6-9-2021

An Equivalence Between Contact Gluing Maps in Sutured Floer Homology: A Conjecture of Zarev

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AN EQUIVALENCE BETWEEN CONTACT GLUING MAPS IN SUTURED FLOER HOMOLOGY: A CONJECTURE OF ZAREV

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
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B.A., Rice University, 2013
August 2021

Deep in the human unconscious is a
pervasive need for a logical universe
that makes sense. But the real
universe is always one step beyond
logic.

—Frank Herbert
Dune

Acknowledgments

I would first like to express my gratitude to my advisor Shea Vela-Vick for introducing me to the subject, for suggesting the present problem, and for his patient support throughout the development of the project. Without his guidance this work would not have been possible. I would also like to thank my advisor Scott Baldridge for his insight and advice about both the day to day and philosophy of being a mathematician.

I am greatly indebted to Mike Wong for his invaluable encouragement and suggestions for improving various versions of the manuscript. His wisdom has contributed to the quality of the exposition and supported me through the last hurdles of completing this project.

This journey would not have been so successful without my coauthor and academic brother, Federico Salmoiraghi. He has been a faithful colleague and friend, and has encouraged me in my academic and personal life. It would take too long to properly thank the graduate students, professors, and staff at LSU who have enriched my life over the years, but I would especially like to thank Tara Fife, Sudipta Ghosh, and Tamanna Chatterjee for their friendship.

Finally, I would like to thank my family for their unwavering love and support. To my parents, Russell and Sharon: you raised me with delight, thought, and integrity; I would not be who I am today without you. I hope to be half the parent to my chil-

dren you have been to me. To my sister, Allison: you have always been there for me in difficult times with encouragement and wisdom; I cannot express enough the admiration and appreciation I have for you. To my brother-in-law, Nick: you have given me sound advice and support as this chapter of life closes and a new one opens, and I am grateful for it.

Contents

Abstract

We show that the contact gluing map of Honda, Kazez, and Matic has a natural algebraic description in bordered sutured Floer homology. In particular, we establish Zarev's conjecture that his gluing map on sutured Floer homology is equivalent, in the appropriate sense, to the contact gluing map. This further solidifies the relationship between bordered Floer theory and contact geometry.

Chapter 1. Introduction

Contact geometry originates from the study of dynamical systems. Like symplectic geometry, it is a natural model for phase spaces in classical mechanics, and has applications to control theory, optics, and thermodynamics. While contact structures exist in all odd dimensions, they have been most studied in dimension 3. A *contact 3-manifold* is a pair (M, ξ), where ξ is a non-integrable 2-plane field on a 3-manifold M. A consequence of the non-integrability condition is that there is no embedded surface whose tangent bundle is contained in ξ.

In recent years, Floer theory has yielded numerous gauge-theoretic invariants for studying 3-manifolds, knots, and contact structures. Among these, Heegaard Floer invariants are particularly easy to compute concretely. First formulated by Ozsváth and Szabó [OS04], the theory associates a chain complex $\widehat{\mathrm{CF}}(Y)$ to a Heegaard splitting of a closed 3-manifold Y. Since the chain homotopy type of the complex is independent of the splitting, its homology $\widehat{\mathrm{HF}}(Y)$ is an invariant of Y called *Heegaard Floer homology.*

There are also invariants adapted to cut-and-paste operations. A *bordered 3-manifold*, is a 3-manifold Y_i with boundary and a fixed identification $\partial Y_i \cong F$, for some surface F. Lipshitz, Ozsváth, and Thurston [LOT18] assign A_∞-modules $\widehat{\mathrm{CFA}}(Y_i), \widehat{\mathrm{CFD}}(Y_i)$ called *bordered modules*, such that if $Y = Y_1 \cup_F Y_2$, then $\widehat{\mathrm{CFA}}(Y_1) \boxtimes \widehat{\mathrm{CFD}}(Y_2) \simeq \widehat{\mathrm{CF}}(Y)$. Here the *box tensor product* $\boxtimes$ is a model for the $\mathcal{A}_\infty$-tensor product. While the other homology theories we discuss can be defined with

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$\mathbb{Z}$-coefficients, the bordered invariants are only defined over $\mathbb{F}_2$-coefficients. As such, we exclusively work with $\mathbb{F}_2$-coefficients through the entirety of this paper.

There is another version of the theory particularly well-suited for contact geometry. A *balanced sutured manifold* (M, Γ) is a 3-manifold M with boundary and a multicurve $\Gamma \subset \partial M$ satisfying certain conditions. Juhász [Juh06] associates a complex $\mathrm{SFC}(M, \Gamma)$ whose homology $\mathrm{SFH}(M, \Gamma)$, called *sutured Floer homology*, is an invariant of (M, Γ). If a contact manifold has a convex boundary, then the dividing set on the boundary determines a balanced sutured manifold. The requirement that the boundary be convex is not restrictive, since any boundary can be made convex after a small perturbation.

For a closed contact manifold (Y, ξ), Ozsváth and Szabó [OS05] defined a *contact invariant* $c(\xi) \in \widehat{\mathrm{HF}}(-Y)$. This invariant vanishes on overtwisted contact structures and is always non-zero when (Y, ξ) is Stein fillable. Honda, Kazez, and Matić [HKM09] define a more general invariant $\mathrm{EH}(\xi) \in \mathrm{SFC}(-M, -\Gamma)$ of contact manifolds with boundary (M, Γ). These and other versions of the contact invariant have profoundly advanced our understanding of contact geometry in dimension three. Ghiggini [Ghi05, Ghi06b, Ghi06a] has used them to study the fillability of closed 3-manifolds. Invariants commonly referred to as LOSS invariants for Legendrian and transverse knots were introduced by Lisca, Ozsváth, Stipsicz, and Szabó [LOSS09]; connections with the invariant defined by Honda, Kazez, and Matić were first shown by Stipscz and Veŕtesi

[SV09], and extended by Etnyre, Vela-Vick, and Zarev [EVZ17].

Much of the power of Heegaard Floer invariants stem from their functorial properties, which come in two flavors. If Y_1 and Y_2 are closed 3-manifolds, then Osváth and Szabó [OS04] showed that a cobordism X with $\partial X = -Y_1 \sqcup Y_2$ induces a map $\mathcal{F}_X : \widehat{\mathrm{HF}}(Y_1) \to \widehat{\mathrm{HF}}(Y_2)$. Juhász [Juh16] defined the notion of a sutured cobordism between balanced sutured manifolds. He also showed that a sutured cobordism (X, Z, ξ) from (M_1, Γ_1) to (M_2, Γ_2) induces a map $\mathcal{F}_{(X,Z,\xi)} : \mathrm{SFH}(M_1, \Gamma_1) \to \mathrm{SFH}(M_2, \Gamma_2)$. These cobordism maps are one sense in which Floer theory is functorial.

The other notion of functoriality is provided by contact gluing maps. Honda, Kazez, and Matić [HKM08] constructed the first such map Φ_ξ, commonly referred to as the *HKM map*, associated to a proper inclusion of sutured manifolds $(M, \Gamma) \subset (M', \Gamma')$ and a compatible contact structure ξ on the complement $M' \setminus \mathrm{Int}(M)$. When $M' \setminus \mathrm{Int}(M)$ has no connected components disjoint from $\partial M'$, the gluing map takes the form

$$\Phi_\xi : \mathrm{SFH}(-M, -\Gamma) \to \mathrm{SFH}(-M', -\Gamma').$$

This map preserves the contact invariant EH in the following way. If ζ is a contact structure on (M, Γ) compatible with ξ, then

$$\Phi_\xi(\mathrm{EH}(\zeta)) = \mathrm{EH}(\zeta \cup \xi).$$

The HKM map is also functorial in the sense that it satisfies identity and composition laws; see Section 6 of [HKM08].

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Because of these properties, HKM maps play an important role in both sutured Floer theory and contact geometry. Of note, Juhász [Juh16] employed them in his construction of cobordism maps on sutured Floer homology and Golla [Gol15] used HKM maps to relate the LOSS invariant for Legendrian knots in S^3 with the invariant defined by Honda, Kazez, and Matić. HKM maps can often track contact invariants under cut-and-paste operations. In this regard, Massot [Mas12] used them to systematically produce isotopy classes of universally tight, torsion-free contact structures with vanishing Ozsváth–Szabó contact invariant. Recently, Juhász and Zemke [JZ20] gave an alternate description of the HKM map in terms of contact handles and used it to prove several results about cobordisms maps in sutured Floer homology.

Outside the realm of contact geometry, Zarev [Zar11] defined a gluing operation $\cup_{\mathcal{F}}$ for sutured manifolds (M_1, Γ_1), (M_2, Γ_2) with suitable sutured subsurfaces $\mathcal{F} \subset \partial M_1$, $\overline{\mathcal{F}} \subset \partial M_2$. He also defined an associated gluing map which takes the form of a pairing

$$\Psi_{\mathcal{F}} : \mathrm{SFH}(M_1, \Gamma_1) \otimes \mathrm{SFH}(M_2, \Gamma_2) \to \mathrm{SFH}((M_1, \Gamma_1) \cup_{\mathcal{F}} (M_2, \Gamma_2)).$$

If M_1 is equipped with a contact structure ξ compatible with Γ_1, then we can define a map which is formally similar to the HKM map

$$\Psi_{\xi} : \mathrm{SFH}(-M_2, -\Gamma_2) \to \mathrm{SFH}((-M_1, -\Gamma_1) \cup_{-\mathcal{F}} (-M_2, -\Gamma_2))$$

by $\Psi_{\xi}(\mathbf{y}) = \Psi_{-\mathcal{F}}(\mathrm{EH}(\xi) \otimes \mathbf{y})$. We will refer to Ψ_{ξ} as the *bordered contact gluing map*, since Zarev used bordered sutured theory to define the map $\Psi_{\mathcal{F}}$.

Due to its algebraic nature, it not clear a priori whether this map deserves the moniker of contact gluing map. Indeed, it is not apparent from the definition that Ψ_ξ sends contact invariants to contact invariants. Despite this, Zarev claimed that the HKM and bordered contact gluing maps are equivalent. The main result of this paper affirms this conjecture; we give a rough formulation here and the precise statement in Theorem 7.0.3.

Theorem 1. *Suppose that Φ_ξ is the* HKM *map for a proper inclusion $(M, \Gamma) \subset (M', \Gamma')$ of sutured manifolds with no isolated components and compatible contact structure ξ. There is a sutured contact manifold (M'', Γ'', ξ'') and graded isomorphisms f, g such that $(M', \Gamma') \cong (M'', \Gamma'') \cup_\mathcal{F} (M, \Gamma)$, the contact structures ξ and ξ'' have equivalent contact handle decompositions, and the following diagram commutes.*

$$
\begin{array}{ccc}
\mathrm{SFH}(-M, -\Gamma) & \xrightarrow{\ \ \ \ \ f\ \ \ \ \ } & \mathrm{SFH}(-M, -\Gamma) \\
\downarrow{\scriptstyle \Phi_\xi} & & \downarrow{\scriptstyle \Psi_{\xi''}} \\
\mathrm{SFH}(-M', -\Gamma') & \xrightarrow{\ \ g\ \ } & \mathrm{SFH}((-M'', -\Gamma'') \cup_{-\mathcal{F}} (-M, -\Gamma))
\end{array}
$$

The strategy for attacking this theorem is as follows. We first describe the HKM maps for contact handle attachments in terms of certain diagrammatic maps for contact handles. We then identify these diagrammatic maps with bordered contact gluing maps for contact handle attachments. Next, we prove that an arbitrary bordered contact gluing map is a composition of such handle attachment maps; along the way, we must

show that Ψ_ξ maps contact invariants to contact invariants and satisfies an appropriate identity law. Combining all these results with the functorial properties of the HKM map yields Theorem 1.

Remark 1.0.1. Note that the hypothesis of Theorem 1 requires that $M' \setminus int(M)$ have no components disjoint from $\partial M'$ (no "isolated components"); this ensures that ξ can be decomposed using only contact 1-and 2-handles.

Remark 1.0.2. It is natural to ask whether the graded isomorphisms in Theorem 1 can be chosen to be identity maps. As of the writing of this paper, the map $\Psi_{\mathcal{F}}$ is only known to be well-defined up to maps induced by graded homotopy equivalence, since it is defined using bordered sutured theory. A stronger version of Theorem 1 would require naturality of bordered sutured Floer homology.

Theorem 1 has several interesting consequences. First, we obtain a bordered version of the HKM map.

Corollary 1.0.3. *Let $\mathcal{M} = (M, \gamma, \mathcal{F}, \mathcal{Z})$ and $\mathcal{M}' = (M', \gamma', \mathcal{F}, \mathcal{Z})$ be bordered sutured manifolds with $M \subset M'$ and $M' \setminus int(M)$ a sutured manifold equipped with a compatible contact structure ξ. Then there exists a map of type-D structures induced by ξ*

$$\Phi_\xi : \widehat{\mathrm{BSD}}(-\mathcal{M}) \to \widehat{\mathrm{BSD}}(-\mathcal{M}'),$$

which is natural with respect to gluing bordered sutured manifolds along $\mathcal{F}$.

Similar statements can be formulated for type-A structures and the various bi-module structures involved in the bordered sutured theory. Of note, Etnyre, Vela-Vick,

and Zarev [EVZ17] used this corollary as a fundamental tool to prove their alternate characterization of the minus version of knot Floer homology. As Mathews [Mat19] points out, this corollary also has interesting implications for the relationship between sutured Floer homology and the contact category defined by Honda and Tian [HT16].

Theorem 1 also provides an alternate proof of Juhasz and Zemke's description of contact gluing maps for contact handle attachments found in [JZ20]. Honda, Kazez, and Matić's original description of their gluing map required Heegaard diagrams to satisfy a number of technical conditions, collectively referred to as contact-compatibility. Even for contact handle attachments, the use of contact-compatible Heegaard diagrams often renders the associated HKM gluing map uncomputable on the level of chain complexes. As a result, most prior applications relied only on formal properties of the HKM maps. However, there are a priori diagram-dependent maps σ_i associated to contact handle attachments which are simple to compute at the level of chain complexes; see chapter 3. It is possible to describe a given HKM map in terms of diagrammatic maps; see Lemma 3.0.8. It turns out that these diagrammatic maps are well-defined up to graded isomorphism. This means that an arbitrary diagrammatic map induces the HKM up to graded ismorphism.

Corollary 1.0.4. *The diagrammatic maps σ_i are well-defined up to graded isomorphism. In particular, any diagrammatic map induces the corresponding* HKM *map up to graded isomorphism.*

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Finally, we use Corollary 1.0.4 to provide an algorithm for computing the HKM map combinatorially following Plamenevskaya's work in [Pla07].

Corollary 1.0.5. *Given a proper inclusion of sutured manifolds $(M, \Gamma) \subset (M', \Gamma')$ with no isolated components and compatible contact structure ξ on $M' \setminus int(M)$, there are nice Heegaard diagrams $\mathcal{H}, \mathcal{H}'$ for $(-M, -\Gamma), (-M', -\Gamma')$ respectively and an inclusion of complexes $\mathrm{SFC}(\mathcal{H}) \to \mathrm{SFC}(\mathcal{H}')$ which can be used to compute the* HKM *map $\Phi_\xi :$ $\mathrm{SFH}(-M, -\Gamma) \to \mathrm{SFH}(-M', -\Gamma')$ up to graded isomorphism.*

Organization

In chapter 3, we introduce the diagrammatic maps and relate them to the corresponding HKM maps; see Lemma 3.0.8. In chapter 4, we review the bordered sutured theory that goes into Zarev's gluing map $\Psi_{\mathcal{F}}$ on sutured Floer homology. We define the associated contact gluing map Ψ_ξ in chapter 5 and prove that it satisfies the composition law Lemma 5.0.13. chapter 6 reviews more background on making concrete computations with the bordered gluing map and using the formula in Lemma 6.0.1 in particular. We relate the diagrammatic maps with the bordered contact gluing map in chapter 7 by Lemma 7.0.1 and Lemma 7.0.2 and use them to prove two versions of Theorem 1. We discuss corollaries and applications in chapter 8.

Chapter 2. General Background

We outline the basic ideas in contact geometry and Floer theory necessary to understand most of the paper.

An important idea in contact greometry is the dichotomy of overtwisted versus tight. A contact 3-manifold (M, ξ) is *overtwisted* if there is an embedded disk $D \subset M$ with $T(\partial D) \subset \xi$. If a contact manifold is not overtwisted, it is called *tight.* Eliashberg [Eli89] showed that the overtwisted contact structures on a given M are equivalent to some well-understood algebro-topological information about M. As such, current research on contact 3-manifolds focuses on tight contact structures.

There is a refinement of classical knot theory in contact geometry. A knot $K \subset (M, \xi)$ is *transverse* if $T_p K \cap \xi_p = 0 \in T_p M$ for all $p \in K$. Knots K, K' are said to be *transversely isotopic* if they are isotopic through transverse knots. A knot $K \subset (M, \xi)$ is *Legendrian* if $T_p K \subset \xi_p$ for all $p \in K$. Knots K, K' are *Legendrian isotopic* if they are isotopic through Legendrian knots. Any knot in a contact manifold is C^0-close to a transverse or Legendrian knot. The construction of partial open books discussed in Subsection 3.0.2 uses a *Legendrian graph*, which is an embedded graph whose edges are Legendrian. The problem of classifying the transverse and Legendrian knots sharing a given knot type is of interest in its own right. Furthermore, by Etnyre and Van Horn-Morris [EV11], a contact structure is determined by its transverse knot theory.

The maps we study in this paper are associated to cutting and gluing operations

on contact manifolds; these operations are the concern of convex surface theory. A surface F in a contact manifold (M, ξ) is *convex* if there is a vector field v on M that is transverse to F and whose flow preserves ξ. This condition is not restrictive, since any surface is C^∞-close to a convex surface. The set of points on a convex surface where the vector field v is tangent to ξ is called the *dividing set* on F. The dividing set does not depend on the choice of v up to isotopy. It is standard practice to confuse a dividing set with its isotopy class, and we do so in this paper. We will also allow our convex surfaces to have corners. In this case, the dividing set has breaks at the corners. However, there is a canonical way to close the curves of the dividing set when one smooths the corners. This is referred to as *smoothing*, and we use it implicitly throughout the paper. See the Edge-rounding Lemma in [Hon00].

The contact invariants under consideration only depend on the isotopy class of the associated contact structure. For contact manifolds with convex boundary, these invariants also do not distinguish between contact structures related by *flexibility* as defined in [BS16]. In a neighborhood of the boundary, the contact structure is determined up to isotopy and flexibility by the dividing set. The upshot is that a dividing set for a convex surface roughly encodes the information needed to glue two contact structures together.

There is a version of Morse theory for contact manifolds with boundary. The theory allows us to decompose a contact manifold into the usual, 0,1,2, and 3-handles

equipped with a certain tight contact structure; these are called *contact handles.* For the most part, we will only need to concern ourselves with the dividing sets on the boundary of these handles. Contact 0-handles and 3-handles are Darboux balls, so the dividing set on each is a single simple closed curve. The dividing sets for contact 1-handles and 2-handles are shown in Figure 3.1; note that we have smoothed each dividing set as discussed previously. See [Ozb11] for more regarding the theory of contact handle decompositions.

Sutured Floer homology was defined by Juhász in [Juh06]. A *balanced sutured manifold* (M, Γ) is a 3-manifold M with boundary with an embedded oriented multicurve $\Gamma \subset \partial M$ such that

- $\overline{\partial M \setminus \Gamma} = R_+ \sqcup R_-$

- the induced orientation on ∂R_+ agrees with the orientation on Γ

- the induced orientation on ∂R_- disagrees with the orientation on Γ

- $\chi(R_+) = \chi(R_-)$, where χ is the Euler characteristic

A *diagram* $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for a balanced sutured manifold consists of an oriented surface with boundary Σ along with two collections of pairwise disjoint embedded curves $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_g\}$ and $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_g\}$. The balanced sutured manifold describe by this diagram is obtained by attaching 2-handles along each curve $\alpha_i \times \{-1\}$ and $\beta_j \times \{1\}$ in $\Sigma \times [-1, 1]$; the suture set Γ is the multicurve $\partial \Sigma \times \{0\}$.

We must impose some technical condition on our diagrams. The *regions* of a

diagram are the closures of components of $\Sigma \setminus (\bigcup \boldsymbol{\alpha} \cup \bigcup \boldsymbol{\beta})$; a region is a *basepoint region* if it has non-empty intersection with $\partial \Sigma$. The *domains* of a diagram are elements of the free abelian group generated by all non-basepoint regions; a *periodic domain* is domain whose boundary is a sum of curves in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. A diagram is *admissible* if each of its periodic domains have both positive and negative coefficients.

One associates a chain complex $\mathrm{SFC}(\mathcal{H})$ to an admissible diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ in the following way. The *generators* $\mathcal{G}$ of $\mathcal{H}$ are unordered g-tuples of points $\mathbf{x} = (x_1, \ldots, x_g)$ in $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$ such that each α_i and β_j intersects $\mathbf{x}$ in a single point. As a vector space, $\mathrm{SFC}(\mathcal{H})$ is the free $\mathbb{F}_2$-module generated by G. The differential is given by

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathcal{G}} \#\mathcal{M}(\mathbf{x}, \mathbf{y})$$

where $\#\mathcal{M}(\mathbf{x}, \mathbf{y})$ is a modulo 2 count of holomorphic disks/bigons which correspond to certain domains. We direct the unacquainted reader to [Juh06] for details.

The homology of this complex is $\mathrm{SFH}(\mathcal{H})$, the *sutured Floer homology* associated the diagram. It is equipped with gradings which we do not need to consider. It is well-defined up to graded homotopy equivalence. Indeed, if we alter the β-curves to get another diagram $\mathcal{H}' = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}')$ for the sutured manifold, there is a graded homotopy equivalence

$$\Psi_{\mathcal{H}, \mathcal{H}'} : \mathrm{SFC}(\mathcal{H}) \to \mathrm{SFC}(\mathcal{H}')$$

which counts holomorphic triangles in the triple diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}')$, provided it is

admissible in the appropriate sense. We compute some such maps in 7.0.3. See [OS04] for details.

Chapter 3. HKM Handle Attachments

The goal of this section is to describe the HKM maps associated to contact i-handle attachments in terms of the diagram-dependent maps σ_i. The ideas described here are familiar to experts, though we record them for the sake of clarity and self-containment.

Remark 3.0.1. Note that the discussion in this section parallels many ideas in [JZ20]. In particular, our padded contact handles are analogous their Morse-type contact handles and Lemma 3.0.8 is analogous to their Theorem 5.8.

We assume familiarity with some basic ideas in contact geometry, particularly in convex surface theory. Throughout this paper, the contact structures we consider are defined up to isotopy. As such, we make implicit use of Giroux's uniqueness and flexibility theorems; see [Gir91]. We also assume familiarity with Heegaard diagrams arising from partial open books, though we review some key features in Subsection 3.0.2 to clarify notation. We refer to such diagrams as *partial open book diagrams*; we will represent a partial open book diagram as in [HKM08] by the quotient of a surface S with embedded β-arcs and α-arcs. The full Heegaard surface Σ is formed by identifying pairs of intervals in ∂S, which we shade black, so that each arc becomes a closed curve after identification[1]. All our pictures represent diagrams $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for $(M, -\Gamma)$ with

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[1]While Σ is usually defined by gluing a copy of a subsurface $Q \subset S$ to S along $\partial Q \cap \partial S$, the condition that the α-arcs in S form a basis guarantees that the diagram $(S \cup Q, \boldsymbol{\beta}, \boldsymbol{\alpha})$ is diffeomorphic to $(S/ \sim, \boldsymbol{\beta}, \boldsymbol{\alpha})$, where $\sim$ is the relation which identifies pairs of intervals in ∂S.
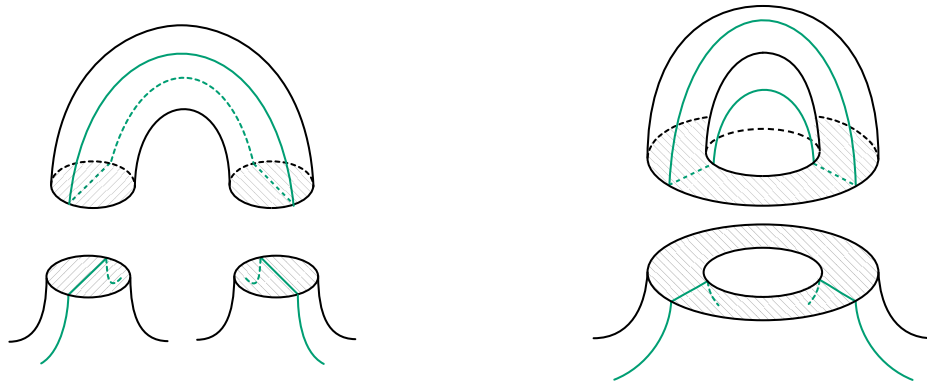
Figure 3.1. Contact 1- and 2-handle attachments

the positive orientation of Σ as depicted in Figure 3.7. We reverse the roles of α-curves
and β-curves, and we think of these as diagrams $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ for $(-M, -\Gamma)$, as is standard
when working with partial open books.

3.0.1. Diagrammatic maps for handles and bypasses

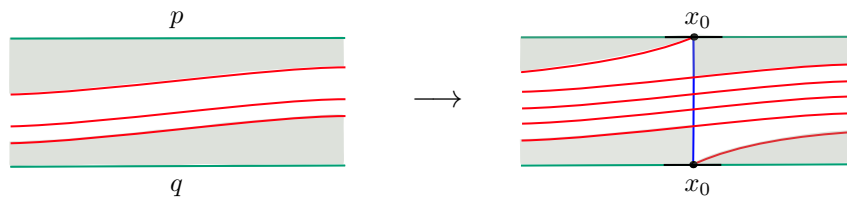Figure 3.2. A diagrammatic contact 1-handle attachment.

Figure 3.3. A diagrammatic contact 2-handle attachment.

We need to define diagrammatic maps associated to contact handle attachments in order to relate the HKM and bordered contact gluing maps. We refer to the work of Giroux [Gir91] and Ozbagci [Ozb11] for the notion of contact handles in dimension three. Contact 1-and 2-handles are shown in Figure 3.1 along with their dividing sets. Denote the result of attaching a contact i-handle h^i to (M, Γ, ξ) by (M_i, Γ_i, ξ_i). Note too that we can attach a contact handle to a balanced sutured manifold (M, Γ). Identify a neighborhood of ∂M with $\partial M \times [-1, 0]$ and let ξ_Γ be the $[-1, 0]$-invariant contact structure compatible with Γ. Then we can attach h^i to $\partial M = \partial M \times \{0\}$, and the induced dividing set will make the result a balanced sutured manifold, denoted (M_i, Γ_i).

Let $\mathcal{H} = (\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ be a diagram for $(-M, -\Gamma)$. Recall that we recover $(-M, -\Gamma)$ from $\mathcal{H}$ by attaching 2-handles along $(\boldsymbol{\beta} \times \{0\}) \sqcup (\boldsymbol{\alpha} \times \{1\})$ to $(\Sigma \times [0, 1], \partial \Sigma \times \{\frac{1}{2}\})$. With this identification, a contact 1-handle attachment along $(\{p\} \times \{\frac{1}{2}\}) \sqcup (\{q\} \times \{\frac{1}{2}\}) \subset \Gamma$, corresponds to attaching a 2-dimensional 1-handle to Σ along $\{p, q\}$ to obtain a new surface Σ_1. Then $\mathcal{H}_1 = (\Sigma_1, \boldsymbol{\beta}, \boldsymbol{\alpha})$ is a diagram for $(-M_1, -\Gamma_1)$; see Figure 3.2. A contact 2-handle attached along a curve δ with $\delta \cap \Gamma = \{p\} \times \{\frac{1}{2}\} \cup \{q\} \times \{\frac{1}{2}\}$ effects the following change at the level of diagrams. Attach a 2-dimensional 1-handle to Σ along $\{p, q\}$ to obtain a new surface Σ_2; call the core of the 1-handle λ. Define arcs $b = \delta \cap R_+(\Gamma)$ and $a = \delta \cap R_-(\Gamma)$, and let $\beta_0 = b \cup \lambda$, $\alpha_0 = a \cup \lambda$. Perturb these curves so that β_0 and α_0 intersect a single time positively in the 1-handle. Then $\mathcal{H}_2 = (\Sigma_2, \boldsymbol{\beta} \cup \beta_0, \boldsymbol{\alpha} \cup \alpha_0)$ is a diagram for $(-M_2, -\Gamma_2)$; see Figure 3.3.

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We refer to these operations on diagrams as *diagrammatic contact handle attachments.* Furthermore, if $\mathcal{H}$ is a partial open book diagram compatible with ξ, then the result of a diagrammatic handle attachment $\mathcal{H}_i$ is a partial open book diagram compatible with $\xi_i = \xi \cup h^i$; see the proof of Proposition 3.0.6 or [HKM09] for a more detailed discussion. Additionally, if $\mathbf{x}$ is the canonical collection of intersection points representing $\mathrm{EH}(\xi)$, then $\mathbf{x}$ represents $\mathrm{EH}(\xi_1)$ and $(\mathbf{x}, x_0)$ represents $\mathrm{EH}(\xi_2)$. In the case of the 2-handle, call x_0 the *preferred intersection point* of $\mathcal{H}_2$; this parallels the terminology in Subsection 3.0.2.

In general, a diagrammatic handle attachment induces an obvious inclusion of the associated sutured Floer complexes.

Definition 3.0.2. Let $\mathcal{H}$ be a Heegaard diagram for $(-M, -\Gamma)$. Let $\mathcal{H}_i$ be a diagram for $(-M_i, -\Gamma_i)$ obtained from $\mathcal{H}$ by a diagrammatic contact i-handle attachment. For $i = 1$, the associated *diagrammatic map* is the map $\sigma_1 : \mathrm{SFC}(\mathcal{H}) \to \mathrm{SFC}(\mathcal{H}_1)$ given by $\sigma_1(\mathbf{y}) = (\mathbf{y})$. For $i = 2$, the associated *diagrammatic map* is the map $\sigma_2 : \mathrm{SFC}(\mathcal{H}) \to \mathrm{SFC}(\mathcal{H}_2)$ given by $\sigma_2(\mathbf{y}) = (\mathbf{y}, x_0)$.

Note that these maps depend on a choice of diagram, and it is not clear a priori whether or not they induce well-defined maps on SFH. We will show in the course of this paper that the diagrammatic maps for i-handle attachments are well-defined up to graded homotopy equivalence and induce maps on homology which are equivalent to the HKM contact gluing maps for the corresponding handle attachments; see

Corollary 1.0.4. Note that Juhász and Zemke have shown in [JZ20] that this is the case, though our proof is independent from theirs.
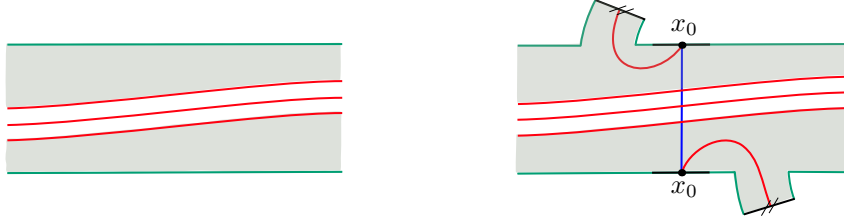

Figure 3.4. A diagrammatic positive trivial bypass.

Figure 3.5. A diagrammatic negative trivial bypass.

In the course of identifying these diagrammatic maps with the HKM map, we will also need maps associated to trivial bypasses. Attaching a bypass can be viewed as attaching a contact 1-handle followed by a contact 2-handle; as such, trivial bypasses have associated diagrammatic maps. A *diagrammatic positive trivial bypass attachment* is depicted in Figure 3.4 and a *diagrammatic negative trivial bypass attachment* is depicted in Figure 3.5.

Definition 3.0.3. Let $\mathcal{H}$ be a Heegaard diagram for $(-M, -\Gamma)$, and let $\mathcal{H}_{12}$ be a diagram for $(-M, -\Gamma)$ obtained from $\mathcal{H}$ by a diagrammatic trivial bypass attachment.

The associated *diagrammatic map* is the map $\tau : \mathrm{SFC}(\mathcal{H}) \to \mathrm{SFC}(\mathcal{H}_{12})$ given by $\tau(\mathbf{y}) = (\mathbf{y}, x_0)$.

We will need the following standard computation; a stronger version can be found in [HKM08].

Lemma 3.0.4. *The map* $\tau : \mathrm{SFC}(\mathcal{H}) \to \mathrm{SFC}(\mathcal{H}_{12})$ *associated to a diagrammatic trivial bypass is a graded isomorphism of complexes.*

Proof. For a positive bypass, x_0 is the only intersection point in the new α-curve; see Figure 3.4. For a negative bypass, x_0 is the only intersection point in the new β-curve; see Figure 3.5. In either case, the map τ thus gives a one-to-one correspondence between generators of $\mathcal{H}$ and generators of $\mathcal{H}_{12}$. Furthermore, holomorphic disks in $\mathcal{H}_{12}$ correspond exactly to holomorphic disks in $\mathcal{H}$. $\qquad\square$

3.0.2. HKM maps for contact handle attachments.

Since HKM maps are only defined for proper inclusions $(M, \Gamma) \subset (M', \Gamma')$ of sutured manifolds, diagrammatic handle attachments do not obviously fit into the HKM framework. We will introduce proper inclusions associated to contact handle attachments which allow us to relate diagrammatic and HKM contact handle attachment maps.

Definition 3.0.5. Given a sutured manifold (M, Γ), let the *padding* be the sutured contact manifold $(P = \partial M \times [0, 1], -\Gamma \sqcup \Gamma, \xi_\Gamma)$, where $-\Gamma \sqcup \Gamma = -\Gamma \times \{0\} \sqcup \Gamma \times \{1\}$

and ξ_Γ is the $[0,1]$-invariant contact structure compatible with $-\Gamma \sqcup \Gamma$. A *padded contact i-handle* $(P_i, -\Gamma \sqcup \Gamma_i, \xi_i)$ is the result of attaching a contact i-handle to $\partial M \times \{1\}$ in $(P, -\Gamma \sqcup \Gamma, \xi_\Gamma)$.

We will often omit the contact structure ξ_i from the notation and write $(P_i, -\Gamma \sqcup \Gamma_i)$ for a padded i-handle. The result of attaching a padded contact handle to (M, Γ) is diffeomorphic to (M_i, Γ_i). Moreover, the inclusion $M \subset M \cup P_i$ has an associated HKM map $\Phi_i : \mathrm{SFH}(-M, -\Gamma) \to \mathrm{SFH}(-M_i, -\Gamma_i)$.
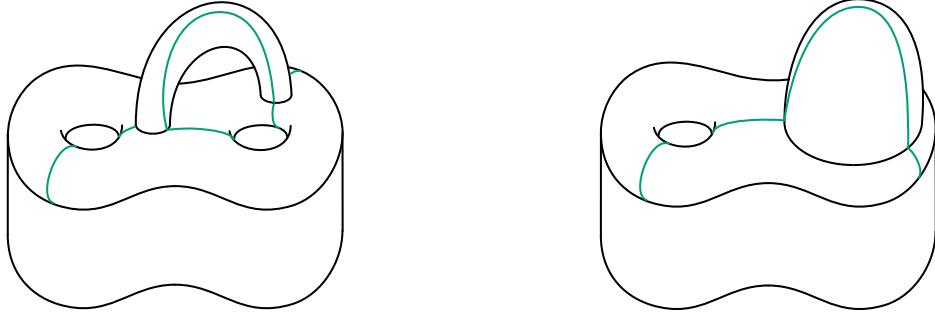

Figure 3.6. Padded contact handles.

Before identifying diagrammatic and HKM handle attachment maps, we will need to review the pertinent features of the definition of the HKM map. To give context and set notation, we first recall how partial open books are constructed following [HKM09]; see also [EO11]. Here and throughout this paper, $N(L)$ denotes a small neighborhood of a submanifold L. If L is a Legendrian graph, $N(L)$ is a standard neighborhood unless otherwise indicated.

Given a sutured contact manifold (M, Γ, ξ), a partial open book decomposition

is determined by a graph $K \subset M$ satisfying the conditions of Theorem 1.1 of [HKM09].

This guarantees that $M \setminus N(K)$ is product disk decomposable so that we can write

$M \setminus N(K) = H_1 = (S \times [-1,1])/_\sim$ and $N(K) = H_2 = (Q \times [1,2])/_\sim$, where $Q \subset$

S and the relations $\sim$ collapse each interval of the form $\{s\} \times [-1,1], s \in \partial S$ and

$\{q\} \times [1,2], q \in \partial Q \cap \partial S$ to a single point; the respective dividing sets are the images

of $\partial S \times [-1,1]$ and $\partial Q \times [1,2]$ under $\sim$. This decomposition determines a monodromy

map $h : Q \times \{2\} \to S \times \{-1\}$ with $h|_{Q \cap \partial S} = \text{Id}_{Q \cap \partial S}$ so that (M, Γ) is diffeomorphic to

$H_1 \cup_{Q \times \{1\}} H_2/_{\sim h}$.

The Heegaard surface for an associated partial open book diagram is the image

of $-S \times \{-1\} \cup Q \times \{1\}$ in the quotient. Curves are determined by a choice of arcs

$\{a_1, \ldots, a_m\}$ in Q which form a *basis* for $(S \setminus Q, R_+(\Gamma))$ in the sense that $(S \times \{1\}) \setminus \cup a_k$

deformation retracts to $R_+(\Gamma)$. For each k, let b_k be a perturbation of a_k such that ∂b_k

is ∂a_k moved in the ∂S direction and $b_k \cap a_k$ is a single intersection point. The sets of full

curves $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are given by the images of the $a_k \times \{1\} \cup a_k \times \{-1\}$ and $b_k \times \{1\} \cup h(b_k) \times \{-1\}$

respectively.

Observe that the image of $\partial\Sigma = ((\partial Q \setminus \partial S) \times \{1\}) \cup (-(\partial S \setminus \partial Q) \times \{-1\})$ cor-

responds to $-\Gamma$ in the quotient, so that the diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a balanced Heegaard

diagram for $(M, -\Gamma)$. Then $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ is a diagram for $(-M, -\Gamma)$; we call any such dia-

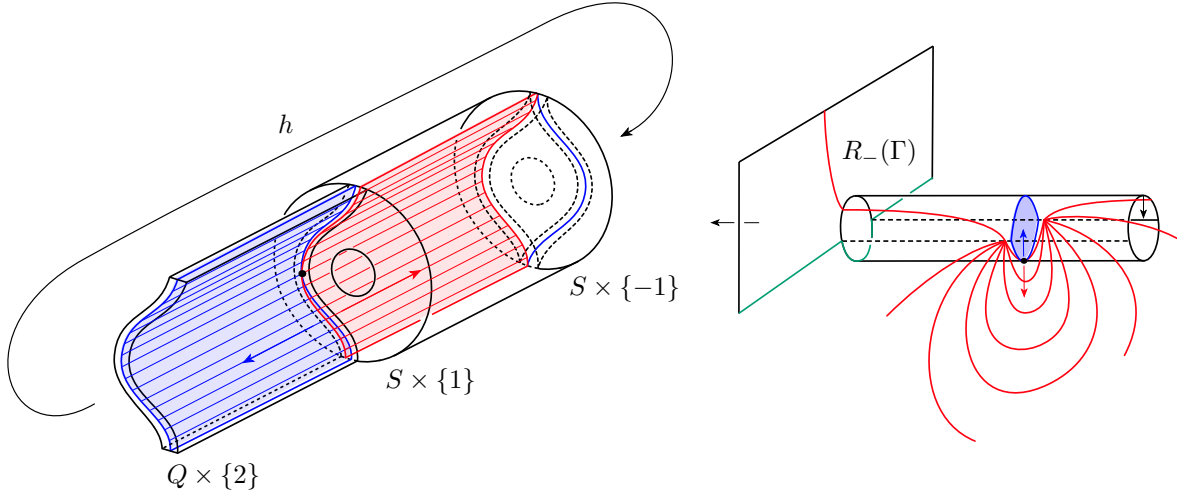gram constructed this way a *partial open book diagram.*

Figure 3.7. On the left a partial open book $M = (S \times [-1,1] \cup Q \times [1,2])/_\sim$. On the right the Heegaard surface embedded in (M, Γ); the black arrows represent the positive orientation of Σ.

When defining the HKM map, one uses diagrams $\mathcal{H} = (\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ for $(-M, -\Gamma)$ and $\mathcal{H}' = (\Sigma' \supset \Sigma, \boldsymbol{\beta} \cup \boldsymbol{\beta}'', \boldsymbol{\alpha} \cup \boldsymbol{\alpha}'')$ for $(-M', -\Gamma')$ which satisfy a number of technical conditions collectively referred to as *contact-compatibility*; in this context, $\mathcal{H}$ is called a *contact-compatible diagram* and $\mathcal{H}'$ is called a *contact-compatible extension* of $\mathcal{H}$. The idea is that the these diagrams will mimic partial open book diagrams to a large degree, with some necessary adaptations. We will need to understand some of these conditions, which we now describe, following section 4 of [HKM08]. Note that there are some notational differences; the roles of objects decorated by single primes and undecorated objects are reversed, and we use Q instead of P.

Parametrize a neighborhood of $\partial M \subset M'$ by $\partial M \times [-1,1]$, where $\partial M = \partial M \times \{0\}$ and $V = \partial M \times [-1,0]$ is contained in M. Note that the sutures Γ determine a $[-1,0]$-

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invariant contact structure ζ on V. The Heegaard splitting for a contact-compatible diagram of $(-M, -\Gamma)$ is determined by a choice of an embedded graph K whose restriction to V satisfies the conditions in Theorem 1.1 of [HKM09] with respect to ζ. (The conditions in Theorem 1.1 of [HKM09] guarantee that a Legendrian graph determines a compatible partial open book decomposition.) In particular, $K|_V$ is Legendrian and the handlebodies $V \setminus N(K)$ and $N(K)|_V$ are each product disk decomposable with respect to the sutures induced by ζ.

The essential choices made in constructing a contact-compatible extension of $\mathcal{H}$ are as follows. Let $W = \partial M \times [0, 1]$, and note that ξ can be perturbed to restrict to the $[0, 1]$-invariant contact structure on W. The extension is determined by a Legendrian graph $K_W \cup K'' \subset M' \setminus \mathrm{Int}(M)$, with $K_W \subset W$ and $K'' \subset M'' = M' \setminus \mathrm{Int}(M \cup W)$. (If one compares with [HKM08], a standard neighborhood of our K_W is the complement of a standard neighborhood of their K''''.) Both K_W and K'' are required to satisfy the conditions of Theorem 1.1 of [HKM09] on W and M'' with respect to ξ. This splits M' into handlebodies $H_1 = M \setminus N(K \cup K_W \cup K'')$ and $H_2 = N(K \cup K_W \cup K'')$; however, H_1 and H_2 need not be product disk decomposable, since K is not necessarily Legendrian for some contact structure on $M \setminus V$. If we restrict our attention to $V \cup W \cup M''$, we have an honest contact manifold with contact structure $\zeta \cup \xi$. The restrictions J_1 and J_2 of the handlebodies H_1 and H_2 to $V \cup W \cup M''$ are product disk decomposable and can be written as $J_1 = S' \times [-1, 1]/_\sim$ and $J_2 = Q' \times [1, 2]/_\sim$, with $Q' \subset S'$ and $\sim$ the

relation which collapses each interval $\{s'\} \times [-1, 1]$, $s' \in \partial S'$ and $\{q'\} \times [-1, 1]$, $q' \in \partial Q'$ to a point, just as for an ordinary partial open book decomposition.

Define the full Heegaard surface Σ' to be the complement of $R_+(\Gamma')$ in ∂H_1. More explicitly, Σ' is given by $\partial(M \setminus (N(K) \cup V)) \cup (S' \times \{-1\} \cup Q' \times \{1\})$. The curves in $\boldsymbol{\beta}'$ and $\boldsymbol{\alpha}'$ are defined to be the curves in $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ together with additional curves $\boldsymbol{\beta}'', \boldsymbol{\alpha}''$ that are generated in the following way. Chose a collection of arcs $\{a_1'', ..., a_m''\}$ on $Q' \times \{1\}$ which form a *basis* for $(S' \setminus Q, R_+(\Gamma'))$ in the sense that $(S' \setminus Q) \setminus \cup_{k=1}^m a_k''$ deformation retracts to $R_+(\Gamma')$; here, S' is identified with $S' \times \{1\}$, while Q is the restriction of $Q' \times \{1\}$ to $V \subset M$. For each k, let b_k'' be a perturbation of a_k'' such that $\partial b_k''$ is $\partial a_k''$ moved in the $\partial S'$–direction and $b_k'' \cap a_k''$ is a single intersection point, as is standard procedure when constructing a partial open book diagram. Call the collection of these intersection points the *preferred intersections* of the extension, and denote it by $\mathbf{x}''$. The arcs $\{b_k''\}, \{a_k''\}$ are completed to full sets of curves $\boldsymbol{\beta}'', \boldsymbol{\alpha}''$ in $\mathcal{H}'$ using the monodromy map $h : Q' \times \{2\} \to S' \times \{-1\}$ obtained from the identification of $V \cup W \cup M''$ with $J_1 \cup_h J_2$. More explicitly, β_i'' and α_i'' are the respective images of $b_i'' \times \{1\} \cup h(b_i'' \times \{2\})$ and $a_i'' \times \{1\} \cup a_i'' \times \{-1\}$ under the relation $\sim$.

3.0.3. Identifying diagrammatic and HKM maps

In this section we identify the diagrammatic and HKM contact handle attachment maps. We will show that a diagrammatic handle attachment can be chosen to respect contact-compatibility and the preferred intersections.
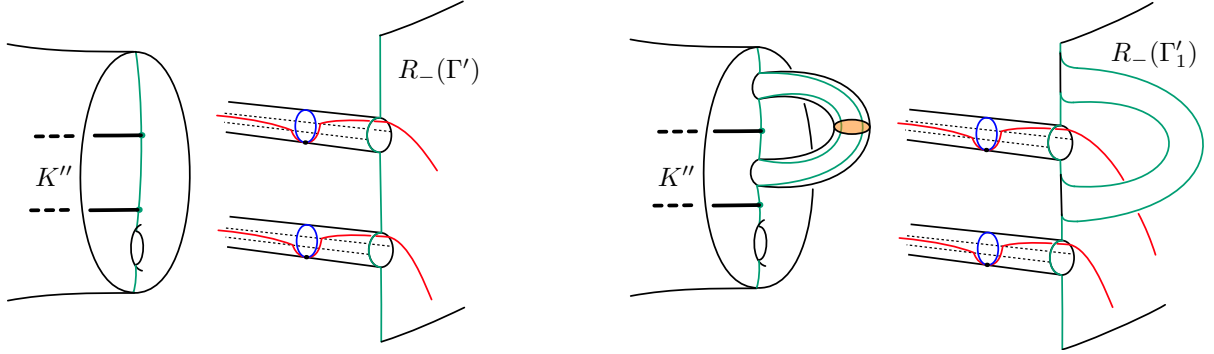
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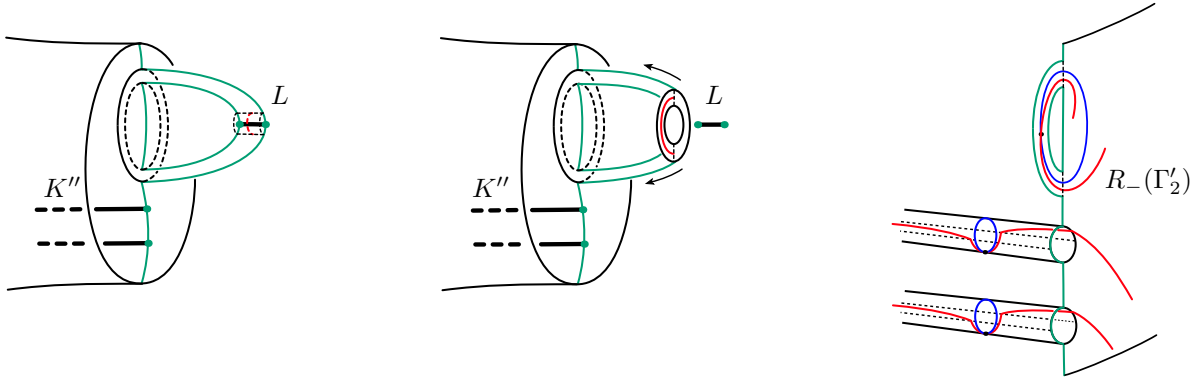Figure 3.8. Left: (M', Γ') and $\mathcal{H}'$; Right: (M_1', Γ_1') and $\mathcal{H}_1'$

Figure 3.9. Left: (M_2', Γ_2'); Center: $M_2' \setminus N(L)$; Right: $\mathcal{H}_2'$

Proposition 3.0.6. *Let $\mathcal{H} = (\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ be a contact-compatible diagram for $(-M, -\Gamma)$*

and $\mathcal{H}' = (\Sigma', \boldsymbol{\beta} \cup \boldsymbol{\beta}'', \boldsymbol{\alpha} \cup \boldsymbol{\alpha}'')$ be a contact compatible extension of $\mathcal{H}$ for $(-M', -\Gamma')$.

Let (M_i', Γ_i') be the result of attaching a contact i-handle to (M', Γ'). Then there is a

contact-compatible extension $\mathcal{H}_i'$ of $\mathcal{H}$ for $(-M_i', -\Gamma_i')$ which is obtained from $\mathcal{H}'$ by a

diagrammatic handle attachment.

Proof. The graphs involved in the definition of a contact-capatible extension are all

required to satisfy a number of conditions; however, K'' need only

1. satisfy the conditions of Theorem 1.1 of [HKM09] with respect to $(M'', \xi|_{M''})$

2. have each component intersect Γ' at least twice

3. have $\partial K''$ disjoint from ∂K_W.

Attaching a contact handle to $\partial M'$ only alters K'' and the choice of basis arcs, so it suffices to verify that there are modifications of K'' and $\{a_k''\}$ which satisfy the properties describe above. The reader may verify that no other conditions in contact-compatibility are affected, and may be safely swept under the rug.

Given a 1-handle attachment (M_1', Γ_1'), the feet of the 1-handle can be isotoped to be disjoint from $\partial K''$, since K'' meets $\partial M'$ in a finite number of points by Theorem 1.1 of [HKM09]. The graph $K'' \subset M' \subset M_1'$ still satisfies the conditions of Theorem 1.1 of [HKM09] with respect to $\xi_1 = \xi \cup h^1$, since one can cut M_1'' along the cocore of h^1 to recover M'', i.e. a system of compressing disks for $M_1'' \setminus N(K'')$ can be obtained by adding the cocore of h^1 to a system of compressing disks for $M'' \setminus N(K'')$. The new Heegaard surface is obtained from Σ' by attaching a strip at the attaching sphere of h^1, since that is the corresponding change on S'; see Figure 3.8. The basis $\{a_k''\}_{k=1}^m$ for $\mathcal{H}'$ is still a basis for the new extension. Since the monodromy is unaffected, the curves also remain the same. This new extension is thus exactly a diagrammatic 1-handle attachment $\mathcal{H}_1'$.

Given a 2-handle attachment (M_2', Γ_2') with attaching curve δ, the attaching region can be isotoped to be disjoint from K'', since K'' meets $\partial M'$ in a finite number of points by Theorem 1.1 of [HKM09]. Let L be a Legendrian approximation of the

cocore of h^2 with $\partial L \subset \Gamma_2'$. The graph $K'' \cup L$ satisfies the conditions of Theorem 1.1 of [HKM09] with respect to the contact structure ξ_2 obtained by attaching the contact 2-handle h^2 to ξ, since $M_2'' \setminus N(K'' \cup L)$ is contactomorphic to $M'' \setminus N(K'')$. Note that $K'' \cup L$ also satisfies the other two conditions listed above. Let S_2' play the role of S' for the new extension. While S_2' is diffeomorphic to S', there is an additional strip in Σ_2' due to the new component $\partial N(L) \setminus \partial h^2$ of Q_2' coming from L; see Figure 3.9. This strip does not deformation retract to a subset of $R_+(\Gamma_2')$, so the arcs $\{a_k''\}_{k=1}^m$ no longer form a basis. Let μ be a meridian of $\partial N(L) \setminus \partial h^2$, and define a_0'' to be the arc $R_+(\partial N(L) \setminus \partial h^2) \cap \mu$. The collection $\{a_k''\}_{k=0}^m$ is now a basis for the new extension.

Since the arc b_0'' is completed to a full β-curve using the core of the new strip attached along $\delta \cap \Gamma'$, this new extension is exactly a diagrammatic 2-handle attachment $\mathcal{H}_2'$. $\qquad\square$

Proposition 3.0.7. *Let $\mathcal{H}$, $\mathcal{H}'$, and $\mathcal{H}_i'$ be as above, and $\phi_\xi : \mathrm{SFC}(\mathcal{H}) \to \mathrm{SFC}(\mathcal{H}')$ be the map $\mathbf{y} \to (\mathbf{y}, \mathbf{x}'')$ which induces the* HKM *map as described above. Then the map $\sigma_i \circ \phi_\xi : \mathrm{SFC}(\mathcal{H}) \to \mathrm{SFC}(\mathcal{H}_i')$ induces the* HKM *map $\Phi_{\xi_i} : \mathrm{SFH}(-M, -\Gamma) \to \mathrm{SFH}(-M_i', -\Gamma_i')$.*

Proof. For $i = 1$, the preferred intersections $\mathbf{x}''$ of $\mathcal{H}'$ are also the preferred intersections of $\mathcal{H}_1'$. The HKM map Φ_{ξ_1} is thus induced by the chain map $\mathbf{y} \to (\mathbf{y}, \mathbf{x}'')$, which factors as $\sigma_1 \circ \phi_\xi$.

For $i = 2$, the preferred intersections of $\mathcal{H}_2'$ are $\mathbf{x}'' \cup x_0$, where $x_0 = b_0'' \cap a_0''$, the

preferred intersection of the diagrammatic 2-handle. The HKM map Φ_{ξ_2} is thus induced by the chain map $\mathbf{y} \to (\mathbf{y}, \mathbf{x}'', x_0)$, which factors as $\sigma_2 \circ \phi_\xi$. $\qquad\square$

We are now ready to relate diagrammatic and HKM maps.

Lemma 3.0.8. *Let (M, Γ) be a balanced sutured manifold. There is a contact-compatible diagram $\mathcal{H}$ for $(-M, -\Gamma)$, a contact-compatible extension $\mathcal{H}'$ for $(-(M \cup P), -\Gamma)$, a diagrammatic handle attachment $\mathcal{H}'_i$ for $(-(M \cup P_i), -\Gamma_i)$, and a graded homotopy equivalence $\phi_{\xi_\Gamma} : \mathrm{SFC}(\mathcal{H}) \to \mathrm{SFC}(\mathcal{H}')$ such that the induced map on homology*

$$(\sigma_i \circ \phi_{\xi_\Gamma})_* : \mathrm{SFH}(-M, -\Gamma) \to \mathrm{SFH}(-(M \cup P_i), -\Gamma_i) \approx \mathrm{SFH}(-M_i, -\Gamma_i)$$

is the HKM *map Φ_i.*

Proof. From the proof of Theorem 6.1 in [HKM08], there is a contact compatible extension $\mathcal{H}'$ of a contact-compatible diagram $\mathcal{H}$ such that the map $\phi_{\xi_\Gamma} : \mathrm{SFC}(\mathcal{H}) \to \mathrm{SFC}(\mathcal{H}')$ which induces Φ_{ξ_Γ} is a graded homotopy equivalence induced by a composition of diagrammatic trivial bypass maps. The lemma then follows immediately from Proposition 3.0.7. $\qquad\square$

Chapter 4. Background on Zarev's Gluing Map

We now turn our attention to Zarev's bordered gluing map $\Psi_{\mathcal{F}}$. In this section we review the requisite bordered sutured Floer theory for understanding the structure of this gluing map, though the interested reader should look in [Zar11] for a full treatment of the topological and algebraic objects involved. Readers familiar with bordered sutured theory and Zarev's gluing map should skip to chapter 5, where we define the associated contact gluing map Ψ_{ξ} and prove that it satisfies the composition law necessary for the proof of Theorem 1.

We deviate from Zarev's notational conventions in the following ways. Throughout, we reserve script letters for partially sutured manifolds, bordered sutured manifolds, and sutured surfaces. We avoid using them for honest sutured manifolds, except when the margins of the page force us to compress notation. We distinguish between suture sets on sutured manifolds and dividing sets on sutured surfaces by using a lower case γ instead of Γ for the dividing set on a sutured surface. As mentioned in Subsection 4.0.1, we denote the mirror of a partially sutured manifold $\mathcal{W}$ by $\overline{\mathcal{W}}$ instead of $-\mathcal{W}$. If $\mathcal{H}$ is a diagram for a sutured manifold (M, Γ), then Zarev treats $\mathrm{SFC}(M, \Gamma)$ as the $\mathcal{A}_{\infty}$–homotopy type of the complex $\mathrm{SFC}(\mathcal{H})$; morphisms between sutured Floer complexes are then defined up to $\mathcal{A}_{\infty}$–homotopy equivalence. We will instead work with representatives of these invariants, treating sutured Floer complexes as honest graded $\mathbb{F}_2$-vector spaces, and graded chain maps as morphisms. Similarly, we treat the

bordered invariants $\widehat{\mathrm{BSA}}$, $\widehat{\mathrm{BSD}}$, and $\widehat{\mathrm{BSAA}}$ as honest Type-A, Type-D, and Type-AA structures, instead of their $\mathcal{A}_\infty$–homotopy classes.

4.0.1. Some basic objects

We first review some topological objects which play an important role in bordered sutured theory. A *sutured surface* is an oriented surface F with boundary and a number of signed points $\Lambda \subset \partial F$ which alternate in sign as one traverses each component of ∂F; each component of ∂F is required to intersect Λ. A sutured surface $\mathcal{F} = (F, \Lambda)$ has associated sutured surfaces $-\mathcal{F} = (-F, -\Lambda)$ and $\overline{\mathcal{F}} = (-F, \Lambda)$. A *dividing set* on a sutured surface is a collection of embedded, oriented arcs and closed curves $\gamma \subset F$ such that $\partial\gamma = -\Lambda$ and such that each component of $F \setminus \gamma$ can be oriented so that the induced orientation on the boundary agrees with the orientation of γ. Components where this orientation agrees with the orientation on F constitute the $R_+(\gamma)$ region, while the other components constitute the $R_-(\gamma)$ region. Unless otherwise indicated, we assume that all sutured surfaces are *non-degenerate*, i.e. they have no closed components.

A *partially sutured manifold* $\mathcal{W} = (W, \gamma, \mathcal{F})$ is a manifold M whose boundary is the union of two sutured surfaces with common boundary, one of which (usually referred to as the "sutured part of the boundary") has a dividing set γ and the other being $\mathcal{F}$. If $\mathcal{W}_1 = (W_1, \gamma_1, \mathcal{F}_1 \sqcup \mathcal{F}_2)$ and $\mathcal{W}_2 = (W_2, \gamma_2, -\mathcal{F}_2 \sqcup \mathcal{F}_3)$ are partially sutured

manifolds, then we can *glue* or *concatenate* them along $\mathcal{F}_2$ to obtain

$$\mathcal{W}_1 \cup_{\mathcal{F}_2} \mathcal{W}_2 = (W_1 \cup_F W_2, \gamma_1 \cup_{\Lambda_2} \gamma_2, \mathcal{F}_1 \sqcup \mathcal{F}_3).$$

If $\mathcal{W} = (W, \gamma, \mathcal{F})$ is a partially sutured manifold, we denote $(-W, -\gamma, -\mathcal{F})$ by $-\mathcal{W}$ and $(-W, \gamma, \overline{\mathcal{F}})$ by $\overline{\mathcal{W}}$. Note that this differs from Zarev's notation, where there is no notation for the former and the latter is denoted $-\mathcal{W}$. In the special case of an honest sutured manifold, i.e. $\mathcal{F} = (\emptyset, \emptyset)$, we have $-(M, \Gamma) = (-M, -\Gamma)$, and $\overline{(M, \Gamma)} = (-M, \Gamma)$.

If (M, Γ) is a sutured manifold, and $\mathcal{W} = (W, \gamma, \mathcal{F})$ is a partially sutured manifold with $W \subset M$, $\mathrm{Int}(F) \subset M$, $\partial W \setminus \mathrm{Int}(F) \subset \partial M$, and $\gamma \subset \Gamma$, we denote the partially sutured manifold $(\overline{M \setminus W}, \Gamma \setminus \mathrm{Int}(\gamma), -\mathcal{F})$ by $(M, \Gamma) \setminus \mathcal{W}$.

A *bordered sutured manifold* $(M, \gamma, \mathcal{F}, \mathcal{Z})$ is a partially sutured manifold whose non-sutured part $\mathcal{F}$ of the boundary is parametrized by an arc diagram $\mathcal{Z}$. We discuss arc diagrams in Subsection 6.0.1

Let $\mathcal{F} = (F, \Lambda)$ be a sutured surface and γ a dividing set on $\mathcal{F}$. The *cap* associated to γ is the partially sutured manifold $(F \times [0, 1], (\Lambda \times [0, 1]) \cup (\gamma \times \{1\}), -\mathcal{F})$. The caps associated to contact 1-and 2-handle attachments are shown in Figure 4.1.

The *positive twisting slice* $\mathcal{TW}_{\mathcal{F},+}$ for $\mathcal{F}$ is the partially sutured manifold $(F \times [0, 1], \gamma, -\mathcal{F} \cup -\overline{\mathcal{F}})$, where γ is obtained by giving $\Lambda \times [0, 1]$ a minimal fractional Dehn twist in the ∂F-direction. The positive twisting slices for contact 1-and 2-handles are shown in Figure 4.2. The *negative twisting slice* $\mathcal{TW}_{\mathcal{F},-}$ for $\mathcal{F}$ is defined the same way,
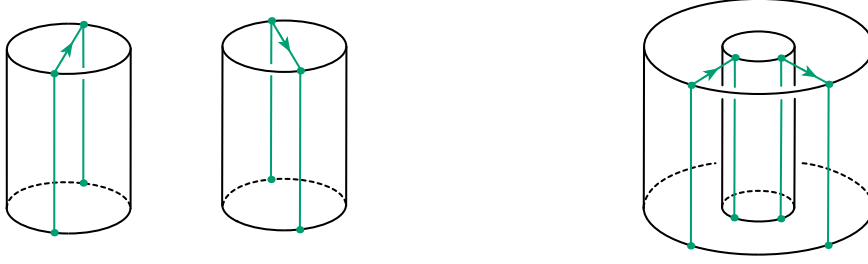
Figure 4.1. Caps for contact 1-and 2-handles. The dividing set γ is shown on the top and sides, while the bottom $-\mathcal{F}$ is undecorated.

except that one twists in the $-\partial F$-direction. Note that $-\mathcal{TW}_{\mathcal{F},+} \cong \mathcal{TW}_{-\mathcal{F},-}$.

Given a partially sutured manifold $\mathcal{W}$ with non-sutured boundary $\mathcal{F}$, the *double* $\mathcal{D}(\mathcal{W})$ of $\mathcal{W}$ is the sutured manifold $\overline{\mathcal{W}} \cup_{\overline{\mathcal{F}}} \mathcal{TW}_{\overline{\mathcal{F}},-} \cup_{-\mathcal{F}} \mathcal{W}$. Suppose $\mathcal{W}$ is the cap for a dividing set γ on $\mathcal{F}$, and consider the the $[0,1]$-invariant contact structure ξ_γ on $F \times [0,1]$ associated to γ. Then ξ_γ induces a dividing set Γ_γ on $F \times [0,1]$ such that $(F \times [0,1], \Gamma_\gamma)$ is diffeomorphic to $\mathcal{D}(\mathcal{W})$ as sutured manifolds. Note that Γ_γ is a dividing set in the sense of convex surface theory, not in the sense of sutured surfaces discussed above.

4.0.2. Gluing along sutured surfaces

We now review the operation of gluing sutured manifolds in the sense of [Zar11] and its connection with contact geometry.

Let $\mathcal{F} = (F, \Lambda)$ be a sutured surface with dividing set γ. Let (M_1, Γ_1) and (M_2, Γ_2) be sutured manifolds with embeddings $(F, \gamma) \subset (\partial M_1, \Gamma_1)$ and $(-F, \gamma) \subset (\partial M_2, \Gamma_2)$. The embedding $F \to \partial M_1$ extends to an embedding $\mathcal{W} \to M_1$, where $\mathcal{W}$ is the cap for the dividing set γ on $\mathcal{F}$. Similarly, $-F \to \partial M_2$ extends to an embedding
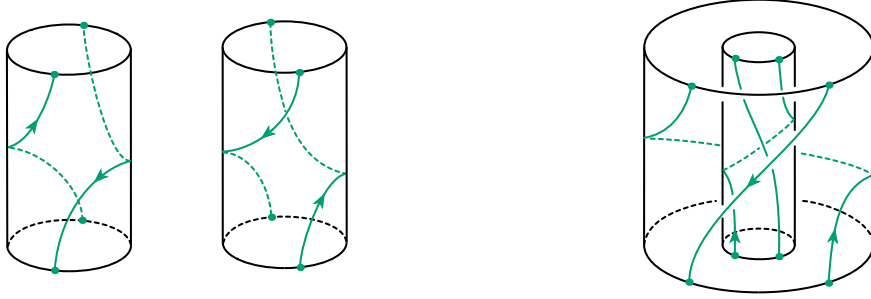
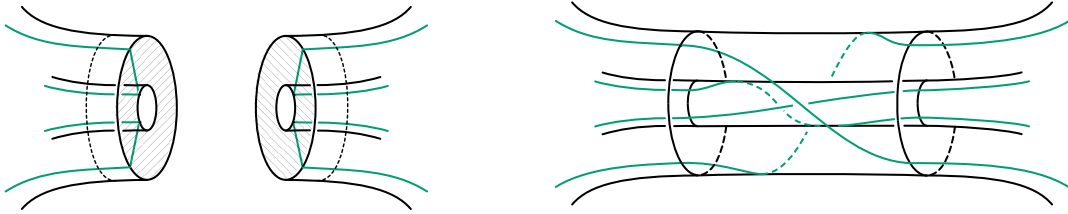Figure 4.2. The positive twisting slices for contact 1-and 2-handles.

Figure 4.3. A bordered gluing operation along an annulus. The caps $\mathcal{W}$ and $\overline{\mathcal{W}}$ are a neighborhood of the shaded region.

$\overline{\mathcal{W}} \to M_2$. Zarev defines the result of gluing (M_1, Γ_1) to (M_2, Γ_2) along $(\mathcal{F}, \gamma)$ to be the manifold

$$(M_1, \Gamma_1) \cup_{\mathcal{F}} (M_2, \Gamma_2) := ((M_1, \Gamma_1) \setminus \mathcal{W}) \cup_{\mathcal{F}} \mathcal{TW}_{\mathcal{F},+} \cup_{-\overline{\mathcal{F}}} ((M_2, \Gamma_2) \setminus \overline{\mathcal{W}}),$$

where $\mathcal{TW}_{\mathcal{F},+}$ is the positive twisting slice associated to $\mathcal{F}$. We refer to the gluing operation above as a *bordered gluing* to distinguish it from a proper inclusion of sutured manifolds; a bordered gluing operation is shown in Figure 4.3. Note that the manifold $(M_1, \Gamma_1) \cup_{\mathcal{F}} (M_2, \Gamma_2)$ is independent of the dividing set γ; however, the bordered gluing map defined below will depend on it.

Bordered gluing operations describe how to glue the sutured manifolds underly-

ing contact structures with convex boundary; however, there is a necessary orientation reversal. Let (M, Γ, ξ) be a connected sutured contact manifold and let $\mathcal{F} = (F, \Lambda)$ be a sutured surface with $(F, \partial F, \Lambda) \subset (M, \partial M, \Gamma)$; $(\text{Int}(F) \subset \text{Int}(M)$; F convex with respect to ξ; and ∂F Legendrian. If $M \setminus F$ has two connected components, then its closure may be decomposed as sutured contact manifolds $(M_1, \Gamma_1, \xi_1) \sqcup (M_2, \Gamma_2, \xi_2)$, where $\xi_i = \xi|_{M_i}$ and Γ_i is the dividing set induced by ξ_i after smoothing corners. Suppose the normal vector for F in M points out of M_1 and into M_2. If $\mathcal{W} = (W, \Gamma_1|_F, -\mathcal{F})$ is the cap for $\mathcal{F}$, then we can write

$$(M, \Gamma) \cong ((M_1, \Gamma_1) \setminus \mathcal{W}) \cup_{\mathcal{F}} \mathcal{T}\mathcal{W}_{\mathcal{F}, -} \cup_{-\overline{\mathcal{F}}} ((M_2, \Gamma_2) \setminus \overline{\mathcal{W}}).$$

We will use the notation $\xi = \xi_1 \cup_{\mathcal{F}} \xi_2$ to describe the relationship among these contact structures.

Note that the twist of dividing sets which arises from smoothing corners is negative, instead of the positive twist used in the bordered gluing operation. This is consistent with the orientation reversal associated to the contact invariant, since we have

$$(-M_1, -\Gamma_1) \cup_{-\mathcal{F}} (-M_2, -\Gamma_2) = ((-M_1, -\Gamma_1) \setminus -\mathcal{W}) \cup_{-\mathcal{F}} \mathcal{T}\mathcal{W}_{-\mathcal{F}, +} \cup_{\overline{\mathcal{F}}} ((-M_2, -\Gamma_2) \setminus -\overline{\mathcal{W}}),$$

which is just

$$-[((M_1, \Gamma_1) \setminus \mathcal{W}) \cup_{\mathcal{F}} \mathcal{T}\mathcal{W}_{\mathcal{F}, -} \cup_{-\overline{\mathcal{F}}} ((M_2, \Gamma_2) \setminus \overline{\mathcal{W}})] = (-M, -\Gamma).$$

Thus, it makes sense for the contact invariants $\text{EH}(\xi_1) \in \text{SFH}(-M_1, -\Gamma_1)$ and

$\mathrm{EH}(\xi_2) \in \mathrm{SFH}(-M_2, -\Gamma_2)$ to be paired by the bordered gluing map to $\mathrm{EH}(\xi_1 \cup_\mathcal{F} \xi_2)$; see Lemma 5.0.12.

Remark 4.0.1. Note that we are using the symbol $\cup_\mathcal{F}$ for the distinct operations of gluing partially sutured manifolds along components of their non-sutured boundaries, gluing balanced sutured manifolds along sutured subsurfaces of their boundaries with dividing sets, and gluing contact structures along convex surfaces. We rely on context to distinguish them.

Zarev shows that if (M_1, Γ_1) and (M_2, Γ_2) are balanced, so is $(M_1, \Gamma_1) \cup_\mathcal{F} (M_2, \Gamma_2)$. He also defines a gluing map

$$\mathrm{SFH}(M_1, \Gamma_1) \otimes \mathrm{SFH}(M_2, \Gamma_2) \xrightarrow{\Psi_{\mathcal{F}, \gamma}} \mathrm{SFH}((M_1, \Gamma_1) \cup_\mathcal{F} (M_2, \Gamma_2)),$$

associated to the bordered gluing operation. To avoid confusion with the HKM contact gluing map, we will refer to this map as the *bordered gluing map*. Also, when $\mathcal{F}$ is embedded in the boundary of a sutured manifold, we will assume unless otherwise stated that the dividing set is the restriction of the sutures to $\mathcal{F}$; in this case we omit the dividing set from the notation and write $\Psi_\mathcal{F}$ for the bordered gluing map.

Remark 4.0.2. The bordered gluing map is well-defined up to maps induced by graded homotopy equivalence. It is not known to be natural with respect to holomorphic triangle maps in the way that the HKM map is.

The bordered gluing operation is a special case of the *join operation* found in [Zar11]. Similarly, the bordered gluing map is a special case of the *join map*. We will

use the fact the the bordered gluing map possesses the following properties of the join.

Theorem 4.0.3 (Zarev). *The bordered gluing map $\Psi_{\mathcal{F}}$ satisfies the following properties:*

- *(Symmetry) Let $(M_1, \Gamma_1), (M_2, \Gamma_2)$ be sutured manifolds with embeddings $\mathcal{F} \subset \partial M_1$, $\overline{\mathcal{F}} \subset \partial M_2$. There are graded isomorphisms f, g such that the following diagram commutes*

$$
\begin{array}{ccc}
\mathrm{SFH}(M_1, \Gamma_1) \otimes \mathrm{SFH}(M_2, \Gamma_2) & \xrightarrow{\ f\ } & \mathrm{SFH}(M_2, \Gamma_2) \otimes \mathrm{SFH}(M_1, \Gamma_1) \\
\Big\downarrow {\scriptstyle \Psi_{\mathcal{F}}} & & \Big\downarrow {\scriptstyle \Psi_{\overline{\mathcal{F}}}} \\
\mathrm{SFH}((M_1, \Gamma_1) \cup_{\mathcal{F}} (M_2, \Gamma_2)) & \xrightarrow{\ g\ } & \mathrm{SFH}((M_2, \Gamma_2) \cup_{\overline{\mathcal{F}}} (M_1, \Gamma_1)))
\end{array}
$$

- *(Associativity) Let $(M_1, \Gamma_1), (M_2, \Gamma_2), (M_3, \Gamma_3)$ be sutured manifolds and $\mathcal{F}_1, \mathcal{F}_2$ be sutured surfaces with embeddings $\mathcal{F}_1 \hookrightarrow \partial M_1$, $(\overline{\mathcal{F}}_1 \sqcup \mathcal{F}_2) \hookrightarrow \partial M_2$, and $\overline{\mathcal{F}}_2 \hookrightarrow \partial M_3$. Write $\mathcal{M}_k = (M_k, \Gamma_k)$. The following diagram commutes up to graded isomorphism:*

$$
\begin{array}{ccc}
\mathrm{SFH}(\mathcal{M}_1) \otimes \mathrm{SFH}(\mathcal{M}_2) \otimes \mathrm{SFH}(\mathcal{M}_3) & \xrightarrow{\ \Psi_{\mathcal{F}_2}\ } & \mathrm{SFH}(\mathcal{M}_1) \otimes \mathrm{SFH}(\mathcal{M}_2 \cup_{\mathcal{F}_2} \mathcal{M}_3) \\
\Big\downarrow {\scriptstyle \Psi_{\mathcal{F}_1}} & & \Big\downarrow {\scriptstyle \Psi_{\mathcal{F}_1}} \\
\mathrm{SFH}(\mathcal{M}_1 \cup_{\mathcal{F}_1} \mathcal{M}_2) \otimes \mathrm{SFH}(\mathcal{M}_3) & \xrightarrow{\ \Psi_{\mathcal{F}_2}\ } & \mathrm{SFH}(\mathcal{M}_1 \cup_{\mathcal{F}_1} \mathcal{M}_2 \cup_{\mathcal{F}_2} \mathcal{M}_3)
\end{array}
$$

- *(Identity) Let (M, Γ) be a sutured manifold with embedded cap $\mathcal{W}$ for $\mathcal{F}$, and let $\mathcal{D}(\mathcal{W})$ be the double of $\mathcal{W}$. There is a distinguished class $[\Delta] \in \mathrm{SFH}(\mathcal{D}(\mathcal{W}))$, called the* diagonal class *such that the map*

$$
\Psi_{\mathcal{F}}(\,\cdot\,, [\Delta]) : \mathrm{SFH}(M, \Gamma) \to \mathrm{SFH}((M, \Gamma) \cup_{\mathcal{F}} \mathcal{D}(\mathcal{W}))
$$

is a graded isomorphism.

Chapter 5. The Bordered Contact Gluing Map

The goal of this section is to define the bordered contact gluing map and prove that it satisfies functorial properties akin to the HKM map.

5.0.1. The bordered contact gluing map

We will need to introduce a technical condition on the dividing set γ in order to ensure that the bordered contact gluing map possesses the desired functorial properties found in Subsection 5.0.2.

Definition 5.0.1. A dividing set γ on a sutured surface $\mathcal{F}$ is *disk-decomposable* if the underlying sutured manifold of the *I*-invariant contact structure for γ on $(F \times I)$ has a product disk decomposition.

Definition 5.0.2. Let (M''', Γ'') and (M, Γ) be sutured manifolds which can be glued along a sutured surface $\mathcal{F}$ with disk-decomposable dividing set γ. Let ξ be a compatible contact structure on M''. The associated *bordered contact gluing map*

$$\Psi_\xi : \text{SFH}(-M, -\Gamma) \to \text{SFH}((-M'', -\Gamma'') \cup_{-\mathcal{F}} (-M, -\Gamma))$$

is defined to be $\Psi_\xi([\mathbf{y}]) = \Psi_{-\mathcal{F}}(\text{EH}(\xi), [\mathbf{y}])$.

If (M'', Γ'') is a contact handle h^i with attaching region $\mathcal{F}_i$, then we denote this map by $\Psi_i : \text{SFH}(-M, -\Gamma) \to \text{SFH}(-M_i, -\Gamma_i)$.

Remark 5.0.3. Note that the dividing sets for contact handle attachments are disk-decomposable; see Subsection 6.0.2.

Remark 5.0.4. Note that some restriction on the dividing set is necessary, since the identity property Lemma 5.0.6 is false when the I-invariant contact structure is overtwisted. We conjecture that the bordered contact gluing map map satisfies Lemma 5.0.6, Lemma 5.0.12, and Lemma 5.0.13 whenever the I-invariant contact structure associated to the dividing set is tight. It would suffice to prove Lemma 5.0.7 for such dividing sets, since the rest of the arguments carry through.

Lemma 5.0.5. *The map Ψ_ξ is well-defined up to graded isomorphism.*

Proof. This is immediate from the fact that $\Psi_{-\mathcal{F}}$ is defined up to graded isomorphism.

$\square$

5.0.2. Properties of the bordered contact gluing map

We will now prove that Ψ_ξ satisfies identity, EH-preservation, and composition laws analogous to the ones satisfied by the HKM-map. The composition law Lemma 5.0.13 is necessary for the proof of Theorem 1, and our proof the composition law requires the other two properties.

Lemma 5.0.6. *Let (M, Γ) be a sutured manifold and $\mathcal{F} \subset \partial M$ be a sutured surface with disk-decomposable dividing set γ. Let $\mathcal{W}$ be the cap for $-\gamma$ on $-\mathcal{F}$ and let ξ_γ be the I-invariant contact structure for $-\gamma$ on the double $\mathcal{D}(\mathcal{W})$. Then*

$$\Psi_{\xi_\gamma} : \mathrm{SFH}(-M, -\Gamma) \to \mathrm{SFH}(\mathcal{D}(\mathcal{W}) \cup_{-\overline{\mathcal{F}}} (-M, -\Gamma))$$

is a graded isomorphism.

Proof. This follows immediately from the identity and symmetry properties of the bordered gluing map along with Lemma 5.0.7. □

Lemma 5.0.7. *Let $\mathcal{W}$ be the cap for a disk-decomposable dividing set γ on a sutured surface $\mathcal{F}$. Let ξ be the I-invariant contact structure on $F \times I$. Then the diagonal class $[\Delta] \in \mathrm{SFH}(\mathcal{D}(\mathcal{W}))$ is the contact class $\mathrm{EH}(\xi)$.*

Proof. The diagonal class is always non-trivial by [Zar11]. Because the dividing set is disk-decomposable, $\mathrm{SFH}(\mathcal{D}(\mathcal{W}))$ has rank 1 and the contact invariant is non-trivial. Thus, the contact invariant and the diagonal class are the same. □

We now wish to prove that the bordered contact gluing map preserves the contact invariant. To do this, we rely on the identification of the bordered contact handle maps with HKM handle maps provided by Lemma 7.0.1 and Lemma 7.0.2. This shows that each bordered contact handle map Ψ_i preserves the contact invariant. We then need to leverage associativity of the bordered gluing map; see Theorem 4.0.3. However, we can only do this with contact handles which have disjoint attaching regions, and this cannot always be achieved when the attaching curve of a 2-handle runs over a 1-handle. To get around this problem, we now introduce a version of the padded handles of Subsection 3.0.2 adapted to our needs. We will prove that the bordered contact maps associated to these modified handle attachments are equivalent to the Ψ_i.

Let (M, Γ) be a balanced sutured manifold and ξ_Γ be the $[-1, 0]$-invariant con-

tact structure in a closed neighborhood $N(\partial M) \cong \partial M \times [-1, 0]$ of the boundary $\partial M \times \{0\}$. Let Σ be a subsurface of ∂M such that $\gamma = \Gamma|_\Sigma$ is a dividing set for the (possibly degenerate) sutured surface $(\Sigma, -\partial\gamma)$. Let ξ_γ be the $[0, 1]$-invariant contact structure on $\Sigma \times [0, 1]$ compatible with γ.

If h^i is a contact handle whose attaching region A is contained in Σ, let L be a Legendrian graph of the form $\{q_k\} \times [0, 1]$ where each q_k is a point in γ disjoint from A and each component of γ contains at least one q_k. The *punctured padding* associated to (Σ, A) and L is the contact manifold $((\Sigma \times [0, 1]) \setminus N(L), \xi_\gamma^L)$, where ξ_γ^L is the restriction of ξ_γ to $(\Sigma \times [0, 1]) \setminus N(L)$. We denote the underlying sutured manifold by (P^L, Γ^L), where Γ^L is the dividing set induced by ξ_γ^L. The *punctured padded contact i-handle* associated to Σ, h^i, and L is $((\Sigma \times [0, 1]) \setminus N(L), \xi_\gamma^L) \cup_{A \times \{1\}} h^i$; we denote the associated sutured contact manifold by $(P_i^L, \Gamma_i^L, \xi_i^L)$. The sutured surface $(\Sigma \setminus N(\{q_j\}), -\partial(\gamma \setminus N(\{q_j\})))$ is denoted $\mathcal{F}^L$. Note that $\xi_\Gamma \cup_A h^i$ is contactomorphic to $\xi_\Gamma \cup_{\mathcal{F}^L} \xi_i^L$ relative to $\partial M \times [-1, -\frac{1}{2}]$.

Remark 5.0.8. Note that a contact handle decomposition of 1-and 2-handles can be described by a punctured padded contact handle decomposition. If a contact structure ξ can be factored into padded handles $(P_{i_1}, \xi_{i_1}), \ldots, (P_{i_n}, \xi_{i_n})$ attached to an I-invariant contact structure (P, ξ_Γ), we can choose Legendrians $L_j \subset P_{i_j}$ so that each L_j can be extended through $P_{i_{j+1}} \cup \ldots \cup P_{i_n}$ without intersecting the attaching region of any contact handle; see Figure 5.1. One can do this in such a way that ξ restricted to $P \cup P_{i_1}^{L_1} \cup \ldots \cup P_{i_n}^{L_n}$ is contactomorphic to ξ. One can also perform a similar construction if each

padded handle (P_{i_j}, ξ_{i_j}) is replaced by $(\Sigma_j \times [j, j+1], \xi_{\gamma_j}) \cup h^{i_j}$, where each Σ_j is a (possibly degenerate) sutured surface with dividing set γ_j. We employ this latter case in Lemma 5.0.12 and Lemma 5.0.13; details are left to the reader.

In order to make use of these modified contact handles, we need to to know that the dividing set $\Gamma^L|_{\mathcal{F}^L} = \gamma \setminus N(\{q_j\})$ is disk-decomposable.

Lemma 5.0.9. *Let Σ be a (possibly degenerate) sutured surface with dividing set γ, and let $\{p_k\}$ be a finite collection of points which has non-empty intersection with each closed component of γ. Let F be the surface $\Sigma \setminus N(\{p_k\})$. Then $(F, -\partial\gamma|_F)$ is a non-degenerate sutured surface and the dividing set $\gamma|_F$ is disk-decomposable.*

Proof. $(F, -\partial\gamma|_F)$ is non-degenerate by construction. Let ξ be the $[0,1]$-invariant contact structure on $F \times [0,1]$ compatible with $\gamma|_F$ and Γ_ξ be the sutures induced by ξ. Note that ξ is tight, since $\gamma|_F$ has no closed components.

We can choose a collection of disjoint embedded arcs $\{a_i\} \subset F \setminus \gamma|_F$ with $\partial a_i \subset \partial F$ such that $F \setminus \cup a_i$ is a number of disks. The collection $\{a_i \times [0,1]\} \subset F \times [0,1]$ determines a product disk decomposition of ξ. (Each $\partial(a_i \times [0,1])$ intersects Γ_ξ twice because of smoothing at the corners.) Thus, $\mathrm{SFH}(F \times [0,1], \Gamma_\xi)$ has rank one. $\qquad\square$

Note that punctured padded handles depend on a choice of subsurface Σ and Legendrian graph L. We will show that this ambiguity does not affect the associated gluing maps. To prove this and the remaining properties of Ψ_ξ, we assume the following

consequence of Lemma 7.0.1 and Lemma 7.0.2; we may do this, since the proof of these these lemmas does not depend on any of the results in this section.
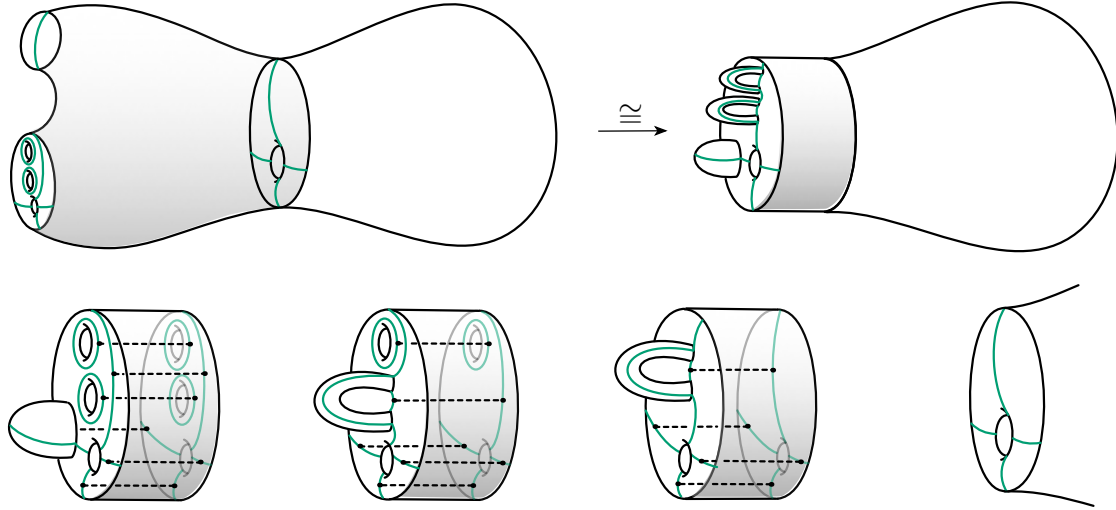

Figure 5.1. Decomposing a contact structure into padded handle attachments. A choice of Legendrians which determine a punctured padded handle decomposition are indicated by dashed lines.

Lemma 5.0.10. *Let (M_i, Γ_i, ξ_i) be a sutured contact manifold obtained from (M, Γ, ξ) by a contact i-handle attachment, and let $\Psi_i : \mathrm{SFH}(-M, -\Gamma) \to (-M_i, -\Gamma_i)$ be the associated bordered contact gluing map. Then $\Psi_i(\mathrm{EH}(\xi)) = \mathrm{EH}(\xi_i)$ up to graded isomorphism.*

Proof. By Lemma 3.0.8, Lemma 7.0.1, and Lemma 7.0.2, the bordered contact gluing map Ψ_i equals the corresponding HKM-map Φ_i up to graded isomorphism. Since Φ_i preserves the contact class, so does Ψ_i up to graded isomorphism. $\square$

Now we show that the bordered contact gluing maps for punctured padded handle attachments and ordinary contact handle attachments agree.

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Lemma 5.0.11. *Let* (M, Γ) *be a sutured manifold,* h^i *be a contact i-handle, and*

$(P_i^L, \Gamma_i^L, \xi_i^L)$ *be a punctured padded contact i-handle. Under the identification*

$(M, \Gamma) \cong (P^L, \Gamma^L) \cup_{\mathcal{F}^L} (M, \Gamma)$ *the corresponding bordered contact gluing maps*

$$\Psi_i, \Psi_{\xi_i^L} : \mathrm{SFH}(-M, -\Gamma) \to \mathrm{SFH}(-M_i, -\Gamma_i)$$

are equal up to graded isomorphism.

Proof. Split $(P_i^L, \Gamma_i^L, \xi_i^L)$ into a contact handle h^i and the I-invariant contact structure

ξ_Γ^L on the punctured padding (P^L, Γ^L). By associativity of the bordered gluing map, the

following diagram commutes up to graded isomorphism:

$$\mathrm{SFH}(-h^i) \otimes \mathrm{SFH}(-P^L, -\Gamma^L) \otimes \mathrm{SFH}(-M, -\Gamma) \xrightarrow{\Psi_{-\mathcal{F}_i}} \mathrm{SFH}(-P_i^L, -\Gamma_i^L) \otimes \mathrm{SFH}(-M, -\Gamma)$$

$$\downarrow{\Psi_{-\mathcal{F}^L}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\Psi_{-\mathcal{F}^L}}$$

$$\mathrm{SFH}(-h^i) \otimes \mathrm{SFH}(-P^L \cup -M, -\Gamma) \xrightarrow{\Psi_{-\mathcal{F}_i}} \mathrm{SFH}(-P_i^L \cup -M, \Gamma_i)$$

By Lemma 5.0.10, the bordered gluing map $\Psi_{-\mathcal{F}_i}$ on the top sends $(\mathrm{EH}(h^i), \mathrm{EH}(\xi_\Gamma^L))$

to $\mathrm{EH}(\xi_i^L)$, up to graded isomorphism. Evaluating on the contact elements yields the

following commutative diagram:

$$\mathrm{SFH}(-M, -\Gamma) \xrightarrow{\mathrm{Id}} \mathrm{SFH}(-M, -\Gamma)$$

$$\downarrow{\Psi_{\xi_\Gamma^L}} \qquad\qquad\qquad\qquad \downarrow{\Psi_{\xi_i^L}}$$

$$\mathrm{SFH}(-P^L \cup -M, -\Gamma) \xrightarrow{\Psi_i} \mathrm{SFH}(-P_i^L \cup -M, \Gamma_i)$$

The map $\Psi_{\xi_\Gamma^L}$ is a graded isomorphism by Lemma 5.0.6, since the dividing set on $\mathcal{F}^L$ is

disk-decomposable by Lemma 5.0.9. Since the diagram commutes up to graded isomor-

phism, the lemma follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

We now show that the bordered contact gluing map preserves the contact class.

Lemma 5.0.12. *Let (M, Γ) and (M'', Γ'') be sutured manifolds with compatible contact structures ζ and ξ respectively, and let $(M', \Gamma') = (M'', \Gamma'') \cup_{\mathcal{F}} (M, \Gamma)$ for some sutured surface $\mathcal{F}$ with disk-decomposable dividing set. Then*

$$\Psi_\xi(\mathrm{EH}(\zeta)) = \mathrm{EH}(\xi \cup_{\mathcal{F}} \zeta)$$

up to graded isomorphism.

Proof. Decompose (M'', Γ'') into a number of punctured padded handles $(P_{i_1}^{L_1}, \Gamma_{i_1}^{L_1}), \ldots,$ $(P_{i_n}^{L_n}, \Gamma_{i_n}^{L_n})$ and let us consider two particular compositions of bordered gluing maps

$$\mathrm{SFH}(-P_{i_n}^{L_n}, -\Gamma_{i_n}^{L_n}) \otimes \ldots \otimes \mathrm{SFH}(-P_{i_1}^{L_1}, -\Gamma_{i_1}^{L_1}) \otimes \mathrm{SFH}(-M, -\Gamma) \to \mathrm{SFH}(-M', -\Gamma')$$

corresponding to gluing in different orders. One composition corresponds to attaching the punctured padded handles sequentially to (M, Γ), while another composition corresponds to attaching the punctured padded handles sequentially to $(P_{i_1}^{L_1}, \Gamma_{i_1}^{L_1})$ to get (M'', Γ'') before finally gluing to (M, Γ). By Theorem 4.0.3, these compositions give rise to the following diagram which commutes up to graded isomorphism:

$$\mathrm{SFH}(-P_{i_n}^{L_n}, -\Gamma_{i_n}^{L_n}) \otimes \ldots \otimes \mathrm{SFH}(-P_{i_1}^{L_1}, -\Gamma_{i_1}^{L_1}) \otimes \mathrm{SFH}(-M, -\Gamma)$$

$$\downarrow \qquad\qquad\qquad \searrow$$

$$\mathrm{SFH}(-M'', -\Gamma'') \otimes \mathrm{SFH}(-M, -\Gamma) \xrightarrow{\quad \Psi_{-\mathcal{F}} \quad} \mathrm{SFH}(-M', -\Gamma')$$

By Lemma 5.0.11 and Lemma 5.0.10, each punctured padded handle attachment map preserves the contact invariant up to graded isomorphism. The vertical arrow thus

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maps $\mathrm{EH}(\xi_{i_n}^{L_n}) \otimes \ldots \otimes \mathrm{EH}(\xi_{i_1}^{L_1}) \otimes \mathrm{EH}(\zeta)$ to $\mathrm{EH}(\xi) \otimes \mathrm{EH}(\zeta)$, while the diagonal arrow maps

$\mathrm{EH}(\xi_{i_n}^{L_n}) \otimes \ldots \otimes \mathrm{EH}(\xi_{i_1}^{L_1}) \otimes \mathrm{EH}(\zeta)$ to $\mathrm{EH}(\xi \cup_{\mathcal{F}} \zeta)$. Since Ψ_ξ is the evaluation of $\Psi_{-\mathcal{F}}$ on

$\mathrm{EH}(\xi)$, it must send $\mathrm{EH}(\zeta)$ to $\mathrm{EH}(\xi \cup_{\mathcal{F}} \zeta)$ up to graded isomorphism. $\square$

We can finally prove the composition law which we need in our proof of Theorem 1.

Lemma 5.0.13. *Let (M_1, Γ_1), (M_2, Γ_2), and (M_3, Γ_3) be sutured manifolds with compatible contact structures ξ on (M_1, Γ_1) and ξ' on (M_2, Γ_2). Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are sutured surfaces with disk decomposable dividing sets such that the bordered gluing $(M_1, \Gamma_1) \cup_{\mathcal{F}} (M_2, \Gamma_2) \cup_{\mathcal{F}'} (M_3, \Gamma_3)$ is defined. Then*

$$\Psi_{\xi \cup_{\mathcal{F}} \xi'} = \Psi_\xi \circ \Psi_{\xi'}$$

up to graded isomorphism.

Proof. Write $\mathcal{M}_k = (M_k, \Gamma_k)$. The following diagram commutes up to graded isomorphism by associativity of Zarev's bordered gluing map.

$$
\begin{array}{ccc}
\mathrm{SFH}(-\mathcal{M}_1 \sqcup -\mathcal{M}_2 \sqcup -\mathcal{M}_3) & \xrightarrow{\Psi_{-\mathcal{F}'}} & \mathrm{SFH}(-\mathcal{M}_1 \sqcup -\mathcal{M}_2 \cup_{-\mathcal{F}'} -\mathcal{M}_3) \\
\Big\downarrow{\scriptstyle \Psi_{-\mathcal{F}}} & & \Big\downarrow{\scriptstyle \Psi_{-\mathcal{F}}} \\
\mathrm{SFH}(-\mathcal{M}_1 \cup_{-\mathcal{F}} -\mathcal{M}_2 \sqcup -\mathcal{M}_3) & \xrightarrow{\Psi_{-\mathcal{F}'}} & \mathrm{SFH}(-\mathcal{M}_1 \cup_{-\mathcal{F}} -\mathcal{M}_2 \cup_{-\mathcal{F}'} -\mathcal{M}_3)
\end{array}
$$

By Lemma 5.0.12, the map $\Psi_{-\mathcal{F}}$ on top sends $\mathrm{EH}(\xi) \otimes \mathrm{EH}(\xi') \otimes [\mathbf{y}]$ to $\mathrm{EH}(\xi \cup_{\mathcal{F}} \xi') \otimes [\mathbf{y}]$ for any cycle $\mathbf{y}$. By evaluating on contact elements, we obtain the following diagram

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which also commutes up to graded isomorphism.

$$\begin{array}{ccc}
\mathrm{SFH}(-\mathcal{M}_3) & \xrightarrow{\Psi_{\xi'}} & \mathrm{SFH}(-\mathcal{M}_2 \cup_{-\mathcal{F}'} -\mathcal{M}_3) \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \Psi_\xi} \\
\mathrm{SFH}(-\mathcal{M}_3) & \xrightarrow{\Psi_{\xi \cup_{\mathcal{F}} \xi'}} & \mathrm{SFH}(-\mathcal{M}_1 \cup_{-\mathcal{F}} -\mathcal{M}_2 \cup_{-\mathcal{F}'} -\mathcal{M}_3)
\end{array}$$

$\square$

Chapter 6. Background on Computing the Bordered Gluing Map

This section contains the background necessary for computing the bordered contact gluing map in the case of a contact handle attachment. We will need this in Subsection 7.0.1 when identifying the bordered map with the diagrammatic maps of Subsection 3.0.1.

6.0.1. Arc diagrams, bordered algebras, and bordered sutured diagrams

Computing the bordered gluing map explicitly requires one to identify a particular generator in a certain Heegaard diagram for the twisting slice $\mathcal{TW}_{-\mathcal{F},+}$; see Lemma 6.0.1. This section is devoted to the background necessary to construct this diagram and identify the generator. Those familiar with Auroux-Zarev diagrams or willing to take this identification on faith may skip to chapter 7 and simply refer to Figure 6.3 and Lemma 6.0.1 when proving Lemma 7.0.2. We first review arc diagrams and describe the bordered algebras associated to contact handles.

An *arc diagram* $\mathcal{Z}$ is a union of oriented intervals $\mathbf{Z} = \sqcup Z_i$, an even number of marked points $\{z_j\} \subset \operatorname{Int}(\mathbf{Z})$, and a 2-to-1 function $\mu : \{z_j\} \to \mathbb{Z}_{>0}$. Points z_j and z_k are said to be *matched* if $\mu(z_j) = \mu(z_k)$. We require that the result of surgery on $\mathbf{Z}$ along all pairs of matched points $\mu^{-1}(l)$ have no closed components. An oriented subinterval of $\mathbf{Z}$ from one marked point to another (not necessarily matched) marked point is called a *Reeb chord*. Arc diagrams also come with a type: α-type or β-type.
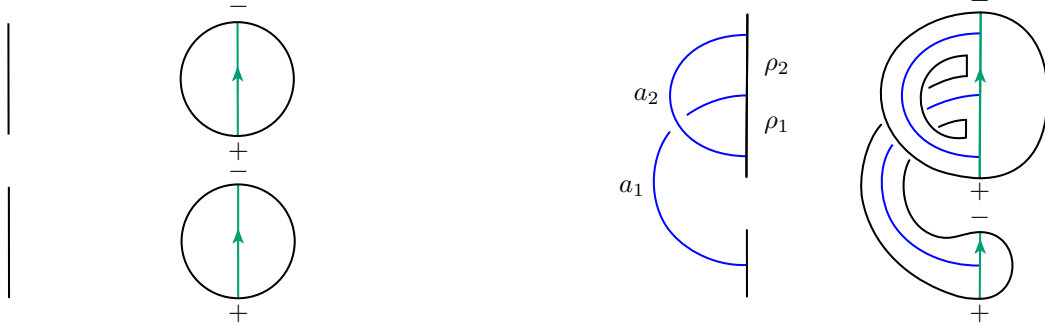
Figure 6.1. Arc diagrams $\mathcal{Z}_1$ (Far left) and $\mathcal{Z}_2$ (Middle right) for contact 1-and 2-handles. The sutured surfaces $\mathcal{F}_1$ (Middle left) and $\mathcal{F}_2$ (Far right) they parametrize are also depicted. Note that these diagrams have α-type, though we have drawn the arcs blue since we will be working with Heegaard diagrams of the form $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$.

Given $\mathcal{Z}$, there is an arc diagram $-\mathcal{Z}$ obtained by changing the orientation of each Z_i, and another arc diagram $\overline{\mathcal{Z}}$ obtained by changing the type.

The *graph* $G(\mathcal{Z})$ of an arc diagram is the ribbon graph obtained by attaching arcs $\{a_l\}$ to $\mathbf{Z}$ so that $\partial a_l = \mu^{-1}(l)$. If $\mathcal{Z}$ is α-type, the arcs are attached on the left with respect to the orientation on the intervals Z_i; if $\mathcal{Z}$ is β-type, the arcs are attached on the right. The graph $\overline{G(\mathcal{Z})}$ is obtained by reversing the cyclic ordering around each vertex in $\cup \partial a_l$. Alternatively, $\overline{G(\mathcal{Z})}$ is $G(\overline{\mathcal{Z}})$ as a ribbon graph.

An arc diagram $\mathcal{Z}$ *parametrizes* a sutured surface $\mathcal{F} = (F, \Lambda)$ if there is an embedding $G(\mathcal{Z}) \subset F$ such that $\partial(\cup Z_i) = \Lambda$ as oriented manifolds, and F deformation retracts onto $G(\mathcal{Z})$. If $\mathcal{Z}$ parametrizes $\mathcal{F}$, then $-\mathcal{Z}$ parametrizes $-\mathcal{F}$ and $\overline{\mathcal{Z}}$ parametrizes $\overline{\mathcal{F}}$.

The bordered algebra $A = A(\mathcal{Z})$ associated to an arc diagram $\mathcal{Z}$ is generated as an $\mathbb{F}_2$-vector space by certain formal sums of strand diagrams; multiplication corre-

sponds to concatenation of strand diagrams. A is a differential graded algebra over the ground ring $I \subset A$ consisting of idempotents of A.

There is a preferred basis of idempotents which is in one-to-one correspondece with collections of arcs of $\mathcal{Z}$; we denote the basis idempotent corresponding to $\{a_{i_1}, \ldots, a_{i_p}\}$ by $\iota_{i_1, \ldots, i_p}$. The basis of idempotents can be extended to a basis of A by adding algebra elements associated to collections of Reeb chords; we will abuse notation and write ρ for the algebra element associated to a single Reeb chord ρ. (We will not need to consider algebra elements associated to collections of multiple Reeb chords.)

The algebra and the idempotents split as $A = \bigoplus_i A(i)$ and $I = \bigoplus_i I(i)$, each indexed by the number of strands in the associated diagrams. (Note that we are following Zarev's convention for indexing summands of the algebra, which deviates from the convention in [LOT15].) Since we only need to understand the bordered algebras for the arc diagrams we have chosen for contact 1-and 2-handles, we omit the general definition here, and explicitly describe the salient features of the algebras we need. The interested reader can find formal definitions in [LOT15] and [Zar11]. Note that the bordered algebras $A(-\mathcal{Z})$ and $A(\overline{\mathcal{Z}})$ are isomorphic to the opposite algebra $A(\mathcal{Z})^{op}$, which is $A(\mathcal{Z})$ with multiplication reversed.

Let $\mathcal{Z}_1$ be the arc diagram on the left of Figure 6.1 which parametrizes the sutured surface $\mathcal{F}_1 = (F_1, \Lambda_1)$ for a contact 1-handle. The associated bordered algebra $A_1 = A(\mathcal{Z}_1)$ has rank 1; the only non-trivial summand is $A_1(0) = I_1(0)$, with basis
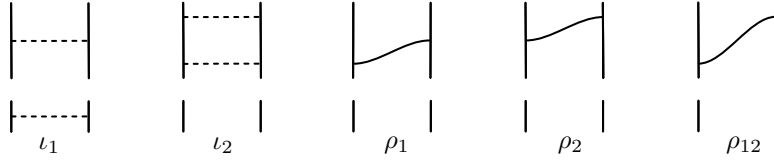
Figure 6.2. Strand diagrams corresponding to basis elements of $A_2(1)$. The two leftmost diagrams correspond to basis idempotents, while the others correspond to Reeb chords. Multiplication is given by concatenating, i.e. $\rho_1 \cdot \rho_2 = \rho_{12}$.

element the idempotent $\iota_\varnothing$ corresponding to the strand diagram with no strands.

Let $\mathcal{Z}_2$ be the arc diagram on the right of Figure 6.1 which parametrizes the sutured surface $\mathcal{F}_2 = (F_2, \Lambda_2)$ for a contact 2-handle. The associated bordered algebra $A_2 = A(\mathcal{Z}_2)$ has rank 9 with three non-trivial summands. Just as for $\mathcal{Z}_1$, the summand $A_2(0) = I_2(0)$ has rank 1, with basis element the idempotent $\iota_\varnothing$ corresponding to the strand diagram with no strands. The summand $A_2(1)$ has rank 5, with basis elements corresponding to the strand diagrams depicted in Figure 6.2. The ground ring $I_2(1)$ is generated by the basis idempotents ι_1, ι_2. Each of these corresponds to a formal sum of two strand diagrams, each such sum is represented by a superposition of the two diagrams with dashed lines. The nontrivial actions by idempotents are

$$\iota_2 \cdot \rho_1 \cdot \iota_1 = \rho_1 \qquad \iota_1 \cdot \rho_2 \cdot \iota_2 = \rho_2 \qquad \iota_2 \cdot \rho_{12} \cdot \iota_2 = \rho_{12},$$

while the only other non-trivial multiplication is $\rho_1 \cdot \rho_2 = \rho_{12}$. Multiplication is given by concatenating strand diagrams. The summand $A_2(2)$ has rank 3; it does not appear in our computations.

Furthermore, each algebra is equipped with a grading and a differential; we will

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not need to consider either.

Suppose $\mathcal{Z}^\alpha$ is an α-type arc diagram and $\mathcal{Z}^\beta$ is a β-type arc diagram. A *bordered sutured Heegaard diagram* $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{Z}^\alpha, \mathcal{Z}^\beta)$ consists of

- an oriented surface Σ without closed components

- a collection $\boldsymbol{\alpha} = \boldsymbol{\alpha}^c \cup \boldsymbol{\alpha}^a$ of pairwise disjoint, properly embedded curves $\boldsymbol{\alpha}^c$ and arcs $\boldsymbol{\alpha}^a$ in Σ

- another collection $\boldsymbol{\beta} = \boldsymbol{\beta}^c \cup \boldsymbol{\beta}^a$ of pairwise disjoint, properly embedded curves $\boldsymbol{\beta}^c$ and arcs $\boldsymbol{\beta}^a$ in Σ

- embeddings $(G(\mathcal{Z}^\alpha), \mathbf{Z}^\alpha) \subset (\Sigma, \partial\Sigma)$ and $(\overline{G(\mathcal{Z}^\beta)}, \mathbf{Z}^\beta) \subset (\Sigma, \partial\Sigma)$ such that $\mathbf{Z}^\alpha$ and $\mathbf{Z}^\beta$ are disjoint, the orientation on each component of $\mathbf{Z}^\alpha \sqcup \mathbf{Z}^\beta$ agrees with the orientation on $\partial\Sigma$, the arcs of $G(\mathcal{Z}^\alpha)$ agree with the arcs of $\boldsymbol{\alpha}^a$, the arcs of $G(\mathcal{Z}^\beta)$ agree with the arcs of $\boldsymbol{\beta}^a$, and no component of $\Sigma \setminus \boldsymbol{\alpha}$ or $\Sigma \setminus \boldsymbol{\beta}$ is disjoint from $\partial\Sigma \setminus (\mathbf{Z}^\alpha \sqcup \mathbf{Z}^\beta)$.

Let $A = A(\mathcal{Z}^\alpha)$ and $B = A(\mathcal{Z}^\beta)$ be the bordered algebras with respective subrings of idempotents $I^\alpha \subset A$, $I^\beta \subset B$. We can choose to consider the bordered module $\widehat{\mathrm{BSAA}}(\mathcal{H})$ as an $(I^\alpha)^{op}, I^\beta$–bimodule generated by tuples of intersection points which occupy each curve in $\boldsymbol{\alpha}^c \cup \boldsymbol{\beta}^c$ exactly once and each arc in $\boldsymbol{\alpha}^a \cup \boldsymbol{\beta}^a$ at most once. If a generator $\mathbf{x}$ occupies arcs which correspond to $a_{i_1}, \dots, a_{i_p} \subset G(\mathcal{Z}^\alpha)$ and $b_{j_1}, \dots, b_{j_q} \subset \overline{G(\mathcal{Z}^\beta)}$, then we have the algebra action $\iota_{i_1,\dots,i_p} \cdot \mathbf{x} \cdot \iota_{j_1,\dots,j_q} = \mathbf{x}$. Since these are the unique basis idempotents which act nontrivially on $\mathbf{x}$ we will also write

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this equation as $\iota_L(\mathbf{x}) \cdot \mathbf{x} \cdot \iota_R(\mathbf{x}) = \mathbf{x}$.

The differential and A_∞-algebra actions are determined by counting certain pseudoholomorphic curves. The A_∞-module actions are maps

$$m_{i|1|j} : (A^{op})^{\otimes i} \otimes \widehat{\mathrm{BSAA}}(\mathcal{H}) \otimes B^{\otimes j} \to \widehat{\mathrm{BSAA}}(\mathcal{H})$$

which satisfy certain relations, where each tensor product is over $(I^\alpha)^{op}$ or I^β as appropriate.

One similarly defines Type-A and Type-D structures $\widehat{\mathrm{BSA}}$, $\widehat{\mathrm{BSD}}$ for diagrams of the form $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{Z}^\alpha, \varnothing)$ or $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \varnothing, \mathcal{Z}^\beta)$. Just as for $\widehat{\mathrm{BSAA}}$, these invariants are generated as $\mathbb{F}_2$-vector spaces over tuples of intersection points which occupy each full curve exactly once and each arc in at most once.

A bordered sutured diagram is *nice*, if every component of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ is one of the following:

1. a region with non-trivial intersection with $(\partial\Sigma) \setminus (\mathbf{Z}^\alpha \sqcup \mathbf{Z}^\beta)$; referred to as

   a *basepoint region* or *suture region*

2. a bigon with no side in $\mathbf{Z}^\alpha \sqcup \mathbf{Z}^\beta$

3. a quadrilateral with at most one side in $\mathbf{Z}^\alpha \sqcup \mathbf{Z}^\beta$.

In this case the differential of $\widehat{\mathrm{BSAA}}(\mathcal{H})$ is given by counting each region which is an embedded bigon or rectangle in the interior of the Heegaard surface, and which is disjoint from the basepoint regions. The A_∞-module actions are given by counting each region which is an embedded rectangle with one side a Reeb chord ρ in $\mathbf{Z}^\alpha \sqcup \mathbf{Z}^\beta \subset$

$\partial\Sigma$ and the other three sides in $\mathrm{Int}(\Sigma)$. More precisely, let $\mathbf{x} = (x_0, z_1, \ldots, z_n)$ and

$\mathbf{y} = (y_0, z_1, \ldots, z_n)$ be generators in a nice bordered diagram $\mathcal{H}$ with basis idempotent

actions $\iota_L(\mathbf{x}) \cdot \mathbf{x} \cdot \iota_R(\mathbf{x}) = \mathbf{x}$ and $\iota_L(\mathbf{y}) \cdot \mathbf{y} \cdot \iota_R(\mathbf{y}) = \mathbf{y}$, and let ρ be a Reeb chord in $\mathbf{Z}^\alpha$.

Suppose there is a rectangular region D which has corners $\{x_0, y_0\} \cup \partial\rho$, goes out of x_0,

goes into y_0, and has no z_k in its interior. Then D contributes a $\mathbf{y}$ term to the action

$m_{0|1|1}(\mathbf{x}, \iota_L(\mathbf{x}) \cdot \rho \cdot \iota_R(\mathbf{y}))$. If ρ is a chord in $\mathbf{Z}^\beta$ instead, then D contributes a $\mathbf{y}$ term to

the action $m_{1|1|0}(\iota_L(\mathbf{y}) \cdot \rho \cdot \iota_R(\mathbf{x}), \mathbf{x})$. The only other non-trivial actions in either case

come from compatible idempotents, i.e. $m_{0|1|1}(\mathbf{x}, \iota_R(\mathbf{x})) = m_{1|1|0}(\iota_L(\mathbf{x}), \mathbf{x}) = \mathbf{x}$.

As an example, we explicitly compute some actions of the $\mathcal{A}_\infty$-bimodule

$\widehat{\mathrm{BSAA}}(\mathcal{H}_{AZ})$ associated to the nice diagram $\mathcal{H}_{AZ} = (\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, \overline{\mathcal{Z}}_2 \sqcup \mathcal{Z}_2)$ on the right

hand side of Figure 6.3. We will give $\widehat{\mathrm{BSAA}}(\mathcal{H}_{AZ})$ a right action by $A(-\mathcal{Z}_2) \approx A(\mathcal{Z}_2)^{op}$

and a left action by $A(\overline{\mathcal{Z}}_2) \approx A(\mathcal{Z}_2)^{op}$. Note that the usual roles of α and β curves are

reversed and $\mathcal{Z}_2$ has α-type.

The region D_1 contributes the action $m_{1|1|0}(\rho_1, z_1) = \rho_1 \cdot z_1 = z_4$. The region

$D_1 \cup D_2$ contributes the action $m_{1|1|0}(\rho_{12}, z_1) = \rho_{12} \cdot z_1 = z_3$. The region D_5 contributes

the action $m_{0|1|1}(z_2, \rho_1) = z_2 \cdot \rho_1 = z_3$. The region $D_3 \cup D_4$ contributes $m_{0|1|1}(z_4, \rho_2) =$

$z_4 \cdot \rho_2 = z_5$. The region $D_1 \cup D_3 \cup D_4$ contributes no action, since it has two sides which

hit the boundary.

In fact, $\widehat{\mathrm{BSAA}}(\mathcal{H}_{AZ})$ is isomorphic to the dual algebra $A(-\mathcal{Z}_2)^\vee$ as an $\mathcal{A}_\infty$-

bimodule; see page 122 of [Zar11] for the general relationship between the actions of

a bimodule M and its dual $M^\vee$. This can be seen by identifying z_1 with $\rho_{12}^\vee$, z_2 with $\rho_1^\vee$, z_3 with $\iota_2^\vee$, z_4 with $\rho_2^\vee$, and z_5 with $\iota_1^\vee$. Under this identification, the actions described above are dual to the multiplications $\rho_2 \cdot \rho_1 = \rho_{12}$, $\iota_2 \cdot \rho_{12} = \rho_{12}$, $\rho_1 \cdot \iota_2 = \rho_1$, and $\rho_2 \cdot \iota_1 = \rho_2$ in $A(-\mathcal{Z}_2) \approx A(\mathcal{Z}_2)^{op}$. The reader may verify that the remaining actions of $\widehat{\mathrm{BSAA}}(\mathcal{H}_{AZ})$ and $A(-\mathcal{Z}_2)^\vee$ correspond exactly.

We will see in the next section that understanding this identification is necessary for computing the bordered gluing map.

6.0.2. Computing the bordered gluing map with Heegaard diagrams

In this section we discuss how to compute the bordered gluing map in the special case of contact handle attachments.

Let $(\mathcal{W}, -\mathcal{Z}) = (W, \gamma, -\mathcal{F}, -\mathcal{Z})$ be the bordered cap for a sutured surface $\mathcal{F} = (F, \Lambda)$, where $\mathcal{F}$ is parametrized by $\mathcal{Z}$. Now, suppose that sutured manifolds (M_1, Γ_1) and (M_2, Γ_2) can be glued along $\mathcal{F}$; this is equivalent to saying that there are embeddings $(F, \Lambda) \subset (\partial M_1, \Gamma_1)$ and $(-F, \Lambda) \subset (\partial M_2, \Gamma_2)$ which extend to embeddings $\mathcal{W} \subset (M_1, \Gamma_1)$ and $\overline{\mathcal{W}} \subset (M_2, \Gamma_2)$. Then $(M_1, \Gamma_1) \setminus \mathcal{W}$ and $(M_2, \Gamma_2) \setminus -\mathcal{W}$ are partially sutured manifolds with sutured surfaces $\mathcal{F}$ and $\overline{\mathcal{F}}$, so that $((M_1, \Gamma_1) \setminus \mathcal{W}, \mathcal{Z})$ and $((M_2, \Gamma_2) \setminus \overline{\mathcal{W}}, \overline{\mathcal{Z}})$ are bordered sutured manifolds. Note that the positive twisting slice $(\mathcal{TW}_{\mathcal{F},+}, -\mathcal{Z} \sqcup -\overline{\mathcal{Z}})$ is also a bordered sutured manifold.

Choose bordered sutured Heegaard diagrams $\mathcal{H}_W$ for $(\mathcal{W}, -\mathcal{Z})$, $\mathcal{H}_U$ for $((M_1, \Gamma_1) \setminus \mathcal{W}, \mathcal{Z})$, and $\mathcal{H}_V$ for $((M_2, \Gamma_2) \setminus \overline{\mathcal{W}}, \overline{\mathcal{Z}})$. These determine Type-D structures $U = \widehat{\mathrm{BSD}}(\mathcal{H}_U)$

and $V = \widehat{\mathrm{BSD}}(\mathcal{H}_V)$, along with a Type-A structure $W = \widehat{\mathrm{BSA}}(\mathcal{H}_W)$. The mirror image $\mathcal{H}_{-\mathcal{W}}$ of $\mathcal{H}_\mathcal{W}$ is a Heegaard diagram for $\overline{\mathcal{W}}$ and the corresponding Type-A structure is $\widehat{\mathrm{BSA}}(\mathcal{H}_{-W}) = W^\vee$. Let $A = A(\mathcal{Z})$. There is a nice diagram $\mathcal{H}_{AZ}$ for $(\mathcal{T}\mathcal{W}_{\mathcal{F},+}, -\mathcal{Z} \sqcup -\overline{\mathcal{Z}})$, discovered independently by Auroux and Zarev, such that $\widehat{\mathrm{BSAA}}(\mathcal{H}_{AZ}) \cong A^\vee$ as $\mathcal{A}_\infty$-bimodules over A. The Auroux-Zarev diagrams associated to contact handle attachments are depicted in Figure 6.3; see Proposition 12.4.2 in [Zar11] for the general construction as well as [Aur10] and [LOT11] for computations.

Now, we have

$$\mathrm{SFC}(M_1, \Gamma_1) \otimes \mathrm{SFC}(M_2, \Gamma_2) \simeq U \boxtimes W \otimes W^\vee \boxtimes V, \text{ and}$$

$$\mathrm{SFC}((M_1, \Gamma_1) \cup_\mathcal{F} (M_2, \Gamma_2)) \simeq U \boxtimes A^\vee \boxtimes V.$$

With these identifications, [Zar11] gives an explicit formula for computing the bordered gluing map when the bordered module $W = \widehat{\mathrm{BSA}}(\mathcal{W})$ has a single generator and trivial structure maps; in this case W is called *elementary*. Note that if $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathcal{Z}^\alpha, \mathcal{Z}^\beta)$ is a bordered sutured diagram with a single generator and all regions adjacent to the intervals $\mathbf{Z}^\alpha \sqcup \mathbf{Z}^\beta \subset \partial\Sigma$ are basepoint regions, then the bordered modules for $\mathcal{H}$ are elementary.

Lemma 6.0.1. *(Zarev) Let U, V, W, A be as above. Suppose that W is an elementary module with unique generator w. The bordered gluing map $\Psi_\mathcal{F}$ is induced by the map*

$$\psi_{\mathcal{F}} : U \boxtimes W \otimes W^\vee \boxtimes V \to U \boxtimes A^\vee \boxtimes V \ \text{defined by}$$

$$\psi_{\mathcal{F}}(\mathbf{u} \boxtimes w \otimes w^\vee \boxtimes \mathbf{v}) = \mathbf{u} \boxtimes \iota_L(w)^\vee \boxtimes \mathbf{v},$$

where $\mathbf{u} \in U, \mathbf{v} \in V$ are arbitrary and $\iota_L(w)$ is the basis idempotent in A which acts non-trivially on w.
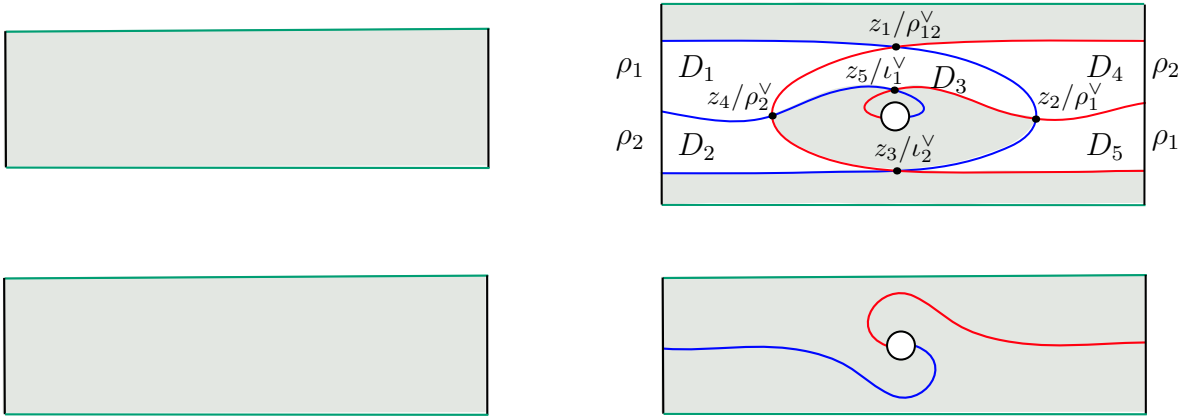

Figure 6.3. Left: $\mathcal{H}_{AZ}$ for $\mathcal{Z}_1$; Right: $\mathcal{H}_{AZ}$ for $\mathcal{Z}_2$. Each point is additionally labeled by the basis element it corresponds to under the identification $\widehat{\mathrm{BSAA}}(\mathcal{H}_{AZ}) \cong A(-\mathcal{Z}_2)^\vee$. The orientations of the Reeb chords agrees with the boundary orientation, i.e. the chords on the right are oriented upward, while the chords on the left are oriented downward.

Figure 6.4 shows how to decompose each contact handle $-h^i$ into a bordered cap $(-\mathcal{W}_i, \mathcal{Z}_i)$ and a bordered contact handle $(-h^i \setminus -\mathcal{W}_i, -\mathcal{Z}_i)$, along with the parametrizations of $\mathcal{F}_i$ and $-\mathcal{F}_i$ in each case. Figure 6.5 shows the diagrams $\mathcal{H}_U \cup \mathcal{H}_W$ corresponding to this decomposition. Suppose that $\mathcal{H}$ is a Heegaard diagram for the manifold $(-M, -\Gamma)$ to which the contact handle $-h^i$ is attached. After some Heegaard moves, we may arrange that a neighborhood of the attaching region in $-M$ be represented by a
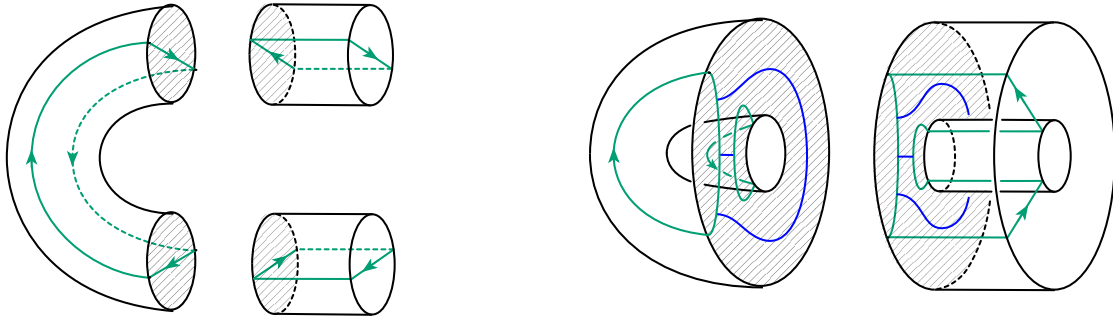
Figure 6.4. Decomposition of a contact 1-handle (Left) and a contact 2-handle (Right) into a cap and a bordered contact handle. The bordered part is shaded and parametrizing arc diagrams are depicted.
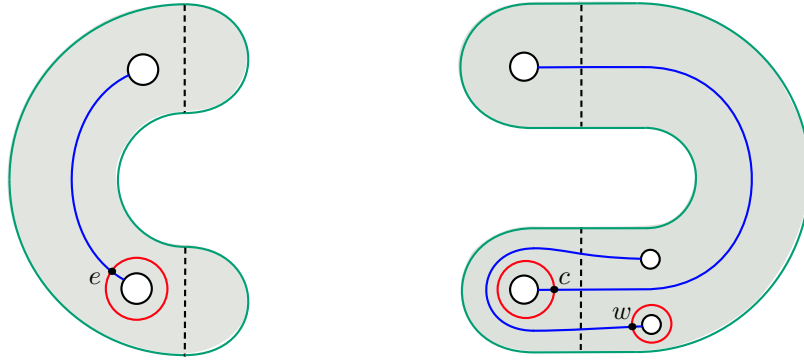

Figure 6.5. $\mathcal{H}_U \cup \mathcal{H}_W$ for a contact 1-handle (Left) and a contact 2-handle (Right).

copy of $\mathcal{H}_{-W}$ embedded in $\mathcal{H}$, so that $\mathcal{H} = \mathcal{H}_{-W} \cup \mathcal{H}_V$. To attach $-h^i$, one forms the concatenation $\mathcal{H}_U \cup \mathcal{H}_{AZ} \cup \mathcal{H}_V$. Note that, for each contact handle, the bordered module $\widehat{\mathrm{BSA}}(\mathcal{H}_W)$ is elementary, so that we can use Lemma 6.0.1 for computations.

Chapter 7. Proof of the Main Theorem

The goal of this section is the proof of Theorem 1. We will start by relating the diagrammatic and bordered contact gluing maps for contact handles.

7.0.1. Diagrammatic and bordered contact gluing maps

Lemma 7.0.1. *Given an admissible diagram $\mathcal{H}$ for $(-M, -\Gamma)$, there is a diagrammatic handle attachment $\mathcal{H}'$ for $(-M_1, -\Gamma_1)$ such that the diagrammatic map σ_1 : $\mathrm{SFC}(\mathcal{H}) \to \mathrm{SFC}(\mathcal{H}')$ induces the bordered contact gluing map Ψ_1 : $\mathrm{SFH}(-M, -\Gamma) \to \mathrm{SFH}(-M_1, -\Gamma_1)$ on homology.*

Proof. Use the diagram $\mathcal{H}_U \cup \mathcal{H}_W$ shown in Figure 6.5 for $-h^1$. Note that any admissible Heegaard diagram $\mathcal{H}$ for $(-M, -\Gamma)$ splits as $\mathcal{H} = \mathcal{H}_{-W} \cup \mathcal{H}_V$, where $\mathcal{H}_{-W}$ is a neighborhood of the attaching region of h^1 in $\mathcal{H}$.

The intersection point e represents the sole generator $\mathrm{EH}(h^1)$ of $\mathrm{SFH}(-h^1)$; it is also the generator of $\widehat{\mathrm{BSD}}(\mathcal{H}_U)$. The diagram $\mathcal{H}_W$ for the cap has no curves, so the generator w of $\widehat{\mathrm{BSD}}(\mathcal{H}_W)$ is the empty element. (In bordered sutured theory, a diagram with no generator is allowed; in this case the bordered invariants have rank 1, generated
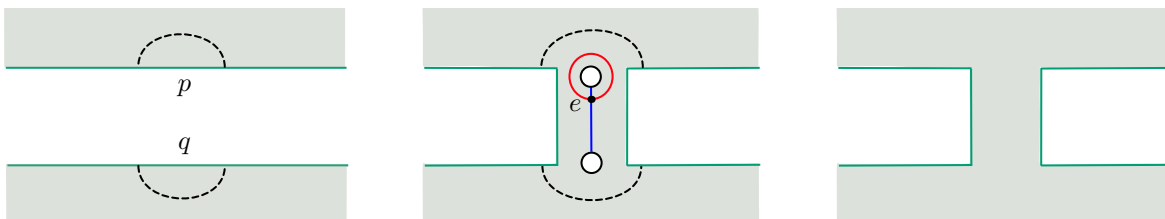

Figure 7.1. Gluing a contact 1-handle.

by the empty set.) We have $\iota_\varnothing \cdot w = w$, where $\iota_\varnothing$ is the unique generator of $A(-\mathcal{Z}_1)$; it corresponds to the strand diagram with no strands. Fix an arbitrary cycle $\mathbf{y} \in \mathrm{SFC}(\mathcal{H})$ and note that it corresponds to $(w^\vee, \mathbf{y}) \in \mathrm{SFC}(\mathcal{H}_{-W} \cup \mathcal{H}_V)$. Since $\widehat{\mathrm{BSD}}(\mathcal{H}_W)$ is an elementary module, we can use Lemma 6.0.1 to see that the bordered contact gluing map is induced by

$$\psi_{-\mathcal{F}}(e \otimes \mathbf{y}) = \psi_{-\mathcal{F}}((e, w) \otimes (w^\vee, \mathbf{y})) = (e, \iota_\varnothing^\vee, \mathbf{y}).$$
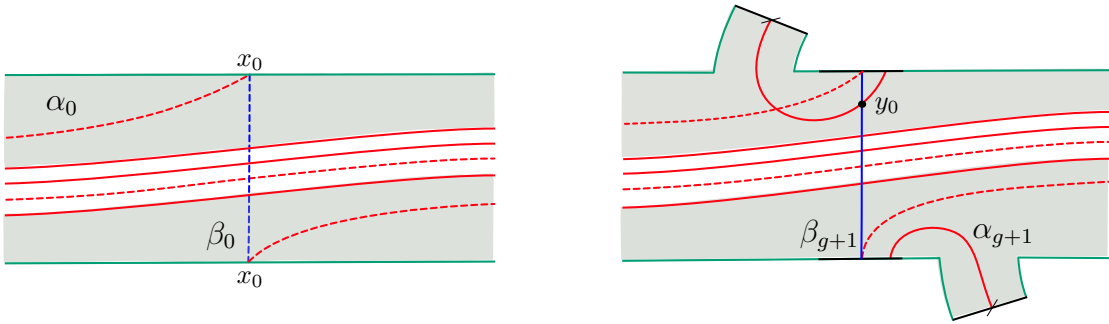

Figure 7.2. (Left) The diagram $\mathcal{H}_1$ with arcs for the contact 2-handle; (Right) Attaching the first trivial bypass.
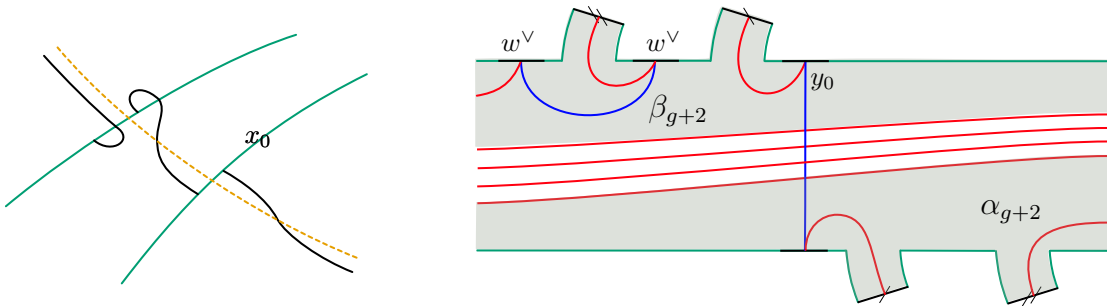

Figure 7.3. (Left) The attaching curve for the 2-handle and the associated trivial bypass arcs in ∂M; (Right) Attaching the second trivial bypass to obtain $\mathcal{H}_2$.

Since the diagram $\mathcal{H}_{AZ}$ for the twisting slice has no curves, $\iota_{\varnothing}^{\vee}$ is the empty set, and we have $(e, \iota^{\vee}, \mathbf{y}) = (e, \mathbf{y})$. Now, destabilizing yields the diagram for $(-M, -\Gamma)$ with a 1-handle attached to the Heegaard surface, exactly as in the case of the diagrammatic map. The map $(e, \mathbf{y}) \to \mathbf{y}$ is a graded homotopy equivalence, so that $\Psi_1[\mathbf{y}] = [\mathbf{y}] = [\sigma_1(\mathbf{y})]$.

Note that $\mathcal{H}' = \mathcal{H}_U \cup \mathcal{H}_{AZ} \cup \mathcal{H}_V$ is admissible, since its periodic domains correspond to the periodic domains of $\mathcal{H}$. $\qquad\square$

Lemma 7.0.2. *Given an admissible diagram $\mathcal{H}_1$ for $(-M, -\Gamma)$, there are admissible diagrams $\mathcal{H}_3$ for $(-M, -\Gamma)$, $\mathcal{H}_4$ for $(-M_2, -\Gamma_2)$, and $\mathcal{H}_6$ for $(-M_2, -\Gamma_2)$; and graded isomorphisms f, g making the following diagram commute.*

$$
\begin{array}{ccc}
\mathrm{SFH}(\mathcal{H}_1) & \xrightarrow{\;f\;} & \mathrm{SFH}(\mathcal{H}_3) \\
\Big\downarrow{\scriptstyle (\sigma_2)_*} & & \Big\downarrow{\scriptstyle \Psi_2} \\
\mathrm{SFH}(\mathcal{H}_6) & \xleftarrow[\;g\;]{} & \mathrm{SFH}(\mathcal{H}_4)
\end{array}
$$

Proof. We will show how to compute Ψ_2 from a given diagram for σ_2 by a sequence of diagrams $\{\mathcal{H}_i\}$. For the most part, the diagram $\mathcal{H}_{i+1}$ will differ from $\mathcal{H}_i$ by a number of Heegaard moves, so we have graded isomorphisms $\Psi_{\mathcal{H}_i, \mathcal{H}_{i+1}} : \mathrm{SFH}(\mathcal{H}_i) \to \mathrm{SFH}(\mathcal{H}_{i+1})$ induced by holomorphic triangle counts; f and g will be compositions of such maps. To ease discussion, we defer the computations of these triangle maps to Subsection 7.0.3.

Let $\mathcal{H}_1$ be an admissible Heegaard diagram for $(-M, -\Gamma)$ and use the diagram $\mathcal{H}_U \cup \mathcal{H}_W$ shown in Figure 6.5 for $-h^2$. The tuple (c, w) represents the unique generator $\mathrm{EH}(h^2)$ of $\mathrm{SFH}(-h^2)$, while w is the sole generator for $\widehat{\mathrm{BSA}}(\mathcal{H}_W)$. Referencing
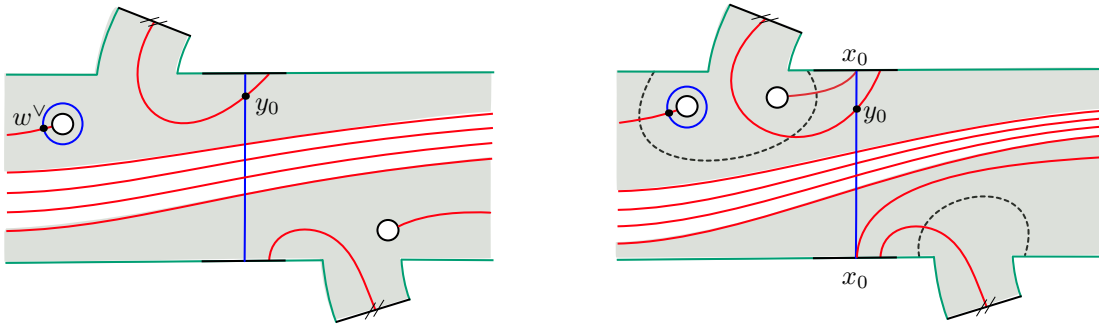
Figure 7.4. Left: $\mathcal{H}_2$; Right: $\mathcal{H}_3$

Figure 6.3, we have $\iota_2 \cdot w = w$. The dashed lines in Figure 7.2 are the curves α_0, β_0 for

the diagrammatic 2-handle attachment. Fix a cycle $\mathbf{y} \in \mathrm{SFC}(\mathcal{H}_1)$.

In order to compute the bordered contact gluing map, we must find a way to

realize $(-M, -\Gamma)$ as $-\overline{\mathcal{W}}_2 \cup_{-\mathcal{F}} ((-M, -\Gamma) \setminus \overline{\mathcal{W}}_2)$ at the level of Heegaard diagrams, where

$\mathcal{W}_2$ is the cap for the 2-handle. For this purpose, we peform two stabilizations. The first

is depicted on the right side of Figure 7.2 so that β_{g+1} agrees with β_0. The second is

depicted in Figure 7.3. This yields a diagram $\mathcal{H}_2$ where α_{g+2} agrees with α_0 outside a

neighborhood of β_0. (One can think of these stabilizations as attaching trivial bypasses

along arcs which approximate the attaching curve for the 2-handle; see Figure 7.3.)

The map $\Psi_{\mathcal{H}_1, \mathcal{H}_2}$ sending $[\mathbf{y}]$ to $[(w^\vee, y_0, \mathbf{y})]$ is induced by a well-defined isomorphism of

graded complexes. Handleslide β_{g+1} over β_{g+2} to obtain the diagram $\mathcal{H}_3$. The triangle

count $\Psi_{\mathcal{H}_2, \mathcal{H}_3}$ sends $[(w^\vee, y_0, \mathbf{y})]$ to $[(w^\vee, y_0, \mathbf{y})]$.

The right hand side of Figure 7.4 shows how to split $\mathcal{H}_3$ as $\mathcal{H}_3 = \mathcal{H}_{-W} \cup \mathcal{H}_V$.

Performing a bordered gluing with the 2-handle yields the diagram $\mathcal{H}_4 = \mathcal{H}_U \cup \mathcal{H}_{AZ} \cup$
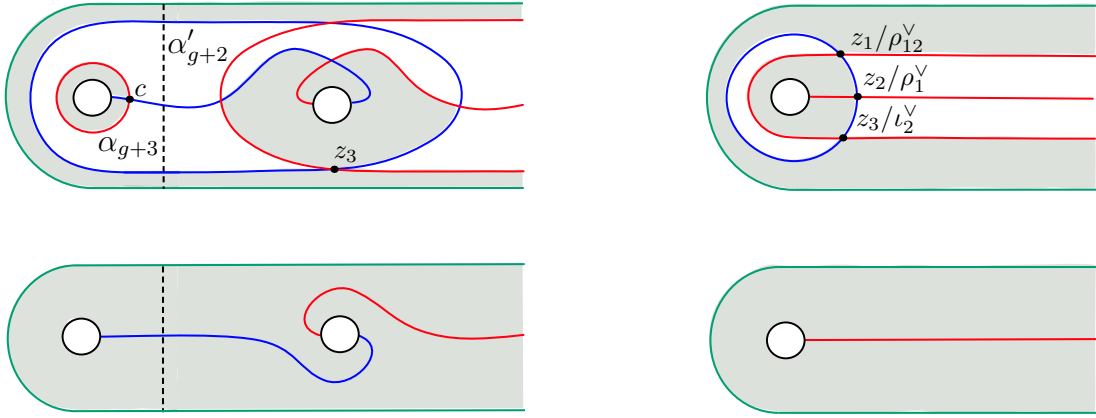
Figure 7.5. Destabilizing from $\mathcal{H}_4$ to $\mathcal{H}_5$; points on the right are additionally labelled by the algebra elements in $A(-\mathcal{Z}_2)^\vee$ they correspond to as in Figure 6.3.

$\mathcal{H}_V$. Note that z_3 corresponds to the basis idempotent ι_2 which acts non-trivially on w; compare with Figure 6.3. The associated bordered gluing map is induced by

$$\psi_{-\mathcal{F}}((c,w) \otimes (w^\vee, y_0, \mathbf{y})) = (c, z_3, y_0, \mathbf{y}).$$

After gluing, we destabilize to obtain $\mathcal{H}_5$ as depicted in Figure 7.6. The part of $\mathcal{H}_5$ which represents $-h^2 \cup \mathcal{TW}_{-\mathcal{F},+}$ before and after the destabilization are shown in Figure 7.5. This destabiliation is equivalent to performing two handleslides followed by a trivial destabilization. The triangle count $\Psi_{\mathcal{H}_4, \mathcal{H}_5}$ sends $[(c, z_3, y_0, \mathbf{y})]$ to $[(z_3, y_0, \mathbf{y})]$.

Since $\mathbf{y}$ is a cycle, we have $\partial(z_1, y_0, \mathbf{y}) = (z_3, y_0, \mathbf{y}) + (z_2, x_0, \mathbf{y})$; this can be seen from Figure 7.6, where the shaded regions correspond to the terms on the right. Thus, $[(z_3, y_0, \mathbf{y})] = [(z_2, x_0, \mathbf{y})]$.

Now, perform a small Hamiltonian isotopy to remove z_1 and z_3, handleslide β_{g+1} over β'_{g+2}, and perform a trivial destabilization to obtain $\mathcal{H}_6$; this is the diagram for a
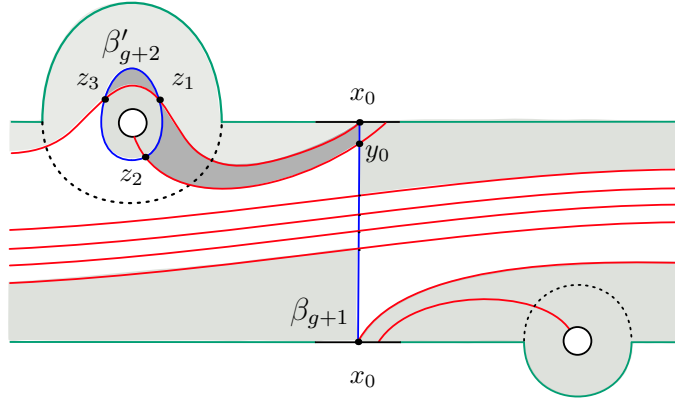
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Figure 7.6. $\mathcal{H}_5$; the regions shaded grey represent terms in $\partial(\mathbf{y}, y_0, z_1)$.
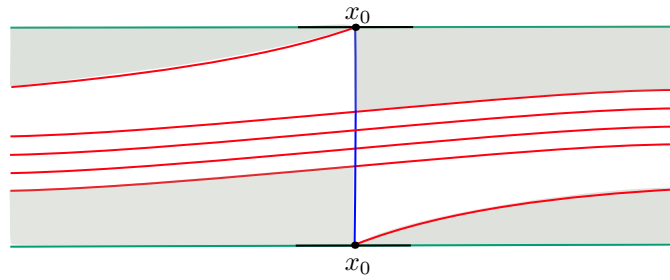

Figure 7.7. $\mathcal{H}_6$

diagrammatic 2-handle attachment on $\mathcal{H}_1$. The associated triangle count $\Psi_{\mathcal{H}_5,\mathcal{H}_6}$ sends $[(z_2, x_0, \mathbf{y})]$ to $[(x_0, \mathbf{y})]$.

By composing these maps, we see that the bordered contact gluing map from $\mathcal{H}_1$ to $\mathcal{H}_6$ is

$$[\mathbf{y}] \xrightarrow{\Psi_{\mathcal{H}_1,\mathcal{H}_2}} [(w^\vee, y_0, \mathbf{y})] \xrightarrow{\Psi_{\mathcal{H}_2,\mathcal{H}_3}} [(w^\vee, y_0, \mathbf{y}))] \xrightarrow{\Psi_2} [(c, z_3, y_0, \mathbf{y})]$$

$$\xrightarrow{\Psi_{\mathcal{H}_4,\mathcal{H}_5}} [(z_3, y_0, \mathbf{y})] = [(z_2, x_0, \mathbf{y})] \xrightarrow{\Psi_{\mathcal{H}_5,\mathcal{H}_6}} [(x_0, \mathbf{y})] = [(\mathbf{y}, x_0)],$$

which is the map on homology induced by σ_2.

Note that all the diagrams we have used are admissible by the following argument. In general, let $\alpha \subset \boldsymbol{\alpha}$ be a curve in a diagram which abuts a suture region on each side. Without loss of generality, suppose that α appears in the boundary of a periodic domain $\mathcal{D}$, and the multiplicity of $\mathcal{D}$ on each region to the left of $\boldsymbol{\alpha}$ is nonnegative. Since there is a basepoint region D on the left where $\mathcal{D}$ has multiplicity zero; let $a = \partial D \cap \alpha$. Let D' be the region with $\partial D' \cap \alpha = -a$. Then $\mathcal{D}$ must have negative multiplicity on D'.

Thus, if one is looking for a periodic domain with no negative coefficients, one can erase any curve (not just an α-curve) which abuts the basepoint region on both sides. In each of the diagrams $\mathcal{H}_2$ through $\mathcal{H}_6$, we can erase a number of such curves to obtain $\mathcal{H}_1$ with a number of strips attached to the boundary; the periodic domains of each diagram obtained this way are exactly the periodic domains of $\mathcal{H}_1$. Since $\mathcal{H}_1$ is admissible by hypothesis, so are $\mathcal{H}_2$ through $\mathcal{H}_6$. $\qquad\square$

7.0.2. Proof of Theorem 1

We are now ready to prove the precise formulation of the main theorem.

Theorem 7.0.3. *Suppose that Φ_ξ is the HKM map for a proper inclusion $(M, \Gamma) \subset (M', \Gamma')$ of sutured manifolds with no isolated components and compatible contact structure ξ. There is a sutured contact manifold (M'', Γ'', ξ''), a sutured surface $\mathcal{F} = (F, \Lambda)$, and graded isomorphisms f, g such that*

1. *$(M', \Gamma') \cong (M'', \Gamma'') \cup_\mathcal{F} (M, \Gamma)$*

2. *$(M' \setminus \mathrm{Int}(M), \xi)$ is contactomorphic to $(M'' \cup_F (\partial M \times [0,1]), \xi'' \cup_\mathcal{F} \xi_\Gamma)$, where ξ_Γ is the $[0,1]$-invariant contact structure compatible with Γ*

3. *the dividing set $\Gamma|_F$ is disk-decomposable as defined in Subsection 5.0.1*

4. *the following diagram commutes*

$$
\begin{array}{ccc}
\mathrm{SFH}(-M, -\Gamma) & \xrightarrow{\quad f \quad} & \mathrm{SFH}(-M, -\Gamma) \\
\downarrow{\scriptstyle \Phi_\xi} & & \downarrow{\scriptstyle \Psi_{\xi''}} \\
\mathrm{SFH}(-M', -\Gamma') & \xrightarrow{\quad g \quad} & \mathrm{SFH}((-M'', -\Gamma'') \cup_{-\mathcal{F}} (-M, -\Gamma))
\end{array}
$$

Proof of Theorem 1. Decompose ξ into a sequence of contact handles $h^{i_1}, \ldots, h^{i_n}$ attached to $\partial M \times \{1\}$ in $(\partial M \times [0,1], -\Gamma \sqcup \Gamma, \xi_\Gamma)$. We can factor the HKM map as $\Phi_\xi = \Phi_{i_n} \circ \ldots \circ \Phi_{i_1}$, where each Φ_{i_j} is the HKM-map for attaching a padded handle $(P_{i_j}, \Gamma_{i_j}, \xi_{i_j})$.

We can also decompose ξ as a sequence of punctured padded handles $(P_{i_1}^{L_1}, \Gamma_{i_1}^{L_1}, \xi_{i_1}^{L_1}), \ldots, (P_{i_n}^{L_n}, \Gamma_{i_n}^{L_n}, \xi_{i_n}^{L_n})$ attached to $(\partial M \times [0,1], -\Gamma \sqcup \Gamma, \xi_\Gamma)$ along a

surface $F \subset \partial M \times \{1\}$. Let (M'', Γ'', ξ'') be the structure obtained by attaching

$(P_{i_2}^{L_2}, \Gamma_{i_2}^{L_2}, \xi_{i_2}^{L_2}), \ldots, (P_{i_n}^{L_n}, \Gamma_{i_n}^{L_n}, \xi_{i_n}^{L_n})$ to $(P_{i_1}^{L_1}, \Gamma_{i_1}^{L_1}, \xi_{i_1}^{L_1})$.

By construction, $(M'', \Gamma'') \cup_{\mathcal{F}} (M, \Gamma)$ can be decomposed as a number of punctured

padded handle attachments to (M, Γ) with bordered contact gluing maps $\Psi_{i_1}, \ldots, \Psi_{i_n}$,

where each Ψ_{i_j} corresponds to $(P_{i_j}^{L_j}, \Gamma_{i_j}^{L_j}, \xi_{i_j}^{L_j})$. By Lemma 5.0.11, these are also the

maps for the h_{i_j} up to graded isomorphism. The bordered contact gluing map $\Psi_{\xi''}$

factors as $\Psi_{\xi''} = \Psi_{i_n} \circ \ldots \circ \Psi_{i_1}$ up to graded isomorphism by Lemma 5.0.13.

By Lemma 3.0.8, Lemma 7.0.1, and Lemma 7.0.2 the gluing maps Φ_{i_j} and Ψ_{i_j}

equal $(\sigma_{i_j})_*$ up to graded isomorphism for each j. By the composition law for the HKM

map and Lemma 5.0.13, $\Phi_\xi = \Psi_{\xi''}$ up to graded isomorphism.

$\square$

While we have couched Theorem 1 in terms of constructing a bordered contact

map from an HKM map, we can also go in the other direction.

Theorem 7.0.4. *Let* $(M'', \Gamma'') \cup_{\mathcal{F}} (M, \Gamma)$ *be a bordered gluing, where the dividing set* $\Gamma|_F$

on $\mathcal{F}$ *is disk-decomposable, and let* ξ'' *be a compatible contact structure on* M''. *There is*

a sutured manifold (M', Γ'), *a proper inclusion* $(M, \Gamma) \subset (M', \Gamma')$, *a compatible contact*

structure ξ *on* $M' \setminus \text{Int}(M)$, *and graded isomorphisms* f', g' *such that*

1. $(M', \Gamma') \cong (M'', \Gamma'') \cup_{\mathcal{F}} (M, \Gamma)$

2. $(M' \setminus \text{Int}(M), \xi)$ *is contactomorphic to* $(M'' \cup_{\mathcal{F}} (\partial M \times [0, 1]), \xi'' \cup_{\mathcal{F}} \xi_\Gamma)$, *where* ξ_Γ

   is the $[0, 1]$-*invariant contact structure compatible with* Γ

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3. the proper inclusion has no isolated components

4. the following diagram commutes

$$
\begin{array}{ccc}
\mathrm{SFH}(-M,-\Gamma) & \xrightarrow{\;\;f'\;\;} & \mathrm{SFH}(-M,-\Gamma) \\
\downarrow{\scriptstyle\Psi_{\xi''}} & & \downarrow{\scriptstyle\Phi_\xi} \\
\mathrm{SFH}((-M'',-\Gamma'')\cup_{-\mathcal{F}}(-M,-\Gamma)) & \xrightarrow{\;\;g'\;\;} & \mathrm{SFH}(-M',-\Gamma')
\end{array}
$$

Proof. Let $(M',\Gamma') = (M'',\Gamma'')\cup_{\mathcal{F}\times\{1\}}(M\cup_{\partial M\times\{0\}}(\partial M\times[0,1]),\Gamma)$ and let $\xi = \xi''\cup_{\mathcal{F}\times\{1\}}\xi_\Gamma$.

It is clear that the first two conditions are satisfied by construction.

Furthermore, any isolated component of the proper inclusion $(M,\Gamma) \subset (M',\Gamma')$ would correspond to a component of M'' whose boundary is entirely contained in F, or a closed component of M''. Since bordered gluings are only allowed along proper sutured subsurfaces, the former case is not allowed, while the latter is excluded by hypothesis.

Since ξ and ξ'' are constructed from I-invariant contact structures by attaching corresponding contact handles, we can argue as in the proof of Theorem 1 that Φ_ξ and $\Psi_{\xi''}$ are equal up to graded isomorphism. $\qquad\square$

7.0.3. Triangle counts

We now verify that the maps in the proof of Lemma 7.0.2 are graded isomorphisms. All these maps are induced by the standard holomorphic triangle counts associated to handleslides. For α-handleslides, triangle counts induce chain maps of the form $\mathrm{SFC}(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha}) \otimes \mathrm{SFC}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\alpha}') \to \mathrm{SFC}(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha}')$, while triangle counts induce maps of

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the form $\mathrm{SFC}(\Sigma, \boldsymbol{\beta}', \boldsymbol{\beta}) \otimes \mathrm{SFC}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}) \to \mathrm{SFC}(\Sigma, \boldsymbol{\beta}', \boldsymbol{\alpha})$ for β-handleslides. We color $\boldsymbol{\alpha}$-curves red and $\boldsymbol{\alpha}'$-curves orange; we color $\boldsymbol{\beta}$-curves blue and $\boldsymbol{\beta}'$-curves purple.

We will decorate generators in each $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}')$ or $(\Sigma, \boldsymbol{\beta}', \boldsymbol{\alpha})$ with primes and leave generators in each $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ undecorated. We relabel after each computation, i.e. the generator $(c', z_3', y_0', \mathbf{y}')$ in $\mathcal{H}_{4,4,5}$ is the generator $(c, z_3, y_0, \mathbf{y})$ in $\mathcal{H}_{4.5,5}$. We denote the highest degree element in each $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\alpha}')$ or $(\Sigma, \boldsymbol{\beta}', \boldsymbol{\beta})$ by Θ and the lowest degree element by η.

We will make use of the following standard fact in our computations. Let D and D' be regions in a Heegaard diagram with a common edge in an oriented α-curve α_0, and let $\mathcal{D}$ be the domain of a Whitney polygon with holomorphic representative. If α_0 abuts a basepoint region on the left and a basepoint region on the right, then the multiplicities of $\mathcal{D}$ on D and D' differ by at most 1. In particular, if the common edge is contained in $\partial \mathcal{D}$ and D is a basepoint region, then $\mathcal{D}$ has multiplicity one on D'.

Proposition 7.0.5. $\Psi_{\mathcal{H}_2, \mathcal{H}_3}[(w, y_0, \mathbf{y})] = [(w', y_0', \mathbf{y}')]$.

Proof. Consider Figure 7.8. Note that any triangle exiting y_0 must have multiplicity 1 on D and multiplicity 0 on D'. There is a unique such triangle, and it sends y_0 to y_0'. We can erase the curves which contain y_0 or y_0', and the new diagram is the diagram for a small Hamiltonian isotopy. The corresponding triangle map is chain homotopic to the identity, and thus sends $[(w, \mathbf{y})]$ to $[(w', \mathbf{y}')]$. We then have $\Psi_{\mathcal{H}_2, \mathcal{H}_3}[(w, y_0, \mathbf{y})] = [(w', y_0', \mathbf{y}')]$. $\square$
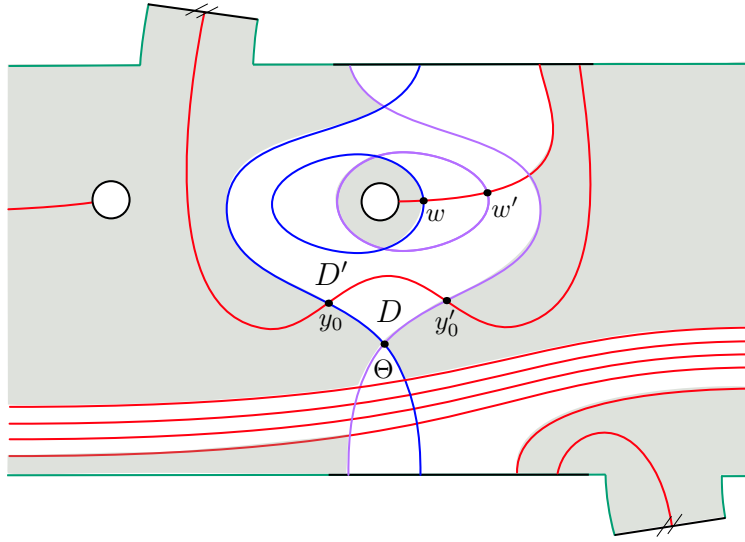
Figure 7.8. The diagram for $\Psi_{\mathcal{H}_2, \mathcal{H}_3}$.

Proposition 7.0.6. $\Psi_{\mathcal{H}_3, \mathcal{H}_4}[(c, z_3, y_0, \mathbf{y})] = [(z'_3, y'_0, \mathbf{y}')]$.

Proof. Let $\mathcal{H}_{4.5}$ be the diagram obtained by handlesliding α'_{g+2} over α_{g+3} and consider

the triple diagram in Figure 7.9 for this move. Any triangle for $\Psi_{\mathcal{H}_4, \mathcal{H}_{4.5}}$ exiting z_3 must

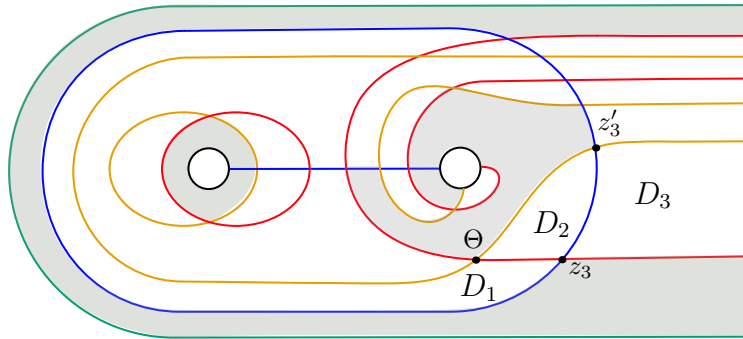have must have multiplicity 0 on D_1 and D_3 and multiplicity 1 on D_2; this follows

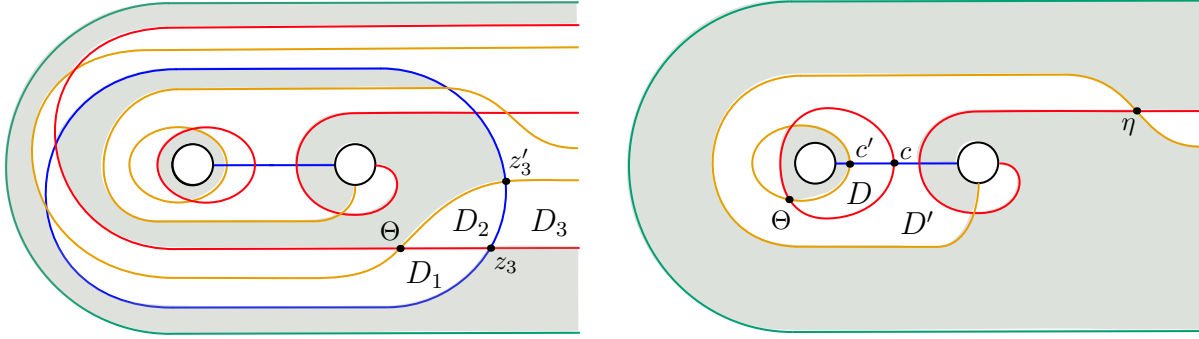Figure 7.9. The diagram for $\Psi_{\mathcal{H}_4, \mathcal{H}_{4.5}}$.

Figure 7.10. The diagrams for $\Psi_{\mathcal{H}_{4.5}, \mathcal{H}_5}$.

from the observation that the curve α'_{g+2} abuts a basepoint region on both the left and the right. (This cannot be seen in Figure 7.9, but it is true in the complete diagram.) There is a unique such triangle; it enters z'_3. Erasing curves yields a diagram for a small Hamiltonian isotopy, so that $\Psi_{\mathcal{H}_4, \mathcal{H}_{4.5}}[(c, z_3, y_0, \mathbf{y})] = [(c', z'_3, y'_0, \mathbf{y}')]$.

To compute $\Psi_{\mathcal{H}_{4.5}, \mathcal{H}_5}$, consider the diagram on the left of Figure 7.10. Any triangle exiting z_3 must have multiplicity 0 on D_1 and D_3 and multiplicity 1 on D_2. There is a unique such triangle; it enters z'_3. Erasing curves yields the diagram on the right of Figure 7.10. Since no triangle in our count can enter η, all triangles must have multiplicity 0 on D'. There is a unique such triangle exiting c; its domain is the region D, and it enters c'. Erasing curves yields a diagram for a small Hamiltonian isotopy, and the corresponding triangle map sends $[(c, z_3, y_0, \mathbf{y})]$ to $[(c', z'_3, y'_0, \mathbf{y}')]$. We now obtain $\mathcal{H}_5$ by performing a trivial destabilization. The associated map sends $[(c, z_3, y_0, \mathbf{y})]$ to $[(z'_3, y'_0, \mathbf{y}')]$ so that $\Psi_{\mathcal{H}_{4.5}, \mathcal{H}_5}[(c, z_3, y_0, \mathbf{y})] = [(z'_3, y'_0, \mathbf{y}')]$. $\qquad\square$

Proposition 7.0.7. $\Psi_{\mathcal{H}_5, \mathcal{H}_6}[(z_2, x_0, \mathbf{y})] = [(x'_0, \mathbf{y}')]$.
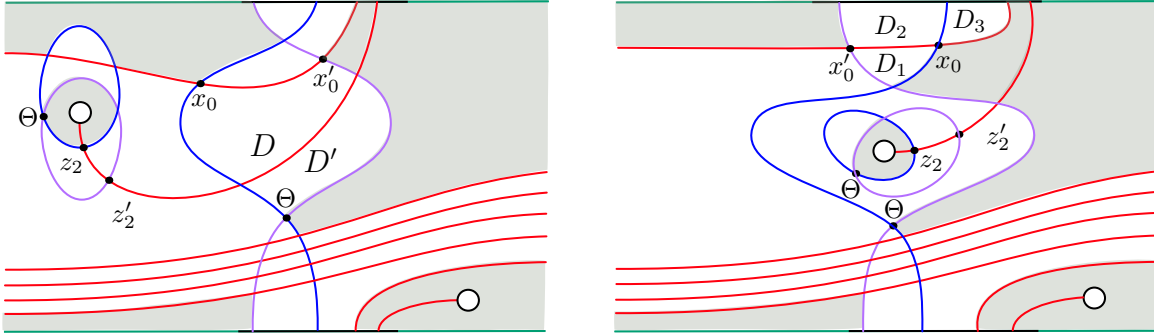
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Figure 7.11. $\Psi_{\mathcal{H}_5, \mathcal{H}_{5.5}}$ and $\Psi_{\mathcal{H}_{5.5}, \mathcal{H}_6}$.

Proof. Let $\mathcal{H}_{5.5}$ be the diagram obtained by performing the obvious finger move on $\mathcal{H}_5$; diagrams for $\Psi_{\mathcal{H}_5, \mathcal{H}_{5.5}}$ and $\Psi_{\mathcal{H}_{5.5}, \mathcal{H}_6}$ are shown in Figure 7.11. On the left, any triangle exiting x_0 must have multiplicity one on D and D'. There is a unique such triangle; it enters x_0'. On the right, any triangle exiting x_0 must have multiplicity one on D_2 and multiplicity zero on D_1 and D_3. There is a unique such triangle; it enters x_0'.

In each diagram in Figure 7.11, erasing curves yields a diagram for a small Hamiltonian isotopy. Each corresponding triangle map sends $[(z_2, x_0, \mathbf{y})]$ to $[(z_2', x_0', \mathbf{y}')]$. We then obtain $\mathcal{H}_6$ by performing a trivial destabilization; the corresponding map sends $[(z_2, x_0, \mathbf{y})]$ to $[(x_0', \mathbf{y}')]$. $\qquad\square$

Note that the diagrams used in Proposition 7.0.5, Proposition 7.0.6, and Proposition 7.0.7 are all admissible, since the admissibility argument at the end of Lemma 7.0.2 holds for triply-periodic domains in triple diagrams. Each of the diagrams can be reduced to a diagram for a small Hamiltonian isotopy by erasing curves which abut su-

ture regions on both sides. Since the diagram for a small Hamiltonian isotopy in an admissible diagram is itself admissible, so is each diagram in Proposition 7.0.5, Proposition 7.0.6, and Proposition 7.0.7.

Chapter 8. Applications

Our first corollary is the extension of contact gluing maps to the bordered sutured category, Corollary 1.0.3 from chapter 1. This corollary is an essential ingredient in [EVZ17].

Corollary 8.0.1. *Let $\mathcal{M} = (M, \gamma, \mathcal{F}, \mathcal{Z})$ and $\mathcal{M}' = (M', \gamma', \mathcal{F}, \mathcal{Z})$ be bordered sutured manifolds with $M \subset M'$ and $M' \setminus int(M)$ a sutured manifold equipped with a compatible contact structure ξ and no isolated components. Let $\mathcal{H}_M$ and $\mathcal{H}_{M'}$ be admissible diagrams for $-\mathcal{M}$ and $-\mathcal{M}'$, respectively. Then there exists a map of type-D structures induced by ξ*

$$\phi_\xi : \widehat{\mathrm{BSD}}(\mathcal{H}_M) \to \widehat{\mathrm{BSD}}(\mathcal{H}_{M'}),$$

satisfying the following property. If $\mathcal{N} = (N, \gamma_N, -\mathcal{F}, -\mathcal{Z})$ is a bordered sutured manifold with diagram $\mathcal{H}_N$ for $-\mathcal{N}$, then the map

$$\phi_\xi \boxtimes \mathrm{Id}_{\widehat{\mathrm{BSA}}(\mathcal{H}_N)} : \widehat{\mathrm{BSA}}(\mathcal{H}_\mathcal{M}) \boxtimes \widehat{\mathrm{BSD}}(\mathcal{H}_\mathcal{N}) \to \widehat{\mathrm{BSA}}(\mathcal{H}_{\mathcal{M}'}) \boxtimes \widehat{\mathrm{BSD}}(\mathcal{H}_\mathcal{N})$$

induces the contact gluing map $\Phi_\xi : \mathrm{SFH}(-\mathcal{M} \cup_{-\mathcal{F}} -\mathcal{N}) \to \mathrm{SFH}(-\mathcal{M}' \cup_{-\mathcal{F}} -\mathcal{N})$ up to graded homotopy equivalence.

Similar statements hold for the type-A and bimodule structures found in the bordered sutured theory.

Proof. Bordered gluing operations and the bordered gluing map extend to the bordered

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sutured category. Using the notation of Subsection 6.0.2, a bordered gluing map

$$\Psi_{\mathcal{F}} : \mathrm{SFH}(M_1, \Gamma_1) \otimes \mathrm{SFH}(M_2, \Gamma_2) \to \mathrm{SFH}((M_1, \Gamma_1) \cup_{\mathcal{F}} (M_2, \Gamma_2))$$

is induced by a map of the form

$$\mathrm{Id}_U \boxtimes \nabla_W \boxtimes \mathrm{Id}_V : U \boxtimes W \otimes W^\vee \boxtimes V \to U \boxtimes A^\vee \boxtimes V.$$

Zarev [Zar11] proves that $\Psi_{\mathcal{F}}$ is well-defined by showing that the algebraic join $\nabla_W : W \otimes W^\vee \to A^\vee$ is well-defined up to $\mathcal{A}_\infty$-homotopy, and using the fact that $(\boxtimes V, \boxtimes \mathrm{Id}_V)$ is a dg-functor from the category of $\mathcal{A}_\infty$-modules to the category of chain complexes; see [LOT15] for explanation.

Note that the bordered gluing operation $\cup_{\mathcal{F}}$ as discussed in Subsection 4.0.2 extends to partially sutured and bordered sutured manifolds. The role of one or both (M_i, Γ_i) can be replaced by $\mathcal{M}_i = (M_i, \gamma_i, \mathcal{F}_i, \mathcal{Z})$ with $(\pm F, \gamma) \subset (\partial M_i) \setminus F_i$. As in Theorem 1, there is a sutured manifold (M'', Γ'') such that $(M'', \Gamma'') \cup_{\mathcal{F}'} \mathcal{M} \cong \mathcal{M}'$, where the dividing set on $\mathcal{F}'$ is disk-decomposible. Furthermore, we can choose $\mathcal{F}'$ such that the dividing set has no closed components, so that we can choose a parametrization of $\mathcal{F}'$ by an arc diagram $\mathcal{Z}'$ so that the dividing set is elementary with respect to $\mathcal{Z}'$. Let A' be the associated bordered algebra. Let $\mathcal{H}_{W'}$ be a diagram for the cap $\mathcal{W}'$ for the dividing set on $\mathcal{F}'$; let $\mathcal{H}_{U'}$ be a diagram for $(-M'', -\Gamma'') \setminus -\mathcal{W}'$; and let $\mathcal{H}_{V'}$ be a diagram for $(-\mathcal{M} \setminus -\overline{\mathcal{W}'}$. Let U', V', and W' be the bordered modules $\widehat{\mathrm{BSD}}(\mathcal{H}_{U'}), \widehat{\mathrm{BSDD}}(\mathcal{H}_{V'})$, and $\widehat{\mathrm{BSA}}(-\mathcal{H}_{W'})$ respectively. Note that W' has a single generator w, since the dividing set

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on $\mathcal{F}'$ is elementary by Proposition 15.1.2 in [Zar11]. We can define a map

$$\psi_{-\mathcal{F}'} = \operatorname{Id}_{U'} \boxtimes \nabla_{W'} \boxtimes \operatorname{Id}_{V'} : U' \boxtimes W' \otimes (W')^{\vee} \boxtimes V' \to U' \boxtimes (A')^{\vee} \boxtimes V'.$$

which is well-defined up to graded homotopy equivalence of Type-D structures, since by Lemma 2.3.13 of [LOT15], $\cdot \boxtimes \operatorname{Id}_{V'}$ is an $\mathcal{A}_{\infty}$-functor.

Similar to Theorem 1, note that there is a compatible contact structure ξ'' on M'' such that ξ is contactomorphic to ξ'' attached to an I-invariant contact structure. Since $U' \boxtimes W'$ is homotopy equivalent to $\operatorname{SFC}(-M'', -\Gamma'')$, we can define $\phi_{\xi} : (W')^{\vee} \boxtimes V \to U' \boxtimes (A')^{\vee} \boxtimes V'$ by

$$\phi_{\xi}(w^{\vee} \boxtimes \mathbf{y}) = \psi_{-\mathcal{F}'}(\mathbf{x} \boxtimes w, w^{\vee} \boxtimes \mathbf{y}),$$

where $\mathbf{x} \boxtimes w$ represents $\operatorname{EH}(\xi'')$.

Denote $\widehat{\operatorname{BSD}}(\mathcal{H}_N)$ by X. Since $(W')^{\vee} \boxtimes V' \boxtimes X \simeq \operatorname{SFC}(\mathcal{H}_M \cup_{\mathcal{F}} \mathcal{H}_N)$ and $U' \boxtimes (A')^{\vee} \boxtimes V' \boxtimes X \simeq \operatorname{SFC}(\mathcal{H}_{M'} \cup_{\mathcal{F}} \mathcal{H}_N)$, the bordered contact gluing map $\Psi_{\xi''}$ is induced by evaluating

$$\operatorname{Id}_{U'} \boxtimes \nabla_{W'} \boxtimes \operatorname{Id}_{V' \boxtimes X}$$

on $\mathbf{x} \boxtimes w$, while $\phi_{\xi} \boxtimes \operatorname{Id}_X$ is the evaluation of

$$\operatorname{Id}_{U'} \boxtimes \nabla_{W'} \boxtimes \operatorname{Id}_{V'} \boxtimes \operatorname{Id}_X$$

on $\mathbf{x} \boxtimes w$. It is clear that these maps are equal up to homotopy equivalence since $\operatorname{Id}_{V'} \boxtimes \operatorname{Id}_X$ is homotopic to $\operatorname{Id}_{V' \boxtimes X}$. (This is from Lemma 2.3.13 in [LOT15].)

Note that by Theorem 1, $\Psi_{\xi''}$ agrees with the contact gluing map Φ_ξ up to graded isomorphism. Also, since $\psi_{-\mathcal{F}'}$ is defined up to graded homotopy equivalence, so is ϕ_ξ, so we can define $\phi_\xi : \widehat{\mathrm{BSD}}(\mathcal{H}_M) \to \widehat{\mathrm{BSD}}(\mathcal{H}_{M'})$ by using homotopy equivalences $W' \boxtimes V' \simeq \widehat{\mathrm{BSD}}(\mathcal{H}_M)$ and $U' \boxtimes (A')^\vee \boxtimes V' \simeq \widehat{\mathrm{BSD}}(\mathcal{H}_{M'})$.

This completes the proof for type-D structures. Similar arguments hold for type-A modules and bimodules. $\qquad\square$

We also obtain an independent proof of Juhász and Zemke's result that diagrammatic maps agree with the corresponding contact gluing maps; see [JZ20]. This is Corollary 1.0.4 in chapter 1.

Corollary 8.0.2. *Given a diagram $\mathcal{H}$ for $(-M, -\Gamma)$, there is a diagram for $(-M_i, -\Gamma_i)$ such that the HKM map $\Phi_i : \mathrm{SFH}(-M, -\Gamma) \to \mathrm{SFH}(-M_i, -\Gamma_i)$ is induced by the diagrammatic map σ_i up to graded isomorphism. Furthermore, given any other diagrammatic attachment for $(-M_i, -\Gamma_i)$, the associated diagrammatic map σ'_i also induces Φ_i up to graded isomorphism.*

Proof. The first statement is immediate from Lemma 3.0.8. By Lemma 7.0.1 and Lemma 7.0.2, both $(\sigma_i)_*$ and $(\sigma'_i)_*$ are equal to bordered contact gluing maps Ψ_i and Ψ'_i respectively, up to graded isomorphism. Since the map $\Psi_{-\mathcal{F}}$ is well-defined, Ψ_i and Ψ'_i are equal up to graded isomorphism. Then $(\sigma_i)_*$ and $(\sigma'_i)_*$ are also equal up to graded isomorphism. $\qquad\square$

As a final application, we use Corollary 8.0.2 to show that the HKM map can be computed using nice diagrams by extending Plamenevskaya's application of the Sarkar–Wang algorithm in [Pla07]. This is Corollary 1.0.5 in chapter 1.

Corollary 8.0.3. *Suppose that* $(-M, -\Gamma) \subset (-M', -\Gamma')$ *is a proper inclusion of sutured manifolds with no isolated components and contact gluing map* Φ_ξ. *Given an admissible diagram* $\mathcal{H}_0$ *for* $(-M, -\Gamma)$, *there is a diagram* $\mathcal{H}_2$ *for* $(-M', -\Gamma')$, *a nice diagram* $\mathcal{H}^{nice}$ *for* $(-M, -\Gamma)$, *a nice diagram* $\mathcal{H}_2^{nice}$ *for* $(-M', -\Gamma')'$, *a map* $\phi_\xi^{nice} : \mathrm{SFC}(\mathcal{H}^{nice}) \to \mathrm{SFC}(\mathcal{H}_2^{nice})$ *of the form* $\mathbf{y} \to (\mathbf{y}, \mathbf{x}_0)$, *and graded isomorphisms* f, g *such that the following diagram commutes:*

$$
\begin{array}{ccc}
\mathrm{SFH}(\mathcal{H}_0) & \xrightarrow{\ f\ } & \mathrm{SFH}(\mathcal{H}^{nice}) \\
\Big\downarrow{\scriptstyle \Phi_\xi} & & \Big\downarrow{\scriptstyle \phi_\xi^{nice}} \\
\mathrm{SFH}(\mathcal{H}_2) & \xrightarrow{\ g\ } & \mathrm{SFH}(\mathcal{H}_2^{nice})
\end{array}
$$

Some discussion is in order before we begin the proof. By Corollary 1.0.4, the HKM map can be factored into diagrammatic maps $\Phi_\xi = (\sigma_{i_n} \circ \ldots \sigma_{i_1})_*$ up to graded isomorphism. Let $\mathcal{H}$ be a diagram for $(-M, -\Gamma)$ and $\mathcal{H}'$ a diagram for $(-M', -\Gamma')$ obtained by diagrammatic handle attachments. The composition $\sigma_{i_n} \circ \ldots \circ \sigma_{i_1}$ takes the form $\phi_\xi(\mathbf{y}) = (\mathbf{y}, \mathbf{x}_0)$, where $\mathbf{x}_0$ is the collection of preferred intersections of the 2-handles. We wish to show that this situation can be realized by nice diagrams and that the preferred intersections in the nice diagrams correspond to $\mathbf{x}_0$ up to graded homotopy equivalence.

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Note that we cannot just perform diagrammatic handle attachments on a given nice diagram for (M, Γ), since a diagrammatic contact 2-handle attachment on a nice diagram need not result in a nice diagram. We will instead find a nice diagram for (M', Γ') and work backwards to obtain a suitable diagram for (M, Γ).

As in [Pla07], the difficulty is showing that the Sarkar–Wang algorithm may be applied in a way that preserves $\mathbf{x}_0$. More precisely, we wish to show there is a diagram which is commutative up to homotopy,

$$
\begin{array}{ccc}
\mathrm{SFC}(\mathcal{H}) & \xrightarrow{\psi} & \mathrm{SFC}(\mathcal{H}^{nice}) \\
\downarrow{\scriptstyle \phi_\xi} & & \downarrow{\scriptstyle \mathbf{z} \to (\mathbf{z}, \mathbf{x}_0^{nice})} \\
\mathrm{SFC}(\mathcal{H}') & \xrightarrow{\psi'} & \mathrm{SFC}((\mathcal{H}^{nice})')
\end{array}
$$

where $\mathbf{x}_0^{nice}$ is the preferred intersection in $(\mathcal{H}^{nice})'$ and ψ, ψ' are triangle maps induced by Heegaard moves in the Sarkar–Wang algorithm. In [Pla07], they key idea is to show that one can apply Sarkar–Wang without performing a finger move around a full β-curve. This implies that no handleslide maps arise, and the preferred intersections are preserved by the maps which arise in the course of the algorithm.

In our setting, we cannot always rule out finger moves around full β-curves in general, but we can rule out finger moves around β-curves coming from contact 2-handle attachments. Recall that attaching a contact 2-handle adds a strip to the Heegaard surface. If a finger move which enters this strip does not cross the corresponding
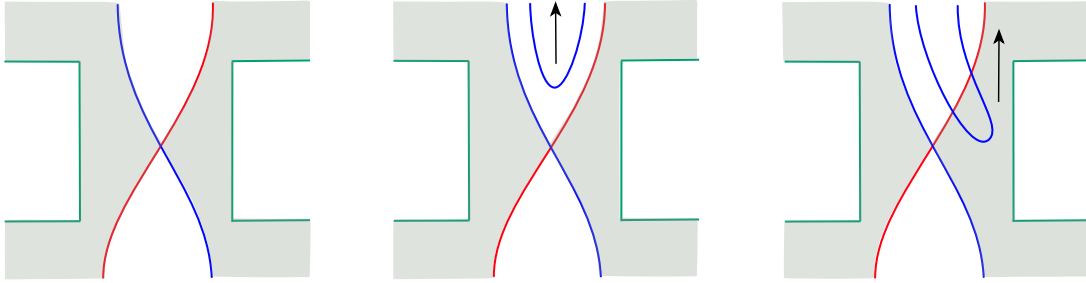
Figure 8.1. Finger moves cannot pass through a strip coming from a 2-handle attachment.
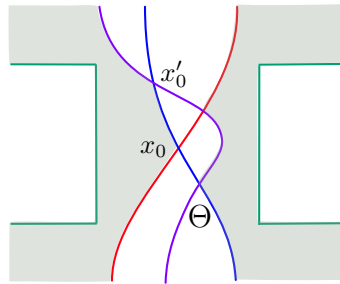

Figure 8.2. A strip coming from a 2-handle attachment in a triple diagram.

α-curve, then it can be isotoped outside the strip. If it crosses the α-curve, it must terminate there, since the α-curve borders a basepoint region on one side. In this case, we can also isotope the finger move outside the strip; see Figure 8.1. This means that we can apply Sarkar–Wang without handlesliding over any of the new β-curves.

This is sufficient for our purposes, since if we perform a handleslide over a β-curve coming from $\mathcal{H}$ or a finger move, then the diagram for the triangle count in the strip looks like Figure 8.2. It is easy to see that these maps all preserve the preferred intersections.

Proof. Let $\mathcal{H}_0 = (\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ be a diagram for $(-M, -\Gamma)$ and apply the Sarkar–Wang algorithm to obtain a nice diagram $\mathcal{H}_0^{nice}$. Decompose $(-M', -\Gamma')$ as $(-M, -\Gamma) \cup_j -h_j^1 \cup_k -h_k^2$. We attach diagrammatic 1-handles to form a diagram $\mathcal{H}_1^{nice}$ for $(-M, -\Gamma) \cup_j -h_j^1$ which is nice, since we have only modified the sutured region.

Now, attach diagrammatic 2-handles to $\mathcal{H}_1^{nice}$ to obtain a diagram $\mathcal{H}_2 = (\Sigma', \boldsymbol{\beta} \cup \boldsymbol{\beta}_0, \boldsymbol{\alpha} \cup \boldsymbol{\alpha}_0)$ for $(-M', -\Gamma')$. Apply Sarkar–Wang again to get a nice diagram $\mathcal{H}_2^{nice} = (\Sigma', \boldsymbol{\beta}' \cup \boldsymbol{\beta}_0', \boldsymbol{\alpha} \cup \boldsymbol{\alpha}_0)$. As discussed above, we can do this so that all finger moves are performed in the complement of the strips for the contact handles, $\Sigma' \setminus \mathrm{Int}(\Sigma)$. Thus, the triangle map $\psi_{\mathcal{H}_2, \mathcal{H}_2^{nice}}$ sends generators of the form $(\mathbf{y}, \mathbf{x}_0)$ to generators of the form $(\mathbf{z}, \mathbf{x}_0^{nice})$.

Now, let $\mathcal{H}^{nice} = (\Sigma, \boldsymbol{\beta}', \boldsymbol{\alpha})$ be the Heegaard diagram for $(-M, -\Gamma)$ obtained by removing all the diagrammatic handles from $\mathcal{H}_2^{nice}$. Note that it is nice, since it differs from $\mathcal{H}_0^{nice}$ by some finger moves which appear in the Sarkar-Wang algorithm. Consider the triple diagram $(\Sigma', \boldsymbol{\beta} \cup \boldsymbol{\beta}_0, \boldsymbol{\beta}' \cup \boldsymbol{\beta}_0', \boldsymbol{\alpha} \cup \boldsymbol{\alpha}_0)$ for $\Psi_{\mathcal{H}_2, \mathcal{H}_2^{nice}}$; let $\mathbf{x}_0$ and $\mathbf{x}_0^{nice}$ be the preferred intersections in $\boldsymbol{\beta}_0 \cap \boldsymbol{\alpha}_0$ and $\boldsymbol{\beta}_0' \cap \boldsymbol{\alpha}_0$ respectively. Since we only care about the image of the contact gluing map, we can restrict the triangle count for $\Psi_{\mathcal{H}_2, \mathcal{H}_2^{nice}}$ to the subspace of $\mathrm{SFC}(\mathcal{H}_2)$ generated by tuples of the form $(\mathbf{y}, \mathbf{x}_0)$. This restricted count is the same as the full triangle count for $\Psi_{\mathcal{H}_0, \mathcal{H}^{nice}}$ arising from $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}', \boldsymbol{\alpha})$, so that for a cycle $\mathbf{y}$ in $\mathrm{SFC}(\mathcal{H}_0)$, we have $\psi_{\mathcal{H}_2, \mathcal{H}_2^{nice}}(\mathbf{y}, \mathbf{x}_0) = (\psi_{\mathcal{H}_0, \mathcal{H}^{nice}}(\mathbf{y}), \mathbf{x}_0^{nice})$. We can define a

map $\phi_\xi^{nice} : \mathrm{SFC}(\mathcal{H}^{nice}) \to \mathrm{SFC}(\mathcal{H}_2^{nice})$ by

$$\phi_\xi^{nice}(\psi_{\mathcal{H}_0,\mathcal{H}^{nice}}(\mathbf{y})) = (\psi_{\mathcal{H}_0,\mathcal{H}^{nice}}(\mathbf{y}), \mathbf{x}_0^{nice}),$$

so that for any cycle $\mathbf{z} \in \mathrm{SFC}(\mathcal{H}^{nice})$ we have

$$(\phi_\xi^{nice})_*[\mathbf{z}] = [(\mathbf{z}, \mathbf{x}_0^{nice})].$$

By construction, $(\phi_\xi^{nice})_*$ has the desired form and is equal to the HKM map Φ_ξ up to graded isomorphism. $\qquad\square$

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Vita

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