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Multipartite quantum correlations and local recoverability

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Characterizing genuine multipartite quantum correlations in quantum physical systems has historically been a challenging problem in quantum information theory. More recently, however, the *total correlation* or *multipartite information* measure has been helpful in accomplishing this goal, especially with the multipartite symmetric quantum (MSQ) discord (Piani *et al.* 2008 *Phys. Rev. Lett.* **100**, 090502. (doi:10.1103/PhysRevLett.100.090502)) and the conditional entanglement of multipartite information (CEMI) (Yang *et al.* 2008 *Phys. Rev. Lett.* **101**, 140501. (doi:10.1103/PhysRevLett.101.140501)). Here, we apply a recent and significant improvement of strong subadditivity of quantum entropy (Fawzi & Renner 2014 (<http://arxiv.org/abs/1410.0664>)) in order to develop these quantities further. In particular, we prove that the MSQ discord is nearly equal to zero if and only if the multipartite state for which it is evaluated is approximately locally recoverable after performing measurements on each of its systems. Furthermore, we prove that the CEMI is a faithful entanglement measure, i.e. it vanishes if and only if the multipartite state for which it is evaluated is a fully separable state. Along the way, we provide an operational interpretation of the MSQ discord in terms of the partial state distribution protocol, which in turn, as a special case, gives an interpretation for the original discord quantity. Finally, we prove an inequality that could potentially improve upon the Fawzi–Renner inequality in the multipartite context, but it remains an open question to determine whether this is so.

1. Introduction

The quantification and characterization of correlations in multiple physical systems has a long history, with

some of the first proposals for information measures being the works of McGill [1] and Watanabe [2]. Of particular interest for us here is the *total correlation* measure proposed by Watanabe [2], which is defined for a set of random variables X_1, \dots, X_l as the sum of the individual entropies less the joint entropy

$$I(X_1 : \dots : X_l) \equiv H(X_1) + \dots + H(X_l) - H(X_1 \dots X_l), \quad (1.1)$$

where $H(\cdot)$ is the Shannon entropy. The total correlation has the salient properties of being non-negative and monotone non-increasing under local operations, meaning that it does not increase under the local discarding of information, i.e. for random variables $X_1, X'_1, \dots, X_l, X'_l$, the following inequality holds:

$$I(X_1 X'_1 : \dots : X_l X'_l) \geq I(X'_1 : \dots : X'_l). \quad (1.2)$$

The generalization of the total correlation to quantum physical systems is straightforward, given simply by replacing Shannon entropies with von Neumann entropies [3]. In the quantum information theory literature, the quantity is known as the *multipartite information*. Specifically, let $\rho_{A_1 A_2 \dots A_l}$ be a multipartite density operator representing the state of systems A_1, \dots, A_l (i.e. $\rho_{A_1 A_2 \dots A_l}$ is a trace one, positive semi-definite operator acting on the tensor-product Hilbert space $\mathcal{H}_{A_1} \otimes \dots \otimes \mathcal{H}_{A_l}$). The multipartite information of this state is defined as

$$I(A_1 : \dots : A_l)_\rho \equiv H(A_1)_\rho + \dots + H(A_l)_\rho - H(A_1 \dots A_l)_\rho, \quad (1.3)$$

with the von Neumann entropy of a density operator σ on system S defined in terms of the natural logarithm as $H(S)_\sigma \equiv H(\sigma) \equiv -\text{Tr}\{\sigma \log \sigma\}$ and the marginal entropies $H(A_i)_\rho$ are defined with respect to the reduced density operator

$$\rho_{A_i} = \text{Tr}_{A_1 \dots A_l \setminus A_i} \{\rho_{A_1 \dots A_l}\}. \quad (1.4)$$

The quantity in (1.3) is also non-negative and monotone non-increasing under the local discarding of information, i.e. the following inequality holds for a multipartite density operator $\rho_{A_1 A'_1 \dots A_l A'_l}$:

$$I(A_1 A'_1 : \dots : A_l A'_l)_\rho \geq I(A'_1 : \dots : A'_l)_\rho. \quad (1.5)$$

The above inequality follows because the multipartite information can be written in terms of the relative entropy $D(\rho \parallel \sigma) \equiv \text{Tr}\{\rho[\log \rho - \log \sigma]\}$ [4] as

$$I(A_1 : \dots : A_l)_\rho = D(\rho_{A_1 \dots A_l} \parallel \rho_{A_1} \otimes \dots \otimes \rho_{A_l}), \quad (1.6)$$

and the relative entropy is monotone non-increasing under quantum operations [5], i.e. $D(\rho \parallel \sigma) \geq D(\mathcal{N}(\rho) \parallel \mathcal{N}(\sigma))$ for any states ρ and σ and quantum channel \mathcal{N} (recall that a quantum channel is a completely positive trace preserving (CPTP) linear map).

Given the inequality in (1.5), we are left to wonder whether one could refine it in a non-trivial way by finding a state-dependent remainder term. This kind of question has been the driving force behind several recent investigations in quantum information theory [6–13], culminating in the following breakthrough inequality of Fawzi & Renner [14]:

$$I(A; B|C)_\rho \geq -\log F(\rho_{ABC}, \mathcal{R}_{C \rightarrow AC}(\rho_{BC})) \quad (1.7)$$

$$\geq \frac{1}{4} \|\rho_{ABC} - \mathcal{R}_{C \rightarrow AC}(\rho_{BC})\|_1^2, \quad (1.8)$$

where $I(A; B|C)_\rho$ is the conditional quantum mutual information of a tripartite state ρ_{ABC} , defined as

$$I(A; B|C)_\rho \equiv H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho, \quad (1.9)$$

and $\mathcal{R}_{C \rightarrow AC}$ is a particular CPTP ‘recovery map’ which acts on system C alone in an attempt to recover the ‘lost’ system A . The quantity $F(\omega, \tau) \equiv \|\sqrt{\omega} \sqrt{\tau}\|_1^2$ is the quantum fidelity between states ω and τ [15], with $\|A\|_1 \equiv \text{Tr}\{\sqrt{A^\dagger A}\}$ the Schatten ℓ_1 norm. The trace distance between two density operators ω and τ is defined in terms of the trace norm as $\|\omega - \tau\|_1$ and characterizes how well one can distinguish the states ω and τ in any physical experiment. The Fawzi–Renner

inequality gives a state-dependent improvement to strong subadditivity (i.e. $I(A;B|C)_\rho \geq 0$) [16,17] and has even been improved upon in recent work of Brandao *et al.* [18]. One can also see the recent work [19] for a simpler proof of (1.7).

The difference of the two multipartite informations in (1.5) is the basis for two distinct measures of quantum correlations: the multipartite symmetric quantum (MSQ) discord [20] and the conditional entanglement of multipartite information (CEMI) [21], which were inspired by the quantum discord [22,23] and the squashed entanglement [24], respectively. We briefly motivate these quantities here and give formal definitions later in the paper. We begin by describing the MSQ discord. Let $A_1 \cdots A_l$ be quantum systems held by spatially separated parties and suppose that each party measures their local system, leading to classical systems $X_1 \cdots X_l$. We could then compute the non-negative information gap $I(A_1 : \cdots : A_l) - I(X_1 : \cdots : X_l)$ and optimize it with respect to all local measurements. Suppose that the state is classical to begin with, meaning that it can be written as

$$\sum_{x_1, \dots, x_l} p(x_1, \dots, x_l) |x_1\rangle\langle x_1|_{A_1} \otimes \cdots \otimes |x_l\rangle\langle x_l|_{A_l}, \quad (1.10)$$

for some joint probability distribution $p(x_1, \dots, x_l)$ and orthonormal bases $\{|x_i\rangle_{A_i}\}$ for $i \in \{1, \dots, l\}$. Then there are local measurements that do not change the state at all after they are performed, and the MSQ discord is equal to zero. If the state cannot be written as above, then it cannot be understood in a classical way, such that there does not exist a set of local measurements that would leave the state undisturbed. In this sense, the MSQ discord is a measure of multipartite quantum correlations between the different parties and it is known that it is a faithful measure [20], meaning that it is zero if and only if the state is multipartite classical as written in (1.10). Other desirable properties for a discord-like measure are described in [25, §2.1].

The CEMI is motivated by the concept of the monogamy of quantum entanglement [21], that if two or more systems are highly entangled then any other systems cannot be too entangled with them. On the other hand, states which are close to being unentangled are highly shareable [26] or extendible [27], such that there could be many other systems sharing the same correlations with them. To define the CEMI, we begin with a multipartite state on the systems $A_1 \cdots A_l$ and try to find a global state on these systems and some others $A'_1 \cdots A'_l$ that is consistent with the original state, meaning that we recover the original state when tracing over $A'_1 \cdots A'_l$. Based on the aforementioned ideas, any classical correlations can be shared with the extension systems $A'_1 \cdots A'_l$, while entanglement cannot be shared. The information gap $I(A_1 A'_1 : \cdots : A_l A'_l) - I(A'_1 : \cdots : A'_l)$ attempts to subtract out the multipartite classical correlations that are shareable, so that what is left is a measure of multipartite quantum entanglement. One then optimizes this quantity by taking an infimum over all extension states. The work of Yang *et al.* [21] fully justified this approach, proving that the CEMI is a proper entanglement measure, bearing many properties which are desirable for such a measure. What was left open was to prove that the CEMI is a faithful entanglement measure, meaning that it is equal to zero if and only if the state on $A_1 \cdots A_l$ is a fully separable (unentangled) state [28] of the following form:

$$\sum_z p(z) \sigma_{A_1}^z \otimes \cdots \otimes \sigma_{A_l}^z, \quad (1.11)$$

where $p(z)$ is a probability distribution and $\sigma_{A_i}^z$ is a quantum state on system A_i .

2. Summary of results

The Fawzi–Renner inequality in (1.7) has a number of implications for entanglement theory and more general quantum correlations: it gives an alternative method [6,29] from [30] for establishing the faithfulness of the squashed entanglement measure [24] and it allows for characterizing quantum states with discord [22,23] nearly equal to zero as being approximate fixed points of entanglement breaking (EB) channels [13, Proposition 29].

The main objective of this paper is to pursue extensions of these ideas for multipartite quantum states and correlation measures. In particular, we first demonstrate that the following ‘local recoverability’ inequality is a consequence of the inequality in (1.7):

$$I(A_1 A'_1 : \dots : A_l A'_l)_\rho - I(A'_1 : \dots : A'_l)_\rho \geq \left[\frac{1}{2l} \|\rho_{A_1 A'_1 \dots A_l A'_l} - (\mathcal{R}_{A'_1 \rightarrow A_1 A'_1}^1 \otimes \dots \otimes \mathcal{R}_{A'_l \rightarrow A_l A'_l}^l)(\rho_{A'_1 \dots A'_l})\|_1 \right]^2, \quad (2.1)$$

where $\mathcal{R}_{A'_1 \rightarrow A_1 A'_1}^1, \dots, \mathcal{R}_{A'_l \rightarrow A_l A'_l}^l$ are local recovery maps. The implication of the above inequality is that if the gap $I(A_1 A'_1 : \dots : A_l A'_l)_\rho - I(A'_1 : \dots : A'_l)_\rho$ is nearly equal to zero, then the full state $\rho_{A_1 A'_1 \dots A_l A'_l}$ is ‘locally recoverable’, i.e. one can approximately recover it by performing the local recovery maps $\mathcal{R}_{A'_1 \rightarrow A_1 A'_1}^1, \dots, \mathcal{R}_{A'_l \rightarrow A_l A'_l}^l$. The converse of this statement is a direct consequence of the Alicki–Fannes inequality [31], with a proof proceeding similarly to the steps in (4.8)–(4.12) and a dimension dependence only on the systems A_1, \dots, A_l . It might be possible to improve upon the inequality in (2.1), i.e. to have the l -independent inequality:

$$I(A_1 A'_1 : \dots : A_l A'_l)_\rho - I(A'_1 : \dots : A'_l)_\rho \geq -\log F(\rho_{A_1 A'_1 \dots A_l A'_l}(\mathcal{R}_{A'_1 \rightarrow A_1 A'_1}^1 \otimes \dots \otimes \mathcal{R}_{A'_l \rightarrow A_l A'_l}^l)(\rho_{A'_1 \dots A'_l})). \quad (2.2)$$

We elaborate more on this possibility in §7.

Regardless of whether the conjectured inequality in (2.2) holds, we can already establish two consequences of the inequality in (2.1).

- (1) The MSQ discord from [20] is nearly equal to zero if and only if the multipartite state $\rho_{A_1 \dots A_l}$ is locally recoverable after performing measurements on each of the systems A_1, \dots, A_l . Equivalently, such a state has MSQ discord nearly equal to zero if and only if it is an approximate fixed point of a tensor product of EB channels. Recall that any EB channel can be written as a composition of a measurement channel followed by a preparation channel [32]. We detail this result in §4.
- (2) The CEMI from [21] is faithful, i.e. it vanishes if and only if a multipartite state $\rho_{A_1 \dots A_l}$ is fully separable. We detail this result in §5.

Additional contributions of this paper are to show explicitly in §5a that the CEMI is an upper bound on the multipartite squashed entanglement from [33,34] and in §6b to give an operational interpretation of the MSQ discord in terms of the partial state distribution protocol from [21]. We conclude in §8 with a summary of results and directions for future work.

3. Local recoverability

In this section, we give a proof of the local recoverability inequality in (2.1). We start with an explicit proof of the following lemma, which is implicit in the partial state distribution protocol of Yang *et al.* [21].

Lemma 3.1. *Let $\rho_{A_1 A'_1 \dots A_l A'_l}$ be a multipartite quantum state. Then we have the following identity:*

$$I(A_1 A'_1 : \dots : A_l A'_l)_\rho - I(A'_1 : \dots : A'_l)_\rho = \sum_{i=1}^l I(A_i; A_1^{i-1} A'_{[l] \setminus \{i\}} | A'_i), \quad (3.1)$$

where $A_1^{i-1} \equiv A_{i-1} \dots A_1$ (interpreted to be empty if $i=1$) and $A'_{[l] \setminus \{i\}}$ is a shorthand indicating all of the A' systems except for A'_i . In addition, the expansion on the right-hand side can proceed in any order.

Proof. Consider that

$$I(A_1A'_1 : \dots : A_lA'_l)_\rho - I(A'_1 : \dots : A'_l)_\rho = \sum_{i=1}^l H(A_iA'_i)_\rho - H(A_1A'_1A_2A'_2 \dots A_lA'_l)_\rho - \left[\sum_{i=1}^l H(A'_i)_\rho - H(A'_1A'_2 \dots A'_l)_\rho \right] \quad (3.2)$$

$$= \sum_{i=1}^l H(A_i|A'_i)_\rho - H(A_1A_2 \dots A_l|A'_1A'_2 \dots A'_l)_\rho \quad (3.3)$$

$$= \sum_{i=1}^l H(A_i|A'_i)_\rho - \sum_{i=1}^l H(A_i|A_1^{i-1}A'_1A'_2 \dots A'_l)_\rho \quad (3.4)$$

$$= \sum_{i=1}^l [H(A_i|A'_i)_\rho - H(A_i|A_1^{i-1}A'_1A'_2 \dots A'_l)_\rho] \quad (3.5)$$

$$= \sum_{i=1}^l I(A_i; A_1^{i-1}A'_{[l] \setminus \{i\}}|A'_i). \quad (3.6)$$

The first equality is an expansion following from definitions. The second equality uses the chain rule for conditional entropy, i.e. $H(A|B) = H(AB) - H(B)$. The third equality follows from an inductive application of the chain rule for conditional entropy. The final equality follows from an expansion for conditional mutual information as $I(A; B|C) = H(A|C) - H(A|CB)$. The statement about expanding in an arbitrary order follows because the expansion in the third equality can proceed in any order. ■

Proof of (2.1). We can now easily prove the inequality in (2.1). From lemma 3.1, we can conclude that

$$I(A_1A'_1 : \dots : A_lA'_l)_\rho - I(A'_1 : \dots : A'_l)_\rho \geq I(A_i; A_{[l] \setminus \{i\}}A'_{[l] \setminus \{i\}}|A'_i) \quad (3.7)$$

for all $i \in \{1, 2, \dots, l\}$ because (a) the expansion there can proceed in any order and (b) the conditional mutual information is non-negative [16,17]. From the inequality in (1.7), we can then conclude that there exists a recovery map $\mathcal{R}_{A'_i \rightarrow A_iA'_i}^i$ such that

$$I(A_1A'_1 : \dots : A_lA'_l)_\rho - I(A'_1 : \dots : A'_l)_\rho \geq I(A_i; A_{[l] \setminus \{i\}}A'_{[l] \setminus \{i\}}|A'_i) \quad (3.8)$$

$$\geq \frac{1}{4} \|\rho_{A_1A'_1 \dots A_lA'_l} - \mathcal{R}_{A'_i \rightarrow A_iA'_i}^i(\rho_{A_{[l] \setminus \{i\}}A'_1 \dots A'_l})\|_1^2, \quad (3.9)$$

which is equivalent to

$$2\sqrt{I(A_1A'_1 : \dots : A_lA'_l)_\rho - I(A'_1 : \dots : A'_l)_\rho} \geq \|\rho_{A_1A'_1 \dots A_lA'_l} - \mathcal{R}_{A'_i \rightarrow A_iA'_i}^i(\rho_{A_{[l] \setminus \{i\}}A'_1 \dots A'_l})\|_1 \quad (3.10)$$

$$= \|\rho_{A_1A'_1 \dots A_lA'_l} - (\mathcal{R}_{A'_i \rightarrow A_iA'_i}^i \circ \text{Tr}_{A_i})(\rho_{A_1A'_1 \dots A_lA'_l})\|_1. \quad (3.11)$$

Using the triangle inequality l times and monotonicity of the trace distance under quantum operations (i.e. that $\|\omega - \tau\|_1 \geq \|\mathcal{N}(\omega) - \mathcal{N}(\tau)\|_1$ for density operators ω and τ and a quantum channel \mathcal{N}), we can then conclude that

$$2l\sqrt{I(A_1A'_1 : \dots : A_lA'_l)_\rho - I(A'_1 : \dots : A'_l)_\rho} \geq \|\rho_{A_1A'_1 \dots A_lA'_l} - (\mathcal{R}_{A'_1 \rightarrow A_1A'_1}^1 \otimes \dots \otimes \mathcal{R}_{A'_l \rightarrow A_lA'_l}^l)(\rho_{A'_1 \dots A'_l})\|_1, \quad (3.12)$$

which is equivalent to (2.1). ■

Remark 3.2. The above proof demonstrates that there are in fact 2^l inequalities that hold, depending on whether one chooses to apply the trace-out-and-recovery maps or not. The inequality then takes on the following form:

$$I(A_1 A'_1 : \dots : A_l A'_l)_\rho - I(A'_1 : \dots : A'_l)_\rho \geq \left[\frac{1}{2^{|j^l|}} \|\rho_{A_1 A'_1 \dots A_l A'_l} - ((\mathcal{R}_{A'_1 \rightarrow A_1 A'_1}^1 \circ \text{Tr}_{A_1})^{j_1} \otimes \dots \otimes (\mathcal{R}_{A'_l \rightarrow A_l A'_l}^l \circ \text{Tr}_{A_l})^{j_l})(\rho_{A_1 A'_1 \dots A_l A'_l})\|_1 \right]^2, \quad (3.13)$$

where $j^l \equiv j_1 \dots j_l$ is a binary string indicating which recovery maps are applied and $|j^l|$ is the number of ones in j^l if j^l is not the all-zeros bit string, with $|j^l|$ otherwise being equal to one.

4. Approximate faithfulness of the multipartite symmetric quantum discord

In this section, we provide a generalization of the approximate faithfulness of quantum discord [13, Proposition 29] to the multipartite case. In particular, recall the MSQ discord from [20]:

$$D(\bar{A}_1 : \dots : \bar{A}_l)_\rho \equiv I(A_1 : \dots : A_l)_\rho - \sup_{\{\mathcal{M}_{A_1 \rightarrow X_1}^1, \dots, \mathcal{M}_{A_l \rightarrow X_l}^l\}} I(X_1 : \dots : X_l)_\omega, \quad (4.1)$$

where $\rho_{A_1 \dots A_l}$ is a multipartite quantum state and $\omega_{X_1 \dots X_l}$ is the state resulting from local measurements of $\rho_{A_1 \dots A_l}$ according to the measurement maps $\mathcal{M}_{A_1 \rightarrow X_1}^1, \dots, \mathcal{M}_{A_l \rightarrow X_l}^l$:

$$\omega_{X_1 \dots X_l} \equiv (\mathcal{M}_{A_1 \rightarrow X_1}^1 \otimes \dots \otimes \mathcal{M}_{A_l \rightarrow X_l}^l)(\rho_{A_1 \dots A_l}). \quad (4.2)$$

The measurement map $\mathcal{M}_{A_i \rightarrow X_i}^i$ is defined as

$$\mathcal{M}_{A_i \rightarrow X_i}^i(\sigma_{A_i}) \equiv \sum_x \text{Tr}\{\Lambda_{A_i}^x \sigma_{A_i}\} |x\rangle\langle x|_{A_i}, \quad (4.3)$$

for some positive semi-definite operators $\Lambda_{A_i}^x$ which sum to the identity and where $\{|x\rangle_{A_i}\}$ is an orthonormal basis for the system A_i .

Proposition 4.1 (Approximate faithfulness). *The MSQ discord is nearly equal to zero if and only if $\rho_{A_1 \dots A_l}$ is an approximate fixed point of a tensor product of EB channels $\mathcal{E}_{A_1}^1, \dots, \mathcal{E}_{A_l}^l$. That is, suppose that there exist EB channels $\mathcal{E}_{A_1}^1, \dots, \mathcal{E}_{A_l}^l$ such that*

$$\|\rho_{A_1 \dots A_l} - (\mathcal{E}_{A_1}^1 \otimes \dots \otimes \mathcal{E}_{A_l}^l)(\rho_{A_1 \dots A_l})\|_1 \leq \varepsilon \quad (4.4)$$

for some $\varepsilon \in [0, 1]$. Then

$$D(\bar{A}_1 : \dots : \bar{A}_l)_\rho \leq (l+1)h_2\left(\frac{\varepsilon}{2}\right) + \varepsilon \sum_{i=1}^l \log(|A_i|), \quad (4.5)$$

where $h_2(\varepsilon)$ is the binary entropy with the property that $\lim_{\varepsilon \searrow 0} h_2(\varepsilon) = 0$. Conversely, suppose that

$$D(\bar{A}_1 : \dots : \bar{A}_l)_\rho \leq \varepsilon \quad (4.6)$$

for some $\varepsilon > 0$. Then there exist EB channels $\mathcal{E}_{A_1}^1, \dots, \mathcal{E}_{A_l}^l$ such that

$$\|\rho_{A_1 \dots A_l} - (\mathcal{E}_{A_1}^1 \otimes \dots \otimes \mathcal{E}_{A_l}^l)(\rho_{A_1 \dots A_l})\|_1 \leq 2l\sqrt{\varepsilon}. \quad (4.7)$$

Proof. The proof of the inequality in (4.5) proceeds exactly as in the proof of [13, Proposition 29]. Consider that every EB channel can be written as a composition of a measurement map and a preparation [32], i.e. $\mathcal{E}_{A_i}^i = \mathcal{P}_{X_i \rightarrow A_i}^i \circ \mathcal{M}_{A_i \rightarrow X_i}^i$. Then

$$D(\bar{A}_1 : \dots : \bar{A}_l)_\rho = I(A_1 : \dots : A_l)_\rho - \sup_{\{\mathcal{M}_{A_1 \rightarrow X_1}^1, \dots, \mathcal{M}_{A_l \rightarrow X_l}^l\}} I(X_1 : \dots : X_l)_\omega \quad (4.8)$$

$$\leq I(A_1 : \dots : A_l)_\rho - I(X_1 : \dots : X_l)_{\otimes_i \mathcal{M}_{A_i \rightarrow X_i}^i(\rho)} \quad (4.9)$$

$$\leq I(A_1 : \dots : A_l)_\rho - I(A_1 : \dots : A_l)_{\otimes_i \mathcal{P}_{X_i \rightarrow A_i}^i \circ \mathcal{M}_{A_i \rightarrow X_i}^i(\rho)} \quad (4.10)$$

$$= I(A_1 : \dots : A_l)_\rho - I(A_1 : \dots : A_l)_{\otimes_i \mathcal{E}_{A_i}^i(\rho)} \quad (4.11)$$

$$\leq (l+1)h_2\left(\frac{\varepsilon}{2}\right) + \varepsilon \sum_{i=1}^l \log(|A_i|). \quad (4.12)$$

The first inequality follows by choosing the measurement maps not to be the optimal ones, but instead the ones making up the first part of the EB channels $\{\mathcal{E}_{A_i}^i\}$. The second inequality follows from the fact that the multipartite information is monotone under local operations (here being the processing of the measured systems according to the preparation maps). The last inequality is a consequence of the Fannes–Audenaert inequality [35], which states that

$$|H(\rho) - H(\sigma)| \leq T \log(d-1) + h_2(T), \quad (4.13)$$

with $T = \frac{1}{2} \|\rho - \sigma\|_1$ and d the dimension of the density operators ρ and σ .

After recalling that any quantum channel (including measurement maps) can be understood as an isometric embedding of the input in a tensor-product Hilbert space followed by a partial trace [36], we can see that (4.7) is a consequence of the inequality in (2.1). Specifically, for a particular set of measurements, we can write

$$I(A_1 : \dots : A_l)_\rho - I(X_1 : \dots : X_l)_\omega = I(X_1 E_1 : \dots : X_l E_l)_\omega - I(X_1 : \dots : X_l)_\omega, \quad (4.14)$$

where

$$\omega_{X_1 E_1 \dots X_l E_l} \equiv (\mathcal{U}_{A_1 \rightarrow X_1 E_1}^{\mathcal{M}^1} \otimes \dots \otimes \mathcal{U}_{A_l \rightarrow X_l E_l}^{\mathcal{M}^l})(\rho_{A_1 \dots A_l}) \quad (4.15)$$

and $\mathcal{U}_{A_i \rightarrow X_i E_i}^{\mathcal{M}^i}$ is an isometric CPTP map, so that

$$\mathcal{U}_{A_i \rightarrow X_i E_i}^{\mathcal{M}^i}(\cdot) \equiv \mathcal{U}_{A_i \rightarrow X_i E_i}^{\mathcal{M}^i}(\cdot) [\mathcal{U}_{A_i \rightarrow X_i E_i}^{\mathcal{M}^i}]^\dagger, \quad (4.16)$$

where $\mathcal{U}_{A_i \rightarrow X_i E_i}^{\mathcal{M}^i}$ is an isometric extension of the measurement map $\mathcal{M}_{A_i \rightarrow X_i}^i$. Then (4.14) follows because the multipartite information is invariant under local isometries, as one can see from its definition in (1.3) and invariance of quantum entropy under isometries. The inequality (4.7) then follows because there exist recovery maps $\mathcal{R}_{X_1 \rightarrow X_1 E_1}^1, \dots, \mathcal{R}_{X_l \rightarrow X_l E_l}^l$ such that

$$I(X_1 E_1 : \dots : X_l E_l)_\omega - I(X_1 : \dots : X_l)_\omega \geq \left[\frac{1}{2l} \|\omega_{X_1 E_1 \dots X_l E_l} - (\mathcal{R}_{X_1 \rightarrow X_1 E_1}^1 \otimes \dots \otimes \mathcal{R}_{X_l \rightarrow X_l E_l}^l)(\omega_{X_1 \dots X_l})\|_1 \right]^2 \quad (4.17)$$

$$\geq \left[\frac{1}{2l} \|\rho_{A_1 \dots A_l} - (\mathcal{P}_{X_1 \rightarrow A_1}^1 \otimes \dots \otimes \mathcal{P}_{X_l \rightarrow A_l}^l)(\omega_{X_1 \dots X_l})\|_1 \right]^2 \quad (4.18)$$

$$= \left[\frac{1}{2l} \|\rho_{A_1 \dots A_l} - (\mathcal{E}_{A_1}^1 \otimes \dots \otimes \mathcal{E}_{A_l}^l)(\rho_{A_1 \dots A_l})\|_1 \right]^2. \quad (4.19)$$

The first inequality is a consequence of (2.1). We define the following CPTP maps:

$$\mathcal{T}_{X_i E_i \rightarrow A_i}^i(\gamma_{X_i E_i}) \equiv [\mathcal{U}_{A_i \rightarrow X_i E_i}^{\mathcal{M}^i}]^\dagger \gamma_{X_i E_i} \mathcal{U}_{A_i \rightarrow X_i E_i}^{\mathcal{M}^i} + \text{Tr}\{(I_{X_i E_i} - \mathcal{U}_{A_i \rightarrow X_i E_i}^{\mathcal{M}^i} [\mathcal{U}_{A_i \rightarrow X_i E_i}^{\mathcal{M}^i}]^\dagger) \gamma_{X_i E_i}\} \sigma_{A_i}^i \quad (4.20)$$

where $\sigma_{A_i}^i$ is some state on system A_i . Observe that

$$(\mathcal{T}_{X_1 E_1 \rightarrow A_1}^1 \otimes \cdots \otimes \mathcal{T}_{X_l E_l \rightarrow A_l}^l)(\omega_{X_1 \dots X_l E_1 \dots E_l}) = \rho_{A_1 \dots A_l}. \quad (4.21)$$

Then the second inequality above follows by defining the preparation maps $\mathcal{P}_{X_i \rightarrow A_i}^i$ as

$$\mathcal{P}_{X_i \rightarrow A_i}^i \equiv \mathcal{T}_{X_i E_i \rightarrow A_i}^i \circ \mathcal{R}_{X_i \rightarrow X_i E_i}^i, \quad (4.22)$$

and noting that the trace distance does not increase under the CPTP map $\mathcal{T}_{X_1 E_1 \rightarrow A_1}^1 \otimes \cdots \otimes \mathcal{T}_{X_l E_l \rightarrow A_l}^l$. (The maps $\mathcal{P}_{X_i \rightarrow A_i}^i$ are preparations because they act on classical registers.) The last equality follows from the definition of $\omega_{X_1 \dots X_l}$ in (4.2) and the fact that any composition of a measurement map followed by a preparation map is EB [32]. ■

5. Faithfulness of the conditional entanglement of multipartite information

The CEMI is an entanglement measure defined in [21]. It bears some similarities with the squashed entanglement [24] and its multipartite version [33,34]. In [21,37], the CEMI was shown to be non-negative, monotone under local operations and classical communication, convex, additive, asymptotically continuous and equal to zero for separable states. It is not known to be monogamous. Given a multipartite state $\rho_{A_1 \dots A_l}$, the CEMI is defined as follows:

$$E_I(A_1 : \dots : A_l)_\rho \equiv \frac{1}{2} \inf_{\rho_{A_1 A'_1 \dots A_l A'_l}} I(A_1 A'_1 : \dots : A_l A'_l)_\rho - I(A'_1 : \dots : A'_l)_\rho, \quad (5.1)$$

where the infimum is over all extensions $\rho_{A_1 A'_1 \dots A_l A'_l}$ of $\rho_{A_1 \dots A_l}$, i.e.

$$\rho_{A_1 \dots A_l} = \text{Tr}_{A'_1 \dots A'_l} \{\rho_{A_1 A'_1 \dots A_l A'_l}\}. \quad (5.2)$$

In this section, we prove that the CEMI is faithful, i.e. equal to zero if and only if the state $\rho_{A_1 \dots A_l}$ is separable. Before doing so, it may be helpful to review the if-part of this theorem from [21]. If $\rho_{A_1 \dots A_l}$ is separable, then it has a decomposition of the following form [28]:

$$\rho_{A_1 \dots A_l} \equiv \sum_x p_X(x) \sigma_{A_1}^{1,x} \otimes \cdots \otimes \sigma_{A_l}^{l,x}, \quad (5.3)$$

for a probability distribution p_X and states $\{\sigma_{A_1}^{1,x}\}, \dots, \{\sigma_{A_l}^{l,x}\}$. In this case, one particular extension of this state has the following form:

$$\sum_x p_X(x) \sigma_{A_1}^{1,x} \otimes |x\rangle\langle x|_{A'_1} \otimes \cdots \otimes \sigma_{A_l}^{l,x} \otimes |x\rangle\langle x|_{A'_l}. \quad (5.4)$$

It is then clear for this particular extension that

$$I(A'_1 : \dots : A'_l) \geq I(A_1 A'_1 : \dots : A_l A'_l), \quad (5.5)$$

because one can produce the systems A_1, \dots, A_l by local preparation maps of the form

$$(\cdot) \rightarrow \sum_x |x\rangle\langle x|_{A'_i} (\cdot) |x\rangle\langle x|_{A'_i} \otimes \sigma_{A_i}^{i,x}. \quad (5.6)$$

Combined with the inequality in (1.5) and the definition of E_I in (5.1), we find that E_I is equal to zero if the state is separable.

We now establish the only-if-part of faithfulness of CEMI, which is a consequence of the following proposition.

Proposition 5.1. *The CEMI of a multipartite state $\rho_{A_1 \dots A_l}$ obeys the following bound:*

$$E_I(A_1 : \dots : A_l)_\rho \geq \frac{1}{16 \cdot (l+1)^4} \left(\sum_{i=2}^l |A_i|^2 \right)^{-2} \|\rho_{A_1 \dots A_l} - \text{SEP}(A_1 : \dots : A_l)\|_1^4, \quad (5.7)$$

where $\|\rho_{A_1 \dots A_l} - \text{SEP}(A_1 : \dots : A_l)\|_1$ is the trace distance from $\rho_{A_1 \dots A_l}$ to the set of multipartite separable states.

Proof. The proof of this proposition proceeds along the lines outlined in [6,29], an analysis which is repeated in both [13,14]. Let ε_ρ denote the value of the following quantity for a particular extension $\rho_{A_1 A'_1 \dots A_l A'_l}$:

$$\varepsilon_\rho = I(A_1 A'_1 : \dots : A_l A'_l)_\rho - I(A'_1 : \dots : A'_l)_\rho. \quad (5.8)$$

From remark 3.2, we know that there exist recovery maps $\mathcal{R}_{A'_1 \rightarrow A_1 A'_1}^1, \dots, \mathcal{R}_{A'_l \rightarrow A_l A'_l}^l$ such that the following inequalities hold:

$$\varepsilon_\rho \geq \left[\frac{1}{2^l} \left\| \rho_{A_1 A'_1 \dots A_l A'_l} - ((\mathcal{R}_{A'_1 \rightarrow A_1 A'_1}^1 \circ \text{Tr}_{A_1})^{j_1} \otimes \dots \otimes (\mathcal{R}_{A'_l \rightarrow A_l A'_l}^l \circ \text{Tr}_{A_l})^{j_l})(\rho_{A_1 A'_1 \dots A_l A'_l}) \right\|_1 \right]^2, \quad (5.9)$$

where $j^l \equiv j_1 \dots j_l$ is a binary string indicating which recovery maps are applied. Setting

$$\delta_\rho \equiv 2\sqrt{\varepsilon_\rho}, \quad (5.10)$$

these inequalities are then equivalent to the following ones:

$$l \cdot \delta_\rho \geq \left\| \rho_{A_1 A'_1 \dots A_l A'_l} - ((\mathcal{R}_{A'_1 \rightarrow A_1 A'_1}^1 \circ \text{Tr}_{A_1})^{j_1} \otimes \dots \otimes (\mathcal{R}_{A'_l \rightarrow A_l A'_l}^l \circ \text{Tr}_{A_l})^{j_l})(\rho_{A_1 A'_1 \dots A_l A'_l}) \right\|_1. \quad (5.11)$$

Let $A_j^k \equiv A_{j,1} \dots A_{j,k}$ for $j \in \{1, \dots, l\}$, and let $\Omega_{A_1^k \dots A_l^k}$ denote the following state, which results from many repeated attempts at local recovery:

$$\begin{aligned} \Omega_{A_1^k \dots A_l^k} &\equiv ((\mathcal{R}_{A'_1 \rightarrow A_{1,k} A'_1}^1 \circ \dots \circ \mathcal{R}_{A'_1 \rightarrow A_{1,1} A'_1}^1) \\ &\otimes \dots \otimes (\mathcal{R}_{A'_l \rightarrow A_{l,k} A'_l}^l \circ \dots \circ \mathcal{R}_{A'_l \rightarrow A_{l,1} A'_l}^l))(\rho_{A_1 A'_1 \dots A_l A'_l}). \end{aligned} \quad (5.12)$$

From (5.11), the triangle inequality, monotonicity of the trace distance under quantum operations, and the fact that the recovery maps $\mathcal{R}_{A'_1 \rightarrow A_{1,k} A'_1}^1, \dots, \mathcal{R}_{A'_l \rightarrow A_{l,k} A'_l}^l$ commute with each other because they act on different systems, we can conclude that all of the following inequalities hold:

$$\|\rho_{A_1 \dots A_l} - \Omega_{A_{1,x_1} \dots A_{l,x_l}}\|_1 \leq lk \cdot \delta_\rho \quad (5.13)$$

for all tuples (x_1, \dots, x_l) , where $x_i \in \{1, \dots, k\}$ and $i \in \{1, \dots, l\}$. We can then symmetrize the systems A_i^k according to the random permutation

$$\bar{\Pi}_{A_i^k}(\cdot) \equiv \frac{1}{k!} \sum_{\pi \in S_k} W_{A_{i,1} \dots A_{i,k}}^\pi(\cdot)(W_{A_{i,1} \dots A_{i,k}}^\pi)^\dagger, \quad (5.14)$$

where $W_{A_{i,1} \dots A_{i,k}}^\pi$ is a unitary representation of the permutation π which acts on the k -partite space $\mathcal{H}_{A_{i,1}} \otimes \dots \otimes \mathcal{H}_{A_{i,k}}$ as

$$W_{A_{i,1} \dots A_{i,k}}^\pi |m_1\rangle_{A_{i,1}} \otimes \dots \otimes |m_k\rangle_{A_{i,k}} = |m_{\pi^{-1}(1)}\rangle_{A_{i,1}} \otimes \dots \otimes |m_{\pi^{-1}(k)}\rangle_{A_{i,k}}. \quad (5.15)$$

This leads to the multipartite extension state

$$\bar{\Omega}_{A_1^k \dots A_l^k} \equiv (\bar{\Pi}_{A_1^k} \otimes \dots \otimes \bar{\Pi}_{A_l^k} \otimes \text{Tr}_{A'_1 \dots A'_l})(\Omega_{A_1^k \dots A_l^k}). \quad (5.16)$$

Combining convexity of the trace norm with the inequalities in (5.13) gives the following inequality:

$$\|\rho_{A_1 \dots A_l} - \bar{\Omega}_{A_{1,1} A_{2,1} \dots A_{l,1}}\|_1 \leq lk \cdot \delta_\rho, \quad (5.17)$$

quantifying the distance between $\rho_{A_1 \dots A_l}$ and the set of multipartite k -extendible states [26,27]. By applying proposition A.2 in the appendix, we know that

$$\|\bar{\Omega}_{A_{1,1} A_{2,1} \dots A_{l,1}} - \text{SEP}(A_1 : \dots : A_l)\|_1 \leq \frac{2}{k} \left(\sum_{i=2}^l |A_i|^2 \right). \quad (5.18)$$

By choosing

$$k = \left\lceil \sqrt{\frac{2}{\delta_\rho}} \left(\sum_{i=2}^l |A_i|^2 \right)^{1/2} \right\rceil \quad (5.19)$$

and combining (5.17) and (5.18) with the triangle inequality, we find that

$$\|\rho_{A_1 \dots A_l} - \text{SEP}(A_1 : \dots : A_l)\|_1 \leq (l+1) \left(\sum_{i=2}^l |A_i|^2 \right)^{1/2} \sqrt{2\delta_\rho} \quad (5.20)$$

$$= 2(l+1) \left(\sum_{i=2}^l |A_i|^2 \right)^{1/2} \sqrt[4]{\varepsilon_\rho}. \quad (5.21)$$

As the inequality holds independently of the particular extension $\rho_{A_1 A'_1 \dots A_l A'_l}$, we can rearrange it and take an infimum over all such extensions to find that

$$E_l(A_1 : \dots : A_l)_\rho \geq \frac{1}{16 \cdot (l+1)^4} \left(\sum_{i=2}^l |A_i|^2 \right)^{-2} \|\rho_{A_1 \dots A_l} - \text{SEP}(A_1 : \dots : A_l)\|_1^4. \quad (5.22)$$

■

Remark 5.2. The above approach follows that given by Li & Winter in [6,29]. The appendix of Li & Winter [29] sketches an approach for the multipartite squashed entanglement (the definition of which is recalled in the next section) but remarked that there were difficulties in completing the proof because in this case the local recovery map acts on the same extension system and it is not clear whether inequalities like those in (5.13) would hold. This difficulty is removed in our setting here (for the CEMI) because the local recovery maps act on different subsystems of the extension system. It still remains an open question to establish faithfulness of the multipartite squashed entanglement.

(a) Conditional entanglement of multipartite information is an upper bound on multipartite squashed entanglement

The conditional multipartite information of $\sigma_{A_1 \dots A_l E}$ is defined as

$$I(A_1 : \dots : A_l | E)_\sigma \equiv H(A_1 | E)_\sigma + \dots + H(A_l | E)_\sigma - H(A_1 \dots A_l | E)_\sigma. \quad (5.23)$$

From this, one can define the multipartite squashed entanglement of a state $\rho_{A_1 \dots A_l}$ as [33,34]

$$E_{\text{sq}}(A_1 : \dots : A_l)_\rho \equiv \frac{1}{2} \inf_{\rho_{A_1 \dots A_l E}} I(A_1 : \dots : A_l | E)_\rho, \quad (5.24)$$

where the infimum is over all extensions $\rho_{A_1 \dots A_l E}$ of $\rho_{A_1 \dots A_l}$. The following proposition generalizes Proposition 3 of [37, p. 4] to the multipartite setting but is however implicit in their concluding statement ‘All conclusions for the bipartite case can be similarly deduced’. (Nevertheless, it seems worthwhile to produce a short explicit proof.)

Proposition 5.3. *The multipartite squashed entanglement $E_{\text{sq}}(A_1 : \dots : A_l)_\rho$ is never larger than the CEMI $E_l(A_1 : \dots : A_l)_\rho$:*

$$E_{\text{sq}}(A_1 : \dots : A_l)_\rho \leq E_l(A_1 : \dots : A_l)_\rho. \quad (5.25)$$

Proof. Consider that [33]

$$I(A_1 : \dots : A_l | E)_\rho = \sum_{i=1}^l I(A_i; A_1^{i-1} | E). \quad (5.26)$$

While, from lemma 3.1, an additional application of the chain rule and strong subadditivity, we have that

$$I(A_1 A'_1 : \dots : A_l A'_l)_\rho - I(A'_1 : \dots : A'_l)_\rho = \sum_{i=1}^l I(A_i; A_1^{i-1} A'_{[l] \setminus \{i\}} | A'_i) \quad (5.27)$$

$$= \sum_{i=1}^l I(A_i; A_1^{i-1} | A'_{[l] \setminus \{i\}} A'_i) + I(A_i; A'_{[l] \setminus \{i\}} | A'_i) \quad (5.28)$$

$$= \sum_{i=1}^l I(A_i; A_1^{i-1} | A'_1 \dots A'_i) + I(A_i; A'_{[l] \setminus \{i\}} | A'_i) \quad (5.29)$$

$$\geq \sum_{i=1}^l I(A_i; A_1^{i-1} | A'_1 \dots A'_i) \quad (5.30)$$

$$= I(A_1 : \dots : A_l | A'_1 \dots A'_l)_\rho \quad (5.31)$$

$$\geq E_{\text{sq}}(A_1 : \dots : A_l)_\rho. \quad (5.32)$$

As the above chain holds independently of the particular extension, this establishes (5.25). ■

6. Partial state distribution and operational interpretations

In this section, we review the partial state distribution protocol from [21] and discuss how it gives an operational interpretation for the MSQ discord (Yang *et al.* [21] already observed that the protocol gives an operational interpretation of the CEMI). The review in this section also serves to prepare for the result and discussion given in §7. Along the way, we also establish optimality for the total quantum communication rate of the partial state distribution protocol.

The core protocol underlying partial state distribution is point-to-point quantum state redistribution (QSR) [38,39], so we begin by briefly reviewing that. Recall that the QSR protocol applies to many copies of a four-party pure state ψ_{JKLM} . That is, suppose that a reference possesses the J systems, a sender systems KL and a receiver systems M . The goal is to transfer the K systems to the receiver using as few noiseless qubit channels and as little entanglement as possible while maximizing the fidelity of the reproduced state. This transfer can happen perfectly in the asymptotic limit of many copies as long as the rate of quantum communication is at least $\frac{1}{2}I(K; J|M)_\psi$, which is half the conditional mutual information evaluated with respect to a single copy of ψ . That is, the main result of [38,39] is that there exists a sequence of encodings $\mathcal{E}_{K^n L^n X_n \rightarrow L^n G_n}$ and decodings $\mathcal{D}_{G_n Y_n M^n \rightarrow K^n M^n}$ such that

$$\lim_{n \rightarrow \infty} \|(\mathcal{D}_{G_n Y_n M^n \rightarrow K^n M^n} \circ \mathcal{E}_{K^n L^n X_n \rightarrow L^n G_n})(\psi_{JKLM}^{\otimes n} \otimes \Phi_{X_n Y_n}) - \psi_{JKLM}^{\otimes n}\|_1 = 0, \quad (6.1)$$

where $\Phi_{X_n Y_n}$ is a maximally entangled state and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \dim(G_n) = \frac{1}{2}I(K; J|M)_\psi. \quad (6.2)$$

The QSR protocol can also generate entanglement, but we are not concerned with this aspect in what follows.

For convenience of presentation, we will review the partial state distribution protocol for the case of four parties (one central sender and three receivers), with it being clear how to extend the idea to more parties. Let the state of interest be $\rho_{A_1 A'_1 A_2 A'_2 A_3 A'_3}$ and let $\phi_{A_1 A'_1 A_2 A'_2 A_3 A'_3 R}$ be a purification of it. Partial state distribution begins with the central sender possessing the systems $RA_1 A_2 A_3$, Receiver 1 system A'_1 , Receiver 2 system A'_2 and Receiver 3 system A'_3 . We assume that the central sender shares unlimited entanglement with each of the receivers

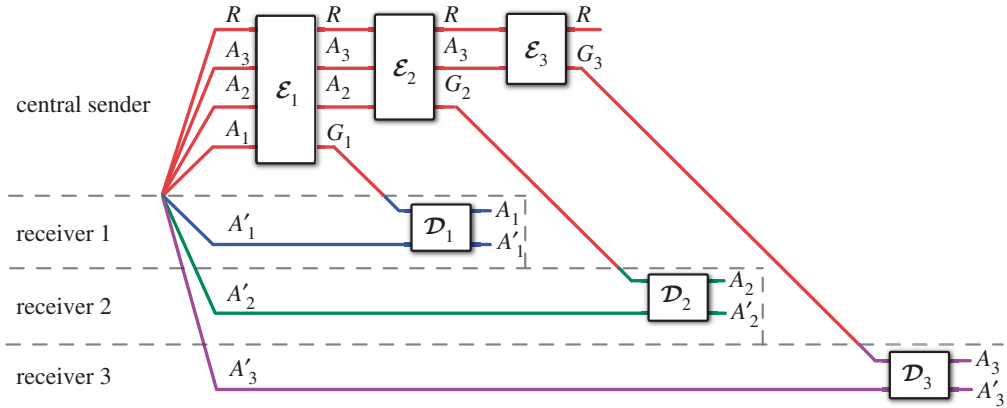


Figure 1. Partial state distribution [21]. The partial state distribution protocol is a way for a central sender to communicate various systems to receivers possessing quantum side information. The protocol makes use of QSR for each round (in the figure, for simplicity, we do not depict the entanglement consumed or generated by the protocol and it is also implicit that the protocol acts on many copies of the initial state). The total quantum communication cost is given by $I(A_1A'_1 : A_2A'_2 : A_3A'_3) - I(A'_1 : A'_2 : A'_3)$. Observe that the decodings can proceed in any order. The generalization of this protocol to more parties is straightforward. (Online version in colour.)

before communication begins. The partial state distribution protocol gives an operational interpretation of the information quantity

$$I(A_1A'_1 : A_2A'_2 : A_3A'_3)_\rho - I(A'_1 : A'_2 : A'_3)_\rho \quad (6.3)$$

as twice the total rate of quantum communication needed by the central sender in order to transfer the system A_1 to Receiver 1, A_2 to Receiver 2 and A_3 to Receiver 3. In order to see this, consider that lemma 3.1 gives the following expansion:

$$\begin{aligned} & I(A_1A'_1 : A_2A'_2 : A_3A'_3)_\rho - I(A'_1 : A'_2 : A'_3)_\rho \\ &= I(A_3 : A_1A_2A'_1A'_2|A'_3)_\rho + I(A_2 : A_1A'_1A'_3|A'_3)_\rho + I(A_1 : A'_2A'_3|A'_1)_\rho. \end{aligned} \quad (6.4)$$

This suggests that we can perform the QSR protocol three times. Indeed, the partial state distribution protocol proceeds as follows and as depicted in figure 1:

- (1) The first round corresponds to the term $I(A_1 : A'_2A'_3|A'_1)_\rho$. The central sender begins with systems $RA_1A_2A_3$, Receiver 1 has system A'_1 and the other receivers have systems $A'_2A'_3$ (which play the role of reference systems in the point-to-point QSR protocol). The sender acts with an encoding $\mathcal{E}_{R^n A_1^n A_2^n A_3^n X_n^1 \rightarrow R^n A_2^n A_3^n G_n^1}$ and transmits system G_n^1 to the receiver. Receiver 1 then performs a decoding $\mathcal{D}_{G_n^1 Y_n^1 A_1^n \rightarrow A_1^n A_1^n}$ to recover the A'_1 systems.
- (2) The second round corresponds to $I(A_2 : A_1A'_1A'_3|A'_3)_\rho$. The central sender begins with systems RA_2A_3 , Receiver 2 has system A'_2 and the other receivers have systems $A_1A'_1A'_3$ (these systems now play the role of reference systems in the point-to-point QSR protocol). The central sender acts with an encoding $\mathcal{E}_{R^n A_2^n A_3^n X_n^2 \rightarrow R^n A_3^n G_n^2}$ and transmits system G_n^2 to the receiver. Receiver 2 then performs a decoding $\mathcal{D}_{G_n^2 Y_n^2 A_2^n \rightarrow A_2^n A_2^n}$ to recover the A'_2 systems.
- (3) The third round corresponds to $I(A_3 : A_1A_2A'_1A'_2|A'_3)_\rho$. The central sender begins with systems RA_3 , Receiver 3 has system A'_3 and the other receivers have systems $A_1A'_1A_2A'_2$ (these now playing the role of reference systems in the point-to-point QSR protocol). The central sender acts with an encoding $\mathcal{E}_{R^n A_3^n X_n^3 \rightarrow R^n G_n^3}$ and transmits system G_n^3 to Receiver 3. Receiver 3 then performs a decoding $\mathcal{D}_{G_n^3 Y_n^3 A_3^n \rightarrow A_3^n A_3^n}$ to recover the A'_3 systems.

As all three protocols perform perfectly in the asymptotic limit, by exploiting the triangle inequality with (6.1) three times, we find that

$$\lim_{n \rightarrow \infty} \left\| (\mathcal{D}_n^3 \circ \mathcal{E}_n^3 \circ \mathcal{D}_n^2 \circ \mathcal{E}_n^2 \circ \mathcal{D}_n^1 \circ \mathcal{E}_n^1) \left(\phi_{A_1 A_1' A_2 A_2' A_3 A_3' R}^{\otimes n} \otimes \bigotimes_{i=1}^3 \Phi_{X_n^i Y_n^i} \right) - \phi_{A_1 A_1' A_2 A_2' A_3 A_3' R}^{\otimes n} \right\|_1 = 0, \quad (6.5)$$

with

$$2 \lim_{n \rightarrow \infty} \frac{1}{n} \log(\dim(G_n^1) \dim(G_n^2) \dim(G_n^3)) \\ = I(A_3 : A_1 A_2 A_1' A_2' | A_3')_\rho + I(A_2 : A_1 A_1' A_3' | A_2')_\rho + I(A_1 : A_2' A_3' | A_1')_\rho. \quad (6.6)$$

Due to the nature of this protocol, observe that we can commute all of the decoding maps to the end and each of these decodings commute with each other as they act on different spaces. That is, we have that

$$\lim_{n \rightarrow \infty} \left\| ((\mathcal{D}_n^3 \otimes \mathcal{D}_n^2 \otimes \mathcal{D}_n^1) \circ \mathcal{E}_n^3 \circ \mathcal{E}_n^2 \circ \mathcal{E}_n^1) \right. \\ \left. \times \left(\phi_{A_1 A_1' A_2 A_2' A_3 A_3' R}^{\otimes n} \otimes \bigotimes_{i=1}^3 \Phi_{X_n^i Y_n^i} \right) - \phi_{A_1 A_1' A_2 A_2' A_3 A_3' R}^{\otimes n} \right\|_1 = 0. \quad (6.7)$$

(We cannot however commute the encodings with each other.)

An interesting observation from [21] is that the information quantity in (6.4) is conservative, corresponding to the different expansions in lemma 3.1 and, operationally, to the fact that we can perform the partial state distribution protocol in any order (we would however require different encodings and decodings in order to do so). Also, Yang *et al.* [21] interpreted the CEMI in terms of the partial state distribution protocol as the total rate of quantum communication needed to transfer the systems A_1 through A_3 to independent receivers who possess the best possible quantum side information in the form of extension systems A_1' , A_2' and A_3' , generalizing the squashed entanglement interpretation from [40] to the multipartite setting.

(a) Optimality

The optimality of the total quantum communication rate in partial state distribution was not discussed in [21], but it follows from a simple argument that exploits the structure of any protocol for partial state distribution and a few salient properties of the multipartite information. A proof proceeds similarly to [41, theorem 13]. Indeed, any general protocol for partial state distribution has the form given in figure 1, with the exception that the encoder can be taken as just one CPTP linear map from the input systems $R^n A_1^n A_2^n A_3^n X_1^n X_2^n X_3^n$ to the systems $R^n G_1^n G_2^n G_3^n$. Let σ denote the global state after the encoder acts. A protocol for partial state distribution has a final state ω after the local decodings which is ε -close in trace distance to the ideal i.i.d. state $\phi_{A_1 A_1' A_2 A_2' A_3 A_3' R}^{\otimes n}$. So we proceed with the following chain of inequalities

$$nI(A_1 A_1' : A_2 A_2' : A_3 A_3')_\phi = I(A_1^n A_1'^n : A_2^n A_2'^n : A_3^n A_3'^n)_{\phi^{\otimes n}} \quad (6.8)$$

$$\leq I(A_1^n A_1'^n : A_2^n A_2'^n : A_3^n A_3'^n)_\omega + f(\varepsilon). \quad (6.9)$$

The first equality is from the additivity of the multipartite information on tensor-power states and the inequality follows from the assumption that ω is ε -close to the ideal state and by applying the Fannes–Audenaert inequality [35] with $f(\varepsilon)$ a function with the property that

$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} (1/n)f(\varepsilon) = 0$. Continuing, we have that

$$I(A_1^n A_1^m : A_2^n A_2^m : A_3^n A_3^m)_\omega \leq I(A_1^m G_1^n Y_1^n : A_2^m G_2^n Y_2^n : A_3^m G_3^n Y_3^n)_\sigma \quad (6.10)$$

$$\leq I(A_1^m Y_1^n : A_2^m Y_2^n : A_3^m Y_3^n)_\sigma + 2 \log(|G_1^n| |G_2^n| |G_3^n|) \quad (6.11)$$

$$= I(A_1^m : A_2^m : A_3^m)_\sigma + 2 \log(|G_1^n| |G_2^n| |G_3^n|) \quad (6.12)$$

$$= nI(A'_1 : A'_2 : A'_3)_\phi + 2 \log(|G_1^n| |G_2^n| |G_3^n|), \quad (6.13)$$

where the first inequality follows from quantum data processing (the local decoders can only decrease the multipartite information). The second inequality follows from

$$I(C_1 D_1 : \dots : C_l D_l) = \sum_{i=1}^l H(C_i D_i) - H(C_1 D_1 \dots C_l D_l) \quad (6.14)$$

$$= \sum_{i=1}^l H(C_i) + H(D_i | C_i) - H(C_1 \dots C_l) - H(D_1 \dots D_l | C_1 \dots C_l) \quad (6.15)$$

$$\leq \sum_{i=1}^l H(C_i) - H(C_1 \dots C_l) + 2 \log(|D_1| \times \dots \times |D_l|) \quad (6.16)$$

$$= I(C_1 : \dots : C_l) + 2 \log(|D_1| \times \dots \times |D_l|). \quad (6.17)$$

The equality in (6.12) follows because the systems are product with respect to the cut $A_1^m A_2^m A_3^m | Y_1^n | Y_2^n | Y_3^n$. The final equality is again additivity. Putting everything together we find that

$$\frac{1}{2} [I(A_1 A'_1 : A_2 A'_2 : A_3 A'_3)_\phi - I(A'_1 : A'_2 : A'_3)_\phi] \leq \frac{1}{n} \log(|G_1^n| |G_2^n| |G_3^n|) + \frac{1}{n} f(\varepsilon). \quad (6.18)$$

Taking the limit as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$ then establishes the information gap in (6.3) as twice the minimum total rate of quantum communication needed in any partial state distribution protocol. This analysis clearly extends to any finite number of parties.

(b) Operational interpretation of the multipartite symmetric quantum discord

The partial state distribution protocol gives a compelling operational interpretation of the MSQ discord, different from and arguably simpler than those considered in previous contexts [42,43]. Suppose that we have a multipartite state $\rho_{A_1 \dots A_l}$ shared by l local parties, each of whom possesses system A_i , where $i \in \{1, \dots, l\}$. Let $\phi_{R A_1 \dots A_l}$ be a state which purifies $\rho_{A_1 \dots A_l}$, where R is an environment system inaccessible to the local parties. Suppose now that a measurement occurs on each of the systems, according to the measurement maps $\mathcal{M}_{A_1 \rightarrow X_1}^1, \dots, \mathcal{M}_{A_l \rightarrow X_l}^l$, producing the state $\omega_{R X_1 \dots X_l}$:

$$\omega_{R X_1 \dots X_l} \equiv (\mathcal{M}_{A_1 \rightarrow X_1}^1 \otimes \dots \otimes \mathcal{M}_{A_l \rightarrow X_l}^l)(\phi_{R A_1 \dots A_l}). \quad (6.19)$$

A measurement corresponds to a loss of information, and one way to represent this is with isometric extensions of the measurement process, so that the full state is

$$\omega_{R X_1 E_1 \dots X_l E_l} \equiv (\mathcal{U}_{A_1 \rightarrow X_1 E_1}^{\mathcal{M}^1} \otimes \dots \otimes \mathcal{U}_{A_l \rightarrow X_l E_l}^{\mathcal{M}^l})(\phi_{R A_1 \dots A_l}), \quad (6.20)$$

and $\mathcal{U}_{A_i \rightarrow X_i E_i}^{\mathcal{M}^i}$ is an isometric extension of the measurement map $\mathcal{M}_{A_i \rightarrow X_i}^i$. As the systems E_1, \dots, E_l are lost to the environment after the measurement process, it becomes the case that the environment possesses the systems R, E_1, \dots, E_l , and each of the local parties possesses one of the measurement outcomes.

With this set-up, we can now see that the (unoptimized) MSQ discord

$$I(A_1 : \dots : A_l)_\rho - I(X_1 : \dots : X_l)_\omega = I(X_1 E_1 : \dots : X_l E_l)_\rho - I(X_1 : \dots : X_l)_\omega \quad (6.21)$$

is equal to the twice the total rate of quantum communication needed for the environment to send the systems E_1, \dots, E_l back to each of the local parties in order to restore the coherence lost in the

measurement processes. Due to the fact that the QSR protocol is dual under time reversal [38,39], the unoptimized MSQ discord is also equal to the twice the total rate of quantum communication needed by the local parties to transmit the systems E_1, \dots, E_l back to the environment, thus additionally characterizing the rate at which coherence is lost in the measurement process. The (optimized) MSQ discord simply includes a further optimization over the measurements themselves in order to minimize the total quantum communication cost.¹

(c) Operational interpretation of the quantum discord

We remark that this approach in terms of partial state distribution gives as a special case a compelling operational interpretation of the original quantum discord, again different from and arguably simpler than those considered previously [42,43]. Indeed, consider a bipartite state ρ_{AB} and a measurement map $\mathcal{M}_{A \rightarrow X}$. The unoptimized quantum discord is defined as

$$I(A; B)_\rho - I(X; B)_{\mathcal{M}(\rho)} = I(XE; B)_{\mathcal{U}(\rho)} - I(X; B)_{\mathcal{U}(\rho)} \quad (6.22)$$

$$= I(E; B|X)_{\mathcal{U}(\rho)}, \quad (6.23)$$

where the first equality follows because every measurement map has an isometric extension $\mathcal{U}_{A \rightarrow XE}^{\mathcal{M}}$ and the mutual information is invariant under local isometries. The second equality is a consequence of the chain rule (this rewriting of discord in terms of conditional mutual information was first explicitly given in [44]). Purifying the original state with a reference system R , we have a pure state on systems $REXB$. After the measurement occurs, it is natural to associate the system E as being 'lost' and thus given to the other environment system R . That is, after the measurement occurs, the systems R and E are with the environment, the system X is with a party who has the measurement outcome, and the system B is with another party who plays no role in the protocol. We can then readily see from the QSR protocol that $I(E; B|X)_{\mathcal{U}(\rho)}$ is twice the rate of quantum communication needed in order to transmit the system E to the party possessing X (assuming that the B system is with a different party who does not play a role in this transfer). We can thus interpret $I(E; B|X)_{\mathcal{U}(\rho)}$ as twice the quantum communication cost needed to restore the coherence that was lost in the measurement process. After this transfer occurs, $I(E; B|X)_{\mathcal{U}(\rho)}$ is also equal to twice the quantum communication rate needed to send the system E back to the environment (this is because state redistribution is dual under time reversal [38,39]). The quantity thus also characterizes the amount of quantum information lost in the measurement process. Optimizing over all measurements gives the optimized discord (keeping in mind that one could potentially optimize over collective measurements and get a *regularized* discord).

7. Potential improvement of the local recoverability inequality

It might be possible to improve upon the local recoverability inequality given in (2.1). Here, we provide what might be a first step, which follows an approach recently given in [18].

Proposition 7.1. *Let $\rho_{A_1 A'_1 \dots A_l A'_l}$ be a multipartite quantum state. Then the following inequality holds:*

$$\begin{aligned} & I(A_1 A'_1 : \dots : A_l A'_l)_\rho - I(A'_1 : \dots : A'_l)_\rho \\ & \geq \lim_{n \rightarrow \infty} \min_{\mathcal{R}^1, \dots, \mathcal{R}^l} \frac{1}{n} D(\rho_{A_1 A'_1 \dots A_l A'_l}^{\otimes n} \| \mathcal{R}_{A_1^n \rightarrow A'_1^n}^1 \otimes \dots \otimes \mathcal{R}_{A_l^n \rightarrow A'_l^n}^l (\rho_{A_1 \dots A'_l}^{\otimes n})), \end{aligned} \quad (7.1)$$

where $\mathcal{R}_{A_1^n \rightarrow A'_1^n}^1, \dots, \mathcal{R}_{A_l^n \rightarrow A'_l^n}^l$ are a sequence of local recovery maps.

Proof. The proof of this lemma is very similar to the proof of Proposition 3 of [18], except that we invoke the partial state redistribution protocol reviewed in §6. Picking up from the notation

¹A subtle point here is that one could more generally include an optimization over collective quantum measurements acting on many copies of the state, which would result in a *regularized* MSQ discord being equal to the total quantum communication cost.

there, and specializing to a state on systems $A_1A'_1A_2A'_2A_3A'_3$, let

$$\varphi_{R^n G_n^1 G_n^2 G_n^3 Y_n^1 Y_n^2 Y_n^3 A_1^m A_2^m A_3^m} \equiv \mathcal{E}_{R^n A_3^m X_n^3 \rightarrow R^n G_n^3}^3 \left(\mathcal{E}_{R^n A_2^m A_3^m X_n^2 \rightarrow R^n A_3^m G_n^2}^2 \left(\mathcal{E}_{R^n A_1^m A_2^m A_3^m X_n^1 \rightarrow R^n A_2^m A_3^m G_n^1}^1 \left(\phi_{A_1 A'_1 A_2 A'_2 A_3 A'_3 R}^{\otimes n} \otimes \bigotimes_{i=1}^3 \phi_{X_n^i Y_n^i} \right) \right) \right) \quad (7.2)$$

denote the state after the encodings. Tracing over R^n and applying the operator inequality $\sigma_{CD} \leq [\dim(D)]^2 \tau_C \otimes \rho_D$ three times, we find that

$$\varphi_{G_n^1 G_n^2 G_n^3 Y_n^1 Y_n^2 Y_n^3 A_1^m A_2^m A_3^m} \leq [\dim(G_n^1)]^2 [\dim(G_n^2)]^2 [\dim(G_n^3)]^2 \tau_{G_n^1} \otimes \tau_{G_n^2} \otimes \tau_{G_n^3} \otimes \tau_{Y_n^1} \otimes \tau_{Y_n^2} \otimes \tau_{Y_n^3} \otimes \rho_{A_1^m A_2^m A_3^m}. \quad (7.3)$$

Now for $i \in \{1, 2, 3\}$, define the perturbed decoding operations

$$\tilde{\mathcal{D}}_n^i \equiv (1 - 2^{-n}) \mathcal{D}_n^i + 2^{-n} \Lambda_{\text{dep}}, \quad (7.4)$$

where Λ_{dep} is the completely depolarizing channel. As these are completely positive and acting on different spaces, we find that

$$\begin{aligned} & (\tilde{\mathcal{D}}_n^3 \otimes \tilde{\mathcal{D}}_n^2 \otimes \tilde{\mathcal{D}}_n^1) (\varphi_{G_n^1 G_n^2 G_n^3 Y_n^1 Y_n^2 Y_n^3 A_1^m A_2^m A_3^m}) \\ & \leq [\dim(G_n^1)]^2 [\dim(G_n^2)]^2 [\dim(G_n^3)]^2 \\ & \quad \times (\mathcal{R}_{A_1^m \rightarrow A_1^m A_1^m}^1 \otimes \mathcal{R}_{A_2^m \rightarrow A_2^m A_2^m}^2 \otimes \mathcal{R}_{A_3^m \rightarrow A_3^m A_3^m}^3) (\rho_{A_1^m A_2^m A_3^m}), \end{aligned} \quad (7.5)$$

where the recovery map $\mathcal{R}_{A_i^m \rightarrow A_i^m A_i^m}^i$ for $i \in \{1, 2, 3\}$ is defined to be the map that first tensors in maximally mixed states on systems G_n^i and Y_n^i and then performs $\tilde{\mathcal{D}}_n^i$. Using operator monotonicity of the logarithm, we find that

$$\begin{aligned} & D(\rho_{A_1 A'_1 A_2 A'_2 A_3 A'_3}^{\otimes n} \| (\mathcal{R}_{A_1^m \rightarrow A_1^m A_1^m}^1 \otimes \mathcal{R}_{A_2^m \rightarrow A_2^m A_2^m}^2 \otimes \mathcal{R}_{A_3^m \rightarrow A_3^m A_3^m}^3) (\rho_{A_1^m A_2^m A_3^m})) \\ & \leq D(\rho_{A_1 A'_1 A_2 A'_2 A_3 A'_3}^{\otimes n} \| (\tilde{\mathcal{D}}_n^3 \otimes \tilde{\mathcal{D}}_n^2 \otimes \tilde{\mathcal{D}}_n^1) (\varphi_{G_n^1 G_n^2 G_n^3 Y_n^1 Y_n^2 Y_n^3 A_1^m A_2^m A_3^m})) \\ & \quad + 2 \log(\dim(G_n^1) \dim(G_n^2) \dim(G_n^3)). \end{aligned} \quad (7.6)$$

Theorem 3 of [45] gives that

$$\lim_{n \rightarrow \infty} \frac{1}{n} D(\rho_{A_1 A'_1 A_2 A'_2 A_3 A'_3}^{\otimes n} \| (\tilde{\mathcal{D}}_n^3 \otimes \tilde{\mathcal{D}}_n^2 \otimes \tilde{\mathcal{D}}_n^1) (\varphi_{G_n^1 G_n^2 G_n^3 Y_n^1 Y_n^2 Y_n^3 A_1^m A_2^m A_3^m})) = 0 \quad (7.7)$$

as a consequence of (6.7). With this, we can conclude the statement in (7.1) by combining the above with (6.6) and (6.4).

It should be clear from here how the general multipartite case proceeds. Letting the number of parties be some positive integer l , we first apply lemma 3.1. Next, we perform the partial state distribution protocol in the same fashion as above. Importantly, all of the encodings take place in a particular order, but the decodings all act on different spaces and thus commute. Finally, we apply the same reasoning at the end to conclude the general statement of the lemma. ■

We leave as an open question whether the following inequality holds:

$$\begin{aligned} & I(A_1 A'_1 : \dots : A_l A'_l)_\rho - I(A'_1 : \dots : A'_l)_\rho \\ & \geq -\log F(\rho_{A_1 A'_1 \dots A_l A'_l} (\mathcal{R}_{A'_1 \rightarrow A_1 A'_1}^1 \otimes \dots \otimes \mathcal{R}_{A'_l \rightarrow A_l A'_l}^l) (\rho_{A'_1 \dots A'_l})), \end{aligned} \quad (7.8)$$

where $\rho_{A_1 A'_1 \dots A_l A'_l}$ is a multipartite quantum state and $\mathcal{R}_{A'_1 \rightarrow A_1 A'_1}^1, \dots, \mathcal{R}_{A'_l \rightarrow A_l A'_l}^l$ are some local recovery maps. At the very least, the inequality holds for classical systems as a consequence of theorem 5 of [29]. By extending the methods of Fawzi & Renner [14] and Brandao *et al.* [18], it might be possible to establish the above inequality.

8. Discussion

We have demonstrated how the inequality in (1.7) implies a relation between the multipartite information gap $I(A_1 A'_1 : \dots : A_l A'_l) - I(A'_1 : \dots : A'_l)$ and local recoverability. Namely, a multipartite state has a multipartite information gap nearly equal to zero if and only if the systems A_1, \dots, A_l are locally recoverable from the respective systems A'_1, \dots, A'_l . This result in turn implies that (i) the MSQ discord of a state $\rho_{A_1 \dots A_l}$ is nearly equal to zero if and only if the state is locally recoverable after measurements occur on each of the systems and (ii) the CEMI is faithful. We have also given a compelling operational interpretation of the MSQ discord as the twice the total quantum communication cost needed to restore the coherence lost from a sequence of local measurements. A similar operational interpretation applies to the original quantum discord quantity as well. Finally, proposition 7.1 gives another lower bound on the multipartite information gap by generalizing an approach recently outlined in [18].

There are several open questions to consider going forward from here. First, it would be interesting if the inequality in (7.8) were true. It is true for classical systems, and to show it for quantum systems, one could consider extending the methods given in [18, Proposition 4] to this multipartite setting. Next, in light of the recent developments in [11–13], one could define a geometric CEMI as follows:

$$E_I^F(A_1 : \dots : A_l)_\rho \equiv -\frac{1}{2} \log \sup_{\substack{\rho_{A_1 A'_1 \dots A_l A'_l} \\ \mathcal{R}^1, \dots, \mathcal{R}^l}} F(\rho_{A_1 A'_1 \dots A_l A'_l}, (\mathcal{R}_{A'_1 \rightarrow A_1 A'_1}^1 \otimes \dots \otimes \mathcal{R}_{A'_l \rightarrow A_l A'_l}^l)(\rho_{A'_1 \dots A'_l})), \quad (8.1)$$

where the optimization is over all extensions of $\rho_{A_1 \dots A_l}$ and all recovery maps $\mathcal{R}^1, \dots, \mathcal{R}^l$. One could also define a multipartite surprisal of measurement recoverability as

$$D^F(\bar{A}_1 : \dots : \bar{A}_l)_\rho \equiv -\log \sup_{\mathcal{E}_{A_1}^1, \dots, \mathcal{E}_{A_l}^l} F(\rho_{A_1 \dots A_l}, (\mathcal{E}_{A_1}^1 \otimes \dots \otimes \mathcal{E}_{A_l}^l)(\rho_{A_1 \dots A_l})), \quad (8.2)$$

where the optimization is over all local EB channels. One could even consider other discord-like quantities of the above form, but involving alternate (pseudo-)distance measures such as the trace distance and relative entropy. We can already conclude that the geometric CEMI is faithful by the results given in this paper, and one could pursue further properties of these quantities in future work.

Data accessibility. This work does not have any experimental data.

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Appendix A. Multipartite de Finetti theorem

We begin by recalling Theorem II.7' of [46]:

Theorem A.1. *Let ζ_{EF} be a k -extendible state, in the sense that there is a state $\theta_{EF_1 \dots F_k}$ that is invariant with respect to permutations of the F systems and such that $\text{Tr}_{F_2 \dots F_k} \{\theta_{EF_1 \dots F_k}\} = \zeta_{EF}$. Then there exists a measure $d\mu(\sigma_F)$ on states σ_F on the F system and a family of states $\{\xi_E^\sigma\}$ parametrized by σ_F such that*

$$\left\| \zeta_{EF} - \int d\mu(\sigma_F) \xi_E^\sigma \otimes \sigma_F \right\|_1 \leq \frac{2|F|^2}{k}. \quad (\text{A } 1)$$

The following proposition follows directly from prior results in the literature, but we state it here and give a brief proof for readers' convenience:

Proposition A.2. Let $\rho_{A_1 A_2 \dots A_l}$ be a multipartite k -extendible state, i.e. there exists a state

$$\omega_{A_1 A_2, 1 \dots A_2, k \dots A_{l, 1} \dots A_{l, k}} \quad (\text{A } 2)$$

that is permutation invariant with respect to the systems $A_{j, 1} \dots A_{j, k}$, for each $j \in \{2, \dots, l\}$, and such that $\rho_{A_1 A_2 \dots A_l} = \text{Tr}_{A_{2, 2} \dots A_{2, k} \dots A_{l, 2} \dots A_{l, k}} \{\omega\}$. Then

$$\|\rho_{A_1 A_2 \dots A_l} - \text{SEP}(A_1 : A_2 : \dots : A_l)\|_1 \leq \frac{2}{k} (|A_2|^2 + \dots + |A_l|^2). \quad (\text{A } 3)$$

Proof. The idea is to proceed similar to the proof of Doherty *et al.* [27, Theorem 1], but here invoking theorem A.1 several times. We consider a particular example with only three parties for simplicity, and it will then be clear how the approach extends to states with more parties. So we begin with a multipartite k -extendible state ρ_{ABC} and its multipartite k -extension $\omega_{AB_1 \dots B_k C_1 \dots C_k}$. We first apply theorem A.1 to $\omega_{AB_1 \dots B_k C}$ (where $C = C_1$), setting $E = AB_1 \dots B_k$ and $F = C$. We can conclude that there exists a measure $d\mu(\sigma_C)$ and a family of states $\{\xi_{AB_1 \dots B_k}^\sigma\}$ such that

$$\left\| \omega_{AB_1 \dots B_k C} - \int d\mu(\sigma_C) \xi_{AB_1 \dots B_k}^\sigma \otimes \sigma_C \right\|_1 \leq \frac{2|C|^2}{k}. \quad (\text{A } 4)$$

Due to the invariance of the state $\omega_{AB_1 \dots B_k C}$ under permutations of the B systems and monotonicity of the trace norm under quantum operations, we can conclude the following inequality:

$$\left\| \omega_{AB_1 \dots B_k C} - \int d\mu(\sigma_C) \bar{\xi}_{AB_1 \dots B_k}^\sigma \otimes \sigma_C \right\|_1 \leq \frac{2|C|^2}{k}, \quad (\text{A } 5)$$

where $\bar{\xi}_{AB_1 \dots B_k}^\sigma \equiv \bar{\Pi}_{B^k}(\xi_{AB_1 \dots B_k}^\sigma)$, with $\bar{\Pi}_{B^k}$ a channel that randomly permutes the B systems (defined in (5.14)). Given that each state $\bar{\xi}_{AB_1 \dots B_k}^\sigma$ is permutation symmetric with respect to the B systems, we can again invoke theorem A.1 to conclude that there exists a measure $d\mu(\tau(\sigma))$ on states $\tau(\sigma)_B$ and a family of states $\{\chi_A^{\tau(\sigma)}\}$ such that

$$\left\| \bar{\xi}_{AB}^\sigma - \int d\mu(\tau(\sigma)) \chi_A^{\tau(\sigma)} \otimes \tau(\sigma)_B \right\|_1 \leq \frac{2|B|^2}{k}. \quad (\text{A } 6)$$

This implies that

$$\left\| \bar{\xi}_{AB}^\sigma \otimes \sigma_C - \int d\mu(\tau(\sigma)) \chi_A^{\tau(\sigma)} \otimes \tau(\sigma)_B \otimes \sigma_C \right\|_1 \leq \frac{2|B|^2}{k} \quad (\text{A } 7)$$

and applying convexity of the trace norm gives

$$\left\| \int d\mu(\sigma_C) \bar{\xi}_{AB}^\sigma \otimes \sigma_C - \iint d\mu(\sigma_C) d\mu(\tau(\sigma)) \chi_A^{\tau(\sigma)} \otimes \tau(\sigma)_B \otimes \sigma_C \right\|_1 \leq \frac{2|B|^2}{k}. \quad (\text{A } 8)$$

Applying monotonicity of the trace norm under partial trace to (A 5) gives

$$\left\| \rho_{ABC} - \int d\mu(\sigma_C) \bar{\xi}_{AB}^\sigma \otimes \sigma_C \right\|_1 \leq \frac{2|C|^2}{k}. \quad (\text{A } 9)$$

We finally combine (A 8) and (A 9) with the triangle inequality to get

$$\left\| \rho_{ABC} - \iint d\mu(\sigma_C) d\mu(\tau(\sigma)) \chi_A^{\tau(\sigma)} \otimes \tau(\sigma)_B \otimes \sigma_C \right\|_1 \leq \frac{2}{k} (|B|^2 + |C|^2). \quad (\text{A } 10)$$

As the state on the right is a convex combination of product states, it is fully separable, so that we can conclude

$$\|\rho_{ABC} - \text{SEP}(A : B : C)\|_1 \leq \frac{2}{k} (|B|^2 + |C|^2). \quad (\text{A } 11)$$

Extending this proof to more parties is done in the obvious way, so that we can conclude (A 3). ■

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