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Rigid Connections on the Projective Line with Elliptic Toral Singularities

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This thesis is dedicated to my mother and father,

Zahra Bakhtavar,

and

Mohammad Mehdi Azhang.

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Abstract

We generalize two studies of rigid G-connections on $\mathbb{P}^1$ which have an irregular singularity at origin and a regular singularity at infinity with unipotent monodromy: one is the work of Kamgarpour-Sage which classifies rigid homogeneous Coxeter G-connections with slope $\frac{r}{h}$, where h is the Coxeter number of G, and the other is the work of Chen, which proves the existence of rigid homogeneous elliptic regular G-connections with slope $\frac{1}{m}$, where m is an elliptic number for G. In our work, similar to Chen, we look for rigid homogeneous elliptic regular G-connections, but we allow the slope to have a numerator greater than 1. However, for the present purpose, we essentially restrict to the case where G is either Sp_{2n} or SO_{2n+1}. For Sp_{2n}, we show that Kamgarpour-Sage connections and Chen connections exhaust all the rigid homogeneous elliptic regular connections; furthermore, we write down explicit matrices for all rigid homogeneous elliptic regular Sp_{2n}-connections. For SO_{2n+1}-connections, having introduced the notion of "generalized Chen connections," we classify all rigid connections of this type. We conjecture that any rigid homogeneous elliptic regular SO_{2n+1}-connection is in this form. We also construct an explicit example of a rigid non-Coxeter SO_9-connection with slope $\frac{3}{4}$, as well as an explicit construction of Chen SO_{2n+1}-connections with slope $\frac{1}{2}$.

Chapter 1. Introduction

1.1. Some historical remarks on rigid connections

A connection can be thought as a system of first order homogeneous linear differential equations. In particular, connections on complex algebraic curves, like $\mathbb{P}^1$, $\mathbb{C}$, or $\mathbb{G}_m = \mathbb{C} \smallsetminus \{0\}$, are equivalent to system of first order *ordinary* differential equations over a complex variable. In the algebraic setting, one might be interested primarily in systems of differential equations whose matrix have entries in the field of rational functions (resp. formal Laurent series); these can be interpreted as GL_n-connections (resp. formal GL_n-connections), as the coefficient matrix lies in $\mathfrak{gl}_n(\mathbb{C}(t))$ (resp. $\mathfrak{gl}_n(\mathcal{K})$.) One can further require that the matrix of the connection lie in the Lie algebra, $\mathfrak{g}$, of an algebraic group G; more precisely, the matrix be in $\mathfrak{g}(\mathbb{C}(t))$ (resp. $\mathfrak{g}(\mathcal{K})$. This gives us, what is known as, G-connections (formal G-connections. They have attracted a lot of attention, not only for their own sake, but also for their role in the geometric Langlands program; in fact, "when G is reductive, G-connections on complex algebraic curves are precisely the *geometric langlands parameters*" [2], [38].

In this dissertation we study connections on $\mathbb{P}^1$ with exactly two singularities: one is an *irregular singularity* and the other is a *regular singularity* with *unipotent monodromy*. For convenience, and following [2], we require that the irregular singularity be at 0, and the regular singularity be at ∞. The motivation for choosing singularities with this specific prescription roots in the work of Frenkel and Gross, [3], in construction of an important *cohomologically rigid* connection on the trivial G-bundle on $\mathbb{P}^1 \smallsetminus \{0, \infty\}$. The role of 0 and ∞ in our work is swapped, compared to the convention adopted by Frenkel and Gross in their original paper, but this is merely a matter of convenience.

The study of rigidity, however, goes long before the Langlands program was invented; it can be tracked in Riemann's work on Gauss's hypergeometric functions, and as mentioned in [2], Riemann's insight was studying Gauss's functions by the corresponding *local system*, i.e. locally constant sheaf, on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. An important idea behind his work is that hypergeometric local system is *physically rigid*, This means that the local system is determined up to isomorphism once the local monodromies at the three singular points are known.

The subject of rigid local systems, or equivalently rigid connections, has been revisited by Katz [29], who has defined the notion of cohomological rigidity. He has proved that physical and cohomological rigidity for l-adic local systems are equivalent. However, in the context of G-connections, as Yun explains in [7], it is only known that physical rigidity implies cohomological rigidity. In this dissertation, we only discuss cohomological rigidity, and for brevity we simply refer to it as "rigidity".

1.2. Our goal, and the organization of the dissertation

As mentioned above, We will look at a specific family of connections on $\mathbb{P}^1$ with an irregular singularity at the origin and a regular singularity with unipotent monodromy at infinity. The connections that are discussed below, all have the same type of singularities, upto a interchanging the role of 0 and ∞.

The Frenkel-Gross connections, which are rigid *homogeneous Coxeter* connections on $\mathbb{P}^1$ with *slope* $\frac{1}{m}$ has been recently generalized in two directions:

(1) rigid homogeneous Coxeter connections on $\mathbb{P}^1$ with slope $\frac{r}{m}$, where r is not nec-

essarily 1 (by Kamgarpour-Sage, [2])

(2) rigid homogeneous *elliptic regular* connections on $\mathbb{P}^1$ with slope $\frac{1}{m}$ (by Chen, [5], building on unpublished work of Yun)

We try to generalize (1) and (2). Our eventual goal is to find all rigid elliptic regular connections with slope $\frac{r}{m}$. however, in this work we still have some restrictions:

(A): we only look at homogeneous ones. However, it is good to know that, a recent joint work of Kulkarni-Livesay-Matherne-Nguyen-Sage, [30], shows that, for SL_n, the homogeneity does not really impose a restriction.

(B): We restrict ourselves, at least for SO_{2n+1}, to what we have called "generalized Chen connections".

(C): We are only considering some of the simple groups: SL_n, Sp_{2n}, and SO_{2n+1}.

The rigid homogeneous elliptic regular connections for SL_n, are already found by the work of Kamgarpour-Sage [2].

So, our contribution, up to now, is only for Sp_{2n}, and SO_{2n+1}:

1) As we will see, in theorem 4.4, for Sp_{2n}, all the rigid homogeneous elliptic regular connections, are either Coxeter (those studied by Kamgarpour-Sage), or they are the "Chen connections". In addition, we write down *explicit* matrices for all rigid homogeneous elliptic regular Sp_{2n}-connections; see example 4.1, and theorem 4.4.

2) For SO_{2n+1}, we will see that there are more of them; we will list all the rigid generalized Chen connections in theorem5.12, but we do not construct them. However, we will see an explicit example of that for SO_9 with slope $\frac{3}{4}$; see example 5.7. Also, we give an explicit construction for Chen connections with slope $\frac{1}{2}$ in section 5.3.

We bring some background theory about connections and rigidity in chapters 2 and

3. We will define Chen connections and generalized Chen connections in 3.3.

Chapter 2. Background on Connections

Our goal in this chapter is to review the definition of connections on vector bundles on $\mathbb{P}^1$, as well as G-connections, where G is a reductive group. Our focus is on the following simple groups: SL_n, Sp_{2n}, and SO_{2n+1}. Of course, it will be also useful to look at the case $G = \mathrm{GL}_n$. We will start with a review of all these groups and their Lie algebras. For a reader who is familiar with these definitions, it might be still helpful to notice our choice of basis, and our matrix presentations of elements of these groups and algebras. After defining connections, we discuss two important invariants: slope and irregularity, both of which are measures to describe how irregular a connection is at a singularity. Then, we will define what we mean by "homogeneous elliptic toral" connections, which are formal connections, as well as "homogeneous elliptic regular" connections, which are global connections on $\mathbb{G}_m = \mathbb{C} \setminus \{0\}$.

2.1. Review of groups and algebras; fixing the choice of basis

In this section, we review the construction of classical groups, in particular, Sp_{2n} and SO_{2n+1}, as well as their Lie algebras, and we mention some facts about them that we are going to use in the subsequent chapters. A reader familiar with this topic can easily skip this section; however it is important to notice our convention about the choice of basis.

Let V be a vector space over $\mathbb{C}$. As usual, $\mathrm{GL}(V)$, stands for the group of all linear automorphisms of V, and $\mathfrak{gl}(V)$ stands for the vector space of all endomorphisms of V. We know that $\mathfrak{gl}(V)$ is the Lie algebra of $\mathrm{GL}(V)$. Also, recall that:

$$\mathrm{SL}(V) := \{g \in \mathrm{GL}(V) \quad | \quad \det(g) = 1\}$$

is a subgroup of $\mathrm{GL}(V)$, and

$$\mathfrak{sl}(V) := \{x \in \mathfrak{gl}(V) \mid \operatorname{tr}(x) = 0\}$$

is a Lie subalgebra of $\mathfrak{gl}(V)$; Furthermore, $\mathfrak{sl}(V)$ is the Lie algebra of $\mathrm{SL}(V)$.

Let B denote the vector space consisting of bilinear forms on V. One can see that $\mathrm{GL}(V)$ acts on B by:

$$(g.\beta)(v_1, v_2) := \beta(g^{-1}v_1, g^{-1}v_2)$$

where $g \in \mathrm{GL}(V)$, $\beta \in B$, and v_1 and v_2 are in V.

This induces an action of $\mathfrak{gl}(V)$ on B; namely, for x in $\mathfrak{gl}(V)$, we have:

$$(x.\beta)(v_1, v_2) = -\beta(xv_1, v_2) - \beta(v_1, xv_2)$$

Given a bilinear form β, the set

$$G_\beta := \{g \in \mathrm{GL}(V) \mid g.\beta = \beta\}$$

is a subgroup of $\mathrm{GL}(V)$.

In a parallel way, the set

$$\mathfrak{g}_\beta := \{x \in \mathfrak{gl}(V) \mid x.\beta = 0\}$$

is a Lie subalgebra of $\mathfrak{gl}(V)$. One can show that $\mathfrak{g}_\beta$ is indeed the Lie algebra of G_β, as the notation suggests.

We are ready to state a proposition that will be very important for our construction of rigid connections.

Proposition 2.1. *Let β be a bilinear form on V, and $\mathfrak{g}_\beta$ be the corresponding Lie subalgebra of $\mathfrak{gl}(V)$. Let x be in $\mathfrak{g}_\beta$, and r be a positive odd integer. Then x^r lies in $\mathfrak{g}_\beta$.*

Proof. By definition of $\mathfrak{g}_\beta$, for all vectors v_1 and v_2 in V, we have $(x.\beta)(v_1, v_2) = 0$. Therefore,

$$\beta(xv_1, v_2) = -\beta(v_1, xv_2)$$

So, we can write

$$\beta(x^3v_1, v_2) = \beta(x(x^2v_1), v_2) = -\beta(x^2v_1, xv_2)$$

$$= +\beta(xv_1, x^2v_2) = -\beta(v_1, x^3v_2)$$

This implies that $(x^3.\beta)(v_1, v_2) = 0$, for all v_1 and v_2, that is, $x^3 \in \mathfrak{g}_\beta$. The rest of the proof follows by an easy induction on r.

□

Choosing various bilinear forms, one can get different Lie groups and Lie algebras. Two nice family of bilinear forms are the symmetric and the skew-symmetric ones; as defined in the following:

Given a bilinear form β, we say that:

1) β is symmetric, if $\beta(v_1, v_2) = \beta(v_2, v_1)$

2) β is skew-symmetric, if $\beta(v_1, v_2) = -\beta(v_2, v_1)$.

The nondegenerate skew-symmetric bilinear forms give rise to the "symplectic group" and the "symplectic algebra," which we discuss in section 2.1.1. The nondegenerate symmetric bilinear forms give rise to the "orthogonal groups and algebras," which we see in section 2.1.2.

Fixing a basis $(e_1, \ldots, e_m)$ for V, any bilinear form β on V, can be presented by a matrix $[\beta]$, where $[\beta]_{ij} := B(e_i, e_j)$.

For a vector $v \in V$, lets denote the matrix presentation of v in the chosen basis by the column vector $[v]$. Then we have:

$$\beta(v_1, v_2) = [v_1]^T \; [\beta] \; [v_2]$$

. where $[v_1]^T$ denotes the transpose of $[v_1]$.

It turns out that β is nondegenarte if and only if $[\beta]$ is non-singular. Moreover, β is symmetric (resp. skew-symmetric) if and only if $[\beta]$ is symmetric (resp. skew-symmetric).

Let $g \in \mathrm{GL}(V)$, and let $[g]$ denote the matrix presentation of g in the basis $(e_1, \ldots, e_m)$. Then the matrix presentation of β in the basis $g.e_1, \ldots, g.e_m$, can be denoted by $[\beta]_g$, and we we have:

$$[\beta]_g = [g]^T \; [\beta] \; [g]$$

where $[g]^T$ denotes the transpose of $[g]$.

having the basis $(e_1, \ldots, e_m)$ fixed, we can describe elements of $\mathfrak{gl}(V)$ by matrices, and in this situation we denote it by $\mathfrak{gl}_m$. Consequently elements of $\mathfrak{g}_\beta$, in matrix form, have the following description:

$$\mathfrak{g}_\beta = \{X \in \mathfrak{gl}_{2n} \;\; | \;\; [\beta]X + X^T[\beta] = 0\}$$

Now we are ready to study the symplectic and orthogonal algebras. We will use some of the terminologies and conventions used in Neal Livesay's thesis, [18].

2.1.1. The symplectic group and the symplectic algebra

A symplectic $\mathbb{C}$-vector space (V, ω) is a vector space V equipped with a nondegenerate skew-symmetric bilinear form ω. One can show that any symplectic vector space is even dimensional and it has a symplectic basis:

$$(e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n)$$

with: $\omega(e_i, f_j) = -\omega(f_j, e_i) = \delta_{ij}$, and $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$.

This is a standard basis that is often used in the literature; however it is more convenient for our purpose to work with a new basis obtained by reordering f_i's in the opposite direction:

$$(e_1, e_2, \ldots, e_n, f_n, f_{n-1}, \ldots, f_1)$$

This basis will be fixed throughout our work. In this basis, ω is represented by the matrix

$$J = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}$$

where K is the $n \times n$ matrix with 1's on the ***anti-diagonal*** :

$$K = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

If A and B are square matrices, and B is obtained by reflecting A with respect to the anti-diagonal, we write:

$$B = A^{\ddagger}$$

For exmple:

$$A = \begin{pmatrix} 1 & 3 \\ 7 & -2 \end{pmatrix} \qquad A^{\ddagger} = \begin{pmatrix} -2 & 3 \\ 7 & 1 \end{pmatrix}$$

and we say that B is the ***anti-transpose*** of A.

The group G_ω and the Lie algebra $\mathfrak{g}_\omega$ associated to β are called the **symplectic group** and the **symplectic algebra**, and they are denoted, respectively, by $\mathrm{Sp}(V)$ and $\mathfrak{sp}(V)$.

In our chosen basis for V, the symplectic algebra, which we now denote by $\mathfrak{sp}_{2n}$, consists of the following matrices:

$$\mathfrak{sp}_{2n} = \{X \in \mathfrak{gl}_{2n} | JX + X^T J = 0\}$$

It follows that elements X of $\mathfrak{sp}_{2n}$ have the following form:

$$\left(\begin{array}{c|c} A & B \\ \hline C & -A^\dagger \end{array}\right)$$

where B and C are ***anti-diagonally symmetric*** $n \times n$ matrices, that is,

$$B^\dagger = B \quad \text{and} \quad C^\dagger = C$$

and A is an arbitrary $n \times n$ matrix.

Remark 2.2. *To see this, one can use the following observation:*

$$K^2 = I$$

and, for any $n \times n$ matrix A,

$$KAK = (A^T)^\dagger = (A^\dagger)^T$$

Block-diagonal v.s. symplectic-block-diagonal:

As it will be noticed, we need two different block diagonal construction, which we define in the following. Notice that given a matrix $A \in \mathfrak{gl}_k$, we can embed it into $\mathfrak{sp}_{2k}$, in the following way:

$$\tilde{A} = \left(\begin{array}{c|c} A & \\ \hline & -A^{\dagger} \end{array}\right)$$

This suggests the following definition:

Definition 2.3. *Suppose that A and B are matrices in $\mathfrak{gl}_k$. We define the* ***block-diagonal combination*** *of A and B to be the following matrix in $\mathfrak{sp}_{4k}$:*

$$\left(\begin{array}{c|c|c|c} A & & & \\ \hline & B & & \\ \hline & & -B^{\dagger} & \\ \hline & & & -A^{\dagger} \end{array}\right)$$

We also define the ***block-diagonal embedding of** A **into** $\mathfrak{sp}_{2km}$* *to be the following matrix:*

$$\left(\begin{array}{c|c|c|c|c|c} A & & & & & \\ \hline & \ddots & & & & \\ \hline & & A & & & \\ \hline & & & -A^{\dagger} & & \\ \hline & & & & \ddots & \\ \hline & & & & & -A^{\dagger} \end{array}\right)$$

Definition 2.4. *Suppose that A and B are matrices in $\mathfrak{sp}_{2k}$, written in the form:*

$$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \qquad B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$$

We define the ***symplectic-block-diagonal combination*** *of A and B, to be the* *following matrix in $\mathfrak{sp}_{4k}$:*

$$A = \left(\begin{array}{c|c|c|c} A_{11} & & & A_{12} \\ \hline & B_{11} & B_{12} & \\ \hline & B_{21} & B_{22} & \\ \hline A_{21} & & & A_{22} \end{array}\right)$$

We also define the ***symplectic-block-diagonal embedding of*** A ***into*** $\mathfrak{sp}_{2km}$ *to be the following matrix:*

$$A = \left(\begin{array}{c|c|c|c|c|c} A_{11} & & & & & A_{12} \\ \hline & \ddots & & & \iddots & \\ \hline & & A_{11} & A_{12} & & \\ \hline & & A_{21} & A_{22} & & \\ \hline & \iddots & & & \ddots & \\ \hline A_{21} & & & & & A_{22} \end{array}\right)$$

2.1.2. The orthogonal group SO_{2n+1} and the orthogonal algebra $\mathfrak{so}_{2n+1}$

Let V be a k-vector space of dimension $2n+1$. Let $(e_1, \ldots, e_{2n+1})$ be an arbitrary basis for V, and Let β be the bilinear form on V whose presentation in the chosen basis, is given by the following symmetric matrix of size $2n+1$:

$$S = \begin{pmatrix} & & & & & & 1 \\ & & & & & \iddots & \\ & & & & 1 & & \\ & & & 1 & & & \\ & & 1 & & & & \\ & \iddots & & & & & \\ 1 & & & & & & \end{pmatrix}$$

It follows that β is a nondegenarate symmetric bilinear form.

Remark 2.5. *All nondegenerate symmetric matrices define equivalent bilinear forms over* $\mathbb{C}$*, or any algebraically closed field; however, it is most convenient for our purposes to work with the matrix* S *above.*

As mentioned before, we define the **orthogonal group** $\mathrm{SO}(V)$ (resp. **orthogonal algebra** $\mathfrak{so}(V)$) to be the group G_β (resp. Lie algebra $\mathfrak{g}_\beta$).

In our fixed basis, we denote the matrix presentation of this group and algebra, by SO_{2n+1} and $\mathfrak{so}_{2n+1}$, respectively. We have:

$$\mathfrak{so}_{2n+1} = \{X \in \mathfrak{gl}_{2n} \mid SX + X^T S = 0\}$$

$$= \{X \in \mathfrak{gl}_{2n} \mid SXS^{-1} = -X^T\}$$

Notice that, as mentioned in remark 2.2, we have:

$$S^2 = I$$

and, for any $n \times n$ matrix A,

$$SAS = (A^\dagger)^T$$

So:

$$\mathfrak{so}_{2n+1} = \{X \in \mathfrak{gl}_{2n} \mid X^{\dagger} = -X\}$$

that is, $\mathfrak{so}_{2n+1}$, in our basis, consists of ***anti-diagonally skew-symmetric*** matrices.

2.1.3. Some important facts about $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n+1}$

Let us record the special case of proposition 2.1 for our favorite algebras in this dissertation:

Corollary 2.6. *Let A be a matrix in $\mathfrak{sp}_{2n}$ (resp. $\mathfrak{so}_{2n+1}$), and let r be a positive odd integer. Then A^r lies in $\mathfrak{sp}_{2n}$ (resp. $\mathfrak{so}_{2n+1}$).*

Proposition 2.7. *Let F be a field and A be a matrix in $\mathfrak{so}_{2n+1}(F)$, and let $p(\lambda)$ be the characteristic polynomial of A. Then $p(\lambda)$ is in the form:*

$$-\lambda^{2n+1} + a_{2n-1}\lambda^{2n-1} + \cdots + a_3\lambda^3 + a_1\lambda$$

that is, the coefficients of p in the terms with even powers of λ are all zero.

Corollary 2.8. *A be a matrix in $\mathfrak{so}_{2n+1}(\mathbb{C})$, and let $p(\lambda)$ be the characteristic polynomial of A. Then $p(\lambda)$ is in the form:*

$$p(\lambda) = -\lambda \prod_{i=1}^{n} (\lambda^2 - b_i)$$

where b_i's are complex numbers.

Proof. By Proposition 2.7, we can write $p(\lambda) = -\lambda g(\lambda^2)$, where g is a monic polynomial of degree n. Since $\mathbb{C}$ is algebraically closed, g completely factors into linear terms and we get the desired form for $p(\lambda)$.

□

2.2. Connections on $\mathbb{P}^1$

2.2.1. Connections on vector bundles over $\mathbb{P}^1$

We will begin with a standard definition of connections on algebraic curves, which, in fact, can be applied to a much broader context. At the first glance, the definition might seem a little forbidding, but one can give a more intuitive definition in terms of covariant derivatives; we refer the reader to the rich literature on this subject. However, in this dissertation we will not explicitly use either of these coordinate-free definitions of connections, rather, we will focus on the presentation of a connection in a given trivialization. As we will see, this will lead us to think about a connection as an equivalent class of differential equations. The reader might be aware that this implies that connections can also be defined as a locally constant sheaf. Another way to think about connections is viewing them as D-modules. As we will see, this will be important in defining rigidity for connections. However, it follows that for the connections of our interest in this dissertation, there is a simple criterion for rigidity, that can help us avoid getting into details of the subject of D-modules.

Let X be a complex algebraic curve and B be a vector bundle of rank n over X. As usual, we denote the space of sections on B by $\Gamma(B)$ and the cotangent space of X by $T^*(X)$.

Definition 2.9. *A connection ∇ on B is a $\mathbb{C}$-linear map*

$$\nabla : \Gamma(B) \to \Gamma(B \otimes T^*(X))$$

such that the Leibniz rule is satisfied; that is, given f, an algebraic function on X, and s, *a section of B, we have:*

$$\nabla(f.s) = f.\nabla(s) + s \otimes df$$

We may also say ∇ is a connection on X, whenever B is understood from the context.

For simplicity, we assume that B is trivializable. Once a trivialization ϕ of B is chosen, a section s of B can be represented by a column vector:

$$\begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

where f_i's are algebraic functions on X. Then, ∇ defines an operator which inputs a section s of B and outputs a column vector consisting of one-forms on X:

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \mapsto \begin{pmatrix} df_1 \\ df_2 \\ \vdots \\ df_n \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} dt$$

where $a_{ij}(t)$'s are algebraic functions on X, that is elements of $\mathcal{O}(X)$, where $\mathcal{O}$ is the structure sheaf of X. As an example, when X is the complex plane, $a_{ij}(t)$'s are polynomials in t. When $X = \mathbb{G}_m$, the punctured complex plane, $a_{ij}(t)$'s are in $\mathbb{C}[t, t^{-1}]$, and when $X = \mathbb{P}^1$, then $a_{ij}(t)$'s are constant functions.

Let's define the matrix A with $A_{ij} = a_{ij}$, then the above operator can be written as:

$$d + A.dt$$

Of course, the matrix A depends on the trivialization ϕ, and to emphasize that we write it as A_ϕ. Now we can consider ∇ as the following differential operator:

$$\nabla = d + A_\phi.dt$$

Sometimes we might abuse notation and write $\nabla = d + A.dt$

Any other trivialization of B can be written as $g.\phi$, where g is in $\mathrm{GL}_n(\mathcal{O}(X))$, the group of invertible $n \times n$ matrices with entries in $\mathcal{O}(X)$. One can check that in the new trivialization, ∇ can be written as:

$$\nabla = d + \left(gA_\phi g^{-1} - \frac{dg}{dt}g^{-1}\right).dt$$

In other words, we have:

$$A_{g.\phi} = gA_\phi g^{-1} - \frac{dg}{dt}g^{-1}$$

It can be easily checked that:

$$A_{(g_2 g_1).\phi} = A_{g_2.(g_1.\phi)}$$

In other words, this defines a left action of $\mathrm{GL}_n(\mathcal{O}(X))$ on $\mathfrak{gl}_n(\mathcal{O}(X))$, which is called the gauge action. So, the set of connections on B can be viewed as

$$\mathfrak{gl}_n(\mathcal{O}(X))/\,\mathrm{GL}_n(\mathcal{O}(X))$$

via the gauge action.

Moreover, we say that two connections are isomorphic, or equivalent, if their matrices in the same trivialization are related by the gauge action via an element of $\mathrm{GL}_n(\mathcal{O}(X))$.

The horizontal sections of B are defined to be the set of sections s of B such that $\nabla s = 0$. With the notations as above, we can think of the horizontal sections as the solutions of the following differential equation:

$$\begin{pmatrix} \frac{df_1}{dt} \\ \frac{df_2}{dt} \\ \vdots \\ \frac{df_n}{dt} \end{pmatrix} = - \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

From this perspective, we can think of a connection on X as the equivalent class of a system of linear ordinary differential equations in the complex variable t. To explain this, we notice that choosing a different trivialization for B, we get a different differential equation, whose matrix is related to the matrix A by the gauge action. we consider the differential equations, obtained in this way, as being equivalent.

2.2.2. Meromorphic connections on $\mathbb{P}^1$

A meromorphic connection on $\mathbb{P}^1$ is defined as a connection on a Zariski open subset X of $\mathbb{P}^1$. Recall that if X is a Zarisky open subset of $\mathbb{P}^1$, then $X = \mathbb{P}^1$, or $X = \mathbb{P}^1 \setminus \{x_1, \dots x_n\}$, where $\{x_1, \dots, x_n\}$ is a finite set of points in $\mathbb{P}^1$. The algebraic functions on X, $\mathcal{O}(X)$, is the set of rational functions whose singularities are only at a subset of $\{x_1, \dots, x_n\}$.

In the same fashion, we can define a meromorphic connection on $\mathbb{C}$. Without loss of generality, one can view a meromorphic connection on $\mathbb{P}^1$ with at least one singularity as a meromorphic connection on $\mathbb{C}$, by applying an automorphism of $\mathbb{P}^1$, putting one of the singularities at infinity. For example, a meromorphic connection on $\mathbb{P}^1$ with one singularity, can be viewed as a connection on the complex plane, and a meromorphic connection with two singularities can be viewed as connection on $\mathbb{G}_m = \mathbb{C} \setminus \{0\}$, the punctured complex plane.

Given an $n \times n$ matrix with entries in rational functions, the equivalent class of the operator $d + A.dt$ defines a meromorphic connection on $\mathbb{P}^1$, or more explicitly, on a trivial vector bundle of rank n on X, where $X \subset \mathbb{P}^1$ is the set obtained by removing the singular points of A from $\mathbb{P}^1$. The gauge action is by elements of $\mathrm{GL}_n(\mathcal{O}(X))$.

Suppose that $\nabla = d + A.dt$ is a meromorphic connection on $\mathbb{P}^1$ which has a singularity at 0. Then ∇ can be rewritten as:

$$\nabla = d + (At).\frac{dt}{t} = d + \tilde{A}.\frac{dt}{t}$$

where $\tilde{A} = At$.

It turns out that writing ∇ in this form, make the analysis of the singularity easier. We call the matrix $\tilde{A}$ the "connection matrix" of ∇ in the given trivialization ϕ, and denote it by $[\nabla]$, or $[\nabla]_\phi$, to emphasize the trivialization.

Remark 2.10. *In the literature,* $[\nabla]$ *usually denotes the one-form* $A.dt = \tilde{A}.\frac{dt}{t}$*, and is called the connection one-form or the connection matrix; but it is more convenient for us to simply consider* $\tilde{A}$ *as the connection matrix, and denote it by* $[\nabla]$.

In general, given a meromorphic connection $\nabla = d + A.dt$ on $\mathbb{P}^1$ and a singularity x of ∇, with $x \neq \infty$, we can define the variable $z = t - x$, and write ∇ in a similar form:

$$\nabla = d + \tilde{B}.\frac{dz}{z}$$

where $\tilde{B}(z) = A(z + x).z$, which can be viewed as a rational function in the variable z, that is, an element of $\mathbb{C}(z)$.

If $x = \infty$, we can define $u = \frac{1}{t}$, and we can still write ∇ in the form

$$\nabla = d + \tilde{C}.\frac{du}{u}$$

where $\tilde{C}(u) = -\frac{A(\frac{1}{u})}{u} \in \mathbb{C}(u)$.

Now, suppose that the set of singular points of ∇ is $\{x_1, \dots, x_n\}$. By the observation above, we can assume that $x_1 = 0$, and write $\nabla = d + \tilde{A}\frac{dt}{t}$. We can find a neighborhood U of 0 in the Euclidean topology of the complex plane such that U contains no singularities of $\tilde{A}$ except 0. This means that the $\tilde{a_{ij}}$'s, the entries of $\tilde{A}$ can be written as Laurent series around 0, all of which converge in $U \setminus \{0\}$. It follows that there is a punctured neighborhood of 0, where the entries of $\tilde{A}^{-1}$ can also be written as convergent Laurent's series.

By ignoring the issue of convergence of Laurent's series, we get the field of "formal Laurent's series," which is denoted by $\mathbb{C}((t))$. As the notation suggests, one can show that $\mathbb{C}((t))$ is equivalent to the quotient field of the ring of "formal power series," $\mathbb{C}[[t]]$.

Throughout this dissertation we denote $\mathbb{C}((t))$ by $\mathcal{K}$, and $\mathbb{C}[[t]]$ by $\mathcal{O}$. Recall that $\mathcal{O}$ is the ring of integers of $\mathcal{K}$. The notation for the ring of integers should not be confused with the notation used for the structure sheaf of X, which is denoted by $\mathcal{O}_X$.

2.2.3. Formal Connections

A *formal connection* ∇ can be defined to be the equivalent class of an expression:

$$d + [\nabla]\frac{dt}{t}$$

where $[\nabla]$ is a matrix with entries in the field of Laurent series, $\mathcal{K}$, and the equivalent classes are defined via the gauge action of $\mathrm{GL}(\mathcal{K})$. Like before, we abuse notation and simply write $\nabla = d + [\nabla]\frac{dt}{t}$.

From a geometric point of view, this can be considered as a connection on a vector bundle $\mathcal{E}$ of rank n on the affine scheme $X = \mathrm{Spec}(\mathcal{K})$. As a topological space, X is a singleton set, $\{p\}$, where $p = \{0\}$ is the only prime ideal of the field $\mathcal{K}$. The structure sheaf $\mathcal{O}_X$ is determined by the global sections, $\mathcal{O}_X(X) = \mathcal{K}$. It follows that the vector bundle $\mathcal{E}$ can be viewed, simply, as an n dimensional $\mathcal{K}$-vector space. One can think of X as the infinitesimal punctured disk, which is also called the *formal punctured disk*.

Suppose that ∇ is a meromorphic connection on $\mathbb{P}^1$, which has a singularity at x. We recall from the previous section that, without loss of generality, we can assume that x is the origin of the complex plane, and write $\nabla = d + [\nabla]\frac{dt}{t}$, where the entries of $[\nabla]$ are in $\mathbb{C}(t) \subset \mathcal{K}$. Therefore, ∇ induces a formal connection $\hat{\nabla}_x$, associated to the singularity x, which inherits the *local* information of the *global connection* ∇.

2.2.4. G-connections: connections on principal bundles over $\mathbb{P}^1$

So far we have defined connections on vector bundles over X, where X is an algebraic curve or the affine scheme $\mathrm{Spec}(\mathcal{K})$. A generalization of vector bundles is the notion of fiber bundles, where the fiber above each point of X can be a space different than a vector space, which is the case for vector bundles. Connections can be defined over fiber bundles, but our focus here is a specific type of fiber bundles, called principal G-bundles, where G is a reductive complex algebraic group. In this case the fiber over each point of X, is a space topologically isomorphic to G, and G acts on each fiber transitively and "freely": the free action of G on fibers means that the action of any non-identity element g of G does not have a fixed point.

The connections on principal G bundles are called G-connections. In particular, one can define G-connections on $\mathbb{P}^1$, $\mathbb{C}$, $\mathbb{G}_m$, or more generally $\mathbb{P}^1 \setminus \{x_1, \dots, x_n\}$, as well as G-connections on the formal punctured disk, $\mathrm{Spec}(\mathcal{K})$, which are called *formal G-connections* . A G-connection on $\mathbb{P}^1 \setminus \{x_1, \dots, x_n\}$ is also called a *meromorphic G-connection on* $\mathbb{P}^1$.

It is possible to give a coordinate-free definition of G-connections on X, however, it suffices for our purposes here to work with a definition that involves the choice of trivialization. As in the case of vector bundles, for simplicity, we assume that our principal bundle P is trivializable, and ϕ is a (global) trivialization for P.

Suppose that G is a complex algebraic group, and let $\mathfrak{g}$ denote its Lie algebra. Given a field extension $F \supset \mathbb{C}$,we can extend scalars to F and define $G(F)$ and $\mathfrak{g}(F)$; this can be seen more easily, if one assumes further that G is a linear group.

Let P be a trivializable (principal) G-bundle over $X = \mathbb{P}^1 \setminus \{x_1, \dots, x_n\}$, and for simplicity assume that $x_1 = 0$. Having the trivialization ϕ fixed, we can define a G-connection ∇ on X as an operator $\nabla = d + [\nabla]_\phi . \frac{dt}{t}$, where $[\nabla]_\phi$ is in $\mathfrak{g}\big(\mathbb{C}(t)\big)$, and the singularities of $[\nabla]_\phi$ are in a subset of $\{x_1, \dots, x_n\}$. As before, we call $[\nabla]_\phi$, the connection matrix of ∇ in the trivialization ϕ.

For g in $G\big(\mathbb{C}(t)\big)$, one can find the presentation of ∇ in the trivialization $g.\phi$, by the gauge action:

$$\nabla = d + \left(g[\nabla]_\phi g^{-1} - \frac{dg}{dt} g^{-1}\right)_{g.\phi} . \frac{dt}{t}$$

We also call ∇ a meromorphic G-connection on $\mathbb{P}^1$.

Comparing the formula for G-connections and connections on vector bundles, one can see that GL_n-connections are equivalent to connections on a vector bundle of rank n.

A formal G-connection ∇ can be defined as the equivalent class of the expression $d + [\nabla]\frac{dt}{t}$, with $[\nabla] \in \mathfrak{g}(\mathcal{K})$, via the gauge action by elements of $G(\mathcal{K})$. As before this can be considered as a connection on the G-bundle $\mathcal{E}$ over the formal punctured disk. Such a G bundle, topologically, can be viewed as a point p together with the fiber G.

As before, a meromorphic G-connection ∇, with a singularity at x, induces the formal G-connection $\hat{\nabla}_x$, which encodes the local behavior of ∇ around the singular point.

2.3. Slope, irregularity, and the formal type of a formal connection

Let B be vector bundle over $\mathbb{P}^1$ and ∇ be a meromorphic connection on B, with a singularity at a point x. For simplicity, we may assume that B is (globally) trivializable.

As mentioned before, ∇ induces a formal connection $\hat{\nabla}_x$. Choosing a local parameter t at x and a local trivialization ϕ, $\hat{\nabla}_x$ can be written as:

$$d + \left(M_{-r}t^{-r} + M_{-r+1}t^{-r+1} + \dots\right)\frac{dt}{t}$$

where $M_i \in \mathfrak{gl}_n(\mathbb{C})$, $M_{-r} \neq 0$ and $r \geq 0$.

If M_{-r} is *nonnilpotent*, then the next proposition shows that r is an invariant of ∇:

Proposition 2.11. *Let $\hat{\nabla}$ be a formal connection with a presentation*

$$d + \left(M_{-r}t^{-r} + M_{-r+1}t^{-r+1} + \dots\right)\frac{dt}{t}$$

Suppose that M_{-r} is nonnilpotent, then any expression of $\hat{\nabla}$ in this form has its first term in degree $-r$ or below.

Definition 2.12. *When M_{-r} is nonnilpotent, $M_{-r}t^{-r}$ is called a leading term of $\hat{\nabla}$ (in the degree grading). $M_{-r}t^{-r}$ is also considered as the leading term of the (global) connection ∇ in the trivialization ϕ at x.*

Remark 2.13. *It follows that when M_{-r} is nonnilpotent, any leading term of ∇ is in degree r, so r is an invariant of ∇.*

Definition 2.14. *With the notations above, when M_{-r} is nonnilpotent, r is called the slope of ∇ at the singularity x. In this situation, if $r > 0$ we say that ∇ is irregular singular at x, and if $r = 0$, we say that ∇ is regular singular at x. Similarly, these notions can be defined for a formal connection $\hat{\nabla}$.*

Given a meromorphic connection ∇ on $\mathbb{P}^1$ with a singularity at x, it is possible that no trivialization at x, gives an expansion of ∇ whose first term is nonnilpotent. In other words, ∇ might not have a leading term in the usual degree grading. In fact there are many interesting connections with this property. However, as we will see in the subsequent sections, by a result of Bremer and Sage, [13], ∇ has always a leading term in a certain family of gradings, called the Moy-Prasad gradings. This fact will help us give a general definition of slope from this perspective. There are other definitions of slope that do not assume this result; for that, we refer to [1].

If ∇ has a regular semisimple leading term, and x is an irregular singular point, then we have the following remarkable result:

Theorem 2.15. *Let $\hat{\nabla}$ be a formal connection expressed as:*

$$d + \left(M_{-r}t^{-r} + M_{-r+1}t^{-r+1} + \dots\right)\frac{dt}{t}$$

Suppose that M_{-r} is regular semisimple (hence, nonnilpotent) and $r < 0$. Then $\hat{\nabla}$ can be written as

$$d + \left(D_{-r}t^{-r} + D_{-r+1}t^{-r+1} + \dots + D_0\right)\frac{dt}{t}$$

with each D_i diagonal (and no terms in positive degrees).

For more details about this result we refer to the book Wasow, [37], on this subject.

Definition 2.16. *In the situation of the above proposition, the diagonal one-form*

$$\left(D_{-r}t^{-r} + D_{-r+1}t^{-r+1} + \cdots + D_0\right)\frac{dt}{t}$$

is called a formal type of $\hat{\nabla}$.

Remark 2.17. *Suppose that* $\hat{\nabla}$ *is a formal connection with the regular semisimple leading term* M_{-r}*, as above. Then the formal type*

$$d + \left(D_{-r}t^{-r} + D_{-r+1}t^{-r+1} + \cdots + D_0\right)\frac{dt}{t}$$

defines an isomorphism class of global connections on $\mathbb{G}_m$*. If* $\hat{\nabla}_{\mathbb{G}_m}$ *is a connection in this class, we call it a globalization of the formal connection* $\hat{\nabla}$*. It turns out that any other possible formal type of* $\hat{\nabla}$ *gives rise to the same isomorphic class of connections on* $\mathbb{G}_m$*. This is true for a more general family of formal connections, called toral connections, as we will see soon.*

Proposition 2.18. *Suppose that* ∇ *can be expressed, at a singularity* x*, by* $d + D\frac{dt}{t}$ *Where* D *is a diagonal matrix. Suppose that* D_{ii} *is a Laurent polynomial starting with degree* $-r_i$*, where* $r_i \geq 0$*, for each* i*. Then* $\sum_i r_i$ *is an invariant of* ∇ *at* x*, which is called the irregularity of* ∇ *at* x*, and we denote it by* $\mathrm{Irr}(\hat{\nabla}_x)$.

Remark 2.19. *The notion of irregularity at a singularity can be defined for any meromorphic connection. For this, we refer to [1].*

Slope and irregularity of a connection ∇ at a singularity x, are two measures that describe how badly the differential equation corresponding to ∇ behaves around x.

2.4. Moy-Prasad filtrations and Moy-Prasad gradings

Suppose that G is a simple complex alebraic group with Lie algebra $\mathfrak{g}$. Moy-Prasad gradings are a family of gradings on $\mathfrak{g}(\mathbb{C}[t, t^{-1}])$, including the obvious degree grading. We will need them for defining the notion of "leading term" of a connection. In particular, they are powerful in defining "homogeneity", in a very useful way, which we will see soon.

More generally, there is a bigger family, but rather, consisting of filtrations on $\mathfrak{g}(\mathcal{K})$, called the Moy-Prasad filtrations, a subfamily of which, produce the Moy-Prasad gradings. Even though Moy-Prasad filtrations are important in the theory that we are employing, we don't explicitly work with them in calculations for our work, except those filtrations associated with Moy-Prasad gradings.

2.4.1. Moy-Prasad gradings

To describe the Moy-Prasad grading, first we need to fix a split maximal torus S in $G(\mathcal{K})$ among those split maximal tori that are in the form of $T(\mathcal{K})$, for some maximal torus T in G. We will refer to S as the "satndard torus." If $\mathfrak{t}$ is the Lie algebra of T, then $\mathfrak{s} = \mathfrak{t}(\mathcal{K})$, is a Cartan subalgebra in $\mathfrak{g}(\mathcal{K})$, and is called the "standard Cartan subalgebra."

Now, fix a basis for the root system Φ of $\mathfrak{g}$, corresponding to $\mathfrak{t}$. Let us denote the simple roots by $\alpha_1, \ldots, \alpha_n$. Also, given a root β in Φ, let e_β be a basis for the associated root space.

Given an n-tuple $x = (x_1, x_2, \ldots x_n)$, with x_i's real numbers, we can define the Moy-Prasad grading associated to x and to the standard Cartan subalgebra, on $\mathfrak{g}(\mathcal{K})$:

- If $y \in \mathfrak{t}$ and $y \neq 0$, then $t^k y$ is homogeneous of degree k.
- If $c \in \mathbb{C}^*$, then $ct^k e_{\alpha_i}$ is homogeneous of degree $x_i + k$.

- More generally, if β is any root, so $b = \sum m_i a_i$ for some integers m_i's, then $ct^k e_\beta$ is homogeneous of degree $(\sum m_i x_i) + k$.

Remark 2.20. *Notice that when $x = 0$, we get the usual degree grading.*

Example 2.21. *Moy-Prasad grading on $\mathfrak{sl}_3(\mathbb{C}[t, t^{-1}])$, with $x = (\frac{1}{3}, \frac{1}{3})$*

Let E_{ij} denote the matrix in $\mathfrak{gl}_3$ with entries, $(E_{ij})_{lm} = \delta_{il}\delta_{jm}$. Let $\mathfrak{t}$ denote the Cartan subalgebra consisting of diagonal matrices in $\mathfrak{sl}_3$, and let the simple roots, α_1 and α_2, be the roots corresponding to the root spaces E_{12} and E_{23}.

Then consider the matrix

$$\omega = \begin{pmatrix} 0 & & t^{-1} \\ 1 & 0 & \\ & 1 & 0 \end{pmatrix}$$

The 1's on the subdiagonal are in the root spaces corresponding to $-\alpha_1$ and $-\alpha_2$, so the grade of those entries are, respectively, $x_1.(-1)$ and $x_2.(-1)$, which gives us $\frac{1}{3}.(-1) = \frac{-1}{3}$. To find grade of the entry t^{-1}, we first notice that the root space, E_{13}, corresponds to the root $\beta = \alpha_1 + \alpha_2$, so we get $(1).\frac{1}{3} + (1).\frac{1}{3} - 1 = \frac{-1}{3}$.

So, ω has homogeneous grade $\frac{-1}{3}$, with respect to the Moy-Prasad grading associated to $x = (\frac{1}{3}, \frac{1}{3})$.

One can easily check that

$$\omega^2 = \begin{pmatrix} 0 & t^{-1} & \\ & 0 & t^{-1} \\ 1 & & 0 \end{pmatrix} \quad \omega^3 = \begin{pmatrix} t^{-1} & & \\ & t^{-1} & \\ & & t^{-1} \end{pmatrix} \quad \omega^4 = \begin{pmatrix} 0 & & t^{-2} \\ t^{-1} & 0 & \\ & t^{-1} & 0 \end{pmatrix} \quad \cdots$$

have homogeneous grade, respectively, $\frac{-2}{3}$, $\frac{-3}{3}$, $\frac{-4}{3}$, and so on.

Similarly,

$$\omega^{-1} = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ t & & 0 \end{pmatrix}$$

has homogeneous grade $\frac{1}{3}$, and

$$\omega^{-2} = \begin{pmatrix} 0 & & 1 \\ t & 0 & \\ & t & 0 \end{pmatrix} \quad \omega^{-3} = \begin{pmatrix} t & & \\ & t & \\ & & t \end{pmatrix} \quad \omega^{-4} = \begin{pmatrix} 0 & t & \\ & 0 & t \\ t^2 & & 0 \end{pmatrix} \quad \cdots$$

have homogeneous grade, respectively, $\frac{2}{3}$ and $\frac{3}{3}$, $\frac{4}{3}$, and so on.

Let's look at the characteristic polynomial of ω:

$$p(\lambda) = -\lambda^3 - t^{-1}$$

Notice that p can be viewed as a homogeneous polynomial in the variables λ and $t^{\frac{-1}{3}}$, moreover the exponent of t in $t^{\frac{-1}{3}}$ is exactly the homogeneous grade of ω. As will see in many more examples, this is not a coincidence; however, some caution is required to formulate this observation.

Remark 2.22. *In all the examples that we discuss in this dissertation, we choose the standard Cartan subalgebra, $\mathfrak{s}$, to be the set of diagonal matrices in the given matrix presentation of $\mathfrak{g}(\mathcal{K})$, given in each example, as we did in Example 2.21.*

We will see more examples of Moy-Prasad grading, specially for $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n+1}$, in section 2.4.4.

Given an n-tuple x, we denote the subspace with homogeneous degree r of

$\mathfrak{g}(\mathbb{C}[t,t^{-1}])$, by $\mathfrak{g}(\mathcal{K})_x(r)$. So, we can write:

$$\mathfrak{g}(\mathcal{K})_x(r) = \begin{cases} \mathfrak{t}t^r + \bigoplus_{\alpha(x)+m=r} \mathfrak{g}_\alpha t^m & \text{if } r \in \mathbb{Z} \\ \bigoplus_{\alpha(x)+m=r} \mathfrak{g}_\alpha t^m & \text{otherwise} \end{cases}$$

The associated Moy-Prasad filtration on $\mathfrak{g}(\mathcal{K})$ is denoted by

$$\{\mathfrak{g}(\mathcal{K})_{x,r} \ \mid \ r \in \mathbb{R}\}$$

where $\mathfrak{g}(\mathcal{K})_{x,r} \subset \mathfrak{g}(\mathcal{K})$ is the direct product of $\mathfrak{g}(\mathcal{K})_x(s)$, for $s \geq r$.

Notice that $\mathfrak{g}(\mathcal{K})_{x,r}$ is a decreasing $\mathbb{R}$-filtration; furthermore, it is an $\mathcal{O}$-submodule of $\mathfrak{g}(\mathcal{K})$, where $\mathcal{O}$ stands for $\mathbb{C}[[t]]$, the ring of integers of $\mathcal{K}$, as mentioned before.

It can be easily seen that this filtration satisfies the following properties:

- it is a decreasing $\mathbb{R}$-filtration of $\mathfrak{g}(\mathcal{K})$
- each $\mathfrak{g}(\mathcal{K})_{x,r}$ is an $\mathcal{O}$-lattice in $\mathfrak{g}(\mathcal{K})$; that is, $\mathfrak{g}(\mathcal{K})_{x,r}$ is an $\mathcal{O}$-submodule of maximal rank in $\mathfrak{g}(\mathcal{K})$
- $\mathfrak{g}(\mathcal{K})_{x,r+1} = t\mathfrak{g}(\mathcal{K})_{x,r}$

We define

$$\mathfrak{g}(\mathcal{K})_{x,r+} := \bigcup_{s>r} \mathfrak{g}(\mathcal{K})_{x,r}$$

Then, one can observe that the set of real numbers r for which $\mathfrak{g}(\mathcal{K})_{x,r+} \neq \mathfrak{g}(\mathcal{K})_{x,r}$ is a discrete subset of $\mathbb{R}$.

In the next section we will discuss the whole family of the Moy-Prasad filtration, including the ones that are not associated with a Moy-Prasad grading.

2.4.2. Moy-Prasad filtrations

To describe Moy-Prasad filtrations precisely, one needs some language of the theory of "buildings"; buildings are defined as simplicial complexes with certain axioms. However, for our purposes, it is enough if we accept some of the terminologies and facts as black boxes, as we will see below, and get a general picture, in which one can make sense of the definitions and propositions that we will discuss here. We avoid the details of the theory of buildings, but refer the interested reader to the book of Abramenko and Brown, [39]. Also, for the precise theory of Moy-Prasad filtrations we refer to the work of Bremer and Sage, in [13].

As mentioned above, a building is a simplicial complex Δ with certain axioms. The facets with maximal dimension are called *alcoves*; in the literature, they are also called *chambers*. The axioms of buildings guarantee the existence of certain subcomplexes of Δ, called *apartments*, that have nice properties and their union is Δ. Each apartment consists of a specific collection of chambers.

Associated to a simple complex algebraic group G, there is a building $\mathcal{B}$, called the *Bruhat-Tits building of* G, whose facets are in one to one correspondence with certain subgroups of $G(\mathcal{K})$, called the *parahoric* subgroups. The chambers are in correspondence with the *Iwahori* subgroups, which are a particular type of parahoric subgroups. The apartments are in correspondence with split maximal tori in $G(\mathcal{K})$. The *standard apartment*, denoted by $\mathcal{A}$, is the apartment corresponding to the standard torus $S = T(\mathcal{K})$, as described in 2.4.1.

Any point x in this building defines a filtration on $\mathfrak{g}(\mathcal{K})$, which we call the Moy-Prasad filtration.

Having fixed a basis of the root system Φ of $\mathfrak{g}$, associated to $\mathfrak{t}$, one gets a correspondence between points in the standard apartment and the n-tuples $x = (x_1, x_2, \dots x_n)$, where x_i's are real numbers. If one restricts to the points in the standard apartment, then the filtration comes from a grading, which is the Moy-Prasad grading associated to x, as we saw in section 2.4.1.

The *fundamental alcove* is the alcove in the standard apartment whose points have positive coordinates, that is, $x_i \geq 0$ for all i. It is sometimes helpful to restrict to those Moy-Prasad gradings that are associated to the points in the fundamental alcove; furthermore, sometimes we prefer to add more restriction and require that x be chosen in the fundamental alcove in a way that the grade associated to the longest root in Φ, say, γ, be less than or equal to 1.; more precisely, if $\gamma = \sum k_i a_i$, then $\sum k_i x_i \leq 1$.

The loop group, $G(\mathcal{K})$, acts on $\mathcal{B}$ by "simplicial automorphisms;" meaning that each automorphism preserves the simplicial structure on $\mathcal{B}$; in particular, chambers are mapped to chambers. The induced action on the set of chambers is transitive.

For any $x \in \mathcal{B}$, the Moy-Prasad filtration associated to x is a filtration on $\mathfrak{g}(\mathcal{K})$, denoted by:

$$\{\mathfrak{g}(\mathcal{K})_{x,r} \ \mid \ r \in \mathbb{R}\}$$

satisfying similar properties to the Moy-Prasad filtrations coming from the standard apartment:

- it is a decreasing $\mathbb{R}$-filtration of $\mathfrak{g}(\mathcal{K})$
- each $\mathfrak{g}(\mathcal{K})_{x,r}$ is an $\mathcal{O}$-lattice in $\mathfrak{g}(\mathcal{K})$
- the filtration is periodic, in the sense that $\mathfrak{g}(\mathcal{K})_{x,r+1} = t\mathfrak{g}(\mathcal{K})_{x,r}$
- the set of real numbers r for which $\mathfrak{g}(\mathcal{K})_{x,r+} \neq \mathfrak{g}(\mathcal{K})_{x,r}$ is a discrete subset of $\mathbb{R}$, where $\mathfrak{g}(\mathcal{K})_{x,r+} \coloneqq \bigcup_{s>r} \mathfrak{g}(\mathcal{K})_{x,r}$, as was defined in section 2.4.1.

Remark 2.23. *Another property of Moy-Prasad filterations is that the filtered subspace of index* 0*, is the parahoric algebra associated to the chamber that contains* x.

As we will see, the quotient space $\mathfrak{g}(\mathcal{K})_{x,r}/\mathfrak{g}(\mathcal{K})_{x,r+}$ has an important role in defining "the leading term of a connection," or more precisely, the "fundamental stratum." First, we need to define a "stratum":

Definition 2.24. *A G-stratum of depth r is a triple (x, r, β) with $x \in \mathcal{B}$, r a non-negative real number, and $\beta \in \mathfrak{g}(\mathcal{K})_{x,-r}/\mathfrak{g}(\mathcal{K})_{x,-r+}$*

Given a G-stratum (x, r, β), we can write $\beta = \hat{\beta} + \mathfrak{g}(\mathcal{K})_{x,-r+}$, for some $\hat{\beta}$ in $\mathfrak{g}(\mathcal{K})_{x,-r}$. We call $\hat{\beta}$, a representative of β. Following [2], we may abuse notation to refer to the stratum as $(x, r, \hat{\beta})$. If x is in the standard apartment, there is a unique homogeneous representative $\beta^\flat$ in $\mathfrak{g}(\mathcal{K})_x(-r)$.

Definition 2.25. *A G-stratum (x, r, β) is called a fundamental stratum if every representative of β is non-nilpotent.*

Proposition 2.26. *Let (x, r, β) be a G-stratum, and x be in the standard apartment. Let $\beta^\flat$ be the homogeneous representative of β. If $\beta^\flat$ is non-nilpotent then any representative of β is non-nilpotent, hence β is a fundamental stratum.*

Later we will see that for x in the standard apartment, many other properties of the representatives of β can be read off from the homogeneous representative $\beta^\flat$.

2.4.3. Slope of a connection via fundamental strata

There are several equivalent ways to define the slope of a connection. One of them, due to Bremer and Sage, [13], is through the notion of fundamental stratum, which we discuss here. For the other definitions of slope we refer to [1].

Recall that, given a connection ∇, and a trivialization ϕ, we can write $\nabla = d + [\nabla]_\phi \frac{dt}{t}$.

Definition 2.27. *Let x be a point in the standard apartment. We say that ∇ contains the stratum (x, r, β) with respect to trivialization ϕ if $[\nabla]_\phi$ is in $\mathfrak{g}(\mathcal{K})_{x,-r}$ and is a representative for β. In this situation, we may also say that (x, r, β) is a stratum in ∇ (with respect to ϕ).*

One can define this notion of stratum containment for any x in the building. We refer to [13] for the general definition.

Recall that the loop group $G(\mathcal{K})$ acts on the Bruhat-Tits building of G. Also, it is a fact that the loop group acts on the set of G-strata. We will not describe this action, but merely use the fact that this action is defined and well-behaved. We denote this action by: $g.(x, r, \beta) = (g.x, r, g.\beta)$.

Theorem 2.28. *Suppose that ∇ contains the stratum (x, r, β) with respect to the trivialization ϕ, then for each g in the loop group, ∇ contains the stratum $(g.x, r, g.\beta)$ with respect to the trivialization $g.\phi$; furthermore, if (x, r, β) is fundamental, so is $(g.x, r, g.\beta)$.*

The following theorem is a result of Bremer and Sage, [13]. We state this theorem for simple groups, however, it is still true for connected reductive groups.

Theorem 2.29. *Let G be a simple complex algebraic group G. Then, every G-connection ∇ contains a fundamental stratum (x, r, β), where x is in the fundamental chamber $C \subset \mathcal{A}$ and $r \in \mathbb{Q}$; the depth r is positive if and only if ∇ is irregular singular. Moreover, the following statements hold:*

- *(1) If ∇ contains the stratum (y, r', β'), then $r' \geq r$*
- *(2) If ∇ is irregular singular, a stratum (y, r', β') contained in ∇ is fundamental if and only if $r' = r$.*

Remark 2.30. *Combining theorem 2.29 and 2.28, we conclude that a G-connection ∇*

contains a fundamental stratum with respect to any trivialization.

Theorem 2.29 allows us to have the following definitions of slope and the leading term of a connection.

Definition 2.31. *Given a G-connection ∇, the slope of ∇ is defined to be the depth r of any fundamental stratum that it contains.*

This definition clearly matches, and generalizes, the definition of slope in section 2.3, by considering $x = 0$ in the fundamental chamber, where we get the usual degree grading.

We can also define the leading term of a connection:

Definition 2.32. *If (x, r, β) is a fundamental stratum of ∇, with respect to the trivialization ϕ, we say that β is the leading term of ∇ with respect to the trivialization ϕ and the Moy-Prasad filtration associated to x.*

2.4.4. Moy-Prasad gradings on $\mathfrak{so}_{2n+1}$

We describe the Moy-Prasad gradings on $\mathfrak{so}_9$, as we need them most for this dissertation. The Moy-Prasad gradings on $\mathfrak{so}_{2n+1}$, for any n, can be described easily in the same way. First, we need to fix our ordered basis for Φ, the root system of $\mathfrak{so}_9$:

$$\alpha_1 = E_{12} - E_{89}, \quad \alpha_2 = E_{23} - E_{78}, \quad \alpha_3 = E_{34} - E_{67}, \quad \alpha_4 = E_{45} - E_{56}$$

We will keep this basis fixed throughout this dissertation. For $\mathfrak{so}_{2n+1}$, we consider the ordered basis obtained by the same algorithm. If $\alpha = b_1\alpha_1 + \cdots + b_4\alpha_4$ is any root, we represent it by its coordinate (b_1, b_2, b_3, b_4). Let's put the coordinate of the root spaces of SO_9, corresponding to positive roots in the corresponding entries in the matrix, except, for brevity, we don't write on the entries on the lower anti-diagonal.

$$\begin{pmatrix}
0 & (1,0,0,0) & (1,1,0,0) & (1,1,1,0) & (1,1,1,1) & (1,1,1,2) & (1,1,2,2) & (1,2,2,2) & * \\
 & 0 & (0,1,0,0) & (0,1,1,0) & (0,1,1,1) & (0,1,1,2) & (0,1,2,2) & * & \\
 & & 0 & (0,0,1,0) & (0,0,1,0) & (0,0,1,0) & * & & \\
 & & & 0 & (0,0,0,1) & * & & & \\
 & & & & * & & & & \\
 & & & * & & 0 & & & \\
 & & * & & & & 0 & & \\
 & * & & & & & & 0 & \\
* & & & & & & & & 0
\end{pmatrix}$$

Now, for a point $x = (x_1, x_2, x_3, x_4)$ in the standard apartment, the grade of each root space e_α, in the Moy-Prasad grading associated to x, will be obtained by the dot product of the coordinate of x and the coordinate of α. The following example exhibits this:

Example 2.33. *Moy-Prasad grading on $\mathfrak{so}_9$, associated to $x = (\frac{-3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.*

For positive roots, we write the grading of each root space in the entries associated to each root space:

$$\begin{pmatrix} 0 & \frac{-3}{4} & \frac{-2}{4} & \frac{-1}{4} & 0 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & * \\ & 0 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & \frac{5}{4} & * & \frac{3}{4} \\ & & 0 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & * & \frac{5}{4} & \frac{2}{4} \\ & & & 0 & \frac{1}{4} & * & \frac{3}{4} & 1 & \frac{1}{4} \\ & & & & * & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 0 \\ & & & * & & 0 & \frac{1}{4} & \frac{2}{4} & \frac{-1}{4} \\ & & * & & & & 0 & \frac{1}{4} & \frac{-2}{4} \\ & * & & & & & & 0 & \frac{-3}{4} \\ * & & & & & & & & 0 \end{pmatrix}$$

The grading of lower triangular entries are negative of the grading of their upper triangular counterpart; these correspond to negative roots. It follows that, for example, a matrix like:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-1} & 0 \\ 0 & 0 & & & 0 & & & 0 & -t^{-1} \\ 0 & 1 & 0 & & & t^{-1} & & & 0 \\ 0 & & 1 & 0 & & & t^{-1} & & 0 \\ 0 & & & 1 & 0 & & & 0 & 0 \\ 1 & & & & -1 & 0 & & & 0 \\ 0 & 0 & & & & -1 & 0 & & 0 \\ 0 & 0 & 0 & & & & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has homogeneous grade equal to $-\frac{1}{4}$, with respect to the given Moy-Prasad grading.

2.5. Toral, elliptic toral, and Coxeter toral connections

2.5.1. The general theory

In the following, we define three classes of *formal* connections: toral connections, elliptic toral connections, and Coxeter toral connections. As we will see, Coxeter toral connections are elliptic toral, and as the name implies, elliptic toral connections are toral.

Recall that for a (finite dimensional) Lie algebra L over an *algebraically closed field* K (of characteristic 0), all Cartan subalgebras are conjugate. However, this result does not necessarily hold if L is not finite dimensional over K, or K is not an algebraically closed field; for example, $K = \mathcal{K}$, the field of Laurent series. The following theorem comes to our help:

Theorem 2.34. *Let G be a simple complex group, and $\mathfrak{g}$ be its Lie algebra. Then there is a bijection between conjugacy classes of Cartan subalgebras in the loop algebra, $\mathfrak{g}(\mathcal{K})$, and the conjugacy classes in the Weyl group. Equivalently, there is a bijection between conjugacy classes of maximal tori in the loop group, $G(\mathcal{K})$, and the conjugacy classes in the Weyl group.*

We refer to [34] for the proof. This can be viewed as an exercise in Galois cohomology, but for classical groups, it is easy to see the bijection explicitly.

Definition 2.35. *Let w be an element of the Weyl group, W, with generators $s_1, \dots s_q$. We can view w as a linear automorphism of the reflection representation. Then, w is called:*

regular, *if it has an eigenvector that is not on any of the reflection hyperplanes (walls).*

elliptic, *if it has no fixed points on the reflection representation, that is, 1 is not an eigenvalue.*

elliptic regular, *if it is both elliptic and regular.*

Coxeter, *if it can be written as the product of all the elements in a generating set of reflections of W, each appearing once; that is, if $s_1, \dots, s_q$ are the generating reflections of W, then $w = s_{\sigma(1)} \dots s_{\sigma(q)}$, for some permutation σ.*

Example 2.36. *Regular, elliptic regular, and Coxeter elements in the Weyl group of* SL_3*:*

Let W denote the Weyl group of SL_3. Recall that, in general, the Weyl group of SL_n, is isomorphic to the symmetric group S_n. So, W is isomorphic to the the group of permutations of three elements; it is generated by two transpositions:

$$s_1 = (12)(3) \text{ and } s_2 = (1)(23)$$

We can list all the 6 elements of W:

- identity: 1
- reflections: s_1, s_2, and $s_3 \coloneqq s_1 s_2 s_1 = s_2 s_1 s_2$
- rotations: $s_1 s_2$ and $s_2 s_1$

The Coxeter elements, by definition, are: $s_1 s_2$ and $s_2 s_1$, which are the rotations here; their order is 3. It can be seen that all the elements of W are regular. The Coxeter elements, in the reflection representation, does not have a fixed point; that means that they are elliptic. The identity element and the reflections (transpositions), s_1, s_2, and s_3 are not elliptic. So, the only elliptic regular elements in SL_3 are the Coxeter ones. As we will see, this is true in general for SL_n. And, the Coxeter elements, in any Weyl group, are always elliptic regular.

Proposition 2.37. *We have the following facts:*

1) two regular elements in W have the same order if and only if they are conjugate.

*2) All the Coxeter elements in W are conjugate to each other, and they form "**the coxeter class**". The order of the elements in this class is **the Coxeter number** h.*

3) The Coxeter class is an elliptic regular class.

We refer to [35] for details. In view of theorem 2.34 and the above definitions, we can define regular tori, elliptic tori, and Coxeter tori in $G(\mathcal{K})$:

Definition 2.38. *Suppose that G is a simple group, with Weyl group W. Let T be a maximal torus in $\mathrm{G}(\mathcal{K})$ and $[T]$ denote its conjugacy class. Then, we say that:*

*T is a **regular maximal torus** if $[T]$ corresponds to a regular conjugacy class in W.*

*T is an **elliptic maximal torus** if $[T]$ corresponds to an elliptic conjugacy class in W.*

*T is an **elliptic regular maximal torus** if $[T]$ corresponds to an elliptic regular conjugacy class in W.*

*T is a **Coxeter maximal torus** if $[T]$ corresponds to the Coxeter class in W.*

In a parallel way, we define the regular, elliptic, and Coxeter Cartan subalgebras.

From the definitions and proposition 2.37, it is clear that Coxeter tori are elliptic regular, and elliptic regular tori are regular.

In the following, we give another description of elliptic maximal tori. But, before that, we need some definitions:

Definition 2.39. *Let G be a linear complex algebraic group. A maximal torus S in $G(\mathcal{K})$ is called an split maximal torus, if it is conjugate to the maximal torus consisting of diago-*

nal matrices in $G(\mathcal{K})$.

Recall that $\mathbb{G}_m := \mathbb{C} \setminus \{0\}$, the group of units in $\mathbb{C}$. An split maximal torus T have "nonobvious" cocharacters, that is, there are rational homomorphisms $\chi : \mathbb{G}_m \to T$ such that the image of χ is not in the center of G. For example, in SL_n, the torus consisting of diagonal matrices, have the following cocharacters:

$$t \mapsto \begin{pmatrix} t^{a_1} & & & & \\ & t^{a_2} & & & \\ & & \ddots & & \\ & & & t^{a_{n-1}} & \\ & & & & t^{a_n} \end{pmatrix}$$

where $a_n = -(a_1 + a_2 + \cdots + a_{n-1})$, and a_i's are not all zero; if we allow all a_i's to be zero then we get the constant homomorphism, whose image is in the center of SL_n.

Remark 2.40. *Under the correspondence in theorem 2.34, the conjugacy class of split maximal tori in $G(\mathcal{K})$ corresponds to the conjugacy class of the identity element, $\{e\}$, in the Weyl group of G. Hence, the class of split maximal tori is a regular class. It is clear from the definitions that this class is not elliptic.*

Remark 2.41. *Any maximal torus $\hat{T}$ in $G(\mathcal{K})$, in the form of $\hat{T} = T(\mathcal{K})$, with T a maximal torus in G, is an split maximal torus. But the converse is not true; split maximal tori are not necessarily in the form of $T(\mathcal{K})$.*

Proposition 2.42. *A maximal torus S in the loop group is elliptic if and only if it has no rational cocharacter, modulo the ones whose image is in the center of G.*

For a torus S in the loop group, being elliptic means that S is as far as possible from being split.

Recall that a matrix A with entries in a field K is called semisimple if it is diagonalizable using conjugation by matrices with entries in an algebraic closure of K. We say that A is *regular semisimple* if it is semisimple and its eigenvalues are distinct.

Proposition 2.43. *Let A be an element of $\mathfrak{g}(\mathcal{K})$. Then A is regular semisimple if and only if the centralizer of A in $\mathfrak{g}(\mathcal{K})$ is a Cartan subalgebra.*

Notice the different use of the word "regular" in the context of matrices versus the context of tori and Cartan subalgebras: the centralizer of a regular semisimple elements of $\mathfrak{g}(\mathcal{K})$ is a Cartan subalgebras, *not necessarily* a regular Cartan subalgebras.

Proposition 2.44. *Let (x, r, β) be a G-stratum. If the centralizer of every representative of β is a Cartan subalgebra in $\mathfrak{g}(\mathcal{K})$, then all of these centralizers are regular Cartan subalgebras, furthermore, they are in the same conjugacy class in $\mathfrak{g}(\mathcal{K})$.*

Remark 2.45. *Combining propositions 2.43 and 2.44, we can say that: every representative of β is a regular semisimple endomorphism, if and only if the centralizer of each representatives is a regular Cartan subalgebra*

Proposition 2.46. *Suppose that ∇ contains the fundamental stratum (x, r, β) with respect to the trivialization ϕ. If x is in the standard apartment then every representative of β is regular semisimple if and only if the homogeneous representative $\beta^\flat$ is regular semisimple.*

Proposition 2.47. *Suppose that ∇ contains the fundamental stratum (x, r, β) with respect to the trivialization ϕ and the centralizer of every representative of β is a Cartan subalgebra in the conjugacy class $[\mathfrak{s}]$. Let ϕ' be another trivialization and (x', r, β') be the fundamental stratum of ∇ with respect to ϕ'. Then the centralizer of any representative of β' also lies in $[\mathfrak{s}]$.*

Definition 2.48. *Let G be a simple group and ∇ be a G-connection.*

1) We say that ∇ is a toral connection if there is a trivialization ϕ and a fundamental stratum (x, r, β) contained in ∇, with respect to ϕ, such that the centralizer of any representative $\hat{\beta}$ is a regular Cartan subalgebra.

2) If, in addition to the condition in (1), each centralizer is also elliptic, we say that ∇ is an elliptic toral connection.

3) If, furthermore, each centralizer is a Coxeter Cartan subalgebra, then we say that ∇ is a Coxeter toral connection.

Suppose that ∇ is a toral connection, and β is the leading term of ∇ with respect to the the trivialization ϕ. If $\hat{\beta}$ is a representative of β, then the centralizer of $\hat{\beta}$ in the loop algebra is a regular Cartan subalgebra, say, $\mathfrak{s}$. It follows from 2.44 and 2.47 that the conjugacy class of $\mathfrak{s}$, denoted by $[\mathfrak{s}]$, is determined by ∇, and does not depend on the choice of the trivialization or the representative of the leading term. We might say ∇ is an $[\mathfrak{s}]$-toral connection, or simply, an $\mathfrak{s}$-toral connection. If S is the corresponding maximal torus to $\mathfrak{s}$, we may also say that ∇ is $[S]$-toral, or S-toral.

So, we can think of the conjugacy class of $\mathfrak{s}$ in the loop algebra, as the "type" of ∇. By theorem 2.34, there is a conjugacy class of W, the Weyl group of G, associated to $[\mathfrak{s}]$, and hence associated to ∇. Proposition 2.37 guarantees that the order of any element in this class is the same, say m; so we can associate the number m to ∇. We can say even more:

Recall from proposition 2.37 that any regular element in W with order m is in the same conjugacy class. Now let $\mathfrak{s}'$ be a regular Cartan subalgebra in $\mathfrak{g}(\mathcal{K})$, and denote its conjugacy class by $[\mathfrak{s}']$. Suppose that the corresponding conjugacy class in W consists of elements of order m, then we can conclude that $\mathfrak{s}'$ and $\mathfrak{s}$ are in the same conjugacy class in

$\mathfrak{g}(\mathcal{K})$. This means that m determines the conjugacy class of $\mathfrak{s}$; so, m can also be thought as the type of ∇.

Proposition 2.49. *Let ∇ be an $[\mathfrak{s}]$-toral connection and let m denote the order of the elements in the conjugacy class corresponding to $[\mathfrak{s}]$ in the Weyl group. Then, the slope of ∇ is $\frac{r}{m}$, where r is an integer relatively prime to m.*

In other words, the type of a toral connection is determined by its slope in lowest terms.

It follows from proposition 2.46 that if we choose x to be in the standard apartment, we can simply look at the homogeneous representative of the leading term to check whether a connection is toral or not. We can state similar results for elliptic toral connections and Coxeter toral connections, which we record in the following proposition:

Proposition 2.50. *Suppose that ∇ is a G-connection with fundamental stratum (x, r, β) with respect to the trivialization ϕ, and suppose that x is in the standard apartment. Let $\beta^\flat$ be the homogeneous representative of β. Then:*

1) ∇ is toral if and only if the centralizer of $\beta^\flat$ is a Cartan subalgebra (equivalantly, ∇ is toral if and only if $\beta^\flat$ is regular semisimple.)

2) ∇ is elliptic toral if and only if the centralizer of $\beta^\flat$ is an elliptic Cartan subalgebra.

3) ∇ is Coxeter toral if and only if the centralizer of $\beta^\flat$ is a Coxeter Cartan subalgebra.

Remark 2.51. *Given an S-**toral** connection ∇,*

1) It follows from proposition 2.42 that ∇ is an elliptic toral G-connection if and only if S has no rational cocharacter modulo the ones with image in the center of the loop *group, $G(\mathcal{K})$.*

2) ∇ is a Coxeter toral G-connection if and only if its slope has denominator h in lowest form, where h is the Coxeter number of G.

Proposition 2.52. *Let ∇ be a toral connection. Then its matrix in every trivialization is regular semisimple.*

In the following, we bring a brief description of regular elliptic tori in $\mathrm{SL}_n(\mathcal{K})$, $\mathrm{Sp}_{2n}(\mathcal{K})$ and $\mathrm{SO}_{2n+1}(\mathcal{K})$, without proof. For the classification of regular classes in the Weyl group, we refer to the paper of Springer, [35]. We will return to this topic for Sp_{2n} and SO_{2n+1} in sections 4.2 and 5.2.

2.5.2. Toral SL_n-connections

Again we recall that W is isomorphic to the symmetric group S_n, and in S_n conjugacy classes consist of permutations with the same length of cycles; more precisely, conjugacy classes are in one to one correspondence with partitions of n. The Coxeter number of the group S_n is n. It turns out that the regular conjugacy classes in W come in two types, one for divisors of n and one for divisors of $n-1$. Suppose that the order of a regular conjugacy class is k and $n = km$, then the cycle type of the class is m disjoint k-cycles. The corresponding maximal torus has m block-diagonal $k \times k$ Coxeter tori. If $km = n-1$, then the cycle type is m disjoint k-cycles and a single 1-cycle. The corresponding maximal torus has m block-diagonal $k \times k$ Coxeter tori with a 1 in the remaining 1×1 block. Note that in both cases, the order of the elements in the class is k. The only elliptic regular class is the Coxeter class,in fact, this is the only elliptic class.

We can rephrase this result in the following proposition:

Proposition 2.53. *The elliptic toral* SL_n*-connections are exactly the Coxeter toral connections. They are the toral connections whose slope has denominator* n *(in lowest form).*

2.5.3. Toral Sp_{2n}-connections

For Sp_{2n}, the weyl group is the group of signed permutations. We recall that the conjugacy classes in the Weyl group are determined by signed partitions of n, that is, each part is either positive or negative. The regular conjugacy classes come in two types:

1) If k is an odd divisor of n, say $n = km$, then the conjugacy class determined by m disjoint positive k-cycles is regular. The order of such elements is k.

The corresponding maximal Cartan subalgebra is the *block-diagonal embedding* of the $\mathfrak{gl}_k$ Coxeter Cartan subalgebra into $\mathfrak{sp}_{2n}$ (refer to definition 2.3.) But, none of these are elliptic.

2) The other type of regular conjugacy class comes from any divisor k of n (even or odd). Again, we write $n = km$. This class is determined by m disjoint negative k-cycles; the order of these elements is $2k$.

The corresponding maximal Caratan subalgebra is the *symplectic-block-diagonal embedding* of $\mathfrak{sp}_{2k}$ Coxeter Cartan subalgebra into $\mathfrak{sp}_{2n}$ (refer to definition 2.4.) All of these are elliptic. Thus:

Proposition 2.54. *The elliptic toral* Sp_{2n}*-connections are the ones whose slope have denominator* $2k$*, for any divisor* k *of* n*. The Coxeter toral connections are those with denominator* $2n$*.*

2.5.4. Toral SO_{2n+1}-connections

Since the Weyl group of SO_{2n+1} is the same as the one for Sp_{2n}, the description of the regular and regular elliptic classes are the same as for Sp_{2n}:

1) even divisors of $2n$ give rise to elliptic regular classes; if the divisor is $2s$, then one gets $\frac{n}{s}$ negative s-cycles.

2) odd divisors of n gives rise to non-elliptic regular classes; if the divisor is s, one gets $\frac{n}{s}$ positive s-cycles.

This implies:

Proposition 2.55. *The elliptic toral SO_{2n+1}-connections are the ones whose slope have denominator $2k$, for any divisor k of n. The Coxeter toral connections are those with denominator $2n$.*

Unlike for Sp_n, the description of maximal Cartan subalgebras does not seem to be easy. So, we are not giving an explicit general form for these connections, except for slope $\frac{1}{2}$; we will see this and more examples in chapter 5.

2.5.5. Adjoint Irregularity for toral connections

As we see in the following sections, given a G-connection ∇, the irregularity of the adjoint connection, $\mathrm{Irr}(\mathrm{ad}_\nabla)$, plays an important role in the rigidity analysis, which will be discussed in the next section. We have an easy formula, when ∇ is toral:

Theorem 2.56. *If ∇ is a toral G-connection with slope s and $|\Phi|$ denotes the number of roots for G, then:*

$$\mathrm{Irr}(\mathrm{ad}_\nabla) = s.|\Phi|$$

We refer the reader to Lemma 19 in [1] for a proof.

For our reference we recall the number of roots for the groups that we need here:

$\mathrm{SL}_n : |\Phi| = n(n-1)$

$\mathrm{Sp}_{2n} : |\Phi| = 2n^2$

$\mathrm{SO}_{2n+1} : |\Phi| = 2n^2$

2.5.6. S-formal types and globalization of toral connections: Coxeter and elliptic regular connections

As mentioned before, we can globalize a toral connection to a connection on $\mathbb{G}_m$, up to isomorphism of connections on $\mathbb{G}_m$, via the notion of formal types, which can be defined for toral connections. We refer to [2], section 2.3, for a precise discussion of S-formal types of S-toral connections, using the notion of *regular stratum*.

Let S be a maximal torus in $\mathrm{G}(\mathcal{K})$, and $\mathfrak{s}$ be its corresponding Lie algebra in $\mathfrak{g}(\mathcal{K})$. Given an S-toral G-connection we can "diagonalize" it into $\mathfrak{s}$, generalizing the idea in 2.15; this means that, we can write:

$$\nabla = d + \Big(D_{-r} + D_{-r+1} + \cdots + D_0\Big)\frac{dt}{t}$$

where D_i's are elements of $\mathfrak{s}(i)$, the i'th graded piece of $\mathfrak{s}$, and D_{-r} is regular semisimple. Of course, r is the slope of ∇. We have not exactly described what we mean by the graded pieces of $\mathfrak{s}$; we refer to [2] for more details.

The 1-form

$$\Big(D_{-r} + D_{-r+1} + \cdots + D_0\Big)\frac{dt}{t}$$

is called the S-formal type of ∇. The expression

$$d + \Big(D_{-r} + D_{-r+1} + \cdots + D_0\Big)\frac{dt}{t}$$

defines an isomorphism class of global connections on $\mathbb{G}_m$. It turns out that for an S-toral connection, the isomorphism class obtained in this way is unique. We will call any element of this class *the globalization of* ∇.

The globalized version of Coxeter toral connections, and more generally, elliptic toral connections will be important for this dissertation, so we give them a name:

Definition 2.57. *We call a globalization of a Coxeter toral connection, a* ***Coxeter connection****, and a globalization of an elliptic toral connection, an* ***elliptic regular connection****.*

2.5.7. Homogeneous toral connections

It is possible to define homogeneous connections in general, however, to avoid unnecessary complexity we restrict ourselves to toral connections. The good thing about an S-toral connection, as we saw in the previous section, is that it admits an S-formal type:

$$\nabla = d + \left(D_{-r} + D_{-r+1} + \cdots + D_0\right)\frac{dt}{t}$$

with D_i's in $\mathfrak{s}(i)$ and D_r rgular semisimple.

Definition 2.58. *Let ∇ be an S-toral connection. We say that ∇ is homogeneous if it has a homogeneous S-formal type, that is:*

$$\nabla = d + D_{-r}\frac{dt}{t}$$

with D_r a rgular semisimple element in $\mathfrak{s}(r)$, the rth graded piece of $\mathfrak{s}$.

Remark 2.59. *Alternatively, homogeneous toral connections can be defined in the following way: a toral connection is homogeneous it can be expressed as $d + A\frac{dt}{t}$ where A is a regular semisimple matrix, which is homogeneous with respect to some Moy-Prasad grading (with x in the standard apartment).*

For a toral connection $\nabla = d + A\frac{dt}{t}$, it turns out that homogeneity is related to the homogeneity of the characteristic polynomial of $A(t)$, in the variables λ and t^{-1}, if ∇ is written in a suitable trivialization.

We can define the globalized version of homogeneous toral connections; in particular:

Definition 2.60. *We call a globalization of a homogeneous Coxeter toral connection, a homogeneous Coxeter connection, and a globalization of a homogeneous elliptic toral connection, a homogeneous elliptic regular connection.*

Chapter 3. Rigid G-Connections

Let ∇ be a meromorphic G-connection on $\mathbb{P}^1$. We say that:

1) ∇ is **physically rigid**, or *rigid*, if any connection ∇' on $\mathbb{P}^1$ with the same formal isomorphism classes at the singularities, is globally isomorphic to ∇, that is $\nabla' = \nabla$; in other words, there is a unique connection with the given local data.

2) ∇ is **cohomologically rigid** if any infinitesimal deformation that preserves the formal isomorphism classes at the singularities preserves the global isomorphism class; in other words, the moduli space of connections with given local data is discrete.

It follows directly from the definition that physical rigidity implies cohomological rigidity. It is known that the converse is true for $G = \mathrm{GL}_n$. For this we refer to [17].

For our purposes in this dissertation, we only focus on cohomological rigidity; with this remark, it should not cause confusion if we abbreviate "cohomological rigidity" to "rigidity."

Following [7] and [2], we give another definition of cohomological rigidity, which is closely related to the definition given above:

Definition 3.1. *Let X be a be a nonempty (Zariski) open subset of $\mathbb{P}^1$, and $j : X \hookrightarrow \mathbb{P}^1$ be the inclusion map. Let ∇ be a connection on X. We say that ∇ is (cohomologically) rigid, if*

$$\mathrm{H}^1(\mathbb{P}^1, j_{!*}\,\mathrm{ad}_\nabla) = 0$$

Remark 3.2. *It can be shown that if ∇ satisfies*

$$\mathrm{H}^i(\mathbb{P}^1, j_{!*}\,\mathrm{ad}_\nabla) = 0$$

for $i = 0, 1, 2$, then ∇ has no infinitesimal deformations. We refer to the work of Yun, [7], *for the details. In practice,* H^0 *and* H^2 *vanish for many connections of interest, eg., for irreducible ones, and hence, for elliptic toral connections.*

To understand the meaning of the cohomology used in the definition of rigidity one needs to notice that connections are D-modules, and we are looking at the de Rham cohomology for holonomic D-modules. But, since we have a simpler consequence of this definition, which we might call "rigidity criterion", we will not need to know about cohomology for our results.

3.1. Rigidity criterion for our connections

It is desirable to understand rigid G-connections on $\mathbb{P}^1$ with the following type of singularities:

- irregular singularity at the origin
- regular singularity at infinity with unipotent monodromy or no singularity at infinity

In this dissertation we restrict ourselves to the case that the irregular singularity at the origin is homogeneous elliptic toral. It turns out that with this restriction, the desired condition at infinity will be automatically satisfied. To see this, let ∇ be a G-connection on $\mathbb{G}_m$, and let t be a parameter at the origin for $\mathbb{G}_m$. Let us denote $[\nabla]$, by the matrix $A(t)$. Recall that $A(t)$ in any trivialization, is an element of $\mathfrak{g}\Big(\mathbb{C}[t, t^{-1}]\Big)$. This implies that that the formal connection $\hat{\nabla}_0$ has the same expression as ∇.

If ∇ is homogeneous elliptic toral at 0, then there is a trivialization that $A(t)$ is a homogeneous element with respect to a Moy-Prasad grading based at a point x in the fundamental alcove.

Let r be the slope of ∇ at 0. We split our analysis into two parts: when $r < 1$ and when $r \geq 1$.

Case 1: $r < 1$

In this case, it follows from the construction of Moy-Prasad gradings that $A(t) = N + M.t^{-1}$, where N is a constant lower triangular matrix and M is a constant upper triangular matrix. Now, to express ∇ at $t = \infty$, we use the parameter $u = \frac{1}{t}$. Using $\frac{dt}{t} = -\frac{du}{u}$, we can write:

$$\hat{\nabla}_\infty = d + B(u).\frac{du}{u}$$

Where $B(u) = -A(t(u)) = -N - M.u$, therefore,

$$\hat{\nabla}_\infty = d + (-N - M.u).\frac{du}{u} = d + (-N.\frac{1}{u} - M).du$$

The coefficient of the term $\frac{1}{u}$ is called the residue term, which is $-N$ here. Another way to think about it is to set $u = 0$ in $B(u) = -N - M.u$. Since N is nilpotent, being lower triangular, we say that ∇ has a nilpotent residue. It is a fact that nilpotent residue at a singularity implies a unipotent momodromy. So, in this case we have unipotent monodromy at infinity, as desired.

Case 2: $r \geq 1$

We write $r = r_0 + k$, where k is in $\mathbb{N}$ and $k \geq 1$. In this case, we can write:

$$A(t) = t^{-k}(N + M.t^{-1})$$

Therefore,

$$\hat{\nabla}_\infty = d + u^k(-N - M.u).\frac{du}{u} = d + u^{k-1}(-N - M.u).du$$

which means that ∇ does not have any singularity at infinity, since $k - 1 \geq 0$. Notice that setting $u = 0$, gives us the residue term 0, so in this case the residue term at infinity is again nilpotent.

The rigidity for our connections, with the above conditions, boils down to the following much easier criterion, which we refer to, as the "**numerical criterion for rigidity**":

Proposition 3.3. *Let ∇ be a connection on* G_1 *with elliptic toral singularity at the origin. Let N denote the nilpotent matrix obtained by evaluating $[\nabla]$ at $t = \infty$. Then, ∇ is rigid if and only if:*

$$\mathrm{Irr}(\mathrm{ad}_\nabla) = \dim C_{\mathfrak{g}}(N)$$

where $C_{\mathfrak{g}}(N)$ denotes the centralizer of N in $\mathfrak{g}$.

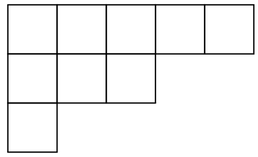
Remark 3.4. *This is proved, in [2], for a Coxeter toral connection ∇. However, the proof only uses that ∇ is elliptic toral.*

We have already discussed an easy way to calculate $\mathrm{Irr}(\mathrm{ad}_\nabla)$ for toral connections. To apply the rigidity criterion, it is left to figure out $\dim C_{\mathfrak{g}}(N)$. First, we need to discuss the partition and the dual partition associated to N:

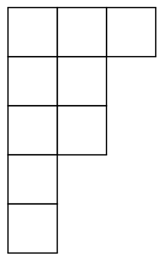
Let N be an $k \times k$ nilpotent matrix in the classical Lie algebra $\mathfrak{g}$. For our purpose here, we restrict ourselves to SL_n, Sp_{2n}, and SO_{2n+1}. Notice that all the eigenvalues of N are zero, and its Jordan form, looked as a matrix in $\mathfrak{gl}_l$, is determined by the size of the Jordan blocks. This associate to N a partition of k, say, $k = p_1 + p_2 \cdots + p_n$, which is denoted by $P = (p_1, p_2, \ldots p_n)$.

Given a partition P, we can define the dual partition, denoted by $\hat{P} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_m)$, which is illustrated in the following example:

Example 3.5. *Let* $P = (5,3,1)$ *be a partition of* 9*. This gives us a Young tableau:*



The dual Young tableau is obtained by reflecting the original tableau around the diagonal:



which gives us the dual partition $\hat{P} = (3,2,2,1,1)$.

Once we find the partition and the dual partition associated to a nilpotent matrix, we can easily calculate $\dim C_{\mathfrak{g}}(N)$ by the following formulas, depending on which Lie algebra $\mathfrak{g}$ is:

$$\dim C_{\mathfrak{sl}_n}(N) = \sum_{i=1}^{m} \hat{p}_i^2 - 1 \tag{3.1}$$

$$\dim C_{\mathfrak{sp}_{2n}}(N) = \frac{1}{2}\sum_{i=1}^{m} \hat{p}_i^2 + \frac{1}{2}(\#\text{ of odd parts in } P) \tag{3.2}$$

$$\dim C_{\mathfrak{so}_{2n+1}}(N) = \frac{1}{2}\sum_{i=1}^{m} \hat{p}_i^2 - \frac{1}{2}(\#\text{ of odd parts in } P) \tag{3.3}$$

For more details, refer to the book of Collingwood and McGovern, [41].

Example 3.6. *Let $N \in \mathfrak{so}_9$ be the following matrix:*

$$N = \begin{pmatrix} 0 &&&&&&&& \\ 0 & 0 &&&&&&& \\ 1 & 0 & 0 &&&&&& \\ 0 & 1 & 0 & 0 &&&&& \\ & 1 & 1 & 0 & 0 &&&& \\ && 1 & 0 & 0 & 0 &&& \\ &&& -1 & -1 & 0 & 0 && \\ &&&& -1 & -1 & 0 & 0 & \\ &&&&& 0 & -1 & 0 & 0 \end{pmatrix}$$

where the empty entries are all zero. The Jordan form is:

$$J = \begin{pmatrix} 0 & 1 &&&&&&& \\ & 0 & 1 &&&&&& \\ && 0 & 1 &&&&& \\ &&& 0 & 1 &&&& \\ &&&& 0 &&&& \\ &&&&& 0 & 1 && \\ &&&&&& 0 & 1 & \\ &&&&&&& 0 & \\ &&&&&&&& 0 \end{pmatrix}$$

The associated partition is $P = (5, 3, 1)$. The number of parts with odd size is 3. As we saw in example 3.5, the dual partition is $\hat{P} = (3, 2, 2, 1, 1)$. So, equation 3.3 yields:

$$\dim C_{\mathfrak{so}_{2n+1}}(N) = \frac{1}{2}(3^2 + 2^2 + 2^2 + 1^2 + 1^2 - 3) = 8$$

We recall that for $\mathfrak{sl}_n$, the partitions of n are in one to one correspondence with the nilpotent conjugacy classes. However for other classical groups the correspondence is a little more complicated. For $\mathfrak{sp}_{2n}$ the nilpotent conjugacy classes are in one to one correspondence with partition of $2n$ that odd parts appear with even multiplicity. And, for $\mathfrak{so}_{2n+1}$, the nilpotent conjugacy classes are in one to one correspondence with partitions of $2n+1$ in which even parts appear with even multiplicity.

The following result tells us that, for finding new rigid G-connections, we are left to look at connections with slope less than one:

Theorem 3.7. *Let G be a simple group. Suppose that ∇ is a G-connection with elliptic toral singularity at the origin, of slope greater than 1. Then ∇ is rigid if and only if $\hat{\nabla}_0$ is Coxeter toral with slope $\frac{h+1}{h}$.*

Remark 3.8. *This is discussed in [2] for homogenous Coxeter connections, however, it can be proved for all elliptic regular connections.*

In the following, we are going to apply the rigidity criterion to G-connections, where for our purpose, G can be either SL_n, Sp_{2n} and SO_{2n+1}. Our original goal has been to look at all simple groups, but we will investigate the other types in later works. The classifications for Sp_{2n} and SO_{2n+1} is the goal of the following chapters. For SL_n the problem is already solved, as the special case of the following theorem of Kamgarpour-Sage, [2], about rigid Coxeter G-connections.

3.2. Classification of rigid homogeneous Coxeter G-connections

Theorem 3.9. *Let G be a simple group, with coxeter number h, and let ∇ be a G-connection with homogeneous Coxeter singularity at the origin, of slope $\frac{r}{h}$. Then ∇ is rigid* *if and only if r satisfies the following conditions for each group:*

SL_n*:* $r|n+1$ *or* $r|n-1$

SO_{2n+1}*:* $r|n+1$ *or* $r|2n+1$

Sp_{2n}*:* $r|2n+1$ *or* $r|2n-1$

SO_{2n+1}*:* $r|2n$ *or* $r|2n-1$

E_7*:* $r=1$ *or* $r=7$

other exceptional groups: $r=1$

Remark 3.10. *For SL_n, this theorem can be stated for all Coxeter connections, without restricting to the homogeneous ones, as mentioned in the introduction. The point is that, for SL_n, the existence of a rigid Coxeter connection which is irregular singular at* 0*, with slope $\frac{r}{n}$, and is regular singular at infinity with nilpotent residue, depends only on r and not on the homogeneity of the formal type at* 0*. For this, we refer to [30]. For other simple groups, this result is only known for homogeneous Coxeter connections.*

Recalling that the only elliptic toral SL_n-connections are the Coxeter toral ones, we have the following result:

Corollary 3.11. ***(classification of rigid homogeneous elliptic regular*** SL_n***-connections)*** *Let ∇ be an SL_n-connection be an elliptic regular connection on $\mathbb{G}_m$. Then ∇ is rigid if and only if it has slope $\frac{r}{n}$, where $r|n+1$ or $r|n-1$.*

Notice that theorem 3.9 implies that, for all simple groups, there is a rigid homogeneous coxeter connection with slope $\frac{1}{h}$. As we will see in the following section, a result of Chen, [5], generalizes this for homogeneous elliptic toral connections with slope $\frac{1}{m}$.

3.3. Chen connections and generalized Chen connections

Theorem 3.12. *Let G be a simple group. Let m be an elliptic regular number, and $\mathfrak{s}$ be a Cartan subalgebra in the conjugacy class associated to m. Then, there exists a rigid connection of the form $\nabla = d + D\frac{dt}{t}$, where D is a regular semisimple element in $\mathfrak{s}$ and has homogeneous degree $\frac{-1}{m}$; in particular, there is a rigid homogeneous elliptic regular connection of slope $\frac{1}{m}$, for any elliptic regular number m.*

This theorem motivates the following definition:

Definition 3.13. *A rigid connection on $\mathbb{G}_m$ which has a homogeneous elliptic toral singularity at the origin, with slope $\frac{1}{m}$, and a regular singularity at infinity, with nilpotent residue, is called a **Chen connection**.*

In the following chapters, we will see examples of these connections, when $G = \mathrm{Sp}_{2n}$ and $G = \mathrm{SO}_{2n+1}$, for the elliptic regular number $m = 2$. For SL_n, we can only have $m = h$, as the only elliptic regular connections are the Coxeter ones. So, Chen connections are classified by theorem 3.9.

Remark 3.14. *For each elliptic regular number m, there is a unique nilpotent orbit associated to Chen connections with slope $\frac{1}{m}$ on $\mathbb{G}_m$.*

To sketch the idea, suppose that we are looking at Chen connections of slope $\frac{1}{m}$. In the theory due to Yun, one starts by fixing a certain point x in the fundamental alcove which, in the language that we are using, gives rise to elliptic toral connections. Now, consider the space of all possible (necessarily nilpotent) residues one gets from slope $\frac{1}{m}$ homogeneous elliptic toral connections associated to this x; this is just the linear subspace of lower triangular matrices with homogeneous degree $\frac{-1}{m}$. It is a fact that there is a Zariski *dense open subset of this subspace which lies in a single nilpotent orbit of $\mathfrak{g}$. Chen considers elliptic regular connections with such residues, and he shows that these connections are rigid. Now suppose that there is another nilpotent orbit in the closure of this single distinguished one. This means that if one has an elliptic regular connection ∇, with the residue in this orbit, the centralizer has strictly larger dimension. this implies that for ∇, the dimension of H^1 is less than zero, which is a contradiction.*

Proposition 3.15. *Let ∇ be a connection on $\mathbb{G}_m$. Let m be an elliptic regular number, and $[\mathfrak{s}]$ be the class of cartan subalgebras associated to m. Then, ∇ is a Chen connection of slope $\frac{1}{m}$ if and only if it as a rigid connection of the form $d + D.\frac{dt}{t}$, where $D \in \mathfrak{s}_0$ is a regular semisimple element of $\mathfrak{s}_0$ with homogeneous grade $\frac{-1}{m}$, and $\mathfrak{s}_0 \in [\mathfrak{s}]$*

The backward direction is immediate, since we already know that an elliptic toral connection with slope $\frac{1}{m}$ defines a global connection on $\mathbb{G}_m$ with nilpotent residue at infinity. To see the forward direction one needs to use remark 3.14; that is, there is only one nilpotent orbit that can be obtained as the orbit of the residue at infinity from a rigid connection $d + D.\frac{dt}{t}$ as above.

For our purposes in this dissertation, we define what we have called "a generalized Chen connection" for $G = \mathrm{Sp}_{2n}$ and $G = \mathrm{SO}_{2n+1}$. At the first glance, it might not seem obvious that the construction is well-defined, however, it turns out that the definition does work.

Definition 3.16. *Suppose that G is either Sp_{2n} or SO_{2n+1}. Let ∇ be a Chen G-connection with slope $\frac{-1}{m}$, and let $[\mathfrak{s}]$ be the conjugacy class of Cartan subalgebras associated to m. Write ∇ in the form $d + D\frac{dt}{t}$, where D is a homogeneous regular semisimple element in $\mathfrak{s}$. For each positive integer r, with $\gcd(r, m) = 1$, let $\nabla^{(r)} = d + D^r\frac{dt}{t}$. We call $\nabla^{(r)}$ a* ***gen-***

eralized Chen connection.

For this definition to make sense, we need to ensure that $[\nabla]^r$ is an element $\mathfrak{sp}_{2n}$ (resp. $\mathfrak{so}_{2n+1}$). Notice that since a Chen connection ∇, by definition, induces the formal elliptic toral connection $\hat{\nabla}_0$, and since we are restricting ourselves to $G = \mathrm{Sp}_{2n}$ (resp. $G = \mathrm{SO}_{2n+1}$), it follows that m is even. Since $\gcd(r, m) = 1$, we know that r is an odd integer. Therefore, by corollary 2.6, $[\nabla]^r$ is indeed in $\mathfrak{sp}_{2n}$ (resp. $\mathfrak{so}_{2n+1}$).

It turns out that any expression of the Chen connection ∇, in the form $d + \tilde{D}^r \frac{dt}{t}$, with $\tilde{D}$ satisfying the same properties as D, specified above, gives rise to the same connection, so the generalized Chen connection obtained from a Chen connection is well-defined.

Remark 3.17. *A generalized Chen connection is not required to be rigid.*

The next proposition ensures us that generalized Chen connections are in the class of connections that we are looking at:

Proposition 3.18. *A generalized Chen connection has a homogeneous elliptic toral singularity at the origin, with slope* $\frac{r}{m}$.

Our main goal in the subsequent chapters is **to classify rigid generalized Chen connections**. Our expectation is that, all rigid homogeneous elliptic regular connections are, conjecturally, rigid generalized Chen connections.

Chapter 4. Rigid Homogeneous Elliptic Regular Sp_{2n}-Connections on $\mathbb{G}_m$

4.1. Homogeneous Coxeter Sp_{2n}-connections

First we begin with the local version of these connections, that is, the homogeneous Coxeter toral ones. Recall that the the Weyl group of Sp_{2n} is the group of signed permutations, and the Coxeter number is $2n$. This means that the Coxeter Sp_{2n}-connections has slope $\frac{r}{2n}$, where $\gcd(r, 2n) = 1$.

We start withthe homogeneous Coxeter toral connections with slope $\frac{1}{2n}$. It can be shown that they can be, essentially, written in the form:

$$\nabla = d + (\omega_n)\frac{dt}{t}$$

where ω_n is the following matrix in $\mathfrak{sp}_{2n}$:

$$\omega_n = \begin{pmatrix} 0 & & & & & & & t^{-1} \\ 1 & 0 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 1 & 0 & & & & \\ & & & 1 & 0 & & & \\ & & & & -1 & \ddots & & \\ & & & & & \ddots & 0 & \\ & & & & & & -1 & 0 \end{pmatrix}$$

where the blank entries are all zero.

One can show that the centralizer of ω_n in $\mathfrak{sp}_{2n}(\mathcal{K})$ is indeed a Coxeter Cartan subalgebra. The connection ∇, defined above, is known as the Frenkel-Gross Sp_{2n}-connection

For defining ω_n, it is possible to choose a different matrix by putting arbitrary constants on the subdiagonal, as long as the matrix stays in $\mathfrak{sp}_{2n}$. It can be shown that other choices of the constants will change the centralizer, only upto conjugation by constant matrices, that is, an element of $\mathrm{Sp}_{2n}(\mathbb{C})$; for more details, refer to [2], section 3. This implies that different choices for the constants in ω_n, give rise to connections that are isomorphic, when viewed as formal connections, or even as global connections on $\mathbb{G}_m$.

Notice that from theorem 3.9, we already know that this is a rigid connection, hence a Chen connection. We know, by corollary 2.6, that if r is an odd integer then $(\omega_n)^r$ lies in $\mathfrak{sp}_{2n}$. It is easy to see that $(\omega_n)^r$ has the following form:

$$(\omega_n)^r = \begin{pmatrix} 0 & & & & & t^{-1} & & \\ & 0 & & & & & \ddots & \\ & & \ddots & & & & & t^{-1} \\ 1 & & & 0 & & & & \\ & 1 & & & 0 & & & \\ & & \ddots & & & \ddots & & \\ & & & -1 & & & 0 & \\ & & & & -1 & & & 0 \end{pmatrix}$$

where the rth subdiagonal entries are constants and the $(n-r)$'th superdiagonal entries are constant multiples of t^{-1}.

Now, we define the connections $\nabla_{(r)}$ by:

$$\nabla_{(r)} = d + (\omega_n)^r . \frac{dt}{t}$$

Notice that, by construction, $\nabla_{(r)}$ is a generalized Chen connection. It follows that

$\nabla_{(r)}$ is a homogeneous Coxeter connection of slope $\frac{r}{2n}$.

4.2. Homogeneous elliptic regular Sp_{2n}-connections

As briefly mentioned in 2.5.3, the homogeneous elliptic toral Sp_{2n}-connections are obtained from the following recipe:

Let n_b be a positive divisor of n, with $n = n_b.q$, and ω_{n_b} be the matrix in $\mathfrak{sp}_{2n_b}$, described above. Let $[\nabla]_{(n_b,r)}$ be the matrix consisting of q *symplectic-block-diagonal* (refer to definition 2.4) copies of ω^r, in each $n_b \times n_b$ block, with the choice of various constants to ensure that $[\nabla]_{(n_b,r)}$ does not have repeated eigenvalues, as we will illustrate in an example below. Then,we write:

$$\nabla_{(n_b,r)} = d + [\nabla]_{(n_b,r)} \cdot \frac{dt}{t}$$

Notice that if we choose $n_b = n$ then we get the Coxeter Sp_{2n}-connections.

Example 4.1. *Notice that we can always choose $n_b = 1$, $q = n$ and $r = 1$. Then:*

$$\omega_{n_b} = \omega_1 = \begin{pmatrix} 0 & t^{-1} \\ 1 & 0 \end{pmatrix}$$

Combining q of these matrices, symplectic-block-diagonally (refer to definition 2.4), *we get:*

$$\begin{pmatrix} &&&&&&& t^{-1} \\ &&&&&& t^{-1} & \\ &&&&& \iddots && \\ &&&& t^{-1} &&& \\ &&& 1 &&&& \\ && \iddots &&&&& \\ & 1 &&&&&& \\ 1 &&&&&&& \end{pmatrix}$$

This matrix has repeated eigen values. To fix this problem, we choose different constants in each of ω_1's:

$$[\nabla]_{(n_b,r)} = [\nabla]_{(1,1)} = \begin{pmatrix} &&&&&&& t^{-1} \\ &&&&&& t^{-1} & \\ &&&&& \iddots && \\ &&&& t^{-1} &&& \\ &&& n &&&& \\ && \iddots &&&&& \\ & 2 &&&&&& \\ 1 &&&&&&& \end{pmatrix}$$

As we will see in the next section, the matrix $[\nabla]_{(n_b,1)}$, constructed above, is rigid, as we might expect from Theorem 3.12; so it is a Chen connection. It turns out from the analysis in the next section, that except Chen connections (with various slopes $\frac{1}{n_b}$), and the rigid homogeneous Coxeter connections given by theorem 3.9, there are no more rigid homogeneous elliptic regular SO_{2n}-connections.

4.3. Rigidity analysis for homogeneous elliptic regular Sp_{2n}-connections

We want to investigate under what conditions $\nabla_{(n_b,r)}$ is rigid. Let $N_{n_b,r}$ denote the matrix obtained by evaluating $[\nabla]_{n_b,r}$ at $t = \infty$. We write $2n_b = kr + n'$ where $0 < n' < r$. Notice that for $r > 1$, we must have $n' > 0$ as $\gcd(r, 2n_b) = 1$. For each block, out of those q blocks, the Jordan form has the following partition:

$$(\underbrace{k+1,\dots,k+1}_{n' \text{ times}},\ \underbrace{k,\dots,k}_{r-n' \text{ times}})$$

which might be abbreviated as:

$$\left((k+1)^{n'}, k^{r-n'}\right)$$

So, we get the following partition of $2n$:

$$\left((k+1)^{n'q}, k^{(r-n')q}\right)$$

Notice that if k is even, we have $n'q$ parts of odd size, and if k is odd, we have $(r-n')q$ parts of odd size. It is worth noticing that this partition is indeed admissible for Sp_{2n}: Recall that for Sp_{2n}, the nilpotent conjugacy classes are in one to one correspondence with partitions of $2n$, in which odd parts appear with even multiplicity. To see that this is the case here, first notice that r has to be odd, as it is relatively prime to $2n_b$. Now, if k is odd, then kr is also odd and the equation $2n_b = kr + n'$ implies that n' is also odd, hence $(r - n')$, the multiplicity of the odd part, is even. On the other hand, if k is even, the equation $2n_b = kr + n'$ implies that n' is even, therefore the number of odd parts in this case, which is $n'q$ is even.

The dual partition is:

$$\left((qr)^k, n'q\right)$$

Therefore, we have

$$\dim C_{\mathfrak{sp}_{2n}}(N_{n_b,r}) = \frac{1}{2}\left(q^2kr^2 + q^2(n')^2 + \begin{cases} n'.q & \text{if } k \text{ is even} \\ \\ (r-n').q & \text{if } k \text{ is odd} \end{cases} \right)$$

and

$$\mathrm{Irr}(\mathrm{ad}_{\nabla_{(n_b,r)}}) = \frac{r}{2n_b}.(2n^2) = \frac{r}{n_b}(q^2n_b^2) = q^2rn_b = \frac{1}{2}q^2(kr+n')r$$

The rigidity criterion, after simplifying some terms, and dividing by $q \neq 0$, gives:

$$q(n')^2 + \begin{cases} n' & \text{if } k \text{ is even} \\ \\ (r-n') & \text{if } k \text{ is odd} \end{cases} = qrn'$$

For simplicity, we break out our analysis into three cases:

- Case 1: $r > 1$ and k is even
- Case 2: $r > 1$ and k is even
- Case 3: $r = 1$

Notice that when $r > 1$ we have $n' > 0$ and when $r = 1$ we have $n' = 0$. Now, we begin analysing each case:

Case 1: $r > 1$, and k is even

The rigidity criteron, after dividing by n', gives:

$$qn' + 1 = qr$$

Therefore, $q(r-n') = 1$, and since $q > 0$, we must have $q = 1$ and $r - n' = 1$. We conclude that, for this case, $\nabla_{(n_b,r)}$ is rigid if and only if it is Coxeter and $n' = r-1$. Let us formulate the conclusion $r - n' = 1$ in terms of a condition on r and n_b: If $r - n' = 1$, then $2n_b =$

$kr + (r-1)$, therefore, $2n_b + 1 = r(k+1)$. This implies:

$$r | 2n_b + 1$$

Notice that this is one of the rigidity criteria for Coxeter Sp_{2n}-connections, which we already know from theorem 3.9. On the other hand if $r|2n_b + 1$, without necessarily assuming that k is even, then for some integer j, we have:

$$2n_b + 1 = rj \implies 2n_b = rj - 1 \implies 2n_b = r(j-1) + (r-1)$$

which means that $k = j-1$ and $n_b = r-1$. Notice that since $j|2n_b+1$, we know that j has to be odd, and therefore k is even. We can summarize our observations, up to this point, as:

Remark 4.2. *Suppose that $r > 1$. If $\nabla_{(n_b,r)}$ is rigid and k is even, then $\nabla_{(n_b,r)}$ is Coxeter and $r|2n_b+1$. Conversely, if $r|2n_b+1$, and $\nabla_{(n_b,r)}$ is Coxeter, then $\nabla_{(n_b,r)}$ is rigid.*

Case 2: $r > 1$ and k is odd

From the rigidity criterion, we have:

$$q(n')^2 + r - n' = qrn' \implies (n'-r)(qn'-1) = 0$$

Therefore $qn' - 1 = 0$, since $n' < r$. This implies that $q = 1$ and $n' = 1$. Again, let us formulate this in terms of a condition on r and n_b: If $n' = 1$, we have $2n_b = kr+1$, therefore, $kr = 2n_b - 1$, which implies that

$$r | 2n_b - 1$$

And gain, notice that this is one of the rigidity criteria for Coxeter Sp_{2n}-connections, given in theorem 3.9. On the other hand, keeping the assumption that $r > 1$, but without assuming that k is odd, if $r|2n_b - 1$, we can write, for some integer j:

$$2n_b - 1 = rj \implies 2n_b = rj + 1$$

This implies that $n' = 1$ and $k = j$ and since $j|2n_b - 1$, we conclude that k is odd. Therefore, we can rephrase what we have observed, in this case, as:

Remark 4.3. *Suppose that $r > 1$. If $\nabla_{(n_b,r)}$ is rigid and k is odd, then $\nabla_{(n_b,r)}$ is Coxeter and $r|2n_b - 1$. Conversely, if $r|2n_b - 1$, and $\nabla_{(n_b,r)}$ is Coxeter, then $\nabla_{(n_b,r)}$ is rigid.*

Case 3: $r = 1$

Here, we have $n' = 0$, and $k = 2n_b$, so k is even. Therefore the rigidity criterion gives $0 = 0$, which is always true for all integers q. So, the connection $\nabla_{(n_b,1)}$ is rigid.

We record our final conclusion as the following classification theorem:

Theorem 4.4. ***Classification of rigid homogeneous elliptic regular*** Sp_{2n}***-connections***

The only rigid homogeneous elliptic regular Sp_{2n}-connections, are the Chen connections of slope $\frac{1}{2k}$ for each $k|n$ and the Coxeter connections with slope $\frac{r}{2n}$, where $r|2n - 1$ or $r|2n + 1$.

Chapter 5. Rigid Homogeneous Elliptic Regular SO_{2n+1} Connections on $\mathbb{G}_m$

5.1. Homogeneous Coxeter SO_{2n+1}-connections

The discussion here is almost identical to section 4.1, where we discussed the symplectic case: the weyl group of SO_{2n+1} is the group of signed permutations, and the Coxeter number is $2n$. The Coxeter toral Sp_{2n}-connections have slope $\frac{r}{2n}$, where $\gcd(r, 2n) = 1$.

Homogeneous Coxeter toral connections with slope $\frac{1}{2n}$, can be written as:

$$\nabla = d + (\omega_n)\frac{dt}{t}$$

where ω_n is the following matrix in $\mathfrak{so}_{2n+1}$:

$$\omega_n = \begin{pmatrix} 0 & & & & & & & t^{-1} & 0 \\ 1 & 0 & & & & & & 0 & -t^{-1} \\ & 1 & \ddots & & & & \iddots & & \\ & & \ddots & 0 & & 0 & & & \\ & & & 1 & 0 & & & & \\ & & & 0 & -1 & 0 & & & \\ & & \iddots & & & \ddots & \ddots & & \\ & 0 & & & & & -1 & 0 & \\ 0 & & & & & & & -1 & 0 \end{pmatrix}$$

The connection ∇ defined above is known as the Frenkel-Gross connection for SO_{2n+1}. Theorem 3.9 tells us that these are rigid connections, hence, they are Chen connections.

As before, we can look at $(\omega_n)^r$ with $\gcd(r, n) = 1$ to construct homogeneous Coxeter connections with slope $\frac{r}{2n}$, which by construction, are generalized Chen connections.

Below, we will see some examples. For simplicity we denote t^{-1} by u.

Example 5.1. *Coxeter* SO_3*-connection with slope* $\frac{1}{2}$

$n = 1$

$r = 1$

$$[\nabla] = \omega_1 = \begin{pmatrix} 0 & u & 0 \\ 1 & 0 & -u \\ 0 & -1 & 0 \end{pmatrix}$$

Slope: $\frac{r}{2n} = \frac{1}{2}$

Characteristic polynomial: $p(\lambda) = -\lambda(\lambda^2 - 2u)$

Eigenvalues: $0, \pm\sqrt{2}u^{\frac{1}{2}}$

$[\nabla]$ at $t = \infty$ $(u = 0)$:

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

The Jordan form of N:

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

One Jordan block of size 3. Associated partition: $P = (3)$

The dual partition: $\hat{P} = (1, 1, 1)$.

$\mathrm{Irr}(\mathrm{ad}_\nabla) = (\frac{1}{2})(2(1^2)) = 1$

$\dim C_{\mathfrak{so}_3}(N) = \frac{1}{2}(1^2 + 1^2 + 1^2 - 1) = 1$

As expected, the rigidity criterion is satisfied, and ∇ is a Chen connection.

Example 5.2. *Coxeter* SO_5*-connection with slope* $\frac{1}{4}$

$n = 2$

$r = 1$

$$[\nabla] = \omega_2 = \begin{pmatrix} 0 & 0 & 0 & u & 0 \\ 1 & 0 & 0 & 0 & -u \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Slope: $\frac{r}{2n} = \frac{1}{4}$

Characteristic polynomial: $p(\lambda) = -\lambda(\lambda^4 + 2u)$

Eigenvalues: $0, \xi^j \sqrt[4]{2} u^{\frac{1}{4}}$ where ξ is a primitive 8-th root of unity, and $j \in \{1, 3, 5, 7\}$.

$[\nabla]$ at $t = \infty$ $(u = 0)$:

$$N = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & -1 & 0 & \\ & & & -1 & 0 \end{pmatrix}$$

The Jordan form of N:

$$J = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

One Jordan block of size 5. Associated partition: $P = (5)$

The dual partition: $\hat{P} = (1, 1, 1, 1, 1)$.

Again, one can easily check that the rigidity criterion is satisfied, as expected.

Example 5.3. *Coxeter* SO_5*-connection with slope* $\frac{3}{4}$

$n = 2$

$r = 3$

$$[\nabla] = (\omega_2)^3 = \begin{pmatrix} 0 & u & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & -u & 0 \\ 1 & 0 & 0 & 0 & -u \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

Slope: $\frac{r}{2n} = \frac{3}{4}$

Characteristic polynomial: $p(\lambda) = -\lambda(\lambda^4 - 2u^3)$

Eigenvalues: $0, \pm\sqrt[4]{2}u^{\frac{3}{4}}, \pm i\sqrt[4]{2}u^{\frac{3}{4}}$

Since the connection matrix is regular semisimple with homogeneous grade $-\frac{3}{4}$, and since the denominator is the Coxeter number for SO_5, the connection is a homogeneous Coxeter connection on $\mathbb{G}_m$ with slope $\frac{3}{4}$. The slope satisfies the rigidity condition $r|n+1$ from theorem 3.9. So it is a rigid connection. Let's verify this via the rigidity criterion as well:

$[\nabla]$ at $t = \infty$ $(u = 0)$:

$$N = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ 1 & & & 0 & \\ 0 & -1 & & & 0 \end{pmatrix}$$

The Jordan form of N:

$$J = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & & & \\ & & 0 & 1 & \\ & & 0 & 0 & \\ & & & & 0 \end{pmatrix}$$

Three Jordan blocks. Associated partition: $P = (2, 2, 1)$

The dual partition is $\hat{P} = (3, 2)$.

$\mathrm{Irr}(\mathrm{ad}_\nabla) = (\frac{3}{4})(2(2^2)) = 6$

$\dim C_{\mathfrak{so}_5}(N) = \frac{1}{2}(3^2 + 2^2 - 1) = 6$

As expected, the rigidity criterion is satisfied.

5.2. Homogeneous elliptic regular SO_{2n+1}-connections

In section 2.5.4, we saw a brief description of regular elliptic classes in the Weyl group, and we mentioned that we are not giving an explicit general form for these connections, except for slope $\frac{1}{2}$, which will be discussed in section 5.3. But, we bring some examples. Let's start with a non-example:

Example 5.4. *Non-toral* SO_5*-connection with slope* $\frac{1}{2}$

$n = 2$

$k = 1$

$r = 1$

$$[\nabla] = \begin{pmatrix} 0 & 0 & 2u & 0 & 0 \\ 0 & 0 & u & 0 & 0 \\ 1 & 1 & 0 & -u & -2u \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

Slope: $\frac{r}{2k} = \frac{1}{2}$

Characteristic polynomial: $p(\lambda) = -\lambda^3(\lambda^2 + 4u)$

Eigenvalues: $0, 0, 0, \pm 2iu^{\frac{1}{2}}$

We have the repeated eigen value 0. It follows from proposition 2.52 that the connection is not toral.

Example 5.5. *Non-Coxeter homogeneous elliptic toral* SO_5*-connection with slope* $\frac{1}{2}$

$n = 2$

$k = 1$

$r = 1$

$$[\nabla] = \begin{pmatrix} 0 & 0 & 0 & u & 0 \\ 0 & 0 & u & 0 & -u \\ 0 & 1 & 0 & -u & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

Slope: $\frac{r}{2k} = \frac{1}{2}$

Characteristic polynomial: $p(x) = -\lambda(\lambda^2 - u)(\lambda^2 - 3u)$

Eigenvalues: $0, \pm u^{\frac{1}{2}}, \pm\sqrt{3}u^{\frac{1}{2}}$

So, the matrix of the connection is regular semisimple. Also, it can be seen that the matrix is homogeneous of grade $\frac{-1}{2}$ with respect to the Moy-Prasad grading associated to $x = (\frac{1}{2}, -\frac{1}{2})$, in the standard apartment. This implies that ∇ is homogeneous toral with slope $\frac{1}{2}$, viewed as a formal connection at the origin. Since the denominator of slope is an even divisor of $2n$, the singularity at origin is homogeneous elliptic toral. This means that ∇ is a homogeneous elliptic regular connection on $\mathbb{G}_m$. Since the denominator of the slope is not equal to the Coxeter number for SO_5, which is $2n = 4$, the connection is not Coxeter. Now let's see if it is rigid:

$[\nabla]$ at $t = \infty$ $(u = 0)$:

$$N = \begin{pmatrix} 0 & & & & \\ 0 & 0 & & & \\ 0 & 1 & 0 & & \\ 1 & 0 & -1 & 0 & \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

The Jordan form of N:

$$J = \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$$

Three Jordan blocks. Associated partition: $P = (3,1,1)$ Dual partition: $\hat{P} = (3,1,1)$.

$\mathrm{Irr}(\mathrm{ad}_\nabla) = (\frac{1}{2}).(2(2^2)) = 4$

$\dim_{\mathfrak{so}_5}(N) = \frac{1}{2}(3^2 + 1^2 + 1^2 - 3) = 4$

Therefore ∇ is rigid.

Example 5.6. *A homogeneous elliptic toral* SO_9*-connection with slope* $\frac{1}{4}$

$n = 4$

$k = 2$

$r = 1$

$$[\nabla] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 \\ 0 & 0 & & & & & & 0 & -u \\ 0 & 1 & 0 & 0 & 0 & u & 0 & & 0 \\ 0 & & 1 & 0 & & & -u & & 0 \\ 0 & & 0 & 1 & 0 & & 0 & & 0 \\ 1 & & 0 & 0 & -1 & 0 & 0 & & 0 \\ 0 & & 0 & 0 & 0 & -1 & 0 & & 0 \\ 0 & 0 & & & & & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice that we have intentionally kept some of the zero entries blank to highlight the central 5×5 block, which is the matrix associated to the Coxeter SO_5-connection with slope $\frac{1}{4}$; we saw that in example 5.2.

Also, notice that we have already seen this matrix in example 2.33. So, we already know that ∇ is homogeneous with respect to the Moy-Prasad grading associated to the point $x = (\frac{-3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ in the standard apartment.

The slope of ∇ is: $\frac{r}{2k} = \frac{1}{4}$.

The characteristic polynomial of $[\nabla]$ is $p(x) = -\lambda(\lambda^8 + u^2)$

So, the eigenvalues are: $0, \xi^j u^{\frac{1}{4}}$, where ξ is a primitive 16-th root of unity, and $j \in \{1, 3, 5, 7, 9, 11, 13, 15\}$

This means that $[\nabla]$ is regular semisimple; since $[\nabla]$ is also the leading term of ∇ with respect to the given Moy-Prasad grading, this implies that ∇ is toral. Since the denominator appearing in the slope is an even divisor of $2n$, we conclude that ∇ is a homogeneous elliptic toral connection.

$[\nabla]$ at $t = \infty$ $(u = 0)$:

$$N = \begin{pmatrix} 0 & & & & & & & & \\ 0 & 0 & & & & & & & \\ & 1 & 0 & & & & & & \\ & & 1 & 0 & & & & & \\ & & & 1 & 0 & & & & \\ 1 & & & & -1 & 0 & & & \\ 0 & 0 & & & & -1 & 0 & & \\ 0 & 0 & 0 & & & & -1 & 0 & \\ 0 & 0 & 0 & -1 & & & & 0 & 0 \end{pmatrix}$$

The Jordan form of N:

$$J = \begin{pmatrix} 0 & 1 & & & & & & & \\ & 0 & 1 & & & & & & \\ & & 0 & 1 & & & & & \\ & & & 0 & 1 & & & & \\ & & & & 0 & 1 & & & \\ & & & & & 0 & 1 & & \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{pmatrix}$$

Three Jordan blocks. Associated partition: $P = (7,1,1)$

The dual partition is $\hat{P} = (3,1,1,1,1,1,1)$

Now, we can easily check that ∇ is not rigid:

$\mathrm{Irr}(\mathrm{ad}_{\nabla}) = (\frac{1}{4}).(2(4^2)) = 8$

$\dim_{\mathfrak{so}_9}(N) = \frac{1}{2}(3^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 - 3) = 6$

So, ∇ is not rigid.

Example 5.7. *Non-Coxeter rigid homogeneous elliptic regular* SO_9*-connection with slope* $\frac{1}{4}$

Using Moy-Prasad, we construct the following matrix:

$n = 4$

$k = 2$

$r = 1$

$$
[\nabla_{(4,\frac{1}{4})}] = \begin{pmatrix}
0 & & & & & & 2u & u & \\
0 & 0 & & & & & & & -u \\
1 & 0 & 0 & & & & & & -2u \\
0 & 1 & 0 & 0 & & & & & \\
 & 1 & 1 & 0 & 0 & & & & \\
 & & 1 & 0 & 0 & 0 & & & \\
 & & & -1 & -1 & 0 & 0 & & \\
 & & & & -1 & -1 & 0 & 0 & \\
 & & & & & 0 & -1 & 0 & 0
\end{pmatrix}
$$

This is homogeneous with respect to the Moy-Prasad grading associated with the following coefficients: $x_1 = \frac{1}{4}$, $x_2 = 0$, $x_3 = \frac{1}{4}$, and $x_4 = 0$.

slope: $\frac{r}{2k} = \frac{1}{4}$

The characteristic polynomial is: $-\lambda(\lambda^8 + 8u\lambda^4 + 3u^2)$

It has nine distinct eigenvalues: $0, \xi^j(-4 \pm \sqrt{13})^{\frac{1}{4}} u^{\frac{1}{4}}$, where ξ is a primitive 8-th root of unity, and $j \in \{1, 3, 5, 7\}$.

It follows that $\hat{\nabla}_0$ is homogeneous elliptic toral with slope $\frac{1}{4}$. Now, let's see if ∇ is rigid:

$[\nabla]$ at $t = \infty$ $(u = 0)$:

$$N = \begin{pmatrix} 0 & & & & & & & & \\ 0 & 0 & & & & & & & \\ 1 & 0 & 0 & & & & & & \\ 0 & 1 & 0 & 0 & & & & & \\ & 1 & 1 & 0 & 0 & & & & \\ & & 1 & 0 & 0 & 0 & & & \\ & & & -1 & -1 & 0 & 0 & & \\ & & & & -1 & -1 & 0 & 0 & \\ & & & & & 0 & -1 & 0 & 0 \end{pmatrix}$$

Notice that we have already seen, in example 3.6, that:

$$\dim C_{\mathfrak{so}_9}(N) = 8$$

We also have:

$$\mathrm{Irr}(\mathrm{ad}_{\nabla_{(4,\frac{1}{4})}}) = \frac{1}{4}.\Big(2(4^2)\Big) = 8$$

So, $\nabla_{(4,\frac{1}{4})}$ is rigid, hence it is a Chen connection.

In the next example we construct the generalized Chen connection obtained by the third power of $[\nabla_{(4,\frac{1}{4})}]$, and we will see that it is a rigid connection. This example is extremely important for us, as we get **the first non-Coxeter rigid homogeneous elliptic regular connection that is not a Chen connection** (ie. $r \neq 1$).

Example 5.8. *Non-Coxeter rigid homogeneous elliptic regular* SO_9*-connection with slope* $\frac{3}{4}$

$n = 4$

$k = 2$

$r = 1$

We define $\nabla_{(4,\frac{3}{4})}$ to be the connection with the matrix:

$$[\nabla_{(4,\frac{3}{4})}] := [\nabla_{(4,\frac{1}{4})}]^3$$

where $\nabla_{(4,\frac{1}{4})}$ is the connection defined in example 5.7. So, we have:

$$[\nabla_{(4,\frac{3}{4})}] = \begin{pmatrix} 0 & -5u & -4u & & & & & & \\ & 0 & & -u & -u & & & & \\ & & 0 & -4u & -5u & -u & & & \\ & & & 0 & & & u & & \\ & & & & 0 & & 5u & u & \\ & & & & & 0 & 4u & u & \\ -1 & & & & & & 0 & & 4u \\ -2 & & & & & & & 0 & 5u \\ 0 & 2 & 1 & & & & & & 0 \end{pmatrix}$$

This is homogeneous with respect to the Moy-Prasad grading associated with the following coefficients: $x_1 = \frac{1}{4}$, $x_2 = 0$, $x_3 = \frac{1}{4}$, and $x_4 = 0$.

slope: $\frac{r}{2k} = \frac{3}{4}$

The characteristic polynomial is: $-\lambda(\lambda^8 + 440u^3\lambda^4 + 27u^6)$

It has nine distinct eigenvalues: $0, \xi^j(-220 \pm \sqrt{220^2 - 27})^{\frac{1}{4}} u^{\frac{3}{4}}$, where ξ is a primitive 8-th root of unity, and $j \in \{1, 3, 5, 7\}$.

Since the connection matrix is regular semisimple of a homogeneous grade, and the denominator of slope is even, it follows that the induced formal connection at 0 is homogeneous elliptic toral, and the global connection on $\mathbb{G}_m$ is homogeneous elliptic regular.

$[\nabla]$ at $t = \infty$ $(u = 0)$:

$$N = \begin{pmatrix} 0 &&&&&&&& \\ & 0 &&&&&&& \\ && 0 &&&&&& \\ &&& 0 &&&&& \\ &&&& 0 &&&& \\ &&&&& 0 &&& \\ -1 &&&&&& 0 && \\ -2 &&&&&&& 0 & \\ & 2 & 1 &&&&&& 0 \end{pmatrix}$$

The Jordan form of N:

$$J = \begin{pmatrix} 0 & 1 & & & & & & & \\ & 0 & & & & & & & \\ & & 0 & 1 & & & & & \\ & & & 0 & & & & & \\ & & & & 0 & & & & \\ & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{pmatrix}$$

Seven Jordan blocks. Associated partition: $P = (2, 2, 1, 1, 1, 1, 1)$

Number of odd parts: 5

The dual partition: $\hat{P} = (7, 2)$

We have:

$$\dim C_{\mathfrak{so}_9}(N) = \frac{1}{2}(7^2 + 2^2 - 5) = 24$$

and,

$$\mathrm{Irr}(\mathrm{ad}_\nabla) = \frac{3}{4}\Big(2(4^2)\Big) = 24$$

So, $\nabla_{(4, \frac{3}{4})}$ is rigid. But since the numerator of slope is not 1, it is not a Chen connection.

5.3. Construction of rigid homogeneous elliptic regular SO_{2n+1}-connections with slope $\frac{1}{2}$

We construct Chen's connections of slope $\frac{1}{2}$; recall that a Chen connection, by our definition, is a rigid homogeneous elliptic regular connection whose numerator of slope is one. We list these connections here, and then we prove that they are indeed rigid homogeneous elliptic regular of slope $\frac{1}{2}$:

For $n = 1$, and $n = 2$, these are the ones that we saw in example 5.1 and 5.5, which we repeat here, and name them $\nabla_{1,1}$ and $\nabla_{2,1}$, respectively.

$$[\nabla_{1,1}] = \begin{pmatrix} 0 & u & 0 \\ 1 & 0 & -u \\ 0 & -1 & 0 \end{pmatrix}$$

$$[\nabla_{2,1}] = \begin{pmatrix} 0 & 0 & 0 & u & 0 \\ 0 & 0 & u & 0 & -u \\ 0 & 1 & 0 & -u & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

We have already seen that, these connections are elliptic toral of slope $\frac{1}{2}$. Let us explain why we are naming them in this fashion, the first entry of the ordered pair in the sub-index of ∇ is n and the second entry is k that appears in the slope, when the slope is written in the form of $\frac{r}{2k}$. Since we are looking at slope $\frac{1}{2}$ here, k will be 1 for all of the connections that we are constructing at the moment. However, in the subsequent sections we will study Chen's connections of slope $\frac{1}{2k}$ for any integer k dividing n.

For higher dimensions, it might not be hard to guess the pattern; we "extend" the super-anti-diagonal and the sub-anti-diagonal using u's and 1's. For example, for $n = 3$, we have:

$$[\nabla_{3,1}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & u & 0 & -u \\ 0 & 0 & 0 & u & 0 & -u & 0 \\ 0 & 0 & 1 & 0 & -u & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial of $[\nabla_{3,1}]$ is:

$$p(\lambda) = -\lambda(\lambda^2 - 2u)(\lambda^2 - (2+\sqrt{2})u)(\lambda^2 - (2-\sqrt{2})u)$$

It is clear that the eigenvalues are distinct, and $[\nabla_{3,1}]$ is regular semisimple. Also, it can be easily checked that $[\nabla_{3,1}]$ is homogeneous of degree $\frac{-1}{2}$ with respect to the Moy-Prasad grading associated with the point $x = (\frac{1}{2}, \frac{-1}{2}, \frac{1}{2})$. It follows that $\nabla_{3,1}$ is homogeneous toral, viewed as a formal connection at the origin. Notice that the slope can also be read easily from the characteristic polynomial, as it is, modulo a factor of λ, homogeneous in the variables λ^2 and u. It can be easily seen that $[\nabla_{n,1}]$ is homogeneous with respect to the Moy-Prasad grading associated to the point x with the following coordinate: $x_n = \frac{1}{2}$, $x_{n-1} = \frac{-1}{2}, \dots, x_1 = (-1)^n\frac{1}{2}$. We will also see that $[\nabla_{n,1}]$'s are regular semisimple. This will prove that these connections are homogeneous toral. Since the denominator, 2, is an elliptic regular number for SO_{2n+1}, being an even divisor of $2n$, this will show that $\nabla_{n,1}$ is a homogeneous elliptic toral connection, viewed as a formal connection at the origin, and is homogeneous elliptic regular viewed as a global connection on $\mathbb{G}_m$. Next, we will show that these connections are rigid.

In the following we show that $[\nabla_{n,1}]$'s are regular semisimple; in fact we find their characteristic polynomials, and see that they are homogeneous in the variables λ^2 and u, which gives us another way to see the slope. To prove that, we will need lemma 5.9 below. It is also helpful to observe that when u is a fixed complex number, then by corollary 2.8, the characteristic polynomial, $p(\lambda)$, of $[\nabla_{n,1}]$, is in the form:

$$p(\lambda) = -\lambda \prod_{i=1}^{n} (\lambda^2 - b_i)$$

where b_i's are complex numbers.

Lemma 5.9. *Let*

$$A = \begin{pmatrix} 0 & \dots & 0 & a_1 & 0 \\ \vdots & & a_2 & 0 & b_1 \\ 0 & \iddots & \iddots & b_2 & 0 \\ a_{2n} & 0 & \iddots & & \vdots \\ 0 & b_{2n} & 0 & \dots & 0 \end{pmatrix}$$

be a matrix in $\mathfrak{gl}_{2n+1}$, with nonzero entries on the first super-anti-diagonal and the first sub-anti-diagonal, and zero entries anywhere else, then the geometric multiplicity of any eigenvalue is one.

Proof. Suppose that λ is an eigenvalue of A, that is,

$$A = \begin{pmatrix} 0 & \dots & 0 & a_1 & 0 \\ \vdots & & a_2 & 0 & b_1 \\ 0 & \iddots & \iddots & b_2 & 0 \\ a_{2n} & 0 & \iddots & & \vdots \\ 0 & b_{2n} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2n} \\ x_{2n+1} \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2n} \\ x_{2n+1} \end{pmatrix}$$

It is enough to show that, having the λ fixed, x_1 determines $x_2, x_3 \ldots x_{2n+1}$

The first row gives us:

$$a_1 x_{2n} = \lambda x_1 \implies x_{2n} = \frac{\lambda}{a_1} x_1$$

Let us denote $\frac{\lambda}{a_1}$ by c_{2n}.

Now, the $2n$'th row gives us:

$$a_{2n} x_1 + b_{2n-1} x_3 = \lambda x_2 n \implies x_3 = \frac{1}{b_{2n-1}} (\lambda c_{2n} - a_{2n}) x_1$$

And now, we denote $\frac{1}{b_{2n-1}}(\lambda c_{2n} - a_{2n})$ by c_3.

Next, we look at the third row, and similarly, we get $x_{2n_2} = c_{2n-2} x_1$, where c_{2n-2} is determined by the entries in A and λ. By induction, one can show that x_{2n-2}, x_3, x_{2n-4}, x_5, ..., x_2, and x_{2n+1} can all be written as $c_i x_1$ where c_i, for each x_i, is determined by the entries of A and λ. This proves the claim. $\square$

Now, we can prove that $[\nabla_{n,1}]$'s are regular semisimple:

Proposition 5.10. *The characteristic polynomial, $p(\lambda)$, of the matrix $[\nabla_{n,1}]$ is in the form:*

$$p(\lambda) = -\lambda \prod_{i=1}^{n} (\lambda^2 - a_i u)$$

where a_i's are distict (nonzero) positive real numbers. In particular, $[\nabla_{n,1}]$ is regular semisimple.

Proof. Let's look at $[\nabla_{n,1}]$, for a fixed n, as a matrix in the variable u; to emphasize this, we rename it $Q(u)$. Notice that its characteristic polynomial $p(\lambda)$ depends on u, and we can look at p as a polynomial in two variables λ and u, and write $p(\lambda, u)$ whenever it is helpful.

We break out the proof in two steps: first, we show that if p is in the form $p(\lambda, u) = -\lambda \prod_{i=1}^{n} (\lambda^2 - a_i u)$, then the a_i's have to be distinct and nonzero. Second, we prove that p is indeed in this form.

Step 1: Notice that $Q(1)$ is a symmetric matrix with entries in real numbers, so it is diagnolizable and the eigenvalues are real numbers. This, combined with Lemma 5.9 implies that the eigenvalues are distinct, that is, $p(\lambda, 1) = -\lambda \prod_{i=1}^{n} (\lambda^2 - a_i)$ has distinct real roots. Therefore, a_i's have to be distinct positive numbers.

Step 2: We already know that $Q(u)$ has an eigenvalue 0, that is, $p(\lambda, u)$ has a factor of λ. So let's write $p(\lambda, u) = -\lambda . g(\lambda, u)$. Furthermore, we know that no term with even degrees in λ appears in $p(\lambda, u)$. This implies that no term with odd degree of λ appears in $g(\lambda, u)$. We will show that g is a homogeneous polynomial in λ^2 and u.

As usual we denote the $n \times n$ identity matrix by I, and the group of permutations of n elemnts by S_n. Also, let's denote the matrix $Q - \lambda I$ by A, and its entries by $a_{i,j}$. So, we can write:

$$p(\lambda, u) = \det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \dots a_{n,\sigma(n)}$$

Each $\sigma \in S_n$ can be written in terms of cycles. We show that only permutations with cycles of length one or two participate in the expression for $\det A$; more precisely, the terms corresponding to the permutations which have cycles of length greater than two have at least one of the $a_{i\sigma(i)}$ equal to zero.

To see this, first we notice that the nonzero terms in A are the following:

- $a_{i,i}$ for all i from 1 to $2n + 1$,
- $a_{i,2n+1-i}$ for $1 \le i \le 2n$ (the super-anti-diagonal entries)

- $a_{i,2n+1-i+2}$ for $2 \le i \le 2n+1$ (the sub-anti-diagonal entries)

Now let's observe that cycles of length two can potentially participate in $\det A$.

The only such possibility would be a cycle of the form:

- $(r_1, 2n+1-r_1)$, for $1 \le r_1 \le 2n$. In this case we have a product of the form

  $$a_{r_1,2n+1-r_1} . a_{2n+1-r_1,r_1}$$

  This is a product of two super-anti-diagonal entries of A. Notice that we have employed the simple calculation: $2n+1-(2n+1-r_1) = r_1$.

- $(r_1, 2n+1-r_1+2)$, for $2 \le r_1 \le 2n+1$. Here we have a product of the form

  $$a_{r_1,2n+1-r_1+2} . a_{2n+1-r_1+2,r_1}$$

  This is a product of two sub-anti-diagonal entries of A. And, again, we have used: $2n+1-(2n+1-r_1+2)+2 = r_1$.

Now, suppose that σ has a cycle of length greater than two, which we denote by $(r_1, r_2, \ldots, r_k)$. This, in particular, means that r_j's are distinct, hence, a_{r_j,r_j+1} is not a diagonal entry of A, for $j \in \mathbb{Z}/k\mathbb{Z}$. Also, suppose that

$$a_{r_1,r_2} . a_{r_2,r_3} . \ldots . a_{r_{k-1},r_k} . a_{r_k,r_1} \neq 0$$

We have two possibilities for a_{r_1,r_2}:

- it is a super-anti-diagonal entry, therefore $r_2 = 2n+1-r_1$. This implies that the next term, a_{r_2,r_3} has to be a sub-anti–diagonal entry, because otherwise, $r_3 = 2n+1-(2n+1-r_1) = r_1$, and the cycle would have length two. This means that

  $$r_3 = 2n+1-(2n+1-r_1)+2 = r_1+2$$

- it is a sub-anti-diagonal entry, therefore $r_2 = 2n+1-r_1+2$. Similar to the previous case, this implies that the next term, a_{r_2,r_3} has to be a super-anti-diagonal entry, because otherwise, $r_3 = 2n+1-(2n+1-r_1+2)+2 = r_1$, and the cycle would have length two. This means that

  $$r_3 = 2n+1-(2n+1-r_1+2) = r_1-2$$

It turns out that the cycle $(r_1, r_2, \dots, r_k)$ is in one of the following forms, based on the parity of k:

If k is even, that is, $k = 2m$, then the cycle is in one of the following two forms:

- $(r_1, r_2, r_1 + 2, r_2 - 2, \dots, r_1 + 2m, r_2 - 2m)$
- $(r_1, r_2, r_1 - 2, r_2 + 2, \dots, r_1 - 2m, r_2 + 2m)$

Notice that these two cycles are essentially the same; meaning that one can write each cycle starting from r_2, and with this, and a possible relabeling of the indexes, both cycles can be written as:

$$(r_1, r_2, r_1 + 2, r_2 - 2, \dots, r_1 + 2m, r_2 - 2m)$$

This implies that $r_1 + 2m + 2 = r_1$, that is, $2m + 2 = 0$. But this is clearly a contradiction, since m is positive.

If k is odd, that is, $k = 2m + 1$, then the cycle is in one of the following two forms:

- $(r_1, r_2, r_1 + 2, r_2 - 2, \dots, r_1 + 2m, r_2 - 2m, r_1 + 2m + 2)$
- $(r_1, r_2, r_1 - 2, r_2 + 2, \dots, r_1 - 2m, r_2 + 2m, r_1 - 2m - 2)$

The first case implies that $r_2 - 2m - 2 = r_1$. We already know that, in this case, $r_2 = 2n + 1 - r_1$, therefore, we have:

$$2n + 1 - r_1 - 2m - 2 = r_1 \implies 2r_1 = 2(n - m) - 1$$

which is impossible.

The second case implies that $r_2 + 2m + 2 = r_1$. Again, we know that, in this case, $r_2 = 2n + 1 - r_1 + 2$, therefore, we have:

$$2n + 1 - r_1 + 2 + 2m + 2 = r_1 \implies 2r_1 = 2(n + m + 2) + 1$$

which is, again, impossible.

This proves the claim that, for σ to be participating in $\det A$, the cycles of σ must have length one or two. A cycle of length 1, say (i), corresponds to the presence of the diagonal entry $a_{i,i} = -\lambda$ as a factor in the term $\text{sgn}(\sigma)a_{1,\sigma(1)}a_{2,\sigma(2)}\dots a_{n,\sigma(n)}$. A cycle of length two contributes a factor u to the same term. So if σ has m cycles of length 2, and hence, $2n+1-2m$ cycles of length 1, then $\text{sgn}(\sigma) = (-1)^m$, and we have:

$$\text{sgn}(\sigma)a_{1,\sigma(1)}a_{2,\sigma(2)}\dots a_{n,\sigma(n)} = (-1)^m(-\lambda)^{2n+1-2m}u^m = (-1)^m(-\lambda)(\lambda^2)^{n-m}u^m$$

This shows that $p(\lambda,u) = \det(A) = -\lambda.g(\lambda,u)$, where g is a homogeneous polynomial in the variables u and λ^2, with degree n.

Step 3: We show that g is in the form $\prod_{i=1}^{n}(\lambda^2 - a_i u)$. This is a standard consequence of the homogeneity of g, which we repeat here for convenience:

Let's denote λ^2 by x; so g is homogeneous in x and u, with degree n. Let's write $x = yu$, then $g(x,u) = u^n.h(y)$, where h is a polynomial in y of degree n. Since $\mathbb{C}$ is algebraically closed, h factors into $(y-a_1)(y-a_2)\dots(y-a_n)$. Writing x as $(x-a_iu)+a_iu$ in g, we can conclude that $x-a_iu$ is a factor of g. Writing $g(x,\lambda) = (x-a_1u).g_1(x,u)$, one can see that $g_1(x,u)$ has to be homogeneous of degree $n-1$. Furthermore, if we write $x = yu$, we get $g_1 = u^{n-1}h_1(y)$, and it is easy to see that g_1 has to factor into $(y-a_2)\dots(y-a_n)$. It follows, by induction, that g factors as $(x-a_1u)\dots(x-a_nu)$, that is:

$$g(\lambda,u) = \prod_{i=1}^{n}(\lambda^2 - a_iu)$$

This completes the proof.

$\square$

So far, we have seen that $\nabla_{n,1}$, is a homogeneous elliptic regular as a connection on $\mathbb{G}_m$, with slope $\frac{1}{2}$ at the singularity at the origin. Now, it is time to prove that $\nabla_{n,1}$ is also rigid, and therefore it is a Chen connection.

Theorem 5.11. *The* SO_{2n+1}*-connections* $\nabla_{n,1}$ *are rigid homogeneous elliptic regular of slope* $\frac{1}{2}$.

Proof. As mentioned above, We are left to show that they are rigid. Notice that our list of $\nabla_{n,1}$'s starts with $n = 1$. We write $n = 2k+1$ or $n = 2k+2$, where k is a non-negative integer. Let $N_{n,1}$ denote $[\nabla_{n,1}]$ at $t = \infty$.

We show that the Jordan form of $N_{n,1}$ has the following partition, based on the parity of n:

$$\begin{cases} P = (3, \underbrace{2, \dots, 2}_{2k \text{ times}}) & \text{if } n = 2k+1 \\ P = (3, \underbrace{2, \dots, 2}_{2k \text{ times}}, 1, 1) & \text{if } n = 2k+2 \end{cases}$$

To show this, first suppose that n is odd. Let's denote our fixed basis for $\mathbb{C}^{2n+1}$ by $\{e_1, \dots, e_{2n+1}\}$, and let's look at the following set of vectors:

$$\begin{aligned} E = &\{e_1, e_2, \dots e_n, \\ &(e_{n+1} - e_{n+3}), (e_{n+2} - e_{n+4}), \dots, (e_{2n-2} - e_{2n}), (e_{2n-1} - e_{2n+1}), \\ &e_{2n}, e_{n+2}\} \end{aligned}$$

Notice that E consists of $2n + 1$ vectors, and one can easily check that they span $\mathbb{C}^{2n+1}$, so E forms a basis. We can see that $N_{n,1}$ has a Jordan block of size 3:

$$e_n \longmapsto e_{n+1} - e_{n+3} \longmapsto -e_{n+2} \longmapsto 0$$

and $n-1$ Jordan blocks of size 2:

$$e_1 \longmapsto e_{2n} \longmapsto 0$$

$$e_2 \longmapsto e_{2n-1} - e_{2n+1} \longmapsto 0$$

$$e_3 \longmapsto e_{2n-2} - e_{2n} \longmapsto 0$$

$$\vdots$$

$$e_{n-1} \longmapsto e_{n+2} - e_{n+4} \longmapsto 0$$

Now, suppose that n is even. Here, we consider the following set of vectors in $\mathbb{C}^{2n+1}$:

$$\begin{aligned} F = \{&e_1, e_2, \dots e_{n-1}, \\ &(e_{n+1} - e_{n+3}), (e_{n+2} - e_{n+4}), \dots, (e_{2n-2} - e_{2n}), (e_{2n-1} - e_{2n+1}), \\ &e_{n+2}, e_{2n+1}, (e_1 + e_2 + \cdots + e_{n+1})\} \end{aligned}$$

Like the previous case, F consists of $2n+1$ vectors, and it spans $\mathbb{C}^{2n+1}$, so it is a basis. And, again, $N_{n,1}$ has a Jordan block of size 3:

$$e_n \longmapsto e_{n+1} - e_{n+3} \longmapsto -e_{n+2} \longmapsto 0$$

and $n-2$ Jordan blocks of size 2:

$$e_2 \longmapsto e_{2n-1} - e_{2n+1} \longmapsto 0$$

$$e_3 \longmapsto e_{2n-2} - e_{2n} \longmapsto 0$$

$$\vdots$$

$$e_{n-1} \longmapsto e_{n+2} - e_{n+4} \longmapsto 0$$

and two Jordan blocks of size 1:

$$e_{2n+1} \longmapsto 0$$

$$e_1 + e_3 + \cdots + e_{n+1} \longmapsto 0$$

We notice that when n is odd, the number of odd parts in the partition is 1, and when n is even, the number of odd parts is 3. It is also good to observe that our partitions are indeed admissible, as the even parts appear with even multiplicity in each case.

The dual partition is:

$$\begin{cases} \hat{P} = (2k+1, 2k+1, 1) & \text{if } n = 2k+1 \\ \hat{P} = (2k+3, 2k+1, 1) & \text{if } n = 2k+2 \end{cases}$$

Recall that the rigidity criterion is: $\dim C_{\mathfrak{g}}(N) = \mathrm{Irr}(\mathrm{ad}_\nabla)$. We have $\mathrm{Irr}(\mathrm{ad}_\nabla) = \frac{1}{2}(2n^2) = n^2$. Therefore, we need to check that whether the following equations are satisfied:

$$\begin{cases} \frac{1}{2}\Big(2(2k+1)^2 + 1 - 1\Big) = (2k+1)^2 & \text{if } n = 2k+1 \\ \frac{1}{2}\Big((2k+3)^2 + (2k+1)^2 + 1 - 3\Big) = (2k+2)^2 & \text{if } n = 2k+2 \end{cases}$$

And, an easy calculation, shows that both are satisfied, that is $\nabla_{n,1}$ is rigid

□

5.4. Rigid generalized Chen connections for SO_{2n+1}

Our main goal in this section is to prove the following theorem:

Theorem 5.12. *Suppose that ∇ is a generalized Chen connection for SO_{2n+1} with slope $\frac{r}{2k}$. Then ∇ is rigid, if and only if it satisfies one of the following conditions:*

1) $k = n$ and $r | n+1$ or $r | 2n+1$

2) $r = 1$

3) $n = 6j - 2$ or $n = 6j + 2$, where j can be any positive integer. Moreover, $r = 3$ and $k = n$.

Some remarks:

1) We have already seen the Coxeter case in section 3.2, by the result of Kamgarpour-Sage.

2) case (2) in the theorem, is the tautological conclusion of the definition of generalized Chen connections and Chen's result in theorem 3.12.

So the new content of this theorem is about non-Coxeter generalized Chen connections with $r > 1$. Hence, in the proof, we will assume that $r > 1$.

Our conjecture is that a stronger version of this theorem holds, and we can classify all rigid homogeneous elliptic regular connections in this way; that is,

Conjecture 5.13. ***(Classification of rigid homogeneous elliptic regular*** SO_{2n+1}***-connections)***

Any rigid homogeneous elliptic regular SO_{2n+1}*-connection is a generalized Chen connection; equivalantly:*

Let ∇ *be a homogeneous elliptic regular* SO_{2n+1}*- connection. Then* ∇ *is rigid if and only if it is in one of the following forms:*

1) the Coxeter connections with slope $\frac{r}{2n}$*, with* $r|n+1$ *or* $r|2n+1$*.*

2) the (non-Coxeter) SO_{2n+1}*-connections,* $\nabla_{(n,k)}$ *with slope* $\frac{1}{2k}$*, with* $k|n$*, and* $k \neq n$*. (non-Coxeter Chen's connections)*

3) provided that $n = 6j - 2$ *or* $n = 6j + 2$*, the (non-Coxeter) homogeneous elliptic regular* SO_{2n+1}*-connections with slope* $\frac{3}{n}$*.*

Some remarks:

In case (2), we could include the Coxeter ones with $k = n$, but these are already counted in (1). Recall that we constructed explicit matrices for slope $\frac{1}{2}$, that is, when $k = 1$. For $k > 1$, We only find their nilpotent orbit at $t = \infty$, as we see in the next subsection.

Conjecture 5.14. *There is a unique nilpotent orbit in* SO_{2n+1} *that satisfies the rigidity criterion, for slope* $\frac{1}{2k}$*, where* $k|n$.

Equivalently:

for each positive integer k *with* $k|n$*, there is a unique partition* P *of* $2n+1$ *that satisfies:*

$$\frac{1}{2}\sum_{i=1}^{m}\hat{p}_i^2 - \frac{1}{2}(\#\ \textit{of odd parts in } P) = \frac{1}{2k}.(2n^2)$$

We already know from Chen's result, in theorem 3.12, that such a partition exists. In the following, we explicitly construct one. Conjecture 5.14 claims that it is unique, but we will not assume this conjecture in our work.

Proposition 5.15. ***(Nilpotent orbit for*** $r=1$***)***

The following partition of $2n+1$ *satisfies the rigidity criterion for slope* $\frac{1}{2k}$ *for* SO_{2n+1}.

Let $n = km$.

$$\begin{cases} P = (2k+1, \underbrace{2k, \dots, 2k}_{(m-1)\ \textit{times}}) & \textit{if } m \textit{ is odd} \\ P = (2k+1, \underbrace{2k, \dots, 2k}_{(m-2)\ \textit{times}}, 2k-1, 1) & \textit{if } m \textit{ is even} \end{cases}$$

Proof. First we notice that this is an "admissible" partition for SO_{2n+1}.

We have:

$$\mathrm{Irr}(\mathrm{ad}_{\nabla}) = \frac{1}{2k}(2n^2) = \frac{(km)^2}{k} = km^2$$

Now, suppose that m is odd. Then, the dual partition is:

$$\hat{P} = (\underbrace{m, \dots, m}_{(2k)\ \text{times}}, 1)$$

Therefore:

$$\frac{1}{2}\sum_{i=1}^{m}\hat{p}_i^2 - \frac{1}{2}(\#\text{ of odd parts in } P) = \frac{1}{2}(2km^2 + 1 - 1) = km^2$$

So, the rigidity criterion is satisfied.

Now, suppose that m is even. Then, the dual partition is:

$$\hat{P} = (m+1, \underbrace{m, \dots, m}_{(2k-2)\text{ times}}, m-1, 1)$$

Therefore:

$$\frac{1}{2}\sum_{i=1}^{m}\hat{p}_i^2 - \frac{1}{2}(\#\text{ of odd parts in } P) =$$

$$\frac{1}{2}\Big((m+1)^2 + (2k-2)m^2 + (m-1)^2 + 1 - 3\Big) = km^2$$

And again, the rigidity criterion is satisfied.

□

5.4.1. Nilpotent orbit for $r > 1$

We begin with the following proposition, which is straightforward to verify:

Proposition 5.16. *Suppose that J is a Jordan block of size l, and $l = tr + s$ with $s < r$. Then J^r decomposes into s blocks of size $t+1$ and $r-s$ blocks of size t.*

Suppose that the matrix N has the Jordan decomposition associated to the partition in proposition 5.15. Then N^r has the following partition:

writing $2k = tr + s$,

$$\begin{cases} (\underbrace{t+1, \dots, t+1}_{ms+1\text{ times}}, \underbrace{t, \dots, t}_{m(r-s)-1\text{ times}}) & \text{if } m \text{ is odd} \\ (\underbrace{t+1, \dots, t+1}_{ms\text{ times}}, \underbrace{t, \dots, t}_{m(r-s)\text{ times}}, 1) & \text{if } m \text{ is even} \end{cases}$$

Note: this still works for the case $s = r-1$, where we have $2k+1 = tr+(s+1) = (t+1)r$. In this case we have 0 blocks of size $t+2$ and $r-0 = s+1$ blocks of size t+1, which is compatible with the case $s < r-1$, where we have $2k+1 = tr+(s+1)$, and we get $s+1$ blocks of size $t+1$ and $r-(s+1)$ blocks of size t.

If m is odd, we have:

$$\text{number of odd parts} = \begin{cases} m(r-s)-1 & \text{if } t \text{ is odd} \\ \\ ms+1 & \text{if } t \text{ is even} \end{cases}$$

and if m is even,

$$\text{number of odd parts} = \begin{cases} m(r-s)+1 & \text{if } t \text{ is odd} \\ \\ ms+1 & \text{if } t \text{ is even} \end{cases}$$

The dual partition for N^r is:

$$\begin{cases} (\underbrace{mr,\ldots,mr}_{t\text{ times}},ms+1) & \text{if } m \text{ is odd} \\ \\ (mr+1,\underbrace{mr,\ldots,mr}_{t-1\text{ times}},ms) & \text{if } m \text{ is even} \end{cases}$$

5.4.2. Proof of Theorem 5.12

Proof. Recall that, we only need to consider the case $r > 1$. We have $\mathrm{Irr}(\mathrm{ad}_\nabla) = \frac{r}{2k}.(2n^2)$. As before, we write: $n = km$ and$2k = rt+s$. So,

$$\mathrm{Irr}(\mathrm{ad}_\nabla) = \frac{r}{2k}.(2n^2) = rkm^2 = \frac{1}{2}rm^2(rt+s)$$

Now we can write the rigidity criterion, for all even and odd cases of m and t.

Case 1: m odd, t odd

Rigidity criterion gives:

$$\frac{1}{2}\Big(m^2r^2t + (ms+1)^2 - (mr - ms - 1)\Big) = \frac{1}{2}rm^2(rt+s)$$

Simplifying, we get:

$$m\Big(ms(r-s) - 3s + r\Big) = 2$$

This implies that $m \mid 2$, and since we have assumed m is odd, the only possibility is $m = 1$; but this is the Coxeter case, which implies that we don't have any non-Coxeter rigid connections when m and t are both odd.

The above equation, and the fact that $m = 1$ impose another restriction:

$$\Big(s(r-s) - 3s + r\Big) = 2$$

Let's verify if this is compatible with the classification of rigid Coxeter connections for SO_{2n+1}, namely, the condition that $r \mid n+1$ or $r \mid 2n+1$.

Rewriting the above equation:

$$s^2 + s(3-r) + (2-r) = 0$$

This factors to:

$$(s+1)(s+2-r) = 0$$

Since $s + 1 > 0$, we get $s = r - 2$. now, we can write $2n = 2km = 2k = rt + (r-2)$, which implies $2(n+1) = r(t+1)$, and since r is odd, we have $r \mid n+1$. On the other hand, if $r \mid n+1$, and $r > 1$, we can write $2(n+1) = r(t+1)$, for some integer t, and we have $2n = rt + (r-2)$, which implies $s = r - 2$.

Case 2: m **odd,** t **even**

$$\frac{1}{2}\Big(m^2r^2t + (ms+1)^2 - (ms+1)\Big) = \frac{1}{2}rm^2(rt+s)$$

Simplifying, we get:

$$ms\Big(m(s-r)+1\Big) = 0$$

Recall that $\gcd(2k, r) = 1$, and $2k = rt + s$, and we are assuming $r > 1$, so, $s \neq 0$. Also, $m \neq 0$, therefore, $m(r-s) = 1$, which implies that $m = 1$ and $r - s = 1$, or equivalently, $s = r - 1$. Since $m = 1$, we are again in the Coxeter case, or in other words we conclude that there is no non-Coxeter rigid connection when m is odd and t is even. Like the previous case, let's verify if the restriction $s = r - 1$ is compatible with the classification of rigid Coxeter connections for SO_{2n+1}:

$$s = r-1 \iff 2n = rt + (r-1) \iff 2n+1 = r(t+1) \iff r \mid 2n+1$$

Case 3: m **even,** t **odd**

$$\frac{1}{2}\Big((mr+1)^2 + (t-1)m^2r^2 + m^2s^2 - \big(m(r-s)+1\big)\Big) = \frac{1}{2}rm^2(rt+s)$$

Simplifying, we get:

$$2mr + m^2s^2 - m(r-s) = m^2rs$$

and since $m \neq 0$, we can divide by m, and after rearranging we get: $r = \big((r-s)m - 1\big)s$, which implies $s \mid r$, but $\gcd(r, s) = 1$, so $s = 1$, and we get $r = (r-1)m - 1$, which can be rearranged to $(r-1)(m-1) = 2$. Since m is even, $m - 1$ is odd, and therefore $m - 1 = 1$ and $r - 1 = 2$, that is, $m = 2$ and $r = 3$.

Case 4: m **even,** t **even**

$$\frac{1}{2}\Big((mr+1)^2+(t-1)m^2r^2+m^2s^2-(ms+1)\Big)=\frac{1}{2}rm^2(rt+s)$$

Dividing by m and simplifying, we get:

$$2r=s\big(1+m(r-s)\big)$$

Since $\gcd(r,s)=1$, we get $s \mid 2$. Since $2k=rt+s$ and t is even, s is even too, so $s=2$. This gives us: $r=1+mr-2m$, which can be rearranged to $(m-1)(r-2)=1$. Since both $m-1$ and $r-2$ are nonnegative, we get $m-1=1$ and $r-2=1$, which gives us: $m=2$ and $r=3$.

$\square$

Bibliography

[1] M. Kamgarpour and D. S. Sage, "A geometric analogue of a conjecture of Gross and Reeder," Amer. J . Math. **141** (2019), 1457–1476.

[2] M. Kamgarpour and D. S. Sage, "Rigid connections on $\mathbb{P}^1$ via the Bruhat-Tits building," Proc. London Math. Soc., to appear, arXiv:1910.00181[math.RT].

[3] E. Frenkel and B. Gross, "A rigid irregular connection on the projective line," Ann. of Math. (2) **170** (2009), 1469–1512.

[4] M. Kamgarpour and L. Yi, "Geometric Langlands for hypergeometric sheaves," arXiv:2006.10870[math.AG].

[5] T.-H. Chen, "Vinberg's θ-groups and rigid connections," Int. Math. Res. Not. IMRN **2017** (2017), 7321–7343.

[6] T.-H. Chen and M. Kamgarpour, "Preservation of depth in local geometric Langlands correspondence," Trans. Amer. Math. Soc. **369** (2017), 1345–1364.

[7] Z. Yun, "Rigidity in automorphic representations and local systems," in *Current Developments in Mathematics 2013*, Int. Press, Somerville MA, 2014, pp. 73–168.

[8] Z. Yun, "Epipelagic representations and rigid local systems," Selecta Math. (N.S.) **22** (2016), 1195–1243.

[9] X. Zhu, "Frenkel-Gross' irregular connection and Heinloth-Ngô-Yun's are the same," Selecta Math. (N.S.) **23** (2017), 245–274.

[10] D. S. Sage, "Regular strata and moduli spaces of irregular singular connections," in *New Trends in Analysis and Interdisciplinary Applications: Selected Contributions of the 10th ISAAC Congress, Macau 2015*, Trends in Mathematics, Birkhaüser/Springer, Cham, 2017, pp. 59–65.

[11] C. Bremer and D. S. Sage, "Isomonodromic deformation of connections with singularities of parahoric formal type," Comm. Math. Phys., **313** (2012), 175–208.

[12] C. Bremer and D. S. Sage, "Moduli spaces of irregular singular connections," Int. Math. Res. Not. IMRN, **2013** (2013), 1800–1872.

[13] C. Bremer and D. S. Sage, "A theory of minimal K-types for flat G-bundles," Int. Math. Res. Not. IMRN, **2018** (2018), 3507–3555.

[14] C. Bremer and D. S. Sage, "Flat G-bundles and regular strata for reductive groups," submitted to Adv. Math., arXiv:1309.6060[math.AG].

[15] C. Bremer and D. S. Sage, "Rigid connections with Iwahori type singularities," in preparation.

[16] W. Crawley-Boevey, "On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero," Duke Math. J. **118** (2003), 339–352.

[17] S. Bloch and H. Esnault, "Local Fourier transforms and rigidity for $\mathscr{D}$-modules," Asian J. Math. **8** (2004), 587–605.

[18] N. Livesay, *Moduli spaces of flat* GSp*-bundles*, Ph.D. thesis, Louisiana State University and Agricultural and Mechanical College, Baton Rouge, 2017.

[19] E. Frenkel and X. Zhu, "Any flat bundle on a punctured disc has an oper structure," Math. Res. Lett. **17** (2010), 27–37.

[20] V. Ginzburg, "Perverse sheaves on a loop group and Langlands duality," arXiv:alg-geom/9511007.

[21] V. Ginzburg, M. Kapranov, and E. Vasserot, "Elliptic algebras and equivariant elliptic cohomology," arXiv:q-alg/9505012.

[22] J. Heinloth, B.-C. Ngô, and Z. Yun, "Kloosterman sheaves for reductive groups," Ann. of Math. (2) **177** (2013), 241–310.

[23] K. Jakob and Z. Yun, "Euphotic representations and rigid automorphic data," arXiv:2008.04029[math.AG].

[24] M. Jimbo, T. Miwa, and K. Ueno, "Monodromy preserving deformations of linear ordinary differential equations with rational coefficients I," Physica D **2** (1981), 306–352.

[25] N. Katz, "On the calculation of some differential Galois groups," Invent. Math. **87** (1987), 13–61.

[26] N. Katz, "On the monodromy groups attached to certain families of exponential sums," Duke Math. J. **54** (1987), 41–56.

[27] N. Katz, *Gauss sums, Kloosterman sums, and monodromy groups*, Annals of Mathematics Studies **116**, Princeton University Press, Princeton, NJ, 1988.

[28] N. Katz, *Exponential sums and differential equations*, Annals of Mathematics Studies **124**, Princeton University Press, Princeton, NJ, 1990.

[29] N. Katz, *Rigid local systems*, Annals of Mathematics Studies **139**, Princeton University Press, Princeton, NJ, 1996.

[30] M. Kulkarni, N. Livesay, J. Matherne, B. Nguyen, and D. S. Sage, "The Deligne-Simpson problem for connections on $\mathbb{G}_m$ with a maximally ramified singularity," in preparation.

[31] A. Moy and G. Prasad, "Unrefined minimal K-types for p-adic groups," Invent. Math. **116** (1994), 393-408.

[32] A. Moy and G. Prasad, "Jacquet functors and unrefined minimal K-types," Comment. Math. Helv. **71** (1996), 98-121.

[33] P. Boalch, "Irregular connections and Kac-Moody root systems," arXiv:0806:1050.

[34] D. Kazhdan and G. Lusztig, "Fixed point varieties on affine flag manifolds," Israel J. Math. **62** (1988), 129–168.

[35] T. A. Springer, "Regular elements of finite reflection groups," Invent. Math., **25** (1974), 159–198.

[36] O. Šuch, "Monodromy of Airy and Kloosterman sheavess," Duke Math. J. **103** (2000), 397–444.

[37] W. Wasow, *Asymptotic expansions for ordinary differential equations*, Wiley Interscience, New York, 1976.

[38] E. Frenkel, *Langlands correspondence for loop groups*, Cambridge University Press, New York, 2007.

[39] P. Abramenko and K. S. Brown, *Buildings; theory and applications*, Springer, New York, 2008.

[40] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1972.

[41] D. H. Collingwood and W. H. McGovern, *Nilpotent orbits in semisimple Lie algebras,* Van Nostrand Reinhold Co., New York, 1993.

Vita

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