & -120{\cdot}13^5 & 3743{\cdot}13^4 \\
\cdots & & & -162{\cdot}13 & 176544{\cdot}13^2 & -15996{\cdot}13^3 & -9030{\cdot}13^4 & 937.13^5 \\
\cdots & & 238{\cdot}13^5 & -396{\cdot}13^4 & 4740{\cdot}13^4 & -192{\cdot}13^5 & -24{\cdot}13^7 & 1871{\cdot}13^5 \\
\cdots & & -902{\cdot}13 & -737836{\cdot}13^2 & 210588{\cdot}13^3 & 10722{\cdot}13^4 & -17806{\cdot}13^5 & -13^6 \\
\cdots & -418{\cdot}13 & 1260{\cdot}13^4 & -16812{\cdot}13^4 & 7416{\cdot}13^5 & 120{\cdot}13^7 & -336{\cdot}13^7 & -1873{\cdot}13^6 \\
\cdots 13 & 51{\cdot}13^3 & 125764{\cdot}13^3 & -77470{\cdot}13^4 & 28214{\cdot}13^5 & 10000{\cdot}13^6 & -815{\cdot}13^7 & -72{\cdot}13^8 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}$$

From the above matrix we can see that the result holds for $r = -13$ to -1. Now fix $r < -13$. Assume the result holds for all negative numbers greater than r. Using Theorem 4.1.1 and (2.2.10), we see that $S_{r,13}(z)$ satisfies the following recursive formula when $r \leq -14$.

$$\begin{aligned}
S_{r,13}(z) &:= \sum_{i=1}^{13} (-1)^{i+1} C_{-i} S_{r+i,13}(z), \\
\text{where } \; C_{-i} &:= (-1)^{i+1} \sum_{\rho = \lceil \frac{14-i}{2} \rceil}^{7} m_{13-i,\rho} g_{13}^{\rho-7}(z), \quad 1 \leq i \leq 13.
\end{aligned} \tag{4.1.11}$$

Notice that $m_{0,7} := 1$ here. Now combining (4.1.10) and (4.1.11), for $r \leq -14$, we have

$$a_{r,p} = \sum_{i=1}^{13} \sum_{t=\lfloor \frac{7r+7i+12}{13} \rfloor}^{0} m_{13-i,p-t+7} \cdot a_{r+i,t}. \tag{4.1.12}$$

Now using (2.2.10), we see that

$$\pi_{13}\left(m_{r,p}\right) \geq \left\lfloor \frac{13p - 7r + 13}{14} \right\rfloor.$$

Then using the induction hypothesis, we have

$$\begin{aligned}
\pi_{13}(a_{r,p}) &= \min_{1\le i\le 13,\ \lfloor\frac{7r+7i+12}{13}\rfloor\le t\le 0}\left\{\pi_{13}(m_{13-i,p-t+7})+\pi_{13}(a_{r+i,t})\right\}.\\
&\ge\left\{\left\lfloor\frac{13(p-t+7)-7(13-i)+13}{14}\right\rfloor+\left\lfloor\frac{13(t)-7(r+i)+13}{14}\right\rfloor\right\},\\
&=\left\{\left\lfloor\frac{13p-13t+7i+13}{14}\right\rfloor+\left\lfloor\frac{13t-7r-7i+13}{14}\right\rfloor\right\},\\
&\ge\left\lfloor\frac{13p-7r+13}{14}\right\rfloor,\quad\text{for all } i,t.
\end{aligned}$$

Here we used the fact that if X, Y are integers then

$$\left\lfloor\frac{X}{14}\right\rfloor+\left\lfloor\frac{Y}{14}\right\rfloor\ge\left\lfloor\frac{X+Y-13}{14}\right\rfloor.$$

□

Lemma 4.1.5. *Let r be a non zero integer. Then we have*

1. $\pi_5\left(S_{r,5}(z)\right)=\pi_5\left(a^{(5)}_{r,\lfloor\frac{r+4}{5}\rfloor}\right)=1$ *iff* $r\not\equiv 1,2 \pmod 5$.
2. $\pi_7\left(S_{r,7}(z)\right)=\pi_7\left(a^{(7)}_{r,\lfloor\frac{2r+6}{7}\rfloor}\right)=1$ *iff* $r\not\equiv 1,4 \pmod 7$.
3. $\pi_{13}\left(S_{r,13}(z)\right)=\pi_{13}\left(a^{(13)}_{r,\lfloor\frac{7r+12}{13}\rfloor}\right)=1$ *iff* $r\not\equiv 10 \pmod{13}$.

For all the other cases, ℓ-adic orders of $S_{r,\ell}(z)$ is greater than or equal to 2.

Proof. The proof follows from induction on r. To prove the induction step, we use the recursive expression for $S_{r,\ell}(z)$ that can be obtained from the modular equations. We demonstrate this by proving (1) when r is a positive multiple of 5.

By Lemma 4, Chapter 8 of [25], we have $S_{5,5}$ only divisible by 5 and $\pi_5(a^{(5)}_{5,1})=1$. Now we assume for $r>1, \pi_5(a^{(5)}_{5r-5,r-1})=1$. Then by the recursive formula for $S_{r,5}(z)$, we have

$$\begin{aligned}
S_{5r,5}(z) =&5^2g_5(z)S_{5r-1,5}(z)+5^2g_5(z)S_{5r-2,5}(z)+15g_5(z)S_{5r-3,5}(z)\\
&+5g_5(z)S_{5r-4,5}(z)+g_5(z)S_{5r-5,5}(z).
\end{aligned}\tag{4.1.13}$$

Then the result follows comparing the coefficient of $g_5^r(z)$. □

Remark 4.1.6. *For $\ell = 5, 7,$ and 13, the ℓ-adic order of $S_{r,\ell}(z)$ is equal to the ℓ-adic order of the coefficient of $g_\ell(z)$ with the least exponent when r is positive. Otherwise it is the ℓ-adic order of the coefficient of $g_\ell(z)$ with the highest exponent. This follows from Lemma 4.1.5 and inequalities* (4.1.4), (4.1.6), *and* (4.1.9).

Let V_ℓ be the vector space of modular functions on $\Gamma_0(\ell)$ where $\ell = 5, 7, 11, 13$ and 17, which are holomorphic everywhere except possibly at 0 and ∞. V_ℓ is mapped to itself by the linear transformation

$$T_\lambda : f(z) \to U_\ell\left(\phi_\ell(z)^\lambda f(z)\right),$$

where λ is an integer. Let $(C_{\mu,\nu}^\lambda)_{\mu,\nu}$ be the matrix of the linear transformation T_λ with respect to a triangular basis of V_ℓ with integral coefficients.

For primes $\ell = 5, 7,$ and 13, we can use $\{g_\ell^\nu(z) | \nu \in \mathbb{Z}^+\}$ as a basis of V_ℓ to calculate the matrix elements when $\lambda > 0$. For $\lambda < 0$, we can take $\{g_\ell^\nu(z) | \nu \in \mathbb{Z}^-\}$ as a basis. For $\ell = 11$, and 17, finding a basis is complicated and they were derived by A.O.L Atkin in [2], and Kim Hughes in [23] respectively.

Now let $\{J_{\ell,\nu}(z) | \nu \in \mathbb{Z}\}$ be an upper triangular basis for V_ℓ with ℓ-integral coefficients. Then we have

$$U_\ell\left(\phi_\ell(z)^\lambda J_{\ell,\mu}(z)\right) = \sum_\nu C_{\mu,\nu}^\lambda J_{\ell,\nu}(z). \tag{4.1.14}$$

Therefore, the Fourier series of $T_\lambda\left(J_{\ell,\mu}(z)\right)$ has all coefficients divisible by ℓ if and only if

$$C_{\mu,\nu}^\lambda \equiv 0 \pmod{\ell} \text{ for all } \nu.$$

Now we define $\theta_\ell(\lambda, \mu) = 1$ if all the coefficients of $U_\ell(\phi_\ell^\lambda J_{\ell,\mu}(z))$ divisible by ℓ. Otherwise we put $\theta_\ell(\lambda, \mu) = 0$.

Lemma 4.1.7. *The values of $\theta_5(\lambda,\mu)$ can be calculated explicitly for all λ and μ using Table 4.1 with the following relations.*

$$\begin{aligned}\theta_5(\lambda,\mu) &= \theta_5(\lambda+5,\mu),\\ \theta_5(\lambda,\mu+1) &= \theta_5(\lambda+6,\mu).\end{aligned} \tag{4.1.15}$$

Table 4.1. Values of $\theta_5(\lambda,\mu)$.

	λ				
μ	0	1	2	3	4
0	0	1	1	0	0

Proof. First notice that using (2.2.8), and (2.3.2), we have that

$$U_5\left(\phi_5^{\lambda+5}(z)g_5^{\mu}(z)\right) = g_5(z)U_5\left(\phi_5^{\lambda}(z)g_5^{\mu}(z)\right) \pmod 5.$$

Now using (4.1.14), we have

$$C_{\mu,\nu}^{\lambda+5} \equiv C_{\mu,\nu-1}^{\lambda} \pmod 5.$$

Hence we have the first equality of (4.1.15). To get the second equality, notice that

$$U_5\left(\phi_5^{\lambda}(z)g_5^{\mu+1}(z)\right) = g_5^{-1}(z)U_5\left(\phi_5^{\lambda+6}(z)g_5^{\mu}(z)\right).$$

Hence we have

$$C_{\mu+1,\nu}^{\lambda} = C_{\mu,\nu+1}^{\lambda+6}.$$

Therefore we have the second equality of (4.1.15). Now using (4.1.4), we can find $\theta_5(\lambda,\mu)$ values for all λ and μ. $\square$

Lemma 4.1.8. *The values of $\theta_7(\lambda,\mu)$ can be calculated explicitly for all λ and μ using Table 4.2 with the following relations.*

$$\begin{aligned} \theta_7(\lambda,\mu) &= \theta_7(\lambda-7,\mu), \\ \theta_7(\lambda,\mu+1) &= \theta_7(\lambda+4,\mu). \end{aligned} \tag{4.1.16}$$

Table 4.2. Values of $\theta_7(\lambda,\mu)$.

	λ						
μ	0	1	2	3	4	5	6
0	0	1	0	0	1	0	0

Proof. First notice that using (2.2.9) and (2.3.2)

$$U_7\left(\phi_7^{\lambda+7}(z)g_7^{\mu}(z)\right) = g_7^2(z)U_7\left(\phi_7^{\lambda}(z)g_7^{\mu}(z)\right) \pmod 7.$$

Now using (4.1.14), we have

$$C_{\mu,\nu}^{\lambda+7} \equiv C_{\mu,\nu-2}^{\lambda} \pmod 7.$$

Hence we have the first equality of (4.1.16). To get the second equality, notice that

$$U_7\left(\phi_7^{\lambda}(z)g_7^{\mu+1}(z)\right) = g_7^{-1}(z)U_7\left(\phi_7^{\lambda+4}(z)g_7^{\mu}(z)\right).$$

Hence we have

$$C_{\mu+1,\nu}^{\lambda} = C_{\mu,\nu+1}^{\lambda+4}.$$

Therefore we have the second equality of (4.1.16). Now using (4.1.6), we can find $\theta_7(\lambda,\mu)$ values for all λ and μ. $\square$

Lemma 4.1.9. *The values of $\theta_{13}(\lambda,\mu)$ can be calculated explicitly for all λ and μ using Table 4.3 with the following relations.*

$$\begin{aligned} \theta_{13}(\lambda,\mu) &= \theta_{13}(\lambda-13,\mu), \\ \theta_{13}(\lambda,\mu+1) &= \theta_{13}(\lambda+2,\mu). \end{aligned} \tag{4.1.17}$$

Table 4.3. Values of $\theta_{13}(\lambda,\mu)$.

	λ												
μ	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	1	0	0

Proof. Again notice that using (2.2.10), and (4.1.14), we have that

$$U_{13}\left(\phi_{13}^{\lambda+13}(z)g_{13}^{\mu}(z)\right) = g_{13}^{7}(z)U_{13}\left(\phi_{13}^{\lambda}(z)g_{13}^{\mu}(z)\right) \pmod{13}.$$

This implies that

$$C_{\mu,\nu}^{\lambda+13} \equiv C_{\mu,\nu-7}^{\lambda} \pmod{13}.$$

Hence we have the first equality of (4.1.17). To get the second equality, notice that

$$U_{13}\left(\phi_{13}^{\lambda}(z)g_{13}^{\mu+1}(z)\right) = g_{13}^{-1}(z)U_{13}\left(\phi_{13}^{\lambda+13}(z)g_{13}^{\mu}(z)\right).$$

Hence we have

$$C_{\mu+1,\nu}^{\lambda} = C_{\mu,\nu+1}^{\lambda+2}.$$

Therefore we have the second equality of (4.1.17). Now using (4.1.9), we can find $\theta_{13}(\lambda,\mu)$ values for all λ and μ. $\square$

4.2. Constructing modular functions

We construct a sequence of modular functions that are the generating functions for the $p_{[1^c\ell^d]}(n)$ restricted to certain arithmetic progressions. This generalizes Gordon's construction for k-colored partitions. Here we use (2.3.2) repeatedly.

Let $L_\ell(c,d,0;z) := 1$ and $\delta_\ell := \frac{\ell^2-1}{24}$,

$$\begin{aligned} L_\ell(c,d,1;z) :&= U_\ell\left(\phi_\ell(z)^c\right), \\ &= U_\ell\left(q^{\delta_\ell\cdot c}\prod_{n=1}^{\infty}\frac{(1-q^{\ell^2\cdot n})^c(1-q^{\ell\cdot n})^d}{(1-q^n)^c(1-q^{\ell\cdot n})^d}\right), \\ &= \prod_{n=1}^{\infty}(1-q^{\ell\cdot n})^c(1-q^n)^d \sum_{m\geq\lceil\frac{\delta_\ell\cdot c}{\ell}\rceil}^{\infty} p_{[1^c\ell^d]}(\ell\cdot m-\delta_\ell\cdot c)q^m. \end{aligned}$$

Similarly we define

$$L_\ell(c,d,2;z) := U_\ell\left(\phi_\ell^d(z)L_\ell(c,d,1;z)\right),$$
$$L_\ell(c,d,2;z) = \prod_{n=1}^{\infty}(1-q^{\ell\cdot n})^d(1-q^n)^c \sum_{m\geq\left\lceil\frac{\delta_\ell\cdot d+\lceil\frac{\delta_\ell\cdot c}{\ell}\rceil}{\ell}\right\rceil}^{\infty} p_{[1^c\ell^d]}(\ell^2 m-\delta_\ell\cdot\ell\cdot d-\delta_\ell\cdot c)q^m.$$

Now, to get an equation for higher powers, we define

$$L_\ell(c,d,r;z) := U_\ell\left(\phi_\ell^{\lambda_{r-1}}(z)L_\ell(c,d,r-1;z)\right), \tag{4.2.1}$$

where

$$\lambda_r = \begin{cases} c & \text{if } r \text{ is even }, \\ d & \text{if } r \text{ is odd.} \end{cases} \tag{4.2.2}$$

Then by a short calculation using (2.3.2), there exist integers n_r and μ_r such that

$$\begin{aligned} L_\ell(c,d,2r;z) &= \prod_{n=1}^{\infty}(1-q^n)^c(1-q^{\ell\cdot n})^d\sum_{m\geq\mu_{2r}} p_{[1^c\ell^d]}(\ell^{2r}m+n_{2r})q^m, \\ L_\ell(c,d,2r-1;z) &= \prod_{n=1}^{\infty}(1-q^{\ell\cdot n})^c(1-q^n)^d\sum_{m\geq\mu_{2r-1}} p_{[1^c\ell^d]}(\ell^{2r-1}m+n_{2r-1})q^m. \end{aligned} \tag{4.2.3}$$

From (4.2.1) and (4.2.3) we can see that

$$n_{2r,\ell}(c,d) = -\delta_\ell\cdot d\cdot\ell^{2r-1}+n_{2r-1},$$

$$n_{2r-1,\ell}(c,d) = -\delta_\ell\cdot c\cdot\ell^{2r-2}+n_{2r-2}.$$

Since $n_{0,\ell}=0$, using above recurrence relations, we have

$$n_{1,\ell}(c,d) = -\delta_\ell \cdot c,$$

$$n_{2,\ell}(c,d) = -\delta_\ell \cdot \ell \cdot d - \delta_\ell \cdot c.$$

Using the summation of a geometric series,

$$\begin{aligned} n_{2r-1,\ell}(c,d) &= -c\left(\frac{\ell^{2r}-1}{24}\right) - \ell \cdot d\left(\frac{\ell^{2r-2}-1}{24}\right). \\ n_{2r,\ell}(c,d) &= -c\left(\frac{\ell^{2r}-1}{24}\right) - \ell \cdot d\left(\frac{\ell^{2r}-1}{24}\right). \end{aligned} \tag{4.2.4}$$

From this we have that

$$24 \cdot n_{2r-1,\ell} \equiv (c + \ell \cdot d) \mod \ell^{2r-1} \text{ and } 24 \cdot n_{2r,\ell} \equiv (c + \ell \cdot d) \mod \ell^{2r}.$$

Therefore, each n_r satisfies

$$24 n_{r,\ell}(c,d) \equiv (c + \ell \cdot d) \mod \ell^r.$$

Now, we need to find μ_r in terms of integers c, d. Using (4.2.1), we have that

$$\mu_{r,\ell} = \left\lceil \frac{\delta_\ell \lambda_{r-1} + \mu_{r-1}}{\ell} \right\rceil. \tag{4.2.5}$$

Notice also that μ_r is the least integer m such that $\ell^r m + n_r \geq 0$, which implies that

$$\begin{aligned} \mu_{2r-1,\ell} &= \left\lceil \frac{\ell \cdot c + d}{24} - \frac{c + \ell \cdot d}{24 \cdot \ell^{2r-1}} \right\rceil, \\ \mu_{2r,\ell} &= \left\lceil \frac{c + \ell \cdot d}{24} - \frac{c + \ell \cdot d}{24 \cdot \ell^{2r}} \right\rceil. \end{aligned} \tag{4.2.6}$$

Following Gordon, we represent these formulas in the following form.

$$\begin{aligned} \mu_{2r-1,\ell} &= \left\lceil \frac{\ell \cdot c + d}{24} \right\rceil + \omega_\ell(c,d) \quad \text{if } |c + \ell \cdot d| < \ell^{2r-1}, \\ \mu_{2r,\ell} &= \left\lceil \frac{c + \ell \cdot d}{24} \right\rceil + \omega_\ell(c,d) \quad \text{if } |c + \ell \cdot d| < \ell^{2r}, \end{aligned} \tag{4.2.7}$$

$$\text{where } \omega_\ell(c,d) = \begin{cases} 1 & \text{if } c + \ell \cdot d < 0 \text{ and } 24 | (c + \ell \cdot d), \\ 0 & \text{otherwise.} \end{cases}$$

4.3. Proofs of congruences

Now we define,

$$A_{r,\ell}(c,d) := \sum_{i=0}^{r-1} \theta_\ell(\lambda_i, \mu_i), \tag{4.3.1}$$

for any positive integer r and integers c, d. We put $A_0 := 0$.

We prove $\pi_\ell(L_\ell(c,d,r;z)) \geq A_r(c,d)$.

Proof. For $\ell = 5, 7$ and 13, we calculate

$$L_\ell(c,d,0;z) := 1,$$

$$L_\ell(c,d,1;z) := U_\ell\left(\phi_\ell(z)^{\lambda_0} g_\ell^{\mu_0}\right) = \sum_{\nu_1=\mu_1} C^{\lambda_0}_{\mu_0,\nu_1} g_\ell^{\nu_1},$$

Here we have $\pi_\ell(L_\ell(c,d,1;z)) \geq \theta_\ell(\lambda_0,\mu_0)$,

$$\begin{aligned} L_\ell(c,d,2;z) :=& U_\ell\left(\phi_\ell^{\lambda_1} L_\ell(c,d,1;z)\right) = U_\ell\left(\phi_\ell(z)^{\lambda_1} \sum_{\nu_1=\mu_1} C^{\lambda_0}_{\mu_0,\nu_1} g_\ell^{\nu_1}\right) \\ =& \sum_{\nu_1=\mu_1} C^{\lambda_0}_{\mu_0,\nu_1} U_\ell\left(\phi_\ell(z)^{\lambda_1} g_\ell^{\nu_1}\right), \\ =& \sum_{\nu_1=\mu_1} C^{\lambda_0}_{\mu_0,\nu_1} \sum_{\nu_2=\mu_2} C^{\lambda_1}_{\nu_1,\nu_2} g_\ell^{\nu_2} = \sum_{\nu_2=\mu_2}\left(C^{\lambda_0}_{\mu_0,\mu_1} C^{\lambda_1}_{\mu_1,\mu_2} + \cdots\right) g_\ell^{\nu_2}. \end{aligned}$$

Here we have $\pi_\ell(L_\ell(c,d,2;z)) \geq \theta_\ell(\lambda_0,\mu_0) + \theta_\ell(\lambda_1,\mu_1)$.

Now by induction we have that,

$$L_\ell(c,d,r;z) := \sum_{\nu_r=\mu_r}\left(C^{\lambda_0}_{\mu_0,\mu_1} C^{\lambda_1}_{\mu_1,\mu_2} \cdots C^{\lambda_{r-1}}_{\mu_{r-1},\nu_r} + \cdots\right) g_\ell^{\nu_r}. \tag{4.3.2}$$

Here λ_r and μ_r are defined in the previous section. Also notice that it is sufficient to consider the first coefficient of the series expansion of $S_{\ell,r}$ to get a lower bound for $\pi_\ell(L_\ell(c,d,r;z))$ by remark 4.1.6.

$\square$

Proof of corollary 1.0.4. Recall (4.3.1),

$$A_{r,\ell}(c,d) = \sum_{i=0}^{r-1} \theta_\ell(\lambda_i, \mu_i).$$

In [33] we used Gordon's argument from Section 4 of [18] to obtained the result for $\ell = 11$. Here we use similar argument for primes $\ell = 5, 7, 13$ with k replaced by $c + \ell d$, . Note that Gordon's calculations had terms involving $11k$, which will be written in a more symmetric shape here using the fact that $\ell(c + \ell d) \equiv \ell c + d \pmod{24}$ for any prime $\ell \neq 2, 3$.

$$\begin{aligned} A_{r,\ell}(c,d) &= \sum_{i=0}^{\log_\ell(c+\ell d)} \theta_\ell(\lambda_i, \mu_i) + \sum_{i=\log_\ell(c+\ell d)}^{r-1} \theta_\ell(\lambda_i, \mu_i) \\ &= \sum_{i=0}^{\log_\ell(c+\ell d)} \theta_\ell(\lambda_i, \mu_i) + N_{1,\ell} \cdot \theta_\ell\left(d, \left\lceil \frac{\ell c + d}{24} \right\rceil + \omega_\ell(c,d)\right) \\ &\qquad\qquad + N_{2,\ell} \cdot \theta_\ell\left(c, \left\lceil \frac{c + \ell d}{24} \right\rceil + \omega_\ell(c,d)\right). \end{aligned}$$

Here $N_{1,\ell}$ is the number of odd integers and $N_{2,\ell}$ is the number of even integers in the interval $\left(\log_\ell |c + \ell d|, r - 1\right)$ respectively.

$$\text{Set} \quad \alpha_\ell := \alpha_\ell(c,d) = \theta_\ell\left(d, \left\lceil \frac{\ell c + d}{24} \right\rceil + \omega_\ell(c,d)\right) + \theta_\ell\left(c, \left\lceil \frac{c + \ell d}{24} \right\rceil + \omega_\ell(c,d)\right). \tag{4.3.3}$$

Now if $r \leq \log_\ell |c + \ell d| + 1$ then $N_{1,\ell} = N_{2,\ell} = 0$,

$$A_r \leq \log_\ell |c + \ell d|.$$

If $r > \log_\ell |c + \ell d| + 1$,

$$\left|N_{1,\ell} - \frac{1}{2}(r - 1 - \log_\ell(c + \ell d))\right| + \left|N_{2,\ell} - \frac{1}{2}(r - 1 - \log_\ell(c + \ell d))\right| < 1.$$

Now consider,

$$\left|A_{r,\ell} - \frac{1}{2}\alpha_\ell(r - 1 - \log_\ell(c + \ell d))\right| < 2 + \log_\ell |c + \ell d|,$$

$$\left|A_{r,\ell} - \frac{\alpha_\ell r}{2}\right| < 2 + \frac{\alpha_\ell}{2} + (1 + \frac{\alpha_\ell}{2}) \log_\ell |c + \ell d|.$$

So we have $A_{r,\ell} = \frac{1}{2}\alpha_\ell r + \mathcal{O}\left(\log|c + \ell d|\right)$.

Now we prove the condition for α_ℓ. As in [33], the proof is complete once we show that α_ℓ only depends on $c + \ell d$ with the period 24 when $\ell = 5, 7$ and 13. Periodicity follows by lemmas 4.1.7, 4.1.8, 4.1.9, and $\alpha_\ell(c + \ell d)$ is invariant under the maps,

$$c \to c + 24 - \ell k \quad \text{and} \quad d \to d + k \quad \text{for each integer k.}$$

When $c + \ell d < 0$ and $24|(c + \ell d)$, using (4.2.7), $\omega_\ell(c, d)$ is 1. Thus, using (4.3.3) we have to change the last column of the table 1.1 for entries when $\ell = 5$ and 7. □

4.4. Using bases for the field of modular functions on $\Gamma_0(11)$ and $\Gamma_0(17)$

Let V_{11} be the vector space of modular functions on $\Gamma_0(11)$, which are holomorphic everywhere except possibly at 0 and ∞. Atkin constructed a basis for V, Gordon in [2] slightly modified these basis elements and defined $J_\nu(z)$. For detailed information about the construction of the basis elements see [2].

Theorem 4.4.1 (Gordon [18])**.** *For all $\nu \in \mathbf{Z}$, we have:*

1. $J_\nu(z) = J_{\nu-5}(z)J_5(z)$,
2. $\{J_\nu(z)| - \infty < \nu < \infty\}$ *is a basis of* V,
3. $\mathrm{ord}_\infty J_\nu(z) = \nu$,
4. $$\mathrm{ord}_0 J_\nu(z) = \begin{cases} -\nu & \text{if } \nu \equiv 0 \pmod 5, \\ -\nu - 1 & \text{if } \nu \equiv 1, 2 \text{ or } 3 \pmod 5, \\ -\nu - 2 & \text{if } \nu \equiv 4 \pmod 5. \end{cases}$$
5. *The Fourier series of $J_\nu(z)$ has integer coefficients, and is of the form $J_\nu(z) = q^\nu + \ldots$.*

Now recall

$$\phi_{11}(z) := q^5 \prod_{n=1}^{\infty} \left(\frac{1-q^{121n}}{1-q^n} \right) = \frac{\eta(121z)}{\eta(z)}.$$

This is a modular function on $\Gamma_0(121)$ by proposition 2.3.4, hence V is mapped to itself by the linear transformation,

$$T_\lambda : f(z) \to U_{11}\left(\phi_{11}(z)^\lambda f(z)\right)$$

where λ is an integer. Let $(C^{\lambda}_{\mu,v})_{\mu,\nu}$ be the matrix of the linear transformation T_λ with respect to the basis elements $J_\nu(z)$.

$$U_{11}\left(\phi_{11}(z)^\lambda J_\mu(z)\right) = \sum_{\nu} C^{\lambda}_{\mu,\nu} J_\nu(z) \tag{4.4.1}$$

Gordon in [18], proved an inequality (equation (17)) about the 11-adic orders of the matrix elements (denoted by $\pi(C_{\mu,\nu})$).

$$\pi_{11}(C^{\lambda}_{\mu,\nu}) \geq \left\lfloor \frac{11\nu - \mu - 5\lambda + \delta_{11}}{10} \right\rfloor. \tag{4.4.2}$$

Here $\delta = \delta(\mu, \nu)$ depends on the residues of μ and $\nu \pmod 5$ according to the Table 4.4.

Table 4.4. Values of $\delta_{11}(\mu, \nu)$.

	ν				
μ	0	1	2	3	4
0	-1	8	7	6	15
1	0	9	8	2	11
2	1	10	4	13	12
3	2	6	5	4	13
4	3	7	6	5	9

We can clearly see from the table that $\delta \geq -1$, so we can rewrite (4.4.2) as,

$$\pi_{11}(C^{\lambda}_{\mu,\nu}) \geq \left\lfloor \frac{11\nu - \mu - 5\lambda - 1}{10} \right\rfloor. \tag{4.4.3}$$

Now by Lemma $2.1(v)$, the Fourier series of $T_\lambda(J_\mu(z))$ has all coefficients divisible by 11 if and only if,

$$C^\lambda_{\mu,\nu} \equiv 0 \pmod{11} \text{ for all } \nu.$$

Similarly we define $\theta_{11}(\lambda,\mu) = 1$ if all the coefficients of $U_{11}(\phi_{11}^\lambda J_\mu(z))$ are divisible by 11. Otherwise we put $\theta_{11}(\lambda,\mu) = 0$.

Now from the recurrences obtained in [18] page 119, we have

$$\theta_{11}(\lambda-11,\mu) = \theta_{11}(\lambda+12,\mu-5) = \theta_{11}(\lambda,\mu). \tag{4.4.4}$$

Therefore the values of $\theta_{11}(\lambda,\mu)$ are completely determined by its values in the range $0 \le \lambda \le 10$ and $0 \le \mu \le 4$. Those values are listed in Table 4.5.

Table 4.5. Values of $\theta_{11}(\lambda,\mu)$.

	λ										
μ	0	1	2	3	4	5	6	7	8	9	10
0	0	1	0	1	0	1	0	1	1	0	0
1	1	1	0	1	0	0	0	1	1	0	0
2	1	1	1	0	0	0	0	1	1	0	0
3	1	0	1	0	0	0	0	1	1	0	0
4	1	0	1	0	1	0	1	1	0	0	0

Kim Hughes calculated the $\theta_{17}(\lambda,\mu)$ values in [23], using a linear bases for modular functions on $\Gamma_0(17)$.

Lemma 4.4.2. *The values of $\theta_{17}(\lambda,\mu)$ can be calculated explicitly for all λ and μ using Table 4.6 with the following relations.*

$$\begin{aligned} \theta_{17}(\lambda,\mu) &= \theta_{17}(\lambda-17,\mu), \\ \theta_{17}(\lambda,\mu) &= \theta_{17}(\lambda+6,\mu-4). \end{aligned} \tag{4.4.5}$$

Table 4.6. Values of $\theta_{17}(\lambda, \mu)$.

	λ																
μ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0

4.5. Proofs for the congruences modulo powers of 11 and 17

Here we follow Gordon's argument to prove $\pi_{11}\left(L_{11}(c,d,r;z)\right) \geq A_r(c,d)$.

Proof of Theorem 1.0.3 for $\ell = 11$. We see that using Proposition 2.3.4, $L_{11}(c,d,r;z) \in V$ for all r. So we have

$$L_{11}(c,d,r;z) = \sum_{\nu} a_{r,\nu} J_\nu. \tag{4.5.1}$$

Now by (2.3.1), (4.2.1) and (4.5.1),

$$a_{r,\nu} = \sum_{\mu \geq \mu_{r-1}} a_{r-1,\mu} C_{\mu,\nu}^{\lambda_{r-1}}. \tag{4.5.2}$$

Now we prove by induction,

$$\pi_{11}(L_{11}(c,d,r;z)) \geq A_r + \left[\frac{\nu - \mu_r}{2}\right] \quad \text{for } \nu \geq \mu_r. \tag{4.5.3}$$

$$\pi_{11}(a_{r,\nu}) \geq A_r + \left[\frac{\nu - \mu_r}{2}\right] \quad \text{for } \nu \geq \mu_r.$$

Since $A_0 = 0$, the result holds for $r = 0$. Now assume the result is true for $r-1$. Using (4.5.3) we have,

$$\pi_{11}(a_{r-1,\nu}) \geq A_{r-1} + \left[\frac{\nu - \mu_{r-1}}{2}\right] \quad \text{for } \nu \geq \mu_{r-1}. \tag{4.5.4}$$

Now from equation (4.5.2) we have,

$$\pi_{11}(a_{r,\nu}) \geq \min_{\mu \geq \mu_{r-1}} \left(\pi_{11}(a_{r-1,\mu}) + \pi_{11}(C_{\mu,\nu}^{\lambda_{r-1}})\right). \tag{4.5.5}$$

From (4.4.3) and (4.5.4) the right hand side of (4.5.5) is at least equal to,

$$A_{r-1} + \left[\frac{\mu - \mu_{r-1}}{2}\right] + \left[\frac{11\nu - \mu - 5\lambda_{r-1} - 1}{10}\right]. \tag{4.5.6}$$

This expression cannot decrease if μ is increased by 2, so its minimum occurs when $\mu = \mu_{r-1} + 1$, therefore at,

$$A_{r-1} + \left[\frac{11\nu - \mu_{r-1} - 5\lambda_{r-1} - 2}{10}\right]. \tag{4.5.7}$$

Now from (4.2.1) we have,

$$\mu_r = \left\lceil\frac{5\lambda_{r-1} + \mu_{r-1}}{11}\right\rceil, \tag{4.5.8}$$

therefore $\mu_r \geq \left(\frac{5\lambda_{r-1} + \mu_{r-1}}{11}\right)$.

Plugging it in (4.5.7), the right hand side of (4.5.4) is at least equal to

$$\begin{aligned} A_{r-1} + \left[\frac{11\nu - 11\mu_r - 2}{10}\right] &= A_{r-1} + 1 + \left[\frac{11(\nu - \mu_r) - 12}{10}\right] \\ &\geq A_r + \left[\frac{\nu - \mu_r}{2}\right] \text{ for all } \nu \geq \mu_r + 2, \end{aligned}$$

since $A_{r-1} + 1 \geq A_r$.

Now consider $\nu = \mu_r$ or $\nu = \mu_r + 1$.

If $\mu = \mu_{r-1}$,

$$\pi_{11}(a_{r-1}, \mu_{r-1}) + \pi_{11}\left(C^{\lambda_{r-1}}_{\mu_{r-1},\nu}\right) \geq A_{r-1} + \theta(\mu_{r-1}, \lambda_{r-1}) = A_r.$$

This also works when $\mu \geq \mu_{r-1} + 2$, since by induction hypothesis,

$$\pi_{11}(a_{r-1}, \mu) \geq A_{r-1} + \left[\frac{\mu - \mu_{r-1}}{2}\right] \geq A_r.$$

Now consider $\mu = \mu_{r-1} + 1$,

Now we need to show,

$$\pi_{11}(a_{r-1},\mu_{r-1}+1)+\pi_{11}(C^{\lambda_{r-1}}_{\mu_{r-1}+1,\nu})\geq A_{r-1}+\left\lceil\frac{11\nu-(\mu_{r-1}+1)-5\lambda_{r-1}+\delta(\mu_{r-1}+1,\nu)}{10}\right\rceil.$$

Since $\nu=\mu_r$ or μ_r+1, it suffices to show that when $\theta(\lambda_{r-1},\mu_{r-1})=1$,

$$\left\lceil\frac{11\mu_r-\mu_{r-1}-1-5\lambda_{r-1}+\delta(\mu_{r-1}+1,\mu_r)}{10}\right\rceil\geq 1,$$

and

$$\left\lceil\frac{11(\mu_r+1)-\mu_{r-1}-1-5\lambda_{r-1}+\delta(\mu_{r-1}+1,\mu_r+1)}{10}\right\rceil\geq 1.$$

Now from (4.4.4), Table 4.4 and Table 4.5 we see that the above claims hold.

We do not provide a proof for $\ell=17$ here, since it is similar to the proof of Theorem 3 in [23]. □

Proof of Corollary 1.0.4 for $\ell=11$ and 17. .

The proof is very similar to the proof of Corollary 1.0.4 for primes $5,7,$ and 13. Therefore, we only prove the condition for α_{11}. The proof is complete once we show that α_{11} only depends $c+11d$ with the period 120. Periodicity follows by the fact that $\alpha_{11}(c+11d)$ is invariant under the each maps,

$$c\to c+120-11k \quad \text{and} \quad d\to d+k \quad \text{for each integer k.}$$

Then notice that, when $\ell=17$, $\alpha_{17}(c+\ell d)$ is invariant under the following maps by Lemma 4.4.2.

$$c\to c+96-17\cdot k \quad \text{and} \quad d\to d+k \quad \text{for each integer k.}$$

□

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Chapter 5. ℓ-adic Module Structures Associated with Partition Functions

5.1. More definitions and basic results

Now we define the operators that we are going to use throughout this chapter.

Definition 5.1.1. *For non negative integers c and d, we define*

$$\begin{aligned} f(z)|D_c(\ell) &:= \Big(f(z)\cdot(\phi_\ell(z))^c\Big)|U(\ell)), \\ f(z)|T_\ell(c,d) &:= \Big(f(z)|D_c(\ell)\Big)|D_d(\ell), \end{aligned} \tag{5.1.1}$$

Recall from Chapter 4, (4.2.1), we have for all $r \geq 0$

$$L_\ell(c,d,r;z) = \begin{cases} 1 & \text{if } r=0, \\ \\ L_\ell(c,d,r-1;z)|D_{\lambda_{r-1}}(\ell), & \text{if } r \geq 1. \end{cases} \tag{5.1.2}$$

where

$$\lambda_r = \begin{cases} c & \text{if } r \text{ is even }, \\ \\ d & \text{if } r \text{ is odd.} \end{cases}$$

Lemma 5.1.2. *For $b \geq 0$, we have that*

$$L_\ell(c,d,b;z) = \begin{cases} \eta^c(z)\eta^d(\ell z)\cdot P_\ell(c,d,b;z) & \text{if } b \text{ is even}, \\ \\ \eta^c(\ell z)\eta^d(z)\cdot P_\ell(c,d,b;z) & \text{if } b \text{ is odd}. \end{cases}$$

Lemma 5.1.3. *If $b \geq 1$ is an integer then, $L_\ell(c,d,b;z)$ is in $M_0^!(\Gamma_0(\ell)) \cap \mathbb{Z}[[q]]$.*

Now we define important variables which play an important role in the weight calculation. Let $r \geq 2$ be an integer. Then we define

$$k_\ell(r,j) := \begin{cases} (1+\lfloor\frac{r}{2}\rfloor)(\ell-1), & j=1, \\ \\ \lfloor\frac{r}{2}\rfloor\ell(\ell-1), & j=2, \\ \\ \ell^{j-1}(\ell-1), & j\geq 3. \end{cases} \tag{5.1.3}$$

We define $k_\ell(r,j) := \ell^{j-1}(\ell-1)$ for $r = 0, 1$. We also set, for $r \geq 0$,

$$\kappa_\ell(r,j) := k_\ell(r,j) + (\ell - 1),$$
$$a_r(j) := \frac{k_\ell(r,j)}{\ell - 1}.$$

We also define the quantity $v(c,d)$. This gives a lower bound for the primes $\ell \geq 5$ such that Theorem 1.0.11 is true. Here we only consider the cases

(1). $c \geq 0$ and $d \geq 0$, but not both c and d are zero.

(2). $c \geq 2, d < 0$, and $2\ell + d \geq 0$,

(3). $c < 0, d < 0, 2\ell + c - 2 \geq 0$, and $2\ell + d - 2 \geq 0$,

(4). $c = 1, d < 0$, and $\ell + d - 1 \geq 0$.

For cases $(1) - (3)$, we define

$$y(c,d) := \begin{cases} 5 + 2\lfloor \frac{c}{2} \rfloor + d & \text{if } d \text{ odd,} \\ 3 + \lfloor \frac{c}{2} \rfloor + \lfloor \frac{d}{2} \rfloor & \text{if } d \text{ even.} \end{cases} \tag{5.1.4}$$

For case (4), we define

$$y(c,d) := \begin{cases} 4 + d & \text{if } d \text{ odd,} \\ 2 + \lfloor \frac{d}{2} \rfloor & \text{if } d \text{ even.} \end{cases} \tag{5.1.5}$$

Then we define

$$v(c,d) := \max\{5, y(c,d)\} \tag{5.1.6}$$

for each c, d discussed in (5.1.4) and (5.1.5).

Remark 5.1.4. *There are other conditions need to satisfy for the prime ℓ. For example, for cases $(1) - (3)$, when c is odd, we need $\ell \geq 5 + c + 2\lfloor \frac{d}{2} \rfloor$. Similarly, when c is even, we need*

$\ell \geq 3 + \lfloor \frac{c}{2} \rfloor + \lfloor \frac{d}{2} \rfloor$. *This follows from the symmetry of the operator* $T_\ell(c,d)$ *introduced in* (5.1.1). *We discuss how to obtain these conditions in Remark 5.2.8.*

Remark 5.1.5. *For* $c = 1, d = 0$*, we can take* $\ell \geq 5$ *as shown in [12]. Now from the symmetry of the operator, we have the same bound when* $d = 1, c = 0$*. For* $d = 0$ *and for any* $c \geq 1$*, we can take* $\ell \geq c + 5$ *as shown in [13]. Again from the symmetry of the operator, we have* $\ell \geq d + 5$ *when* $c = 0$ *and for any* $d \geq 1$*.*

We then calculate lower bounds for the ℓ-adic orders of the several modular forms under the Atkin-Lehner operator. Notice that similar calculations have been used in [12] and [13].

Lemma 5.1.6. *Let* k *be a positive integer.*

1. *Let* $f \in M_{2k}(SL_2(\mathbb{Z})) \cap \mathbb{Z}[[q]]$*. Then*

$$\pi_\ell \left(f|_{2k} W(\ell) \right) \geq k.$$

2. *Let* $A_\ell(z) := \frac{\eta^\ell(z)}{\eta(\ell z)} \in M_{\frac{\ell-1}{2}}(\Gamma_0(\ell)) \cap \mathbb{Z}[[q]]$*. Then*

$$\pi_\ell \left(A_\ell^{2k}|_{k(\ell-1)} W(\ell) \right) \geq k \left(\frac{\ell+1}{2} \right).$$

Proposition 5.1.7. *For all* $j \geq 1$*, we have,*

$$E_{\ell-1}^{\ell^{j-1}}(z) \equiv 1 \pmod{\ell^j}, \quad A_\ell^{2\ell^{j-1}}(z) \equiv 1 \pmod{\ell^j}$$

Proposition 5.1.8 (Proposition 2.3, [12])**.** *Let* $k \geq 1$*, and let* $f(z) \in M_k^{'} \cap \mathbb{Z}_{(\ell)}[[q]]$*,*

1. *We have* $f(z)|T_{\ell,k} \equiv f(z)|U(\ell) \pmod{\ell^{k-1}}$
2. *The operator* $U(\ell)$ *stabilizes* $M_k^{'} \cap \mathbb{Z}_{(\ell)}[[q]]$ *modulo* ℓ^{k-1}*.*

5.2. Important lemmas about operators

5.2.1. U(ℓ) and $D_r(\ell)$ preserve modular forms

Definition 5.2.1. *we will write* $f \in^j S$ *if f is congruent modulo* ℓ^j *to a modular form in a space S.*

Lemma 5.2.2. *Let $\ell \geq v(c,d)$ be a prime, $c, d,$ be non negative integers (not both zero) and $m \geq 1$. If $b \geq 0$ is even, then we have that $L_\ell(c,d,b;z) \in^j M_{k_\ell(d,j)}$ (for $b > 0$, $L_\ell(c,d,b;z)$ is a cusp form). If $b \geq 1$ is odd, then $L_\ell(c,d,b;z) \in^j M_{k_\ell(c,j)}$.*

Here we only prove Lemma 5.2.2 for $c, d \geq 1$. Boylan and Webb proved the case where $c = 1$ and $d = 0$ in [12]. For $c = 0$ and $d = 1$, the result holds for sending b to $b+1$ in Corollary 3.5 of [12]. For $d = 0$, and $c \geq 1$ the result follows from Lemma $3.4(2)$ in [13]. Again sending b to $b+1$ in Lemma $3.4(2)$, the result follows for $c = 0$, and $d \geq 1$.

For $b = 0$, since $L_\ell(c,d,0;z) = 1$ and $E_{\ell-1}^{\ell^{j-1}} \equiv 1 \pmod{\ell^j}$.Thus $L_\ell(c,d,0;z) \in^j M_{k_\ell(d,j)}$ for $j \geq 1$.

Now we need to prove the induction step for even b to b+2 and even b to b+1. This is given in next lemma.

Lemma 5.2.3. *Let $\ell \geq v(c,d)$ be a prime, $c, d \geq 0$, and suppose that we have $\psi(z) \in^j M_{k_\ell(d,j)}$ for $1 \leq j \leq m$. Then for all $1 \leq j \leq m$, we have $\psi(z)|T_\ell(c,d) \in^j S_{k_\ell(d,j)}$. Furthermore, $\psi(z)|D_c(\ell) \in^j S_{k_\ell(c,j)}$ for $1 \leq j \leq m$.*

Proof. By the symmetry, it is enough to show that if $\psi(z)|D_c(\ell) \in^j S_{k_\ell(c,j)}$.

We first prove the case when $j = 1$.

Consider $\psi(z)|D_c(\ell)$. Let $f(z) = \psi(z)\phi_\ell(z)^c$, then using Lemma 2.4.3 and using that $\phi(z) \equiv \Delta(z)^{\frac{\ell^2-1}{24}} \pmod{\ell}$, we have

$$\begin{aligned}
\omega_\ell\left(f(z)|U(\ell)\right) \leq & \ell + \frac{(\ell-1)(1+\lfloor\frac{d}{2}\rfloor) + c(\frac{\ell^2-1}{2}) - 1}{\ell} \\
= & \frac{\ell^2 - 1 + (\ell-1)(1+\lfloor\frac{d}{2}\rfloor) + c(\frac{\ell^2-1}{2})}{\ell} \\
= & (\ell-1)\left(1 + \frac{c}{2} + \frac{2 + \frac{c}{2} + \lfloor\frac{d}{2}\rfloor}{\ell}\right) \\
\leq & (\ell-1)(1+\lfloor\frac{c}{2}\rfloor),
\end{aligned}$$

if

$$\ell > \begin{cases} 4 + c + 2\lfloor \frac{d}{2} \rfloor & \text{if } c \text{ odd,} \\ \\ 2 + \frac{c}{2} + \lfloor \frac{d}{2} \rfloor & \text{if } c \text{ even.} \end{cases}$$

Hence $\psi(z)|D_c(\ell) \in^1 M_{k_\ell(c,1)}$ and $\psi(z)|T_\ell(c,d) \in^1 M_{k_\ell(d,1)}$, if

$$\ell > \begin{cases} 4 + d + 2\lfloor \frac{c}{2} \rfloor & \text{if } d \text{ odd,} \\ \\ 2 + \frac{d}{2} + \lfloor \frac{c}{2} \rfloor & \text{if } d \text{ even.} \end{cases}$$

Now for $j \geq 2$, we use the decomposition in (5.2.1). Notice that similar decompositions have been used in [12] and [13].

$$\begin{aligned} \psi(z)|D_c(\ell) \equiv & \frac{g_{j-1}(z)E_{\ell-1}(z)^{a_d(j)}}{A_\ell(z)^{2a_d(j-1)}}|D_c(\ell) \\ & + \left(g_{j-1}(z)E_{\ell-1}(z)^{a_d(j)-a_d(j-1)} - \frac{g_{j-1}(z)E_{\ell-1}(z)^{a_d(j)}}{A_\ell(z)^{2a_d(j-1)}}\right)|D_c(\ell) \\ & + \left(g_j(z) - g_{j-1}(z)E_{\ell-1}(z)^{a_d(j)-a_d(j-1)}\right)|D_c(\ell) \pmod{\ell^j}. \end{aligned} \tag{5.2.1}$$

Here $g_j(z) \in M_{k_\ell(d,j)}$ such that $\psi(z) \equiv g_j(z) \pmod{\ell^j}$ for all j.

Now lets show each part satisfies the required condition. For the first part, we define,

$$f_j(z) := \frac{g_{j-1}(z)\phi_\ell(z)^c}{A_\ell(z)^{2a_d(j-1)}}|U(\ell). \tag{5.2.2}$$

Recall $h(z) := E_{\ell-1}(z) - \ell^{\ell-1}E_{\ell-1}(\ell z) \in M_{\ell-1}(\Gamma_0(\ell))$ and $h(z) \equiv 1 \pmod{\ell}$. Hence we have,

$$h(z)^{a_c(j)} \equiv 1 \pmod{\ell^j}.$$

Now we use Lemma 2.3.10, to show that

$$f_j(z) \equiv \mathrm{Tr}(f_j(z)h(z)^{a_c(j)}) \pmod{\ell^j}. \tag{5.2.3}$$

Table 5.1. Lower bounds of the ℓ-adic orders

	$g_{j-1}(z)\|_{k_\ell(d,j-1)}\gamma_k$	$(A_\ell(z)^{2a_d(j-1)}\|_{k_\ell(d,j-1)}\gamma_k)^{-1}$	$\phi_\ell(z)^c\|_0\gamma_k$
$1\le k\le \ell-1$	0	0	$\frac{-c}{2}$
$k=0$	$k_\ell(d,j-1)$	$-\ell a_d(j-1)$	-c

Therefore, we will show that,

$$\pi_\ell\left(\mathrm{Tr}(f_j(z)h(z)^{a_c(j)})) - f_j(z)\right) \geq j. \tag{5.2.4}$$

Consider

$$f_j(z)|_0W(\ell) = \frac{g_{j-1}(z)\phi_\ell(z)^c}{A_\ell(z)^{2a_d(j-1)}}|U(\ell)|_0W(\ell) = \frac{1}{\ell}\sum_{k=0}^{\ell-1}\frac{g_{j-1}(z)\phi_\ell(z)^c}{A_\ell(z)^{2a_d(j-1)}}|_0\left(\begin{smallmatrix}1 & k\\ 0 & \ell\end{smallmatrix}\right)|_0\left(\begin{smallmatrix}0 & -1\\ \ell & 0\end{smallmatrix}\right)$$

Let $\gamma_k = \left(\begin{smallmatrix}1 & k\\ 0 & \ell\end{smallmatrix}\right)\left(\begin{smallmatrix}0 & -1\\ \ell & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix}k\ell & -1\\ \ell^2 & 0\end{smallmatrix}\right)$. For that, we calculate the lower bounds of the valuations of the factors of $f_j(z)|_0W(\ell)$. We list these values in Table 5.1. We do not provide details here since these calculations are given in [12]. Using the table values, we have that

$$\begin{aligned}\pi_\ell\left(f_j(z)|_0W(\ell)\right) &\geq -1 + k_\ell(d,j-1) - \ell a_d(j-1) - c,\\ &= -1 + k_\ell(d,j-1) - \frac{\ell k_\ell(d,j-1)}{\ell-1} - c,\\ &= -1 - a_d(j-1) - c.\end{aligned} \tag{5.2.5}$$

Now from Lemma 2.3.10, we have

$$\begin{aligned}&\pi_\ell\left(\mathrm{Tr}(f_j(z)h(z)^{a_c(j)}) - f_j(z)\right)\\ &\geq \min\left(j + \mathrm{ord}_\ell(f_j(z)), a_c(j) + 1 + \mathrm{ord}_\ell(f_j(z)|_0W(\ell))\right),\\ &= a_c(j) - a_d(j-1) - c,\\ &\geq j,\end{aligned} \tag{5.2.6}$$

if

$$\ell > \begin{cases} 4 + c + 2\lfloor\frac{d}{2}\rfloor & \text{if } c \text{ odd},\\ 2 + \frac{c}{2} + \lfloor\frac{d}{2}\rfloor & \text{if } c \text{ even}.\end{cases}$$

Hence by proposition 5.1.8, the first summand is satisfied the required property, now consider we second summand,

$$g_{j-1}(z)E_{\ell-1}(z)^{a_d(j)-a_d(j-1)} - \frac{g_{j-1}(z)E_{\ell-1}(z)^{a_d(j)}}{A_\ell(z)^{2a_d(j-1)}}$$
$$= E_{\ell-1}(z)^{a_d(j)-a_d(j-1)}\left(g_{j-1}(z) - \frac{g_{j-1}(z)E_{\ell-1}(z)^{a_d(j-1)}}{A_\ell(z)^{2a_d(j-1)}}\right).$$

Now we define

$$B_{j,\ell}(z) := g_{j-1}(z) - \frac{g_{j-1}(z)E_{\ell-1}(z)^{a_d(j-1)}}{A_\ell(z)^{2a_d(j-1)}} \equiv 0 \pmod{\ell^{j-1}}$$

Hence, we have

$$\frac{B_{j,\ell}(z)}{\ell^{j-1}}E_{\ell-1}(z)^{a_d(j)-a_d(j-1)} \equiv \frac{B_{j,\ell}(z)}{\ell^{j-1}} \pmod{\ell},$$

and

$$\frac{B_{j,\ell}(z)}{\ell^{j-1}}|D_c(\ell) \equiv \frac{B_{j,\ell}(z)}{\ell^{j-1}}\Delta(z)^{c(\frac{\ell^2-1}{24})}|U(\ell)E_{\ell-1}(z)^{t_j(\ell)} \pmod{\ell},$$

where, $t_j(\ell) := \begin{cases} \ell(\ell - \lfloor \frac{d}{2} - \frac{c}{2} \rfloor) - \frac{c}{2} & \text{if } j = 3, \\ \ell^{j-2}(\ell-1) - \frac{c}{2}(\ell+1) & \text{if } j > 3. \end{cases}$

We use Lemma 2.3.8 to show that $B_{j,\ell}(z)\Delta(z)^{c(\frac{\ell^2-1}{24})}|U(\ell)$ congruent modulo ℓ^j to a form on $SL_2(\mathbb{Z})$ with the same weight.

For $j = 2$, we have $B_{j,\ell}(z) \in^2 S_{k_\ell(d,1)+c\frac{(\ell^2-1)}{2}}$. Therefore, we have

$$\frac{B_{j,\ell}(z)}{\ell^{j-1}}|D_c(\ell) \in^2 S_{k_\ell(c,2)}.$$

Hence the second summand is also satisfied the required condition.

Now consider the third summand,

$$\left(g_j(z) - g_{j-1}(z)E_{\ell-1}(z)^{a_d(j)-a_d(j-1)}\right)|D_c(\ell)$$

Let $F_{j,\ell}(z) := g_j(z) - g_{j-1}(z)E_{\ell-1}(z)^{a_d(j)-a_d(j-1)} \equiv \pmod{\ell^{j-1}}$. Hence we have

$$\frac{F_{j,\ell}(z)}{\ell^{j-1}}|D_c(\ell) \equiv \frac{F_{j,\ell}(z)}{\ell^{j-1}}\Delta(z)^{c(\frac{\ell^2-1}{24})}|U(\ell) \pmod{\ell}, \tag{5.2.7}$$

Then we apply Lemma 2.4.3. Since $F_{j,\ell}(z) \in M_{k_\ell(d,j)}$, we have

$$\omega_\ell\left(\frac{F_{j,\ell}(z)}{\ell^{j-1}}\Delta(z)^{c(\frac{\ell^2-1}{24})}|U(\ell)\right) \leq \ell + \frac{k_\ell(d,j)+c(\frac{\ell^2-1}{2})-1}{\ell}.$$

Then we have

$$\omega_\ell\left(\frac{F_{j,\ell}(z)}{\ell^{j-1}}\Delta(z)^{c(\frac{\ell^2-1}{24})}|U(\ell)\right) = 1 + \lfloor\frac{c}{2}\rfloor + \frac{a_d(j)}{\ell},$$

if $2+c<\ell$ for c odd, and $1+\frac{c}{2}<\ell$ for c even.

Then by easy calculation, we have

$$F_{j,\ell}(z)|D_c(\ell) \equiv F_{j,\ell}(z)\Delta(z)^{c(\frac{\ell^2-1}{24})}|U(\ell)E_{\ell-1}(z)^{m_j(\ell)} \pmod{\ell^j}, \tag{5.2.8}$$

where

$$m_j(\ell) := \begin{cases} \lfloor\frac{c}{2}\rfloor(\ell-1)-1-\lfloor\frac{d}{2}\rfloor & \text{if } j=2, \\ \ell^{j-2}(\ell-1)-1-\lfloor\frac{c}{2}\rfloor & \text{if } j\geq 3. \end{cases}$$

Hence the third term has the required weight modulo ℓ^j.

This shows that $\psi(z)|D_c(\ell) \in^j S_{k_\ell(c,j)}$ for all $j \geq 1$. Similarly we can show that $\psi(z)|T_\ell(c,d) \in^j S_{k_\ell(d,j)}$ for all $j\geq 1$ if we have

$$\ell > \begin{cases} 4+d+2\lfloor\frac{c}{2}\rfloor & \text{if } d \text{ odd,} \\ 2+\frac{d}{2}+\lfloor\frac{c}{2}\rfloor & \text{if } d \text{ even.} \end{cases}$$

□

Now we consider the cases where the integers c or d is negative. Notice that in these cases, we have negative powers of $\phi_\ell(z)$ according to (4.2.1), hence we loose the holomorphicity at he cusp ∞. To avoid this issue, we define the sequence of functions $L_\ell(c,d,b;z)$ a little differently.

First consider the case where $c \geq 2,\ d < 0$ and $2\ell + d \geq 0$.

$$L_\ell\left(c,d,0;z\right) := 1,$$

$$L_\ell\left(c,d,1;z\right) := \phi_\ell^c(z)|U(\ell),$$

$$L_\ell\left(c,d,2;z\right) := A_\ell^2(z)\phi_\ell^{2\ell+d}(z)L_\ell\left(c,d,1;z\right)|U(\ell)$$

Now for $b \geq 3$, we define,

$$L_\ell\left(c,d,b;z\right) := \begin{cases} L_\ell\left(c,d,b-1;z\right)\phi_\ell^{c-2}(z)|U(\ell) & \text{if } b \text{ is odd} \\ L_\ell\left(c,d,b-1;z\right)\phi_\ell^{2\ell+d}(z)|U(\ell) & \text{if } b \text{ is even.} \end{cases} \tag{5.2.9}$$

For $c = 1$ and $\ell + d \geq 1$, we define

$$\begin{aligned} &L_\ell\left(1,d,0;z\right) := 1, \\ &L_\ell\left(1,d,1;z\right) := \phi_\ell(z)|U(\ell), \\ &L_\ell\left(1,d,2;z\right) := A_\ell(z)A_\ell(\ell z)\phi_\ell^{\ell+d}(z)L_\ell\left(1,d,1;z\right)|U(\ell) \end{aligned} \tag{5.2.10}$$

Now for $b \geq 3$, we define,

$$L_\ell\left(1,d,b;z\right) := \begin{cases} L_\ell\left(1,d,b-1;z\right)\phi_\ell^{\ell}(z)|U(\ell) & \text{if } b \text{ is odd} \\ L_\ell\left(1,d,b-1;z\right)\phi_\ell^{\ell+d-1}(z)|U(\ell) & \text{if } b \text{ is even.} \end{cases} \tag{5.2.11}$$

Finally, we consider $c < 0, d < 0, 2\ell + c \geq 2$, and $2\ell + d \geq 2$. In this case we define $L_\ell(c,d,b;z)$ by

$$L_\ell\left(c,d,0;z\right) := 1,$$

$$L_\ell\left(c,d,1;z\right) := A_\ell^2(z)\phi_\ell^{2\ell+c}(z)|U(\ell),$$

$$L_\ell\left(c,d,2;z\right) := A_\ell^2(z)\phi_\ell^{2\ell+d-2}(z)L_\ell\left(c,d,1:z\right)|U(\ell)$$

Now for $b \geq 3$, we define,

$$L_\ell(c,d,b;z) := \begin{cases} L_\ell(c,d,b-1;z)\,\phi_\ell^{2\ell+c-2}(z)|U(\ell) & \text{if } b \text{ is odd} \\ \\ L_\ell(c,d,b-1;z)\,\phi_\ell^{2\ell+d-2}(z)|U(\ell) & \text{if } b \text{ is even.} \end{cases}$$

Next lemma shows the relation between $L_\ell(c,d,b;z)$ when c and d are non negative and $L_\ell(c,d,b:z)$ when c or d is negative. Here we show it only for $c \geq 2$ and $d < 0$. It can be shown similarly that this is true for the other cases.

Lemma 5.2.4. *$L_\ell(c,d,b;z)$ defined in* (5.2.9) *and $L_\ell(c,d,b;z)$ defined in* (4.2.3) *generate partitions $p_{[1^c\ell^d]}(n)$ in the same arithmetic progressions for $c \geq 2$ and $d < 0$.*

Proof. By the definition of $L_\ell(c,d,b;z)$ given in equation (5.2.9), we have the following relation.

$$\begin{aligned} &L_\ell(c,d,b;z) = \\ &\quad \prod_{n=1}^{\infty}(1-q^{\ell n})^c(1-q^n)^d \sum_{n\geq\mu_1}^{\infty} p_{[1^c\ell^d]}(\ell n + n_1')\,q^n, && \text{if } b=1, \\ &\quad \prod_{n=1}^{\infty}(1-q^{\ell n})^{2\ell+d}(1-q^n)^{c-2} \sum_{n\geq\mu_b}^{\infty} p_{[1^c\ell^d]}\left(\ell^b n + n_b'\right)q^n, && \text{if } b \text{ even}, \\ &\quad \prod_{n=1}^{\infty}(1-q^{n})^{2\ell+d}(1-q^{\ell n})^{c-2} \sum_{n\geq\mu_b}^{\infty} p_{[1^c\ell^d]}\left(\ell^b n + n_b'\right)q^n, && \text{if } b(>1) \text{ odd}. \end{aligned} \tag{5.2.12}$$

Comparing coefficients we have,

$$\begin{aligned} n_{2b}' &= -\ell^{2b-1}(2\ell+d)\cdot\delta_\ell + n_{2b-1}' = -2\ell^{2b}\delta_\ell - d\cdot\delta_\ell\cdot\ell^{2b-1} + n_{2b-1}', \\ n_{2b-1}' &= -(c-2)\delta_\ell\ell^{2b-2} + n_{2b-2}' = -c\delta_\ell\ell^{2b-2} + 2\ell^{2b-2}\delta_\ell + n_{2b-2}'. \end{aligned} \tag{5.2.13}$$

Therefore, we have

$$\begin{aligned} n_{2b-1}' &= -c\left(\frac{\ell^{2b}-1}{24}\right) - \ell\cdot d\left(\frac{\ell^{2b-2}-1}{24}\right), \\ n_{2b}' &= -2\ell^{2b}\cdot\delta_\ell - c\left(\frac{\ell^{2b}-1}{24}\right) - \ell\cdot d\left(\frac{\ell^{2b}-1}{24}\right). \end{aligned} \tag{5.2.14}$$

We have calculated the quantities n_b in [34] page 18 with respect to $L_\ell(c,d,b;z)$ that we defined when $c>0$ and $d>0$. Since equation (5.2.9) generate partitions $p_{[1^c\ell^d]}(n)$ a little differently than in [34], we have two different values for n_b and n_b' when b is even. However, comparing (5.2.14) and (4.4) in [34], we see that they both agree modulo ℓ^b. $\square$

The following lemma describes the modular properties of the sequence of generating functions of all other cases.

Lemma 5.2.5. *The following are true.*

1. For $c\geq 1$, $d<0$:

(i). For $c\geq 2$ and $2\ell+d\geq 0$, we have

$$L_\ell(c,d,b;z)\in^j M_{\kappa_\ell(2\ell+d,j)}\cap\mathbb{Z}_{(\ell)}[[q]] \text{ for } b\geq 2 \text{ and } b \text{ even.}$$

$$L_\ell(c,d,b;z)\in^j M_{\kappa_\ell(c-2,j)}\cap\mathbb{Z}_{(\ell)}[[q]] \text{ for } b\geq 3 \text{ and } b \text{ odd.}$$

(ii). For $c=1$ and $\ell+d\geq 1$, we have

$$L_\ell(1,d,b;z)\in^j M_{\kappa_\ell(\ell+d-1,j)}\cap\mathbb{Z}_{(\ell)}[[q]] \text{ for } b\geq 2 \text{ and } b \text{ even.}$$

$$L_\ell(1,d,b;z)\in^j M_{\kappa_\ell(\ell,j)}\cap\mathbb{Z}_{(\ell)}[[q]] \text{ for } b\geq 3 \text{ and } b \text{ odd.}$$

2. For $c<0, d<0, 2\ell+c\geq 2$, and $2\ell+d\geq 2$, we have

$$L_\ell(c,d,b,z)\in^j M_{\kappa_\ell(2\ell+d-2,j)+(\ell-1)} \text{ for } b\geq 4 \text{ and } b \text{ even.}$$

$$L_\ell(c,d,b,z)\in^j M_{\kappa_\ell(2\ell+c-2,j)+(\ell-1)} \text{ for } b\geq 3 \text{ and } b \text{ odd.}$$

We need the following lemma.

Lemma 5.2.6. *The following are true.*

1. For $\ell\geq 5, c\geq 1, d<0$:

(i). For $c\geq 2, 2\ell+d\geq 0$, let $\psi(z)\in^j M_{k_\ell(c,j)}$. Then for all $j\geq 1$, we have

$$\psi(z)A_\ell^2(z)\phi_\ell(z)^{2\ell+d}|U(\ell)\in^j M_{\kappa_\ell(2\ell+d,j)}.$$

(ii). For $c = 1, \ell + d \geq 1$, let $\psi(z) \in^j M_{k_\ell(1,j)}$. Then for all $j \geq 1$, we have

$$\psi(z)A_\ell(z)A_\ell(\ell z)\phi_\ell(z)^{\ell+d-1}|U(\ell) \in^j M_{\kappa_\ell(\ell+d-1,j)}$$

2. For $\ell \geq 5$, let $c < 0, d < 0, 2\ell + d \geq 2$, and $2\ell + c \geq 2$. Let $\psi(z) \in^j M_{\kappa_\ell(2\ell+c,j)}$. Then for all $j \geq 1$, we have

$$\psi(z)A_\ell^2(z)\phi_\ell^{2\ell+d-2}(z)|U(\ell) \in^j M_{\kappa_\ell(2\ell+d-2,j)+(\ell-1)}.$$

Proof. We prove the case where $c \geq 1$ and $d < 0$ of Lemma 5.2.6 here. The other case can be proved similarly. For $j = 1$, from Proposition 5.1.7, we have

$$A_\ell(z) \equiv 1 \pmod{\ell}.$$

Hence, applying Lemma 2.4.3, we have

$$\begin{aligned}
\omega_\ell\Big(\psi(z)|D_{2\ell+d}(\ell)\Big) \leq & \ell + \frac{(1+\lfloor\frac{c}{2}\rfloor)(\ell-1) + \frac{(2\ell+d)(\ell^2-1)}{2} - 1}{\ell}, \\
& = (\ell-1)\left(\frac{1+\lfloor\frac{c}{2}\rfloor + \ell + 1 + \frac{(2\ell+d)(\ell+1)}{2}}{\ell}\right) \\
& \leq (\ell-1)\left(2 + \lfloor\frac{2\ell+d}{2}\rfloor\right),
\end{aligned} \tag{5.2.15}$$

if

$$\ell > \begin{cases} 4 + d + 2\lfloor\frac{c}{2}\rfloor & \text{if } d \text{ odd}, \\ 2 + \frac{d}{2} + \lfloor\frac{c}{2}\rfloor & \text{if } d \text{ even}. \end{cases}$$

Hence $\psi(z)|D_{2\ell+d}(\ell) \in^1 M_{\kappa_\ell(2\ell+d,1)}$ and the result is true for $j = 1$. Now for $j \geq 2$, consider the following decomposition.

$$\begin{aligned}
\psi_j(z)A_\ell^2(z)\phi_\ell(z)^{2\ell+d}|U(\ell) \equiv & \frac{g_{j-1}(z)E_{\ell-1}(z)^{a_c(j)}}{A_\ell(z)^{2a_c(j-1)}}A_\ell^2(z)\phi_\ell(z)^{2\ell+d}|U(\ell) \\
+ & \left(g_{j-1}(z)E_{\ell-1}(z)^{a_c(j)-a_c(j-1)} - \frac{g_{j-1}(z)E_{\ell-1}(z)^{a_c(j)}}{A_\ell(z)^{2a_c(j-1)}}\right)A_\ell^2(z)\phi_\ell(z)^{2\ell+d}|U(\ell) \\
+ & \left(g_j(z) - g_{j-1}(z)E_{\ell-1}(z)^{a_c(j)-a_c(j-1)}\right)A_\ell^2(z)\phi_\ell(z)^{2\ell+d}|U(\ell) \pmod{\ell^j}.
\end{aligned} \tag{5.2.16}$$

Now lets show each part satisfies the required condition. For the first part, we define

$$s_j(z) := \frac{g_{j-1}(z)}{A_\ell(z)^{2a_c(j-1)}} A_\ell^2(z)\phi_\ell(z)^{2\ell+d}|U(\ell), \tag{5.2.17}$$

In the proof of lemma 5.2.2, we used Lemma 2.3.10, to show that a similar form is congruent to a form with same weight on $SL_2(\mathbb{Z})$. Here we use the same method. Therefore, we show that

$$s_j(z) \equiv \mathrm{Tr}\left(s_j(z)h(z)^{a_{2\ell+d}(j)}\right) \pmod{\ell^j}. \tag{5.2.18}$$

Hence we need to show that

$$\pi_\ell\left(\mathrm{Tr}(s_j(z)h(z)^{a_{2\ell+d}(j)}) - s_j(z)\right) \geq j. \tag{5.2.19}$$

We consider the left hand side of (5.2.19). By Lemma 2.3.10 and Table 5.1, we have

$$\begin{aligned}
&\geq a_{2\ell+d}(j) + \pi_\ell\left(\frac{g_{j-1}(z)\phi_\ell(z)^{2\ell+d}}{A_\ell(z)^{2a_c(j-1)-2}}|U(\ell)|_{\ell-1}W(\ell)\right),\\
&= a_{2\ell+d}(j) + k_\ell(c, j-1) - (2\ell+d) - \ell a_c(j-1) + \ell + \frac{\ell-1}{2},\\
&= a_{2\ell+d}(j) - a_c(j-1) - (2\ell+d) + \frac{3\ell-1}{2},\\
&\geq j,
\end{aligned} \tag{5.2.20}$$

if

$$\ell > \begin{cases} 4 + d + 2\lfloor \frac{c}{2} \rfloor & \text{if } d \text{ odd},\\ 2 + \frac{d}{2} + \lfloor \frac{c}{2} \rfloor & \text{if } d \text{ even}. \end{cases}$$

Notice that there is additional $\frac{\ell-1}{2}$ term on the right side of (5.2.20) coming from the $U(\ell)$ operator by (2.3.4). Hence $s_j(z) \in^j M_{\kappa_\ell(2\ell+d,j)}$ for $j \geq 2$.

Now we consider the second summand.

Let $\beta_j(z) := g_{j-1}E_{\ell-1}^{a_c(j)-a_c(j-1)} - \dfrac{g_{j-1}E_{\ell-1}^{a_c(j)}}{A_\ell^{2a_c(j-1)}}.$

$$\beta_j(z) = E_{\ell-1}^{a_c(j)-a_c(j-1)} \left(g_{j-1} - \frac{g_{j-1}E_{\ell-1}^{a_c(j-1)}}{A_\ell^{2a_c(j-1)}} \right) \equiv 0 \pmod{\ell^{j-1}}.$$

We can see that $\beta_j(z) \in M_{k_\ell(c,j)}(\Gamma_0(\ell))$. Now the second summand can be written as

$$\frac{\beta_j(z)}{\ell^{j-1}} A_\ell^2(z)\phi_\ell^{2\ell+d}(z)|U(\ell) \equiv \frac{\beta_j(z)}{\ell^{j-1}} \Delta(z)^{\frac{\ell^2-1}{24}(2\ell+d)}|U(\ell)E_{\ell-1}(z) \pmod{\ell}.$$

This implies

$$\beta_j(z)A_\ell^2(z)\phi^{2\ell+d}(z)|U(\ell) \equiv \beta_j(z)\Delta(z)^{(\frac{\ell^2-1}{24})(2\ell+d)}|U(\ell)E_{\ell-1}(z) \pmod{\ell^j}.$$

Using a similar method in the proof of Proposition 3.3 in [12], we see that

$$\frac{g_{j-1}(z)E_{\ell-1}(z)^{a_c(j-1)}}{A_\ell(z)^{2a_c(j-1)}} \cdot \Delta(z)^{(\frac{\ell^2-1}{24})(2\ell+d)}|U(\ell) \in^j S_{k_\ell(c,j-1)+(\frac{\ell^2-1}{2})(2\ell+d)}.$$

For $j=2$, we see that the filtration of $\beta_j(z)A_\ell^2(z)\phi^{2\ell+d}(z)|U(\ell)$ is $\lfloor \frac{2\ell+d}{2} \rfloor \ell(\ell-1) + \ell - 1$.

Hence the middle term satisfies the required condition.

Then we consider the third summand.

Let $F_j(z) := g_j - g_{j-1}E_{\ell-1}^{a_c(j)-a_c(j-1)} \equiv 0 \pmod{\ell^{j-1}}$.

Hence we can write the third summand as

$$\frac{F_j(z)}{\ell^{j-1}} A_\ell^2(z)\phi_\ell^{2\ell+d}(z)|U(\ell) \equiv \frac{F_j(z)}{\ell^{j-1}} \Delta(z)^{(\frac{\ell^2-1}{24})(2\ell+d)}|U(\ell) \cdot E_{\ell-1}(z) \pmod{\ell}.$$

Here we have $\frac{F_j(z)}{\ell^{j-1}}\Delta(z)^{(\frac{\ell^2-1}{24})(2\ell+d)} \in S_{k_\ell(c,j)+\frac{(\ell^2-1)(2\ell+d)}{2}}$. Now we apply Lemma 2.3.8.

$$\omega_\ell \left(\frac{F_j(z)}{\ell^{j-1}} \Delta(z)^{(\frac{\ell^2-1}{24})(2\ell+d)}|U(\ell) \right) \le \ell + \frac{k_\ell(c,j) + (\frac{\ell^2-1}{2})(2\ell+d) - 1}{\ell}$$

Therefore $\frac{F_j(z)}{\ell^{j-1}}\Delta(z)^{(\frac{\ell^2-1}{24})(2\ell+d)}$ is congruent modulo ℓ to a form on $SL_2(\mathbb{Z})$ with the weight $(\ell-1)(1+\lfloor \frac{c}{2} \rfloor + \lfloor \frac{2\ell+d}{2} \rfloor)$ for $j=2$ and $(\ell-1)(\ell^{j-2}+1+\lfloor \frac{2\ell+d}{2} \rfloor)$ for $j \ge 3$.

Let $S_j(z)$ be such forms. Then

$$\frac{F_j(z)}{\ell^{j-1}} \Delta(z)^{(\frac{\ell^2-1}{24})(\ell+d)} \equiv S_j(z) \equiv S_j(z)E_{\ell-1}^{t_j+1} \pmod{\ell}$$

where

$$t_j = \begin{cases} \lfloor \frac{2\ell+d}{2} \rfloor (\ell-1) - 1 - \lfloor \frac{c}{2} \rfloor \text{ for } j = 2, \\ \\ \ell^{j-2}(\ell-1) - 1 - \lfloor \frac{2\ell+d}{2} \rfloor \text{ for } j \geq 3. \end{cases}$$

Now we prove part (b). Again using (5.2.15) and replacing $2\ell + d$ with $\ell + d - 1$, we see that the result is true for $j = 1$ when $3 + d < \ell$.

Then for $j \geq 2$, we use the following decomposition.

$$\begin{aligned} \psi_j(z) A_\ell(z) A_\ell(\ell z) \phi_\ell^{\ell+d-1} | U(\ell) \equiv & \frac{g_{j-1} E_{\ell-1}^{a_c(j)}}{A_\ell^{2a_c(j-1)}} A_\ell(z) A_\ell(\ell z) \phi_\ell^{\ell+d-1} | U(\ell) \\ & + \left(g_{j-1} E_{\ell-1}^{a_c(j)-a_c(j-1)} - \frac{g_{j-1} E_{\ell-1}^{a_c(j)}}{A_\ell^{2a_c(j-1)}} \right) A_\ell(z) A_\ell(\ell z) \phi_\ell^{\ell+d-1} | U(\ell) \\ & + \left(g_j - g_{j-1} E_{\ell-1}^{a_c(j)-a_c(j-1)} \right) A_\ell(z) A_\ell(\ell z) \phi_\ell^{\ell+d-1} | U(\ell) \pmod{\ell^j}. \end{aligned} \tag{5.2.21}$$

The rest of the proof is similar to the proof given for part (a) except the first part of the decomposition.

Let $s_j'(z) = \frac{g_{j-1} E_{\ell-1}^{a_c(j)}}{A_\ell^{2a_c(j-1)}} A_\ell(z) A_\ell(\ell z) \phi_\ell^{\ell+d-1} | U(\ell)$. Then we need to do the following changes in (5.2.20).

$$\begin{aligned} & \geq a_{\ell+d-1}(j) + \pi_\ell \left(\frac{g_{j-1} \phi_\ell^{\ell+d-1} A_\ell(\ell z)}{A_\ell(z)^{2a_1(j-1)-1}} | U(\ell) |_{\ell-1} W(\ell) \right), \\ & = a_{\ell+d-1}(j) + k_\ell(1, j-1) - (\ell+d-1) - \ell a_1(j-1) + \ell + \frac{\ell-1}{2}, \\ & = a_{\ell+d-1}(j) - a_1(j-1) + \frac{3\ell-1}{2} - (\ell+d-1), \\ & \geq j, \end{aligned} \tag{5.2.22}$$

if

$$\ell > \begin{cases} \frac{3}{2} + \frac{d}{2} & \text{if } d \text{ odd}, \\ \\ 3 + d & \text{if } d \text{ even}. \end{cases}$$

$\square$

Now we need the following lemma.

Lemma 5.2.7. *The following are true.*

1. For $\ell \geq 5, c \geq 1, d < 0$:

(i). let $c \geq 2, d < 0, 2\ell + d \geq 0$, and $\psi(z) \in^j M_{\kappa_\ell(2\ell+d,j)}$. Then $\psi(z)|D_{c-2}(\ell) \in^j M_{\kappa_\ell(c-2,j)}$ for all $j \geq 1$.

(ii). For $c = 1, \ell + d \geq 1$, let $\psi(z) \in^j M_{\kappa_\ell(\ell+d-1,j)}$. Then for all $j \geq 1$, we have $\psi(z)|D_\ell(\ell) \in^j M_{\kappa_\ell(\ell,j)}$.

2. For $\ell \geq 5$, let $c < 0, d < 0, 2\ell+d \geq 2$, and $2\ell+c \geq 2$. Let $\psi(z) \in^j M_{\kappa_\ell(2\ell+d-2,j)+(\ell-1)}$. Then for all $j \geq 1$, we have $\psi(z)|D_{2\ell+c-2}(\ell) \in^j M_{\kappa_\ell(2\ell+c-2,j)+(\ell-1)}$.

Proof. The proof follows similarly to the proof of Lemma 5.2.3. For example, the proof follows from changing c to $c-2$ and d to $2\ell+d$ in the proof of Lemma 5.2.3. □

Remark 5.2.8. *As we see in the proofs of Lemmas 5.2.3 and 5.2.6, we need $\ell \geq v(c,d)$. Other conditions stated with $v(c,d)$ follow from each case of Lemma 5.2.7. These conditions can be obtained in a similar way as for the case where $c, d \geq 2$. For example, when $c \geq 2, d < 0$, and $2\ell + d > 0$, we need to have*

$$\ell > \begin{cases} 4 + c + 2\lfloor \frac{d}{2} \rfloor & \text{if } c \text{ is odd,} \\ 2 + \frac{c}{2} + \lfloor \frac{d}{2} \rfloor & \text{if } c \text{ is even.} \end{cases}$$

Proof of Lemma 5.2.5. We prove the case where $c \geq 2, d < 0$, and $2\ell + d \geq 0$. All other cases are similar. From Lemma 5.2.6, we see that $L_\ell(c,d,2;z) \in^j M_{\kappa_\ell(2\ell+d,j)}$. Changing $k_\ell(d,j)$ with $\kappa_\ell(d,j)$ in Lemma 5.2.3, we see that $L_\ell(c,d,b;z) \in^j M_{\kappa_\ell(c-2,j)}$ when $b \geq 2$. □

Let e be a non-negative integer, m be a positive integers, and $\ell \geq 5$ be a prime. Then we define $M(e,\ell,m)$ denotes the $\mathbb{Z}/\ell^m\mathbb{Z}$-module of modular forms in $M_{k_\ell(e,j)} \cap \mathbb{Z}_{(\ell)}[[q]]$ with coefficients modulo ℓ^m. Then Lemma 5.2.2 implies that we have the following nesting property.

$$M(d,\ell,m) \supseteq \Lambda_\ell^{\mathsf{even}}(c,d,0,m) \supseteq \Lambda_\ell^{\mathsf{even}}(c,d,2,m) \supseteq \cdots \Lambda_\ell^{\mathsf{even}}(c,d,2b,m) \supseteq \cdots$$

$$M(c,\ell,m) \supseteq \Lambda_\ell^{\mathsf{odd}}(c,d,1,m) \supseteq \Lambda_\ell^{\mathsf{odd}}(c,d,3,m) \supseteq \cdots \Lambda_\ell^{\mathsf{odd}}(c,d,2b-1,m) \supseteq \cdots$$

From Lemma 5.2.5, for each case we have a corresponding nesting property. For example, when $c \geq 2, d < 0$, and $2\ell + d \geq 0$, we have

$$M(2\ell+d,\ell,m) \supseteq \triangle_\ell^{\mathsf{even}}(c,d,2,m) \supseteq \triangle_\ell^{\mathsf{even}}(c,d,4,m) \supseteq \cdots \triangle_\ell^{\mathsf{even}}(c,d,2b,m) \supseteq \cdots$$

$$M(c,\ell,m) \supseteq \triangle_\ell^{\mathsf{odd}}(c,d,1,m) \supseteq \triangle_\ell^{\mathsf{odd}}(c,d,3,m) \supseteq \cdots \triangle_\ell^{\mathsf{odd}}(c,d,2b-1,m) \supseteq \cdots$$

As pointed out in [12] and [13], since $M(e,\ell,m)$ has finite rank, these sequences stabilized to a finite rank module, we call $\Omega_\ell^{\mathsf{odd}}(c,d,m)$ for b is odd, and $\Omega_\ell^{even}(c,d,m)$ for b is even. In Section 5.3, we calculate an upper bound for the rank of the stabilized modules.

5.3. Bound for the rank of the stabilized modules

Lemma 5.3.1 (Lemma 4.1,[13]). *Let A be a finite local ring, M be a finitely generated A-module, and $T : M \longrightarrow M$ be an $A-$isomorphism.*

1. *There exists an integer $n > 0$ such that T^n is the identity map on M.*
2. *For all $\mu \in M$ and $n \geq 0$, we have*

$$\mu \in A[T^n(\mu), T^{n+1}(\mu), \cdots]$$

Theorem 5.3.2. *If $\ell \geq v(c,d)$ is prime and $m \geq 1$, then there exist injective $\mathbb{Z}/\ell^m\mathbb{Z}$-module homomorphisms,*

$$\Pi_\ell^{even}(c,d) : \Omega_\ell^{even}(c,d,m) \hookrightarrow S_{k_\ell(d,1)} \cap \mathbb{Z}_{(\ell)}[[q]].$$

$$\Pi_\ell^{odd}(c,d) : \Omega_\ell^{odd}(c,d,m) \hookrightarrow S_{k_\ell(c,1)} \cap \mathbb{Z}_{(\ell)}[[q]].$$

Satisfying the property that for all $\mu_1 \in \Omega_\ell^{\textit{even}}(c,d,m)$ and $\mu_2 \in \Omega_\ell^{\textit{odd}}(c,d,m)$ with $ord_\ell(\mu_i) = j_i < m$.

$$\Pi_\ell^{\textit{even}}(c,d)(\mu_1) \equiv \mu_1 \pmod{\ell^{j_1+1}},$$

$$\Pi_\ell^{\textit{odd}}(c,d)(\mu_2) \equiv \mu_2 \pmod{\ell^{j_2+1}}.$$

Definition 5.3.3. *We consider the following two sub modules of $S_{k_\ell(d,m)} \cap \mathbb{Z}_{(\ell)}[[q]]$,*

$$S_o^{\textit{odd}}(c,d,m) := \left\{ f(z) E_{\ell-1}^{\frac{k_\ell(c,m)-k_\ell(c,1)}{\ell-1}} : f(z) \in S_{k_\ell(c,1)} \cap \mathbb{Z}_{(\ell)}[[q]] \right\}$$

$$S_1^{\textit{odd}}(c,d,m) := \left\{ g(z) : g(z) = \sum_{j=m_0}^{\infty} a_j q^j \in S_{k_\ell(c,m)} \cap \mathbb{Z}_{(\ell)}[[q]] \right.$$
$$\left. \textit{with } m_0 > \textit{dim}(S_{k_\ell(c,1)}) \right\}$$

We can construct a basis $\{f_1 = q + \cdots, \cdots, f_n = q^n + \cdots\}$ for $S_{k_\ell(c,m)} \cap \mathbb{Z}_{(\ell)}[[q]]$ with $f_k(z) \in S_o^{\textit{odd}}(c,d,m)$ for $k \leq \dim(S_{k_\ell(c,1)})$ and $f_k(z) \in S_1^{\textit{odd}}(c,d,m)$ otherwise. Therefore we have,

$$S_{k_\ell(c,m)} \cap \mathbb{Z}_{(\ell)}[[q]] = S_o^{\textit{odd}}(c,d,m) \oplus S_1^{\textit{odd}}(c,d,m).$$

Let $S^{\textit{odd}}(c,d) \subset S_{k_\ell(c,m)} \cap \mathbb{Z}_{(\ell)}[[q]]$ be the largest $\frac{\mathbb{Z}}{\ell^m \mathbb{Z}}$-module such that $T_\ell(d,c)$ is an isomorphism on $S^{\textit{odd}}(c,d) \pmod{\ell^m}$. Similarly, we define $S^{\textit{even}}(c,d)$.

Lemma 5.3.4 (Lemma 4.4, [12])**.** *Suppose that $f(z) \in S^{\textit{odd}}(c,d)$ and $\mathrm{ord}_\ell(f) = i < m$, and that $f(z) = f_0(z) + f_1(z)$ with $f_i(z) \in S_i^{\textit{odd}}(c,d,m)$. Then we have $\mathrm{ord}_\ell(f_1) > i$ and $\mathrm{ord}_\ell(f_0) = i$.*

Corollary 5.3.5 (Corollary 4.7. [12])**.** *Let $f(z), g(z) \in S^{\textit{odd}}(c,d)$, and suppose that $f(z) = f_0(z) + f_1(z)$ and $g(z) = g_0(z) + g_1(z)$ with $f_i, g_i \in S_i^{\textit{odd}}(c,d,m)$. Suppose further that $f_0(z) \equiv g_0(z) \pmod{\ell^m}$. Then we have $f(z) \equiv g(z) \pmod{\ell^m}$.*

Proof. We construct injective homomorphisms ϕ_1, ϕ_2 and ϕ_3 similar to [12],and [13]. Since $S^{odd}(c,d)$ is the largest $\frac{\mathbb{Z}}{\ell^m\mathbb{Z}}$-module such that $T_\ell(d,c)$ is an isomorphism on $S^{\mathsf{odd}}(c,d)$ $\pmod{\ell^m}$, we see that $\Omega_\ell^{odd}(c,d,m) \subseteq S^{\mathsf{odd}}(c,d)$. We define ϕ_1 be the inclusion map $\Omega_\ell^{odd}(c,d,m) \hookrightarrow S^{\mathsf{odd}}(c,d)$.

Let $f(z) \in S^{\mathsf{odd}}(c,d)$ with $f(z) = f_0(z) + f_1(z)$ where $f_i \in S_i^{odd}(c,d,m)$. We assume that $\nu_\ell(f) = i < m$ then by Lemma 4.4, we have $f(z) \equiv f_0 \pmod{\ell^{\nu_\ell(f)} + 1}$. Now we define $\phi_2 : f(z) \to f_0(z) \pmod{\ell^{\nu_\ell(f)+1}}$ and $\phi_2 : S^{odd}(c,d) \to S_0^{odd}(c,d,m)$. Here we see that ϕ_2 is injective by corollary 5.3.5.

Now let $f(z) \in S_0^{odd}(c,d,m)$. Then $f(z) = g(z)E_{\ell-1}^{\frac{k_\ell(c,m)-k_\ell(c,1)}{\ell-1}}$ for some $g(z) \in S_{k_\ell(c,1)} \cap \mathbb{Z}_{(\ell)}[[q]]$. We define $\phi_3 : S_0^{odd}(c,d,m) \to S_{k_\ell(c,1)} \cap \mathbb{Z}_{(\ell)}[[q]]$ by sending $f(z) \to g(z)$. So we have,

$$\Pi_\ell^{odd} : \Omega_\ell^{odd}(c,d,m) \xrightarrow{\phi_1} S^{odd}(c,d) \xrightarrow{\phi_2} S_0^{odd}(c,d,m) \xrightarrow{\phi_3} S_{k_\ell(c,1)} \cap \mathbb{Z}_{(\ell)}[[q]].$$

Further, if we suppose $\mu_1 \in \Omega_\ell^{odd}(c,d,m)$ has $\nu_\ell(\mu_1) < m$, then we have,

$$\Pi_\ell^{odd}(\mu_1)E_{\ell-1}^{\frac{k_\ell(c,m)-k_\ell(c,1)}{\ell-1}} \equiv \mu_1 \pmod{\ell^{\nu_\ell(\mu_1)+1}}.$$

□

Remark 5.3.6. *First notice that applying $D_c(\ell)$ to a form $f(z) := \sum_{n=0}^{\infty} a_n q^n$ satisfying*

$$f|D_c(\ell) = \sum_{n \geq \lfloor \frac{c(\ell^2-1)}{24\ell} \rfloor}^{\infty} a_n q^n.$$

We have isomorphsms $D_c(\ell) : S^{even}(c,d) \to S^{odd}(c,d)$ and $D_d(\ell) : S^{odd}(c,d) \to S^{even}(c,d)$, hence

$$\textit{rank}_{\mathbb{Z}/\ell^m\mathbb{Z}}\left(\Omega_\ell^{even}(c,d,m)\right) = \textit{rank}_{\mathbb{Z}/\ell^m\mathbb{Z}}\left(\Omega_\ell^{odd}(c,d,m)\right) = r_\ell(c,d),$$

where $r_\ell(c,d) \leq \textit{dim}\left(S_{k_\ell(e,1)}\right) - \left\lfloor \frac{e(\ell^2-1)}{24\ell} \right\rfloor$ where $e = \max\{c,d\}$.

Corollary 5.3.7. *For the cases where c or d is negative, we have the following upper bounds for $r_\ell(c,d)$.*

$$r_\ell(c,d) \leq \begin{cases} \dim\left(S_{(\lfloor \frac{e}{2} \rfloor+2)(\ell-1)}\right) - \left\lfloor \frac{e(\ell^2-1)}{24\ell} \right\rfloor & \text{if } c = 1, d < 0, \text{ and } \ell + d \geq 1, \\ \dim\left(S_{(\lfloor \frac{e}{2} \rfloor+2)(\ell-1)}\right) - \left\lfloor \frac{e(\ell^2-1)}{24\ell} \right\rfloor & \text{if } c \geq 2, d < 0, \text{ and } 2\ell + d \geq 0, \\ \dim\left(S_{(\lfloor \frac{e}{2} \rfloor+3)(\ell-1)}\right) - \left\lfloor \frac{e(\ell^2-1)}{24\ell} \right\rfloor & \text{if } c < 0, d < 0, \text{ and } 2\ell + c, 2\ell + d \geq 2, \end{cases} \tag{5.3.1}$$

and

$$e \in \begin{cases} \{\ell, \ell + d - 1\} & \text{if } c = 1, d < 0, \text{ and } \ell + d \geq 1, \\ \{c - 2, 2\ell + d\} & \text{if } c \geq 2, d < 0, \text{ and } 2\ell + d \geq 0, \\ \{2\ell + d - 2, 2\ell + c - 2\} & \text{if } c < 0, d < 0, \text{ and } 2\ell + c, 2\ell + d \geq 2. \end{cases}$$

Then we choose e between the two values such that the quantity on the right hand side of (5.3.1) *is the minimum.*

Proof. The proof follows replacing c and d with corresponding integers for each case. For example, when $c \geq 2, d < 0$, and $2\ell + d \geq 0$, we replace c with $c - 2$ and d with $2\ell + d$ in Theorem $5.3.2$. $\square$

Chapter 6. Applications

6.1. Applications for Theorem 1.0.3

Corollary 6.1.1. *For any integer $m \geq 0$ and for each positive integer r, we have*

$$b_5\left(5^{2r}m + \frac{5^{2r}-1}{6}\right) \equiv 0 \pmod{5^r},$$

$$b_7\left(7^{2r}m + \frac{7^{2r}-1}{4}\right) \equiv 0 \pmod{7^r},$$

$$b_{11}\left(11^{2r}m + 5\cdot\frac{11^{2r}-1}{12}\right) \equiv 0 \pmod{11^r}.$$

Corollary 6.1.2. *For each positive integer r, and for some integer m, we have*

$$b_{13}\left(13^{2r}m + \frac{13^{2r}-1}{2}\right) \not\equiv 0 \pmod{13},$$

$$b_{17}(17m-12) \not\equiv 0 \pmod{17}.$$

Proof. The proof of Corollaries 6.1.1, and 6.1.2, follows from the entries of Table 6.1. The Calculation of A_r follows from Table 3 in [33] when $\ell = 11$. □

Table 6.1. Calculations for ℓ-regular partitions

ℓ	n_{2r}	μ_r	A_r
5	$\frac{5^{2r}-1}{6}$	$\mu_{2r-2}=0,\ \mu_{2r-1}=1$	$A_{2r}=r$
7	$\frac{3\cdot 7^{2r}-1}{4}$	$\mu_{2r-2}=0,\ \mu_{2r-1}=1$	$A_{2r}=r$
11	$\frac{5\cdot 11^{2r}-5}{12}$	$\mu_{2r-2}=0,\ \mu_{2r-1}=1$	$A_{2r}=r$
13	$n_1=-7$	$\mu_{2r-2}=0,\ \mu_{2r-1}=1$	$A_r=0$
17	$n_1=-12$	$\mu_{2r-2}=0,\ \mu_{2r-1}=1$	$A_r=0$

6.1.1. Congruences for ℓ-core partitions

Corollary 6.1.3. *For integers $m \geq 0$ and for each positive integer r, we have*

$$a_5(5^r m - 1) \equiv 0 \pmod{5^r},$$

$$a_7(7^r m - 2) \equiv 0 \pmod{7^r},$$

$$a_{11}(11^r m - 5) \equiv 0 \pmod{11^r}$$

Table 6.2. Calculations for ℓ-core partitions.

ℓ	n_r	μ_r	A_r
5	$n_r = -1$	$\mu_{2r} = 0, \mu_{2r-1} = 1$	$A_{2r} = r$
7	$n_r = -2$	$\mu_{2r} = 0, \mu_{2r-1} = 1$	$A_r = r$
11	$n_r = -5$	$\mu_0 = 0, \mu_{2r} = -4, \mu_{2r+1} = 1$	$A_r = r$
13	$n_{2r} = -7$	$\mu_0 = 0, \mu_{2r} = -6, \mu_{2r-1} = 1$	$A_r = 0$
17	$n_1 = -12$	$\mu_0 = 0, \mu_{2r} = -11, \mu_{2r-1} = 1$	$A_r = 0$

Corollary 6.1.4. *For each positive integer r and for some integer m, we have*

$$a_{13}\left(13^{2r}m - 7\right) \not\equiv 0 \pmod{13},$$

$$a_{17}\left(17m - 12\right) \not\equiv 0 \pmod{17}.$$

Proof. Similarly, the proof of Corollaries 6.1.3 and 6.1.4 follows from the entries of Table 6.2, and the calculation of A_r follows from Table 3 in [33] when $\ell = 11$. $\square$

6.2. Congruences for ℓ-colored generalized Frobenius partitions

In 2018, Chan, Wang, and Yang in [16] studied these partitions using the theory of modular forms and derived new representations. Using their work and Theorem 1.0.3, we proved the congruences for $c\phi_k(n)$ for $k = 5, 7$, and 11.

Corollary 6.2.1. *For all positive integers r, and for all m, we have*

$$c\phi_5\left(5^{2r}m + \frac{19 \cdot 5^{2r} + 5}{24}\right) \equiv 0 \pmod{5^{2r-1}}, \tag{6.2.1}$$

$$c\phi_7\left(7^{2r}m + \frac{17 \cdot 7^{2r} + 7}{24}\right) \equiv 0 \pmod{7^{r}}, \tag{6.2.2}$$

$$c\phi_{11}\left(11^{2r}m - \frac{11^{2r+1} - 11}{24}\right) \equiv 0 \pmod{11^{2r-1}}. \tag{6.2.3}$$

We also use Corollary 1.0.6 to prove an incongruence for $k = 13$.

Corollary 6.2.2. *For each positive integer r, and for some integer m, we have*

$$c\phi_{13}\left(13^{2r}m - \frac{13 - 13^{2r+1}}{24}\right) \not\equiv 0 \pmod{13}. \tag{6.2.4}$$

Proof. We combine Theorem 1.0.3, and Theorem 1.1 to prove congruences of k-colored generalized Frobenius partitions. First notice that using (3.4.3), we have that

$$C\Phi_5(q) = \sum_{n=0}^{\infty} \Big(p_{[1^0 5^1]}(n) + 25 p_{[1^6 5^{-5}]}(n-1) \Big) q^n. \tag{6.2.5}$$

Now using (4.2.4), we have

$$n_{2r,5}(0,1) = \frac{5 - 5^{2r+1}}{24}, \quad n_{2r,5}(6,-5) = \frac{19 \cdot 5^{2r} - 19}{24}. \tag{6.2.6}$$

Observe that using (4.2.2), and (4.2.6), we have that

$$\mu_{r,5}(0,1) = \begin{cases} 0 \text{ when } r = 1, \\ 1 \text{ when } r \geq 2. \end{cases} \quad \lambda_r(0,1) = \begin{cases} 0 \text{ when } r \text{ is even}, \\ 1 \text{ when } r \text{ is odd}. \end{cases} \tag{6.2.7}$$

$$\mu_{r,5}(6,-5) = \begin{cases} 2 \text{ when } r \text{ is odd}, \\ 0 \text{ when } r \text{ is even}. \end{cases} \quad \lambda_r(6,-5) = \begin{cases} 6 \text{ when } r \text{ is even}, \\ -5 \text{ when } r \text{ is odd}. \end{cases}$$

Now using $\theta_5(\lambda, \mu)$ values from Table 4.1, we have

$$A_{2r}(0,1) = 2r - 1 \quad , A_{2r}(6,-5) = 2r.$$

Therefore, using Theorem 1.0.3, for all $m \geq 1$, and $r \geq 1$, we have

$$\begin{aligned} p_{[1^0 5^1]}\left(5^{2r} m + \frac{5 - 5^{2r+1}}{24}\right) &\equiv 0 \pmod{5^{2r-1}}, \\ p_{[1^6 5^{-5}]}\left(5^{2r} m + \frac{19 \cdot 5^{2r} - 19}{24}\right) &\equiv 0 \pmod{5^{2r}}. \end{aligned} \tag{6.2.8}$$

Now using (6.2.5), and the fact that (6.2.8) is true when m replaced by $m+1$, we see that (6.2.1) is true.

Now we prove (6.2.2). From (3.4.4), we have

$$C\Phi_7(q) = \sum_{n=0}^{\infty} \Big(p_{[1^0 7^1]}(n) + 7^2 p_{[1^4 7^{-3}]}(n-1) + 7^3 p_{[1^8 7^{-7}]}(n-2) \Big) q^n. \tag{6.2.9}$$

Now using (4.2.4), we have

$$
\begin{aligned}
n_{2r,7}(0,1) =& \frac{7-7^{2r+1}}{24}, \qquad n_{2r,5}(4,-3) = \frac{17\cdot 7^{2r}-17}{24}, \\
n_{2r,7}(8,-7) =& \frac{41\cdot 7^{2r}-41}{24}.
\end{aligned}
\tag{6.2.10}
$$

To find $\mu_{r,\ell}(c,d)$, we use (4.2.2), and (4.2.6). Thus, we have

$$
\mu_{r,7}(0,1) = \begin{cases} 0 \text{ when } r=1, \\ 1 \text{ when } r \geq 2. \end{cases} \qquad \lambda_r(0,1) = \begin{cases} 0 \text{ when } r \text{ is even}, \\ 1 \text{ when } r \text{ is odd}. \end{cases} \tag{6.2.11}
$$

$$
\mu_{r,7}(4,-3) = \begin{cases} 2 \text{ when } r \text{ is odd}, \\ 0 \text{ when } r \text{ is even}. \end{cases} \quad \lambda_r(4,-3) = \begin{cases} 4 \text{ when } r \text{ is even}, \\ -3 \text{ when } r \text{ is odd}. \end{cases}
$$

$$
\mu_{r,7}(8,-7) = \begin{cases} 3 \text{ when } r \text{ is odd}, \\ 0 \text{ when } r \text{ is even}. \end{cases} \quad \lambda_r(8,-7) = \begin{cases} 8 \text{ when } r \text{ is even}, \\ -7 \text{ when } r \text{ is odd}. \end{cases}
$$

Now using (6.2.10), and values of $\theta_7(\lambda,\mu)$ from Table 4.2, we have

$$
A_{2r}(0,1) = r, \quad A_{2r}(4,-3) = r, \quad A_{2r}(8,-7) = r. \tag{6.2.12}
$$

Therefore, using (6.2.12), and Theorem 1.0.3, for all $m \geq 1$, and $r \geq 1$, we have

$$
\begin{aligned}
p_{[1^0 7^1]}\left(7^{2r}m + \frac{7-7^{2r+1}}{24}\right) &\equiv 0 \pmod{7^r}, \\
p_{[1^4 7^{-3}]}\left(7^{2r}m + \frac{17\cdot 7^{2r}-17}{24}\right) &\equiv 0 \pmod{7^r}, \\
p_{[1^8 7^{-7}]}\left(7^{2r}m + \frac{41\cdot 7^{2r}-41}{24}\right) &\equiv 0 \pmod{7^r}.
\end{aligned}
\tag{6.2.13}
$$

Now using (6.2.9), and the fact that (6.2.13) is true when m replaced by $m+1$, and $m+2$, we can see that (6.2.2) is true.

Now we prove (6.2.3). From (3.4.5), we have

$$C\Phi_{11}(q) = \sum_{n=0}^{\infty} \Big(p_{[1^0 11^1]}(n) + 11 \cdot p(11n-5) \Big) q^n. \tag{6.2.14}$$

Now using (4.2.4), we have

$$n_{2r,11}(0,1) = \frac{11 - 11^{2r+1}}{24}, \quad n_{2r+1,11}(1,0) = \frac{-11^{2r+1}+1}{24} \tag{6.2.15}$$

To find $\mu_{r,\ell}(c,d)$, we use (4.2.2), and (4.2.6). Thus, we have

$$\mu_{r,11}(0,1) = \begin{cases} 0 \text{ when } r = 1, \\ \\ 1 \text{ when } r \geq 2. \end{cases} \quad \lambda_r(0,1) = \begin{cases} 0 \text{ when } r \text{ is even}, \\ \\ 1 \text{ when } r \text{ is odd}. \end{cases} \tag{6.2.16}$$

$$\mu_{r,11}(1,0) = 1, \quad \text{for all } r, \quad \lambda_r(4,-3) = \begin{cases} 1 \text{ when } r \text{ is even}, \\ \\ 0 \text{ when } r \text{ is odd}. \end{cases}$$

Now using (6.2.10), and the values of $\theta_{11}(\lambda,\mu)$ from Table 3 in [33], we have

$$A_{2r}(0,1) = 2r - 1, \quad A_{2r}(1,0) = 2r. \tag{6.2.17}$$

Therefore using (6.2.17) and Theorem 1.0.3, for all $m \geq 1$, and $r \geq 1$, we have

$$\begin{aligned} p_{[1^0 11^1]}\left(11^{2r}m + \frac{11 - 11^{2r+1}}{24}\right) &\equiv 0 \pmod{11^{2r}}, \\ p\left(11^{2r+1}m + \frac{1 - 11^{2r+2}}{24}\right) &\equiv 0 \pmod{11^{2r+1}}. \end{aligned} \tag{6.2.18}$$

Now using (6.2.14) and (6.2.18), we can see that (6.2.3) is true.

Then we prove (6.2.4). From (3.4.6), we have

$$C\Phi_{13}(q) \equiv \sum_{n=0}^{\infty} p_{[1^0 13^1]}(n) q^n \pmod{13}. \tag{6.2.19}$$

For $c = 0, d = 1$, we see that using Table 4.3, $\theta(\lambda_1, \mu_1) = 0$ by equation (4.2.4). Hence, we have

$$n_{1,13}(0,1) = 0.$$

Hence, by Corollary 1.0.6, for some m, we have

$$p_{[1^0 13^1]}\left(13^{2r}m - \frac{13 - 13^{2r+1}}{24}\right) \not\equiv 0 \pmod{13}.$$

$\square$

6.3. Applications for Theorem 1.0.11

In this section, we calculate examples for our results. Here we discuss three examples, one for when $c, d > 1$, one for $c = 1, d < 0$, and finally for $c < 0$, and $d < 0$.

We first consider $c = 2, d = 8$ case:

We apply Theorem 1.0.11 to this case. Remark 5.1.4 and (5.1.6) give a lower bound for the prime ℓ such that Theorem 1.0.11 is true. Since c and d are even, we see that the Theorem 1.0.11 is true for primes $\ell > 7$. Hence the rank of the stabilized module satisfies

$$r_\ell\left(\Omega_\ell^{odd}(2,8)\right) \leq \dim(S_{5(\ell-1)}) - \lfloor \frac{8(\ell^2-1)}{24\ell} \rfloor.$$

We calculate the right hand side in Table 6.3. We have the right hand side is zero for primes 11 and is 1 for primes 13 and 17. For $\ell = 7$, the bound given by the theorem is not accurate and the generating functions $L_7(2,8,b;z) \in^1 S_{36} \cap \mathbb{Z}_{(7)}[[q]]$ when $b \geq b_\ell(2,8,1)$. Since the bound is 1, Theorem 1.0.11 cannot be used to prove the existence of Ramanujan congruences for $\ell = 7$.

Hence using Theorem 1.0.3, for all $m, r \geq 1$, we have

$$p_{[1^2 11^8]}\left(11^{2r}m - \frac{15(11^{2r}-1)}{4}\right) \equiv 0 \pmod{11^{2r-1}}.$$

Table 6.3. Bounds for the ranks when $c = 2, d = 8$.

ℓ	$\dim\left(S_{5(\ell-1)}\right)$	$\lfloor \frac{8(\ell^2-1)}{24\ell} \rfloor$	Bound
7	3	2	1
11	3	3	0
13	5	4	1
17	6	5	1

Table 6.4. Bounds for ranks when $c = 1$ and $d = -1$.

ℓ	$\dim\left(S_{(2+\lfloor \frac{\ell}{2} \rfloor)(\ell-1)}\right)$	$\lfloor \frac{\ell(\ell^2-1)}{24\ell} \rfloor$	Bound
5	1	1	0
7	2	2	0
11	5	5	0
13	8	7	1
17	13	12	1

We also see that there are no Ramanujan congruences for 13. Calculating coefficients using a computer, we see that

$$p_{[1^2 13^8]}(13^3 m - 3108) \equiv p_{[1^2 13^8]}(13m - 14) \pmod{13}.$$

We then consider the case where $c = 1$ and $d = -1$. In this case, we get so called ℓ-regular partitions. Using Remark 5.2.8 and (5.1.6), Theorem 1.0.11 is true primes $\ell \geq 5$. We now use the theorem to find bounds for the rank of the stabilized module.

$$\begin{aligned} r\left(\Omega_\ell^{\mathsf{ODD}}(1,-1)\right) &\leq \dim\left(S_{\kappa_\ell(\ell,1)}\right) - \left\lfloor \frac{\ell(\ell^2-1)}{24\ell} \right\rfloor, \\ &= \dim\left(S_{(2+\lfloor \frac{\ell}{2} \rfloor)(\ell-1)}\right) - \left\lfloor \frac{\ell(\ell^2-1)}{24\ell} \right\rfloor. \end{aligned}$$

We state our calculations in Table 6.4. Hence we see that we have congruences for primes $5, 7$, and 11. This confirms our findings in Corollary 6.1.1.

We now consider the case where $c = -3$ and $d = -2$. Then from Corollary 5.3.7, we have the stabilized module has rank at most

$$\dim\left(S_{\kappa_\ell(2\ell-5,1)+(\ell-1)}\right) - \left\lfloor \frac{(2\ell-5)(\ell^2-1)}{24\ell} \right\rfloor.$$

Table 6.5. Bounds for ranks when $c = -3$ and $d = -2$.

ℓ	$\dim\left(S_{(3+\lfloor\frac{2\ell-5}{2}\rfloor)(\ell-1)}\right)$	$\lfloor\frac{(2\ell-5)(\ell^2-1)}{24\ell}\rfloor$	Bound
5	1	1	0
7	3	2	1
11	8	7	1
13	13	11	2

Then we calculate the right hand sides for primes $\ell = 5, 7, 11,$ and 13 in Table 6.5.

Hence we see that we have a congruence for 5. When compared to the other cases, in this case, we see that we need to choose the minimum number between $2\ell - 4$ and $2\ell - 5$ to get some what better upper bound for rank of the stabilized module. However, using Theorem 1.0.3, we have

$$\begin{aligned} p_{[1^{-3}5^{-2}]}(5n+3) &\equiv 0 \pmod{5}, \\ p_{[1^{-3}7^{-2}]}(7n+6) &\equiv 0 \pmod{7}, \\ p_{[1^{-3}11^{-2}]}(11n+4) &\equiv 0 \pmod{11}, \\ p_{[1^{-3}13^{-2}]}(13n+8) &\equiv 0 \pmod{13}. \end{aligned} \tag{6.3.1}$$

Therefore we see that our bound for the rank is not the optimal bound.

6.4. Improvements

6.4.1. ℓ-adic module structures related to ℓ-core partitions

For $c = 1$ and $d < 0$, we defined the sequence of modular functions $L_\ell(c, d, b; z)$ for $p_{[1^c\ell^d]}(n)$ in (5.2.10) and (5.2.11). However, this construction only works for $\ell + d \geq 1$. For ℓ-core partitions, we define the generating sequence of modular functions $L_\ell(1, -\ell, b; z)$ in the

Table 6.6. Bounds for ℓ-core partitions

ℓ	$\dim\left(S_{(3+\lfloor\frac{2\ell-1}{2}\rfloor)(\ell-1)}\right)$	$\lfloor\frac{(2\ell-1)(\ell^2-1)}{24\ell}\rfloor$	Bound
5	2	1	1
7	4	3	1
11	10	9	1
13	15	13	2
17	25	23	2

following way.

$$\begin{aligned} &L_\ell\left(1,-\ell,0;z\right) := 1,\\ &L_\ell\left(1,-\ell,1;z\right) := \phi_\ell(z)|U(\ell),\\ &L_\ell\left(1,-\ell,2;z\right) := A_\ell^2(z)A_\ell^2(\ell z)\phi_\ell^{\ell-2}(z)L_\ell\left(1,-\ell,1;z\right)|U(\ell),\\ &L_\ell\left(1,-\ell,b;z\right) := \begin{cases} \phi_\ell(z)^{2\ell-1}L_\ell(1,-\ell,b-1;z)|U(\ell) & \text{if } b\geq 3 \text{ odd},\\ \phi_\ell(z)^{\ell-2}L_\ell(1,-\ell,b-1;z)|U(\ell) & \text{if } b\geq 3 \text{ even}.\end{cases}\end{aligned} \tag{6.4.1}$$

Using a similar calculation that we used to prove Lemma 4.1.5, we see that

$$L_\ell(1,-\ell,b;z) \in^j \begin{cases} M_{\kappa_\ell(2\ell-1,j)+(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]] & \text{for } b\geq 2 \text{ odd},\\ M_{\kappa_\ell(\ell-2,j)+(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]] & \text{for } b\geq 2 \text{ even}.\end{cases}$$

Hence similarly as Corollary 5.3.7, we have the rank of the stabilized module $\Omega_\ell(1,-\ell,m)$ has rank at most

$$\dim(S_{\kappa_\ell(e,j)+(\ell-1)}) - \left\lfloor \frac{e(\ell^2-1)}{24\ell} \right\rfloor, \text{ where } e = \max\{2\ell-1, \ell-2\}.$$

We calculated the bound for $\ell \leq 17$ in Table 6.4.1.

Then comparing the results we obtained in Table 6.2, we see that we do not have the optimal bound in this case. Therefore, there may be a better way to define these functions so that we have the correct bound.

Remark 6.4.1. *Recall that* $b_\ell(c,d,m)$ *is the least non negative integer such that* $L_\ell(c,d,b;z) \in \Omega_\ell(c,d,m)$ *for all* $b \geq b_\ell(c,d,m)$. *Authors in [12], [13], and [17], found a bound for* $b_\ell(c,d,m)$ *for each corresponding cases. For example, Boylan and Webb shows that under certain assumptions,* $b_\ell(1,0,m) \geq 2m-1$.

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