10-19-2015

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Phys. Rev. A **92**, 042321 — Published 19 October 2015
DOI: 10.1103/PhysRevA.92.042321

Fidelity of recovery, geometric squashed entanglement, and measurement recoverability

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This paper defines the fidelity of recovery of a tripartite quantum state on systems A, B, and C as a measure of how well one can recover the full state on all three systems if system A is lost and a recovery operation is performed on system C alone. The surprisal of the fidelity of recovery (its negative logarithm) is an information quantity which obeys nearly all of the properties of the conditional quantum mutual information $I(A;B|C)$, including non-negativity, monotonicity with respect to local operations, duality, invariance with respect to local isometries, a dimension bound, and continuity. We then define a (pseudo) entanglement measure based on this quantity, which we call the geometric squashed entanglement. We prove that the geometric squashed entanglement is a 1-LOCC monotone (i.e., monotone non-increasing with respect to local operations and classical communication from Bob to Alice), that it vanishes if and only if the state on which it is evaluated is unentangled, and that it reduces to the geometric measure of entanglement if the state is pure. We also show that it is invariant with respect to local isometries, subadditive, continuous, and normalized on maximally entangled states. We next define the surprisal of measurement recoverability, which is an information quantity in the spirit of quantum discord, characterizing how well one can recover a share of a bipartite state if it is measured. We prove that this discord-like quantity satisfies several properties, including non-negativity, faithfulness on classical-quantum states, invariance with respect to local isometries, a dimension bound, and normalization on maximally entangled states. This quantity combined with a recent breakthrough of Fawzi and Renner allows to characterize states with discord nearly equal to zero as being approximate fixed points of entanglement breaking channels (equivalently, they are recoverable from the state of a measuring apparatus). Finally, we discuss a multipartite fidelity of recovery and several of its properties.

I. INTRODUCTION

The conditional quantum mutual information (CQMI) is a central information quantity that finds numerous applications in quantum information theory [1, 2], the theory of quantum correlations [3, 4], and quantum many-body physics [5, 6]. For a quantum state ρ_{ABC} shared between three parties, say, Alice, Bob, and Charlie, the CQMI is defined as

$$I(A;B|C)_\rho \equiv H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho, \tag{1}$$

where $H(F)_\sigma \equiv -\mathrm{Tr}\{\sigma_F \log \sigma_F\}$ is the von Neumann entropy of a state σ_F on system F and we unambiguously let $\rho_C \equiv \mathrm{Tr}_{AB}\{\rho_{ABC}\}$ denote the reduced density operator on system C, for example. The CQMI captures the correlations present between Alice and Bob from the perspective of Charlie in the independent and identically distributed (i.i.d.) resource limit, where an asymptotically large number of copies of the state ρ_{ABC} are shared between the three parties. It is non-negative [7, 8], non-increasing with respect to the action of local quantum operations on systems A or B, and obeys a duality relation for a four-party pure state ψ_{ABCD}, given by $I(A;B|C)_\psi = I(B;A|D)_\psi$. It finds operational meaning as twice the optimal quantum communication cost in the state redistribution protocol [1, 2]. It underlies the squashed entanglement [4], which is a measure of entanglement that satisfies all of the axioms desired for such a measure [9–11], and furthermore underlies the quantum discord [3], which is a measure of quantum correlations different from those due to entanglement.

In an attempt to develop a version of the CQMI, which could potentially be relevant for the "one-shot" or finite resource regimes, we along with Berta [12] recently proposed Rényi generalizations of the CQMI. We proved that these Rényi generalizations of the CQMI retain many of the properties of the original CQMI in (1). While the application of these particular Rényi CQMIs in one-shot state redistribution remains to be studied, (however, see the recent progress on one-shot state redistribution in [13, 14]) we have used them to define a Rényi squashed entanglement and a Rényi quantum discord [15], which retain several properties of the respective, original, von Neumann entropy based quantities.

One contribution of [12] was the conjecture that the proposed Rényi CQMIs are monotone increasing in the Rényi parameter, as is known to be the case for other Rényi entropic quantities. That is, for a tripartite state ρ_{ABC}, and for a Rényi conditional mutual information

$\widetilde{I}_\alpha(A;B|C)_\rho$ defined as [12, Section 6]

$$\widetilde{I}_\alpha(A;B|C)_\rho \equiv \frac{1}{\alpha-1}\log\left\|\rho_{ABC}^{1/2}\rho_{AC}^{(1-\alpha)/2\alpha}\rho_{C}^{(\alpha-1)/2\alpha}\rho_{BC}^{(1-\alpha)/2\alpha}\right\|_{2\alpha}^{2\alpha}, \quad (2)$$

[12, Section 8] conjectured that the following inequality holds for $0 \leq \alpha \leq \beta$:

$$\widetilde{I}_\alpha(A;B|C)_\rho \leq \widetilde{I}_\beta(A;B|C)_\rho. \quad (3)$$

Proofs were given for this conjectured inequality when the Rényi parameter α is in a neighborhood of one and when $1/\alpha + 1/\beta = 2$ [12, Section 8].

We also pointed out implications of the conjectured inequality for understanding states with small conditional quantum mutual information [12, Section 8] (later stressed in [16]). In particular, we pointed out that the following lower bound on the conditional quantum mutual information holds as a consequence of the conjectured inequality in (3) by choosing $\alpha = 1/2$ and $\beta = 1$:

$$I(A;B|C)_\rho \geq -\log F\left(\rho_{ABC}, \mathcal{R}^P_{C\to AC}(\rho_{BC})\right) \quad (4)$$

$$\geq \frac{1}{4}\left\|\rho_{ABC} - \mathcal{R}^P_{C\to AC}(\rho_{BC})\right\|_1^2, \quad (5)$$

where $\mathcal{R}^P_{C\to AC}$ is a quantum channel known as the Petz recovery map [17–20], defined as

$$\mathcal{R}^P_{C\to AC}(\cdot) \equiv \rho_{AC}^{1/2}\rho_C^{-1/2}(\cdot)\rho_C^{-1/2}\rho_{AC}^{1/2}. \quad (6)$$

The fidelity is a measure of how close two quantum states are and is defined for positive semidefinite operators P and Q as

$$F(P,Q) \equiv \left\|\sqrt{P}\sqrt{Q}\right\|_1^2. \quad (7)$$

Throughout we denote the root fidelity by $\sqrt{F}(P,Q) \equiv \|\sqrt{P}\sqrt{Q}\|_1$. The trace distance bound in (4) was conjectured previously in [21] and a related conjecture (with a different lower bound) was considered in [22].

The conjectured inequality in (4) revealed that (if it is true) it would be possible to understand tripartite states with small conditional mutual information in the following sense: *If one loses system A of a tripartite state ρ_{ABC} and is allowed to perform the Petz recovery map on system C alone, then the fidelity of recovery in doing so will be high.* The converse statement was already established in [12, Proposition 35] and independently in [23, Eq. (8)]. Indeed, suppose now that a tripartite state ρ_{ABC} has large conditional mutual information. Then if one loses system A and attempts to recover it by acting on system C alone, then the fidelity of recovery will not be high no matter what scheme is employed (see [12, Proposition 35] for specific parameters). These statements are already known to be true for a classical system C, but the main question is whether the inequality in (4) holds for a quantum system C.

II. SUMMARY OF RESULTS

When studying the conjectured inequality in (4), we can observe that a simple lower bound on the RHS is in terms of a quantity that we call the *surprisal of the fidelity of recovery*:

$$-\log F\left(\rho_{ABC}, \mathcal{R}^P_{C\to AC}(\rho_{BC})\right) \geq I_F(A;B|C)_\rho \quad (8)$$

$$\equiv -\log F(A;B|C)_\rho, \quad (9)$$

where the *fidelity of recovery* is defined as

$$F(A;B|C)_\rho \equiv \sup_{\mathcal{R}} F\left(\rho_{ABC}, \mathcal{R}_{C\to AC}(\rho_{BC})\right). \quad (10)$$

That is, rather than considering the particular Petz recovery map, one could consider optimizing the fidelity with respect to all such recovery maps. One of the main objectives of the present paper is to study the fidelity of recovery in more detail.

Note: After the completion of this work, we learned of the recent breakthrough result of [23], in which the inequality $I(A;B|C)_\rho \geq -\log F(A;B|C)_\rho$ was established for any tripartite state $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$. Thus, for states with small conditional mutual information (near to zero), the fidelity of recovery is high (near to one). Note that our arXiv posting of the present work (arXiv:1410.1441) appeared one day after the arXiv posting of [23]. Furthermore note that the main result of [23] is now an easy corollary of the more general result in [24].

A. Properties of the surprisal of the fidelity of recovery

Our conclusions for $I_F(A;B|C)_\rho$ are that it obeys many of the same properties as the conditional mutual information $I(A;B|C)_\rho$:

1. **(Non-negativity)** $I_F(A;B|C)_\rho \geq 0$ for any tripartite quantum state, and for finite-dimensional ρ_{ABC}, $I_F(A;B|C)_\rho = 0$ if and only if ρ_{ABC} is a short quantum Markov chain, as defined in [20]. A short quantum Markov chain is a tripartite state ρ_{ABC} for which $I(A;B|C)_\rho = 0$, and such a state necessarily has a particular structure, as elucidated in [20].

2. **(Monotonicity)** $I_F(A;B|C)_\rho$ is monotone with respect to quantum operations on systems A or B, in the sense that

$$I_F(A;B|C)_\rho \geq I_F(A';B'|C)_\omega, \quad (11)$$

where $\omega_{ABC} \equiv (\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'})(\rho_{ABC})$ and $\mathcal{N}_{A\to A'}$ and $\mathcal{M}_{B\to B'}$ are quantum channels acting on systems A and B, respectively.

3. **(Local isometric invariance)** $I_F(A;B|C)_\rho$ is invariant with respect to local isometries, in the sense that
$$I_F(A;B|C)_\rho = I_F\left(A';B'|C'\right)_\sigma, \qquad (12)$$
where
$$\sigma_{A'B'C'} \equiv (\mathcal{U}_{A\to A'} \otimes \mathcal{V}_{B\to B'} \otimes \mathcal{W}_{C\to C'})\left(\rho_{ABC}\right) \qquad (13)$$
and $\mathcal{U}_{A\to A'}$, $\mathcal{V}_{B\to B'}$, and $\mathcal{W}_{C\to C'}$ are isometric quantum channels. An isometric channel $\mathcal{U}_{A\to A'}$ has the following action on an operator X_A:
$$\mathcal{U}_{A\to A'}(X_A) = U_{A\to A'} X_A U^\dagger_{A\to A'}, \qquad (14)$$
where $U_{A\to A'}$ is an isometry, satisfying $U^\dagger_{A\to A'} U_{A\to A'} = I_A$.

4. **(Duality)** For a four-party pure state ψ_{ABCD}, the following duality relation holds
$$I_F(A;B|C)_\psi = I_F(A;B|D)_\psi. \qquad (15)$$

5. **(Dimension bound)** The following dimension bound holds
$$I_F(A;B|C)_\rho \leq 2\log|A|, \qquad (16)$$
where $|A|$ is the dimension of the system A. If the system A is classical, so that we relabel it as X, then
$$I_F(X;B|C)_\rho \leq \log|X|. \qquad (17)$$
By a classical system X, we mean that ρ_{XBC} has the following form:
$$\rho_{XBC} = \sum_x p_X(x)|x\rangle\langle x| \otimes \rho^x_{BC}, \qquad (18)$$
for some probability distribution $p_X(x)$, orthonormal basis $\{|x\rangle\}$, and set $\{\rho^x_{BC}\}$ of density operators.

6. **(Continuity)** If two quantum states ρ_{ABC} and σ_{ABC} are close to each other in the sense that $F\left(\rho_{ABC}, \sigma_{ABC}\right) \approx 1$, then $I_F(A;B|C)_\rho \approx I_F(A;B|C)_\sigma$.

7. **(Weak chain rule)** The chain rule for conditional mutual information of a four-party state ρ_{ABCD} is as follows:
$$I\left(AC;B|D\right)_\rho = I\left(A;B|CD\right)_\rho + I\left(C;B|D\right)_\rho. \qquad (19)$$
We find something weaker than this for I_F, which we call the weak chain rule for I_F:
$$I_F\left(AC;B|D\right)_\rho \geq I_F\left(A;B|CD\right)_\rho. \qquad (20)$$

Let us note here that, by inspecting the definitions, the fidelity of recovery $F(A;B|C)_\rho$ and $I_F(A;B|C)_\rho$ are clearly not symmetric under the exchange of the A and B systems, unlike the conditional mutual information $I(A;B|C)_\rho$. Thus we might also refer to $I_F(A;B|C)_\rho$ as the conditional information that B has about A from the perspective of C.

B. Geometric squashed entanglement

Our next contribution is to define a (pseudo) entanglement measure of a bipartite state that we call the *geometric squashed entanglement*. To motivate this quantity, recall that the squashed entanglement of a bipartite state ρ_{AB} is defined as
$$E^{\text{sq}}(A;B)_\rho \equiv \frac{1}{2}\inf_{\omega_{ABE}} \left\{I(A;B|E)_\omega : \rho_{AB} = \text{Tr}_E\left\{\omega_{ABE}\right\}\right\}, \qquad (21)$$
where the infimum is over all extensions ω_{ABE} of the state ρ_{AB} [4]. The interpretation of $E^{\text{sq}}\left(A;B\right)_\rho$ is that it quantifies the correlations present between Alice and Bob after a third party (often associated to an environment or eavesdropper) attempts to "squash down" their correlations. In light of the above discussion, we define the geometric squashed entanglement simply by replacing the conditional mutual information with I_F:
$$E^{\text{sq}}_F(A;B)_\rho \equiv \frac{1}{2}\inf_{\omega_{ABE}} \left\{I_F(A;B|E)_\omega : \rho_{AB} = \text{Tr}_E\left\{\omega_{ABE}\right\}\right\}. \qquad (22)$$
We also employ the related quantity throughout the paper:
$$F^{\text{sq}}(A;B)_\rho \equiv \sup_{\omega_{ABE}} \left\{F(A;B|E)_\rho : \rho_{AB} = \text{Tr}_E\left\{\omega_{ABE}\right\}\right\}, \qquad (23)$$
with the two of them being related by
$$E^{\text{sq}}_F(A;B)_\rho = -\frac{1}{2}\log F^{\text{sq}}(A;B)_\rho. \qquad (24)$$

We prove the following results for the geometric squashed entanglement:

1. **(1-LOCC Monotone)** The geometric squashed entanglement of ρ_{AB} does not increase with respect to local operations and classical communication from Bob to Alice. That is, the following inequality holds
$$E^{\text{sq}}_F(A;B)_\rho \geq E^{\text{sq}}_F\left(A';B'\right)_\omega, \qquad (25)$$
where $\omega_{AB} \equiv \Lambda_{AB\to A'B'}\left(\rho_{AB}\right)$ and $\Lambda_{AB\to A'B'}$ is a quantum channel realized by local operations and classical communication from Bob to Alice. (Due to the asymmetric nature of the fidelity of recovery, we do not seem to be able to prove that the geometric squashed entanglement is an LOCC monotone.) The geometric squashed entanglement is also convex, i.e.,
$$\sum_x p_X(x) E^{\text{sq}}_F(A;B)_{\rho^x} \geq E^{\text{sq}}_F(A;B)_{\overline{\rho}}, \qquad (26)$$
where
$$\overline{\rho}_{AB} \equiv \sum_x p_X(x)\rho^x_{AB}, \qquad (27)$$

p_X is a probability distribution and $\{\rho^x_{AB}\}$ is a set of states.

2. (**Local isometric invariance**) $E_F^{\text{sq}}(A;B)_\rho$ is invariant with respect to local isometries, in the sense that
$$E_F^{\text{sq}}(A;B)_\rho = E_F^{\text{sq}}(A';B')_\sigma, \tag{28}$$
where
$$\sigma_{A'B'} \equiv (\mathcal{U}_{A\to A'} \otimes \mathcal{V}_{B\to B'})(\rho_{AB}) \tag{29}$$
and $\mathcal{U}_{A\to A'}$ and $\mathcal{V}_{B\to B'}$ are isometric quantum channels.

3. **(Faithfulness)** The geometric squashed entanglement of ρ_{AB} is equal to zero if and only if ρ_{AB} is a separable (unentangled) state. In particular, we prove the following bound by appealing directly to the argument in [22]:
$$E_F^{\text{sq}}(A;B)_\rho \geq \frac{1}{512\,|A|^4}\left\|\rho_{AB} - \text{SEP}(A:B)\right\|_1^4, \tag{30}$$
where the trace distance to separable states is defined by
$$\left\|\rho_{AB} - \text{SEP}(A:B)\right\|_1 \equiv \inf_{\sigma_{AB}\in\text{SEP}(A:B)} \left\|\rho_{AB} - \sigma_{AB}\right\|_1. \tag{31}$$

4. **(Reduction to geometric measure)** The geometric squashed entanglement of a pure state $|\phi\rangle_{AB}$ reduces to the well known geometric measure of entanglement [25] (see also [26] and references therein):
$$E_F^{\text{sq}}(A;B)_\psi = -\frac{1}{2}\log\sup_{|\varphi\rangle_A} \langle\phi|_{AB}(\varphi_A\otimes\phi_B)|\phi\rangle_{AB} \tag{32}$$
$$= -\log\left\|\phi_A\right\|_\infty. \tag{33}$$
Recall that the geometric measure of $|\phi\rangle_{AB}$ is known to be equal to
$$-\log\sup_{|\varphi\rangle_A,|\psi\rangle_B} \langle\phi|_{AB}(\varphi_A\otimes\psi_B)|\phi\rangle_{AB} = -\log\left\|\phi_A\right\|_\infty, \tag{34}$$
where $\|A\|_\infty$ is the infinity norm of an operator A, equal to its largest singular value. (Note that the above quantity is often referred to as the *logarithmic geometric measure of entanglement*. Here, for brevity, we simply refer to it as the geometric measure.)

5. **(Normalization)** The geometric squashed entanglement of a maximally entangled state Φ_{AB} is equal to $\log d$, where d is the Schmidt rank of Φ_{AB}. It is larger than $\log d$ when evaluated for a private state [27, 28] of $\log d$ private bits.

6. **(Subadditivity)** The geometric squashed entanglement is subadditive for tensor-product states, i.e.,
$$E_F^{\text{sq}}(A_1A_2;B_1B_2)_\omega \leq E_F^{\text{sq}}(A_1;B_1)_\rho + E_F^{\text{sq}}(A_2;B_2)_\sigma, \tag{35}$$
where $\omega_{A_1B_1A_2B_2} \equiv \rho_{A_1B_1}\otimes\sigma_{A_2B_2}$.

7. **(Continuity)** If two quantum states ρ_{AB} and σ_{AB} are close in trace distance, then their respective geometric squashed entanglements are close as well.

C. Surprisal of measurement recoverability

The quantum discord $D(\overline{A};B)_\rho$ is an information quantity which characterizes quantum correlations of a bipartite state ρ_{AB}, by quantifying how much correlation is lost through the act of a quantum measurement [3, 29] (we give a full definition later on). By a chain of reasoning detailed in Section VI which begins with the original definition of quantum discord, we define the surprisal of measurement recoverability of a bipartite state as follows:
$$D_F(\overline{A};B)_\rho \equiv -\log\sup_{\mathcal{E}_A} F(\rho_{AB},\mathcal{E}_A(\rho_{AB})), \tag{36}$$
where the supremum is over the convex set of entanglement breaking channels [30]. Since every entanglement breaking channel can be written as a concatenation of a measurement map followed by a preparation map, $D_F(\overline{A};B)_\rho$ characterizes how well one can recover a bipartite state after performing a quantum measurement on one share of it. Equivalently, the quantity captures how close ρ_{AB} is to being a fixed point of an entanglement breaking channel.

We establish several properties of $D_F(\overline{A};B)_\rho$, which are analogous to properties known to hold for the quantum discord [31]:

1. **(Non-negativity)** This follows trivially because the fidelity between two quantum states is always a real number between zero and one.

2. **(Local isometric invariance)** $D_F\left(\overline{A};B\right)_\rho$ is invariant with respect to local isometries, in the sense that
$$D_F(\overline{A};B)_\rho = D_F(\overline{A'};B')_\sigma, \tag{37}$$
where
$$\sigma_{A'B'} \equiv (\mathcal{U}_{A\to A'}\otimes\mathcal{V}_{B\to B'})(\rho_{AB}) \tag{38}$$
and $\mathcal{U}_{A\to A'}$ and $\mathcal{V}_{B\to B'}$ are isometric quantum channels.

3. **(Faithfulness)** $D_F(\overline{A};B)_\rho$ is equal to zero if and only if ρ_{AB} is a classical-quantum state (classical on system A).

4. **(Dimension bound)** $D_F(\overline{A};B)_\rho \leq \log|A|$.
5. **(Normalization)** $D_F(\overline{A};B)_\Phi$ for a maximally entangled state Φ_{AB} is equal to $\log d$, where d is the Schmidt rank of Φ_{AB}.
6. **(Monotonicity)** The surprisal of measurement recoverability is monotone with respect to quantum operations on the unmeasured system, i.e.,
$$D_F(\overline{A};B)_\rho \geq D_F\left(\overline{A};B'\right)_\sigma, \tag{39}$$
where $\sigma_{AB'} \equiv \mathcal{N}_{B\to B'}(\rho_{AB})$.
7. **(Continuity)** If two quantum states ρ_{AB} and σ_{AB} are close in trace distance, then the respective $D_F\left(\overline{A};B\right)$ quantities are close as well.

Finally, we use $D_F(\overline{A};B)_\rho$ and a recent result of Fawzi and Renner [23] to establish that the quantum discord of ρ_{AB} is nearly equal to zero if and only if ρ_{AB} is an approximate fixed point of entanglement breaking channel (i.e., if it is possible to nearly recover ρ_{AB} after performing a measurement on the system A). We then argue that several discord-like measures appearing throughout the literature [31] have a more natural physical grounding if they are based on how far a given bipartite state is from being a fixed point of an entanglement breaking channel.

III. PRELIMINARIES

Norms, states, extensions, channels, and measurements. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators acting on a Hilbert space $\mathcal{H}$. We restrict ourselves to finite-dimensional Hilbert spaces throughout this paper. For $\alpha \geq 1$, we define the α-norm of an operator X as
$$\|X\|_\alpha \equiv \text{Tr}\{(\sqrt{X^\dagger X})^\alpha\}^{1/\alpha}. \tag{40}$$
Let $\mathcal{B}(\mathcal{H})_+$ denote the subset of positive semi-definite operators. We also write $X \geq 0$ if $X \in \mathcal{B}(\mathcal{H})_+$. An operator ρ is in the set $\mathcal{S}(\mathcal{H})$ of density operators (or states) if $\rho \in \mathcal{B}(\mathcal{H})_+$ and $\text{Tr}\{\rho\} = 1$. The tensor product of two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ is denoted by $\mathcal{H}_A \otimes \mathcal{H}_B$ or $\mathcal{H}_{AB}$. Given a multipartite density operator $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we unambiguously write $\rho_A = \text{Tr}_B\{\rho_{AB}\}$ for the reduced density operator on system A. We use ρ_{AB}, σ_{AB}, τ_{AB}, ω_{AB}, etc. to denote general density operators in $\mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$, while ψ_{AB}, φ_{AB}, ϕ_{AB}, etc. denote rank-one density operators (pure states) in $\mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ (with it implicit, clear from the context, and the above convention implying that ψ_A, φ_A, ϕ_A are mixed if ψ_{AB}, φ_{AB}, ϕ_{AB} are pure and entangled).

We also say that pure-state vectors $|\psi\rangle$ in $\mathcal{H}$ are states. Any bipartite pure state $|\psi\rangle_{AB}$ in $\mathcal{H}_{AB}$ is written in Schmidt form as
$$|\psi\rangle_{AB} \equiv \sum_{i=0}^{d-1} \sqrt{\lambda_i}\, |i\rangle_A |i\rangle_B, \tag{41}$$
where $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ form orthonormal bases in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, $\lambda_i > 0$ for all i, $\sum_{i=0}^{d-1} \lambda_i = 1$, and d is the Schmidt rank of the state. By a maximally entangled state, we mean a bipartite pure state of the form
$$|\Phi\rangle_{AB} \equiv \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_A |i\rangle_B. \tag{42}$$

A state $\gamma_{ABA'B'}$ is a private state [27, 28] if Alice and Bob can extract a secret key from it by performing local von Neumann measurements on the A and B systems of $\gamma_{ABA'B'}$, such that the resulting secret key is product with any purifying system of $\gamma_{ABA'B'}$. The systems A' and B' are known as "shield systems" because they aid in keeping the key secure from any eavesdropper possessing the purifying system. Interestingly, a private state of $\log d$ private bits can be written in the following form [27, 28]:
$$\gamma_{ABA'B'} = U_{ABA'B'}\left(\Phi_{AB} \otimes \rho_{A'B'}\right) U^\dagger_{ABA'B'}, \tag{43}$$
where
$$U_{ABA'B'} = \sum_{i,j} |i\rangle\langle i|_A \otimes |j\rangle\langle j|_B \otimes U^{ij}_{A'B'}. \tag{44}$$

The unitaries can be chosen such that $U^{ij}_{A'B'} = V^j_{A'B'}$ or $U^{ij}_{A'B'} = V^i_{A'B'}$. This implies that the unitary $U_{ABA'B'}$ can be implemented either as
$$U_{ABA'B'} = \sum_i |i\rangle\langle i|_A \otimes I_B \otimes V^i_{A'B'} \tag{45}$$
or
$$U_{ABA'B'} = I_A \otimes \sum_i |i\rangle\langle i|_B \otimes V^i_{A'B'}. \tag{46}$$

The trace distance between two quantum states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ is equal to $\|\rho - \sigma\|_1$. It has a direct operational interpretation in terms of the distinguishability of these states. That is, if ρ or σ is prepared with equal probability and the task is to distinguish them via some quantum measurement, then the optimal success probability in doing so is equal to $(1 + \|\rho - \sigma\|_1/2)/2$.

A linear map $\mathcal{N}_{A\to B} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is positive if $\mathcal{N}_{A\to B}(\sigma_A) \in \mathcal{B}(\mathcal{H}_B)_+$ whenever $\sigma_A \in \mathcal{B}(\mathcal{H}_A)_+$. Let id_A denote the identity map acting on a system A. A linear map $\mathcal{N}_{A\to B}$ is completely positive if the map $\text{id}_R \otimes \mathcal{N}_{A\to B}$ is positive for a reference system R of arbitrary size. A linear map $\mathcal{N}_{A\to B}$ is trace-preserving if $\text{Tr}\{\mathcal{N}_{A\to B}(\tau_A)\} = \text{Tr}\{\tau_A\}$ for all input operators $\tau_A \in \mathcal{B}(\mathcal{H}_A)$. If a linear map is completely positive and trace-preserving (CPTP), we say that it is a quantum channel or quantum operation. An extension of a state $\rho_A \in \mathcal{S}(\mathcal{H}_A)$ is some state $\Omega_{RA} \in \mathcal{S}(\mathcal{H}_R \otimes \mathcal{H}_A)$ such that $\text{Tr}_R\{\Omega_{RA}\} = \rho_A$. An isometric extension $U^{\mathcal{N}}_{A\to BE}$

of a channel $\mathcal{N}_{A\to B}$ acting on a state $\rho_A \in \mathcal{S}(\mathcal{H}_A)$ is a linear map that satisfies the following:

$$\mathrm{Tr}_E\left\{U^{\mathcal{N}}_{A\to BE}\rho_A(U^{\mathcal{N}}_{A\to BE})^\dagger\right\} = \mathcal{N}_{A\to B}\left(\rho_A\right), \tag{47}$$

$$U^\dagger_{\mathcal{N}}U_{\mathcal{N}} = I_A, \tag{48}$$

$$U_{\mathcal{N}}U^\dagger_{\mathcal{N}} = \Pi_{BE}, \tag{49}$$

where Π_{BE} is a projection onto a subspace of the Hilbert space $\mathcal{H}_B \otimes \mathcal{H}_E$.

IV. FIDELITY OF RECOVERY

In this section, we formally define the fidelity of recovery for a tripartite state ρ_{ABC}, and we prove that it possesses various properties, demonstrating that the quantity $I_F\left(A;B|C\right)_\rho$ defined in (8) is similar to the conditional mutual information.

Definition 1 (Fidelity of recovery) *Let ρ_{ABC} be a tripartite state. The fidelity of recovery for ρ_{ABC} with respect to system A is defined as follows:*

$$F(A;B|C)_\rho \equiv \sup_{\mathcal{R}_{C\to AC}} F\left(\rho_{ABC}, \mathcal{R}_{C\to AC}\left(\rho_{BC}\right)\right). \tag{50}$$

This quantity characterizes how well one can recover the full state on systems ABC from system C alone if system A is lost.

Proposition 2 (Non-negativity) *Let ρ_{ABC} be a tripartite state. Then $I_F\left(A;B|C\right)_\rho \geq 0$, and for finite-dimensional ρ_{ABC}, $I_F(A;B|C)_\rho = 0$ if and only if ρ_{ABC} is a short quantum Markov chain, as defined in [20].*

Proof. The inequality $I_F(A;B|C)_\rho \geq 0$ is a consequence of the fidelity always being less than or equal to one. Suppose that ρ_{ABC} is a short quantum Markov chain as defined in [20]. As discussed in that paper, this is equivalent to the equality

$$\rho_{ABC} = \mathcal{R}^P_{C\to AC}\left(\rho_{BC}\right), \tag{51}$$

where $\mathcal{R}^P_{C\to AC}$ is the Petz recovery channel. So this implies that

$$F\left(\rho_{ABC}, \mathcal{R}^P_{C\to AC}\left(\rho_{BC}\right)\right) = 1, \tag{52}$$

which in turn implies that $F(A;B|C)_\rho = 1$ and hence $I_F(A;B|C)_\rho = 0$. Now suppose that $I_F\left(A;B|C\right)_\rho = 0$. This implies that

$$\sup_{\mathcal{R}_{C\to AC}} F\left(\rho_{ABC}, \mathcal{R}_{C\to AC}\left(\rho_{BC}\right)\right) = 1. \tag{53}$$

Due to the finite-dimensional assumption, the space of channels over which we are optimizing is compact. Furthermore, the fidelity is continuous in its arguments. This is sufficient for us to conclude that the supremum is achieved and that there exists a channel $\mathcal{R}_{C\to AC}$ for which $F\left(\rho_{ABC}, \mathcal{R}_{C\to AC}\left(\rho_{BC}\right)\right) = 1$, implying that

$$\rho_{ABC} = \mathcal{R}_{C\to AC}\left(\rho_{BC}\right). \tag{54}$$

From a result of Petz [18], this implies that the Petz recovery channel recovers ρ_{ABC} perfectly, i.e.,

$$\rho_{ABC} = \mathcal{R}^P_{C\to AC}\left(\rho_{BC}\right), \tag{55}$$

and this is equivalent to ρ_{ABC} being a short quantum Markov chain [20]. ■

Proposition 3 (Monotonicity) *The fidelity of recovery is monotone with respect to local operations on systems A and B, in the sense that*

$$F(A;B|C)_\rho \leq F\left(A';B'|C\right)_\tau, \tag{56}$$

where $\tau_{A'B'C} \equiv \left(\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'}\right)(\rho_{ABC})$. The above inequality is equivalent to

$$I_F(A;B|C)_\rho \geq I_F\left(A';B'|C\right)_\tau. \tag{57}$$

Proof. For any recovery map $\mathcal{R}_{C\to AC}$, we have that

$$F\left(\rho_{ABC}, \mathcal{R}_{C\to AC}\left(\rho_{BC}\right)\right) \leq F\left(\left(\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'}\right)(\rho_{ABC}), \left(\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'}\right)\left(\mathcal{R}_{C\to AC}\left(\rho_{BC}\right)\right)\right) \tag{58}$$

$$= F\left(\left(\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'}\right)(\rho_{ABC}), \left(\mathcal{N}_{A\to A'} \circ \mathcal{R}_{C\to AC}\right)\left(\mathcal{M}_{B\to B'}\left(\rho_{BC}\right)\right)\right) \tag{59}$$

$$\leq \sup_{\mathcal{R}_{C\to A'C}} F\left(\left(\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'}\right)(\rho_{ABC}), \mathcal{R}_{C\to A'C}\left(\mathcal{M}_{B\to B'}\left(\rho_{BC}\right)\right)\right) \tag{60}$$

$$= F\left(A';B'|C\right)_{(\mathcal{N}\otimes\mathcal{M})(\rho)}, \tag{61}$$

where the first inequality is due to monotonicity of the fidelity with respect to quantum operations. Since the chain of inequalities holds for all $\mathcal{R}_{C\to AC}$, it follows that

$$F(A;B|C)_\rho = \sup_{\mathcal{R}_{C\to AC}} F\left(\rho_{ABC}, \mathcal{R}_{C\to AC}\left(\rho_{BC}\right)\right) \tag{62}$$

$$\leq F\left(A';B'|C\right)_{(\mathcal{N}\otimes\mathcal{M})(\rho)}. \tag{63}$$

■

Remark 4 *The physical interpretation of the above monotonicity with respect to local operations is as follows: for a tripartite state ρ_{ABC}, suppose that system A is lost. Then it is easier to recover the state on systems ABC from C alone if there is local noise applied to systems A or B or both, before system A is lost (and thus before attempting the recovery).*

Proposition 5 (Local isometric invariance) *Let ρ_{ABC} be a tripartite quantum state and let*

$$\sigma_{A'B'C'} \equiv (\mathcal{U}_{A\to A'} \otimes \mathcal{V}_{B\to B'} \otimes \mathcal{W}_{C\to C'})(\rho_{ABC}), \quad (64)$$

where $\mathcal{U}_{A\to A'}$, $\mathcal{V}_{B\to B'}$, and $\mathcal{W}_{C\to C'}$ are isometric quantum channels. Then

$$F(A;B|C)_\rho = F(A';B'|C')_\sigma, \quad (65)$$
$$I_F(A;B|C)_\rho = I_F(A';B'|C')_\sigma. \quad (66)$$

Proof. We prove the statement for fidelity of recovery. We first need to define some CPTP maps that invert the isometric channels $\mathcal{U}_{A\to A'}$, $\mathcal{V}_{B\to B'}$, and $\mathcal{W}_{C\to C'}$, given that $\mathcal{U}^\dagger_{A\to A'}$, $\mathcal{V}^\dagger_{B\to B'}$, and $\mathcal{W}^\dagger_{C\to C'}$ are not necessarily quantum channels. So we define the CPTP linear map $\mathcal{T}^{\mathcal{U}}_{A'\to A}$ as follows:

$$\mathcal{T}^{\mathcal{U}}_{A'\to A}(\omega_{A'}) \equiv \mathcal{U}^\dagger_{A\to A'}(\omega_{A'}) + \operatorname{Tr}\left\{\left(\mathrm{id}_{A'} - \mathcal{U}^\dagger_{A\to A'}\right)(\omega_{A'})\right\}\tau_A, \quad (67)$$

where τ_A is some state on system A. We define the maps $\mathcal{T}^{\mathcal{V}}_{B'\to B}$ and $\mathcal{T}^{\mathcal{W}}_{C'\to C}$ similarly. All three maps have the property that

$$\mathcal{T}^{\mathcal{U}}_{A'\to A} \circ \mathcal{U}_{A\to A'} = \mathrm{id}_A, \quad (68)$$
$$\mathcal{T}^{\mathcal{V}}_{B'\to B} \circ \mathcal{V}_{B\to B'} = \mathrm{id}_B, \quad (69)$$
$$\mathcal{T}^{\mathcal{W}}_{C'\to C} \circ \mathcal{W}_{C\to C'} = \mathrm{id}_C. \quad (70)$$

Let $\mathcal{R}_{C\to AC}$ be an arbitrary recovery map. Then

$$
\begin{aligned}
&F(\rho_{ABC}, \mathcal{R}_{C\to AC}(\rho_{BC})) \\
&= F((\mathcal{U}_{A\to A'} \otimes \mathcal{V}_{B\to B'} \otimes \mathcal{W}_{C\to C'})(\rho_{ABC}), (\mathcal{U}_{A\to A'} \otimes \mathcal{V}_{B\to B'} \otimes \mathcal{W}_{C\to C'})(\mathcal{R}_{C\to AC}(\rho_{BC}))) && (71)\\
&= F(\sigma_{A'B'C'}, (\mathcal{U}_{A\to A'} \otimes \mathcal{W}_{C\to C'})(\mathcal{R}_{C\to AC}(\mathcal{V}_{B\to B'}(\rho_{BC})))) && (72)\\
&= F\left(\sigma_{A'B'C'}, (\mathcal{U}_{A\to A'} \otimes \mathcal{W}_{C\to C'})\left(\mathcal{R}_{C\to AC}\left(\mathcal{T}^{\mathcal{W}}_{C'\to C}(\mathcal{V}_{B\to B'} \otimes \mathcal{W}_{C\to C'})(\rho_{BC})\right)\right)\right) && (73)\\
&\le \sup_{\mathcal{R}_{C'\to A'C'}} F(\sigma_{A'B'C'}, \mathcal{R}_{C'\to A'C'}((\mathcal{V}_{B\to B'} \otimes \mathcal{W}_{C\to C'})(\rho_{BC}))) && (74)\\
&= F(A';B'|C')_\sigma. && (75)
\end{aligned}
$$

The first equality follows from invariance of fidelity with respect to isometries. The second equality follows because $\mathcal{R}_{C\to AC}$ and $\mathcal{V}_{B\to B'}$ commute. The third equality follows from (70). The inequality follows because

$$(\mathcal{U}_{A\to A'} \otimes \mathcal{W}_{C\to C'}) \circ \mathcal{R}_{C\to AC} \circ \mathcal{T}^{\mathcal{W}}_{C'\to C} \quad (76)$$

is a particular CPTP recovery map from C' to $A'C'$. The last equality is from the definition of fidelity of recovery.

Given that the inequality

$$F(\rho_{ABC}, \mathcal{R}_{C\to AC}(\rho_{BC})) \le F(A';B'|C')_\sigma \quad (77)$$

holds for an arbitrary recovery map $\mathcal{R}_{C\to AC}$, we can conclude that $F(A;B|C)_\rho \le F(A';B'|C')_\sigma$.

For the other inequality, let $\mathcal{R}_{C'\to A'C'}$ be an arbitrary recovery map. Then

$$
\begin{aligned}
&F\left(\sigma_{A'B'C'},\mathcal{R}_{C'\to A'C'}\left(\sigma_{B'C'}\right)\right)\\
&\leq F\left(\left(\mathcal{T}^{\mathcal{U}}_{A'\to A}\otimes\mathcal{T}^{\mathcal{V}}_{B'\to B}\otimes\mathcal{T}^{\mathcal{W}}_{C'\to C}\right)\left(\sigma_{A'B'C'}\right),\left(\mathcal{T}^{\mathcal{U}}_{A'\to A}\otimes\mathcal{T}^{\mathcal{V}}_{B'\to B}\otimes\mathcal{T}^{\mathcal{W}}_{C'\to C}\right)\left(\mathcal{R}_{C'\to A'C'}\left(\sigma_{B'C'}\right)\right)\right) &(78)\\
&= F\left(\rho_{ABC},\left(\mathcal{T}^{\mathcal{U}}_{A'\to A}\otimes\mathcal{T}^{\mathcal{W}}_{C'\to C}\right)\left(\mathcal{R}_{C'\to A'C'}\left(\mathcal{T}^{\mathcal{V}}_{B'\to B}\left(\sigma_{B'C'}\right)\right)\right)\right) &(79)\\
&= F\left(\rho_{ABC},\left(\mathcal{T}^{\mathcal{U}}_{A'\to A}\otimes\mathcal{T}^{\mathcal{W}}_{C'\to C}\right)\left(\mathcal{R}_{C'\to A'C'}\left(\left(\mathcal{T}^{\mathcal{V}}_{B'\to B}\circ\mathcal{V}_{B\to B'}\otimes\mathcal{W}_{C\to C'}\right)\left(\rho_{BC}\right)\right)\right)\right) &(80)\\
&= F\left(\rho_{ABC},\left(\mathcal{T}^{\mathcal{U}}_{A'\to A}\otimes\mathcal{T}^{\mathcal{W}}_{C'\to C}\right)\left(\mathcal{R}_{C'\to A'C'}\left(\mathcal{W}_{C\to C'}\left(\rho_{BC}\right)\right)\right)\right) &(81)\\
&\leq \sup_{\mathcal{R}_{C\to AC}} F\left(\rho_{ABC},\mathcal{R}_{C\to AC}\left(\rho_{BC}\right)\right) &(82)\\
&= F(A;B|C)_{\rho}. &(83)
\end{aligned}
$$

The first inequality is from monotonicity of the fidelity with respect to quantum channels. The first equality is a consequence of (68)-(70). The second equality is from the definition of $\sigma_{B'C'}$. The third equality follows from (70). The last inequality follows because $\left(\mathcal{T}^{\mathcal{U}}_{A'\to A}\otimes\mathcal{T}^{\mathcal{W}}_{C'\to C}\right)\circ\mathcal{R}_{C'\to A'C'}\circ\mathcal{W}_{C\to C'}$ is a particular recovery map from C to AC. Given that the inequality $F\left(\sigma_{A'B'C'},\mathcal{R}_{C'\to A'C'}\left(\sigma_{B'C'}\right)\right)\leq F(A;B|C)_{\rho}$ holds for an arbitrary recovery map $\mathcal{R}_{C'\to A'C'}$, we can conclude that $F\left(A';B'|C'\right)_{\sigma}\leq F\left(A;B|C\right)_{\rho}$. ■

Remark 6 *The only property of the fidelity used to prove Propositions 3 and 5 is that it is monotone with respect to quantum operations. This suggests that we can construct a fidelity-of-recovery-like measure from any "generalized divergence" (a function that is monotone with respect to quantum operations).*

Proposition 7 (Duality) *Let ϕ_{ABCD} be a four-party pure state. Then*

$$F(A;B|C)_{\phi}=F(A;B|D)_{\phi}, \tag{84}$$

which is equivalent to

$$I_F(A;B|C)_{\phi}=I_F(A;B|D)_{\phi}. \tag{85}$$

Proof. By definition,

$$F(A;B|C)_{\phi}=\sup_{\mathcal{R}^1_{C\to AC}}F\left(\phi_{ABC},\mathcal{R}^1_{C\to AC}\left(\phi_{BC}\right)\right). \tag{86}$$

Let $\mathcal{U}^{\mathcal{R}^1}_{C\to ACE}$ be an isometric channel which extends $\mathcal{R}^1_{C\to AC}$. Since ϕ_{ABCD} is a purification of ϕ_{ABC} and $\mathcal{U}^{\mathcal{R}^1}_{C\to ACE}\left(\phi_{BCA'D}\right)$ is a purification of $\mathcal{R}^1_{C\to AC}\left(\phi_{BC}\right)$, we can apply Uhlmann's theorem for fidelity to conclude that

$$\sup_{\mathcal{R}^1_{C\to AC}}F\left(\phi_{ABC},\mathcal{R}^1_{C\to AC}\left(\phi_{BC}\right)\right)=\sup_{\mathcal{U}_{D\to A'DE}}\sup_{\mathcal{U}^{\mathcal{R}^1}_{C\to ACE}}F\Big(\mathcal{U}_{D\to A'DE}\left(\phi_{ABCD}\right),\mathcal{U}^{\mathcal{R}^1}_{C\to ACE}\left(\phi_{BCA'D}\right)\Big). \tag{87}$$

Now consider that

$$F(A;B|D)_{\phi}=\sup_{\mathcal{R}^2_{D\to AD}}F\left(\phi_{ABD},\mathcal{R}^2_{D\to AD}\left(\phi_{BD}\right)\right). \tag{88}$$

Let $\mathcal{U}^{\mathcal{R}^2}_{D\to ADE}$ be an isometric channel which extends $\mathcal{R}^2_{D\to AD}$. Since ϕ_{ABCD} is a purification of ϕ_{ABD} and $\mathcal{U}^{\mathcal{R}^2}_{D\to ADE}\left(\phi_{BDA'C}\right)$ is a purification of $\mathcal{R}^2_{D\to AD}\left(\phi_{BD}\right)$, we can apply Uhlmann's theorem for fidelity to conclude that

$$\sup_{\mathcal{R}^2_{D\to AD}}F\left(\phi_{ABD},\mathcal{R}^2_{D\to AD}\left(\phi_{BD}\right)\right)=\sup_{\mathcal{U}_{C\to A'CE}}\sup_{\mathcal{U}^{\mathcal{R}^2}_{D\to ADE}}F\Big(\mathcal{U}_{C\to A'CE}\left(\phi_{ABCD}\right),\mathcal{U}^{\mathcal{R}^2}_{D\to ADE}\left(\phi_{BDA'C}\right)\Big). \tag{89}$$

By inspecting the RHS of (87) and the RHS of (89), we see that the two expressions are equivalent so that the statement of the proposition holds. Figure 1 gives a graphical depiction of this proof which should help in determining which systems are "connected together" and furthermore highlights how the duality between the re-

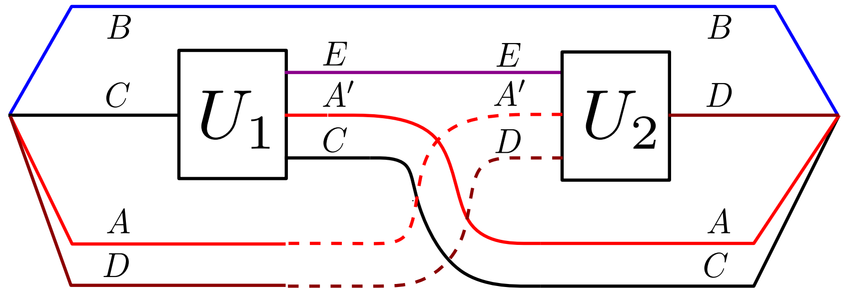

FIG. 1: This figure helps to illustrate the main idea behind the proof of Proposition 7 and furthermore highlights the dual role played by an isometric extension of the recovery map on C and an Uhlmann isometry acting on system D (and vice versa). When reading the figure from left to right, the isometry on the left corresponds to the recovery map and the isometry on the right corresponds to the one from Uhlmann's theorem, and the overlap between the left and right is understood as $F(A;B|C)$. When reading the figure from right to left, the isometries switch their roles and the overlap is understood as $F(A;B|D)$. This picture clarifies in a diagrammatic way how we get the duality relation $F(A;B|C) = F(A;B|D)$.

covery map and the map from Uhlmann's theorem is reflected in the duality for the fidelity of recovery. ■

Remark 8 *The physical interpretation of the above duality is as follows: beginning with a four-party pure state ϕ_{ABCD}, suppose that system A is lost. Then one can recover the state on systems ABC from system C alone just as well as one can recover the state on systems ABD from system D alone.*

Proposition 9 (Continuity) *The fidelity of recovery is a continuous function of its input. That is, given two tripartite states ρ_{ABC} and σ_{ABC} such that $F(\rho_{ABC}, \sigma_{ABC}) \geq 1 - \varepsilon$ where $\varepsilon \in [0,1]$, it follows that*

$$|F(A;B|C)_\rho - F(A;B|C)_\sigma| \leq 8\sqrt{\varepsilon}, \tag{90}$$

$$|I_F(A;B|C)_\rho - I_F(A;B|C)_\sigma| \leq |A|^x \, 8\sqrt{\varepsilon}, \tag{91}$$

where $x = 1$ if system A is classical and $x = 2$ otherwise.

Proof. One of the main tools for our proof is the purified distance [32, Definition 4], defined for two quantum states as

$$P(\rho, \sigma) \equiv \sqrt{1 - F(\rho, \sigma)}, \tag{92}$$

and which for our case implies that

$$P(\rho_{ABC}, \sigma_{ABC}) \leq \sqrt{\varepsilon}. \tag{93}$$

From the monotonicity of the purified distance with respect to quantum operations [32, Lemma 7], it follows that

$$P(\mathcal{R}_{C\to AC}(\rho_{BC}), \mathcal{R}_{C\to AC}(\sigma_{BC})) \leq \sqrt{\varepsilon}, \tag{94}$$

where $\mathcal{R}_{C\to AC}$ is an arbitrary CPTP linear recovery map. By the triangle inequality for purified distance [32, Lemma 5], it follows that

$$\inf_{\mathcal{R}_{C\to AC}} P(\rho_{ABC}, \mathcal{R}_{C\to AC}(\rho_{BC}))$$
$$\leq P(\rho_{ABC}, \mathcal{R}_{C\to AC}(\rho_{BC})) \tag{95}$$
$$\leq P(\rho_{ABC}, \sigma_{ABC}) + P(\sigma_{ABC}, \mathcal{R}_{C\to AC}(\sigma_{BC})) + P(\mathcal{R}_{C\to AC}(\sigma_{BC}), \mathcal{R}_{C\to AC}(\rho_{BC})) \tag{96}$$
$$\leq 2\sqrt{\varepsilon} + P(\sigma_{ABC}, \mathcal{R}_{C\to AC}(\sigma_{BC})). \tag{97}$$

Given that $\mathcal{R}_{C\to AC}$ is arbitrary, we can conclude that

$$\inf_{\mathcal{R}_{C\to AC}} P(\rho_{ABC}, \mathcal{R}_{C\to AC}(\rho_{BC})) \leq 2\sqrt{\varepsilon} + \inf_{\mathcal{R}_{C\to AC}} P(\sigma_{ABC}, \mathcal{R}_{C\to AC}(\sigma_{BC})), \tag{98}$$

which is equivalent to

$$\sqrt{1 - F(A;B|C)_\rho} \leq 2\sqrt{\varepsilon} + \sqrt{1 - F(A;B|C)_\sigma}. \tag{99}$$

Squaring both sides gives

$$1 - F(A;B|C)_\rho$$
$$\leq 4\varepsilon + 4\sqrt{\varepsilon}\sqrt{1 - F(A;B|C)_\sigma} + 1 - F(A;B|C)_\sigma$$
$$\leq 8\sqrt{\varepsilon} + 1 - F(A;B|C)_\sigma, \tag{100}$$

where the second inequality follows because $\varepsilon \in [0,1]$ and the same is true for the fidelity. Rewriting this gives

$$F(A;B|C)_\sigma \leq 8\sqrt{\varepsilon} + F(A;B|C)_\rho. \tag{101}$$

The same approach gives the other inequality:

$$F(A;B|C)_\rho \leq 8\sqrt{\varepsilon} + F(A;B|C)_\sigma. \tag{102}$$

By dividing (101) by $F(A;B|C)_\rho$ (which by Proposition 13 is never smaller than $1/|A|^2$) and taking a logarithm, we find that

$$\log\left(\frac{F(A;B|C)_\sigma}{F(A;B|C)_\rho}\right) \leq \log\left(1 + \frac{8\sqrt{\varepsilon}}{F(A;B|C)_\rho}\right) \tag{103}$$
$$\leq \frac{8\sqrt{\varepsilon}}{F(A;B|C)_\rho} \tag{104}$$
$$\leq |A|^x \, 8\sqrt{\varepsilon}. \tag{105}$$

where we used that $\log(y+1) \leq y$ and the dimension bound from Proposition 13. Applying this to the other inequality in (102) gives that

$$\log\left(\frac{F(A;B|C)_\rho}{F(A;B|C)_\sigma}\right) \leq |A|^x \, 8\sqrt{\varepsilon}, \tag{106}$$

from which we can conclude (91). ■

Proposition 10 (Weak chain rule) *Given a four-party state ρ_{ABCD}, the following inequality holds*

$$I_F\left(AC;B|D\right)_\rho \geq I_F\left(A;B|CD\right)_\rho . \tag{107}$$

Proof. The inequality is equivalent to

$$F\left(AC;B|D\right)_\rho \leq F\left(A;B|CD\right)_\rho , \tag{108}$$

which follows from the fact that it is easier to recover A from CD than it is to recover both A and C from D alone. Indeed, let $\mathcal{R}_{D\to ACD}$ be any recovery map. Then

$$\begin{aligned} &F\left(\rho_{ABCD}, \mathcal{R}_{D\to ACD}\left(\rho_{BD}\right)\right) \\ &= F\left(\rho_{ABCD}, \left(\mathcal{R}_{D\to ACD}\circ \mathrm{Tr}_C\right)\left(\rho_{BCD}\right)\right) &(109)\\ &\leq \sup_{\mathcal{R}_{CD\to ACD}} F\left(\rho_{ABCD}, \left(\mathcal{R}_{CD\to ACD}\right)\left(\rho_{BCD}\right)\right) &(110)\\ &= F\left(A;B|CD\right)_\rho . &(111)\end{aligned}$$

Since the chain of inequalities holds for any recovery map $\mathcal{R}_{D\to ACD}$, we can conclude (108) from the definition of $F\left(AC;B|D\right)_\rho$. ■

Proposition 11 (Conditioning on classical info.) *Let ω_{ABCX} be a state for which system X is classical:*

$$\omega_{ABCX} = \sum_x p_X(x)\omega^x_{ABC}\otimes |x\rangle\langle x|_X , \tag{112}$$

where $\{|x\rangle_X\}$ is an orthonormal basis, p_X is a probability distribution, and each ω^x_{ABC} is a state. Then the following inequalities hold

$$\sqrt{F}\left(A;B|CX\right)_\omega \geq \sum_x p_X\left(x\right)\sqrt{F}(A;B|C)_{\omega^x}, \tag{113}$$

$$I_F\left(A;B|CX\right)_\omega \leq \sum_x p_X(x) I_F(A;B|C)_{\omega^x}. \tag{114}$$

Proof. We first prove the inequality in (113). For any set of recovery maps $\mathcal{R}^x_{C\to CA}$, define $\mathcal{R}_{CX\to CXA}$ as follows:

$$\mathcal{R}_{CX\to CXA}\left(\tau_{CX}\right) \equiv \sum_x \mathcal{R}^x_{C\to CA}\left(\langle x|_X\left(\tau_{CX}\right)|x\rangle_X\right)\ |x\rangle\langle x|_X , \tag{115}$$

so that it first measures the system X in the basis $\{|x\rangle\langle x|_X\}$, places the outcome in the same classical register, and then acts with the particular recovery map $\mathcal{R}^x_{C\to CA}$. Then

$$\begin{aligned} &\left[\sum_x p_X(x)\sqrt{F}\left(\omega^x_{ABC}, \mathcal{R}^x_{C\to CA}\left(\omega^x_{BC}\right)\right)\right]^2 \\ &= F\left(\sum_x p_X(x)\omega^x_{ABC}\otimes|x\rangle\langle x|_X , \sum_x p_X(x)\mathcal{R}^x_{C\to CA}\left(\omega^x_{BC}\right)\otimes|x\rangle\langle x|_X\right) &(116)\\ &= F\left(\sum_x p_X(x)\omega^x_{ABC}\otimes|x\rangle\langle x|_X , \mathcal{R}_{CX\to CXA}\left(\sum_x p_X(x)\omega^x_{BC}\otimes|x\rangle\langle x|_X\right)\right) &(117)\\ &\leq F\left(A;B|CX\right)_\omega . &(118)\end{aligned}$$

Since the inequality holds for any set of individual recovery maps $\{\mathcal{R}^x_{C\to CA}\}$, we obtain (113).

Finally, we recover (114) by applying a negative logarithm to the inequality in (113) and exploiting convexity of $-\log$. ■

Proposition 12 (Conditioning on a product system) *Let $\rho_{ABC} = \sigma_{AB}\otimes\omega_C$. Then*

$$F(A;B|C)_\rho = F(A;B)_\sigma \equiv \sup_{\tau_A} F\left(\sigma_{AB}, \tau_A\otimes\sigma_B\right), \tag{119}$$

$$I_F(A;B|C)_\rho = I_F(A;B)_\sigma \equiv -\log F(A;B)_\sigma . \tag{120}$$

Proof. Consider that, for any recovery map $\mathcal{R}_{C\to AC}$

$$\begin{aligned} &F\left(\sigma_{AB}\otimes\omega_C, \mathcal{R}_{C\to AC}\left(\sigma_B\otimes\omega_C\right)\right) \\ &= F\left(\sigma_{AB}\otimes\omega_C, \sigma_B\otimes\mathcal{R}_{C\to AC}\left(\omega_C\right)\right) &(121)\\ &\leq F\left(\sigma_{AB}, \sigma_B\otimes\mathcal{R}_{C\to A}\left(\omega_C\right)\right) &(122)\\ &\leq \sup_{\tau_A} F\left(\sigma_{AB}, \sigma_B\otimes\tau_A\right). &(123)\end{aligned}$$

The first inequality follows because fidelity is monotone with respect to a partial trace over the C system. The second inequality follows by optimizing the second argument to the fidelity over all states on the A system. Since the inequality holds independent of the recovery map $\mathcal{R}_{C\to AC}$, we find that

$$F(A;B|C)_\rho \leq F(A;B)_\sigma . \tag{124}$$

To prove the other inequality $F(A;B)_\sigma \leq F\left(A;B|C\right)_\rho$, consider for any state τ_A that

$$\begin{aligned}
&F\left(\sigma_{AB}, \tau_A \otimes \sigma_B\right) \\
&= F\left(\sigma_{AB} \otimes \omega_C, \tau_A \otimes \sigma_B \otimes \omega_C\right) && (125)\\
&= F\left(\sigma_{AB} \otimes \omega_C, \left(\mathcal{P}_A^\tau \otimes \mathrm{id}_C\right)\left(\sigma_B \otimes \omega_C\right)\right) && (126)\\
&\leq \sup_{\mathcal{R}_{C\to AC}} F\left(\sigma_{AB} \otimes \omega_C, \mathcal{R}_{C\to AC}\left(\sigma_B \otimes \omega_C\right)\right). && (127)
\end{aligned}$$

The first equality follows because fidelity is multiplicative with respect to tensor-product states. The second equality follows by taking $(\mathrm{id}_C \otimes \mathcal{P}_A^\tau)$ to be the recovery map that does nothing to system C and prepares τ_A on system A. The inequality follows by optimizing over all recovery maps. Since the inequality is independent of the prepared state, we obtain the other inequality

$$F(A;B)_\sigma \leq F(A;B|C)_\rho. \tag{128}$$

The equality $I_F(A;B|C)_\rho = I_F(A;B)_\sigma$ follows by applying a negative logarithm to $F(A;B|C)_\rho = F(A;B)_\sigma$. We note in passing that the quantity on the RHS in (120) is closely related to the sandwiched Rényi mutual information of order 1/2 [33–36]. ∎

Proposition 13 (Dimension bound) *The fidelity of recovery obeys the following dimension bound:*

$$F(A;B|C)_\rho \geq \frac{1}{|A|^2}, \tag{129}$$

which is equivalent to

$$I_F(A;B|C)_\rho \leq 2\log|A|. \tag{130}$$

If the system A is classical, so that we relabel it as X, then the following hold

$$F(X;B|C)_\rho \geq \frac{1}{|X|}, \tag{131}$$

$$I_F(X;B|C)_\rho \leq \log|X|. \tag{132}$$

Examples of states achieving these bounds are $\Phi_{AB} \otimes \sigma_C$ for (129)-(130) *and $\overline{\Phi}_{XB} \otimes \sigma_C$ for* (131)-(132)*, where*

$$\overline{\Phi}_{XB} \equiv \frac{1}{|X|}\sum_x |x\rangle\langle x|_X \otimes |x\rangle\langle x|_B. \tag{133}$$

Proof. Consider that the following inequality holds, simply by choosing the recovery map to be one in which we do not do anything to system C and prepare the maximally mixed state $\pi_A \equiv I_A/|A|$ on system A:

$$\begin{aligned}
F(A;B|C)_\rho &\geq F\left(\rho_{ABC}, \pi_A \otimes \rho_{BC}\right) && (134)\\
&= \frac{1}{|A|} F\left(\rho_{ABC}, I_A \otimes \rho_{BC}\right) && (135)\\
&\geq \frac{1}{|A|}\left[\mathrm{Tr}\left\{\sqrt{\rho_{ABC}}\sqrt{I_A \otimes \rho_{BC}}\right\}\right]^2. && (136)
\end{aligned}$$

Taking a negative logarithm and letting ϕ_{ABCD} be a purification of ρ_{ABC}, we find that

$$\begin{aligned}
&I_F(A;B|C)_\rho \\
&\leq \log|A| - 2\log \mathrm{Tr}\left\{\sqrt{\rho_{ABC}}\sqrt{I_A \otimes \rho_{BC}}\right\} && (137)\\
&= \log|A| - H_{1/2}(A|BC)_\rho && (138)\\
&= \log|A| + H_{3/2}(A|D)_\rho && (139)\\
&\leq \log|A| + H_{3/2}(A)_\rho && (140)\\
&\leq 2\log|A|. && (141)
\end{aligned}$$

The first equality follows by recognizing that the second term is a conditional Rényi entropy of order 1/2 [37, Definition 3] (see Appendix A for a definition). The second equality follows from a duality relation for this conditional Rényi entropy [37, Lemma 6]. The second inequality is a consequence of the quantum data processing inequality for conditional Rényi entropies [37, Lemma 5] (with the map taken to be a partial trace over system D). The last inequality follows from a dimension bound which holds for any Rényi entropy.

To see that $\Phi_{AB} \otimes \sigma_C$ has $I_F(A;B|C) = 2\log|A|$, we can apply Propositions 25 and 24.

For classical A system, we follow the same steps up to (138), but then apply Lemma 42 in Appendix A to conclude that $H_{1/2}(A|BC) \geq 0$ for a classical A. This gives (131)-(132). To see that $\overline{\Phi}_{XB} \otimes \sigma_C$ has $I_F(X;B|C) = \log|X|$, we apply Proposition 12 and then evaluate

$$\begin{aligned}
&F\left(\overline{\Phi}_{XB}, \tau_X \otimes \overline{\Phi}_B\right) \\
&= \left\|\left(\sum_x \frac{1}{\sqrt{|X|}}|x\rangle\langle x|_X \otimes |x\rangle\langle x|_B\right)\left(\sqrt{\tau_X} \otimes \frac{1}{\sqrt{|X|}} I_B\right)\right\|_1^2 \\
&= \left[\frac{1}{|X|}\left\|\left(\sum_x |x\rangle\langle x|_X \otimes |x\rangle\langle x|_B\right)\left(\sqrt{\tau_X} \otimes I_B\right)\right\|_1\right]^2 \\
&= \left[\frac{1}{|X|}\sum_x \left\||x\rangle\langle x|_X \sqrt{\tau_X}\right\|_1\right]^2 \\
&= \left[\frac{1}{|X|}\sum_x \sqrt{\langle x|\tau|x\rangle}\right]^2 \\
&\leq \frac{1}{|X|}\sum_x \langle x|\tau|x\rangle \\
&= \frac{1}{|X|}. && (142)
\end{aligned}$$

Choosing τ_X maximally mixed then achieves the upper bound, i.e.,

$$\begin{aligned}
\sup_{\tau_X} F\left(\overline{\Phi}_{XB}, \tau_X \otimes \overline{\Phi}_B\right) &= F\left(\overline{\Phi}_{XB}, \pi_X \otimes \overline{\Phi}_B\right) && (143)\\
&= \frac{1}{|X|}. && (144)
\end{aligned}$$

∎

The following proposition gives a simple proof of the main result of [23] when the tripartite state of interest is pure:

Proposition 14 (Approximate q. Markov chain) *The conditional mutual information $I(A;B|C)_\psi$ of a pure tripartite state ψ_{ABC} has the following lower bound:*

$$I(A;B|C)_\psi \geq -\log F(A;B|C)_\psi. \tag{145}$$

Proof. Let φ_D be a pure state on an auxiliary system D, so that $|\psi\rangle_{ABC} \otimes |\varphi\rangle_D$ is a purification of $|\psi\rangle_{ABC}$. Consider the following chain of inequalities:

$$\begin{aligned} I(A;B|C)_\psi &= I(A;B|D)_{\psi\otimes\varphi} && (146)\\ &= I(A;B)_\psi && (147)\\ &\geq -\log F\left(\psi_{AB}, \psi_A \otimes \psi_B\right) && (148)\\ &\geq -\log F(A;B)_\psi && (149)\\ &= -\log F(A;B|D)_{\psi\otimes\varphi} && (150)\\ &= -\log F(A;B|C)_\psi. && (151)\end{aligned}$$

The first equality follows from duality of conditional mutual information. The second follows because system D is product with systems A and B. The first inequality follows from monotonicity of the sandwiched Rényi relative entropies [33, Theorem 7]:

$$\widetilde{D}_\alpha(\rho\|\sigma) \leq \widetilde{D}_\beta(\rho\|\sigma), \tag{152}$$

for states ρ and σ and Rényi parameters α and β such that $0 \leq \alpha \leq \beta$. Recall that the sandwiched Rényi relative entropy is defined as [33, 34]

$$\widetilde{D}_\alpha(\rho\|\sigma) \equiv \frac{2\alpha}{\alpha-1}\log\left\|\sigma^{(1-\alpha)/2\alpha}\rho^{1/2}\right\|_{2\alpha} \tag{153}$$

whenever $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$, and it is equal to $+\infty$ otherwise. The following limit is known [33, 34]:

$$\lim_{\alpha\to 1}\widetilde{D}_\alpha(\rho\|\sigma) = D(\rho\|\sigma), \tag{154}$$

where the quantum relative entropy is defined as $D(\rho\|\sigma) \equiv \text{Tr}\{\rho[\log\rho - \log\sigma]\}$ whenever $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$, and it is equal to $+\infty$ otherwise. To arrive at (148), we apply (152) with the choices $\alpha = 1/2$, $\beta = 1$, $\rho = \psi_{AB}$, and $\sigma = \psi_A \otimes \psi_B$. The second inequality follows by optimizing over states on system A and applying the definition in (120). The second-to-last equality follows from Proposition 12 and the last from Proposition 7. ■

V. GEOMETRIC SQUASHED ENTANGLEMENT

In this section, we formally define the geometric squashed entanglement of a bipartite state ρ_{AB}, and we prove that it obeys the properties claimed in Section II.

Definition 15 (Geometric squashed entanglement) *The geometric squashed entanglement of a bipartite state ρ_{AB} is defined as follows:*

$$E_F^{\text{sq}}(A;B)_\rho \equiv -\frac{1}{2}\log F^{\text{sq}}(A;B)_\rho, \tag{155}$$

where

$$\begin{aligned} &F^{\text{sq}}(A;B)_\rho \\ &\quad\equiv \sup_{\omega_{ABE}}\left\{F(A;B|E)_\rho : \rho_{AB} = \text{Tr}_E\left\{\omega_{ABE}\right\}\right\} \end{aligned} \tag{156}$$

The geometric squashed entanglement can equivalently be written in terms of an optimization over "squashing channels" acting on a purifying system of the original state (cf. [4, Eq. (3)]):

Proposition 16 *Let ρ_{AB} be a bipartite state and let $|\psi\rangle_{ABE'}$ be a fixed purification of it. Then*

$$F^{\text{sq}}(A;B)_\rho = \sup_{\mathcal{S}_{E'\to E}} F(A;B|E)_{\mathcal{S}(\psi)}, \tag{157}$$

where the optimization is over quantum channels $\mathcal{S}_{E'\to E}$.

Proof. We first prove the inequality $F^{\text{sq}}(A;B)_\rho \geq \sup_{\mathcal{S}_{E'\to E}} F(A;B|E)_{\mathcal{S}(\psi)}$. Indeed, for a given purification $\psi_{ABE'}$ and squashing channel $\mathcal{S}_{E'\to E}$, the state $\mathcal{S}_{E'\to E}\left(\psi_{ABE'}\right)$ is an extension of ρ_{AB}. So it follows by definition that

$$F(A;B|E)_{\mathcal{S}(\psi)} \leq F^{\text{sq}}(A;B)_\rho. \tag{158}$$

Since the choice of squashing channel was arbitrary, the first inequality follows.

We now prove the other inequality

$$F^{\text{sq}}(A;B)_\rho \leq \sup_{\mathcal{S}_{E'\to E}} F(A;B|E)_{\mathcal{S}(\psi)}. \tag{159}$$

Let ω_{ABE} be an extension of ρ_{AB}. Let φ_{ABEE_1} be a purification of ω_{ABE}, which is in turn also a purification of ρ_{AB}. Since all purifications are related by isometries acting on the purifying system, we know that there exists an isometry $U^\omega_{E'\to EE_1}$ (depending on ω) such that

$$|\varphi\rangle_{ABEE_1} = U^\omega_{E'\to EE_1}|\psi\rangle_{ABE'}. \tag{160}$$

Furthermore, we know that

$$\begin{aligned} \omega_{ABE} &= \text{Tr}_{E_1}\left\{U^\omega_{E'\to EE_1}\psi_{ABE'}\left(U^\omega_{E'\to EE_1}\right)^\dagger\right\} && (161)\\ &\equiv \mathcal{S}^\omega_{E'\to E}\left(\psi_{ABE'}\right), && (162)\end{aligned}$$

where we define the squashing channel $\mathcal{S}^\omega_{E'\to E}$ from the isometry $U^\omega_{E'\to EE_1}$. So this implies that

$$\begin{aligned} F(A;B|E)_\omega &= F(A;B|E)_{\mathcal{S}^\omega(\psi)} && (163)\\ &\leq \sup_{\mathcal{S}_{E'\to E}} F(A;B|E)_{\mathcal{S}(\psi)}. && (164)\end{aligned}$$

Since the inequality above holds for all extensions, the inequality in (159) follows. ■

The following statement is a direct consequence of Proposition 3:

Corollary 17 *The geometric squashed entanglement is monotone with respect to local operations on both systems A and B:*

$$E_F^{\text{sq}}(A;B)_\rho \geq E_F^{\text{sq}}\left(A';B'\right)_\tau, \tag{165}$$

where $\tau_{A'B'} \equiv \left(\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'}\right)(\rho_{AB})$ *and* $\mathcal{N}_{A\to A'}$ *and* $\mathcal{M}_{B\to B'}$ *are local quantum channels. This is equivalent to*

$$F^{\text{sq}}(A;B)_\rho \leq F^{\text{sq}}\left(A';B'\right)_\tau. \tag{166}$$

Proof. Let ω_{ABE} be an arbitrary extension of ρ_{AB} and let

$$\theta_{A'B'E} \equiv \left(\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'}\right)(\omega_{ABE}). \tag{167}$$

Then by the monotonicity of fidelity of recovery with respect to local quantum operations, we find that

$$F(A;B|E)_\omega \leq F\left(A';B'|E\right)_\theta \leq F^{\text{sq}}\left(A';B'\right)_\tau. \tag{168}$$

Since the inequality holds for an arbitrary extension ω_{ABE} of ρ_{AB}, we can conclude that (166) holds and (165) follows by definition. ■

Proposition 18 *The geometric squashed entanglement is invariant with respect to local isometries, in the sense that*

$$E_F^{\text{sq}}(A;B)_\rho = E_F^{\text{sq}}(A';B')_\sigma, \tag{169}$$

where

$$\sigma_{A'B'} \equiv \left(\mathcal{U}_{A\to A'} \otimes \mathcal{V}_{B\to B'}\right)(\rho_{AB}) \tag{170}$$

and $\mathcal{U}_{A\to A'}$ *and* $\mathcal{V}_{B\to B'}$ *are isometric quantum channels.*

Proof. From Corollary 17, we can conclude that

$$E_F^{\text{sq}}(A;B)_\rho \geq E_F^{\text{sq}}(A';B')_\sigma. \tag{171}$$

Now let $\mathcal{T}^{\mathcal{U}}_{A'\to A}$ and $\mathcal{T}^{\mathcal{V}}_{B'\to B}$ be the quantum channels defined in (67). Again using Corollary 17, we find that

$$\begin{aligned} E_F^{\text{sq}}(A';B')_\sigma &\geq E_F^{\text{sq}}(A;B)_{(\mathcal{T}^{\mathcal{U}}\otimes\mathcal{T}^{\mathcal{V}})(\sigma)} && (172)\\ &= E_F^{\text{sq}}(A;B)_\rho, && (173)\end{aligned}$$

where the equality follows from (68)-(69). ■

Proposition 19 *The geometric squashed entanglement obeys the following classical communication relations:*

$$\begin{aligned} E_F^{\text{sq}}\left(AX_A;B\right)_\rho &\leq E_F^{\text{sq}}\left(AX_A;BX_B\right)_\rho && (174)\\ &= E_F^{\text{sq}}\left(A;BX_B\right)_\rho, && (175)\end{aligned}$$

for a state $\rho_{X_AX_BAB}$ *defined as*

$$\rho_{X_AX_BAB} \equiv \sum_x p_X(x)\,|x\rangle\langle x|_{X_A} \otimes |x\rangle\langle x|_{X_B} \otimes \rho^x_{AB}. \tag{176}$$

These are equivalent to

$$\begin{aligned} F^{\text{sq}}\left(AX_A;B\right)_\rho &\geq F^{\text{sq}}\left(AX_A;BX_B\right)_\rho && (177)\\ &= F^{\text{sq}}\left(A;BX_B\right)_\rho. && (178)\end{aligned}$$

Proof. From monotonicity with respect to local operations, we find that

$$F^{\text{sq}}\left(AX_A;BX_B\right)_\rho \leq F^{\text{sq}}\left(AX_A;B\right)_\rho, \tag{179}$$

$$F^{\text{sq}}\left(AX_A;BX_B\right)_\rho \leq F^{\text{sq}}\left(A;BX_B\right)_\rho. \tag{180}$$

We now give a proof of the following inequality:

$$F^{\text{sq}}\left(A;BX_B\right)_\rho \leq F^{\text{sq}}\left(AX_A;BX_B\right)_\rho. \tag{181}$$

Let

$$\rho_{X_AX_BX_EABE} = \sum_x p_X(x)\,|x\rangle\langle x|_{X_A} \otimes |x\rangle\langle x|_{X_B} \otimes |x\rangle\langle x|_{X_E} \otimes \rho^x_{ABE}, \tag{182}$$

where ρ^x_{ABE} extends ρ^x_{AB}. Observe that $\rho_{X_AX_BX_EABE}$ is an extension of $\rho_{X_AX_BAB}$ and ρ_{X_BABE} is an arbitrary extension of ρ_{X_BAB}. Let $\mathcal{R}_{E\to AE}$ be an arbitrary recovery channel and let $\mathcal{R}_{EX_E\to AX_AEX_E}$ be a channel that copies the value in X_E to X_A and applies $\mathcal{R}_{E\to AE}$ to system E. Consider that

$$F\left(\rho_{ABX_BE},\mathcal{R}_{E\to AE}\left(\rho_{BX_BE}\right)\right) \tag{183}$$
$$=\left[\sum_x p_X(x)\sqrt{F}\left(\rho^x_{ABE},\mathcal{R}_{E\to AE}\left(\rho^x_{BE}\right)\right)\right]^2 \tag{184}$$
$$=F\left(\sum_x p_X(x)\left|xxx\right\rangle\left\langle xxx\right|_{X_AX_BX_E}\otimes\rho^x_{ABE},\sum_x p_X(x)\left|xxx\right\rangle\left\langle xxx\right|_{X_AX_BX_E}\otimes\mathcal{R}_{E\to AE}\left(\rho^x_{BE}\right)\right) \tag{185}$$
$$=F\left(\rho_{AX_ABX_BEX_E},\mathcal{R}_{EX_E\to AX_AEX_E}\left(\rho_{BX_BEX_E}\right)\right) \tag{186}$$
$$\leq F^{\text{sq}}\left(AX_A;BX_B\right)_\rho. \tag{187}$$

The first two equalities are a consequence of the following property of fidelity:

$$\sqrt{F}\left(\tau_{ZS},\omega_{ZS}\right)=\sum_z p_Z\left(z\right)\sqrt{F}\left(\tau^z_S,\omega^z_S\right), \tag{188}$$

where

$$\tau_{ZS}\equiv\sum_z p_Z\left(z\right)\left|z\right\rangle\left\langle z\right|_Z\otimes\tau^z_S, \tag{189}$$

$$\omega_{ZS}\equiv\sum_z p_Z\left(z\right)\left|z\right\rangle\left\langle z\right|_Z\otimes\omega^z_S. \tag{190}$$

The third equality follows from the description of the map $\mathcal{R}_{EX_E\to AX_AEX_E}$ given above. The last inequality is a consequence of the definition of F^{sq} because $\rho_{AX_ABX_BEX_E}$ is a particular extension of ρ_{ABX_BE} and $\mathcal{R}_{EX_E\to AX_AEX_E}$ is a particular recovery map. Given that the chain of inequalities holds for all recovery maps $\mathcal{R}_{E\to AE}$ and extensions ρ_{ABX_BE} of ρ_{ABX_B}, we can conclude that

$$F^{\text{sq}}\left(A;BX_B\right)_\rho\leq F^{\text{sq}}\left(AX_A;BX_B\right)_\rho. \tag{191}$$

■

Remark 20 *The inequalities in Proposition 19 demonstrate that the geometric squashed entanglement is monotone non-increasing with respect to classical communication from Bob to Alice, but not necessarily the other way around. The essential idea in establishing the inequality* $F^{\text{sq}}\left(A;BX_B\right)_\rho\leq F^{\text{sq}}\left(AX_A;BX_B\right)_\rho$ *is to give a copy of the classical data to the party possessing the extension system and to have the recovery map give a copy to Alice. It is unclear to us whether the other inequality* $F^{\text{sq}}\left(AX_A;B\right)_\rho\leq F^{\text{sq}}\left(AX_A;BX_B\right)_\rho$ *could be established, given that the recovery operation only goes from an extension system to Alice, and so it appears that we have no way of giving a copy of this classical data to Bob.*

The following theorem is a direct consequence of Corollary 17 and Proposition 19:

Theorem 21 (1-LOCC monotone) *The geometric squashed entanglement is a 1-LOCC monotone, in the sense that it is monotone non-increasing with respect to local operations and classical communication from Bob to Alice.*

Theorem 22 (Convexity) *The geometric squashed entanglement is convex, i.e.,*

$$\sum_x p_X(x)E^{\text{sq}}_F(A;B)_{\rho^x}\geq E^{\text{sq}}_F(A;B)_{\overline{\rho}}, \tag{192}$$

where

$$\overline{\rho}_{AB}\equiv\sum_x p_X(x)\rho^x_{AB}. \tag{193}$$

Proof. Let ρ^x_{ABE} be an extension of each ρ^x_{AB}, so that

$$\omega_{XABE}\equiv\sum_x p_X(x)\left|x\right\rangle\left\langle x\right|_X\otimes\rho^x_{ABE} \tag{194}$$

is some extension of $\overline{\rho}_{AB}$. Then the definition of $E^{\text{sq}}_F(A;B)_{\overline{\rho}}$ and Proposition 11 give that

$$2E^{\text{sq}}_F(A;B)_{\overline{\rho}}\leq I_F\left(A;B|EX\right)_\omega \tag{195}$$
$$\leq\sum_x p_X(x)I_F(A;B|E)_{\rho^x}. \tag{196}$$

Since the inequality holds independent of each particular extension of ρ^x_{AB}, we can conclude (192). ■

Theorem 23 (Faithfulness) *The geometric squashed entanglement is faithful, in the sense that*

$$E^{\text{sq}}_F(A;B)_\rho=0 \text{ if and only if } \rho_{AB} \text{ is separable.} \tag{197}$$

This is equivalent to

$$F^{\text{sq}}(A;B)_\rho=1 \text{ if and only if } \rho_{AB} \text{ is separable.} \tag{198}$$

Furthermore, we have the following bound holding for all states ρ_{AB}*:*

$$E^{\text{sq}}_F(A;B)_\rho\geq\frac{1}{512\left|A\right|^4}\left\|\rho_{AB}-\text{SEP}(A:B)\right\|^4_1. \tag{199}$$

Proof. We first prove the if-part of the theorem. So, given by assumption that ρ_{AB} is separable, it has a decomposition of the following form:

$$\rho_{AB}=\sum_x p_X(x)\left|\psi_x\right\rangle\left\langle\psi_x\right|_A\otimes\left|\phi_x\right\rangle\left\langle\phi_x\right|_B. \tag{200}$$

Then an extension of the state is of the form

$$\rho_{ABE} = \sum_x p_X(x) \left|\psi_x\right\rangle \left\langle\psi_x\right|_A \otimes \left|\phi_x\right\rangle \left\langle\phi_x\right|_B \otimes \left|x\right\rangle \left\langle x\right|_E . \tag{201}$$

Clearly, if the system A becomes lost, someone who possesses system E could measure it and prepare the state $\left|\psi_x\right\rangle_A$ conditioned on the measurement outcome. That is, the recovery map $\mathcal{R}_{E\to AE}$ is as follows:

$$\mathcal{R}_{E\to AE}(\sigma_E) = \sum_x \left\langle x\right| \sigma_E \left|x\right\rangle \ \left|\psi_x\right\rangle \left\langle\psi_x\right|_A \otimes \left|x\right\rangle \left\langle x\right|_E . \tag{202}$$

So this implies that

$$F\left(\rho_{ABE}, \mathcal{R}_{E\to AE}(\rho_{BE})\right) = 1, \tag{203}$$

and thus $F^{\text{sq}}(A;B)_\rho = 1$.

The only-if-part of the theorem is a direct consequence of the reasoning in [22]. We repeat the argument from [22] here for the convenience of the reader. The reasoning from [22] establishes that the trace distance between ρ_{AB} and the set $\text{SEP}(A:B)$ of separable states on systems A and B is bounded from above by a function of $-1/2 \log F^{\text{sq}}(A;B)_\rho$ and $|A|$. This will then allow us to conclude the only-if-part of the theorem.

Let

$$\varepsilon \equiv -\frac{1}{2} \log F^{\text{sq}}(A;B)_\rho \tag{204}$$

for some bipartite state ρ_{AB} and let

$$\varepsilon_{\omega,\mathcal{R}} \equiv -\frac{1}{2} \log F(\omega_{ABE}, \mathcal{R}_{E\to AE}(\omega_{BE})), \tag{205}$$

for some extension ω_{ABE} and a recovery map $\mathcal{R}_{E\to AE}$. By definition, we have that

$$\varepsilon = \inf_{\omega,\mathcal{R}_{E\to AE}} \varepsilon_{\omega,\mathcal{R}}. \tag{206}$$

Then consider that

$$\varepsilon_{\omega,\mathcal{R}} \geq \frac{1}{8} \left\|\omega_{ABE} - \mathcal{R}_{E\to AE}(\omega_{BE})\right\|_1^2, \tag{207}$$

where the inequality follows from a well known relation between the fidelity and trace distance [38]. Therefore, by defining $\delta_{\omega,\mathcal{R}} = \sqrt{8\varepsilon_{\omega,\mathcal{R}}}$ we have that

$$\begin{aligned} \delta_{\omega,\mathcal{R}} &\geq \left\|\omega_{ABE} - \mathcal{R}_{E\to AE}(\omega_{BE})\right\|_1 && (208) \\ &= \left\|\omega_{ABE} - (\mathcal{R}_{E\to A_2E} \circ \text{Tr}_{A_1})(\omega_{A_1BE})\right\|_1 , && (209) \end{aligned}$$

where the systems A_1 and A_2 are defined to be isomorphic to system A. Now consider applying the same recovery map again. We then have that

$$\delta_{\omega,\mathcal{R}} \geq \left\|(\mathcal{R}_{E\to A_3E} \circ \text{Tr}_{A_2})(\omega_{A_2BE}) - \bigcirc_{i=2}^{3} \left(\mathcal{R}_{E\to A_iE} \circ \text{Tr}_{A_{i-1}}\right)(\omega_{A_1BE})\right\|_1 , \tag{210}$$

which follows from the inequality above and monotonicity of the trace distance with respect to the quantum operation $\mathcal{R}_{E\to A_3E}\circ\text{Tr}_{A_2}$. Combining via the triangle inequality, we find for $k \geq 2$ that

$$\left\|\omega_{ABE} - \bigcirc_{i=2}^{3} \left(\mathcal{R}_{E\to A_iE} \circ \text{Tr}_{A_{i-1}}\right)(\omega_{A_1BE})\right\|_1 \leq 2\delta_{\omega,\mathcal{R}} \leq k\delta_{\omega,\mathcal{R}}. \tag{211}$$

We can iterate this reasoning in the following way: For $j \in \{4,\ldots,k\}$ (assuming now $k \geq 4$), apply the maps $\mathcal{R}_{E\to A_jE}\circ\text{Tr}_{A_{j-1}}$ along with monotonicity of trace distance to establish the following inequalities:

$$\left\|\left[\bigcirc_{i=3}^{j} \left(\mathcal{R}_{E\to A_iE} \circ \text{Tr}_{A_{i-1}}\right)(\omega_{A_2BE})\right] - \left[\bigcirc_{i=2}^{j} \left(\mathcal{R}_{E\to A_iE} \circ \text{Tr}_{A_{i-1}}\right)(\omega_{A_1BE})\right]\right\|_1 \leq \delta_{\omega,\mathcal{R}}. \tag{212}$$

Apply the triangle inequality to all of these to establish the following inequalities for $j \in \{1,\ldots,k\}$:

$$\left\|\omega_{ABE} - \bigcirc_{i=2}^{j} \left(\mathcal{R}_{E\to A_iE} \circ \text{Tr}_{A_{i-1}}\right)(\omega_{A_1BE})\right\|_1 \leq k\delta_{\omega,\mathcal{R}}, \tag{213}$$

with the interpretation for $j = 1$ that there is no map applied. From monotonicity of trace distance with respect to quantum operations, we can then conclude the following inequalities for $j \in \{1,\ldots,k\}$:

$$\left\|\rho_{AB} - \text{Tr}_E\left\{\bigcirc_{i=2}^{j} \left(\mathcal{R}_{E\to A_iE} \circ \text{Tr}_{A_{i-1}}\right)(\omega_{A_1BE})\right\}\right\|_1 \leq k\delta_{\omega,\mathcal{R}}. \tag{214}$$

Let $\gamma_{A_1A_2\cdots A_kBE}$ denote the following state:

$$\gamma_{A_1A_2\cdots A_kBE} \equiv \mathcal{R}_{E\to A_kE}\left(\cdots\left(\mathcal{R}_{E\to A_2E}(\omega_{A_1BE})\right)\right). \tag{215}$$

(See Figure 2 for a graphical depiction of this state.)Then the inequalities in (214) are equivalent to the following

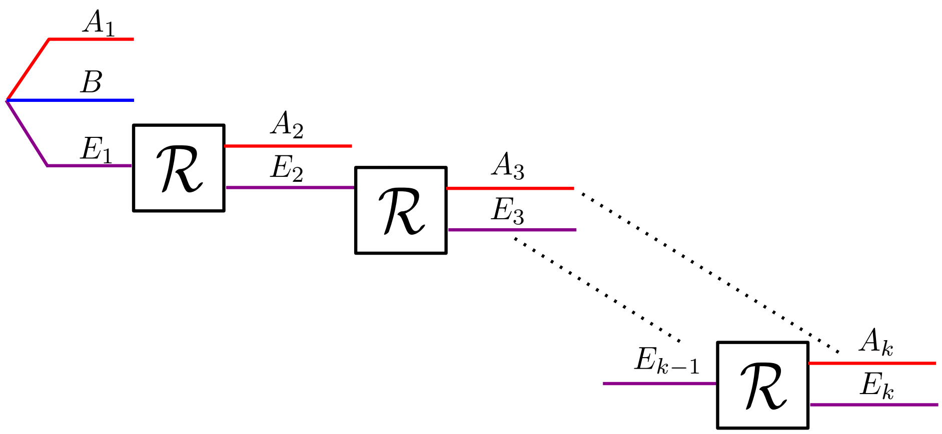

FIG. 2: This figure illustrates the global state after performing a recovery map k times on system E.

inequalities for $j \in \{1 \ldots, k\}$:

$$\left\|\rho_{AB} - \gamma_{A_j B}\right\|_1 \leq k\delta_{\omega,\mathcal{R}}, \tag{216}$$

which are in turn equivalent to the following ones for any permutation $\pi \in S_k$:

$$\left\|\rho_{AB} - \operatorname{Tr}_{A_2 \cdots A_k}\left\{W^{\pi}_{A_1 A_2 \cdots A_k} \gamma_{A_1 A_2 \cdots A_k B}\left(W^{\pi}_{A_1 A_2 \cdots A_k}\right)^{\dagger}\right\}\right\|_1 \leq k\delta_{\omega,\mathcal{R}}, \tag{217}$$

with $W^{\pi}_{A_1 A_2 \cdots A_k}$ a unitary representation of the permutation π. We can then define $\overline{\gamma}_{A_1 \cdots A_k B}$ as a symmetrized version of $\gamma_{A_1 \cdots A_k B}$:

$$\overline{\gamma}_{A_1 \cdots A_k B} \equiv \frac{1}{k!} \sum_{\pi \in S_k} W^{\pi}_{A_1 A_2 \cdots A_k} \gamma_{A_1 \cdots A_k B}\left(W^{\pi}_{A_1 A_2 \cdots A_k}\right)^{\dagger}. \tag{218}$$

The inequalities in (217) allow us to conclude that

$$k\delta_{\omega,\mathcal{R}} \geq \frac{1}{k!} \sum_{\pi \in S_k} \left\|\rho_{AB} - \operatorname{Tr}_{A_2 \cdots A_k}\left\{W^{\pi}_{A_1 A_2 \cdots A_k} \gamma_{A_1 A_2 \cdots A_k B}\left(W^{\pi}_{A_1 A_2 \cdots A_k}\right)^{\dagger}\right\}\right\|_1 \tag{219}$$

$$\geq \left\|\rho_{AB} - \operatorname{Tr}_{A_2 \cdots A_k}\left\{\frac{1}{k!} \sum_{\pi \in S_k} W^{\pi}_{A_1 A_2 \cdots A_k} \gamma_{A_1 A_2 \cdots A_k B}\left(W^{\pi}_{A_1 A_2 \cdots A_k}\right)^{\dagger}\right\}\right\|_1 \tag{220}$$

$$= \left\|\rho_{AB} - \overline{\gamma}_{A_1 B}\right\|_1, \tag{221}$$

where the second inequality is a consequence of the convexity of trace distance. So what the reasoning in [22] accomplishes is to construct a k-extendible state $\overline{\gamma}_{A_1 B}$ that is $k\delta_{\omega,\mathcal{R}}$-close to ρ_{AB} in trace distance.

Following [22], we now recall a particular quantum de Finetti result in [39, Theorem II.7']. Consider a state $\omega_{A_1 \cdots A_k B}$ which is permutation invariant with respect to systems $A_1 \cdots A_k$. Let $\omega_{A_1 \cdots A_n B}$ denote the reduced state on n of the k A systems where $n \leq k$. Then, for large k, $\omega_{A_1 \cdots A_n B}$ is close in trace distance to a convex combination of product states of the form $\int \sigma_A^{\otimes n} \otimes \tau(\sigma)_B \; d\mu(\sigma)$, where μ is a probability measure on the set of mixed states on a single A system and $\{\tau(\sigma)\}_\sigma$ is a family of states parametrized by σ, with the approximation given by

$$\frac{2\left|A\right|^2 n}{k} \geq \left\|\omega_{A_1 \cdots A_n B} - \int \sigma_A^{\otimes n} \otimes \tau(\sigma)_B \; d\mu(\sigma)\right\|_1. \tag{222}$$

Applying this theorem in our context (choosing $n = 1$) leads to the following conclusion:

$$\frac{2\left|A\right|^2}{k} \geq \left\|\overline{\gamma}_{A_1 B} - \int \sigma_{A_1} \otimes \tau(\sigma)_B \; d\mu(\sigma)\right\|_1 \tag{223}$$

$$\geq \left\|\overline{\gamma}_{A_1 B} - \mathrm{SEP}(A_1 : B)\right\|_1, \tag{224}$$

because the state $\int \sigma_{A_1} \otimes \tau(\sigma)_B \; d\mu(\sigma)$ is a particular separable state.

We can now combine (221) and (224) with the triangle inequality to conclude the following bound

$$\|\rho_{AB} - \text{SEP}(A:B)\|_1 \leq \frac{2|A|^2}{k} + k\delta_{\omega,\mathcal{R}}. \tag{225}$$

By choosing k to diverge slower than $\delta_{\omega,\mathcal{R}}^{-1}$, say as $k = |A|\sqrt{2/\delta_{\omega,\mathcal{R}}}$, we obtain the following bound:

$$\begin{aligned} &\|\rho_{AB} - \text{SEP}(A:B)\|_1 \\ &\leq |A| \sqrt{8\delta_{\omega,\mathcal{R}}} && (226) \\ &= (512)^{1/4} |A| \, \varepsilon_{\omega,\mathcal{R}}^{1/4}. && (227) \end{aligned}$$

Since the above bound holds for all extensions and recovery maps, we can obtain the tightest bound by taking an infimum over all of them. By substituting with (204) and (205), we find that

$$\|\rho_{AB} - \text{SEP}(A:B)\|_1 \leq (512)^{1/4} |A| \left(-1/2 \log F^{\text{sq}}(A;B)_\rho\right)^{1/4}, \tag{228}$$

or equivalently

$$\begin{aligned} E_F^{\text{sq}}(A;B)_\rho &= -1/2 \log F^{\text{sq}}(A;B)_\rho && (229) \\ &\geq \frac{1}{512\,|A|^4} \|\rho_{AB} - \text{SEP}(A:B)\|_1^4. && (230) \end{aligned}$$

This proves the converse part of the faithfulness of the geometric squashed entanglement. ■

Proposition 24 (Reduction to geometric measure) *Let ϕ_{AB} be a bipartite pure state. Then*

$$\begin{aligned} E_F^{\text{sq}}(A;B)_\phi &= -\frac{1}{2} \log \sup_{|\varphi\rangle_A} \langle\phi|_{AB} (\varphi_A \otimes \phi_B) |\phi\rangle_{AB} && (231) \\ &= -\log \|\phi_A\|_\infty . && (232) \end{aligned}$$

Proof. Any extension of a pure bipartite state is of the form $\phi_{AB} \otimes \omega_E$, where ω_E is some state. Applying Proposition 12, we find that

$$\begin{aligned} F(A;B|E)_{\phi\otimes\omega} &= F(A;B)_\phi && (233) \\ &= \sup_{\sigma_A} F(\phi_{AB}, \sigma_A \otimes \phi_B) && (234) \\ &= \sup_{|\varphi\rangle_A} \langle\phi|_{AB} (\varphi_A \otimes \phi_B) |\phi\rangle_{AB} . && (235) \end{aligned}$$

The last equality follows due to a convexity argument applied to

$$F(\phi_{AB}, \sigma_A \otimes \phi_B) = \langle\phi|_{AB} \, \sigma_A \otimes \phi_B \, |\phi\rangle_{AB} . \tag{236}$$

Since the equality holds independent of any particular extension of ϕ_{AB}, we obtain (231) upon applying a negative logarithm and dividing by two. The other equality (232) follows because

$$\begin{aligned} &\langle\phi|_{AB} (\varphi_A \otimes \phi_B) |\phi\rangle_{AB} \\ &= \langle\phi|_{AB} (\varphi_A \phi_A \otimes I_B) |\phi\rangle_{AB} && (237) \\ &= \text{Tr}\{|\phi\rangle\langle\phi|_{AB} (\varphi_A \phi_A \otimes I_B)\} && (238) \\ &= \text{Tr}\{\phi_A \varphi_A \phi_A\} && (239) \\ &= \langle\varphi|_A \, \phi_A^2 \, |\varphi\rangle_A . && (240) \end{aligned}$$

Taking a supremum over all unit vectors $|\varphi\rangle_A$ then gives

$$E_F^{\text{sq}}(A;B)_\phi = -\frac{1}{2} \log \left\|\phi_A^2\right\|_\infty , \tag{241}$$

which is equivalent to (232). ■

Proposition 25 (Normalization) *For a maximally entangled state Φ_{AB} of Schmidt rank d,*

$$E_F^{\text{sq}}(A;B)_\Phi = \log d. \tag{242}$$

Proof. This follows directly from (232) of Proposition 24 because $\Phi_A = I_A/d$. ■

Proposition 26 *For a private state $\gamma_{ABA'B'}$ of $\log d$ private bits, the geometric squashed entanglement obeys the following bound:*

$$E_F^{\text{sq}}(AA';BB')_\gamma \geq \log d. \tag{243}$$

Proof. The proof is in a similar spirit to the proof of [40, Proposition 4.19], but tailored to the fidelity of recovery quantity. Recall (43)-(46). Any extension $\gamma_{ABA'B'E}$ of a private state $\gamma_{ABA'B'}$ takes the form:

$$\gamma_{ABA'B'E} = U_{ABA'B'} (\Phi_{AB} \otimes \rho_{A'B'E}) U_{ABA'B'}^\dagger , \tag{244}$$

where $\rho_{A'B'E}$ is an extension of $\rho_{A'B'}$. This is because the state Φ_{AB} is not extendible. Then consider that

$$F(AA';BB'|E)_\gamma = \sup_{\mathcal{R}} F(\gamma_{ABA'B'E}, \mathcal{R}_{E\to AA'E}(\gamma_{BB'E})) , \tag{245}$$

where $\mathcal{R}_{E\to AA'E}$ is a recovery map. From (43)-(46), we can write

$$\gamma_{ABA'B'E} = \frac{1}{d} \sum_{i,j} |i\rangle\langle j|_A \otimes |i\rangle\langle j|_B \otimes V_{A'B'}^i \rho_{A'B'E} \left(V_{A'B'}^j\right)^\dagger , \tag{246}$$

which implies that

$$\gamma_{BB'E} = \frac{1}{d} \sum_i |i\rangle\langle i|_B \otimes \text{Tr}_{\hat{A}'} \left\{ V_{\hat{A}'B'}^i \rho_{\hat{A}'B'E} \left(V_{\hat{A}'B'}^i\right)^\dagger \right\} . \tag{247}$$

So then consider the fidelity of recovery for a particular recovery map $\mathcal{R}_{E\to AA'E}$:

$$F\left(\gamma_{ABA'B'E},\mathcal{R}_{E\to AA'E}\left(\gamma_{BB'E}\right)\right)$$
$$= F\left(U_{ABA'B'}\left(\Phi_{AB}\otimes\rho_{A'B'E}\right)U_{ABA'B'}^{\dagger},\frac{1}{d}\sum_{i}|i\rangle\langle i|_{B}\otimes\mathcal{R}_{E\to AA'E}\left(\operatorname{Tr}_{\hat{A}'}\left\{V_{\hat{A}'B'}^{i}\rho_{\hat{A}'B'E}\left(V_{\hat{A}'B'}^{i}\right)^{\dagger}\right\}\right)\right) \tag{248}$$
$$= F\left(\left(\Phi_{AB}\otimes\rho_{A'B'E}\right),U_{ABA'B'}^{\dagger}\left[\frac{1}{d}\sum_{i}|i\rangle\langle i|_{B}\otimes\mathcal{R}_{E\to AA'E}\left(\operatorname{Tr}_{\hat{A}'}\left\{V_{\hat{A}'B'}^{i}\rho_{\hat{A}'B'E}\left(V_{\hat{A}'B'}^{i}\right)^{\dagger}\right\}\right)\right]U_{ABA'B'}\right), \tag{249}$$

where the second equality follows from invariance of the fidelity with respect to unitaries. Then consider that

$$U_{ABA'B'}^{\dagger}\left[\frac{1}{d}\sum_{i}|i\rangle\langle i|_{B}\otimes\mathcal{R}_{E\to AA'E}\left(\operatorname{Tr}_{\hat{A}'}\left\{V_{\hat{A}'B'}^{i}\rho_{\hat{A}'B'E}\left(V_{\hat{A}'B'}^{i}\right)^{\dagger}\right\}\right)\right]U_{ABA'B'}$$
$$=\left(I_{A}\otimes\sum_{j}|j\rangle\langle j|_{B}\otimes\left(V_{A'B'}^{j}\right)^{\dagger}\right)\left[\frac{1}{d}\sum_{i}|i\rangle\langle i|_{B}\otimes\mathcal{R}_{E\to AA'E}\left(\operatorname{Tr}_{\hat{A}'}\left\{V_{\hat{A}'B'}^{i}\rho_{\hat{A}'B'E}\left(V_{\hat{A}'B'}^{i}\right)^{\dagger}\right\}\right)\right]\times$$
$$\left(I_{A}\otimes\sum_{j'}|j'\rangle\langle j'|_{B}\otimes V_{A'B'}^{j'}\right) \tag{250}$$
$$=\frac{1}{d}\sum_{i}|i\rangle\langle i|_{B}\otimes\left(V_{A'B'}^{i}\right)^{\dagger}\mathcal{R}_{E\to AA'E}\left(\operatorname{Tr}_{\hat{A}'}\left\{V_{\hat{A}'B'}^{i}\rho_{\hat{A}'B'E}\left(V_{\hat{A}'B'}^{i}\right)^{\dagger}\right\}\right)V_{A'B'}^{i}. \tag{251}$$

If we trace over systems $A'B'$, the fidelity only goes up, so consider that the state above becomes as follows after taking this partial trace:

$$\frac{1}{d}\sum_{i}|i\rangle\langle i|_{B}\otimes\operatorname{Tr}_{A'B'}\left\{\left(V_{A'B'}^{i}\right)^{\dagger}\mathcal{R}_{E\to AA'E}\left(\operatorname{Tr}_{\hat{A}'}\left\{V_{\hat{A}'B'}^{i}\rho_{\hat{A}'B'E}\left(V_{\hat{A}'B'}^{i}\right)^{\dagger}\right\}\right)V_{A'B'}^{i}\right\}$$
$$=\frac{1}{d}\sum_{i}|i\rangle\langle i|_{B}\otimes\operatorname{Tr}_{A'B'}\left\{\mathcal{R}_{E\to AA'E}\left(\operatorname{Tr}_{\hat{A}'}\left\{V_{\hat{A}'B'}^{i}\rho_{\hat{A}'B'E}\left(V_{\hat{A}'B'}^{i}\right)^{\dagger}\right\}\right)\right\} \tag{252}$$
$$=\frac{1}{d}\sum_{i}|i\rangle\langle i|_{B}\otimes\operatorname{Tr}_{A'}\left\{\mathcal{R}_{E\to AA'E}\left(\operatorname{Tr}_{\hat{A}'B'}\left\{V_{\hat{A}'B'}^{i}\rho_{\hat{A}'B'E}\left(V_{\hat{A}'B'}^{i}\right)^{\dagger}\right\}\right)\right\} \tag{253}$$
$$=\frac{1}{d}\sum_{i}|i\rangle\langle i|_{B}\otimes\operatorname{Tr}_{A'}\left\{\mathcal{R}_{E\to AA'E}\left(\operatorname{Tr}_{\hat{A}'B'}\left\{\rho_{\hat{A}'B'E}\right\}\right)\right\} \tag{254}$$
$$=\frac{1}{d}\sum_{i}|i\rangle\langle i|_{B}\otimes\operatorname{Tr}_{A'}\left\{\mathcal{R}_{E\to AA'E}\left(\rho_{E}\right)\right\} \tag{255}$$
$$=\pi_{B}\otimes\mathcal{R}_{E\to AE}\left(\rho_{E}\right), \tag{256}$$

where π_B is a maximally mixed state on system B. So an upper bound on (249) is given by

$$F\left(\Phi_{AB}\otimes\rho_{E},\pi_{B}\otimes\mathcal{R}_{E\to AE}\left(\rho_{E}\right)\right)$$
$$\leq F\left(\Phi_{AB},\pi_{B}\otimes\mathcal{R}_{E\to A}\left(\rho_{E}\right)\right) \tag{257}$$
$$=1/d^{2}. \tag{258}$$

Since this upper bound is universal for any recovery map and any extension of the original state, we obtain the following inequality:

$$\sup_{\substack{\gamma_{ABA'B'E}:\\ \gamma_{ABA'B'}=\operatorname{Tr}_{E}\{\gamma_{ABA'B'E}\}}} F\left(AA';BB'|E\right)_{\gamma}\leq 1/d^{2}. \tag{259}$$

After taking a negative logarithm, we recover the statement of the proposition. ■

Proposition 27 (Subadditivity) *Let* $\omega_{A_1B_1A_2B_2}\equiv\rho_{A_1B_1}\otimes\sigma_{A_2B_2}$. *Then*

$$E_{F}^{\text{sq}}\left(A_{1}A_{2};B_{1}B_{2}\right)_{\omega}\leq E_{F}^{\text{sq}}\left(A_{1};B_{1}\right)_{\rho}+E_{F}^{\text{sq}}\left(A_{2};B_{2}\right)_{\sigma}, \tag{260}$$

which is equivalent to

$$F^{\text{sq}}\left(A_{1};B_{1}\right)_{\rho}\cdot F^{\text{sq}}\left(A_{2};B_{2}\right)_{\tau}\leq F^{\text{sq}}\left(A_{1}A_{2};B_{1}B_{2}\right)_{\rho\otimes\tau}. \tag{261}$$

Proof. Let $\rho_{A_1B_1E_1}$ be an extension of $\rho_{A_1B_1}$ and let $\tau_{A_2B_2E_2}$ be an extension of $\tau_{A_2B_2}$. Let $\mathcal{R}^{1}_{E_1\to A_1E_1}$ and

$\mathcal{R}^2_{E_2\to A_2E_2}$ be recovery maps. Then

$$F\left(\rho_{A_1B_1E_1},\mathcal{R}^1_{E_1\to A_1E_1}\left(\rho_{B_1E_1}\right)\right)\cdot F\left(\tau_{A_2B_2E_2},\mathcal{R}^2_{E_2\to A_2E_2}\left(\tau_{B_2E_2}\right)\right)$$

$$= F\left(\rho_{A_1B_1E_1}\otimes\tau_{A_2B_2E_2},\mathcal{R}^1_{E_1\to A_1E_1}\left(\rho_{B_1E_1}\right)\otimes\mathcal{R}^2_{E_2\to A_2E_2}\left(\tau_{B_2E_2}\right)\right) \tag{262}$$

$$\leq \sup_{\omega_{A_1A_2B_1B_2E}}\sup_{\mathcal{R}_{E\to A_1A_2E}}\left\{F\left(\omega_{A_1A_2B_1B_2E},\mathcal{R}_{E\to A_1A_2E}\left(\omega_{B_1B_2E}\right)\right):\rho_{A_1B_1}\otimes\tau_{A_2B_2}=\operatorname{Tr}_E\left\{\omega_{A_1A_2B_1B_2E}\right\}\right\} \tag{263}$$

$$= F^{\rm sq}\left(A_1A_2;B_1B_2\right)_{\rho\otimes\tau}. \tag{264}$$

Since the inequality holds for all extensions $\rho_{A_1B_1E_1}$ and $\tau_{A_2B_2E_2}$ and recovery maps $\mathcal{R}^1_{E_1\to A_1E_1}$ and $\mathcal{R}^2_{E_2\to A_2E_2}$, we can conclude that

$$F^{\rm sq}\left(A_1;B_1\right)_\rho\cdot F^{\rm sq}\left(A_2;B_2\right)_\tau\leq F^{\rm sq}\left(A_1A_2;B_1B_2\right)_{\rho\otimes\tau}. \tag{265}$$

By taking negative logarithms and dividing by 1/2, we arrive at the subadditivity statement for $E_F^{\rm sq}$. ■

Proposition 28 (Continuity) *The geometric squashed entanglement is a continuous function of its input. That is, given two bipartite states ρ_{AB} and σ_{AB} such that $F\left(\rho_{AB},\sigma_{AB}\right)\geq 1-\varepsilon$ where $\varepsilon\in[0,1]$, then the following inequalities hold*

$$\left|F^{\rm sq}(A;B)_\rho-F^{\rm sq}(A;B)_\sigma\right|\leq 8\sqrt{\varepsilon}, \tag{266}$$

$$\left|E_F^{\rm sq}(A;B)_\rho-E_F^{\rm sq}(A;B)_\sigma\right|\leq 4\left|A\right|^2\sqrt{\varepsilon}. \tag{267}$$

Proof. This is a direct consequence of the continuity of fidelity of recovery (Proposition 9). Letting σ_{ABE} be an arbitrary extension of σ_{AB}, [32, Corollary 9] implies that there exists an extension ρ_{ABE} of ρ_{AB} such that

$$F\left(\rho_{ABE},\sigma_{ABE}\right)\geq 1-\varepsilon. \tag{268}$$

By Proposition 9, we can conclude that

$$F(A;B|E)_\sigma\leq F(A;B|E)_\rho+8\sqrt{\varepsilon} \tag{269}$$

$$\leq F^{\rm sq}(A;B)_\rho+8\sqrt{\varepsilon}. \tag{270}$$

Given that the extension of σ_{AB} is arbitrary, we can conclude that

$$F^{\rm sq}(A;B)_\sigma\leq F^{\rm sq}(A;B)_\rho+8\sqrt{\varepsilon}. \tag{271}$$

A similar argument gives that

$$F^{\rm sq}(A;B)_\rho\leq F^{\rm sq}(A;B)_\sigma+8\sqrt{\varepsilon}, \tag{272}$$

from which we can conclude (266). We then obtain (267) by the same line of reasoning that led us to (91). ■

VI. FIDELITY OF RECOVERY FROM A QUANTUM MEASUREMENT

In this section, we propose an alternative measure of quantum correlations, the *surprisal of measurement recoverability*, which follows the original motivation behind the quantum discord [3]. However, our measure has a clear operational meaning in the "one-shot" setting, being based on how well one can recover a bipartite quantum state if one system is measured. We begin by recalling the definition of the quantum discord and proceed from there with the motivation behind the newly proposed measure.

Definition 29 (Quantum discord) *The quantum discord of a bipartite state ρ_{AB} is defined as the difference between the quantum mutual information of ρ_{AB} and the classical correlation [41] of ρ_{AB}:*

$$D(\overline{A};B)_\rho\equiv I(A;B)_\rho-\sup_{\{\Lambda^x\}}I\left(X;B\right)_\sigma \tag{273}$$

$$=\inf_{\{\Lambda^x\}}\left[I(A;B)_\rho-I\left(X;B\right)_\sigma\right], \tag{274}$$

where $\{\Lambda^x\}$ is a POVM with $\Lambda^x\geq 0$ for all x and $\sum_x\Lambda^x=I$ and σ_{XB} is defined as

$$\sigma_{XB}\equiv\sum_x\left|x\right\rangle\left\langle x\right|_X\otimes\operatorname{Tr}_A\left\{\Lambda^x_A\rho_{AB}\right\}. \tag{275}$$

We now recall how to write the quantum discord in terms of conditional mutual information as done explicitly in [42] (see also [12] and [15]). Let $\mathcal{M}_{A\to X}$ denote the following measurement map:

$$\mathcal{M}_{A\to X}\left(\omega_A\right)\equiv\sum_x\operatorname{Tr}\left\{\Lambda^x_A\omega_A\right\}\left|x\right\rangle\left\langle x\right|_X. \tag{276}$$

Using this, we can write (275) as $\sigma_{XB}=\mathcal{M}_{A\to X}(\rho_{AB})$. Now, to every measurement map $\mathcal{M}_{A\to X}$, we can find an isometric extension of it, having the following form:

$$U^{\mathcal{M}}_{A\to XE}\left|\psi\right\rangle_A\equiv\sum_x\left|x\right\rangle_X\left|x,y\right\rangle_E\left\langle\varphi_{x,y}\right|_A\left|\psi\right\rangle_A, \tag{277}$$

where the vectors $\left\{\left|\varphi_{x,y}\right\rangle_A\right\}$ are part of a rank-one refinement of the POVM $\{\Lambda^x_A\}$:

$$\Lambda^x_A=\sum_y\left|\varphi_{x,y}\right\rangle\left\langle\varphi_{x,y}\right|. \tag{278}$$

(In the above, we are taking a spectral decomposition of the operator Λ^x_A.) Thus,

$$\mathcal{M}_{A\to X}\left(\omega_A\right)=\operatorname{Tr}_E\left\{\mathcal{U}^{\mathcal{M}}_{A\to XE}\left(\omega_A\right)\right\}, \tag{279}$$

where

$$\mathcal{U}^{\mathcal{M}}_{A\to XE}(\omega_A) \equiv U^{\mathcal{M}}_{A\to XE}(\omega_A)\left(U^{\mathcal{M}}_{A\to XE}\right)^{\dagger}. \tag{280}$$

Let σ_{XEB} denote the following state:

$$\sigma_{XEB} = \mathcal{U}^{\mathcal{M}}_{A\to XE}(\rho_{AB}). \tag{281}$$

We can use the above development to rewrite the objective function of the quantum discord in (274) as follows:

$$I(A;B)_\rho - I(X;B)_\sigma = I(XE;B)_\sigma - I(X;B)_\sigma \tag{282}$$
$$= I(E;B|X)_\sigma . \tag{283}$$

So this means that we can rewrite the discord in terms of the conditional mutual information as

$$D(\overline{A};B) = \inf_{\{\Lambda^x\}} I(E;B|X)_\sigma , \tag{284}$$

with the state σ_{XEB} understood as described above, as arising from an isometric extension of a measurement map applied to the state ρ_{AB}. We are now in a position to define the surprisal of measurement recoverability:

Definition 30 (Surprisal of meas. recoverability) *We define the following information quantity:*

$$D_F(\overline{A};B)_\rho \equiv \inf_{\{\Lambda^x\}} I_F(E;B|X)_\sigma , \tag{285}$$

where we have simply substituted the conditional mutual information in (284) *with* I_F. *Writing out the right-hand side of* (285) *carefully, we find that*

$$D_F(\overline{A};B) = -\log \sup_{\substack{\mathcal{U}^{\mathcal{M}}_{A\to XE},\\ \mathcal{R}_{X\to XE}}} F\left(\mathcal{U}^{\mathcal{M}}_{A\to XE}(\rho_{AB}), \mathcal{R}_{X\to XE}\left(\mathcal{M}_{A\to X}(\rho_{AB})\right)\right), \tag{286}$$

where $\mathcal{M}_{A\to X}$ *is defined in* (276), $U^{\mathcal{M}}_{A\to XE}$ *is defined in* (277), *and* $\mathcal{U}^{\mathcal{M}}_{A\to XE}$ *is defined in* (280).

This quantity has a similar interpretation as the original discord, as summarized in the following quote from [3]:

> "A vanishing discord can be considered as an indicator of the superselection rule, or — in the case of interest — its value is a measure of the efficiency of einselection. When [the discord] is large for any measurement, a lot of information is missed and destroyed by any measurement on the apparatus alone, but when [the discord] is small almost all the information about [the system] that exists in the [system–apparatus] correlations is locally recoverable from the state of the apparatus."

Indeed, we can rewrite D_F as characterizing how well a bipartite state ρ_{AB} is preserved when an entanglement-breaking channel [30] acts on the A system:

Proposition 31 *For a bipartite state* ρ_{AB}, *we have the following equality:*

$$D_F(\overline{A};B) = -\log \sup_{\mathcal{E}_A} F(\rho_{AB}, \mathcal{E}_A(\rho_{AB})), \tag{287}$$

where the optimization on the right-hand side is over the convex set of entanglement-breaking channels acting on the system A.

Proof. We begin by establishing that

$$\sup_{\substack{\mathcal{U}^{\mathcal{M}}_{A\to XE},\\ \mathcal{R}_{X\to XE}}} F\left(\mathcal{U}^{\mathcal{M}}_{A\to XE}(\rho_{AB}), \mathcal{R}_{X\to XE}\left(\mathcal{M}_{A\to X}(\rho_{AB})\right)\right) \leq \sup_{\mathcal{E}_A} F(\rho_{AB}, \mathcal{E}_A(\rho_{AB})). \tag{288}$$

Let $\mathcal{M}_{A\to X}$ be any measurement map, let $U^{\mathcal{M}}_{A\to XE}$ be an isometric extension for it, and let $\mathcal{R}_{X\to XE}$ be any recovery map. Let $\mathcal{T}_{XE\to A}$ denote the following quantum channel:

$$\mathcal{T}_{XE\to A}(\gamma_{XE}) \equiv \left(U^{\mathcal{M}}\right)^{\dagger}\gamma_{XE}U^{\mathcal{M}} + \mathrm{Tr}\left\{\left(I - U^{\mathcal{M}}\left(U^{\mathcal{M}}\right)^{\dagger}\right)\gamma_{XE}\right\}\sigma_A, \tag{289}$$

where σ_A is some state on the system A. Observe that

$$\left(\mathcal{T}_{XE\to A}\circ\mathcal{U}^{\mathcal{M}}_{A\to XE}\right)(\rho_{AB}) = \rho_{AB}. \tag{290}$$

Then consider that

$$F\left(\mathcal{U}^{\mathcal{M}}_{A\to XE}(\rho_{AB}), \mathcal{R}_{X\to XE}\left(\mathcal{M}_{A\to X}(\rho_{AB})\right)\right)$$
$$\leq F\left(\mathcal{T}_{XE\to A}\left(\mathcal{U}^{\mathcal{M}}_{A\to XE}(\rho_{AB})\right), \mathcal{T}_{XE\to A}\left(\mathcal{R}_{X\to XE}\left(\mathcal{M}_{A\to X}(\rho_{AB})\right)\right)\right) \tag{291}$$
$$= F\left(\rho_{AB}, \mathcal{T}_{XE\to A}\left(\mathcal{R}_{X\to XE}\left(\mathcal{M}_{A\to X}(\rho_{AB})\right)\right)\right) \tag{292}$$
$$\leq \sup_{\mathcal{E}_A} F(\rho_{AB}, \mathcal{E}_A(\rho_{AB})). \tag{293}$$

The first inequality is a consequence of the monotonicity of fidelity with respect to quantum operations and the last follows because any entanglement breaking channel can be written as a concatenation of a measurement followed by a preparation. In the third line, the measurement is $\mathcal{M}_{A\to X}$ and the preparation is $\mathcal{T}_{XE\to A}\circ\mathcal{R}_{X\to XE}$.

We now prove the other inequality:

$$\sup_{\substack{U^{\mathcal{M}}_{A\to XE},\\ \mathcal{R}_{X\to XE}}} F\left(\mathcal{U}^{\mathcal{M}}_{A\to XE}\left(\rho_{AB}\right),\mathcal{R}_{X\to XE}\left(\mathcal{M}_{A\to X}(\rho_{AB})\right)\right) \geq \sup_{\mathcal{E}_A} F\left(\rho_{AB},\mathcal{E}_A\left(\rho_{AB}\right)\right). \quad (294)$$

Let $\mathcal{E}_A$ be any entanglement-breaking channel, which consists of a measurement $\mathcal{M}_{A\to X}$ followed by a preparation $\mathcal{P}_{X\to A}$. Let $U^{\mathcal{M}}_{A\to XE}$ be an isometric extension of the measurement map. Then consider that

$$\begin{aligned} &F(\rho_{AB},\mathcal{E}_A(\rho_{AB})) \\ &= F\left(\rho_{AB},\mathcal{P}_{X\to A}\left(\mathcal{M}_{A\to X}(\rho_{AB})\right)\right) && (295)\\ &= F\left(\mathcal{U}^{\mathcal{M}}_{A\to XE}\left(\rho_{AB}\right),\mathcal{U}^{\mathcal{M}}_{A\to XE}\left(\mathcal{P}_{X\to A}\left(\mathcal{M}_{A\to X}\left(\rho_{AB}\right)\right)\right)\right) && (296)\\ &\leq \sup_{\substack{U^{\mathcal{M}}_{A\to XE},\\ \mathcal{R}_{X\to XE}}} F\left(\mathcal{U}^{\mathcal{M}}_{A\to XE}(\rho_{AB}),\mathcal{R}_{X\to XE}\left(\mathcal{M}_{A\to X}(\rho_{AB})\right)\right), && (297)\end{aligned}$$

where the inequality follows because $\mathcal{U}^{\mathcal{M}}_{A\to XE}\circ\mathcal{P}_{X\to A}$ is a particular recovery map. So (294) follows and this concludes the proof. ■

The proof follows the interpretation given in the quote above: the measurement map $\mathcal{M}_{A\to X}$ is performed on the A system of the state ρ_{AB}, which is followed by a recovery map $\mathcal{P}_{X\to A}$ that attempts to recover the A system from the state of the measuring apparatus. Since the measurement map has a classical output, any recovery map acting on such a classical system is equivalent to a preparation map. So the quantity $D_F(\overline{A};B)$ captures how difficult it is to recover the full bipartite state after some measurement is performed on it, following the original spirit of the quantum discord. However, the quantity $D_F(\overline{A};B)$ defined above has the advantage of being a "one-shot" measure, given that the fidelity has a clear operational meaning in a "one-shot" setting. If $D_F\left(\overline{A};B\right)$ is near to zero, then $F\left(\rho_{AB},\left(\mathcal{P}_{X\to A}\left(\mathcal{M}_{A\to X}\left(\rho_{AB}\right)\right)\right)\right)$ is close to one, so that it is possible to recover the system A by performing a recovery map on the state of the apparatus. Conversely, if $D_F(\overline{A};B)$ is far from zero, then the measurement recoverability is far from one, so that it is not possible to recover system A from the state of the measuring apparatus.

The observation in Proposition 31 leads to the following proposition, which characterizes quantum states with discord nearly equal to zero.

Proposition 32 (Approximate faithfulness) *A bipartite quantum state ρ_{AB} has quantum discord nearly equal to zero if and only if it is an approximate fixed point of an entanglement breaking channel. More precisely, we have the following: If there exists an entanglement breaking channel $\mathcal{E}_A$ and $\varepsilon\in[0,1]$ such that*

$$\left\|\rho_{AB}-\mathcal{E}_A(\rho_{AB})\right\|_1\leq\varepsilon, \quad (298)$$

then the quantum discord $D(\overline{A};B)_\rho$ obeys the following bound

$$D(\overline{A};B)_\rho\leq 4h_2(\varepsilon)+8\varepsilon\log|A|, \quad (299)$$

where $h_2(\varepsilon)$ is the binary entropy with the property that $\lim_{\varepsilon\searrow 0}h_2(\varepsilon)=0$. Conversely, if the quantum discord $D(\overline{A};B)_\rho$ obeys the following bound for $\varepsilon\in[0,1]$:

$$D(\overline{A};B)_\rho\leq\varepsilon, \quad (300)$$

then there exists an entanglement breaking channel $\mathcal{E}_A$ such that

$$\left\|\rho_{AB}-\mathcal{E}_A(\rho_{AB})\right\|_1\leq 2\sqrt{\varepsilon}. \quad (301)$$

Proof. We begin by proving (298)-(299). Since any entanglement breaking channel $\mathcal{E}_A$ consists of a measurement map $\mathcal{M}_{A\to X}$ followed by a preparation map $\mathcal{P}_{X\to A}$, we can write $\mathcal{E}_A=\mathcal{P}_{X\to A}\circ\mathcal{M}_{A\to X}$. Then consider that

$$\begin{aligned} D(\overline{A};B)_\rho &= I(A;B)_\rho-\sup_{\{\Lambda^x\}}I\left(X;B\right)_\sigma && (302)\\ &\leq I(A;B)_\rho-I\left(X;B\right)_{\mathcal{M}(\rho)} && (303)\\ &\leq I(A;B)_\rho-I(A;B)_{\mathcal{P}\circ\mathcal{M}(\rho)} && (304)\\ &= I(A;B)_\rho-I(A;B)_{\mathcal{E}(\rho)} && (305)\\ &\leq 4h_2(\varepsilon)+8\varepsilon\log|A|. && (306)\end{aligned}$$

The first inequality follows because the measurement given by $\mathcal{M}_{A\to X}$ is not necessarily optimal. The second inequality is a consequence of the quantum data processing inequality, in which quantum mutual information is non-increasing with respect to the local operation $\mathcal{P}_{X\to A}$. The last equality follows because $\mathcal{E}_A=\mathcal{P}_{X\to A}\circ\mathcal{M}_{A\to X}$. The last inequality is a consequence of the Alicki-Fannes inequality [9].

We now prove (300)-(301). The Fawzi-Renner inequality $I(A;B|C)_\rho\geq-\log F(A;B|C)_\rho$ which holds for any tripartite state ρ_{ABC} [23], combined with other observations recalled in this section connecting discord with conditional mutual information, gives us that there exists an entanglement breaking channel $\mathcal{E}_A$ such that

$$\begin{aligned} D(\overline{A};B)_\rho &\geq -\log F\left(\rho_{AB},\mathcal{E}_A(\rho_{AB})\right) && (307)\\ &\geq -\log\left(1-\frac{1}{4}\left\|\rho_{AB}-\mathcal{E}_A\left(\rho_{AB}\right)\right\|_1^2\right) && (308)\\ &\geq \frac{1}{4}\left\|\rho_{AB}-\mathcal{E}_A\left(\rho_{AB}\right)\right\|_1^2, && (309)\end{aligned}$$

where the second inequality follows from well known relations between trace distance and fidelity [38] and the last from $-\log(1-x)\geq x$, valid for $x\leq 1$. This is sufficient to conclude (300)-(301). ■

Remark 33 *The main conclusion we can take from Proposition 32 is that quantum states with discord nearly equal to zero are such that they are recoverable after performing some measurement on one share of them, making* *precise the quote from [3] given above. In prior work [43, Lemma 8.12], quantum states with discord exactly equal to zero were characterized as being entirely classical on the system being measured, but this condition is perhaps too restrictive for characterizing states with discord approximately equal to zero.*

Remark 34 *In prior work, discord-like measures of the following form have been widely considered throughout the literature [31]:*

$$\inf_{\chi_{AB}\in CQ} \Delta\left(\rho_{AB}, \chi_{AB}\right), \tag{310}$$

$$\inf_{\chi_{AB}\in CC} \Delta\left(\rho_{AB}, \chi_{AB}\right), \tag{311}$$

where CQ and CC are the respective sets of classical-quantum and classical-classical states and Δ is some suitable (pseudo-)distance measure such as relative entropy, trace distance, or Hilbert-Schmidt distance. The larger message of Proposition 32 is that it seems more reasonable from the physical perspective argued in this section and in the original discord paper [3] to consider discord-like measures of the following form:

$$\inf_{\mathcal{E}_A} \Delta\left(\rho_{AB}, \mathcal{E}_A\left(\rho_{AB}\right)\right), \tag{312}$$

$$\inf_{\mathcal{E}_A,\mathcal{E}_B} \Delta\left(\rho_{AB}, \left(\mathcal{E}_A\otimes\mathcal{E}_B\right)\left(\rho_{AB}\right)\right), \tag{313}$$

where the optimization is over the convex set of entanglement breaking channels and Δ is again some suitable (pseudo-)distance measure as mentioned above. One can understand these measures as being a special case of the proposed measures in [44], but we stress here that we arrived at them independently through the line of reasoning given in this section.

We now establish some properties of the surprisal of measurement recoverability:

Proposition 35 (Local isometric invariance) *$D_F(\overline{A};B)_\rho$ is invariant with respect to local isometries, in the sense that*

$$D_F(\overline{A};B)_\rho = D_F(\overline{A'};B')_\sigma, \tag{314}$$

where

$$\sigma_{A'B'} \equiv \left(\mathcal{U}_{A\to A'}\otimes\mathcal{V}_{B\to B'}\right)\left(\rho_{AB}\right) \tag{315}$$

and $\mathcal{U}_{A\to A'}$ and $\mathcal{V}_{B\to B'}$ are isometric CPTP maps.

Proof. Let $\mathcal{E}_A$ be some entanglement-breaking channel. Let $\mathcal{T}^{\mathcal{U}}_{A'\to A}$ and $\mathcal{T}^{\mathcal{V}}_{B'\to B}$ denote the CPTP maps defined in (67). Then from invariance of fidelity with respect to isometries and the identities in (68)-(69), we find that

$$\begin{aligned}
&F(\rho_{AB},\mathcal{E}_A(\rho_{AB})) \\
&= F\left(\left(\mathcal{U}_{A\to A'}\otimes\mathcal{V}_{B\to B'}\right)(\rho_{AB}), \left(\mathcal{U}_{A\to A'}\otimes\mathcal{V}_{B\to B'}\right)(\mathcal{E}_A(\rho_{AB}))\right) &(316)\\
&= F\left(\left(\mathcal{U}_{A\to A'}\otimes\mathcal{V}_{B\to B'}\right)(\rho_{AB}), \left(\mathcal{U}_{A\to A'}\circ\mathcal{E}_A\circ\mathcal{T}^{\mathcal{U}}_{A'\to A}\right)\left[\left(\mathcal{U}_{A\to A'}\otimes\mathcal{V}_{B\to B'}\right)(\rho_{AB})\right]\right) &(317)\\
&\leq \sup_{\mathcal{E}_{A'}} F\left(\left(\mathcal{U}_{A\to A'}\otimes\mathcal{V}_{B\to B'}\right)(\rho_{AB}), \mathcal{E}_{A'}\left(\left(\mathcal{U}_{A\to A'}\otimes\mathcal{V}_{B\to B'}\right)(\rho_{AB})\right)\right). &(318)
\end{aligned}$$

Since the inequality is true for any entanglement breaking channel $\mathcal{E}_A$, we find after applying a negative logarithm that

$$D_F(\overline{A};B)_\rho \geq D_F\left(\overline{A};B\right)_{(\mathcal{U}\otimes\mathcal{V})(\rho)}. \tag{319}$$

Now consider that

$$\begin{aligned}
&F\left(\left(\mathcal{U}_{A\to A'}\otimes\mathcal{V}_{B\to B'}\right)(\rho_{AB}), \mathcal{E}_{A'}\left[\left(\mathcal{U}_{A\to A'}\otimes\mathcal{V}_{B\to B'}\right)(\rho_{AB})\right]\right) \\
&= F\left(\mathcal{U}_{A\to A'}(\rho_{AB}), \left(\mathcal{E}_{A'}\circ\mathcal{U}_{A\to A'}\right)(\rho_{AB})\right) &(320)\\
&\leq F\left(\left(\mathcal{T}^{\mathcal{U}}_{A'\to A}\circ\mathcal{U}_{A\to A'}\right)(\rho_{AB}), \left(\mathcal{T}^{\mathcal{U}}_{A'\to A}\circ\mathcal{E}_{A'}\circ\mathcal{U}_{A\to A'}\right)(\rho_{AB})\right) &(321)\\
&= F\left(\rho_{AB}, \left(\mathcal{T}^{\mathcal{U}}_{A'\to A}\circ\mathcal{E}_{A'}\circ\mathcal{U}_{A\to A'}\right)(\rho_{AB})\right) &(322)\\
&\leq \sup_{\mathcal{E}_A} F\left(\rho_{AB}, \mathcal{E}_A\left(\rho_{AB}\right)\right). &(323)
\end{aligned}$$

Since the inequality is true for any entanglement breaking channel $\mathcal{E}_{A'}$, we find after applying a negative logarithm that

$$D_F(\overline{A};B)_\rho \leq D_F\left(\overline{A};B\right)_{(\mathcal{U}\otimes\mathcal{V})(\rho)}, \tag{324}$$

which gives the statement of the proposition. ■

Proposition 36 (Exact faithfulness) *The surprisal of measurement recoverability* $D_F\left(\overline{A};B\right)_\rho$ *is equal to zero if and only if* ρ_{AB} *is a classical-quantum state, having the form*

$$\rho_{AB}=\sum_x p_X(x)\left|x\right\rangle\left\langle x\right|_A\otimes\rho_B^x, \tag{325}$$

for some orthonormal basis $\{|x\rangle\}$*, probability distribution* $p_X(x)$*, and states* $\{\rho_B^x\}$*.*

Proof. Suppose that the state is classical-quantum. Then it is a fixed point of the entanglement breaking map $\sum_x |x\rangle\langle x|_A(\cdot)|x\rangle\langle x|_A$, so that the fidelity of measurement recovery is equal to one and its surprisal is equal to zero. On the other hand, suppose that $D_F(\overline{A};B)_\rho=0$. Then this means that there exists an entanglement breaking channel $\mathcal{E}_A$ of which ρ_{AB} is a fixed point (since $F\left(\rho_{AB},\mathcal{E}_A\left(\rho_{AB}\right)\right)=1$ is equivalent to $\rho_{AB}=\mathcal{E}_A\left(\rho_{AB}\right)$), and furthermore, applying the fixed point projection

$$\overline{\mathcal{E}_A}\equiv\lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K}\mathcal{E}_A^k \tag{326}$$

leaves ρ_{AB} invariant. The map $\overline{\mathcal{E}_A}$ has been characterized in [45, Theorem 5.3] to be an entanglement breaking channel of the following form:

$$\overline{\mathcal{E}_A}\left(\cdot\right)=\sum_x \operatorname{Tr}\left\{\Lambda_A^x\left(\cdot\right)\right\}\sigma_A^x, \tag{327}$$

where the states σ_A^x have orthogonal support, $\Lambda_A^x\geq 0$, and $\sum_x\Lambda_A^x=I$. Applying this channel to ρ_{AB} then gives a classical-quantum state, and since ρ_{AB} is invariant with respect to the action of this channel to begin with, it must have been classical-quantum from the start. ■

Proposition 37 (Dimension bound) *The surprisal of measurement recoverability obeys the following dimension bound:*

$$D_F(\overline{A};B)_\rho\leq\log|A|\,, \tag{328}$$

or equivalently,

$$\sup_{\mathcal{E}_A}F\left(\rho_{AB},\mathcal{E}_A\left(\rho_{AB}\right)\right)\geq\frac{1}{|A|}. \tag{329}$$

Proof. The idea behind the proof is to consider an entanglement breaking channel $\mathcal{E}_A$ that completely dephases the system A. Let $\overline{\Delta}_A$ denote such a channel, so that

$$\overline{\Delta}_A\left(\cdot\right)\equiv\sum_i|i\rangle\left\langle i\right|_A\left(\cdot\right)|i\rangle\left\langle i\right|_A, \tag{330}$$

where $\{|i\rangle_A\}$ is some orthonormal basis spanning the space for the A system. Let a spectral decomposition of ρ_{AB} be given by

$$\rho_{AB}=\sum_x p_X(x)\left|\psi^x\right\rangle\left\langle\psi^x\right|_{AB}, \tag{331}$$

where p_X is a probability distribution and $\{|\psi^x\rangle_{AB}\}$ is a set of pure states. We then find that

$$\begin{aligned} &D_F(\overline{A};B)_\rho\\ &\leq-\log F\left(\rho_{AB},\overline{\Delta}_A(\rho_{AB})\right) &(332)\\ &=-2\log\sqrt{F}\left(\rho_{AB},\overline{\Delta}_A\left(\rho_{AB}\right)\right) &(333)\\ &\leq\sum_x p_X(x)\left[-2\log\sqrt{F}\left(\psi_{AB}^x,\overline{\Delta}_A\left(\psi_{AB}^x\right)\right)\right] &(334)\\ &=\sum_x p_X(x)\left[-\log\left\langle\psi^x\right|_{AB}\overline{\Delta}_A\left(\psi_{AB}^x\right)\left|\psi^x\right\rangle_{AB}\right] &(335)\\ &=\sum_x p_X(x)\left[-\log\sum_i\left[\left\langle i\right|_A\psi_A^x\left|i\right\rangle_A\right]^2\right] &(336)\\ &\leq\log|A|\,. &(337)\end{aligned}$$

The second inequality follows from joint concavity of the root fidelity $\sqrt{F}$ and convexity of $-\log$. The last equality is a consequence of a well known expression for the entanglement fidelity of a channel (see, e.g., [46, Theorem 9.5.1]). The last inequality follows by recognizing

$$-\log\sum_i\left[\left\langle i\right|_A\psi_A^x\left|i\right\rangle_A\right]^2 \tag{338}$$

as the Rényi 2-entropy of the probability distribution $\langle i|_A\psi_A^x|i\rangle_A$ and from the fact that all Rényi entropies are bounded from above by the logarithm of the alphabet size of the distribution, which in this case is $\log|A|$. ■

Given that the Rényi 2-entropy of the marginal of a bipartite pure state is an entanglement measure, the following proposition demonstrates that the surprisal of measurement recoverability reduces to an entanglement measure when evaluated for pure states.

Proposition 38 (Pure states) *Let* ψ_{AB} *be a pure state. Then*

$$D_F(\overline{A};B)_\psi=-\log\operatorname{Tr}\left\{\psi_A^2\right\}. \tag{339}$$

Proof. For a pure state ψ_{AB}, consider that

$$\begin{aligned} &D_F(\overline{A};B)_\psi\\ &=-\log\sup_{\mathcal{E}_A}F\left(\psi_{AB},\mathcal{E}_A\left(\psi_{AB}\right)\right) &(340)\\ &=-\log\sup_{\substack{|\phi_x\rangle,|\varphi_x\rangle:\\ \||\phi_x\rangle\|_2=1,\\ \sum_x|\varphi_x\rangle\langle\varphi_x|=I}}\sum_x\left|\left\langle\varphi_x\right|_A\psi_A\left|\phi_x\right\rangle_A\right|^2, &(341)\end{aligned}$$

where the optimization in the second line is over pure-state vectors $|\phi_x\rangle$ and corresponding measurement vectors $|\varphi_x\rangle$ satisfying $\sum_x|\varphi_x\rangle\langle\varphi_x|=I$. The second equality follows from the formula for entanglement fidelity (see, e.g., [46, Theorem 9.5.1]) and the fact that the Kraus operators of an entanglement-breaking channel have the special form $\{|\phi_x\rangle\langle\varphi_x|\}_x$ with $|\phi_x\rangle$ pure quantum states and $\sum_x|\varphi_x\rangle\langle\varphi_x|=I$ [30]. Consider for all such choices, we have that

$$\sum_x |\langle\varphi_x|_A \psi_A |\phi_x\rangle_A|^2$$
$$= \sum_x \langle\varphi_x|_A \psi_A |\phi_x\rangle \langle\phi_x|_A \psi_A |\varphi_x\rangle_A \quad (342)$$
$$\leq \sum_x \langle\varphi_x|_A \psi_A^2 |\varphi_x\rangle_A \quad (343)$$
$$= \sum_x \mathrm{Tr}\left\{|\varphi_x\rangle \langle\varphi_x|_A \psi_A^2\right\} \quad (344)$$
$$= \mathrm{Tr}\left\{\psi_A^2\right\}, \quad (345)$$

where the inequality follows from the operator inequality $|\phi_x\rangle\langle\phi_x|_A \leq I_A$. However, a particular choice of Kraus operators $\{|\phi_x\rangle\langle\varphi_x|\}_x$ is $\{|\psi^x\rangle\langle\psi^x|\}_x$, where $\{|\psi^x\rangle\}_x$ is the set of eigenvectors of ψ_A. For this choice, we find that

$$\sum_x |\langle\psi^x|_A \psi_A |\psi^x\rangle_A|^2 = \mathrm{Tr}\left\{\psi_A^2\right\}, \quad (346)$$

so that we can conclude that

$$\sup_{|\phi_x\rangle,|\varphi_x\rangle:\sum_x|\varphi_x\rangle\langle\varphi_x|=I} \sum_x |\langle\varphi_x|_A \psi_A |\phi_x\rangle_A|^2 = \mathrm{Tr}\left\{\psi_A^2\right\}. \quad (347)$$

■

Proposition 39 (Normalization) *The surprisal of measurement recoverability $D_F\left(\overline{A};B\right)_\Phi$ is equal to $\log d$ for a maximally entangled state Φ_{AB} with Schmidt rank d.*

Proof. This is a direct consequence of Proposition 38 and the fact that $\Phi_A = I_A/d$. ■

Proposition 40 (Monotonicity) *The surprisal of measurement recoverability is monotone with respect to quantum operations on the unmeasured system, i.e.,*

$$D_F(\overline{A};B)_\rho \geq D_F\left(\overline{A};B'\right)_\sigma, \quad (348)$$

where $\sigma_{AB'} \equiv \mathcal{N}_{B\to B'}(\rho_{AB})$.

Proof. Intuitively, this follows because it is easier to recover from a measurement when the state is noisier to begin with. Indeed, let $\mathcal{E}_A$ be an entanglement breaking channel. Then

$$F(\rho_{AB},\mathcal{E}_A(\rho_{AB})) \leq F\left(\sigma_{AB'},\mathcal{E}_A\left(\sigma_{AB'}\right)\right) \quad (349)$$
$$\leq \sup_{\mathcal{E}_A} F\left(\sigma_{AB'},\mathcal{E}_A\left(\sigma_{AB'}\right)\right), \quad (350)$$

where the first inequality is due to the fact that $\mathcal{E}_A$ commutes with $\mathcal{N}_{B\to B'}$ and monotonicity of the fidelity with respect to quantum channels. Since the inequality holds for all entanglement breaking channels, we can conclude that

$$\sup_{\mathcal{E}_A} F\left(\rho_{AB},\mathcal{E}_A\left(\rho_{AB}\right)\right) \leq \sup_{\mathcal{E}_A} F\left(\sigma_{AB'},\mathcal{E}_A\left(\sigma_{AB'}\right)\right). \quad (351)$$

Taking a negative logarithm gives the statement of the proposition. ■

With a proof nearly identical to that for Proposition 28, we find that $D_F(\overline{A};B)_\rho$ is continuous:

Proposition 41 (Continuity) *$D_F(\overline{A};B)$ is a continuous function of its input. That is, given two bipartite states ρ_{AB} and σ_{AB} such that $F\left(\rho_{AB},\sigma_{AB}\right) \geq 1-\varepsilon$ where $\varepsilon \in [0,1]$, then the following inequalities hold*

$$\left|\sup_{\mathcal{E}_A} F\left(\rho_{AB},\mathcal{E}_A\left(\rho_{AB}\right)\right) - \sup_{\mathcal{E}_A} F\left(\sigma_{AB},\mathcal{E}_A\left(\sigma_{AB}\right)\right)\right| \leq 8\sqrt{\varepsilon}, \quad (352)$$

$$\left|D_F(\overline{A};B)_\rho - D_F\left(\overline{A};B\right)_\sigma\right| \leq |A|\, 8\sqrt{\varepsilon}. \quad (353)$$

VII. MULTIPARTITE FIDELITY OF RECOVERY

We state here that it is certainly possible to generalize the fidelity of recovery to the multipartite setting. Indeed, by following the same line of reasoning mentioned in the introduction (starting from the Rényi conditional multipartite information [12, Section 10.1] and understanding the $\alpha = 1/2$ quantity in terms of several Petz recovery maps), we can define the multipartite fidelity of recovery for a multipartite state $\rho_{A_1\cdots A_l C}$ as follows:

$$F\left(A_1;A_2;\cdots;A_l|C\right)_\rho = \sup_{\substack{\mathcal{R}^1_{C\to A_1C},\\ \ldots,\\ \mathcal{R}^{l-1}_{C\to A_{l-1}C}}} F\left(\rho_{A_1\cdots A_lC}, \mathcal{R}^1_{C\to A_1C}\circ\cdots\circ\mathcal{R}^{l-1}_{C\to A_{l-1}C}\left(\rho_{A_lC}\right)\right).$$

The interpretation of this quantity is as written: systems A_1 through A_{l-1} of the state $\rho_{A_1\cdots A_lC}$ are lost, and one attempts to recover them one at a time by performing a sequence of recovery maps on system C alone. We can then define a quantity analogous to the multipartite conditional mutual information as follows:

$$I_F(A_1;A_2;\cdots;A_l|C)_\rho \equiv -\log F(A_1;A_2;\cdots;A_l|C)_\rho, \quad (354)$$

and one can easily show along the lines given for the bipartite case that the resulting multipartite quantity is non-negative, monotone with respect to local operations, and obeys a dimension bound.

We leave it as an open question to develop fully a multipartite geometric squashed entanglement, defined by replacing the conditional multipartite mutual information in the usual definition [47] with I_F given above. One could also explore multipartite versions of the surprisal of measurement recoverability.

VIII. CONCLUSION

We have defined the fidelity of recovery $F(A;B|C)_\rho$ of a tripartite state ρ_{ABC} to quantify how well one can recover the full state on all three systems if system A is lost and the recovery map can act only on system C. By taking the negative logarithm of the fidelity of recovery, we obtain an entropic quantity $I_F(A;B|C)_\rho$ which obeys nearly all of the entropic relations that the conditional mutual information does. The quantities $F(A;B|C)_\rho$ and $I_F(A;B|C)_\rho$ are rooted in our earlier work on seeking out Rényi generationalizations of the conditional mutual information [12]. Whereas we have not been able to prove that all of the aforementioned properties hold for the Rényi conditional mutual informations from [12], it is pleasing to us that it is relatively straightforward to show that these properties hold for $I_F(A;B|C)_\rho$.

Another contribution was to define the geometric squashed entanglement $E_F^{\text{sq}}(A;B)_\rho$, inspired by the original squashed entanglement measure from [4]. We proved that $E_F^{\text{sq}}(A;B)_\rho$ is a 1-LOCC monotone, is invariant with respect to local isometries, is faithful, reduces to the well known geometric measure of entanglement [25, 26] when the bipartite state is pure, normalized on maximally entangled states, subadditive, and continuous. The geometric squashed entanglement could find applications in "one-shot" scenarios of quantum information theory, since it is fundamentally a one-shot measure based on the fidelity. (The fidelity is said to be a "one-shot" quantity because it has an operational meaning in terms of a single experiment: it is the probability with which a purification of one state could pass a test for being a purification of the other state.)

Our final contribution was to define the surprisal of measurement recoverability $D_F(\overline{A};B)_\rho$, a quantum correlation measure having physical roots in the same vein as those used to justify the definition of the quantum discord. We showed that it is non-negative, invariant with respect to local isometries, faithful on classical-quantum states, obeys a dimension bound, and is continuous. Furthermore, we used this quantity to characterize quantum states with discord nearly equal to zero, finding that such states are approximate fixed points of an entanglement breaking channel.

From here, there are several interesting lines of inquiry to pursue. It is clear that generally $I_F(A;B|C) \neq I_F(B;A|C)$: can we quantify how large the gap can be between them in general? Can we prove a stronger chain rule for the fidelity of recovery? If something along these lines holds, it might be helpful in establishing that the geometric squashed entanglement is monogamous or additive. (At the very least, we can say that geometric squashed entanglement is additive with respect to pure states, given that it reduces to the geometric measure of entanglement which is clearly additive by inspecting (232).) Is it possible to improve our continuity bounds to attain "asymptotic continuity"? Can one show that geometric squashed entanglement is nonlockable [40]? Preliminary evidence from considering the strongest known locking schemes from [48] suggests that it might not be lockable. We are also interested in a multipartite geometric squashed entanglement, but we face similar challenges as those discussed in [49] for establishing its faithfulness.

Acknowledgements. We are grateful to Gerardo Adesso, Mario Berta, Todd Brun, Marco Piani, and Masahiro Takeoka for helpful discussions about this work. KS acknowledges support from NSF Grant No. CCF-1350397, the DARPA Quiness Program through US Army Research Office award W31P4Q-12-1-0019, and the Graduate School of Louisiana State University for the 2014-2015 Dissertation Year Fellowship. MMW acknowledges support from the APS-IUSSTF Professorship Award in Physics, startup funds from the Department of Physics and Astronomy at LSU, support from the NSF under Award No. CCF-1350397, and support from the DARPA Quiness Program through US Army Research Office award W31P4Q-12-1-0019.

Appendix A: Appendix

Given a state ρ, a positive semidefinite operator σ, and $\alpha \in [0,1) \cup (1,\infty)$, we define the Rényi relative entropy as

$$D_\alpha(\rho\|\sigma) \equiv \frac{1}{\alpha-1} \log \operatorname{Tr}\left\{\rho^\alpha \sigma^{1-\alpha}\right\}, \tag{A1}$$

whenever the support of ρ is contained in the support of σ, and it is equal to $+\infty$ otherwise. The conditional Rényi entropy of a bipartite state ρ_{AB} is defined as

$$H_\alpha(A|B)_\rho \equiv -D_\alpha(\rho_{AB}\|I_A \otimes \rho_B). \tag{A2}$$

(See, e.g., [37] for details of these definitions.) This leads us to the following lemma:

Lemma 42 *Let ρ_{XB} be a classical-quantum state, i.e., such that*

$$\rho_{XB} \equiv \sum_x p(x) |x\rangle \langle x|_X \otimes \rho_B^x, \tag{A3}$$

where $p(x)$ is a probability distribution and $\{\rho_B^x\}$ is a set of quantum states. For $\alpha \in [0,1) \cup (1,2]$,

$$H_\alpha(X|B) \geq 0. \tag{A4}$$

Proof. This follows because it is possible to copy classical information, and conditional entropy increases with respect to the loss of a classical copy. Consider the following extension of ρ_{XB}:

$$\rho_{X\hat{X}B} \equiv \sum_x p(x) |x\rangle \langle x|_X \otimes |x\rangle \langle x|_{\hat{X}} \otimes \rho_B^x. \tag{A5}$$

Then we show that $H_\alpha(X|\hat{X}B) = 0$ for all $\alpha \in [0,1) \cup (1,\infty)$. Indeed, consider that

$$H_\alpha(X|\hat{X}B)$$
$$= \frac{1}{1-\alpha} \log \operatorname{Tr} \left\{ \left(\sum_x p(x) \left|x\right\rangle \left\langle x\right|_X \otimes \left|x\right\rangle \left\langle x\right|_{\hat{X}} \otimes \rho_B^x \right)^\alpha \left[I_X \otimes \left(\sum_{x'} p\left(x'\right) \left|x'\right\rangle \left\langle x'\right|_{\hat{X}} \otimes \rho_B^{x'} \right)^{1-\alpha} \right] \right\} \tag{A6}$$
$$= \frac{1}{1-\alpha} \log \operatorname{Tr} \left\{ \sum_x p^\alpha\left(x\right) \left|x\right\rangle \left\langle x\right|_X \otimes \left|x\right\rangle \left\langle x\right|_{\hat{X}} \otimes \left(\rho_B^x\right)^\alpha \sum_{x'} p^{1-\alpha}\left(x'\right) I_X \otimes \left|x'\right\rangle \left\langle x'\right|_{\hat{X}} \otimes \left(\rho_B^{x'}\right)^{1-\alpha} \right\} \tag{A7}$$
$$= \frac{1}{1-\alpha} \log \operatorname{Tr} \left\{ \sum_x p(x) \left|x\right\rangle \left\langle x\right|_X \otimes \left|x\right\rangle \left\langle x\right|_{\hat{X}} \otimes \rho_B^x \right\} \tag{A8}$$
$$= 0. \tag{A9}$$

Then for $\alpha \in [0,1) \cup (1,2]$, the desired inequality is a consequence of quantum data processing [37, Lemma 5]:

$$H_\alpha(X|B) \geq H_\alpha(X|\hat{X}B) = 0. \tag{A10}$$

■

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