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Louisiana State University and Agricultural & Mechanical College

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Three-dimensional large-scale coherent structures in a developing plane mixing layer

Seo, Taewon, Ph.D.
The Louisiana State University and Agricultural and Mechanical Col., 1993
THREE-DIMENSIONAL LARGE-SCALE COHERENT STRUCTURES IN A DEVELOPING PLANE MIXING LAYER

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy in The Department of Mechanical Engineering

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List of Symbols

\( A_{mn} \) : Complex amplitude of wave mode \( mn \)

\( C_{1,2,3} \) : The complex constants

\( c \) : The phase velocity

\( cc \) : The complex conjugate

\( D_{1,2,3} \) : The complex constants

\( E_{mn} \) : Energy density of wave mode \( mn \)

\( EW\!W_{mn} \) : The nonlinear wave mode-mode interaction

\( I_{aM} \) : The mean flow energy advection integral coefficient

\( I_{aWmn} \) : The wave-mode energy advection integral coefficient

\( I_{pWmn} \) : The wave-mode pressure work integral coefficient

\( I_{MWmn} \) : The wave-mode production integral coefficient

\( I_{vWmn} \) : The wave-mode viscous dissipation integral coefficient

\( I_{pq}^{ijkl} \) : The mode-mode energy exchange integral coefficient

\( L \) : The spanwise wave-length

\( m \) : The frequency mode

\( n \) : The mode dimensionality
\( P \): The pressure

\( P_{WMmn} \): The phase shift interacting with mean flow

\( P_{pMmn} \): The phase shift induced by the pressure field

\( P_{ijkl}^{pq} \): The phase shift induced by interaction between wave modes

\( PW_{Wmn} \): The nonlinear wave mode-mode phase interaction

\( \bar{Q} \): The mean flow component

\( \tilde{q} \): The large-scale deterministic component

\( \hat{q} \): The eigenfunctions

\( q' \): The small-scale turbulence

\( R \): The velocity ratio

\( Re_0 \): The initial Reynolds number

\( S_{ij}^{pq} \Delta_{kl} \): The modal stress of ij and pq acting on a rate of strain of kl

\( t \): Time

\( U_{\pm\infty} \): Free stream velocity at \( \pm\infty \)

\( \overline{U} \): The average mean velocity

\( u_{ij} \): The velocity component of large-scale structures

\( u' \): The complex amplitude function

\( V \): The dependent variable vector

\( \mathbf{V}^0 \): The asymptotic solutions

\( \bar{v}_{ij} \): The velocity component of large-scale structures

\( v' \): The complex amplitude function
\( \vec{w}_{ij} \) : The velocity component of large-scale structures

\( w_l \) : The complex amplitude function

\( x \) : The streamwise coordinate

\( y \) : The cross-stream coordinate

\( z \) : The spanwise coordinate

\( \alpha \) : Complex eigenvalue

\( \beta \) : Frequency parameter

\( \delta \) : Maximum slope thickness

\( \phi_{mn} \) : Spanwise phase angle of wave mode mn

\( \gamma \) : Spanwise wave number

\( \nu \) : The kinematic viscosity

\( \theta_{ijkl}^{pq} \) : Argument of mode-mode interaction

\( \rho \) : Fluid density

\( \sum_{i,j}^{pq} \Delta_{kl} \) : The mode-mode interaction integral

\( \Psi_{ijkl}^{pq} \) : The total phase angle difference

\( \psi_{mn} \) : The phase angle of wave mode mn
Abstract

A theoretical investigation of the role of three-dimensional large-scale structures and their mutual interactions in a developing, plane mixing layer subjected to external forcing is presented. Large-scale structures are decomposed into 3 fundamental and 2 subharmonic wave modes. Two fundamentals and one subharmonic are three-dimensional. Linear stability of the three-dimensional, viscous shear layer is formulated and solved as the basis for the solution of a nonlinear formulation based on an energy method. This method leads to a set of nonlinearly-coupled, ordinary differential equations governing the evolution of modal energies and phases, and of the shear layer thickness.

A parametric study is carried out examining effects of a multitude of initial conditions. It is found that the evolution of the forced three-dimensional shear layer and the associated local entrainment can be influenced greatly by the initial amplitudes and phases of the large-scale modes. The presence of three-dimensional modes may have a profound effect on shear layer growth when forced at amplitudes comparable or larger than those of the two-dimensional ones. This effect is more pronounced at low spanwise wave numbers. Nonlinear interactions between the fundamentals
and subharmonics indicate that subharmonics of the most amplified frequency of the shear layer are usually produced during the early stages of flow development, while its harmonics are always produced far downstream, regardless of initial conditions. Experimental observations regarding the appearance of three-dimensional large-scale structures, spanwise wave-length doubling and three-dimensional vortex merging phenomena have been given qualitative interpretation under the light of the theoretical results.

Measurements of initial conditions and individual two- and three-dimensional mode energies are presently not available. Therefore, quantitative comparisons of our results with experiments have not been possible. However, this study provides guideline for future experiments that can clarify the influence of initial conditions and three-dimensional structures on the flow evolution. The results of this study also provide useful parametric information for the control, through multi-mode forcing, of shear layers in practical applications, aiming at mixing and transport augmentation.
Chapter 1

Introduction

This research considers spatially developing, free shear layers at moderate Reynolds numbers, that are formed by the merging of two free streams initially separated by a splitter plate. The schematic diagram of the flow for a developing shear layer is sketched in Figure 1.1. Intensive mixing occurs in the velocity gradient region between two free streams and such layers are often referred to as mixing layers. Shear layers are of practical importance in many fields where rapid transition to turbulence is desirable in order to prevent boundary layer separation or to promote rapid mixing. This flow can be found in combustion chambers, ribbed heat exchanger ducts, and chemical flow reactors. Shear layers also generate most of the broadband noise in propulsion systems and understanding of their development aids in aerodynamic noise control. The ability to control mixing, structure and growth of these shear flows can have a considerable impact on many engineering applications. In addition to practical applications, free shear layers are rich in fundamental mechanisms believed to be important in the transition process to turbulence.
Figure 1.1: Schematic diagram of a developing mixing layer
During the past two decades, free shear layers have attracted increasing attention after the discovery of the large-scale coherent structures which are the prominent features of such flows during their early development and transition to fully turbulent flow. Winant & Browand (1974) observed the repeated formation and pairing of vortices in the spatially developing mixing layer at moderate Reynolds numbers. They also found that the growth of a mixing layer is controlled by the pairing mechanism of these vortical structures. Brown & Roshko (1974) demonstrated that the large-scale coherent structures are intrinsic features of a turbulent mixing layer within a wide range of Reynolds number. It has been observed that the large-scale coherent structures are strongly related to mixing, entrainment and noise production (Liu 1974; Fiedler 1991), and that these structures play an important role in the laminar-turbulent transition process (Freymuth 1966; Miksad 1972).

Townsend (1956) first discussed the ideas of the large-scale coherent structures. According to Townsend's (1956) Large Eddies Hypothesis, the flow is composed of a mean motion and fluctuations consisting of the large-eddy motion and the main small-scale turbulent motion. The flow-visualization studies of Kline et al (1967) revealed the presence of the large-scale coherent structures of turbulent boundary layers. In a plane mixing layer, large-scale spanwise coherent vortical structures have been observed forming at a wide variety of Reynolds numbers (Winant & Browand 1974). It is now evident that they are an essential feature in the early stages of development of the turbulent mixing layer. Kovasznay, Kibens & Blackwelder (1970)
developed the concept of conditional sampling, which can distinguish and measure flow quantities with different characteristic time scales, to sort out the large-scale coherent structure from the mean flow or to separate large eddies from the surrounding undisturbed flow. The large-scale coherent structures in turbulent shear flows have been studied extensively by using well-controlled forcing in experiments allowing fluctuations measured at a point to be split into coherent structures and incoherent turbulence by applying conditional averaging (Kovasznay, Kibens & Blackwelder 1970). Hussain & Reynolds (1970) conducted such experiments on well-controlled coherent oscillations in turbulent channel flow, and Reynolds & Hussain (1972) used conditional averaging technique to obtain large-eddy features. Crow & Champagne (1971), Browand & Laufer (1975) and Strange & Crighton (1983) did experiments in a well-controlled manner for the round jet, Gaster et al (1985) and Ho & Huang (1982) for the free shear layer.

Apart from experiments, several analytical and numerical methods are utilized to understand and detail the behavior of large-scale structures and their interactions. The direct numerical simulation of large-scale coherent structures has shown promising results with the development of fast numerical schemes. Riley & Metcalfe (1980) used a pseudo-spectral method to solve the Navier-Stokes equation in a shear layer at low Reynolds numbers. Ashurst (1988) studied the evolution of two-dimensional structures in a temporally growing plane shear layer by numerical simulation based on the discrete vortex dynamics method. Maekawa, Mansour & Buell (1992) used
direct numerical simulation to study the development and large-scale structure of forced spatially developing wakes.

Coherent structures in a sense are the embodiment of our desire to find order in apparent disorder. It is a well-organized large-scale component of the flow with a phase-correlated vorticity (Hussain 1983, 1986; Fiedler 1988). Thus, "turbulence" can be thought of as combination of the large-scale coherent structure and small-scale turbulence. One of the presently prevailing theories is that coherent structures originate as a result of instabilities of instantaneous disturbed flows. Thus from a fundamental point of view, one of the purposes of hydrodynamic stability theory is to understand the transition from laminar to turbulent flow. Schubauer & Skramstad (1947) first showed experimentally that such an approach is representative of the physical characteristics of transitional flows. Temporally growing space-periodic disturbances were initially used to study the flow stability using linearized flow governing equations. Gaster (1962) clearly showed that spatially developing disturbances better represent the stability characteristics of a developing shear layer. Michalke (1965) first calculated the spatial stability of a parallel shear layer. His results showed the amplification rates for different frequencies for an inviscid shear layer with a mean velocity profile approximated by a hyperbolic tangent. He concluded as did Freymuth (1966) that, at least for small amplitude disturbances, the growth of fluctuations in a free boundary layer can more precisely be described by the stability theory of spatially growing disturbances. Monkewitz & Huerre (1982)
have studied the theoretical dependence of spatially growing waves on free-stream velocity ratio for the hyperbolic-tangent profile. The equivalent problem for axisymmetric jets has been investigated by Michalke & Hermann (1982). Recently, Strykowski & Niccum (1992) have studied the influence of velocity and density ratio of spatially developing mixing layers. Michalke (1972) has given a good survey on the instability of free shear layers. Gaster, Kit & Wygnanski (1985) modelled the large-scale coherent structures that occur in a forced turbulent mixing layer at moderately high Reynolds numbers by linear inviscid stability theory. Comparisons with experiments of Weisbrot & Wygnanski (1988) have clearly shown that linear stability of parallel flow is very successful in predicting the shape of the large-scale structure velocities across the shear layer even when they are fully turbulent. The linear instability theory of plane-parallel shear flows also gives a satisfactory description of the initial growth of very small disturbances. In the initial small-amplitude region, Miksad's (1972) experiment is in good agreement with spatially growing linear instability theory. However, the linear stability theory tends to overestimate the streamwise development of the amplitude for a large-scale structure (Wygnanski & Petersen 1985). The streamwise location where the maximum of the amplitude occurs in experiments does not coincide with the location where theory gives a zero amplification rate. Although this can be improved to some extent by including the effects of flow divergence (Crighton & Gaster 1976; Strange & Crighton 1983), there still is a discrepancy between the experimental measurements and linear stability
theory in this matter. It has been shown that the discrepancy between theory and experiment seems to be caused by the neglect of the nonlinear self-interaction of the disturbance, which always exists for finite perturbations no matter how small.

Evidence of the growth of subharmonic modes of the fundamental most amplified large-scale structure have been observed by Freymuth (1966) and Miksad (1972). Kelly (1967) using a weakly nonlinear analysis showed that a mechanism for subharmonic amplification from its interaction with the fundamental exists, and implied that the phase relation between the two modes must be favorable for this mechanism to take effect. Quantitative measurements of the disturbance amplitudes suggesting mode-mode interactions between the fundamental disturbance and its subharmonic, at half the frequency of the fundamental, in mixing layers indicate that the subharmonic peaks further downstream than the fundamental (Ho & Huang 1982; Huang 1985). Their studies have shown that the peaking of the subharmonic coincides with the phenomenon of vortex merging and the doubling of the shear layer thickness. Their shear layer is essentially one undergoing transition and each individual wave mode undergoes a life cycle of amplification and decay. A subsequent experimental study on the two-dimensional mixing layer was given by Zhang, Ho & Monkewitz (1985), who found that the amplification rates of the subharmonic mode varied by \( \pm 30\% \) depending on its phase relative to the fundamental. Hajj, Miksad and Powers (1992) recently showed with their experiments that fundamental-subharmonic parametric resonance mechanisms lead to the transfer of energy to the subharmonic
from the fundamental while the latter also redistributes energy to other modes via these wave interactions. It is the consensus of all experimental investigations that observed step-like growth of the mixing layer is associated with the growth of the subharmonic large-scale structure especially when external forcing is involved. Experiments have also shown that the development of the large-scale structures strongly depends on initial conditions (Zhang, Ho & Monkewitz 1985; Weisbrot & Wygnanski 1988). This fact is of great practical importance since it allows one to control the development of the shear layer by adjusting the initial conditions. It is impossible to predict or justify the above observations with linear stability arguments and this has led to the development of nonlinear analyses and theories with various degrees of success.

Although the two-dimensional large-scale coherent structures are the dominant feature of two-dimensional mean shear flows, there is increasing observational evidence for the existence of three-dimensional coherent structures, in the form of spanwise standing waves manifested by the appearance of organized streamwise vorticity (Miksad 1972; Konrad 1976; Breidenthal 1978, 1981; Bernal 1981; Roshko 1981; Jimenez 1983; Huang 1985 and others). The three-dimensional non-uniformity in a two-dimensional shear layer was first observed by Bradshaw, Ferriss & Johnson (1964) in an axisymmetric jet discussed by Bradshaw (1966) and Miksad (1972). Konrad (1976) and Breidenthal (1978, 1981) made the first concrete experimental observations of streamwise streaks in free shear layers. Wygnanski et al (1979)
did experiments on the perseverance of the two-dimensionality and found weak three-dimensional structures. Huang (1985) concluded that the rapid growth of the spanwise energy during the vortex formation seems to indirectly suggest that the spanwise motion redistributes the energy to the three-dimensional structure.

It is also an experimental fact that the spanwise wavelength of the three-dimensional coherent modes increases further downstream (Bernal 1981; Jimenez 1983; Huang 1985; Bell & Mehta 1992), as if evolving through the emergence of a spanwise subharmonic formation, much in the same spirit as the subharmonic formation in terms of frequency and streamwise wavelength for two-dimensional coherent modes (Winant & Browand 1974). Quantitative observations (Jimenez 1983; Huang 1985) indicate that the combined spanwise, three-dimensional modes develop downstream in a nonequilibrium fashion resembling that of the two-dimensional modes. The wave envelope amplifies and eventually decays. As the amplitude grows, three-dimensional and nonlinear interactions occur in the form of secondary instabilities. Disturbance growth is very rapid in this case, and breakdown to turbulence occurs.

Bernal (1981) presented conclusive evidence that the streamwise streaks were actual counter-rotating pairs of axial vortices superimposed onto the spanwise structure. Jimenez, Cogollos & Bernal (1985) through a three-dimensional reconstruction via digital image processing of motion pictures of a plane turbulent mixing layer confirmed Bernal's (1981) findings. Their three-dimensional graphics reconstruction shows that after the transition to three-dimensionality, the mixing layer
exhibits an array of counter-rotating pairs of streamwise vortices. Lasheras, Cho & Maxworthy (1986) showed that the plane, free shear layer is unstable to three-dimensional perturbations in the upstream conditions for low Reynolds numbers (less than 250 based on half the maximum slope thickness). This instability was found to result in the formation of a well-organized array of streamwise vortices on the braids between consecutive spanwise vortices. The same picture of the three-dimensional structure was observed at high Reynolds numbers (up to 20,000) by Bernal and Roshko (1986). The formation of streamwise vortices has also been detected in recent numerical calculations using spectral methods (Metcalfe et al 1987) and three-dimensional inviscid vortex dynamics methods (Ashurst & Meiburg 1988).

All experimental observations indicate that the three-dimensional coherent structure appears near and after the location of the first roll-up of the shear layer into the spanwise two-dimensional structure. Lasheras & Choi (1988) observed that the exact location of the appearance of the streamwise structure varied widely from experiment to experiment but was always within the braid connecting adjacent spanwise vortex cores. This variability is reminiscent of a situation sensitive to initial conditions. This has been experimentally confirmed by Lasheras et al (1986). Konrad (1976) and Breidenthal (1978) have argued that the streamwise location of appearance of the three-dimensional structure depends on the Reynolds number and Bernal & Roshko (1986) found dependence on the velocity ratio as well. Regarding the spanwise length scale of the three-dimensional structure there is some
disagreement. Lasheras & Choi (1988) who forced three-dimensional disturbances using corrugated splitter plates did not find a preferred spanwise length scale that would indicate a most amplified wave length. On the other hand most of the other experimental studies that did not involve forcing tend to agree that the initial average length scale is of the same order as the Kelvin-Helmholtz wave length. This is partially in agreement with the non-parallel theory of Pierrehumbert & Windall (1982) which however has been seriously disputed (Corcos & Lin 1984; Lin & Corcos 1984). There is also disagreement on the further streamwise evolution of the streamwise structures. Breidenthal (1978), Lasheras et al (1986) and Lasheras & Choi (1988) do not observe changes in the spanwise length scale downstream, while Konrad (1976), Jimenez (1983), Jimenez et al (1985), Huang & Ho (1990) and Bell & Mehta (1992) have observed doubling of the spanwise wave length in a step-like manner in the neighborhood of the two-dimensional vortex pairing location.

Considering all the observations there is little doubt that nonlinear mechanisms must be important in the whole process of the evolution of three- and two-dimensional structures in the shear layer. Therefore, we will apply to the three-dimensional shear layer problem a nonlinear analysis, based on the instability interpretation of the large-scale structure. This interpretation has been proven plausible by most experimental observations and measurements to date, as outlined previously in here and as discussed by a number of review articles on the subject (Ho & Huerre 1984; Thomas 1991; Mankbadi 1992).
A very effective analysis, which introduces the nonlinear effects into the evolution of finite amplitude wave-like disturbances in shear flows, is based on the so-called integral energy method. The energy method was first introduced by Stuart (1958, 1960, 1962) for nonlinear stability analysis. The method was further developed by Ko, Kubota & Lees (1970) in the analysis of the nonlinear development of a laminar wake, and this work was extended to the compressible laminar wake by Liu & Lees (1970). Applications to turbulent flows were given by Liu (1971) for a wake, by Liu (1974) for a plane mixing layer, and Merkine & Liu (1975) for a plane jet. The nonlinear interactions between the mean flow, large-scale single-wave coherent structure and fine-grained turbulence in a turbulent two-dimensional free shear layer were studied by Liu & Merkine (1978) for the temporal case and by Alper & Liu (1978) for the spatial case. Studies on the nonlinear subharmonic-fundamental mode interactions in a laminar developing mixing layer were first given by Nikitopoulos & Liu (1984, 1987), who found that the initial amplitudes as well as the initial phase angle difference between wave modes played a significant role in the development of the shear layer and the large-scale structure. They also showed that the predicted streamwise development of the energy levels of wave modes and their relation to the observed vortex pairing process were in good agreement with experiments by Ho & Huang (1982) both quantitatively and qualitatively. A comparison of the results of Nikitopoulos & Liu (1990) with these experiments is shown in Appendix A.
The sensitivity of the shear layer on the initial subharmonic-fundamental phase difference first predicted by Nikitopoulos & Liu (1984) was subsequently confirmed by the experiments of Zhang et al (1985). The same conclusions were reached by Mankbadi (1985) for the case of an axisymmetric jet. More recently Lee (1988) has applied this method to interpret the evolution of both axisymmetric and helical wave modes in the axisymmetric jet.

As discussed above, the integral energy method has been quite successful in predicting and interpreting, both qualitative and quantitative, observations of the behavior of the large-scale structure in the first stages of transition in free shear layers. In this work, we will use the integral energy method supplemented by equations for the phases of the large-scale modes, developed for two-dimensional modes by Nikitopoulos & Liu (1990), to study the three-dimensional large-scale coherent wave mode interactions in a developing shear layer. It has been argued before on the basis of recent observations that three-dimensional modes can be of considerable importance for the development of the free shear layer particularly when forcing is used to control its downstream behavior. Although some thought and interpretations have been given by Liu (1988, 1989) on the role of the three-dimensional coherent structure, this subject still remains unexplored.

The objective of our study is to investigate the nonlinear effects arising from the interactions between two- and three-dimensional large-scale coherent structures in a spatially developing, viscous mixing layer. Our attention will be focused on
the energy exchange mechanisms between the various modes, and the effects of the
nonlinear evolution of the phases of the interacting modes. The importance of a
multitude of initial conditions that may affect the evolution of the flow is also going
to be examined. An attempt will be made to relate our results to existing experi­
mental observations and to gain insight into ways of controlling the development of
mixing layers through three-dimensional multi-mode forcing.

Before proceeding with the formulation of our problem it should be noted that
in our analysis we are going to ignore the effects of small-scale turbulence. It is
well known that in the early stages of development of shear layers the interactions
between the large-scale structures themselves and with the mean flow are dominant,
while small-scale turbulence acts as an agent for dissipation of a small fraction
of the large-scale and mean energy. This is true in general unless the initial free
stream turbulence level is very high, and has been firmly established by Alper &
Liu (1978) among others. Furthermore, experimental evidence (Lasheras et al 1986;
Bernal & Roshko 1986) shows that the basic characteristics of evolution of the two-
and three-dimensional large-scale structures are essentially the same for very wide
range of Reynolds numbers from laminar to fully turbulent initial shear layer. With
this in mind our analysis is limited to the initial stages of development of the shear
layer before the breakdown of large-scale structure into small-scale turbulence which
dictates the growth of the flow thereafter.
Chapter 2
Mathematical Formulation of the Problem

The objective of this study is to investigate the role of three-dimensional large-scale structures in the developing shear layer subjected to external forcing. To achieve this, one has to look into the mutual interactions between the three-dimensional large-scale structure, the mean flow and the purely two-dimensional large-scale structure. Furthermore, we aim to investigate the effects of subharmonic forcing of the three-dimensional coherent structure.

The problem in its generality is fairly complex and to analyze it a number of simplifying assumptions are in order. Throughout this analysis we will consider the free shear layer (mixing layer) depicted in Figure 1.1. The mean flow, according to the classical assumption, will be considered as parallel with a zero spanwise component. Furthermore, the sole mean flow velocity component will have the typical self-similar, hyperbolic tangent velocity profile (see Figure 1.1). The large-scale structures are going to be viewed as normal wave modes, products of hydrodynamic
instability, which are periodic in space and time and grow with the streamwise coordinate alone (spatial problem).

The nonlinear mutual interactions between the mean flow and a number of large-scale wave modes are going to be studied using the integral energy method. The formulation of the problem according to this method is going to be presented in the following sections of this chapter. In order to study the effects of subharmonic forcing, five wave modes are going to be included in this analysis (Liu 1988). The flow will be decomposed into a mean component and the five wave modes. The equations of motion, subject to the usual boundary layer type approximations (Schlichting 1979; Ko, Kubota & Lees 1970), are going to be derived for each component of the flow. From these, a set of integral equations for the energy content and phase of each wave component are going to be derived together with an equation for the growth of the mean flow. In the course of the analysis additional assumptions will have to be made and these will be justified as they are implemented.

2.1 Decomposition and Equations of Motion

We begin with the continuity and Navier-Stokes equations for an incompressible homogeneous fluid. The coordinate system used, as shown in Figure 1.1, has x, y and z as the streamwise, cross-stream and spanwise coordinates respectively, and u, v and w are the corresponding velocity components. All flow quantities and the coordinates are nondimensionalized by the following reference parameters:
• Length: \( \delta_0 \) (the initial maximum slope thickness at the onset of the shear layer),

• Velocity: \( \overline{U} = (U_{-\infty} + U_{+\infty})/2 \) (average velocity of the two free streams),

• Time: \( \delta_0/\overline{U} \),

• Pressure: \( \rho \overline{U}^2 \)

Thus the equations of motion become

(I) Continuity equation:

\[
\frac{\partial u_i}{\partial x_i} = 0 
\]  \hspace{1cm} (2.1)

(II) Momentum equations:

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re_0} \frac{\partial^2 u_i}{\partial x_j^2} \]  \hspace{1cm} (2.2)

where \( Re_0 = \delta_0 \overline{U}/\nu \) is the Reynolds number characteristic of the shear layer mean flow, and \( \nu \) is the kinematic viscosity.

All flow variables can be decomposed into three components (Reynolds & Hussain 1972) rather than the usual two as in the classical Reynolds decomposition. An arbitrary quantity \( q(\vec{x}, t) \) is thus expressed as

\[
q(\vec{x}, t) = Q(\vec{x}) + \bar{q}(\vec{x}, t) + q'(\vec{x}, t) \]  \hspace{1cm} (2.3)

where \( Q(\vec{x}) \) is the time averaged mean quantity, \( \bar{q}(\vec{x}, t) \) represents the large-scale deterministic component (coherent structure) and \( q'(\vec{x}, t) \) the small-scale random
component (fine-grained turbulence). The interactions between the mean flow, a monochromatic, two-dimensional, large-scale coherent structure, and the fine-grained turbulence have been previously studied by applying combined conditional- and time-averaging techniques by Reynolds & Hussain (1972), Mankbadi & Liu (1981) and Liu & Kaptanoglu (1987) among others. Furthermore the experimental evidence (Weisbrot & Wygnanski 1988; Ho & Huang 1982) from numerous studies attest to the fact that the early development of the shear layer is dominated by the large-scale structure even if the shear layer is turbulent initially (Bell & Mehta 1992) with high Reynolds number.

The focus of the present study is on the mutual interaction of multiple large-scale structure modes with the mean flow and with each other; therefore we will limit our analysis to the first two components of the decomposition as expressed by equation (2.3) and leave out the fine-grained turbulence component. Thus without fine-grained turbulence, each flow quantity is split into two major components; one representing the steady mean flow and one representing the large-scale coherent structures as a group (Nikitopoulos & Liu 1987, 1990; Lee 1988):

\[ q(\vec{x}, t) = Q(\vec{x}) + \tilde{q}(\vec{x}, t). \]  

(2.4)

Several different terms such as "large-scale coherent structure", "wave component", "wave mode", or "orderly structure" will be used hereafter for \( \tilde{q} \).

In the following analysis the wave mode component is going to be expressed as a group of normal wave modes interpreted in general as traveling waves in the
streamwise direction and standing waves in the spanwise direction, while the mean flow will have zero spanwise component \((W = 0)\) and will be independent of the spanwise coordinate. Consequently we will employ a combined time and spanwise space average of our flow quantities expressed in general by

\[
\bar{q}(\bar{x}) = Q(\bar{x}) = \lim_{T \to \infty} \frac{1}{LT} \int_{0}^{L} \int_{0}^{T} q(\bar{x}, t) dt dz
\]

(2.5)

where \(L\) is the spanwise wave-length of the coherent structure and is equal to \(2\pi/\gamma\), where \(\gamma\) is the spanwise wave number.

The governing equations for the mean flow and the wave component are derived from equations (2.1) and (2.2) using the decomposition equation (2.4) and the averaging equation (2.5). Thus, it is easy to show that each component satisfies the continuity equation separately

\[
\frac{\partial U_i}{\partial x_i} = \frac{\partial \bar{u}_i}{\partial x_i} = 0,
\]

(2.6)

and that the momentum equations become:

1. Mean flow:

\[
\frac{\bar{D}U_i}{Dt} = -\frac{\partial P}{\partial x_i} + \frac{1}{Re_0} \frac{\partial^2 U_i}{\partial x^2_j} - \frac{\partial}{\partial x_j} (\bar{u}_i \bar{u}_j),
\]

(2.7)

2. Wave mode:

\[
\frac{\bar{D}\bar{u}_i}{Dt} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{Re_0} \frac{\partial^2 \bar{u}_i}{\partial x^2_j} - \frac{\partial}{\partial x_j} (\bar{u}_i \bar{u}_j - \bar{u}_i \bar{u}_j)
\]

(2.8)

The last term of equation (2.7) is the Reynolds stress induced by the large-scale coherent structures. Equation (2.8) forms the starting point for studying nonlinear
interactions between coherent wave modes themselves. The second term on the left-hand-side of the equation (2.8) is the advection of mean flow momentum by the coherent motion and forms the basic mechanism of shear flow hydrodynamic instabilities (Lin 1955). The last term on the right-hand-side of equation (2.8) introduces nonlinear effects through products of coherent wave modes. In here, \( \overline{D}/Dt = \partial/\partial t + U_j \partial/\partial x_j \) is the Eulerian derivative following the mean flow.

For the purposes of this study the large-scale velocity components \( \tilde{u}_j \) and pressure \( \tilde{p} \) are assumed to be Fourier analyzable, and are decomposed into five wave modes which are periodic in time \( t \) and the spanwise direction \( z \). Thus the decomposition of the large-scale structure yields:

\[
\begin{align*}
\tilde{u} &= \tilde{u}_{10} + \tilde{u}_{20} + \tilde{u}_{11} + \tilde{u}_{21} + \tilde{u}_{22} \\
\tilde{v} &= \tilde{v}_{10} + \tilde{v}_{20} + \tilde{v}_{11} + \tilde{v}_{21} + \tilde{v}_{22} \\
\tilde{p} &= \tilde{p}_{10} + \tilde{p}_{20} + \tilde{p}_{11} + \tilde{p}_{21} + \tilde{p}_{22} \\
\tilde{w} &= \tilde{w}_{11} + \tilde{w}_{21} + \tilde{w}_{22}
\end{align*}
\]

with

\[
\begin{bmatrix}
\tilde{u}_{mn}(\vec{x},t) \\
\tilde{v}_{mn}(\vec{x},t) \\
\tilde{p}_{mn}(\vec{x},t)
\end{bmatrix} = 
\begin{bmatrix}
u'_{mn}(x,y) \\
v'_{mn}(x,y) \\
p'_{mn}(x,y)
\end{bmatrix} e^{-i\beta_{mn}t} \cos(n\gamma z + \phi_{mn}) + \text{c.c.}
\]

and

\[
\tilde{w}_{mn}(\vec{x},t) = \omega_{mn}(x,y) e^{-i\beta_{mn}t} \sin(n\gamma z + \phi_{mn}) + \text{c.c.}
\]
where c.c. denotes the complex conjugate, \( u'_{mn}, v'_{mn}, p'_{mn} \) and \( w_{mn} \) are complex amplitudes functions of the streamwise and cross-stream coordinates, \( \beta_{mn} \) is the frequency of the wave mode \( mn \), and \( n\gamma \) is the wave number of the same. The spanwise phase angle exists only when the three-dimensional wave modes exist; \( \phi_{21} \) and \( \phi_{22} \) are the relative phase angles to \( \phi_{11} \). We have set \( \phi_{11} \), the reference spanwise phase angle, to be equal to zero. We use the subscripts \( mn \) to describe the coherent modes, where \( m \) indicates the frequency mode (\( m=1 \): subharmonic frequency, \( m=2 \): fundamental frequency), and \( n \) indicates the mode dimensionality (\( n=1, 2 \) denote the spanwise standing wave modes of wave number \( \gamma \) and \( 2\gamma \) respectively; and \( n=0 \) indicates two-dimensional wave modes). Thus a large-scale wave mode represents a two-dimensional disturbance if \( n \) is equal to zero, and a three-dimensional disturbance otherwise. The large-scale wave modes with \( mn=10, 11 \) will be called subharmonics and those with \( mn=20, 21, 22 \) will be called fundamentals. The fundamental frequency is 2 times that of the subharmonic. According to equations (2.13) and (2.14), the large-scale structures have a life cycle during which they grow and decay spatially and are periodic in time. The spanwise component, \( \tilde{w}_{mn} \), includes the sine function to satisfy the wave mode continuity equations.

To study the nonlinear interactions between the two- and three-dimensional fundamental and subharmonic modes in the spatially developing shear layer, the selected five wave modes represent the minimum number that needs to be considered (Liu 1988; Lee 1988). The fundamental wave modes 20, 21 and 22 engage in direct
quadratic interactions with the subharmonic wave modes 10 and 11. The interac-
tions between these five modes belong to the family of binary-frequency interactions
previously studied by Nikitopoulos & Liu (1987) for the two-dimensional shear layer.

The continuity equation and the momentum equations for each wave mode \( mn \)
are derived by substituting the decompositions (2.9) ~ (2.12) into the general large-
scale structure continuity and momentum equations. It should be noted that mode
\( mn (|\tilde{u}_i|_{mn}) \) can be affected by the nonlinear mode interaction through products
\( ([\tilde{u}_i]_{kl}[\tilde{u}_j]_{pq}) \) that satisfy the conditions \(|k \pm p| = m\) and \(|l \pm q| = n\). These are the
same relations given by Cohen and Wygnanski (1987 a,b) and by Mankbadi (1992)
as conditions for the fulfillment of resonance type interactions between wave modes.
These conditions are necessary for the existence of such resonance interaction but not
sufficient for the determination of their effectiveness in transferring energy between
the modes involved. This point will be discussed further later on. For the sake of
brevity we are going to show here the continuity and momentum equations of the
10 and 11 wave modes only. In order to avoid complicated expressions the following
equations are shown for \( \phi_{21} = \phi_{22} = 0 \). In general the every term containing modes
2n (n \( \neq 0 \)) is implicitly multiplied be \( \cos(n\gamma z + \phi_{2n}) \) or \( \sin(n\gamma z + \phi_{2n}) \) rather that
\( \cos(n\gamma z) \) or \( \sin(n\gamma z) \). The continuity and momentum equations of the remaining
wave modes are similar to those of 10 and 11 wave modes and can be found in
Appendix C.
The continuity equation of mode 10 is

\[
\frac{\partial u_{10}}{\partial x} + \frac{\partial v_{10}}{\partial y} = 0 \quad (2.15)
\]

The x-momentum equation of mode 10 is

\[
-i\beta_{10}u_{10} + U \frac{\partial u_{10}}{\partial x} + v_{10} \frac{\partial U}{\partial y} = -\frac{\partial p_{10}}{\partial x} + \frac{1}{Re_0} \left( \frac{\partial^2 u_{10}}{\partial x^2} + \frac{\partial^2 u_{10}}{\partial y^2} \right)
- \frac{\partial}{\partial x} \left( 2u_{10}^*u_{20} + u_{11}^*u_{21} \right)
- \frac{\partial}{\partial y} \left( u_{10}^*v_{20} + u_{20}^*v_{10} + u_{11}^*v_{21}/2 + u_{21}^*v_{11}/2 \right) \quad (2.16)
\]

The y-momentum equation of mode 10 is

\[
-i\beta_{10}v_{10} + U \frac{\partial v_{10}}{\partial x} = -\frac{\partial p_{10}}{\partial y} + \frac{1}{Re_0} \left( \frac{\partial^2 v_{10}}{\partial x^2} + \frac{\partial^2 v_{10}}{\partial y^2} \right)
- \frac{\partial}{\partial x} \left( u_{10}^*v_{20} + u_{20}^*v_{10} + u_{11}^*v_{21}/2 + u_{21}^*v_{11}/2 \right)
- \frac{\partial}{\partial y} \left( 2v_{10}^*v_{20} + v_{11}^*v_{21} \right) \quad (2.17)
\]

The continuity equation of mode 11 is

\[
\frac{\partial u_{11}}{\partial x} + \frac{\partial v_{11}}{\partial y} + \gamma w_{11} = 0 \quad (2.18)
\]

The x-momentum equation of 11 mode is

\[
-i\beta_{11}u_{11} + U \frac{\partial u_{11}}{\partial x} + v_{11} \frac{\partial U}{\partial y} = -\frac{\partial p_{11}}{\partial x}
+ \frac{1}{Re_0} \left( \frac{\partial^2 u_{11}}{\partial x^2} + \frac{\partial^2 u_{11}}{\partial y^2} - \gamma^2 w_{11} \right)
- \frac{\partial}{\partial x} \left( 2u_{10}^*u_{21} + 2u_{20}^*u_{11} + u_{11}^*u_{22} \right)
- \frac{\partial}{\partial y} \left( u_{10}^*v_{21} + v_{10}^*u_{21} + u_{20}^*v_{11} + u_{11}^*u_{20} + \frac{u_{11}^*v_{22}}{2} + \frac{v_{11}^*u_{22}}{2} \right)
- \gamma \left( u_{10}^*w_{21} + w_{11}^*u_{20} + \frac{u_{11}^*w_{22}}{2} - \frac{w_{11}^*u_{22}}{2} \right) \quad (2.19)
\]
The y-momentum equation of mode 11 is

\[-i\beta_{11}w_{11} + U \frac{\partial v_{11}}{\partial x} = -\frac{\partial p_{11}}{\partial y} + \frac{1}{Re_0} \left( \frac{\partial^2 v_{11}}{\partial x^2} + \frac{\partial^2 v_{11}}{\partial y^2} - \gamma^2 v_{11} \right) \]

\[-\frac{\partial}{\partial x} \left( w_{10}^* v_{21} + w_{21} v_{10}^* + w_{20} v_{11}^* + \frac{w_{11}^* v_{22}^*}{2} + \frac{v_{11}^* u_{22}^*}{2} \right) \]

\[-\frac{\partial}{\partial y} (2 w_{10}^* v_{21} + 2 v_{20} v_{11}^* + v_{11}^* v_{22}^*) \]

\[-\gamma (w_{10}^* w_{21} + w_{11}^* v_{22}) + \frac{w_{11}^* w_{22}}{2} - \frac{v_{11}^* v_{22}}{2} \] (2.20)

The z-momentum equation of mode 11 is

\[-i\beta_{11}w_{11} + U \frac{w_{11}}{\partial x} = \gamma p_{11} + \frac{1}{Re_0} \left( \frac{\partial^2 w_{11}}{\partial x^2} + \frac{\partial^2 w_{11}}{\partial y^2} - \gamma^2 w_{11} \right) \]

\[-\frac{\partial}{\partial x} \left( w_{10}^* w_{21} + w_{20} w_{11}^* - \frac{w_{22} w_{11}^*}{2} + \frac{w_{11}^* w_{22}^*}{2} \right) \]

\[-\frac{\partial}{\partial y} (w_{10}^* w_{21} + v_{20} w_{11}^* - \frac{v_{22} w_{11}^*}{2} + \frac{w_{11}^* w_{22}^*}{2}) \]

\[+\gamma \left( w_{11}^* w_{22} \right) \] (2.21)

where \((\ )^*\) is the complex conjugate of \((\ )\).

### 2.2 Mechanical Energy Equations

We can explain the physical mechanisms of the large-scale coherent structure interactions using the mechanical energy equations. We shall obtain the mechanical energy equations directly for the various scales of motion, in the usual way, by multiplying the relevant \(i^{th}\)-component momentum equation by the corresponding \(i^{th}\)-component velocity and summing. The mean flow energy equation is obtained by multiplying equation (2.7) by \(U_i\) and manipulating appropriately.
The first group of terms on the right-hand-side of equation (2.22) include the pressure work, the transport of mean flow energy by the Reynolds stresses of the large-scale coherent structures and the viscous diffusion of the mean kinetic energy. The second term represents the energy exchange mechanism between the mean flow and the large-scale coherent wave mode that is effected through the large-scale Reynolds stresses acting on the mean flow rate of strain. The third term represents the rate of viscous dissipation of the mean kinetic energy.

The energy equation for a single large-scale coherent wave mode can be obtained from equation (2.8) in the usual way by multiplying $\tilde{u}_i$ and averaging. After straightforward manipulation we obtain

\[
\frac{\bar{D}}{Dt} \left( \frac{\bar{u}_i^2}{2} \right) = \frac{\partial}{\partial x_j} \left[ \bar{p} \bar{u}_j + \frac{\bar{u}_j \bar{u}_i^2}{2} - \frac{1}{Re_0} \frac{\partial}{\partial x_j} \left( \frac{\bar{u}_i^2}{2} \right) \right] + \frac{1}{Re_0} \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_j} \frac{\bar{u}_i}{\partial x_j} - \frac{1}{Re_0} \left( \frac{\partial \bar{u}_i}{\partial x_j} \right)^2 \tag{2.23}
\]
energy exchanges between large-scale wave modes and is of considerable importance because it provides the only means of growth (or decay) of the large-scale structure beyond that dictated by the interaction with the mean flow. The last term finally represents the viscous dissipation of the modal kinetic energy.

Mechanical energy equations can be obtained for the mean flow and each one of the five modes considered in the present study by introducing the decomposition equations (2.9) ~ (2.12) into equations (2.22) and (2.23) respectively. Alternatively and in view of equations (2.13) and (2.14) the same mechanical energy equations involving the complex wave mode amplitudes can be recovered by using the momentum equation for each mode and performing the following operation:

\[
\begin{align*}
(\tilde{\bar{u}}_{mn} (\bar{u}_{mn} \text{ momentum equation}) + \tilde{\bar{u}}_{mn} (\bar{u}_{mn} \text{ momentum equation}))^v + \\
(\tilde{\bar{v}}_{mn} (\bar{v}_{mn} \text{ momentum equation}) + \tilde{\bar{v}}_{mn} (\bar{v}_{mn} \text{ momentum equation}))^v + \\
(\tilde{\bar{w}}_{mn} (\bar{w}_{mn} \text{ momentum equation}) + \tilde{\bar{w}}_{mn} (\bar{w}_{mn} \text{ momentum equation}))^v,
\end{align*}
\]

and then averaging the result. Repeated indices do not imply summation here.

Thus the mechanical energy equation for wave mode 10 is

\[
U \frac{d}{dx} [|u_{10}|^2 + |v_{10}|^2] =
\]

\[Rate \ of \ change \ of \ modal \ energy\]

\[-2 \text{Re}(u_{10}v_{10}^*) \frac{\partial U}{\partial y}\]

\[Energy \ exchange \ with \ mean \ flow\]
\[-\frac{\partial}{\partial x}[2\text{Rel}(w_{10}^2v_{10}^2)] - \frac{\partial}{\partial y}[2\text{Rel}(v_{10}^2v_{10}^2)]\]

*Pressure work*

\[+ \frac{1}{Re_0} \left[ \frac{\partial^2}{\partial x^2}(|w_{10}|^2 + |v_{10}|^2) + \frac{\partial^2}{\partial y^2}(|w_{10}|^2 + |v_{10}|^2) \right]\]

*Viscous diffusion*

\[-\frac{2}{Re_0} \left[ |\frac{\partial w_{10}}{\partial x}|^2 + |\frac{\partial v_{10}}{\partial y}|^2 + |\frac{\partial w_{10}}{\partial y}|^2 + |\frac{\partial v_{10}}{\partial y}|^2 \right]\]

*Viscous dissipation*

\[-\frac{\partial}{\partial x}[2\text{Rel}\left(\frac{w_{10}^2v_{20}^2}{2} + \frac{v_{10}^2w_{20}^2}{2}\right)] - \frac{\partial}{\partial y}[2\text{Rel}\left(\frac{w_{10}^2v_{20}^2}{2} + \frac{v_{10}^2v_{20}^2}{2}\right)]\]

*Convective transport terms*

\[-2\text{Rel}\left(\frac{w_{10}v_{20}w_{10}v_{10}}{2} + \frac{w_{10}v_{10}w_{20}v_{20}}{2}\right) + (w_{10}v_{10}\frac{\partial w_{20}}{\partial y} + v_{10}v_{10}\frac{\partial v_{20}}{\partial y})\]

*Energy interaction between 10 and 20*

\[-2\text{Rel}\left[\frac{1}{2}(w_{10}w_{21}\frac{\partial w_{11}}{\partial x} + v_{10}w_{21}\frac{\partial v_{11}}{\partial y}) + \frac{1}{2}(w_{10}v_{21}\frac{\partial w_{11}}{\partial y} + v_{10}v_{21}\frac{\partial v_{11}}{\partial y})\right] + \frac{\gamma}{2}(w_{10}w_{21}v_{11} + v_{10}w_{11}v_{21})\cos(\phi_{21})\]

*Energy interaction between 10 and 11*

\[-2\text{Rel}\left[\frac{1}{2}(w_{10}w_{11}\frac{\partial w_{21}}{\partial x} + v_{10}w_{11}\frac{\partial v_{21}}{\partial y}) + \frac{1}{2}(w_{10}v_{11}\frac{\partial w_{21}}{\partial y} + v_{10}v_{11}\frac{\partial v_{21}}{\partial y})\right] + \frac{\gamma}{2}(w_{10}w_{11}v_{21} + v_{10}w_{21}v_{11})\cos(\phi_{21})\]

*Energy interaction between 10 and 21, (2.24)*

and for wave mode 11:

\[\frac{U}{2} \frac{d}{dx}[|w_{11}|^2 + |v_{11}|^2 + |w_{11}|^2]\]

*Rate of change of modal energy*
Energy exchange with the mean flow

\[- \frac{\partial}{\partial x} [2 \text{Re} \left( \frac{u_{11}^2 + v_{11}^2}{2} \right)] - \frac{\partial}{\partial y} [2 \text{Re} \left( \frac{v_{11}^2 + w_{11}^2}{2} \right)]\]

Pressure work

\[+ \frac{1}{2 \text{Re}_0} \left[ \frac{\partial^2}{\partial x^2} \left( |u_{11}|^2 + |v_{11}|^2 + |w_{11}|^2 \right) \right. \]

Viscous diffusion

\[- \frac{1}{\text{Re}_0} \left[ \frac{\partial |u_{11}|^2}{\partial x} + \frac{\partial |v_{11}|^2}{\partial x} + \frac{\partial |w_{11}|^2}{\partial x} + \frac{\partial |u_{11}|^2}{\partial y} + \frac{\partial |v_{11}|^2}{\partial y} + \frac{\partial |w_{11}|^2}{\partial y} \right] \]

Viscous dissipation

\[- \frac{\partial}{\partial x} [2 \text{Re} \left( \frac{u_{10}^* u_{11}^2}{4} + \frac{u_{22}^* u_{11}^2}{8} \cos(\phi_{22}) \right)] - \frac{\partial}{\partial y} [2 \text{Re} \left( \frac{u_{20}^* u_{11}^2}{4} + \frac{u_{22}^* u_{11}^2}{8} \cos(\phi_{22}) \right)]\]

Convective transport terms

\[+ 2 \text{Re} \left[ \frac{1}{2} (u_{10}^* u_{21}^* \frac{\partial u_{11}}{\partial x} + v_{10}^* u_{21}^* \frac{\partial v_{11}}{\partial x}) + \frac{1}{2} (u_{10}^* v_{21}^* \frac{\partial u_{11}}{\partial y} + v_{10}^* v_{21}^* \frac{\partial v_{11}}{\partial y}) \right] \]

Energy interaction between 11 and 10

\[- 2 \text{Re} \left[ \frac{1}{2} (u_{11}^* \frac{\partial u_{20}^*}{\partial x} + u_{11}^* \frac{\partial v_{20}^*}{\partial x}) + \frac{1}{2} (w_{11}^* \frac{\partial u_{20}^*}{\partial y} + v_{11}^* \frac{\partial v_{20}^*}{\partial y}) \right] \]
Energy interaction between 11 and 20

\[-2 \text{Re}\left[ \frac{1}{2} (u_{11} u_{10} \frac{\partial w_{21}^*}{\partial x} + v_{11} u_{10} \frac{\partial v_{21}^*}{\partial x} + w_{11} u_{10} \frac{\partial w_{21}^*}{\partial x}) \right] + \frac{1}{2} \left( u_{11} v_{10} \frac{\partial w_{21}^*}{\partial y} + v_{11} v_{10} \frac{\partial v_{21}^*}{\partial y} + w_{11} v_{10} \frac{\partial w_{21}^*}{\partial y} \right) \right] \cos(\phi_{21})

Energy interaction between 11 and 21

\[-2 \text{Re}\left[ \frac{1}{4} (u_{11}^2 \frac{\partial w_{22}^*}{\partial x} + v_{11} u_{11} \frac{\partial v_{22}^*}{\partial x} + w_{11} u_{11} \frac{\partial w_{22}^*}{\partial x}) \right] + \frac{1}{4} \left( u_{11} v_{11} \frac{\partial w_{22}^*}{\partial y} + v_{11}^2 \frac{\partial v_{22}^*}{\partial y} + w_{11} v_{11} \frac{\partial w_{22}^*}{\partial y} \right) \right] - \frac{\gamma}{2} \left( u_{11} u_{11} w_{22} + v_{11} u_{11} u_{11} + w_{11}^2 w_{22} \right) \cos(\phi_{22})

Energy interaction between 11 and 22 (2.25)

where \( \text{Re}(\quad) \) is the real part of the complex quantity \((\quad)\). The mechanical energy equations of the remaining modes can be found in the Appendix D. The terms representing individual mechanisms of energy exchange and redistribution have been indicated for each one of modes 10 and 11. These mechanisms and their significance, particularly those of mutual mode interactions, are going to be discussed later after the energy equations have been developed further.

The mechanical energy equation for the mean flow is given by

\[
\frac{\overline{D}}{Dt} \frac{U^2}{2} = -\frac{1}{\text{Re}_0} \left( \frac{\partial U}{\partial y} \right)^2 + 2 \text{Re}(u_{10} v_{10} + u_{20} v_{20} + \frac{u_{11} v_{11}^*}{2} + \frac{w_{11} v_{11}^*}{2} + \frac{w_{22} v_{22}^*}{2}) \frac{\partial U}{\partial y}. \tag{2.26}
\]

The pressure work term has been dropped since in a free shear flow there is no mean pressure gradient while the disturbance induced transport term has also been dropped according to the boundary layer approximation.
2.3 Phase Angle Equations

The phase angle differences between wave modes are very important for the nonlinear mode-mode energy exchange terms. The pairing process is dependent on the phase angle difference between the fundamental and subharmonic wave modes. Ho & Huang (1982) have demonstrated that the spreading rates can be manipulated by controlling the vortex-pairing process. Because this pairing process is dependent on the phase angle difference between the fundamental and subharmonic wave modes, we would expect that the spreading rate is dependent on the phase angle differences.

The travelling wave modes will generally not be in phase. It is well understood from previous work (Nikitopoulos & Liu 1987; Lee 1988) and can be easily deduced from the energy interaction terms between wave modes (see equation (2.23)) that the relative phases between modes are most important in determining the direction and magnitude of intra-modal energy exchanges. Furthermore the modal phases depend on the cross-stream coordinate and develop with the downstream coordinate in a nonlinear manner. The original studies on nonlinear coherent structure interactions conducted by Nikitopoulos and Liu (1984), Mankbadi (1985) and Nikitopoulos and Liu (1987) assumed that the streamwise evolution of the phases are dictated by linear theory, while allowing nonlinear evolution of the wave amplitudes. More recent studies by Mankbadi (1992) and Nikitopoulos and Liu (1990) have shown that nonlinear effects on the phase evolution can be of considerable importance. Comparisons with experimental data (Mankbadi 1992) confirmed the importance
of the nonlinear evolution of the phases. It is therefore necessary to derive the equations governing the evolution of the phase angles of each mode separately. We have derived such equations from the modal momentum equations by performing the following operation:

\[
(\tilde{u}_{mn} (\tilde{u}_{mn} \text{ momentum equation}) - \tilde{u}_{mn}^* (\tilde{u}_{mn} \text{ momentum equation})) + \\
(\tilde{v}_{mn} (\tilde{v}_{mn} \text{ momentum equation}) - \tilde{v}_{mn}^* (\tilde{v}_{mn} \text{ momentum equation})) + \\
(\tilde{w}_{mn} (\tilde{w}_{mn} \text{ momentum equation}) - \tilde{w}_{mn}^* (\tilde{w}_{mn} \text{ momentum equation})),
\]

and repeated indices do not imply summation.

Thus the phase angle equation for wave mode 10 is

\[
2\beta_{10} [\left| u_{10} \right|^2 + \left| v_{10} \right|^2]
\]

**Temporal phase**

\[
+ U [2Im(u_{10} \partial u_{10}^* / \partial x + v_{10} \partial v_{10}^* / \partial x)]
\]

**Rate of change of phase**

\[
+ 2Im(u_{10}v_{10}^*) \frac{\partial U}{\partial y}
\]

**Phase shift from interacting with the mean flow**

\[
- \frac{\partial}{\partial x} [2Im(u_{10}p_{10}^*)] - \frac{\partial}{\partial y} [2Im(v_{10}p_{10}^*)]
\]

**Pressure field induced phase shift**

\[
+ \frac{1}{Re_0} [2Im(\partial (u_{10} \partial u_{10}^* / \partial x + v_{10} \partial v_{10}^* / \partial x) + 2Im(\partial (u_{10} \partial u_{10}^* / \partial y + v_{10} \partial v_{10}^* / \partial y))]
\]

**Viscosity induced phase shift**

\[
- \frac{\partial}{\partial x} [2Im(\frac{u_{10}^2 u_{20}^*}{2} + \frac{v_{10}^2 u_{20}^*}{2})] - \frac{\partial}{\partial y} [2Im(\frac{u_{10}^2 v_{20}^*}{2} + \frac{v_{10}^2 v_{20}^*}{2})]
\]
Phase shift induced by convective transport

\[ -2\text{Im}[\left( w_{10}^2 \frac{\partial u'_{20}}{\partial x} + w_{10}v_{10} \frac{\partial v'_{20}}{\partial y} \right) + \left( w_{10}v_{10} \frac{\partial u'_{20}}{\partial y} + v_{10}^2 \frac{\partial v'_{20}}{\partial x} \right)] \]

Phase shift due to interaction between 10 and 20

\[ -2\text{Im}\left[ \frac{1}{2} \left( w_{10}w_{21}^* \frac{\partial u_{11}}{\partial x} + v_{10}u_{21}^* \frac{\partial v_{11}}{\partial x} \right) + \frac{1}{2} \left( w_{10}v_{21}^* \frac{\partial u_{11}}{\partial y} + v_{10}v_{21}^* \frac{\partial v_{11}}{\partial y} \right) \right] - \frac{\gamma}{2} \left( w_{10}w_{11}^* w_{21}^* + v_{10}w_{11}^* v_{21}^* \right) \cos(\phi_{21}) \]

Phase shift due to interaction between 10 and 11

\[ -2\text{Im}\left[ \frac{1}{2} \left( w_{10}w_{12}^* \frac{\partial u_{11}}{\partial x} + v_{10}u_{12}^* \frac{\partial v_{11}}{\partial x} \right) + \frac{1}{2} \left( w_{10}v_{12}^* \frac{\partial u_{11}}{\partial y} + v_{10}v_{12}^* \frac{\partial v_{11}}{\partial y} \right) \right] - \frac{\gamma}{2} \left( w_{10}w_{11}^* v_{12}^* + v_{10}w_{11}^* v_{12}^* \right) \cos(\phi_{21}) \]

Phase shift due to interaction between 10 and 21, (2.27)

where \( \text{Im}(\ ) \) is the imaginary part of the complex quantity ( ), and

for wave mode 11 :

\[ 2\beta_{11}[(|u_{11}|^2 + |v_{11}|^2 + |w_{11}|^2)/2] \]

Temporal phase

\[ + \frac{U}{2} \left[ 2\text{Im}(u_{11} \frac{\partial u'_{11}}{\partial x} + v_{11} \frac{\partial v'_{11}}{\partial x} + w_{11} \frac{\partial w'_{11}}{\partial x}) \right] + \]

Rate of change of phase

\[ 2\text{Im}(w_{11}v_{11}/2) \frac{\partial U}{\partial y} = \]

Phase shift from interacting with the mean flow

\[ - \frac{\partial}{\partial x} [2\text{Im}(\frac{u_{11}p_{11}^*}{2})] - \frac{\partial}{\partial y} [2\text{Im}(\frac{v_{11}p_{11}^*}{2})] \]

Pressure field induced phase shift
Viscosity induced phase shift

\begin{align*}
&+ \frac{1}{Re_0} \left[ \text{Im} \left( \frac{\partial}{\partial x} (w_{11} \frac{\partial u_{11}^*}{\partial x} + v_{11} \frac{\partial v_{11}^*}{\partial x} + w_{11} \frac{\partial w_{11}^*}{\partial x}) \right) \\
&+ \text{Im} \left( \frac{\partial}{\partial y} (w_{11} \frac{\partial u_{11}^*}{\partial y} + v_{11} \frac{\partial v_{11}^*}{\partial y} + w_{11} \frac{\partial w_{11}^*}{\partial y}) \right) \right] \\
&\left. \right. \\
\end{align*}

Phase shift induced by convective transport

\begin{align*}
&- \frac{\partial}{\partial x} \left[ 2 \text{Im} \left( \frac{u_{10}^* u_{21}^*}{4} + \frac{u_{22} u_{11}^*}{8} \cos(\phi_{22}) \right) \right] - \frac{\partial}{\partial y} \left[ 2 \text{Im} \left( \frac{v_{10}^* v_{21}^*}{4} + \frac{v_{22} v_{11}^*}{8} \cos(\phi_{22}) \right) \right] \\
&- \frac{\partial}{\partial x} \left[ 2 \text{Im} \left( \frac{u_{10}^* u_{21}^*}{4} - \frac{u_{22} u_{11}^*}{8} \cos(\phi_{22}) \right) \right] - \frac{\partial}{\partial y} \left[ 2 \text{Im} \left( \frac{v_{10}^* v_{21}^*}{4} - \frac{v_{22} v_{11}^*}{8} \cos(\phi_{22}) \right) \right] \\
&- \frac{\partial}{\partial x} \left[ 2 \text{Im} \left( \frac{u_{10}^* u_{21}^*}{4} + \frac{u_{22} u_{11}^*}{8} \cos(\phi_{22}) \right) \right] - \frac{\partial}{\partial y} \left[ 2 \text{Im} \left( \frac{v_{10}^* v_{21}^*}{4} - \frac{v_{22} v_{11}^*}{8} \cos(\phi_{22}) \right) \right] \\
&- \frac{\partial}{\partial y} \left[ 2 \text{Im} \left( \frac{u_{10}^* u_{21}^*}{4} - \frac{u_{22} u_{11}^*}{8} \cos(\phi_{22}) \right) \right] - \frac{\partial}{\partial y} \left[ 2 \text{Im} \left( \frac{v_{10}^* v_{21}^*}{4} + \frac{v_{22} v_{11}^*}{8} \cos(\phi_{22}) \right) \right] \\
&\left. \right. \\
\end{align*}

Phase shift due to interaction between 11 and 20

\begin{align*}
&+ 2 \text{Im} \left[ \frac{1}{2} (w_{11}^* \frac{\partial u_{11}^*}{\partial x} + v_{11}^* \frac{\partial v_{11}^*}{\partial x}) + \frac{1}{2} (w_{10}^* v_{21}^* \frac{\partial u_{11}^*}{\partial y} + v_{10}^* v_{21}^* \frac{\partial v_{11}^*}{\partial y}) \right] \\
&- \frac{\gamma}{2} \left( u_{10}^* v_{21}^* + v_{10}^* w_{21}^* \right) \cos(\phi_{21}) \\
&\left. \right. \\
\end{align*}

Phase shift due to interaction between 11 and 21

\begin{align*}
&- 2 \text{Im} \left[ \frac{1}{2} (w_{11}^* \frac{\partial w_{11}^*}{\partial x} + v_{11}^* \frac{\partial w_{11}^*}{\partial y}) + \frac{1}{2} (w_{11}^* \frac{\partial w_{11}^*}{\partial y} + v_{11}^* \frac{\partial w_{11}^*}{\partial y}) \right] \\
&\left. \right. \\
\end{align*}

Phase shift due to interaction between 11 and 10

\begin{align*}
&- 2 \text{Im} \left[ \frac{1}{2} (w_{11}^* \frac{\partial u_{20}^*}{\partial x} + v_{11}^* \frac{\partial v_{20}^*}{\partial x}) + \frac{1}{2} (w_{11}^* \frac{\partial v_{20}^*}{\partial y} + v_{11}^* \frac{\partial w_{20}^*}{\partial y}) \right] \\
&\left. \right. \\
\end{align*}
Phase shift due to interaction between $11$ and $22$ \( (2.28) \)

The terms from the bottom to the twelfth line in equation (2.28) describe the non-linear variation in the phase angle between the fundamental and subharmonic wave modes, and the phase angle equations of the remaining modes can be found in the Appendix E.

2.4 Shape Assumptions

Before proceeding any further assumptions must be made regarding the shape of the self-similar mean flow profile and the nature of the spatial dependence of the two- and three-dimensional wave modes.

2.4.1 Mean Flow

The profile of the mean flow streamwise velocity component will be assumed to have the shape of a hyperbolic tangent function. This profile has been verified experimentally (Wygnanski & Fiedler 1970; Ho & Huang 1982) and used by many researchers (Kelly 1967; Monkewits & Huerre 1982; Nikitopoulos & Liu 1987). It is completely satisfactory except in the immediate vicinity of the splitter plate where the flow has a wake type profile. However, Miksad (1972) showed that this region does not affect the subsequent development of the shear layer significantly. Hence in the developed mixing region, the mean velocity profile will be taken as

\[
U = 1 - R \tanh(\eta) \tag{2.29}
\]
where $R = (U_{-\infty} - U_{+\infty})/(U_{-\infty} + U_{+\infty})$ is the velocity ratio of the shear layer, $\eta = y/\delta(x)$ is the rescaled cross-stream coordinate, and $\delta(x)$ is half the local shear layer maximum slope thickness measured in units of initial thickness $\delta_0$. The maximum slope thickness $\delta$ is defined (Winant & Browand 1974) as

$$\delta_w = \frac{U(-\infty) - U(+\infty)}{(dU/dy)_{max}} = 2\delta$$

(2.30)

Using $\eta$ instead of $y$ allows us to consider non-parallel effects implicitly because $\delta$ varies nonlinearly with $x$ as shown experimentally by Ho & Huang (1982) and theoretically by Nikitopoulos & Liu (1987). In this sense our analysis can be considered as weakly non-parallel. The mean velocity and velocity gradient of the profile family expressed by equation (2.29) for various velocity ratios $R$ are shown in Figure 2.1 and Figure 2.2 respectively. Nikitopoulos & Liu (1987) and Monkewitz & Huerre (1982) investigated the mixing layer for $R = 0.31$, Monkewitz & Huerre (1982) and Huang (1985) for $R = 0.68$ and Strykowski & Niccum (1992) for $R = 1$.

2.4.2 Large-scale Coherent Structure

Following earlier work by Stuart (1958), Ko, Kubota & Lees (1970), Liu (1981) and Nikitopoulos & Liu (1984, 1987, 1990), the Fourier amplitudes $u_{mn}$, $v_{mn}$, $w_{mn}$ and $p_{mn}$ in equations (2.13) and (2.14) are assumed to be separable into an unknown
Figure 2.1: Mean velocity profile with various velocity ratios R
Figure 2.2: Gradient of the mean velocity profile with various velocity ratios $R$.
finite complex amplitude and corresponding vertical shape functions:

\[
\begin{bmatrix}
 u_{mn}(x, \eta) \\
 v_{mn}(x, \eta) \\
 p_{mn}(x, \eta) \\
 w_{mn}(x, \eta)
\end{bmatrix} = A_{mn}(x) 
\begin{bmatrix}
 \dot{u}_{mn}(\eta) \\
 \dot{v}_{mn}(\eta) \\
 \dot{p}_{mn}(\eta) \\
 \dot{w}_{mn}(\eta)
\end{bmatrix}. \tag{2.31}
\]

The shape functions \( \dot{u}_{mn}, \dot{v}_{mn}, \dot{p}_{mn} \) and \( \dot{w}_{mn} \) are assumed to be identical to the eigenfunctions of the local linear stability solution for the wave mode \( mn \) (Alper & Liu 1978; Nikitopoulos & Liu 1984, 1987, 1990). It has been shown experimentally by Weisbrot & Wygnanski (1988) that this assumption is valid for both forced and unforced shear layers. The shape functions have an implicit dependence on \( x \) resulting from the local scaling of length with the maximum slope thickness \( \delta(x) \). In the general three-dimensional case they are also dependent on the dimensionless wave number \( \gamma \) and dimensionless frequency \( \beta \).

For our nonlinear analysis the amplitude \( A_{mn} \) can be written as

\[
A_{mn} = |A_{mn}(x)| e^{i\psi_{mn}(x)} \tag{2.32}
\]

in terms of its magnitude \( |A_{mn}(x)| \) and phase angle \( \psi_{mn}(x) \). The amplitude \( A_{mn} \) (magnitude and phase angle) is going to be determined jointly with the maximum slope thickness \( \delta(x) \) from the mean energy equation, wave mode energy equations and phase angle equations. Previous investigations employing nonlinear or weakly nonlinear analyses have used factors, involving the streamwise complex wave number \( \alpha \), and equal to \( e^{i\alpha x} \) or \( e^{i \int_0^x \alpha_r d\xi} \) (Crighton & Gaster 1976) in the amplitude definitions similar to the equation (2.32), to take into account the behavior of the
amplitude accepted for the linear stability analysis which is valid while the amplitude is still very small and nonlinear effects are not important. We shall not introduce such factors in here since we have governing equations for both the magnitude and phase of the complex amplitude.

Before we proceed any further it is appropriate to present the assumed form for the large-scale structure that has thus far been introduced step by step through equations (2.13), (2.14), (2.31) and (2.32). Each large-scale wave mode thus has the form:

\[
\begin{bmatrix}
\tilde{u}_{mn} \\
\tilde{v}_{mn} \\
\tilde{p}_{mn}
\end{bmatrix}
= |A_{mn}(x)| e^{i\psi_{mn}(x)} e^{-i\beta_{mn}t} \cos(n\gamma z + \phi_{mn}) + \text{c.c.} \tag{2.33}
\]

and

\[
\tilde{w}_{mn} = |A_{mn}(x)| e^{i\psi_{mn}(x)} \tilde{w}_{mn}(\eta)e^{-i\beta_{mn}t} \sin(n\gamma z + \phi_{mn}) + \text{c.c.} \tag{2.34}
\]

where \(m \in [1, 2]\) and \(n \in [0, 2]\) with the physical interpretation given in section 2.1. \(\tilde{u}_{mn}(\eta), \tilde{v}_{mn}(\eta), \tilde{p}_{mn}(\eta)\) and \(\tilde{w}_{mn}(\eta)\) are the complex eigenfunctions to be calculated by the local linear stability analysis. The phase angles \(\psi_{mn}(x)\) play a particularly important role in the nonlinear interactions between various wave modes which are dependent on the corresponding phase angle differences. In Nikitopoulos & Liu (1987), these phase angles were assumed to vary according to linear stability along the streamwise direction. In here, we will give the initial phase angles and calculate them along the streamwise direction by solving the wave mode phase angle equations (in the same manner as Lee 1988; Nikitopoulos & Liu 1990).
All fundamental modes have the same frequency $\beta_f$ and all subharmonic modes have the frequency $\beta_s$, the latter being one half of the fundamental frequency.

$$\beta_f = \beta_{20} = \beta_{21} = \beta_{22} \quad (2.35)$$

$$\beta_s = \beta_{10} = \beta_{11} \quad (2.36)$$

$$\beta_f = 2\beta_s \quad (2.37)$$

Because the shape functions are the complex eigenfunctions of the linear stability analysis, there is a complex constant which has to be fixed at a certain reference point. So we need normalization conditions in order for the eigenfunctions to be used consistently for the nonlinear problem later. We will normalize the eigenfunctions of each wave mode $mn$ in a way that allows us to relate the amplitude $|A_{mn}(x)|$ to the corresponding average energy contents across the shear layer. Namely:

$$E_{mn}(x) = \frac{1}{2} \int_{-\infty}^{\infty} (\bar{u}_{mn}^2 + \bar{v}_{mn}^2 + \bar{w}_{mn}^2) dy = |A_{mn}(x)|^2 \delta(x) \quad (2.38)$$

Thus the eigenfunctions of 2-dimensional modes are normalized to satisfy

$$\int_{-\infty}^{\infty} [\hat{u}_{mn}^2 + \hat{v}_{mn}^2] d\eta = 1, \quad (2.39)$$

and those of 3-dimensional modes

$$\int_{-\infty}^{\infty} [\hat{u}_{mn}^2 + \hat{v}_{mn}^2 + \hat{w}_{mn}^2] d\eta = 2. \quad (2.40)$$

### 2.5 Governing Integral Equations

Having established the shape assumptions on the mean flow and the large-scale structure we will proceed to further develop the mean and the large-scale modal
mechanical energy equations and those for the modal phase angles presented in section 2.2 and 2.3. Upon substitution of equations (2.31) and (2.32) into these equations we will integrate them across the shear layer (from $-\infty$ to $+\infty$) and recover a set of nonlinear coupled ordinary differential equations for the thickness of the shear layer $\delta(x)$, the modal energy contents $E_{mn}(x)$ and phases $\psi_{mn}(x)$.

### 2.5.1 The Integral Energy Equation for the Mean Flow

After integrating equation (2.22) across the shear layer and using the usual boundary layer approximations and the stretched coordinate we obtain

$$
\frac{1}{2} \frac{d\delta}{dx} \left[ \int_{-\infty}^{0} U(U^2 - U_{-\infty}^2) d\eta + \int_{0}^{\infty} U(U^2 - U_{+\infty}^2) d\eta \right] = \\
- \frac{1}{Re_\delta} \int_{-\infty}^{\infty} \left( \frac{\partial U}{\partial \eta} \right)^2 d\eta + \int_{-\infty}^{\infty} 2Rel(w_{mn}^*, v_{mn}^*) \frac{\partial U}{\partial \eta} d\eta 
$$

(2.41)

where $Re_\delta$ is based on $\delta$, and

$$
Rel(w_{mn}^*, v_{mn}^*) = |A_{10}|^2 Rel(\hat{u}_{10} \hat{v}_{10}^*) + |A_{20}|^2 Rel(\hat{u}_{20} \hat{v}_{20}^*) + \\
|A_{11}|^2 Rel(\hat{u}_{11} \hat{v}_{11}^*) + |A_{21}|^2 Rel(\hat{u}_{21} \hat{v}_{21}^*) + |A_{22}|^2 Rel(\hat{u}_{22} \hat{v}_{22}^*) 
$$

(2.42)

The last term of the right-hand-side of equation (2.41) represents the nonlinear interactions of each wave mode with the mean flow. The sign of this term therefore controls the direction of the energy exchange. For instance, if

$$
-2Rel(\hat{u}_{mn} \hat{v}_{mn}^*) \frac{\partial U}{\partial y} > 0, 
$$

(2.43)

then energy is transferred from the mean flow to the large-scale wave modes $mn$ and if this term has the opposite sign, the energy is transferred from the large-scale wave mode $mn$ to the mean flow.
2.5.2 Integral Wave Mode Energy Equations

To help understand the physical mechanisms of the wave mode interactions, we obtained the mechanical energy equations for the five large-scale coherent wave modes. These equations which were derived in section 2.2, will be integrated across the shear layer. Thus all the transport terms involving $\frac{\partial}{\partial y}$ will vanish because of the boundary conditions. We will then adopt the following approximation for partial streamwise derivatives that are in the integrants, namely:

$$\frac{\partial}{\partial x}(u_{mn}, v_{mn}, w_{mn}) = i\alpha_{mn}(u_{mn}, v_{mn}, w_{mn})$$  \hspace{1cm} (2.44)

This is consistent with the local linear stability approximation that has been used for the mode eigenfunctions. It has been shown to be adequate and has been used in many previous works (Mankbadi 1985; Lee 1988; Nikitopoulos & Liu 1987). Finally, we will apply the shape assumptions for $u_{mn}$, $v_{mn}$, and $w_{mn}$ outlined in the previous section. The resulting integral equations for all modes have the following general form:

(i). For two-dimensional modes ($n=0$)

\[ \frac{d|A_{m0}|^2}{dx} \left[ \int_{-\infty}^{\infty} (|\hat{u}_{m0}|^2 + |\hat{v}_{m0}|^2)Udy \right. \]
\[ + \left. \int_{-\infty}^{\infty} 2\text{Rel}(\hat{u}_{m0}\hat{v}_{m0})dy \right] = \]
\[ - |A_{m0}|^2 \int_{-\infty}^{\infty} 2\text{Rel}(\hat{u}_{m0}\hat{v}_{m0}) \frac{\partial U}{\partial y}dy \]
\[ - \frac{2|A_{m0}|^2}{Re_0} \left[ \frac{\alpha_{m0}}{\delta} + \int_{-\infty}^{\infty} (|\hat{u}_{m0}|^2 + |\hat{v}_{m0}|^2)dy \right] \]
\[ + 2\text{Rel}[EWW_{m0}] \]  \hspace{1cm} (2.45)
(ii). For three-dimensional modes (n \neq 0)

\[
\frac{d|A_{mn}|^2}{dx} \int_{-\infty}^{\infty} \frac{1}{2} (|\dot{u}_{mn}|^2 + |\dot{v}_{mn}|^2 + |\dot{w}_{mn}|^2) Ud\gamma = \\
+ \int_{-\infty}^{\infty} \text{Re} \{\dot{u}_{mn} \dot{\nu}_{mn}/2\} d\gamma = \\
-|A_{mn}|^2 \int_{-\infty}^{\infty} \text{Re} \{\dot{u}_{mn} \dot{\nu}_{mn}\} \frac{\partial U}{\partial \gamma} d\gamma = \\
-\frac{|A_{mn}|^2}{R \in \delta} \left[ \frac{2|\alpha_{mn}|^2}{\delta} + \frac{2\gamma^2 n^2}{\delta} \right] \\
+ \int_{-\infty}^{\infty} \left[ \frac{\partial \dot{u}_{mn}}{\partial \gamma} \right]^2 + \left[ \frac{\partial \dot{v}_{mn}}{\partial \gamma} \right]^2 + \left[ \frac{\partial \dot{w}_{mn}}{\partial \gamma} \right]^2 d\gamma \\
+ 2 \text{Re} \{EWW_{mn}\} \tag{2.46}
\]

where $EWW_{mn}$ represents collectively the interaction terms between mode mn and all others.

These are given for each mode separately as follows

\[
EWW_{10} = -S_{10}^{10} D_{20} - S_{10}^{21} D_{11} \cos(\phi_{21}) - S_{10}^{11} D_{21} \cos(\phi_{21}) \tag{2.47}
\]

\[
EWW_{20} = S_{10}^{10} D_{20} + S_{11}^{11} D_{20} \tag{2.48}
\]

\[
EWW_{11} = S_{10}^{21} D_{11} \cos(\phi_{21}) - S_{11}^{11} D_{20} - S_{10}^{10} D_{21} \cos(\phi_{21}) - S_{11}^{11} D_{22} \cos(\phi_{22}) \tag{2.49}
\]

\[
EWW_{21} = S_{10}^{11} D_{21} \cos(\phi_{21}) + S_{11}^{10} D_{21} \cos(\phi_{21}) \tag{2.50}
\]

\[
EWW_{22} = S_{11}^{11} D_{22} \cos(\phi_{22}) \tag{2.51}
\]

The notation $S_{ij}^{pq} D_{kl}$ has been chosen to remind us that an individual energy interaction between mode ij and kl is generated by a wave induced stress $S_{ij}^{pq}$, which results from the correlation between modes ij and pq, acting on a rate of strain, $D_{kl}$,
of mode kl. The mode-mode interaction terms involve the amplitudes and phases of the interacting waves:

\[ S_{ij}^{pq} \Delta_{kl} = |A_{ij}| |A_{pq}| \left| A_{kl} \right| e^{-i(-)^{l}\psi_{ij} + (-)^{k}\psi_{kl} + (-)^{r}\psi_{pq}} \Sigma_{ij}^{pq} \Delta_{kl}. \]  

(2.52)

where \( \Sigma_{ij}^{pq} \Delta_{kl} \) are integral coefficients presented in Appendix F. These integral coefficients are of great significance in determining the direction of the energy flow between modes together with the phase angles. The strength of the energy exchange depends primarily on the magnitude of the amplitudes of the modes involved. All the transport terms in the modal energy equations except those due to the pressure field have been neglected for the sake of further simplification, since they are considerably smaller than the other terms, particularly the wave-wave interaction ones. This is fairly common practice justified by Nikitopoulos & Liu (1987) and Lee (1988) among others. It should be noted that summing equations (2.45) and (2.46) over all modes will render the balance for the total energy content of the large-scale structure the wave-wave interaction terms canceling each other out.

2.5.3 Integral Wave Mode Phase Angle Equations

The same procedure applied to the original energy equations is applied to the phase angle equations derived in section 2.2. The result is a set of integral equations governing the evolution of the phases. The general form of these equations is:

(i). For two-dimensional modes \((n=0)\)

\[ \beta_{m0}|A_{m0}|^2 \delta - \frac{d\psi_{m0}}{dx} |A_{m0}|^2 \int_{-\infty}^{\infty} [ |\dot{u}_{m0}|^2 + |\dot{v}_{m0}|^2 ] U dy = \]
\[ -|A_{m0}|^2 \int_{-\infty}^{\infty} \text{Im}(\hat{u}_{m0}\hat{\nu}_{m0}) \frac{\partial U}{\partial y} dy \]

\[ -\frac{d|A_{m0}|^2}{dx} \int_{-\infty}^{\infty} \text{Im}(\hat{u}_{m0}\hat{\nu}_{m0}) dy + 2\frac{\alpha_{rm0}\alpha_{im0}|A_{m0}|^2}{Re_0} \]

\[ + \text{Im}[PW_{Wm0}] \quad (2.53) \]

(ii). For three-dimensional modes \((n \neq 0)\)

\[ \beta_{mn}|A_{mn}|^2 \delta - \frac{1}{2} \frac{d\psi_{mn}}{dx}|A_{mn}|^2 \int_{-\infty}^{\infty} [\hat{u}_{mn}]^2 + [\hat{v}_{mn}]^2 + [\hat{w}_{mn}]^2 U dy = \]

\[ -\frac{1}{2} |A_{mn}|^2 \int_{-\infty}^{\infty} \text{Im}(\hat{u}_{mn}\hat{\nu}_{mn}) \frac{\partial U}{\partial y} dy \]

\[ -\frac{1}{2} \frac{d|A_{mn}|^2}{dx} \int_{-\infty}^{\infty} \text{Im}(\hat{u}_{mn}\hat{\nu}_{mn}) dy + \frac{\alpha_{rmn}\alpha_{imn}|A_{mn}|^2}{Re_0} \]

\[ + \text{Im}[PW_{Wmn}] \quad (2.54) \]

where \(PW_{Wmn}\) represents collectively the phase changes induced on mode \(mn\) because of the interaction with the remaining modes.

These are given for each mode separately

\[ PW_{W10} = -S_{10}^{10} D_{20} - S_{10}^{21} D_{11} \cos(\phi_{21}) - S_{10}^{11} D_{21} \cos(\phi_{21}) \quad (2.55) \]

\[ PW_{W20} = -S_{10}^{10} D_{20} - S_{11}^{11} D_{20} \quad (2.56) \]

\[ PW_{W11} = S_{10}^{21} D_{11} \cos(\phi_{21}) - S_{11}^{11} D_{20} - S_{10}^{10} D_{21} \cos(\phi_{21}) - S_{11}^{11} D_{22} \cos(\phi_{22}) \quad (2.57) \]

\[ PW_{W21} = -S_{10}^{11} D_{21} \cos(\phi_{21}) - S_{11}^{10} D_{21} \cos(\phi_{21}) \quad (2.58) \]

\[ PW_{W22} = -S_{11}^{11} D_{22} \cos(\phi_{22}) \quad (2.59) \]

It should be noted from the signs of these terms that phase changes induced on the subharmonic modes by their interaction with fundamental modes are matched
in direction by the phase change in the fundamental. Consequently the mutual interaction between fundamental and subharmonic modes may not affect their relative phases significantly. Contrary to this the relative phase of the two subharmonic modes can be greatly affected by their interaction with each other (10 and 11). The magnitude of the relative phase between modes also depends on the wave amplitudes. Depending on their magnitude, considerable relative phase shifts can be effected despite the fact that wave-wave interactions tend to shift individual wave phases in the same directions. The even (fundamental) modes do not have a mechanism to interact with each other unless higher order modes are present, and therefore they do not influence each other’s amplitude or phase evolution directly. Interactions involving three-dimensional mode pairs are also considerably affected by the spanwise phase angles $\phi_{21}$ and $\phi_{22}$.

Tables 2.1 and 2.2 together with Figure 2.3 give the picture of the interactions between the various modes. Each subharmonic mode interacts directly with the two-dimensional fundamentals. The three-dimensional subharmonic has the privilege of interacting with all fundamentals while the two-dimensional subharmonic interacts with the two-dimensional and long wave-length three-dimensional fundamentals. The subharmonics also communicate with each other. The short-wave three-dimensional fundamental can also only be influenced by the long-wave three-dimensional subharmonic and none other. So mode 22 can only receive energy from the other modes, if the phases are favorable, through mode 11. It is important to
note that the long-wave fundamental $21$ acts as an agent for the direct interaction between the two-dimensional and three-dimensional subharmonics. If $21$ wave mode is very weak, the interaction between them is going to be severely weakened. Mode $11$ can interact with the two fundamentals ($20$ and $22$) without requiring the presence of other modes, while the two-dimensional subharmonic ($10$) can do this only with the two-dimensional fundamental ($20$).

To gain a clearer picture of the nonlinear evolution of the two-dimensional and three-dimensional modes as well as the mean flow with which they all interact, we must solve the governing equations developed in this chapter. In order to do this we must first be able to calculate all the integral coefficients in these equations. This requires that we solve the two- and three-dimensional, viscous linear stability problem for each mode.
Table 2.1: Nonlinear interactions between the fundamentals and subharmonics in the integral wave mode energy equations

<table>
<thead>
<tr>
<th>Mode</th>
<th>$-S_{10}^{10} D_{20}$</th>
<th>$-S_{10}^{11} D_{21}$</th>
<th>$-S_{10}^{21} D_{11}$</th>
<th>$S_{11}^{10} D_{20}$</th>
<th>$S_{11}^{21} D_{11}$</th>
<th>$-S_{11}^{10} D_{21}$</th>
<th>$-S_{11}^{11} D_{22}$</th>
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<td>20</td>
<td>$S_{10}^{10} D_{20}$</td>
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<td>$S_{10}^{21} D_{11}$</td>
<td>$-S_{11}^{10} D_{20}$</td>
<td>$-S_{11}^{10} D_{21}$</td>
<td>$-S_{11}^{11} D_{22}$</td>
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<td>21</td>
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<td>$S_{10}^{10} D_{21}$</td>
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<td>$S_{11}^{10} D_{21}$</td>
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<tr>
<td>22</td>
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<td>$S_{11}^{10} D_{22}$</td>
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<table>
<thead>
<tr>
<th>Angle</th>
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<th>$\cos(\phi_{21})$</th>
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<th>$\cos(\phi_{22})$</th>
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Table 2.2: Nonlinear interactions between the fundamentals and subharmonics in the integral phase angle equations

<table>
<thead>
<tr>
<th>Mode</th>
<th>$-S_{10}^{10} D_{20}$</th>
<th>$-S_{10}^{11} D_{21}$</th>
<th>$-S_{10}^{21} D_{11}$</th>
<th>$-S_{11}^{10} D_{20}$</th>
<th>$S_{11}^{21} D_{11}$</th>
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<th>$-S_{11}^{11} D_{22}$</th>
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<td>$-S_{11}^{10} D_{21}$</td>
<td>$-S_{11}^{11} D_{22}$</td>
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<td>$S_{11}^{10} D_{22}$</td>
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<th>Angle</th>
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Figure 2.3: Diagram of the nonlinear energy exchanges among the wave modes.
Chapter 3

The Hydrodynamic, Viscous, Linear Stability of the Three-Dimensional Shear Layer

This chapter is concerned with the hydrodynamic instability of the free shear layer. The development of mixing layers downstream of a splitter plate is initially dominated by a linear instability mechanism. A great deal of work has been done on this subject and for general reviews of the hydrodynamic stability of parallel flows such as the free shear layer, the reader is referred to Lin (1955), Betchov & Criminale (1967) and Drazin & Reid (1981) among others. The linear viscous stability theory has received little attention in comparison with that of the inviscid theory. This is primarily due to the fact that inviscid theory is adequate to reveal the conditions under which instability occurs, particularly for flows with inflectional profiles such as the free shear layer. Morris (1976) studied the stability of three axisymmetric jet profiles which represented the flow field of an incompressible jet. Lee (1988) investigated the eigenvalues and eigenfunctions of the stability for a round jet by
solving the eigenvalue problem with a shooting solution matching method instead of one-directional integration method.

The instability mechanism of free boundary layers such as the shear layer is an inviscid one, caused by induction effects, and viscosity has only a damping influence (Lin 1955). For large Reynolds numbers the flow in a free boundary layer is nearly parallel. Thus results obtained by means of the inviscid linearized stability theory for unidirectional flow may then be applied to free boundary layers at large Reynolds numbers. Stability calculations for free shear layers at finite Reynolds numbers by Lessen (1950), Esch (1957) and Betchov & Szewczyk (1963) have in fact shown that for large Reynolds numbers the neutral curve approaches asymptotically the neutral value predicted by the inviscid theory and that for smaller Reynolds numbers the amplification of disturbances is always smaller than the inviscid one. The same result was obtained by Tatsumi & Kakutani (1958), who dealt with the instability of a plane jet.

So the solution of the inviscid stability problem is usually sufficient for the understanding of the basic stability characteristics of the free shear layer. Numerous studies have been carried out investigating the inviscid linear spatial and temporal stability of free shear layers. Michalke (1965,1969) has studied both the two- and three-dimensional inviscid problem while Monkewitz and Huerre (1982) have provided inviscid solutions for different shear layer profiles. Numerous solutions are also known for the temporal problem which is in general easier to solve. Because of
the considerable attention payed to linear stability analysis since its justification by the landmark experiments of Schubauer & Skramstad (1947) a number of methods have been developed to overcome the numerical problems that arise when trying to obtain solutions particularly for semi-infinite or infinite flow domains. The available techniques are well summarized in Betchov & Criminale (1967) and Drazin & Reid (1981).

Hydrodynamic stability has been recognized as one of the fundamental mechanism to understand the transition from laminar to turbulent flow. It is concerned with when and how laminar flows break down, their subsequent development, and their eventual transition to turbulence. It has many applications in engineering, in meteorology and oceanography, and in astrophysics and geophysics. One may say that instability occurs because there is some disturbance of equilibrium of external forces, inertia and viscous stresses of a fluid. The small disturbance may upset this equilibrium and create instability. The instability can be depicted in terms of interactions of vortex lines, which are convected and stretched by the motion of the fluid.

Our interest here lies in obtaining solutions for the viscous spatial two- and three-dimensional stability corresponding to the classical self-similar velocity profile of a free shear layer, which we will then use in solving the nonlinear problem as formulated in the previous chapter. The doubly unbounded domain combined with the effects of viscosity make the numerical solution of the viscous linear problem
quite challenging. In the following sections we will briefly outline the classical formulation of the problem, describe the methods used to obtain solutions and present some results of our calculations.

### 3.1 Governing Equations

We consider the two- and three-dimensional free shear flow of an incompressible viscous fluid and assume that the unperturbed flow has a sole mean velocity component parallel to the streamwise axis with a self-similar profile when presented in terms of the usual boundary layer, stretched coordinate $\eta = y/\delta$ where $\delta$ is the local half-maximum slope thickness of the shear layer as defined previously. The flow local Reynolds number is defined as $Re = U\delta/\nu$.

We superpose a small three-dimensional velocity perturbation with components $\tilde{u}, \tilde{v}, \tilde{w}$ and $\tilde{p}$, functions of $x, \eta, z$ and time $t$. For the spatial linear stability problem, we can assume

$$\tilde{u}(x, \eta, z, t) = \tilde{u}(\eta)e^{i\alpha(x-ct)} \cos(\gamma z), \quad (3.1)$$

$$\tilde{v}(x, \eta, z, t) = \tilde{v}(\eta)e^{i\alpha(x-ct)} \cos(\gamma z), \quad (3.2)$$

$$\tilde{p}(x, \eta, z, t) = \tilde{p}(\eta)e^{i\alpha(x-ct)} \cos(\gamma z), \quad (3.3)$$

and

$$\tilde{w}(x, \eta, z, t) = i\tilde{w}(\eta)e^{i\alpha(x-ct)} \sin(\gamma z) \quad (3.4)$$
where $\alpha$ is the complex wave number, $c$ is the phase velocity, and $\beta$ is the real
frequency equal to $\alpha c$. The imaginary part of the wave number $\alpha(=\alpha_r + i\alpha_i)$
determines the stability of the flow; the flow is stable if $\alpha_i$ has a positive value,
neutral if $\alpha_i$ is equal to zero, and unstable if $\alpha_i$ has a negative value. $\dot{u}(\eta)$, $\dot{v}(\eta)$,
$\dot{w}(\eta)$ and $\dot{p}(\eta)$ are complex amplitude functions and $\gamma$ is the spanwise wave number.

The eigenfunctions $\dot{u}$, $\dot{v}$, $\dot{w}$ and $\dot{p}$ with the normalization discussed in the previous
chapter are the same as the vertical shape functions $\dot{u}_{mn}$, $\dot{v}_{mn}$, $\dot{w}_{mn}$ and $\dot{p}_{mn}$ for the
corresponding $mn$ wave mode. In the linear region, the amplitude of the disturbance
is proportional to $e^{-\alpha_i \eta}$.

When we substitute equations (3.1) ~ (3.4) into the Navier-Stokes equations in
boundary layer form and linearize, we will get

$$i \alpha \dot{u} + \frac{d \dot{v}}{d \eta} + i \gamma \dot{w} = 0,$$  \hspace{1cm} (3.5)

$$i \alpha (U - \frac{\beta}{\alpha}) \dot{u} + \frac{d U}{d \eta} \dot{v} + i \alpha \dot{p} = \frac{1}{Re} (\dot{u}'' - \overline{\alpha^2 \dot{u}}),$$  \hspace{1cm} (3.6)

$$i \alpha (U - \frac{\beta}{\alpha}) \dot{v} + \frac{d \dot{p}}{d \eta} = \frac{1}{Re} (\dot{v}'' - \overline{\alpha^2 \dot{v}}),$$  \hspace{1cm} (3.7)

$$i \alpha (U - \frac{\beta}{\alpha}) \dot{w} + i \gamma \dot{p} = \frac{1}{Re} (\dot{w}'' - \overline{\alpha^2 \dot{w}}),$$  \hspace{1cm} (3.8)

where $\overline{\alpha^2} = \alpha^2 + \gamma^2$.

If we are considering a two-dimensional disturbance ($\gamma = 0$), these equations (3.5)
~ (3.8) can be combined into one fourth-order ordinary differential equation for
one unknown function which is the well known Orr-Sommerfeld equation.

For the general three dimensional disturbance the governing equations (3.5) ~
(3.8) constitute a sixth-order system for the variables $\dot{u}$, $\dot{w}'$, $\dot{v}$, $\dot{p}$, $\dot{w}$, $\dot{w}'$ and the
corresponding boundary conditions are:

\[ \text{at } \eta \rightarrow \pm \infty \quad \dot{u}, \dot{v}, \dot{p}, \dot{w} \rightarrow 0 \]  

(3.9)

For the spatial problem, the system of ordinary differential equations (3.5) ~ (3.8) together with the boundary conditions given by equation (3.9) poses an eigenvalue problem with \( \alpha \) as the complex eigenvalue, \( \dot{u}(\eta), \dot{v}(\eta), \dot{p}(\eta) \) and \( \dot{w}(\eta) \) as eigenfunctions, and \( \beta, \gamma \) and Re as real parameters. To obtain optimum solutions, we have also used an alternate formulation. This is obtained by multiplying equation (3.6) by \( \alpha \) and equation (3.8) by \( \gamma \) and adding, and then multiplying equation (3.8) by \( \alpha \) and equation (3.6) by \( \gamma \) and subtracting, to get equations for the variables \( \alpha \dot{u} + \gamma \dot{w} \) and \( \alpha \dot{w} - \gamma \dot{u} \):

\[
i\alpha(U - \beta/\alpha)(\alpha \dot{u} + \gamma \dot{w}) + \alpha \left( \frac{dU}{d\eta} \dot{v} + i(\bar{\alpha}^2)\dot{p} \right) = \frac{1}{\text{Re}} [D^2 - (\bar{\alpha}^2)](\alpha \dot{u} + \gamma \dot{w}) \tag{3.10}
\]

\[
i\alpha(U - \beta/\alpha)(\gamma \dot{u} - \alpha \dot{w}) + \gamma \left( \frac{dU}{d\eta} \dot{v} \right) = \frac{1}{\text{Re}} [D^2 - (\bar{\alpha}^2)](\gamma \dot{u} - \alpha \dot{w}), \tag{3.11}
\]

with \( \bar{\alpha}^2 = \alpha^2 + \gamma^2, D(\alpha \dot{u} + \gamma \dot{w}) = \alpha d\dot{u}/d\eta + \gamma d\dot{w}/d\eta \) and \( D(\gamma \dot{u} - \alpha \dot{w}) = \gamma d\dot{u}/d\eta - \alpha d\dot{w}/d\eta \).

Equations (3.5), (3.7) and (3.10) are then a fourth-order system for the dependent variables \( \alpha \dot{u} + \gamma \dot{w}, D(\alpha \dot{u} + \gamma \dot{w}), \dot{v} \) and \( \dot{p} \). Therefore, we can determine the eigenvalue \( \alpha \) from the fourth-order system of equations, and if the eigenfunctions \( \dot{u} \) and \( \dot{w} \) are needed, they are obtained by solving the second-order equation (3.11). This formulation has been used to initially determine the \( \alpha \), while the six equation
formulation has been used to obtain the final and complete solution. We found this combined use of both formulations gives accurate solutions in a more efficient way.

Since we seek solutions for both the two- and the three-dimensional problems we will present the solution procedure for both of them in parallel. For the numerical computations, equations (3.5) ~ (3.8) may be written as four coupled first-order differential equations for four dependent variables for two- or three-dimensional disturbances, and six first-order differential equations for six dependent variables for the three-dimensional disturbances, as mentioned before. If we use

\begin{align}
D\dot{u} &\equiv \frac{d\dot{u}}{d\eta}, \\
D\dot{w} &\equiv \frac{d\dot{w}}{d\eta},
\end{align}

(3.12) (3.13)

and if we define the dependent variable vector \( \mathbf{V} \) as

\[
\mathbf{V} = \begin{bmatrix}
\dot{u} \\
D\dot{u} \\
\dot{v} \\
\dot{p}
\end{bmatrix}
\]

(3.14)

and for the three dimensional disturbance

\[
\mathbf{V} = \begin{bmatrix}
(\alpha \dot{u} + \gamma \dot{w}) \\
D(\alpha \dot{u} + \gamma \dot{w}) \\
\dot{v} \\
\dot{p}
\end{bmatrix},
\]

(3.15)
or

\[
V = \begin{bmatrix}
\hat{u} \\
D\hat{u} \\
\hat{v} \\
\hat{p} \\
\hat{w} \\
D\hat{w}
\end{bmatrix}
\]

(3.16)

depending on the formulation used, then equations (3.5) \sim (3.7) can be written as

\[
\frac{dV}{d\eta} = MV
\]

(3.17)

where \(M\) is a \(4 \times 4\) matrix for the two-dimensional case:

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & \text{Re} \frac{dU}{d\eta} & i\alpha \text{Re} \\
-i\alpha & 0 & 0 & 0 \\
0 & -\frac{i\alpha}{\text{Re}} & -\frac{1}{\text{Re}} \Gamma & 0
\end{bmatrix}
\]

(3.18)

with \(\Gamma = \alpha^2 + i\alpha \text{Re}(U - \frac{\partial}{\partial x})\), and either a \(4 \times 4\):

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & \alpha \text{Re} \frac{dU}{d\eta} & i\alpha^2 \text{Re} \\
0 & 0 & 0 & 0 \\
0 & -\frac{i}{\text{Re}} & -\frac{\Omega}{\text{Re}} & 0
\end{bmatrix}
\]

(3.19)

or a \(6 \times 6\) matrix for the three-dimensional case:

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \text{Re} \frac{dU}{d\eta} & i\alpha \text{Re} & 0 & 0 \\
0 & 0 & 0 & 0 & -i\gamma & 0 \\
0 & -\frac{i\alpha}{\text{Re}} & -\frac{1}{\text{Re}} \Omega & 0 & 0 & -\frac{i\gamma}{\text{Re}} \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & i\gamma \text{Re} & \Omega & 0
\end{bmatrix}
\]

(3.20)
with \( \Omega = \alpha^2 + i\alpha Re(U - \frac{\beta}{a}) \). The matrix 3.19 represents the transformed four equation system and the matrix 3.20 represents the complete six equation system.

The eigenvalue problem posed by ordinary differential equations (3.17) together with the boundary conditions (3.9) will be solved numerically for the eigenvalue \( \alpha \) and the eigenfunctions \( \hat{u}, \hat{v}, \hat{w} \) and \( \hat{p} \). In order to do this we need to look at the asymptotic behavior of the solutions at the boundaries of our infinite domain.

### 3.2 Asymptotic Solutions

In the numerical solution of eigenvalue problems on infinite intervals, a common method of proceeding is to replace the infinite interval, \(( -\infty, +\infty )\), by a finite one, say \((-\eta_\infty, \eta_\infty)\), where \( \eta_\infty \) is sufficiently large. The main problem then is to determine the appropriate boundary conditions to be imposed at \( \eta = \pm \eta_\infty \).

To get the asymptotic solutions, we solve the governing equations (3.17) in their limiting forms when \( \eta \to \pm \infty \) where the mean velocity and its derivatives are equal to zero.

The solution of the system which is now one of ordinary differential equations with constant coefficients is straightforward and the details can be found in Mack (1984). So, we will only present the final results here.
(i). Two-dimensional case

At $\eta \to -\infty$ the asymptotic solution is

$$\begin{bmatrix} \hat{u} \\ D\hat{u} \\ \hat{v} \\ \hat{p} \end{bmatrix} = C_1 \begin{bmatrix} i \\ i\alpha \\ 1 \\ -i(U_{-\infty} - \frac{\beta}{\alpha}) \end{bmatrix} e^{\alpha \eta} + C_2 \begin{bmatrix} \frac{i\kappa}{\alpha} \\ \frac{i\alpha^2}{\alpha} \\ 1 \\ 0 \end{bmatrix} e^{\kappa \eta} \tag{3.21}$$

or

$$V^0_- = C_1 V^0_{-1} + C_2 V^0_{-2} \tag{3.22}$$

and at $\eta \to +\infty$ the asymptotic solution is

$$\begin{bmatrix} \hat{u} \\ D\hat{u} \\ \hat{v} \\ \hat{p} \end{bmatrix} = D_1 \begin{bmatrix} -i \\ i\alpha \\ 1 \\ i(U_{+\infty} - \frac{\beta}{\alpha}) \end{bmatrix} e^{-\alpha \eta} + D_2 \begin{bmatrix} \frac{-i\kappa}{\alpha} \\ \frac{i\alpha^2}{\alpha} \\ 1 \\ 0 \end{bmatrix} e^{-\kappa \eta} \tag{3.23}$$

or

$$V^0_+ = D_1 V^0_{+1} + D_2 V^0_{+2}, \tag{3.24}$$

where $\kappa = \sqrt{\alpha^2 + i\alpha \Re(U_{+\infty} - \frac{\beta}{\alpha})}$, $V^0_{\pm 1}$ is the inviscid solution, and $V^0_{\pm 2}$ is the viscous solution. These two solutions are independent and defined by comparison to (3.21) and (3.23). The coefficients $C_1$, $C_2$, $D_1$ and $D_2$ are complex constants to be determined. These asymptotic solutions will be used as initial conditions to solve the system of equations (3.17) for the two-dimensional case.
(ii). The three-dimensional case

At $\eta \to -\infty$ the asymptotic solution is

$$
\begin{bmatrix}
\hat{u} \\
D\hat{u} \\
\hat{v} \\
\hat{p} \\
D\hat{w}
\end{bmatrix} = C_1 
\begin{bmatrix}
\frac{-i\alpha}{\alpha} \\
\frac{i\alpha}{\alpha^2} \\
\frac{1}{\alpha} \\
\frac{1}{\alpha} \\
i\gamma
\end{bmatrix}
\begin{bmatrix}
e^{-\alpha\eta} + C_2 \\
e^{-\alpha\eta} + C_3 \\
e^{-\kappa\eta} + C_3 \\
e^{-\kappa\eta} + C_3 \\
e^{-\kappa\eta}
\end{bmatrix}
\begin{bmatrix}
\frac{-i\alpha}{\alpha} \\
\frac{-i\alpha}{\alpha^2} \\
\frac{1}{\alpha} \\
\frac{1}{\alpha} \\
i\gamma
\end{bmatrix}
$$

or

$$
V_\eta^0 = C_1 V_\eta^0 + C_2 V_\eta^0 + C_3 V_\eta^0
$$

and at $\eta \to +\infty$ the asymptotic solution is

$$
\begin{bmatrix}
\hat{u} \\
D\hat{u} \\
\hat{v} \\
\hat{p} \\
D\hat{w}
\end{bmatrix} = D_1 
\begin{bmatrix}
\frac{-i\alpha}{\alpha} \\
\frac{i\alpha}{\alpha^2} \\
\frac{1}{\alpha} \\
\frac{1}{\alpha} \\
i\gamma
\end{bmatrix}
\begin{bmatrix}
e^{-\alpha\eta} + D_2 \\
e^{-\alpha\eta} + D_3 \\
e^{-\kappa\eta} + D_3 \\
e^{-\kappa\eta} + D_3 \\
e^{-\kappa\eta}
\end{bmatrix}
\begin{bmatrix}
\frac{-i\alpha}{\alpha} \\
\frac{-i\alpha}{\alpha^2} \\
\frac{1}{\alpha} \\
\frac{1}{\alpha} \\
i\gamma
\end{bmatrix}
$$

or

$$
V_\eta^0 = D_1 V_\eta^0 + D_2 V_\eta^0 + D_3 V_\eta^0
$$

where $\kappa = \sqrt{\alpha^2 + i\alpha Re(U_{\pm\infty} - \frac{\beta}{\alpha})}$, $V_\eta^0$ is the inviscid solution, and $V_\eta^0$, $V_\eta^0$, $V_\eta^0$ are the primary and secondary viscous solutions respectively. These are also independent and defined by comparison to (3.25) and (3.27). The coefficients $C_1$, $C_2$, $C_3$, $D_1$, $D_2$, $D_3$ are determined by the boundary conditions and the physics of the problem.
$D_1$, $D_2$ and $D_3$ are complex constants to be determined. These asymptotic solutions will be used as initial conditions to solve the system of equation (3.17) for the three-dimensional case.

The general solutions of equation (3.17) can be expressed as a sum of two independent solutions for the two-dimensional disturbance and three independent solutions for the three-dimensional disturbance.

### 3.3 Numerical Methods

The numerical solution method used for the eigenvalue problem in hand is a classical shooting method. We have used several techniques to resolve the serious numerical complications introduced by the infinite domain and the parasitic contamination of the solution, particularly at high Reynolds number. The general process requires that the appropriate system of equations (3.17) is integrated from $-\eta_\infty$ to 0 and from $+\eta_\infty$ to 0, and the two computed solutions are matched at a matching point (in here $\eta_m = 0$) as shown in Figure 3.1. Matching can be achieved only for the correct value of $\alpha$ (the eigenvalue) and thus the matching condition becomes the dispersion relation which is solved iteratively by a numerical root-finding method. It is found that this matching method gives more accurate results than the one-directional integration method, $-\eta_\infty$ to $+\eta_\infty$ or $+\eta_\infty$ to $-\eta_\infty$. In fact, in the case of an unbounded flow unidirectional integration usually fails to give a solution altogether.

A Runge-Kutta-Fehlberg method (Burden & Faires 1985) is used to integrate the ordinary differential equations. This scheme controls the step size by keeping an
Two-direction integration

\[ \begin{array}{c}
V^0_+ V^0_+ V^0_+ \\
V^0_{+2} V^0_{+2} V^0_{+3}
\end{array} \quad \eta_\infty \]

\[ \begin{array}{c}
V_+ V_+ V_+ \\
V_{+1} V_{+2} V_{+3}
\end{array} \quad \eta = 0 \]

\[ \begin{array}{c}
V^0_- V^0_- V^0_- \\
V^0_{-1} V^0_{-2} V^0_{-3}
\end{array} \quad -\eta_\infty \]

Figure 3.1: The diagram for the two directional numerical method at the matching point

estimate of the local error below the user specified tolerance (in this study \(1 \times 10^{-08}\)).

The estimate of the local error is obtained by comparing the two values evaluated by a fourth-order method and a fifth-order method.

If the Reynolds number of a developing shear layer is small (less than 90), a standard superposition method (Bellman & Kalaba 1965) can be used to solve the system of equations (3.17) with two-point boundary values. The two or three independent solutions (depending on whether we are solving the two- or three-dimensional problem) are integrated from \(-\eta_\infty\) to 0 and from \(+\eta_\infty\) to 0 and eigenvalues are determined by matching \(\dot{u}, D\dot{u}, \dot{v}, \dot{p}, \dot{w}\) and \(D\dot{w}\) at the matching point. The independent asymptotic solutions \(V^0_{\pm 1}, V^0_{\pm 2}\) and \(V^0_{\pm 3}\) (for the three-dimensional case) are used as initial conditions for the integrations.
For the correct eigenvalue, the general solution at the matching point ($\eta_m = 0$) must satisfy

$$V_-(\eta_m) = C_1V_{-1}(\eta_m) + C_2V_{-2}(\eta_m) = D_1V_{+1}(\eta_m) + D_2V_{+2}(\eta_m) = V_+(\eta_m) \quad (3.29)$$

for the two-dimensional case, or

$$V_-(\eta_m) = C_1V_{-1}(\eta_m) + C_2V_{-2}(\eta_m) + C_3V_{-3}(\eta_m) = D_1V_{+1}(\eta_m) + D_2V_{+2}(\eta_m)D_3V_{+3}(\eta_m) = V_+(\eta_m) \quad (3.30)$$

for the three-dimensional case.

Rewriting equations (3.29) and (3.30) in matrix form, we have

$$GC = \begin{bmatrix} V_{-1} & V_{-2} & -V_{+1} & -V_{+2} \\ \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ D_1 \\ D_2 \end{bmatrix} = 0 \quad (3.31)$$

and

$$GC = \begin{bmatrix} V_{-1} & V_{-2} & V_{-3} & -V_{+1} & -V_{+2} & -V_{+3} \\ \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ D_1 \\ D_2 \\ D_3 \end{bmatrix} = 0 \quad (3.32)$$

This matching condition is satisfied if the determinant of the matrix $G$, which is a function of $\alpha, \beta, \gamma$ and $\text{Re}$, is equal to zero:

$$\Lambda(\alpha, \beta, \gamma, \text{Re}) \equiv \text{det}(G) = 0 \quad (3.33)$$
This constitutes the dispersion relation and the eigenvalue $\alpha$ is calculated from it using the iterative technique.

The following iterative technique is used to find the eigenvalue. The numerical integrations are carried out once with an initial estimated value $\alpha_1$, and a second time with a value, $\alpha_2$, which is obtained by slightly changing $\alpha_1$ (e.g. $\alpha_2 = 1.001\alpha_1$). If $A_1$ or $A_2$ is not less than a prescribed tolerance (e.g. $10^{-08}$), the Lagrangian interpolation scheme (Carnahan et al. 1969) is used in order to find a new estimate $\alpha_{n+1}$:

$$\alpha_{n+1} = \sum_{i=1}^{n} \prod_{j=1, i\neq j}^{n} \frac{-\Lambda_j}{\Lambda_i - \Lambda_j} \alpha_i \text{ for } n \geq 2 \quad (3.34)$$

When the determinant condition is satisfied, the constants of equations (3.29) or (3.30) are calculated by Gaussian elimination with pivoting after setting one of them to arbitrarily be equal to 1. The iteration for $\alpha$, is then continued until the relative error $|V_- - V_+|/|V_+|$ is less than $10^{-05}$, and of course $A$ is less than $10^{-08}$. Using this method, we can compute eigenvalues with eight or more significant digits and eigenfunctions with at least five significant digits.

The simple and mathematically exact superposition method does not work when the Reynolds number is large (larger than 90 for our case) because the coefficient matrix of equation (3.17) has eigenvalues whose real parts are well separated. The matrix $G$ in equation (3.31) will thus be poorly conditioned at the matching point consequently the convergence criteria will never be met to the desired accuracy. In other words, if the Reynolds number is higher, the two or three independent
solutions for the two- and three-dimensional case respectively will lose their linear independence because of contaminating error particularly in the viscous region of the mixing layer. Therefore, the eigenvalue and eigenfunctions obtained by the simple superposition method will not be accurate at all. An orthonormalization method is used in order to keep the independence of these sets of solutions.

The orthonormalization method was originally proposed by Godunov (1961), and developed by Bellman & Kalaba (1965), Conte (1966), and Davey (1973) among others. This method has been successfully applied by Morris (1976). Although there is a computer code (Scott & Watts 1975) employing this method, we developed our own computer program because the existing code can not be used for the eigenvalue problem and particularly for the matching procedure with two-directional integrations which is necessary for our infinite domain (see Figure 3.1).

For the simplest case of a two-dimensional governing equation, as shown before, there are two solutions, \( V_{\pm 1}^0 \) and \( V_{\pm 2}^0 \), from either one side at \( \eta = \pm \eta_\infty \). In our case, \( V_{\pm 1}^0 \) is the inviscid and \( V_{\pm 2}^0 \) is the viscous solution. \( V_{\pm 2}^0 \) continue to grow more rapidly with decreasing \( |\eta| \) than do \( V_{\pm 1}^0 \). The parasitic error will follow \( V_{\pm 2}^0 \), and when the difference in the magnitude of \( V_{\pm 2}^0 \) and \( V_{\pm 1}^0 \) becomes sufficiently large, \( V_{\pm 1}^0 \) will no longer be linearly independent of \( V_{\pm 2}^0 \). The two solutions are constantly checked for linear independence and when this is lost, the Gram-Schmidt orthonormalization algorithm is applied. The same procedure is followed with the three independent solutions of the three-dimensional case.
The orthonormalization method with the Gram-Schmidt recursion formulas (Conte 1966) with pivoting (Scott & Watts 1977) is applied in our calculations in order to orthonormalize the set of solution vectors. The points where the orthonormalizations are performed are determined by a test suggested by Scott & Watts (1977). This test is based upon the Gram-Schmidt procedure with pivoting.

A flowchart of the program developed for the solution of the eigenvalue problem, outlining the most significant steps and checkpoints is given in Figure 3.2. For the three-dimensional case the eigenvalue is initially calculated from the modified four equation formulation represented by equations (3.15), (3.17) and (3.20). Then the full six equation formulation is used to yield all the eigenfunctions and make sure that the full solution vectors are matched at $\eta_m = 0$. We have found that this procedure is more efficient since it considerably reduces the number of operations for the determination of the eigenvalue.

The computations are carried out on a VAX 8800 using double precision at SNCC (System Network Computer Center) of Louisiana State University. As a verification of the numerical methods, the computed eigenvalues have been successfully compared at high Reynolds number with the inviscid numerical results of Michalke (1965, 1969) for the mean velocity profile he used.
Figure 3.2: Flow chart for linear stability problem of the developing mixing layer
3.4 Some Numerical Results and Discussion

In this section, we will briefly present some results of the linear viscous stability analysis of the two- and three-dimensional disturbances and discuss them in the light of the known inviscid results.

The computed two-dimensional spatial amplification rates for the hyperbolic tangent profile with velocity ratio $R = 0.31$ are plotted versus nondimensional frequency $\beta$ in Figure 3.3 for several values of the Reynolds number. The inviscid solution is also shown. The latter was obtained numerically by integrating the second-order inviscid stability equation (or Rayleigh equation) with the appropriate choice of integration contour to accommodate the singularity at $U = c$ (Nikitopoulos 1982). The amplification rate can be seen slowly to approach the inviscid result with increasing Reynolds number. According to Figure 3.3, the amplification rate increases as the Reynolds number increases at fixed frequency $\beta$. When the Reynolds number is 1,000, the values of $-\alpha_i$ and $\alpha_r$ are almost equal to those obtained by the inviscid analysis. The eigenvalues $\alpha = \alpha(Re, \beta)$ are given in the Table 3.1 for the Reynolds numbers 100, 1,000 and the inviscid case. The eigenvalues $\alpha$ are consistent with those obtained by Nikitopoulos (1982). As long as the amplification rate $-\alpha_i$ is positive, the large-scale disturbances are amplified. When $\alpha_i$ becomes zero, the disturbance is neither amplified nor damped and is neutrally stable.

For higher frequencies beyond the neutral value the shear layer becomes stable (see Table 3.1 and Figure 3.3) and the disturbances are damped. For the inviscid
Table 3.1: The eigenvalues for a developing two-dimensional mixing layer when Reynolds numbers are 100, 1,000 and ∞

<table>
<thead>
<tr>
<th>β</th>
<th>Re=100</th>
<th>Re=1000</th>
<th>Inviscid</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>α_r</td>
<td>-α_i</td>
<td>α_r</td>
</tr>
<tr>
<td>0.1</td>
<td>0.095987</td>
<td>0.021104</td>
<td>0.094767</td>
</tr>
<tr>
<td>0.4</td>
<td>0.400308</td>
<td>0.047278</td>
<td>0.399245</td>
</tr>
<tr>
<td>0.5</td>
<td>0.502459</td>
<td>0.043922</td>
<td>0.501723</td>
</tr>
<tr>
<td>1.0</td>
<td>0.992245</td>
<td>-0.031923</td>
<td>0.999173</td>
</tr>
<tr>
<td>1.5</td>
<td>1.436096</td>
<td>-0.167145</td>
<td>1.458884</td>
</tr>
</tbody>
</table>

Table 3.2: The eigenvalues for a developing three-dimensional mixing layer when Reynolds number is 1,000.

<table>
<thead>
<tr>
<th>β</th>
<th>γ = 0.1</th>
<th>γ = 0.2</th>
<th>γ = 0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>α_r</td>
<td>-α_i</td>
<td>α_r</td>
</tr>
<tr>
<td>0.1</td>
<td>0.095385</td>
<td>0.022071</td>
<td>0.096428</td>
</tr>
<tr>
<td>0.2</td>
<td>0.194417</td>
<td>0.039721</td>
<td>0.195188</td>
</tr>
<tr>
<td>0.3</td>
<td>0.295969</td>
<td>0.050792</td>
<td>0.296354</td>
</tr>
<tr>
<td>0.4</td>
<td>0.398730</td>
<td>0.056141</td>
<td>0.398707</td>
</tr>
<tr>
<td>0.5</td>
<td>0.501430</td>
<td>0.055868</td>
<td>0.501189</td>
</tr>
<tr>
<td>0.6</td>
<td>0.603474</td>
<td>0.050886</td>
<td>0.603095</td>
</tr>
<tr>
<td>1.0</td>
<td>0.999095</td>
<td>-0.004464</td>
<td>0.998522</td>
</tr>
<tr>
<td>1.5</td>
<td>1.458638</td>
<td>-0.125385</td>
<td>1.457374</td>
</tr>
<tr>
<td>2.0</td>
<td>1.888502</td>
<td>-0.269658</td>
<td>1.893700</td>
</tr>
</tbody>
</table>
Figure 3.3: Amplification rates $-\alpha_i$ versus nondimensional frequency $\beta$ for the case of two-dimensional disturbance with several Reynolds numbers.

Figure 3.4: Amplification rates $-\alpha_i$ versus nondimensional frequency $\beta$ for the case of three-dimensional disturbance with various spanwise wave number $\gamma$ and $Re = 1,000$. 
case the neutral frequency is 1. From Figure 3.3 we see that the neutral stability point is moved to lower frequencies as the Reynolds number is decreased because of the damping effect of viscosity.

For temporally growing disturbances it is known that the growth rate of three-dimensional disturbances can not exceed that of two-dimensional ones (Squires theorem). Inviscid calculations in the amplification region for the tanh profile with $R = -1$ done by Michalke (1969) have confirmed that the same is true for spatially growing disturbances. Naturally the viscous solutions will obey the same rule. In Figure 3.4 the spatial growth rates $-\alpha_i$ for the three-dimensional disturbances are plotted as functions of the nondimensional frequency $\beta$ for various spanwise wave numbers $\gamma$ for a Reynolds number equal to 1000. Eigenvalues $\alpha = \alpha(Re, \beta, \gamma)$ are given in the Table 3.2 for this Reynolds number. We see that the growth rates for $\gamma \neq 0$ are always smaller than for $\gamma = 0$ which corresponds to two-dimensional disturbances and three-dimensional disturbances are less unstable than two-dimensional disturbances, as expected. From Figure 3.4 it is also seen that neutral stability occurs at higher frequencies, $\beta$, as the spanwise wave number $\gamma$ decreases, since the smaller scales are more dissipative.

The calculated maximum amplification rates $-\alpha_{i_{\text{max}}}$ are shown in Figure 3.5 as functions of the Reynolds number for various spanwise wave numbers $\gamma$. As the Reynolds number is increased, the maximum growth rate increases monotonically and asymptotically approaches the inviscid limit. The growth rate is of course
always largest in the two-dimensional case \((\gamma = 0)\) which is the lower limit of \(\gamma\).

For any given spanwise wave number, there is a critical Reynolds number below which the three-dimensional disturbance is damped for all frequencies. As it can be deduced from Figure 3.5, the critical Reynolds number increases with increasing \(\gamma\). For values of the spanwise wave number \(\gamma\) larger than approximately 0.5, this critical Reynolds number increases considerably. For example, the critical Reynolds number for \(\gamma = 0.8\) is approximately 350. For the two-dimensional case \((\gamma = 0)\), we have calculated the critical Reynolds number to be approximately 12.5 as shown in Figure 3.6 where the neutral stability curve is shown in frequency versus Reynolds number space. Below this value, all disturbances are damped. In Region I the two-dimensional disturbance is unstable and stable in Region II. Of course, it is well known that at very low Reynolds numbers the parallel mean flow assumption may not be exactly valid and therefore the critical Reynolds number calculated here may not be exactly correct. However, Lessen & Ko (1966) calculated a critical Reynolds number of the same order for jet flows by the nonparallel flow theory.

The variation of the maximum amplification rate with spanwise wave number \(\gamma\) for several Reynolds numbers is shown in Figure 3.7. The maximum amplification rate increases when spanwise wave number \(\gamma\) decreases. The decrease of \((-\alpha_{\text{max}})\) with \(\gamma\) at a fixed Reynolds number is quite linear with the exception of low spanwise wave numbers \((\gamma \leq 0.2)\). This is in good agreement with the inviscid results of Michalke (1969). Three-dimensional wave modes of long wave-length are more likely
to occur in a shear layer if their amplification rates can be boosted through combined forcing and interaction with other modes. It is understood that the amplification rate shown here result from interaction with the mean flow alone.

In addition to the calculated and presented eigenvalues for different frequencies, spanwise wave numbers and Reynolds numbers, we have also examined the calculated eigenvalues to make sure that our code renders accurate results. The computed eigenfunctions are normalized by the following condition at the matching point (in here, \( \eta = 0 \));

\[
\hat{v}_{mn}(\eta = 0) = 1.
\]  

(3.35)

This normalization precedes (2.39) and (2.40). This also renders the phase of the cross-stream component of the disturbance equal to \( \psi_{mn} \) at the maximum slope location. Thus this will be our phase reference point. In Figure 3.8, the eigenfunctions \( \hat{u}, \hat{v} \) and \( \hat{p} \) for the two-dimensional disturbance are shown near the most amplified frequencies for two Reynolds numbers, 50 and 1,000. In Figures 3.9 and 3.10 we show the eigenfunctions \( \hat{u}, \hat{v}, \hat{p} \) and \( \hat{w} \) as functions of \( \eta \) for the three-dimensional case near the most amplified mode. We also show the eigenfunctions for the two- and three-dimensional disturbances in Figures 3.11 ~ 3.16 at near the neutral stability and within the damped region for Reynolds numbers 50 and 1,000. The calculated eigenfunctions have a strong peak as the frequency decreases. The trends in the amplified region are consistent with those of the inviscid analyses of Michalke (1965, 1969) and of the experiments by Gaster, Kit & Wygnanski (1985), Cohen &
Figure 3.5: The maximum amplification rates $-\alpha_{i\text{max}}$ versus Reynolds number for various spanwise wave numbers $\gamma$. 
Figure 3.6: The neutral stability curve for two-dimensional disturbances

Figure 3.7: The maximum amplification rates $-\alpha_{\text{max}}$ versus spanwise wave numbers $\gamma$ for several Reynolds numbers
Wygnanski (1987 a) and Weisbrot & Wygnanski (1988). In Figures 3.11 ~ 3.16, we can see the sharp change around the center line of the shear layer for high Reynolds number, which tells us that there exists a critical point. At near this point viscous effects are expected to be important. Variations of all these quantities are smoother and more diffused for lower Reynolds numbers as expected.

In Figures 3.17 ~ 3.22, we show the Reynolds stresses induced by the large-scale wave modes for Re=1000 and various frequencies. The magnitudes of the normal stresses decay slowly with distance away from the center of the shear layers. The maxima of Reynolds stress \((\overline{u\overline{v}})\) occur around the center of the mixing layers as shown in Figures 3.17 ~ 3.22 and the trend of our results has a good agreement with the experimental observation by Weisbrot and Wygnanski (1988).

As we see in Figures 3.17 and 3.18, in the amplified frequency region the Reynolds stress induced by the large-scale wave modes is positive. A positive Reynolds stress implies that the energy of the mean flow will transfer to the large-scale wave mode. When the frequency is increased, the Reynolds stresses induced by large-scale structures decrease becomes zero at the point of neutral stability and in the damped region the Reynolds stress will be negative for the two-dimensional disturbances. In the negative Reynolds stress region the large-scale coherent structure loses energy to the mean flow and by the viscous dissipation and the growth rate of the mixing layer is stopped.
Figure 3.8: The eigenfunctions near the most amplified frequency $\beta_{max} = 0.45$ for $Re=1,000$ and $\beta_{max} = 0.35$ for $Re=50$ spanwise wave number $\gamma = 0$
Figure 3.9: The eigenfunctions near the most amplified frequency $\beta_{max} = 0.455$ for $Re=1,000$ and $\beta_{max} = 0.36$ for $Re=50$ spanwise wave number $\gamma = 0.3$
Figure 3.10: The eigenfunctions near the most amplified frequency $\beta_{\text{max}} = 0.455$ for $\text{Re}=1,000$ and $\beta_{\text{max}} = 0.36$ for $\text{Re}=50$ spanwise wave number $\gamma = 0.3$
Figure 3.11: The eigenfunctions near neutral frequency $\beta = 0.97$ for $Re=1,000$ and $\beta = 0.73$ for $Re=50$ spanwise wave number $\gamma = 0$
Figure 3.12: The eigenfunctions near neutral frequency $\beta = 0.93$ for $\text{Re}=1,000$ and $\beta = 0.70$ for $\text{Re}=50$ spanwise wave number $\gamma = 0.3$
Figure 3.13: The eigenfunctions near neutral frequency $\beta = 0.93$ for $Re=1,000$ and $\beta = 0.70$ for $Re=50$ spanwise wave number $\gamma = 0.3$
Figure 3.14: The eigenfunctions damped frequency $\beta = 1.50$ for $Re=1,000$ and $\beta = 1.50$ for $Re=50$ spanwise wave number $\gamma = 0$. 
Figure 3.15: The eigenfunctions damped frequency $\beta = 1.50$ for $Re=1,000$ and $\beta = 1.50$ for $Re=50$ spanwise wave number $\gamma = 0.3$
Figure 3.16: The eigenfunctions damped frequency $\beta = 1.50$ for Re=1,000 and $\beta = 1.50$ for Re=50 spanwise wave number $\gamma = 0.3$
Figure 3.17: The Large-scale Reynolds stresses for \( \text{Re}=1000, \beta = 0.45 \) and \( \gamma = 0.0 \)

Figure 3.18: The Large-scale Reynolds stresses for \( \text{Re}=1000, \beta = 0.455 \) and \( \gamma = 0.3 \)
Figure 3.19: The Large-scale Reynolds stresses for Re=1000, $\beta = 0.97$ and $\gamma = 0.0$

Figure 3.20: The Large-scale Reynolds stresses for Re=1000, $\beta = 0.93$ and $\gamma = 0.3$
Figure 3.21: The Large-scale Reynolds stresses for $Re=1000$, $\beta = 1.50$ and $\gamma = 0.0$

Figure 3.22: The Large-scale Reynolds stresses for $Re=1000$, $\beta = 1.50$ and $\gamma = 0.3$
Chapter 4

The Nonlinear Development of Modal Amplitudes and Phases

Having solved the viscous, linear stability problem the integral coefficients in the integral equations developed in Chapter 2 can be evaluated from the linear stability eigenfunctions. The integral equations will be summarized below in section 4.1. The calculated integral coefficients will be discussed in section 4.2, and the initial conditions and the computational procedure for the nonlinear analysis will be discussed in section 4.3.

4.1 The Integral Equations

Now, with the equations of motion developed into equations for the energies and phases of the modes and the shape functions assumed and determined, every tool is at hand to attempt to solve the integral mean energy equation (2.41), the integral wave mode energy equations (2.45) and (2.46), and those for the modal phases (2.53) and (2.54). Using the stretched cross-stream coordinate $\eta$ in these equations and the energy content of the wave mode $mn$ defined as $E_{mn}(x) = |A_{mn}(x)|^2 \delta(x)$,
the following set of equations are obtained after some manipulation:

1. **Mean flow**:

\[
I_{M} \frac{d\delta}{dx} = \frac{1}{Re_\delta} \Phi_M - \frac{1}{\delta} \sum_{mn} E_{mn} I_{MN} W_{mn} \tag{4.1}
\]

2. **Wave mode energies**:

- **10**:

\[
I_{a_{10}} \frac{dE_{10}}{dx} = -\frac{1}{\delta} E_{10} I_{M_{10}} - \frac{1}{\delta Re_\delta} E_{10} I_{v_{10}} + EW_{10}
\]

\[
EW_{10} = -\frac{1}{\delta^2} [E_{10} \sqrt{E_{20}} I_{1020} + \sqrt{E_{10} E_{11} E_{21}} \cos(\phi_{21})(I_{1011}^{21} + I_{1021}^{11})] \tag{4.2}
\]

- **20**:

\[
I_{a_{20}} \frac{dE_{20}}{dx} = -\frac{1}{\delta} E_{20} I_{M_{20}} - \frac{1}{\delta Re_\delta} E_{20} I_{v_{20}} + EW_{20}
\]

\[
EW_{20} = \frac{1}{\delta^2} [E_{10} \sqrt{E_{20}} I_{1020}^{10} + E_{11} \sqrt{E_{20}} I_{1120}^{11}] \tag{4.3}
\]

- **11**:

\[
I_{a_{21}} \frac{dE_{21}}{dx} = -\frac{1}{\delta} E_{21} I_{M_{21}} - \frac{1}{\delta Re_\delta} E_{21} I_{v_{21}} + EW_{21}
\]

\[
EW_{21} = \frac{1}{\delta^2} [\sqrt{E_{10} E_{11} E_{21}} \cos(\phi_{21})(I_{1011}^{21} - I_{1121}^{10}) - E_{11} \sqrt{E_{20}} I_{1120}^{11} - E_{11} \sqrt{E_{22}} I_{1122}^{11} \cos(\phi_{22})] \tag{4.4}
\]

- **21**:

\[
I_{a_{21}} \frac{dE_{21}}{dx} = -\frac{1}{\delta} E_{21} I_{M_{21}} - \frac{1}{\delta Re_\delta} E_{21} I_{v_{21}} + EW_{21}
\]

\[
EW_{21} = \frac{1}{\delta^2} [\sqrt{E_{10} E_{11} E_{21}} \cos(\phi_{21})(I_{1021}^{11} + I_{1121}^{10})] \tag{4.5}
\]
\[ I_{apW22} \frac{dE_{22}}{dx} = -\frac{1}{\delta} E_{22} I_{MW22} - \frac{1}{\delta Re} E_{22} I_{w22} + EWW_{22} \]
\[ EWW_{22} = \frac{1}{\delta^2} [E_{11} \sqrt{E_{22}} I_{1122}^{11} \cos(\phi_{22})] \quad (4.6) \]

3. Wave phase angles :

- 10 :
\[ P_{aw10} \frac{d\psi_{10}}{dx} = \beta_{10} + \frac{P_{WM10}}{\delta} + \frac{1}{E_{10}} \frac{dE_{10}}{dx} P_{pw10} - 2 \alpha_{10} \alpha_{110}^{10} \delta Re^\delta + PWW_{10} \]
\[ PWW_{10} = \frac{1}{\delta^2} \left[ \sqrt{E_{22}} P_{1020}^{10} + \sqrt{E_{11} E_{21}} \cos(\phi_{21})(P_{1021}^{21} + P_{1021}^{11}) \right] \quad (4.7) \]

- 20 :
\[ P_{aw20} \frac{d\psi_{20}}{dx} = \beta_{20} + \frac{P_{WM20}}{\delta} + \frac{1}{E_{20}} \frac{dE_{20}}{dx} P_{pw20} - 2 \alpha_{20} \alpha_{120}^{10} \delta Re^\delta + PWW_{20} \]
\[ PWW_{20} = \frac{1}{\delta^2} \left[ \frac{E_{10}}{\sqrt{E_{20}}} P_{1020}^{10} + \frac{E_{11}}{\sqrt{E_{20}}} P_{1120}^{11} \right] \quad (4.8) \]

- 11 :
\[ P_{aw11} \frac{d\psi_{11}}{dx} = \beta_{11} + \frac{P_{WM11}}{\delta} + \frac{1}{E_{11}} \frac{dE_{11}}{dx} P_{pw11} - \frac{\alpha_{11} \alpha_{111}^{10}}{\delta Re^\delta} + PWW_{11} \]
\[ PWW_{11} = \frac{1}{\delta^2} \left[ \sqrt{E_{22}} P_{1122}^{11} \cos(\phi_{22}) + \sqrt{E_{22}} P_{1120}^{11} \right. \\
\left. + \sqrt{E_{11}} E_{21} \left( P_{1021}^{10} - P_{1121}^{21} \right) \cos(\phi_{21}) \right] \quad (4.9) \]

- 21 :
\[ P_{aw21} \frac{d\psi_{21}}{dx} = \beta_{21} + \frac{P_{WM21}}{\delta} + \frac{1}{E_{21}} \frac{dE_{21}}{dx} P_{pw21} - \frac{\alpha_{21} \alpha_{121}^{10}}{\delta Re^\delta} + PWW_{21} \]
\[ PWW_{21} = \frac{1}{\delta^2} \left[ \frac{E_{10} E_{11}}{\sqrt{E_{21}}} \cos(\phi_{21})(P_{1021}^{10} + P_{1121}^{11}) \right] \quad (4.10) \]
\[ P_{aw22} \frac{d\psi_{22}}{dx} = \beta_{22} + \frac{P_{WM22}}{\delta} + \frac{1}{E_{22}} \frac{dE_{22}}{dx} P_{pW22} - \frac{\alpha_{22} \alpha_{i22}}{\delta Re_{\delta}} + PWW_{22} \]

\[ PWW_{22} = \frac{1}{\delta^2} \left[ \frac{E_{11}}{E_{22}} \right] P_{1122}^{11} \cos(\phi_{22}) \]  

The integral coefficients in the above system of equations are given below. The mean flow energy advection integral coefficient \( I_{aM} \) is

\[ I_{aM} = -\frac{1}{2} \left[ \int_{-\infty}^{0} (1 - R \tanh \eta)((1 - R \tanh \eta)^2 - (1 + R)^2) \, d\eta \right. \]
\[ + \left. \int_{0}^{\infty} (1 - R \tanh \eta)((1 - R \tanh \eta)^2 - (1 - R)^2) \, d\eta \right] \]

\[ = R^2 (3 - 2 \ln 2), \]  

and the mean viscous dissipation integral coefficient \( \Phi_M \) is

\[ \Phi_M = \int_{-\infty}^{\infty} \frac{R^2}{\cosh^4 \eta} \, d\eta = \frac{4R^2}{3}, \]  

where \( R \) is the velocity ratio.

The integral coefficient \( I_{apWmn} \) consists of a wave-mode energy advection and pressure transport component:

\[ I_{apWmn} = I_{awmn} + I_{pwmn}, \]  

where the wave-mode energy advection integral coefficient \( I_{awmn} \) is

\[ I_{awmn} = \begin{cases} \int_{-\infty}^{\infty} U(|\hat{u}_{mn}|^2 + |\hat{v}_{mn}|^2) \, d\eta & \text{if } n = 0, \\ \frac{1}{2} \int_{-\infty}^{\infty} U(|\hat{u}_{mn}|^2 + |\hat{v}_{mn}|^2 + |\hat{w}_{mn}|^2) \, d\eta & \text{if } n \neq 0, \end{cases} \]

\[ \text{(4.15)} \]
and the wave-mode pressure work integral coefficient $I_{pWmn}$ is

$$I_{pWmn} = \begin{cases} 
\int_{-\infty}^{\infty} 2 \text{Rel}(\hat{u}_{mn} \hat{p}_{mn}) \, d\eta & \text{if } n = 0, \\
\int_{-\infty}^{\infty} \text{Rel}(\hat{u}_{mn} \hat{p}_{mn}) \, d\eta & \text{if } n \neq 0.
\end{cases} \quad (4.16)$$

The wave-mode production integral coefficient $I_{MWmn}$ is

$$I_{MWmn} = \begin{cases} 
\int_{-\infty}^{\infty} 2 \text{Rel}(\hat{u}_{mn} \hat{v}_{mn}) \frac{\partial U}{\partial \eta} \, d\eta & \text{if } n = 0, \\
\int_{-\infty}^{\infty} \text{Rel}(\hat{u}_{mn} \hat{v}_{mn}) \frac{\partial U}{\partial \eta} \, d\eta & \text{if } n \neq 0,
\end{cases} \quad (4.17)$$

and the wave-mode viscous dissipation integral coefficient $I_{vWmn}$ is

$$I_{vWmn} = \begin{cases} 
2|\alpha_{mn}|^2 + 2 \int_{-\infty}^{\infty} (|\frac{\partial \hat{u}_{mn}}{\partial \eta}|^2 + |\frac{\partial \hat{v}_{mn}}{\partial \eta}|^2) \, d\eta & \text{if } n = 0, \\
2|\alpha_{mn}|^2 + 2\gamma^2 n^2 + \int_{-\infty}^{\infty} (|\frac{\partial \hat{u}_{mn}}{\partial \eta}|^2 + |\frac{\partial \hat{v}_{mn}}{\partial \eta}|^2 + |\frac{\partial \hat{w}_{mn}}{\partial \eta}|^2) \, d\eta & \text{if } n \neq 0.
\end{cases} \quad (4.18)$$

The remaining terms of each wave energy equations 4.2 ~ 4.6 are mode-mode energy exchange terms, which are equal to $2 \text{Rel}(S_{ij}^{pq} D_{kl})$ in equation (2.52). The integral $I_{ijkl}^{pq}$ is defined as

$$I_{ijkl}^{pq} = 2 \text{Rel}(\Sigma_{ij}^{pq} \Delta_{kl} e^{-i((-)^p \psi_{pq} + (-)^q \psi_{ij} + (-)^r \psi_{kl})}) \quad (4.19)$$
In the phase angle equations, $P_{awmn}$ is equal to the wave mode advection integral $I_{awmn}$. The integral influencing the mode phase shift from its interaction with the mean flow is

$$P_{WMmn} = \begin{cases} \int_{-\infty}^{\infty} Im(\dot{u}_{mn}\dot{v}_{mn}) \frac{\partial U}{\partial \eta} d\eta & \text{if } n=0, \\ \frac{1}{2} \int_{-\infty}^{\infty} Im(\ddot{u}_{mn}\ddot{v}_{mn}) \frac{\partial U}{\partial \eta} d\eta & \text{if } n \neq 0, \end{cases} \quad (4.20)$$

and the one influencing the phase shift induced by the pressure field is

$$P_{pMmn} = \begin{cases} \int_{-\infty}^{\infty} Im(\dot{u}_{mn}\ddot{p}_{mn}) d\eta & \text{if } n=0, \\ \frac{1}{2} \int_{-\infty}^{\infty} Im(\ddot{u}_{mn}\ddot{p}_{mn}) d\eta & \text{if } n \neq 0. \end{cases} \quad (4.21)$$

The last group of terms in each wave phase angle equation represents the phase shift induced by interaction between wave modes, which is equal to $Im(S_{ij}^{pq} D_{kl})$ in equation (2.52). $P_{ijkl}^{pq}$ can be written as

$$P_{ijkl}^{pq} = Im(\sum_{ij}^{pq} \Delta_{kl} e^{-i((-)^r\psi_{ij}+(-)^\psi_{ij}+(-)^t\psi_{kl})}) \quad (4.22)$$

If we rewrite the mode-mode interaction integral $\sum_{ij}^{pq} \Delta_{kl}$ which is given in Appendix F, as

$$\sum_{ij}^{pq} \Delta_{kl} = |\sum_{ij}^{pq} \Delta_{kl}| e^{i\theta_{ijkl}} \quad (4.23)$$
then the total phase angle difference involved in the relevant interaction term will be

\[ \Phi_{ijkl}^{pq} = (-)^{p+1} \psi_{pq}^{ij} + (-)^{i+1} \psi_{ij} + (-)^{k+1} \psi_{kl} + \theta_{ijkl}^{pq} \]  \hspace{1cm} (4.24)

This total phase angle controls both the corresponding energy and phase interaction between modes \( ij \) and \( kl \). When \( \Phi_{ijkl}^{pq} = m\pi \), where \( m \) is an integer, we see from (4.19) and (4.22) that the energy interaction is strongest (maximum \( I_{ijkl}^{pq} \)) while the phases of modes \( ij \) and \( kl \) are not affected at all because \( P_{ijkl}^{pq} \) is zero. This relationship introduces an additional condition to those described in Chapter 2, for the existence of resonance type interactions between a pair of wave modes \( ij \) and \( kl \). Thus the necessary and sufficient conditions for the resonance interaction mechanism to be in effect become:

\[ |i \pm k| = p \]  \hspace{1cm} (4.25)

\[ |j \pm l| = q \]  \hspace{1cm} (4.26)

\[ \Phi_{ijkl}^{pq} = (-)^{p+1} \psi_{pq}^{ij} + (-)^{i+1} \psi_{ij} + (-)^{k+1} \psi_{kl} + \theta_{ijkl}^{pq} = m\pi \]  \hspace{1cm} (4.27)

The last condition can be shown to be equivalent to that of Cohen and Wygnanski (1987) if the phases are determined on the basis of linear stability, and their initial values as well as the argument \( \theta_{ijkl}^{pq} \) are identically zero. Thus, Cohen and Wygnanski's condition, which is one imposed on the complex wave numbers \( \alpha_{ij} \) and \( \alpha_{kl} = \alpha_{pq} \), and in the case of subharmonic-fundamental resonance interaction requires that the phase velocities of the interacting waves are not equal, is not, appropriately general and can not explain the dependence of the outcome of
the resonance interaction on the initial phase angles. This dependence is supported by numerous experimental studies (Arbey and Ffowcs-Williams 1984; Zhang et al 1985; Ng and Bradley 1988; Raman and Rice 1989; Corke 1990). In particular the experimental results of Corke (1990) indicate that resonance is possible even if the phase velocities are not equal. Condition 4.27 is an integral one since it is arrived at after the interaction terms have been averaged in time as well as the spanwise and cross-stream directions. In this sense this condition indicates that the resonant interaction is possible when the relevant wave Reynolds stress is aligned on the average, with the corresponding wave rate of strain. This alignment of course is not necessary to be true locally and at all instances, as long as the net integral condition is met. On the other hand, when \( \Psi_{ijkl}^{pq} = (2m + 1)\pi/2 \), the corresponding energy interaction between modes is eliminated while the phases of the modes involved can be strongly influenced (maximum \( P_{ijkl}^{pq} \)). Thus the condition for maximum energy interaction between two modes \( ij \) and \( kl \), and therefore maximum local amplification rate modification requires \( \Psi_{ijkl}^{pq} \) to be a multiple of \( \pi \).

The overall strength of the interaction between modes, however, strongly depends on the strength of the modes involved. This is evident from equations 4.2 ~ 4.11. Therefore, as long as the participating modes are strong enough a mode \( ij \) will gain energy from \( kl \) as long as \( \frac{4n+1}{2}\pi \leq \Psi_{ijkl}^{pq} \leq \frac{4n+3}{2}\pi \), while it will lose energy to \( kl \) if \( \frac{4n-1}{2}\pi \leq \Psi_{ijkl}^{pq} \leq \frac{4n+1}{2}\pi \). Thus the resonance band for the phase angles can be rather broad as long as the strength of the participating modes is favorable, and the overall
exchange between modes as they evolve depends on the residence time of the total interaction phase angle within the above mentioned bands as well as the strength of the modes involved.

Regarding the influence of the modal interactions on the phase of mode $ij$, it is worth pointing out that the interaction terms can become more significant as mode $ij$ becomes weaker (note interaction terms in equations 4.7 ~ 4.11 with $E_{ij}^{-1}$). This is always true for the fundamentals while the subharmonics are influenced in this way only through three wave interactions. The rate of change of the subharmonic phases from their direct 2-mode interactions with the fundamentals are independent of the subharmonic energies and are controlled by the energy of the fundamentals.

4.2 Discussion of the Integral Coefficients of the Nonlinear Equations

In this section the integral coefficients in the integral equations of the mean flow, wave mode energies and phase angles will be presented as functions of the dimensionless frequency for the profile family with $R = 0.31$. Amplification rates $\alpha$, and eigenfunctions for various Reynolds numbers, Strouhal numbers and spanwise wave numbers have been presented in Chapter 3 where the linear calculations were examined and found to be consistent with observed and expected from theory trends. In this section we will examine the linear amplification rates and integral coefficients in conjunction with specific initial values of the above mentioned parameters. This is more pertinent to the developing nature of the shear layer in which our interest lies.
The initial values of the Strouhal number, $\beta_0$, and the spanwise wave number, $\gamma_0$, will always be referred to the subharmonics, and the corresponding values for spatial and spanwise fundamentals being twice these values. The initial Reynolds number, $Re_0$, is referred to the initial shear layer thickness.

The mean flow energy advection integral coefficient $I_{aM}$ and the mean flow viscous dissipation integral coefficient $\Phi_M$ are functions of velocity ratio $R$. If we assume that the integral coefficients $I_{aM}$ and $\Phi_M$ of the nonlinear equation of the mean flow are constants and there is no interaction between the mean flow and the wave modes that are $E_{mn}I_{MWmn} = 0$, then we see from equation 4.1 that the shear layer growth is proportional to the square root of the streamwise direction that is $\delta \propto \sqrt{x}$ which is the classical result.

In Figure 4.1 we examine the linear amplification rates of two- and three-dimensional fundamentals (20,21,22) and their subharmonics (10,11). For a given set of initial conditions ($Re_0, \beta_0, \gamma_0$) the frequencies, spanwise wave numbers and Reynolds numbers for both the subharmonic and fundamental will change as the flow is marching downstream. It is noted here that the ratio of the frequency to the Reynolds number is not changed as they both vary downstream and is therefore equal to the initial value. In Figure 4.1 this ratio for the subharmonic frequency to the Reynolds number is 0.0022 ($\beta_s/Re$) and the fundamental 0.0044 ($\beta_f/Re$). The ratio of the initial spanwise wave number over either the initial Reynolds number or the initial frequency is also preserved as these quantities vary downstream. In Figure 4.1 the
corresponding ratio $\beta_s/\gamma_0$ is 2.2. In this case for the fundamental mode, the initial frequency ($\beta_f$) is set at the maximum amplification rate since the shear layer will naturally respond to that frequency. As we see in Figure 4.1, it is obvious that in the absence of mode interactions the fundamentals will grow faster at first since they have twice the frequency of the subharmonics and higher amplification rates initially. The subharmonics reach their peak amplification while the fundamentals have just crossed neutral stability and begin being damped.

The advection and pressure integral coefficients for this case are shown in Figures 4.2 and 4.3 when the initial Reynolds number, frequency and spanwise wave number are 100, 0.22 and 0.1, respectively. The advection integral $I_{aWmn}$ is nearly constant at a value about 1 and is almost identical for subharmonics and fundamentals because of the normalization conditions (see equations 2.39 and 2.40). The wave mode pressure work integral $I_{pWmn}$ in Figure 4.3 is two orders of magnitude smaller than the advection integral and is less than zero for most frequencies, thus diminishing the growth of the waves. It asymptotically approaches zero at high frequencies. For all subharmonic wave modes, the integral has small positive values in the damped region. Far downstream the pressure transport by the large-scale wave mode is weak because the wave mode pressure work integral approaches to zero. The variation of these integral coefficients shown in Figures 4.2 and 4.3 is typical for a wide variety of $Re_0$, $\gamma_0$ and $\beta_0$ and it is not necessary to examine them further.
Figure 4.1: The growth rate of subharmonic modes (10,11) and fundamental modes (20,21,22) for $Re_0 = 100$, $\beta_0 = 0.22$ and $\gamma_0 = 0.1$
Figure 4.2: The wave mode energy advection integral $I_{aWmn}$ versus frequency $\beta$ for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0 = 100$, $\beta_0 = 0.22$ and $\gamma_0 = 0.1$

From the mean flow in equation (4.1), we can see that the thickness of the shear layer will increase as long as energy from the mean flow is transferred to the wave modes. This corresponds to the range of frequencies for which the wave mode is amplified according to the local linear analysis (see Figure 4.1). Thus the energy density of the wave modes will grow by extracting energy from the mean flow. When the energy transferred to each wave mode by the mean flow is balanced by viscous dissipation, the point of neutral stability has been reached.

The wave mode production integrals $I_{MWmn}$ are shown in Figures 4.4 ~ 4.10 for several cases. They tell us how the mean flow interacts with the large-scale wave modes. The sign of $I_{MWmn}$ controls the direction of the energy transfer between the mean flow and the wave mode. It is negative when the wave receives energy from
Figure 4.3: The wave mode pressure energy integral $I_{pWmn}$ versus frequency $\beta$ for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=100$, $\beta_0=0.22$ and $\gamma_0 = 0.1$

the mean flow and positive when the opposite is true. Wave modes are amplified by extracting energy from the mean flow in the region where the wave mode is unstable, and is damped by losing energy to the mean flow and to viscous dissipation when the wave mode is stable. A predominantly positive Reynolds stress in the region of high mean velocity gradient implies a feed of energy from the mean flow into the large-scale wave mode, while a predominantly negative Reynolds stress will result into a feed of energy from the large-scale wave mode back into the mean flow. During the early stages of pairing, the two-dimensional mode Reynolds stress is predominantly positive while the Reynolds stress becomes negative far downstream. This has been confirmed by Metcalfe et al (1987) and Cohen & Wygnanski (1987a, 1987b). In the damped region, the three-dimensional wave modes are less effective
in returning energy to the mean flow than two-dimensional wave modes. In fact a	hree-dimensional mode may keep drawing energy from the mean flow while it is
damped. The Reynolds stress of a three-dimensional mode does not always change
sign in the entire region of high mean strain rate past the neutral stability point
(see Figures 3.21 and 3.22). In Figure 4.4, we see that the three-dimensional sub-
harmonic (11) and the three-dimensional short wave-length wave mode (22) still
extract energy from the mean flow even if they are damped. Only wave mode 21
among the three-dimensional wave modes is able to return energy back to the mean
flow in the damped region. However this occurs well after it has passed the neutral
state. The three-dimensional wave modes lose a large amount of energy by viscous
dissipation compared to the two-dimensional wave modes in the damped region. In
this capacity they have the characteristic behavior of small-scale turbulence par-
ticularly at short wave-lengths (large wave numbers). This is shown by the higher
values of the viscous dissipation integrals for the three-dimensional modes in Fig-
ures 4.11 ~ 4.17. Viscosity becomes an increasingly dominant factor in the damping
of the two-dimensional modes as they attain higher frequencies. As a consequence
the mean-wave interaction integral reaches a maximum in the damped region and
declines at higher frequencies (see Figures 4.4 ~ 4.10). This decline is accompanied
by rapid increase of the viscous dissipation integral depicted in Figures 4.11 ~ 4.17.

Figures 4.4 ~ 4.6 show that as the initial spanwise wave number is increased,
the three-dimensional wave modes extract less energy from the mean flow in the
unstable region and are not able to return energy back to the mean flow although they have passed the neutral stability point. The three-dimensional wave mode can return energy back to the mean flow in the damped region only for low initial spanwise wave number (long wave-length). When the initial spanwise wave number increases, the three-dimensional wave modes interacting with the mean flow are less pronounced than the two-dimensional ones. As \( \gamma_0 \) increases, the three-dimensional wave modes have a shorter range to be amplified. A comparison of Figures 4.4 ~ 4.6 shows that the three-dimensional wave modes have an overall weaker energy interaction with the mean flow in the amplified region as the initial spanwise wave number increases.

We present the wave mode production integrals \( I_{MW_{mn}} \) in Figures 4.4, 4.7 and 4.8 for three different initial Reynolds numbers. As the initial Reynolds number increases, the transfer of energy back to the mean flow in the damped region is reduced and damping is increasingly effected by viscous dissipation (see Figures 4.11, 4.14 and 4.15). In the unstable (amplified) region the energy interaction with the mean flow is very weakly dependent upon the initial Reynolds number as shown in Figures 4.4, 4.7 and 4.8. As we see in Figures 4.14 and 4.15, the viscous dissipation is less important when the initial Reynolds number increases.

We also show \( I_{MW_{mn}} \) in Figures 4.4, 4.9 and 4.10 for three different initial frequencies. As the initial frequency increases to high values, the three-dimensional wave modes can return energy back to the mean flow (Figure 4.10) because the local
spanwise wave numbers and Reynolds numbers are lower and viscous dissipation is less pronounced (see Figures 4.11, 4.16 and 4.17) at corresponding local frequencies. As the initial frequency decreases, the three-dimensional wave mode interaction with the mean flow is less pronounced than the two-dimensional ones for the same reason. In general, lower initial frequency has a longer amplification region and therefore the wave mode has the potential to extract more energy from the mean flow.

As expected the wave mode viscous dissipation integrals $I_{wmn}$ are always greater than zero, and the wave mode always loses its energy by viscous dissipation. As mentioned during the previous discussion and as we can see from Figures 4.11 ~ 4.17, in the damped region a wave mode is strongly affected by viscosity and the large-scale viscous dissipation integral starts to increase drastically after the neutral stability point. In contrast, the viscous dissipation is almost negligible in the amplified region. The three-dimensional wave modes especially are more strongly affected by viscosity since they represent a smaller scale of motion.

In the phase angle equations 4.7 ~ 4.11, $P_{awmn}$ is equal to the wave mode advection integral coefficients $I_{awmn}$ and it is always greater than zero as shown in Figure 4.2. The integral coefficients representing the phase shift due to the interaction with the mean flow are presented for several cases in Figures 4.18 ~ 4.20. According to the phase angle equations 4.7 ~ 4.11, when $P_{WMmn}$ is greater than zero, the phase is increased and in the opposite case the phase is reduced because of the interaction with the mean flow.
Figure 4.4: The wave mode production integral $I_{MW_{mn}}$ versus frequency $\beta$ for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=100$, $\beta_0=0.22$ and $\gamma_0=0.1$

For the two-dimensional modes $P_{WM_{mn}}$ is always less than zero in the amplified region while for the three-dimensional modes $P_{WM_{mn}}$ is predominantly negative in the same region (see Figures 4.18 ~ 4.20). It is also noted that for the two-dimensional modes $P_{WM_{mn}}$ changes sign temporarily in the early damped stage and becomes negative again tending to what seems to be a constant value at high frequencies.

For the three-dimensional modes $P_{WM_{mn}}$ is positive in the damped region tending to a constant value at high local frequencies in Figures 4.18 and 4.19. Mode 21 displays behavior similar to the two-dimensional modes at high initial frequency and low spanwise wave number as shown in Figure 4.20. These trends are characteristic for a wide range of initial parameter values. From equations 4.7 ~ 4.11 it is seen
**Figure 4.5:** The wave mode production integral $I_{MW_{mn}}$ versus frequency $\beta$ for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=100, \beta_0=0.22$ and $\gamma_0 = 0.2$.

**Figure 4.6:** The wave mode production integral $I_{MW_{mn}}$ versus frequency $\beta$ for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=100, \beta_0=0.22$ and $\gamma_0 = 0.4$. 

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Repetitions in the caption text were corrected and the names of the figures were updated accordingly.
Figure 4.7: The wave mode production integral $I_{MWmn}$ versus frequency $\beta$ for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=50$, $\beta_0=0.22$ and $\gamma_0 = 0.1$

Figure 4.8: The wave mode production integral $I_{MWmn}$ versus frequency $\beta$ for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=200$, $\beta_0=0.22$ and $\gamma_0 = 0.1$
Figure 4.9: The wave mode production integral $I_{MWmn}$ versus frequency $\beta$ for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=100$, $\beta_0=0.1$ and $\gamma_0=0.1$.

Figure 4.10: The wave mode production integral $I_{MWmn}$ versus frequency $\beta$ for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=100$, $\beta_0=0.4$ and $\gamma_0=0.1$. 
Figure 4.11: The wave mode viscous dissipation integral $I_{vW_{mn}}/Re$ versus frequency $\beta$ for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=100$, $\beta_0=0.22$ and $\gamma_0 = 0.1$

Figure 4.12: The wave mode viscous dissipation integral $I_{vW_{mn}}/Re$ versus frequency $\beta$ for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=100$, $\beta_0=0.22$ and $\gamma_0 = 0.2$
Figure 4.13: The wave mode viscous dissipation integral $I_{\omega_{mn}}/Re$ versus frequency $\beta$ for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=100$, $\beta_0=0.22$ and $\gamma_0 = 0.4$

Figure 4.14: The wave mode viscous dissipation integral $I_{\omega_{mn}}/Re$ versus frequency $\beta$ for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=50$, $\beta_0=0.22$ and $\gamma_0 = 0.1$
Figure 4.15: The wave mode viscous dissipation integral $I_{vw_{mn}}/Re$ versus frequency $\beta$ for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=200$, $\beta_0=0.22$ and $\gamma_0 = 0.1$

Figure 4.16: The wave mode viscous dissipation integral $I_{vw_{mn}}/Re$ versus frequency $\beta$ for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=100$, $\beta_0=0.1$ and $\gamma_0 = 0.1$
that the mean flow effect on the phase represented by term $P_{WM_{mn}}/\delta$ becomes weak as the shear layer grows.

The integral coefficients representing the pressure induced phase shift are presented for a variety of cases in Figures 4.21 ~ 4.23. This integral does not control the direction of the modal phase change by itself. The relevant term in equations 4.2 ~ 4.6 contains the modal energy amplification rate so that the direction of the phase change induced by the pressure field also depends on whether the particular mode is amplified or damped. According to the phase equations 4.7 ~ 4.11 and given that $P_{pM_{mn}}$ is always positive in the amplification region (see Figures 4.21 ~ 4.23) the pressure induced phase change will be positive while the wave is amplified. This is true provided that the interaction with other modes does not reverse the
energy growth. In such a case the pressure induced phase shift will also be reversed. For the two-dimensional modes \( P_{pMmn} \) becomes negative in the damped region and thus, unless nonlinear effects dictate otherwise, the pressure induced phase shift remains positive. Much like the mean flow interaction integral discussed previously, \( P_{pMmn} \) does not always change sign in the damped region for the three-dimensional modes. For most cases it remains positive and thus the direction of the pressure induced phase shift is reversed when the three-dimensional modes become damped. The three-dimensional modes, particularly mode 21, retain a positive pressure phase change in the damped region when the initial frequency is sufficiently high and the spanwise wave number is sufficiently low (see Figure 4.23 for all three-dimensional modes and Figure 4.21 for mode 21 only). The pressure induced integral coefficient asymptotically tends to zero at high frequencies \( \beta \), its decay occurring sooner for the three-dimensional modes.

The magnitudes and arguments of the mode-mode interaction integrals are presented for several cases in Figures 4.24 ~ 4.37. The magnitudes themselves of the mode-mode interaction integrals are not the only factor that determines the strength of interactions because the modal energies and shear layer thickness are involved in calculating the whole mode-mode interaction terms (see equations 4.2 ~ 4.6). Typically the magnitude of these terms is controlled by a factor \( (E_{ij}E_{kl}E_{pq})^{\frac{1}{2}}/\delta^{\frac{3}{2}} \) and thus, the strength of the participating modal energies as well as the shear layer thickness. Far downstream as all modes are in decay these interactions will weaken.
Figure 4.18: Phase integral coefficients: Interaction with the mean flow for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=100$, $\beta_0=0.22$ and $\gamma_0 = 0.1$

Figure 4.19: Phase integral coefficients: Interaction with the mean flow for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=100$, $\beta_0=0.22$ and $\gamma_0 = 0.2$
Figure 4.20: Phase integral coefficients: Interaction with the mean flow for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=100$, $\beta_0=0.4$ and $\gamma_0 = 0.1$

Figure 4.21: Phase integral coefficients: Interaction with the pressure field for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=100$, $\beta_0=0.22$ and $\gamma_0 = 0.1$
Figure 4.22: Phase integral coefficients: Interaction with the pressure field for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=100$, $\beta_0=0.22$ and $\gamma_0 = 0.2$

Figure 4.23: Phase integral coefficients: Interaction with the pressure field for 2-dimensional wave modes (10,20) and 3-dimensional wave modes (11,21,22) for $Re_0=100$, $\beta_0=0.4$ and $\gamma_0 = 0.1$
considerably. The same occurs in the regions of rapid growth of the shear layer, although the net effect can be balanced by the simultaneous growth of one or more of the interacting modes. Similarly the magnitude of the effect of the mode-mode interactions on the modal phases is controlled by factor $\left( \frac{E_{pq}E_{kl}}{E_{pq}} \right)^{\frac{1}{2}} \frac{1}{\delta^2}$ as well as the magnitude of the interaction integrals (see equations 4.7 ~ 4.11). If mode $E_{pq}$ becomes very weak the mode interaction can bring about considerable phase variations which in turn can decisively influence the energy interactions. It is noted that the mode-mode interaction of the purely two-dimensional modes is predominantly weaker than the mode-mode interaction which involves the three-dimensional modes in the entire frequency range. As the initial spanwise wave number is increased, all the three-dimensional mode interaction integrals overall increase in magnitude with the exception of that involving the short wave-length fundamental, which decreases again when $\gamma_0=0.4$ (compare Figures 4.24 ~ 4.26). The maxima of this integral also move to lower frequencies as $\gamma_0$ is increased. A much weaker dependence on the initial Reynolds number can be seen from the comparison of Figures 4.24, 4.27 and 4.28. A comparison of Figures 4.24, 4.29 and 4.30 shows that the magnitude of the mode interaction coefficients is overall increased at low initial frequencies. This is because both the Reynolds number and the spanwise wave number are higher at the same nominal local frequency $2\beta$ when the initial frequency is lower. Overall it seems that the three-dimensional modes have the potential for more vigorous mutual interactions, particularly of the fundamental-subharmonic type, when the
initial wave number is high, i.e., when shorter spanwise wave-lengths are involved. This is an interesting result particularly since shorter wave-lengths are more heavily damped in the absence of such nonlinear interactions.

The arguments of the mode-mode interaction integrals are of paramount importance because they control the direction of the energy exchange between modes in combination with the phase angles of the interacting modes. This has been discussed to some extent in the previous sections. As we see in Figures 4.31 ~ 4.37, where the mode-mode interaction integral arguments are presented for various cases, sharp changes exist when one of the wave modes involved is in the neighborhood of neutral stability. This is, however, not true for all interactions. The changes are sharper as the corresponding Reynolds number is higher (compare Figures 4.31, 4.34 and 4.35) and the initial frequency higher (compare Figures 4.36 and 4.37). As a result, nonlinear effects can become important where critical layers appear and rapidly change the nature of the relevant mode-mode interaction. The argument of the mode-mode interaction integral between 10 and 20 wave modes in the damped region of the fundamental seems to be constant at a value near $\pi/2$ (see Figures 4.31 ~ 4.37).

In fact the arguments of the mode-mode interaction integrals seem to be nearly constant throughout wide ranges of frequency. When the phase of mode-mode interaction integral coefficient is such as to render $\Psi_{ijkl}^{pq}$ equal to multiple of $\pi$, the product of the Reynolds stresses induced by the wave modes and the strain rates
will be maximum in an integral sense, and the corresponding energy interaction will be strongest.

### 4.3 Computational Method for the Nonlinear Problem

The equations (4.1) \( \sim \) (4.11) for the five modal energy contents \( E_{mn}(x) \), the five modal phase angles \( \psi_{mn}(x) \) and the mean flow thickness \( \delta(x) \) can now be solved, subject to an equal number of initial conditions. All the integral coefficients are implicitly functions of \( x \) since they depend on the dimensionless frequencies \( \beta_{mn} \) and spanwise wave numbers \( n\gamma \), that are themselves functions of \( \delta(x) \). These integral coefficients can be calculated a priori from the local eigenfunctions of the viscous linear stability analysis. The integral interval \((-\infty, +\infty)\) will be divided into the
Figure 4.25: Magnitude of the wave mode-mode interaction integral versus frequency \( \beta \) for \( Re_0=100, \beta_0 = 0.22 \) and \( \gamma_0 = 0.2 \)

Figure 4.26: Magnitude of the wave mode-mode interaction integral versus frequency \( \beta \) for \( Re_0=100, \beta_0 = 0.22 \) and \( \gamma_0 = 0.4 \)
Figure 4.27: Magnitude of the wave mode-mode interaction integral versus frequency $\beta$ for $Re_0=50$, $\beta_0 = 0.22$ and $\gamma_0 = 0.1$

Figure 4.28: Magnitude of the wave mode-mode interaction integral versus frequency $\beta$ for $Re_0=200$, $\beta_0 = 0.22$ and $\gamma_0 = 0.1$
Figure 4.29: Magnitude of the wave mode-mode interaction integral versus frequency $\beta$ for $Re_0=100$, $\beta_0 = 0.1$ and $\gamma_0 = 0.1$

Figure 4.30: Magnitude of the wave mode-mode interaction integral versus frequency $\beta$ for $Re_0=100$, $\beta_0 = 0.4$ and $\gamma_0 = 0.1$
Figure 4.31: Argument of the wave mode-mode interaction integral versus frequency \( \beta \) for \( Re_0=100, \beta_0 = 0.22 \) and \( \gamma_0 = 0.1 \)

Figure 4.32: Argument of the wave mode-mode interaction integral versus frequency \( \beta \) for \( Re_0=100, \beta_0 = 0.22 \) and \( \gamma_0 = 0.2 \)
Figure 4.33: Argument of the wave mode-mode interaction integral versus frequency $\beta$ for $Re_0=100$, $\beta_0 = 0.22$ and $\gamma_0 = 0.4$

Figure 4.34: Argument of the wave mode-mode interaction integral versus frequency $\beta$ for $Re_0=50$, $\beta_0 = 0.22$ and $\gamma_0 = 0.1$
Figure 4.35: Argument of the wave mode-mode interaction integral versus frequency $\beta$ for $Re_0=200$, $\beta_0 = 0.22$ and $\gamma_0 = 0.1$.

Figure 4.36: Argument of the wave mode-mode interaction integral versus frequency $\beta$ for $Re_0=100$, $\beta_0 = 0.1$ and $\gamma_0 = 0.1$. 
Figure 4.37: Argument of the wave mode-mode interaction integral versus frequency $\beta$ for $Re_0=100$, $\beta_0=0.4$ and $\gamma_0=0.1$

three subintervals $(-\infty, -\eta_\infty)$, $(-\eta_\infty, \eta_\infty)$ and $(\eta_\infty, \infty)$. In the domains $(-\infty, -\eta_\infty)$ or $(\eta_\infty, \infty)$, the asymptotic solutions will be used. Simpson’s method (Burden & Faires 1985) are applied to integrate in the interval $(-\eta_\infty, \eta_\infty)$. The nonlinear ordinary differential equations (4.1) ~ (4.11) will be integrated by the Adams-Moulton Predictor-Corrector method (Burden & Faires 1985), and for convenience the integral coefficients that are known discretely are fitted with a cubic spline interpolation scheme (Burden & Faires 1985).
Chapter 5

Results and Discussion of the Nonlinear Analysis

There are many wave modes in a naturally occurring flow including two- and three-dimensional ones. In this Chapter we are going to present the results of the nonlinear analysis when the large-scale coherent structure is assumed to be composed of five wave modes 10, 20, 11, 21 and 22. As mentioned before, the wave modes 10 and 11 are the subharmonics and the wave modes 20, 21 and 22 are the fundamentals. As shown in Figure 2.3, the three-dimensional subharmonic wave mode interacts with all others, and the two-dimensional subharmonic wave mode interacts with all modes but the 22 wave mode. The fundamental wave modes can not interact directly with each other, but the subharmonic wave modes can.

The streamwise development of a shear layer has been determined by solving the integral mean energy equation 4.1, the five integral wave mode energy equations 4.2 ~ 4.6, and the five integral wave mode phase angle equations 4.7 ~ 4.11 with their corresponding initial conditions. Tables 5.1 and 5.2 summarize the various cases we
have examined and provide the necessary information regarding the initial energies, spanwise phases and spanwise wave number. For many of the cases examined the initial phase angles $\psi_{mn}$ have been chosen on the premise of maximizing the initial energy transfer between the fundamentals and the subharmonics. As mentioned in the previous Chapter, the maximum amount possible, for the given initial energies, of energy will be exchanged initially between the fundamentals and the subharmonics when the initial values of the total phase angle differences ($\Psi_{ijkl}^{pq}$) are a multiple of $\pi$.

For given initial values of the total phase angle differences ($\Psi_{ijkl}^{pq}$), the initial values of the phase angles ($\psi_{mn}$) can be determined from the equation 4.24 as mentioned in the previous Chapter. By the way, the total phase angle differences $\Psi_{1021}^{11}$ and $\Psi_{1121}^{10}$ can not exactly satisfy the resonance conditions 4.27 controlling the exchanges between 10 and 21, and 11 and 21 respectively unless $\theta_{1021}^{11}$ and $\theta_{1121}^{10}$ satisfy the following condition:

$$\theta_{1021}^{11} - \theta_{1121}^{10} = 2n\pi.$$ 

This condition is a function of the frequency parameter (Strouhal number), spanwise wave number and Reynolds number. Figure 5.1 shows the difference ($\theta_{1021}^{11} - \theta_{1121}^{10}$) as a function of $\beta$ at a given initial spanwise wave number ($\gamma_0$) and frequency ($\beta_0$) for three different initial Reynolds numbers ($Re_0$). Figure 5.2 shows the same at a given initial Reynolds number and frequency for three different initial spanwise wave numbers. It is seen that this condition is usually met further downstream (at higher $\beta$) than the initial one. Considering the discussion carried out in Chapter
Figure 5.1: The difference \((\theta_{1021}^{11} - \theta_{1121}^{10})\) as a function of \(\beta\) at a given initial spanwise wave number \((\gamma_0)\) and frequency \((\beta_0)\) for three different initial Reynolds numbers \((Re_0)\).

4, regarding the broader significance of the resonance condition we have not sought to satisfy this condition initially. All initial total phase angles except \(\Psi_{1021}^{11}\) and \(\Psi_{1121}^{10}\), in the majority of the cases examined, have been chosen to exactly satisfy the resonance conditions.

5.1 The Purely Two-Dimensional Wave Mode Interactions

In this section we will present the results of the effects of the initial phase of mode 20, \(\psi_{201}\), by varying the initial total phase angle difference, \(\Psi_{1020}^{10}\), for the purely two-dimensional wave mode interactions. In doing this we will set \(\psi_{101} = 0\), using the phase of mode 10 as a reference. This case was studied by Nikitopoulos & Liu
Figure 5.2: The difference $(\theta_{121}^{11} - \theta_{1121}^{10})$ as a function of $\beta$ at a given initial Reynolds number ($Re_0$) and frequency ($\beta_0$) for three different initial spanwise wave numbers ($\gamma_0$). (1987) under the assumption that the modal phases vary according to inviscid linear stability. We will use this simple case as an introduction to the more complex case of five two- and three-dimensional wave modes, pointing out the main features and mechanisms of the problem that are common to both cases. We will therefore not vary all the parameters as was done in Nikitopoulos & Liu (1987) but only examine the effect of the initial phase angles for two subharmonic energy levels. Figure 5.3 shows the results of the nonlinear analysis for the effect of the initial total phase angle difference ($\psi_{1020}^{10}$) on the development of the shear layer when the large-scale coherent structures are assumed to be composed of two wave modes (10 and 20), and the initial energy densities of the subharmonic 10 and the fundamental 20 wave
modes are $10^{-04}$ and $10^{-02}$ respectively. In the initial region, where $E_{10}$ is much smaller than $E_{20}$, the shear layer thickness grows mainly because of energy transfer from the mean flow to the fundamental as shown in Figure 5.4. The growth rate of the shear layer, due to energy drain of the mean flow, is depicted in Figure 5.4(c) and it can be seen that this growth rate reaches a first peak as the energy drain from the fundamental is maximized. Figure 5.4(c) is also indicative of the entrainment ignoring the effect of viscosity in the growth rate of the shear layer. The growth of the fundamental in the initial region is thus responsible for increased entrainment and mixing. The fundamental energy grows first, reaches the maximum point where it becomes neutral, and starts to decay because of loss of energy to the mean flow (see Figures 5.3(b) and 5.4(b)). The subharmonic also follows approximately the same course of the fundamental but peaks further downstream. As the fundamental saturates, the growth of the shear layer declines (Figure 5.4(c)) steadily until the subharmonic becomes strong enough to counteract the trend by extracting enough energy from the mean flow. As the subharmonic grows stronger, it extracts an increasing amount of energy from the mean flow leading to the second peak in Figure 5.4(c). Thus the presence of the subharmonic is responsible for a significant increase in the growth of the shear layer and the subsequent increase in entrainment. In the absence of the subharmonic, a laminar shear layer would of course continue to grow, after the fundamental saturation point, because of viscosity, at much slower rate. In the case of a turbulent shear layer the growth in the same region would be
dictated by local small-scale turbulence energy level (Liu & Merkine 1976). However, the entrainment tied to the growth of the shear layer without the presence of the subharmonic is considerably lower than that caused by the presence of the subharmonic. Thus subharmonic forcing provides a definite advantage in terms of local growth and entrainment enhancement in the early stages of shear layer development.

The peak of the fundamental wave mode energy is related with the first plateau of the shear layer thickness, and the peak of the subharmonic energy is related with the second plateau in the shear layer thickness further downstream. According to Ho & Huang (1982) and Huang (1985) the position of the peak of the fundamental is near the location of the first roll-up of the shear layer. When the fundamental and subharmonic are of comparable strength beginning of vortex-merging has been observed. Vortex-merging is completed when the subharmonic energy has peaked.

As mentioned above, the wave modes will grow to reach a maximum point by receiving energy from the mean flow and will decay by returning energy back to the mean flow. This process is weakly nonlinear since the coupling between the mean flow thickness and the modal energies is of that nature. An additional nonlinear effect is introduced through the energy interaction between the two modes. This interaction is controlled by the total phase angle $\Psi_{1020}^{10}$ and the modal energies as outlined previously. In the initial region of the shear layer, as shown in Figures 5.3 and 5.4, the different initial phase angles and the interaction between modes do not affect the development of the shear layer thickness $\delta$ and the fundamental energy density
Figure 5.3: Effect of the initial phase angle $\psi_{20i}$ on the development of the shear layer for two wave mode when $E_{10i} = 10^{-04}$ and $E_{20i} = 10^{-02}$. 
Figure 5.4: Effect of the initial phase angle $\psi_{20i}$ on the development of the energy integral terms for two wave mode when $E_{10i} = 10^{-04}$ and $E_{20i} = 10^{-02}$. Terms have been multiplied by a factor $10^3$. 
because the fundamental energy production \( E_{20} I_{MW20}/\delta \) is much greater than both the subharmonic energy production \( E_{10} I_{MW10}/\delta \), and the modal interaction term, because of the small energy content of the subharmonic. In the same region the development of the subharmonic energy density \( E_{10} \) is affected by the value of the initial total phase angle because the nonlinear mode-mode interaction can be a sizable fraction of the subharmonic energy production (see Figure 5.4(e)). The total phase angle difference \( \Psi_{1020}^{10} \) controls the direction of energy transfer of the nonlinear wave mode interaction. The direction of the wave mode-mode energy interaction when \( \Psi_{1020}^{10} = \pi \), initially, is almost exactly opposite to that when \( \Psi_{1020}^{10} = 0 \) in the initial region (see enlargement of this region in Figure 5.4(e)). The initial values 0 and \( \pi \) maximize the initial energy exchange between wave modes while they do not affect the initial variation of the phases. The signs of the interactions are opposite for these two values. In similar manner, the initial values \( \pi/2 \) and \( 3\pi/2 \) maximize the nonlinear phase shift and eliminate the initial energy exchange between wave modes. As the subharmonic grows, the effects of the different initial phase angle on the mean flow and the fundamental begin to show. The evolution of the energy exchange between the fundamental and the subharmonic which is presented in Figure 5.4(e) shows that the direction of this energy transfer may change several times downstream. For example in the case with \( \Psi_{1020}^{10} = 0 \) initially the initial transfer is unfavorable to the subharmonic. After 50 initial shear layer thicknesses, however, this transfer has become favorable to the subharmonic and becomes stronger as the
subharmonic and the fundamental become comparable in strength. Shortly after, this interaction rapidly changes sign and the subharmonic starts losing energy to the fundamental once more. The rapid reversal of this energy transfer occurs as the subharmonic energy becomes larger than that of the fundamental and is caused by the local variation of the total phase angle \( \Psi_{1020}^{10} \) which at that location crosses from the third to the fourth quadrant. As a result of the strong overall favorable interaction with the fundamental, the subharmonic is able to extract an increased amount of energy from the mean flow (notice the higher second peaks in Figure 5.4(c)) and the shear layer growth is faster (Figure 5.3(c)). In the case of initial \( \Psi_{1020}^{10} = 3\pi/2 \) the subharmonic loses energy to the fundamental throughout, with the exception of a very weak gain up to \( x/\delta_0 \approx 25 \). As a result the subharmonic has a weaker interaction with the mean flow (notice smallest second peak in Figure 5.4(c)) and the growth of the shear layer is impaired (Figure 5.4(c)). The viscous dissipation terms for the subharmonic (Figure 5.4(d)) and the fundamental (Figure 5.4(f)) are always weaker than the interaction with the mean flow but comparable to the modal interaction. Overall when \( \Psi_{1020}^{10} \) is 0 and \( \pi/2 \) initially, the subharmonic extracts a considerable amount of energy from the fundamental (see Figure 5.4(e)) and the subharmonic grows fast in this region (see Figure 5.3(a)). As a result, under these conditions the shear layer thickness \( \delta \) grows much fast while the subharmonic is growing. In Figure 5.3 we have also shown as a reference the result of the decoupled case which does not include the nonlinear wave mode-mode interaction terms in the
integral equations. After reaching its peak, in the decoupled case, the fundamental energy density $E_{20}$ decreases monotonically and very fast because of the lack of the nonlinear wave mode-mode interaction downstream.

In Figures 5.5 and 5.6 we show the results for different initial values of the phase angle ($\Psi_{1026}^{10}$) when $E_{10i} = 10^{-06}$ and $E_{20i} = 10^{-02}$ i.e. when the subharmonic is initially weaker. It is obvious that in the initial region where the fundamental extracts most of the energy from the mean flow, because the initial energy density of the fundamental is much stronger than that of the subharmonic, the trends are in general the same as in the previous case where the subharmonic was stronger. The initially weaker subharmonic has a weaker overall interaction with the fundamental as can be seen from a comparison between Figures 5.4(e) and 5.6(e). The subharmonic also has a weaker interaction with the mean flow if Figures 5.4(c) and 5.6(c) are compared. The effect of the initial phase angle difference on the streamwise development of the shear layer, however, is not weaker. The fundamental persists further downstream after it saturates and is overtaken by the subharmonic at a much later location than before. This essentially moves vortex-merging and the associated step-like growth of the shear layer downstream, and results in an overall much slower growth of the shear layer and locally reduced entrainment. This is noted by the fact that the peak of the energy extracted from the mean flow because of the subharmonic is almost half as strong in Figure 5.6(c) than it is in Figure 5.4(c). Between the peaks attributed to the fundamental (first) and the subharmonic (second) in
Figure 5.6(c) it is seen that energy is returned to the mean flow by the fundamental and in this region a slow growth of the shear layer (Figure 5.5(c)) is sustained by viscous diffusion. This slower growth of the shear layer is responsible for sustaining the fundamental after it has saturated and allowing the subharmonic to grow over a wider range of the streamwise coordinate (compare the regions of positive energy exchange with the mean flow in Figures 5.4(a) and 5.6(a)). This occurs because slower growth of the shear layer thickness is reflected in slower increases in the local frequency parameter ($\beta$) which controls the values of all the interaction integrals.

Both cases examined, as well as various others not presented here, have indicated that the direction of the interaction between the two wave modes can be reversed several times in the course of the evolution of the flow depending on the total initial phase angle. However, far downstream where the subharmonic dominates and after it has reached its peak growth (i.e. most amplified Strouhal number value) the final interaction is in favor of the fundamental which is thus preserved instead of being extinguished through continuous loss of energy to the mean flow and through dissipation. This is clearly shown by the comparison with the decoupled case in Figures 5.3(b) and 5.5(b). The conclusion reached by this fact is that the mechanism is there, and a strong one indeed, that will produce harmonics of the most amplified frequency.

The fundamental-subharmonic interaction as presented in the two-mode case is a second order effect and certainly weaker that the interaction of each mode with
the mean flow. However, in the more complex and more realistic situation where multiple modes are involved the collective effects of this kind of interaction can be considerable and therefore the mode-mode nonlinear interaction is a significant mechanism for the bi-directional redistribution of energy between the various scales of motion of a transitional flow.

5.2 Three-Dimensional Shear Layer: Effect of the Spanwise Phase Angles ($\phi_{mn}$)

In this section we will present the results of the effects of the spanwise phase angles ($\phi_{21}$ and $\phi_{22}$) for two different sets of the initial energy densities of the five wave modes and for two different spanwise wave numbers (one low and one high). Comparisons will be made in each case with corresponding two-dimensional cases (1a) and (4b) which are used as references. The magnitude of the initial energy density of the single fundamental wave mode 20 in these two-mode cases is equal to the total initial energy density of the combined fundamentals in the five-mode cases ($20 + 21 + 22$). Likewise, the single subharmonic wave mode 10 in these two-mode cases is equal to the initial energy density of the combined subharmonics in the five mode cases ($10 + 11$). In this way we can see how the presence of three-dimensional wave modes affect the development of the shear layer relative to the purely two-dimensional case. By changing the spanwise phase angles we can influence the directions and magnitudes of the nonlinear wave mode-mode interactions of the three-dimensional wave modes according to the wave mode energy and phase
Figure 5.5: Effect of the initial phase angle $\psi_{20i}$ on the development of the shear layer for two wave mode when $E_{10i} = 10^{-08}$ and $E_{20i} = 10^{-02}$.
Terms have been multiplied by a factor $10^3$

Figure 5.6: Effect of the initial phase angle $\psi_{20}$ on the development of the energy integral terms for two wave mode when $E_{10i} = 10^{-06}$ and $E_{20} = 10^{-02}$. 
angle equations presented in Chapter 4. In all cases the energy levels of the subharmonics are initially lower than those of the fundamentals as it is common in low level subharmonic forcing experiments. The fundamentals are usually in the proximity of the most amplified frequency for an unforced shear layer initially. For all cases presented, the initial frequency of the fundamentals is equal to 0.44 which corresponds to the most amplified frequency of the two-dimensional fundamental, while the initial frequency of the subharmonics is $\beta_0 = 0.22$. The results for three values of the spanwise phase angles are going to be presented in all cases; namely, $\phi_{21} = 0, \pi/2, \pi$ and $\phi_{22} = 0, \pi/2, \pi$.

Angle $\phi_{21}$ controls the direct interactions of mode 21 with 11 and 10 as well as the indirect interaction between 10 and 11 as has been shown in Tables 2.1 and 2.2. Angle $\phi_{22}$ only controls the direct interaction between 11 and 22 and thus is expected to be less important as a parameter especially since 22 is the mode with the lowest growth potential, being high frequency and short wave length. The values 0 and $\pi$ have been chosen because they maximize the relevant energy exchanges and nonlinear phase shifts. The signs of the interactions are opposite for these two values. At $\pi/2$ the corresponding interactions are eliminated throughout the evolution of the modes and this gives the opportunity to assess the effects of the nonlinear interactions by eliminating them. The initial modal phase angles $\psi_{mni}$ have been set to satisfy the nonlinear resonance condition 4.27, outlined in section 4.1, in a way that ensures that all the energy interactions between the subharmonics
and the fundamentals are initially maximized and in favor of the subharmonics, when \( \phi_{21} \) and \( \phi_{22} \) are zero.

Figure 5.7 shows the modal-energy and shear-layer-growth results of the nonlinear analysis for the three spanwise phase angles, \( \phi_{21} \), examined at \( \phi_{22} = 0 \). The initial energy densities of the fundamentals are equally strong and those of the subharmonics are also equally strong and the initial spanwise wave number, \( \gamma_i \), is low. Figure 5.8 shows the energy interaction of each mode with the mean flow as expressed by \( E_{mn} I_{MWW_{mn}}/\delta \) in equations 4.2 ~ 4.6. Figure 5.8(e) shows the net energy exchange with the mean flow, expressed by term \( \sum_{mn} (E_{mn} I_{MWW_{mn}}/\delta) \). Because the early growth of the shear layer is governed by this exchange with the large-scale structure rather than the mean viscous dissipation, this term can be interpreted as being representative of the entrainment.

The modal energy dissipation by viscosity, given by \( E_{mn} I_{vW_{mn}}/(Re_0 \delta^2) \), is presented in Figure 5.9. Finally, Figure 5.10 presents the net mode-mode energy exchanges for each mode \( EWW_{mn} \). These represent the net exchange of energy of each mode with all the others collectively. By convention when any of the modal energy exchanges in Figure 5.10 is positive, the corresponding mode gains energy. The same presentation format described for this case is used throughout this section and in many subsequent ones. Figure 5.11 presents the total phase angle differences of wave mode-mode interactions. The total phase angle differences, \( \Psi_{ijkl}^{pq} \), control the direction of the energy transfer of the wave mode-mode interactions.
Table 5.1: The initial conditions for the amplitudes ($E_{mni}$) spanwise phase angles ($\phi_{mni}$) and spanwise wave number ($\gamma_i$) which we examined

<table>
<thead>
<tr>
<th>case</th>
<th>$E_{10i}$</th>
<th>$E_{20i}$</th>
<th>$E_{11i}$</th>
<th>$E_{21i}$</th>
<th>$E_{22i}$</th>
<th>$\phi_{21i}$</th>
<th>$\phi_{22i}$</th>
<th>$\gamma$</th>
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<tr>
<td>1</td>
<td>$10^{-04}$</td>
<td>$10^{-02}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
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<td>$3 \times 10^{-02}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
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<td>$10^{-02}$</td>
<td>$10^{-04}$</td>
<td>$10^{-02}$</td>
<td>$10^{-02}$</td>
<td>0</td>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>3</td>
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<td>$10^{-02}$</td>
<td>$10^{-06}$</td>
<td>$10^{-06}$</td>
<td>$10^{-06}$</td>
<td>0</td>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>3a</td>
<td>$0.99 \times 10^{-04}$</td>
<td>$2.9998 \times 10^{-02}$</td>
<td>$10^{-06}$</td>
<td>$10^{-06}$</td>
<td>$10^{-06}$</td>
<td>0</td>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>4</td>
<td>$10^{-08}$</td>
<td>$10^{-04}$</td>
<td></td>
<td></td>
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Table 5.2: The initial conditions for the amplitudes \( E_{mni} \) spanwise phase angles \( \phi_{mni} \) and spanwise wave number \( \gamma_i \) which we examined

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As mentioned previously the initial phase angles have been chosen so that the energy will be initially transferred from the fundamentals to the subharmonics when $\phi_{21} = 0$ and it is reversed when $\phi_{21} = \pi$. When $\phi_{21} = \pi/2$, the subharmonic wave mode 10 can only interact with the fundamental wave mode 20 while the three-dimensional subharmonic can only interact with the two-fundamentals 20 and 22. The fundamental wave mode 20 does not depend on the spanwise phase angle directly.

This format of presentation has been used throughout this section and most of the subsequent ones.

As it is evident from Figure 5.8, the amplitudes of the fundamental wave modes which are initially stronger (20, 21 and 22) grow mainly due to the extraction of energy from the mean flow in the initial region. In this region, the magnitude of the fundamental energy production ($E_{mn}I_{MW_{mn}}/\delta$) is larger than viscous dissipation terms ($E_{mn}I_{vW_{mn}}/(Re_0\delta^2)$) and the nonlinear wave mode-mode interaction terms ($EWW_{mn}$) as shown in Figures 5.8 ~ 5.10. From the linear stability study the fundamental wave mode production integrals $I_{MW_{20}}$, $I_{MW_{21}}$ and $I_{MW_{22}}$ are comparable in the amplified region according to Figure 4.4. The three-dimensional modes, particularly the short wave length one, have their growth rate reduced by higher viscous dissipation. Following this, all fundamental wave modes do not grow equally until they reach their peak values. Their growth however is not visibly affected by the different initial $\phi_{21}$, since the energy exchange with the subharmonics is weak
Figure 5.7: Effect of the spanwise phase angle $\phi_{21}$: Development of the modal energy densities $E_{mn}$ and the growth of the shear layer $\delta$. Two- and three-dimensional modes are of equal initial strength. Long spanwise wave length ($\gamma = 0.1$). cases (2): $\phi_{21} = 0$, (7): $\pi/2$, (21): $\pi$, (1a): 2-D
Figure 5.8: Effect of the spanwise phase angle $\phi_{21}$: Energy interaction with the mean flow. Two- and three-dimensional modes are of equal initial strength. Long spanwise wave length ($\gamma = 0.1$). cases (2): $\phi_{21} = 0$, (7): $\pi/2$, (21): $\pi$, (1a): 2-D

Terms have been multiplied by a factor of $10^3$. 
Figure 5.9: Effect of the spanwise phase angle $\phi_{21}$: Viscous dissipation. Two- and three-dimensional modes are of equal initial strength. Long spanwise wave length ($\gamma = 0.1$). cases (2): $\phi_{21} = 0$, (7): $\pi/2$, (21): $\pi$, (1a): 2-D

Terms have been multiplied by a factor $10^3$.
Figure 5.10: Effect of the spanwise phase angle $\phi_{21}$: Net energy interaction between modes. Two- and three-dimensional modes are of equal initial strength. Long spanwise wave length ($\gamma = 0.1$). cases (2): $\phi_{21} = 0$, (7): $\pi/2$, (21): $\pi$, (1a): 2-D.
Terms have been divided by a factor of $\pi$.

Figure 5.11: Effect of the spanwise phase angle $\phi_{21}$: Total phase angle difference $\Psi^P_{ijkl}$. Two- and three-dimensional modes are of equal initial strength. Long spanwise wave length ($\gamma = 0.1$). cases (2): $\phi_{21} = 0$, (7): $\pi/2$, (21): $\pi$, (1a): 2-D
relative to the fundamental high energy contents. This is not so for subharmonics, however, that are initially much weaker than the fundamentals and their initial interaction with the mean flow is weak (Figure 5.8(a), (c)) and comparable to the collective energy exchanges with the fundamentals (Figure 5.10(a), (c)). As a result the growth rate of the subharmonics is influenced by this interaction and the spanwise angle $\phi_{21}$ can have a notable effect on this initial evolution of the subharmonics. At $\phi_{21} = 0$ the energy exchanges favor the subharmonics at the onset, and this is sustained through the saturation of the fundamentals. Both subharmonics gain while all fundamentals are losing energy through their interaction. At $\phi_{21} = \pi$ the initial energy exchanges between modes 11, 10 and 21 are reversed and favor the fundamental. This is immediately obvious from the mode interaction energy in Figures 5.10(a), (c) and (d) where magnified windows of the initial evolution are shown due to the small initial magnitude of these interactions. It is also evident in the reduced growth rate of the subharmonics depicted in Figures 5.7(a) and (c). The fundamentals, being stronger, have not been affected.

It is worth to observe that, although the transfer of energy directly controlled by $\phi_{21}$ in this case is initially from the subharmonics to the fundamental 21, this situation quickly reverses, as the phase angles of the modes involved change, and energy soon flows from the fundamental 21 to the subharmonics (Figure 5.10) all the way through the saturation of the former. As to the growth of the shear layer, it remains unaltered in the initial region which is dominated by the fundamentals.
because their growth is not affected by the small amounts of energy exchanges with the weaker subharmonics.

Further downstream the nonlinear wave mode-mode interactions become stronger as we can see in Figure 5.10 and the development of the fundamentals depends strongly on the nonlinear wave mode-mode interaction. The second peaks of the fundamentals in Figure 5.7 are due to the energy transfer from the subharmonics to the fundamentals. As a result of the nonlinear coupling between wave modes the collective energy interaction of 20 with other modes can be considerably modified in an indirect way depending on the value of $\phi_{21}$. This is evident in Figure 5.10(b) where for $\phi_{21} = 0$ the interaction with other modes is overall in favor of mode 20 throughout the domain of its evolution (with the exception of a short initial region), while for $\phi_{21} = \pi$ this interaction is unfavorable in the initial half of the domain of 20 evolution and becomes favorable thereafter.

The three-dimensional wave mode 21 interacts with the subharmonics depending on the spanwise phase angle $\phi_{21}$. Therefore when $\phi_{21}$ is equal to $\pi/2$, the three-dimensional 21 wave mode is decoupled from them and is not involved in the development of the shear layer in the downstream region because it decays due to the lack of the nonlinear wave mode-mode interaction as we can see in Figures 5.7(d) and 5.10(d). When $\phi_{21}$ equals zero, the maximum energy transfers initially from the fundamentals to the subharmonics and the subharmonic wave modes grow strongly in the initial region as shown in Figure 5.7(a) and (c). Therefore the shear layer
thickness grows faster when the subharmonics become dominant due to their increased strength. When $\phi_{21}$ equals $\pi$, the initial energy transfer is the opposite of case $\phi_{21} = 0$. As a result, the amplitudes of the subharmonics do grow slower in the initial region because they initially lose energy to the fundamental wave mode 21. However, the fundamental wave modes 20 and 22 still lose initially energy to the subharmonics no matter what value of $\phi_{21}$ is. So the three-dimensional subharmonic wave mode 11 gains more energy from the fundamental wave modes 20 and 22 than it loses to the fundamental wave mode 21 and soon recovers as shown in Figure 5.10(c). The same is not true, however, for mode 10. Its interaction with 21 is strong enough to reduce the gain coming from the interaction with mode 20 and quickly starts losing energy as seen in Figure 5.10(a).

As shown in Figure 5.7(f), although the downstream energy extraction from the mean flow is largest in the two-wave-mode case (Figure 5.8(e)), the shear layer thickness grows slower than when $\phi_{21}$ equals 0 because of the, initially favorable to the subharmonics, nonlinear mode-mode interactions. The three-dimensional 21 wave mode among the fundamentals loses most energy initially to the subharmonics when $\phi_{21}$ equals 0 (see Figure 5.10). As a result, the amplitudes of the subharmonics grow faster in the initial region when $\phi_{21}$ equals 0 as shown in Figure 5.7. When $\phi_{21}$ increases, the shear layer growth somewhat retards in the downstream region because both the energy interaction between the subharmonics and the mean flow, and the nonlinear wave mode-mode interactions decrease. It is observed from Figure 5.7(f)
that for the five-mode cases where $\phi_{21}$ is 0 and $\pi$, the growth of the shear layer is different than the case where only two-dimensional modes are present. In fact the shear layer grows faster and earlier because of the presence of the three-dimensional modes in the region of subharmonic dominance. The role of the nonlinear interactions with mode 21 in the growth of the shear layer is also outlined in Figure 5.7(f) where it can be seen that for $\phi_{21} = \pi/2$, i.e. in the absence of wave interactions due to 21, the growth of the shear layer is not considerably different than the two-dimensional mode case. It can therefore be concluded that the three-wave interactions due to 21 are probably more important than the direct two-wave interactions between the subharmonics 10 and 11 and the fundamentals 20 and 22. Far downstream after the saturation of the subharmonics the growth of the three-dimensional shear layer is slightly faster because some or all of the three-dimensional modes can still extract energy from the mean flow even after they have started to decay overall. This was pointed out in Chapter 4 when the integral coefficients of the interaction with the mean flow were discussed (see for example Figure 4.4). The modification of the growth of the shear layer in the early stages of subharmonic dominance has been observed experimentally by Bell & Mehta (1992).

Figure 5.11 shows the evolution of the total interaction phase angles $\Psi_{ijkl}^{pq}$ for each interaction between modes. The discontinuities observed are due to $2\pi$ jumps of the interaction integral angles $\theta_{ijkl}^{pq}$ that are restricted to a range of $2\pi$ in the calculations. Therefore these discontinuities have no real effect except for the visual one. Since
\( \Psi_{ijkl}^{pq} \) stays effectively in the same quadrant across these discontinuities the variation of \( \Psi_{ijkl}^{pq} \) within that quadrant is actually continuous. We have included these total phase angles because they provide an easy method of determining the direction of the energy interactions they influence. For interactions involving modes 21 and 22 when 
\[-\pi/2 \leq \phi_{21}, \phi_{22} \leq \pi/2 \]
and for all interactions not involving 21 and 22, the interaction between \( ij \) and \( kl \) is in favor of mode \( ij \) if 
\[(4m + 1)\pi/2 \leq \Psi_{ijkl}^{pq} \leq (4m + 3)\pi/2, \]
where \( m \) is an integer. Likewise for \( \pi/2 \leq \phi_{21}, \phi_{22} \leq 3\pi/2 \) the interaction is in favor of \( ij \) if 
\[(4m - 1)\pi/2 \leq \Psi_{ijkl}^{pq} \leq (4m + 1)\pi/2. \]
We will include the evolution of the total interaction phase angles for every case examined hereafter, even if they are not referred to in our discussion, in order to give the opportunity to the reader to keep track of the direction of each energy coupling between modes, if he/she so wishes. In Figure 5.11 and for \( \phi_{21} = \pi/2 \) one can clearly see the evolution of the total phase angles involving mode 21 in the absence of any related nonlinear phase and energy interactions. They are quite different than those resulting from the coupled cases \( (\phi_{21} = 0 \text{ and } \phi_{21} = \pi) \). An other feature worth noting is that the signature of the total phase angles for the three interactions involving 21 are very similar. This is to be expected since they all involve the same three modes and the corresponding interaction angles \( \theta_{ijkl}^{pq} \) as calculated previously from local linear stability considerations are not considerably different.

Figures 5.12 ~ 5.14 show the results of the nonlinear analysis for the three spanwise phase angles, \( \phi_{21} \), examined when the initial energy densities of the three
dimensional wave modes are stronger than those of the two-dimensional wave modes and the initial spanwise wave number, \( \gamma_i \), is low.

Because the three-dimensional wave modes are stronger than the two-dimensional ones, the shear layer grows mainly because of the energy interaction of the three-dimensional fundamental wave modes (21 and 22) with the mean flow in the initial region, as shown in Figures 5.13(b), (d) and (f). In this region, since the amplitude of 20 wave mode is much weaker than those of 21 and 22, this mode extracts a very small amount of energy from the mean flow as shown in Figure 5.13(b). As a result, the fundamental 20 wave mode does not affect the development of the shear layer in the initial region. However, the fundamental 20 wave mode gains energy from the nonlinear wave mode interactions with the subharmonics in the downstream region and therefore the amplitude of the fundamental 20 wave mode has a second peak in the downstream region (see Figure 5.12(b)). When \( \phi_{21} \) is equal to \( \pi/2 \), the magnitudes of the energy interaction of the subharmonic wave mode 10 with the mean flow (\( E_{10} I_{MW10}/\delta \)) and the nonlinear wave mode-mode interaction (\( EWW_{10} \)) are very small throughout the whole region because the two-dimensional subharmonic is the weakest initially and is subjected to an early energy drain through its initial interactions with the other modes, as shown in Figures 5.13 and 5.14. As a result, when \( \phi_{21} = \pi/2 \), the subharmonic 10 does not influence the shear layer growth until much further downstream. However, at \( \phi_{21} = \pi/2 \) the three-dimensional subharmonic wave mode 11 gains considerably more energy from the mean flow because
Figure 5.12: Effect of the spanwise phase angle $\phi_{21}$: Development of the modal energy densities $E_{mn}$ and the growth of the shear layer $\delta$. Three-dimensional modes initially stronger than two-dimensional modes. Long spanwise wave length ($\gamma = 0.1$). cases (5): $\phi_{21} = 0$, (8): $\pi/2$, (23): $\pi$, (4b): 2-D
Terms have been multiplied by a factor of $10^3$.

Figure 5.13: Effect of the spanwise phase angle $\phi_{21}$: Energy interaction with the mean flow. Three-dimensional modes initially stronger than two-dimensional modes. Long spanwise wave length ($\gamma = 0.1$). cases (5): $\phi_{21} = 0$, (8): $\pi/2$, (23): $\pi$, (4b): 2-D
Figure 5.14: Effect of the spanwise phase angle $\phi_{21}$: Net energy interaction between modes. Three-dimensional modes initially stronger than two-dimensional modes. Long spanwise wave length ($\gamma = 0.1$). cases (5): $\phi_{21} = 0$, (8): $\pi/2$, (23): $\pi$, (4b): 2-D
Figure 5.15: Effect of the spanwise phase angle $\phi_{21}$: Total phase angle difference $\Psi_{ijkt}^{pq}$. Three-dimensional modes initially stronger than two-dimensional modes. Long spanwise wave length ($\gamma = 0.1$). cases (5): $\phi_{21} = 0$, (8): $\pi/2$, (23): $\pi$, (4b): 2-D
the initial interaction with 20 and 22 is not overall unfavorable. As a consequence of the reduced amplitude of the subharmonic wave mode 10 the shear layer growth (Figure 5.12(f)) is considerably slowed down, the overall energy extracted from the mean flow is less and the entrainment reduced (Figure 5.13(e)). On the other hand, the subharmonics gain initially energy, mainly from the three-dimensional fundamental wave mode 21, through the nonlinear wave mode-mode interactions as shown in Figure 5.14(d) when $\phi_{21} = 0$ and $\pi$. Consequently the two-dimensional subharmonic is much stronger and contributes to a more vigorous extraction of energy from the mean flow (Figure 5.13(a)) reflected by a faster growth of the shear layer in the region of subharmonic dominance. It is noteworthy that, although the initially favorable energy exchange, through mode-mode interactions with 21, regarding the subharmonics is reversed as $\phi_{21}$ is changed from 0 to $\pi$, the subharmonics quickly recover, because the nonlinear variation of the total phase angles (Figure 5.15) quickly changes the energy flow direction in favor of the subharmonics. Thus in both cases of $\phi_{21} = 0$ and $\pi$ the overall nonlinear energy interaction is in favor of the subharmonics as it is evident from Figures 5.14(a) and (c). As before, far downstream the nonlinear interactions are invariably in favor of the fundamentals, which experience renewed growth evidenced by second downstream peaks of their energy.

The comparison with the purely two-dimensional case shows that the shear layer growth is somewhat reduced initially because of the stronger three-dimensional modes that have reduced capacity of extracting energy from mean flow. Further
downstream when the subharmonics become dominant the growth distribution is modified considerably. In the case of \( \phi_{21} = 0 \) and \( \pi \) the thickness of the shear layer is larger compared to the purely two-dimensional one, while at \( \phi_{21} = \pi/2 \) the growth of the shear layer is severely delayed until further downstream. Finally it is seen that the effect of \( \phi_{21} \) is somewhat more pronounced in this case where the three-dimensional modes are stronger, compared to the one previously examined where the two- and three-dimensional modes were of equal strength.

The two cases we have discussed thus far involved three-dimensional modes of rather low initial spanwise wave number (\( \gamma_i = 0.1 \)) in the sense that the three-dimensional fundamentals were near their maximum amplification point according to linear stability. We will now consider the case where the initial spanwise wave number is higher (\( \gamma_i = 0.4 \)) and the three-dimensional fundamental wave mode 22 (\( \gamma_i = 0.8 \)) is initially in the neighborhood of neutral stability. For this high spanwise wave number we shall examine the same two cases, in every other respect, that we dealt with for the low wave number. It should also be pointed out that initial phase angles at which the results have been obtained for \( \gamma_i = 0.4 \) satisfy the resonance conditions for maximum initial interactions of the subharmonics with the fundamentals. Figure 5.16 shows the results of the nonlinear analysis for the three different spanwise phase angles, \( \phi_{21} \), under consideration. The three-dimensional wave modes are initially stronger than the two-dimensional wave modes and the initial spanwise wave number, \( \gamma_i \), is high.
Figure 5.16: Effect of the spanwise phase angle $\phi_{21}$: Development of the modal energy densities $E_{mn}$ and the growth of the shear layer $\delta$. Three-dimensional modes initially stronger than two-dimensional modes. Short spanwise wave length ($\gamma = 0.4$). cases (13): $\phi_{21} = 0$, (15): $\pi/2$, (24): $\pi$, (4b): 2-D
Terms have been multiplied by a factor of $10^3$.

Figure 5.17: Effect of the spanwise phase angle $\phi_{21}$: Energy interaction with the mean flow. Three-dimensional modes initially stronger than two-dimensional modes. Short spanwise wave length ($\gamma = 0.4$). cases (13): $\phi_{21} = 0$, (15): $\pi/2$, (24): $\pi$, (4b): 2-D
Figure 5.18: Effect of the spanwise phase angle $\phi_{21}$: Net energy interaction between modes. Three-dimensional modes initially stronger than two-dimensional modes. Short spanwise wave length ($\gamma = 0.4$). cases (13): $\phi_{21} = 0$, (15): $\pi/2$, (24): $\pi$, (4b): 2-D.
Figure 5.19: Effect of the spanwise phase angle $\phi_{21}$: Total phase angle difference $\psi_{ijkl}^{pq}$. Three-dimensional modes initially stronger than two-dimensional modes. Short spanwise wave length ($\gamma = 0.4$). cases (13): $\phi_{21} = 0$, (15): $\pi/2$, (24): $\pi$, (4b): 2-D.

Terms have been divided by a factor of $\pi$. 
As we know from the results of the linear stability analysis, when the initial spanwise wave number \( \gamma_i \) increases, the neutral stability point moves upstream. As expected, the three-dimensional wave modes do not gain considerable amounts of energy from the mean flow because they are very close to the neutral state from their onset. Thus the three-dimensional fundamental wave modes do not extract enough energy from the mean flow to cause vigorous growth of the shear layer. This is obvious from the comparison with the purely two-dimensional case of equal initial energy content, which displays a much more vigorous growth initially, attributed to the absorption of energy from the mean flow by the fundamental. The difference in the overall energy extracted from the mean flow is shown in Figure 5.17(e). Further downstream the total energy extraction from the mean flow by the subharmonics, indeed primarily the two-dimensional one, is higher when \( \phi_{21} = 0 \) and \( \pi \). When \( \phi_{21} = \pi/2 \) the only mode-mode interaction of mode 10 with the two-dimensional fundamental 20 is initially favorable but very weak and becomes unfavorable far downstream. Consequently its growth and strength are considerably diminished (Figure 5.16(a)), its capacity to extract energy from the mean flow weakened (Figure 5.17(a)) and the growth of the shear layer far downstream severely delayed (Figure 5.16(f)). This is not so in the cases where \( \phi_{21} = 0 \) or \( \pi \). In both cases the initial net energy transfer due to mode-mode interactions, although very small, is favorable to the subharmonics (Figures 5.18(a) and (c)) and as a consequence their growth is enhanced resulting in more effective energy drain of the mean flow.
(Figure 5.17(e)) when they prevail downstream and, consequently, faster growth of
the shear layer (Figure 5.16(f)). As seen before in the corresponding case for the
low initial wave number very little difference exists between the $\phi_{21} = 0$ and $\pi$ cases.
This is again so because at $\phi_{21} = \pi$ when the initial mode-mode energy interactions
with $21$ are unfavorable to the subharmonics, a very sharp change (due to nonlinear
phase interaction effects) in the total phase angles shown in Figures 5.19(a), (e)
and (f) brings them to a quadrant where these interactions become favorable. This
causes the evolution from that point on to appear as if the initial energy interac-
tions were identical. The downstream evolution of the shear layer is dominated by
mode 10 since 11, although initially stronger, remains small due to its higher initial
spanwise wave number. The considerable difference in the shear layer evolution ob-
served when $\phi_{21} = \pi/2$, that decouples the fundamental $21$ from the other modes,
underlines the importance of the nonlinear interactions associated with this mode.
The comparison with the purely two-dimensional case leads to similar conclusions
to those reached for the corresponding short initial spanwise wave-length case, and
needs no further discussion.

Figure 5.20 shows the results of the nonlinear analysis for the three spanwise
phase angles, $\phi_{21}$, examined when the initial energies of the fundamentals are equally
strong, the initial energies of the subharmonics are also equally strong, and the initial
spanwise wave number, $\gamma_1$, is high. The energy interaction of each wave mode with
the mean flow, the net exchange of energy of each mode with all others collectively
and the evolution of the total interaction phase angles are shown in Figures 5.21, 5.22 and 5.23 respectively.

The trends are consistent with those discussed in the previous cases. A few things are worth noting though. The growth of the shear layer is overall faster than the previous case because the overall higher energy content of the subharmonics (10+11) and the fundamentals (20+21+22) combined. This was also true for the low wave number case. As in the previous case the far downstream evolution of the shear layer is primarily controlled by the two-dimensional subharmonic. The observed differences in the growth of the mean flow with varying $\phi_{21}$ are primarily due to the modification of the two-dimensional subharmonic growth because of its interaction with the three-dimensional modes. The cases $\phi_{21} = 0$ and $\pi$ are no longer close to identical because the nonlinear phase interaction is quite different (see Figure 5.23) when the initial amplitudes are different. This underlines the strong dependence of the flow evolution on these initial amplitudes.

Figure 5.24 shows results of the nonlinear analysis when the spanwise phase angle $\phi_{22}$ is 0 and $\pi/2$. The initial amplitudes of the fundamentals are stronger than those of the subharmonics and the initial spanwise wave number, $\gamma_i$, is low. All subharmonics are of equal strength and the same is true for all fundamentals.

It is immediately evident that $\phi_{22}$ does not seem to affect the evolution of the modes (other than 22) considerably. This is natural since it only affects a single interaction mechanism, that between 11 and 22, and because the short wave-length
Figure 5.20: Effect of the spanwise phase angle $\phi_{21}$ : Development of the modal energy densities $E_{mn}$ and the growth of the shear layer $\delta$. Two- and three-dimensional modes of equal initial strength. Short spanwise wave length ($\gamma = 0.4$). cases (12): $\phi_{21} = 0$, (14): $\pi/2$, (22): $\pi$, (1a): 2-D
Terms have been multiplied by a factor of $10^3$.

Figure 5.21: Effect of the spanwise phase angle $\phi_{21}$: Energy interaction with the mean flow. Two- and three-dimensional modes of equal initial strength. Short spanwise wave length ($\gamma = 0.4$). cases (12): $\phi_{21} = 0$, (14): $\pi/2$, (22): $\pi$, (1a): 2-D
Figure 5.22: Effect of the spanwise phase angle $\phi_{21}$: Net energy interaction between modes. Two- and three-dimensional modes of equal initial strength. Short spanwise wave length ($\gamma = 0.4$). cases (12): $\phi_{21} = 0$, (14): $\pi/2$, (22): $\pi$, (1a): 2-D
Terms have been divided by a factor of $\pi$.

Figure 5.23: Effect of the spanwise phase angle $\phi_{21}$: Total phase angle difference $\Psi_{ijkl}^{pq}$. Two- and three-dimensional modes of equal initial strength. Short spanwise wave length ($\gamma = 0.4$). cases (12): $\phi_{21} = 0$, (14): $\pi/2$, (22): $\pi$, (1a): 2-D
Figure 5.24: Effect of the spanwise phase angle $\phi_{22}$: Development of the modal energy densities $E_{mn}$ and the growth of the shear layer $\delta$. Two- and three-dimensional modes of equal initial strength. Long spanwise wave length ($\gamma = 0.1$). cases (2): $\phi_{22} = 0$, (9): $\pi/2$, (1a): 2-D
fundamental has the smallest capacity of all the modes to exchange energy with the mean flow. It can be seen that $\phi_{22}$ has a weak indirect effect on all other modes except for 11 and 22 and that this indirect effect is transmitted to all other modes through the three-dimensional subharmonic. This is also true for the case (which is not presented here) where the three-dimensional modes are stronger than the two-dimensional ones, although the $\phi_{22}$ effect is slightly more pronounced. At higher initial spanwise wave numbers we found the $\phi_{22}$ effect to be even weaker as expected. When $\phi_{22} = \pi/2$, the interactions associated with it is eliminated. Far downstream the shear layer growth of the three-dimensional five-mode cases are faster than that of the two-dimensional two-mode case because of the effect of the nonlinear three-dimensional wave mode-mode interactions.

In Figures 5.25 ~ 5.27, we are presenting the combined effect of $\phi_{21}$ and $\phi_{22}$ for the high initial energy densities of the three-dimensional wave modes and the low initial spanwise wave number ($\gamma_i$). In this case only the fundamental wave mode 20 does not directly depend on any spanwise phase angle. When $\phi_{21} = \phi_{22} = \pi$, the three-dimensional fundamental wave modes initially gain energy from the subharmonics (see Figures 5.27(d) and (e)).

The opposite is true when $\phi_{21} = \phi_{22} = 0$. In the case where $\phi_{21} = \phi_{22} = \pi/2$ modes 21 and 22 are decoupled from the rest and all interactions associated with them are eliminated. Consequently only the binary interactions between the two subharmonics and the two-dimensional fundamental 20 mode remain in effect. The
importance of the eliminated interactions is evident from the strong modification of the growth of the subharmonics (Figures 5.25(a) and (c)) and the considerable reduction and delay of the growth of the shear layer far downstream where the subharmonics are dominant (Figure 5.25(f)). This is brought about as mentioned several times before because of the reduction of energy extraction from the mean flow by the weaker subharmonics. Note that in the case where $\phi_{21} = \phi_{22} = \pi$ the evolution is quite different from the case where $\phi_{21} = \pi$ and $\phi_{22} = 0$ that has been presented before. This shows that under certain circumstances even a variation in $\phi_{22}$ can have a noticeable effect although in the majority of cases its impact is not significant as argued previously.

In closing this section we can conclude that the effect of the spanwise phase angle $\phi_{21}$ on the evolution of the wave modes and the shear layer far downstream where the subharmonics become dominant, can be considerable. This effect is dependent on the initial modal energies and the initial spanwise wave number. Although only limited variation of the modal energies has been examined the presented cases are adequate to demonstrate their importance. It is also obvious from the cases where mode 21 was decoupled from all others and all interactions that require its participation turned off ($\phi_{21} = \pi/2$), that the nonlinear, mode-mode, three wave interactions are very important in determining the flow evolution. Comparisons with purely two-dimensional cases of identical net energy content of the fundamentals and subharmonics have shown that the presence of the three-dimensional modes may contribute
to increases or decreases of the shear layer growth depending on the values of the parameters that control the nonlinear mode interactions and consequently can either enhance or attenuate entrainment and mixing. An other observation can be made on the plateaus of the shear layer thickness that are attributed to the saturation of the fundamentals (first plateau) or the subharmonics (second plateau). These plateaus are more or less indifferent to all the parameters varied in this section. This is not surprising since it is known from simple two-dimensional binary mode interaction studies (Nikitopoulos & Liu 1987), that the plateau levels depend only on the initial frequency parameter which we have not varied in this section. This quasi invariance of the plateaus to initial conditions, other than the initial frequency parameter, reflects that the integral entrainment, up to each plateau, which is due to the presence of the large-scale structure, is unaffected. Local variations of the entrainment rate, however, can be significant, and the downstream distance from the onset of the shear layer where this almost invariant integral entrainment has been realized, can be significantly altered.

5.3 Three-Dimensional Shear Layer: Effect of the Initial Phase Angles ($\psi_{mni}$)

To analyze the effects of the initial phase angles ($\psi_{mni}$) for five wave modes, we fixed all initial phase angles to be zero, with the exception of the one phase angle whose effect on the evolution of the shear layer we would like to examine. We tested most of the cases outlined in Tables 5.1 and 5.2 and chose to present the most representative
Figure 5.25: Effect of the spanwise phase angle $\phi_{22}$: Development of the modal energy densities $E_{mn}$ and the growth of the shear layer $\delta$. Three-dimensional modes initially stronger than two-dimensional modes. Long spanwise wavelength ($\gamma = 0.1$). cases (5): $\phi_{22} = 0$, (11a): $\pi/2$, (11): $\pi$, (4b): 2-D
Figure 5.26: Effect of the spanwise phase angle $\phi_{22}$: Energy interaction with the mean flow. Three-dimensional modes initially stronger than two-dimensional modes. Long spanwise wave length ($\gamma = 0.1$). cases (5): $\phi_{22} = 0$, (11a): $\pi/2$, (11): $\pi$, (4b): 2-D.
Figure 5.27: Effect of the spanwise phase angle $\phi_{22}$: Net energy interaction between modes. Three-dimensional modes initially stronger than two-dimensional modes. Long spanwise wave length ($\gamma = 0.1$). cases (5): $\phi_{22} = 0$, (11a): $\pi/2$, (11): $\pi$, (4b): 2-D.
Terms have been divided by a factor of \( \pi \).

Figure 5.28: Effect of the spanwise phase angle \( \phi_{22} \) : Total phase angle difference \( \Psi_{ijkl}^{pq} \). Three-dimensional modes initially stronger than two-dimensional modes. Long spanwise wave length (\( \gamma = 0.1 \)). cases (5): \( \phi_{22} = 0 \), (11a): \( \pi / 2 \), (11): \( \pi \), (4b): 2-D
ones. The three-dimensional 11 wave mode can interact with all other modes, so the evolution of the shear layer is expected to be more strongly dependent on the initial phase angle $\psi_{11i}$. The three-dimensional 21 wave mode among the fundamentals can interact with both subharmonics and takes place in the indirect interaction between 11 and 10. So, the initial phase angle $\psi_{21i}$ can also be important in the flow development. Mode 22 interacts only with its subharmonic 11 and has the lowest potential for growth because it has the highest frequency and shortest wave-length of the interacting modes. Therefore the initial phase angle associated with it, $\psi_{22i}$, is less likely to be an important factor. The two-dimensional fundamental 20 interacts with both subharmonics. In the case of the purely two-dimensional fundamental-subharmonic interaction presented previously in section 5.1 we have made a first examination of the initial phase angle $\psi_{20i}$. Therefore, we will primarily examine in detail the results of the effects of the initial phase angles $\psi_{11i}$ and $\psi_{21i}$.

Figures 5.29 ~ 5.34 show the results of the growth of the shear layer thickness $\delta$ and the amplitudes of the energy contents when the initial phase angle $\psi_{11i}$ varies from $-\pi$ to $\pi$ and the initial frequency of the subharmonics, Reynolds number and spanwise wave number are 0.22, 100 and 0.1, respectively, for case 2. All other initial phase angles and the spanwise phase angles have been kept to zero. Figures 5.35 ~ 5.39 show the energy interaction terms $(E_{mn} I_{\delta} W_{mn}/\delta)$ of each mode with the mean flow and Figures 5.40 ~ 5.44 show the net nonlinear energy interactions of each of the wave modes with all remaining ones. According to Figures 5.30 ~ 5.32 where the
evolution of the fundamentals is shown, their growth is independent of $\psi_{11i}$ in the initial region while the subharmonics are weak. This behavior is typical as we have seen from the cases examined thus far where the subharmonics are initially weaker than the fundamentals. However, during the decay stage of the fundamentals, i.e. when they either return energy to the mean flow (20 and 21) and/or dissipated by viscosity (22), $\psi_{11i}$ has a pronounced effect on modifying and even reversing the decay of the fundamentals due to the nonlinear wave mode-mode interactions. This effect is dependent on the strength of the subharmonics, since it is obvious that the interactions are stronger in the downstream region where the subharmonics are stronger (Figures 5.30 ~ 5.34 and 5.40 ~ 5.42). As shown in Figures 5.35 ~ 5.37, the interaction between the fundamentals and the mean flow has not been affected by the initial phase angle $\psi_{11i}$ in the initial region of fundamental dominance. Further downstream the fundamental interactions with the mean flow are modified because the fundamentals themselves are modified from their mutual interactions with the stronger subharmonics. This is more evident for mode 20 and least evident for 22 because the latter has a higher decay rate, independently from its interaction with 11, because of increased viscous effects. The nonlinear wave mode-mode interaction between the fundamentals and the subharmonics is naturally strongly affected by the initial phase angle $\psi_{11i}$ as shown in Figures 5.40 ~ 5.42. When the initial phase angle $\psi_{11i}$ approaches either $\pi$ or $-\pi$, the fundamentals have the strongest nonlinear interaction with the subharmonics as shown in Figures 5.40 ~ 5.42. However when
the initial phase angle $\psi_{11i}$ is near zero, the nonlinear interaction of the three-
dimensional fundamental wave modes with the subharmonics is much weaker as
shown in Figures 5.41 and 5.42. This happens primarily because the strength of
the three-dimensional 11 wave mode is considerably smaller (see Figure 5.34) in
the entire flow development region. The reason for this dramatic reduction of 11
energy is the early sustained unfavorable overall interaction with the other modes
that weakens it while it is not very strong. This is hard to see from Figure 5.44 but it
is quite evident in Figure 5.47(c) where a representative case is given for $\psi_{11i} = \pi/5$.

Figures 5.33 ~ 5.44 show that the initial phase angle $\psi_{11i}$ has a significant effect
on the growth of the subharmonics. The strength of the three-dimensional 11 wave
mode decreases as $\psi_{11i}$ approaches to $\pi/5$ because of the already mentioned unfa-
vorable interaction with the other modes and because it extracts significantly less
energy from the mean flow. Mode 10 also has its energy growth reduced because of
unfavorable mode interactions in the early stages of its evolution (Figure 5.33) but
not as drastically as 11. As a result, the shear layer thickness $\delta$ grows considerably
slower near $\psi_{11i} = \pi/5$ (see Figure 5.29), even though the growth rate of the subhar-
monic 10 wave mode becomes eventually higher far downstream. Therefore, in this
case, the three-dimensional subharmonic 11 wave mode plays a most important role
in controlling the downstream growth of the shear layer as the initial phase angle
$\psi_{11i}$ is varied. Except for a rather narrow range of $\psi_{11i}$ near $\pi/5$, for the widest range
of $\psi_{11i}$, the fundamental-subharmonic interaction displays an overall typical pattern.
Figure 5.29: Effect of the initial phase angle $\psi_{11i}$ on the growth of the shear layer $\delta$ for case 2

The initial mode interaction favors the subharmonics while downstream as the fundamentals begin to decay and the subharmonics grow stronger the energy flow is reversed in favor of the former. The dramatic reduction in the three-dimensional subharmonic strength in the neighborhood of $\psi_{11i} = \pi/5$ is a product of local collective resonance which in this case is detrimental to the subharmonic growth and that of the shear layer.

To further outline the effect of $\psi_{11i}$, we are going to briefly discuss a "best" and "worst" case based on the three-dimensional plots. The "best" is taken when $\psi_{11i} = \pi$ or $-\pi$ (the results are identical) and the "worst" case when $\psi_{11i} = \pi/5$ according to previous results.
Figure 5.30: Effect of the initial phase angle $\psi_{11i}$ on the development of energy density $E_{20}$ for case 2.

Figure 5.31: Effect of the initial phase angle $\psi_{11i}$ on the development of energy density $E_{21}$ for case 2.
Figure 5.32: Effect of the initial phase angle $\psi_{11}$ on the development of energy density $E_{22}$ for case 2

Figure 5.33: Effect of the initial phase angle $\psi_{11}$ on the development of energy density $E_{10}$ for case 2
Figure 5.34: Effect of the initial phase angle $\psi_{11i}$ on the development of energy density $E_{11}$ for case 2

Figure 5.35: Effect of the initial phase angle $\psi_{11i}$ on the development of $E_{20} I_{MW20}$ for case 2
Figure 5.36: Effect of the initial phase angle $\psi_{11i}$ on the development of $E_{21}I_{MW_{21}}$ for case 2

Figure 5.37: Effect of the initial phase angle $\psi_{11i}$ on the development of $E_{22}I_{MW_{22}}$ for case 2
Figure 5.38: Effect of the initial phase angle $\psi_{11i}$ on the development of $E_{10}\cdot I_{MW10}$ for case 2

Figure 5.39: Effect of the initial phase angle $\psi_{11i}$ on the development of $E_{11}\cdot I_{MW11}$ for case 2
Figure 5.40: Effect of the initial phase angle $\psi_{11}$ on the development of $EWW_{20}$ for case 2
Figure 5.41: Effect of the initial phase angle $\psi_{11}$ on the development of $EWW_{21}$ for case 2

Figure 5.42: Effect of the initial phase angle $\psi_{11}$ on the development of $EWW_{22}$ for case 2
Figure 5.45 shows the results of the nonlinear analysis of the amplitudes of each energy content \( E_{mn} \) and the growth of the shear layer \( \delta \) for these two cases. Figures 5.46 and 5.47 show the energy interactions with the mean flow \( E_{mn} I_{MW_{mn}/\delta} \) and the nonlinear wave mode-mode interactions \( E W W_{mn} \) respectively. In the initial region, there is no difference in the growth of the fundamentals and the shear layer because the fundamental wave modes extract almost equal amount of energy from the mean flow in both cases. The growth of the three-dimensional subharmonic wave mode 11 is considerably suppressed in the "worst" case and it has a second peak because of the nonlinear wave mode-mode interactions far downstream and the lower local Strouhal numbers caused by the slower growth of the shear layer. Lower local Strouhal numbers imply prolonged positive interaction with the mean flow as long as the local Strouhal number is less than that of a neutral disturbance. For the same reason, in the "worst" case, the two-dimensional subharmonic wave mode 10 gains considerably more energy from the mean flow in the downstream region as shown in Figure 5.46 as well as through the nonlinear wave mode-mode interactions as shown in Figure 5.47. In the "worst" case, the fundamental wave modes 20 and 21 initially gain energy from the subharmonics while the three-dimensional short wave length wave mode (22) loses energy to the long-wave subharmonic 11. On the other hand, in the "best" case the subharmonics initially gain energy from the fundamental wave modes. Further downstream the direction of this interaction is unilaterally reversed. In the downstream region, the purely two-dimensional subharmonic wave
Figure 5.43: Effect of the initial phase angle $\psi_{11i}$ on the development of $EWW_{10}$ for case 2

Figure 5.44: Effect of the initial phase angle $\psi_{11i}$ on the development of $EWW_{11}$ for case 2
mode governs the development of the shear layer in the "worst" case. The overall effect of $\psi_{11i}$ on the downstream evolution of the shear layer growth is considerable (Figure 5.45(f)) as is the effect on entrainment (Figure 5.46(e)).

Figures 5.49 ~ 5.54 show the results of the nonlinear analysis when the initial phase angle $\psi_{21i}$ varies from $-\pi$ to $\pi$ for case 2 (Table 5.1). As expected, in the initial region of a shear layer the different initial values of $\psi_{21i}$ do not affect the development of the shear layer thickness and the fundamental energy densities, because the fundamental energy production terms are much larger than the nonlinear wave mode-mode interaction terms. The initial growth of the initially weak subharmonics however is strongly dependent on the initial phase angle $\psi_{21i}$. As shown in Figures 5.51 and 5.52, the three-dimensional subharmonic energy density $E_{11i}$ has the smallest peak values at $\psi_{21i} = -\pi/5$ while the two-dimensional subharmonic wave mode 10 has its maximum peak at the same initial phase angle further downstream, after undergoing a reduced early growth due to interaction with other modes (see Figure 5.57(a)). The development of the mean flow (Figure 5.49) is a function of the initial phase angle $\psi_{21i}$ only in the region where its growth is governed by the subharmonics. The slowest growth of the shear layer thickness occurs in the neighborhood $\psi_{21i} = -\pi/5$ where the energy gain of the three-dimensional subharmonic from the mean flow is small because of the unfavorable nonlinear wave mode-mode interaction. The growth of the shear layer is extremely sensitive to $\psi_{21i}$ variations only within a narrow range of its values, much like the case of $\psi_{11i}$. 
Figure 5.45: Effect of the spanwise phase angle $\psi_{11}$: Development of the modal energy densities $E_{m,n}$ and the growth of the shear layer $\delta$. Two- and three-dimensional modes of equal initial strength.
Terms have been multiplied by a factor of $10^3$.

Figure 5.46: Effect of the spanwise phase angle $\psi_{11}$: Energy interaction with the mean flow. Two- and three-dimensional modes of equal initial strength.
Figure 5.47: Effect of the spanwise phase angle $\psi_{11}$: Net energy interaction between modes. Two- and three-dimensional modes of equal initial strength.

Terms have been multiplied by a factor $10^3$. 
Terms have been divided by a factor of $\pi$.

Figure 5.48: Effect of the spanwise phase angle $\psi_{11}$: Total phase angle difference $\psi_{ijkl}^{PS}$. Two- and three-dimensional modes of equal initial strength.
Figure 5.49: Effect of the initial phase angle $\psi_{21}$ on the growth of the shear layer $\delta$ for case 2

Figure 5.50: Effect of the initial phase angle $\psi_{21}$ on the development of energy density $E_{20}$ for case 2
Figure 5.51: Effect of the initial phase angle $\psi_{21}$ on the development of energy density $E_{10}$ for case 2

Figure 5.52: Effect of the initial phase angle $\psi_{21}$ on the development of energy density $E_{11}$ for case 2
Figure 5.53: Effect of the initial phase angle $\psi_{21}$ on the development of energy density $E_{21}$ for case 2

Figure 5.54: Effect of the initial phase angle $\psi_{21}$ on the development of energy density $E_{22}$ for case 2
Figure 5.55 shows the results for the amplitudes of each energy content ($E_{mn}$) and the growth of the shear layer ($\delta$) for "best" and "worst" cases, when the initial phase angle $\psi_{11i} = \pi$ and $-\pi/5$ respectively, as done previously for $\psi_{11i}$. Figures 5.56 and 5.57 show the energy interactions with the mean flow ($E_{mn}I_{MW_{mn}}/\delta$) and the nonlinear wave mode-mode interactions ($E_{WW_{mn}}$) respectively. It is seen that in the initial region, the subharmonics of the "worst" case grow much slower because of the nonlinear wave mode-mode interactions with the fundamentals as shown in Figure 5.57. In the same region, the fundamentals do not depend on the initial phase angles, so the magnitudes of the energy interactions between the fundamental wave modes and the mean flow are not different between the two cases. The subharmonics gain energy from the nonlinear wave mode-mode interactions with the fundamentals in the initial region for the "best" case as seen in Figure 5.57. The fundamental wave mode 20 gains energy from all the subharmonics and the three-dimensional fundamental wave mode 22 gains energy from the three-dimensional subharmonic wave mode 11 almost throughout the whole region for the "worst" case. In this case only the three-dimensional fundamental wave mode 21 loses energy to the subharmonics in the initial region as shown in Figure 5.57(d). Further downstream, the growth of the shear layer is considerably faster for the "best" case than for the "worst" case because the subharmonics for the "best" case extract more energy from the mean flow (Figure 5.56) due to the initially favorable nonlinear interactions with the fundamentals (see Figure 5.57).
Figures 5.58 and 5.59 show the effects of the initial phase angles $\psi_{20i}$ and $\psi_{22i}$ on the evolution of the shear layer. According to Figures 5.58 and 5.59 the growth of the shear layer is much less sensitive to $\psi_{20i}$ and $\psi_{22i}$ variations, but the most pronounced effect is again confined to a narrow range of their values.

In this section we have examined the effects of the initial phase angles $\psi_{mni}$ on the evolution of the various wave modes and the mean flow. The main characteristics in all cases were that:

1. The fundamentals and the growth of the shear layer are unaffected by the choice of initial phase angles in the initial region where the fundamentals dominate,

2. The development of the shear layer is very sensitive to the initial phase angles further downstream when the subharmonics become dominant. This high sensitivity of the shear layer growth is limited to a narrow band of $\psi_{mni}$ values,

3. When the subharmonics are dominant far downstream, the energy transfer due to their nonlinear interaction with the fundamentals is almost unilaterally in favor of the latter and independent of $\psi_{mni}$,

4. The evolution of the fundamentals is mostly affected when they are not dominant and they are preserved much further downstream than their interaction with the mean flow could afford them in the absence of the nonlinear interactions with the subharmonics,
5. The three-dimensional subharmonic 11 is much more sensitive to the nonlinear interactions with other modes compared to the two-dimensional one, because (a) 11 is less capable to draw energy from the mean flow, (b) is subject to stronger viscous dissipation and (c) has the ability to interact with all other modes.

The quantitative aspects of these observations can be strongly modified depending on the values of those initial parameters that have not been varied in this section (initial energy densities, initial spanwise wave numbers etc). This is also true since the effect of each initial phase angle has been examined for a single set of values of the remaining ones. However, the qualitative aspects of our observations are representative of the general evolution of the flow, on the basis of a number of additional cases that have been examined but are not presented here.

5.4 Three-Dimensional Shear Layer: Effect of the Initial Spanwise Wave Number ($\gamma_i$)

The effect of the initial spanwise wave number on the five mode problem has been examined to some extent in section 5.2. However, for completeness, we will discuss here a single case for four different initial spanwise wave numbers; namely $\gamma_i = 0$, 0.1, 0.3 and 0.4. At $\gamma_i = 0$ the large-scale structure is two-dimensional (infinitely long spanwise wave-length) and only two-modes are involved. In all cases the total initial energy of the subharmonics is preserved as is the total initial energy of the fundamentals.
Figure 5.55: Effect of the spanwise phase angle $\psi_{21}$: Development of the modal energy densities $E_{m,n}$ and the growth of the shear layer $\delta$. Two- and three-dimensional modes of equal initial strength.
Figure 5.60 shows the results of the nonlinear analysis for the four initial spanwise wave numbers, $\gamma_i$, examined. The initial energy densities of the three-dimensional wave modes are stronger than those of the two-dimensional wave modes and the spanwise phase angles are $\phi_{21} = \phi_{22} = 0$. As mentioned before, we have also set the initial values of the total phase angle differences ($\Psi_{ijkl}^{pq}$) controlling the interactions between the fundamentals and subharmonics, to $\pi$, so that a maximum amount of energy will be transferred initially from the fundamentals to the subharmonics. As the initial spanwise wave number $\gamma_i$ increases, the points where the three-dimensional modes become neutral move upstream as indicated by the linear stability analysis shown in Figure 3.4. Thus the three-dimensional wave modes are less amplified and quickly damped at higher wave numbers. Although their evolution is nonlinear, it is evident that the amplitudes of the three-dimensional wave modes are overall weaker while the initial spanwise wave number increases as shown in Figure 5.60. The energy extracted by each of the three-dimensional wave-modes from the mean flow is considerably decreased with increasing initial wave number, as shown in Figures 5.61(c), (d) and (f). The viscous dissipation term of each wave mode ($E_{mn}I_{Wmn}/(\delta Re_{0})$) is shown in Figure 5.62. The three-dimensional modes experience much higher viscous dissipation initially when the spanwise wave number is high. Further downstream the amount of energy dissipated is larger for the small initial spanwise wave number modes, because the dissipation is proportional to the modal energy and in this case the three-dimensional modes are much stronger. In
Terms have been multiplied by a factor of $10^3$.

Figure 5.56: Effect of the spanwise phase angle $\psi_{21}$: Energy interaction with the mean flow. Two- and three-dimensional modes of equal initial strength.
Terms have been multiplied by a factor $10^3$.

Figure 5.57: Effect of the spanwise phase angle $\psi_{21}$: Net energy interaction between modes. Two- and three-dimensional modes of equal initial strength.
Figure 5.58: Effect of the initial phase angle $\psi_{20}$ on the growth of the shear layer $\delta$ for case 2

Figure 5.59: Effect of the initial phase angle $\psi_{22}$ on the growth of the shear layer $\delta$ for case 2
all cases in the initial region the three-dimensional wave modes 21 and 22 are the ones mainly extracting energy from the mean flow. In the same region the fundamental wave mode 20 does not extract a comparable amount of energy from the mean flow because it was initially weaker. Thus, the shear layer thickness $\delta$ grows mainly because of the three-dimensional fundamental wave modes 21 and 22 in the initial region. As $\gamma_i$ increases, the initial growth of the shear layer retards because the dominant modes 22 and 21 are less effective in extracting energy from the mean flow as shown Figure 5.61. The two-dimensional subharmonic 10 and fundamental 20 wave modes do not depend explicitly on the initial spanwise wave number $\gamma_i$. However, they seem to be considerably affected by the variation of $\gamma_i$. This is so, primarily because of the initial reduction of the shear layer growth experienced at low spanwise initial wave numbers. As a consequence of the reduced shear layer thickness the local Strouhal numbers remain at lower values further downstream enabling the two-dimensional fundamental and subharmonic to grow stronger from a longer favorable interaction with the mean flow at an earlier stage of the shear layer development (see Figures 5.61(a) and (b)). The increase in the two-dimensional mode strength contributes to a stronger interaction between them. It is seen from Figures 5.63(a) and (b) that the interaction between the two-dimensional modes is much stronger in the presence of three-dimensional modes with high initial spanwise wave numbers even when compared to the purely two-dimensional case where they start out a lot stronger initially. It is fair to say that the direct interaction between
10 and 20 in the case of high initial spanwise wave number is the only significant one, all others involving three-dimensional modes being weak. All three-dimensional mode-mode interactions are stronger for low $\gamma_i$.

In the downstream region and for low $\gamma_i$, the three-dimensional wave mode 11 contributes to the growth of the shear layer as its strength is comparable to the two-dimensional subharmonic 10. In contrast, at high $\gamma_i$, the downstream evolution of the shear layer thickness is dominated by the two-dimensional subharmonic and experiences a very sharp growth due to its increased strength.

5.5 Three-Dimensional Shear Layer: Effect of the Initial Energy Density ($E_{mni}$)

Previous investigations of two-dimensional large-scale structures have shown that the initial energy contents of existing and/or forced modes are of considerable importance in the evolution of the shear layer. This was also pointed out and demonstrated earlier when the two-dimensional case was examined as an introduction to the three-dimensional nonlinear calculations. We also have discussed initial energy effects in the section dealing with the influence of the spanwise phase angles $\phi_{21}$ and $\phi_{22}$. In there we examined two basic cases; one case involved all subharmonics of equal strength and all fundamentals of equal strength, while the other had the three-dimensional modes stronger than the two-dimensional ones. In this section we will try to outline the effect of the variation of the total initial energy corresponding to each frequency, and the distribution of the initial energy between the two- and
Figure 5.60: Effect of the spanwise wave number $\gamma_i$: Development of the modal energy densities $E_{mn}$ and the growth of the shear layer $\delta$. Three-dimensional modes initially stronger than two-dimensional modes. $\phi_{21i} = \phi_{22i} = 0$. cases (5): $\gamma_i = 0$, (25): 0.3, (13): 0.4, (1a): 2-D
Figure 5.61: Effect of the spanwise wave number $\gamma_i$: Energy interaction with the mean flow. Three-dimensional modes initially stronger than two-dimensional modes. $\phi_{2i} = \phi_{22i} = 0$. cases (5): $\gamma_i = 0$, (25): 0.3, (13): 0.4, (1a): 2-D

Terms have been multiplied by a factor of $10^3$. 
Terms have been multiplied by a factor $10^3$

Figure 5.62: Effect of the spanwise wave number $\gamma_i$: Viscous dissipation. Three-dimensional modes initially stronger than two-dimensional modes. $\phi_{21i} = \phi_{22i} = 0$. cases (5): $\gamma_i = 0$, (25): 0.3, (13): 0.4, (1a): 2-D
Figure 5.63: Effect of the spanwise wave number $\gamma_i$: Net energy interaction between modes. Three-dimensional modes initially stronger than two-dimensional modes. $\phi_{21i} = \phi_{22i} = 0$. cases (5): $\gamma_i = 0$, (25): 0.3, (13): 0.4, (1a): 2-D
Figure 5.64: Effect of the spanwise wave number $\gamma_i$: Total phase angle difference $\Psi_{ijkl}^{pq}$. Three-dimensional modes initially stronger than two-dimensional modes. $\phi_{21i} = \phi_{22i} = 0$. cases (5): $\gamma_i = 0$, (25): 0.3, (13): 0.4, (1a): 2-D.
three-dimensional modes of equal frequency, both for the fundamental and subharmonic components by presenting a total of five cases. The purpose is to illustrate the relative importance of the initial strength of the subharmonic and fundamental components and that of the initial strength of three- and two-dimensional structures. The initial conditions other than the energy levels, for all the cases presented, are the typical ones used thus far and are shown on the Figure legends.

The strongest mode at the onset of the shear layer we have examined here, and throughout all previous sections, is of the order of $10^{-02}$ or roughly 1% of the mean flow energy based on the average velocity of the shear layer. This order of magnitude is typical of what is encountered in experiments. High excitation levels are known to cause mean flow distortion and this would alter the problem considerably. Therefore we have kept the highest initial energy levels of the order $10^{-02}$.

Figures 5.65 ~ 5.67 show the influence of the initial strength of the subharmonics. Both the three- and two-dimensional subharmonics are of equal strength and in the two cases shown the total initial subharmonic energies are different by two orders of magnitude. It is clear from Figure 5.65(f), as in the two-dimensional case, the stepwise growth of the shear layer is moved downstream since the subharmonics become dominant further downstream. It is also obvious that the growth rate of the shear layer during the dominance of the subharmonics is reduced when the subharmonics are weaker initially. This is visible when examining the second peaks of the energy exchanged with the mean flow (Figure 5.66). The peak for the case of the initially
weaker subharmonics is not only further downstream but also lower than that of the initially stronger subharmonic. The fundamentals, that are affected only after the subharmonics have become of comparable strength, are modified downstream by the different initial subharmonic energies. It is observed that the two-dimensional fundamental and the long wave-length one experience a much slower decay in the case of initially weaker subharmonics. This is primarily due to the delayed growth of the shear layer. If the shear layer thickness is lower, so is the local Strouhal number and the local spanwise wave number. Consequently under these conditions the fundamentals do not loose as much energy to the mean flow. For the short wave-length fundamental the effect is not as pronounced. With regard to the mode-mode interactions, they are initially weaker as the initial subharmonics are weaker and become noticeable further downstream when the fundamentals are favored. As a result the fundamental second peaks are broader, are also moved downstream, and the fundamentals survive longer.

In Figures 5.68 and 5.69 we explore the effect of the initial relative strength of the two- and three-dimensional subharmonics at a fixed initial subharmonic total energy. Three cases are presented. In case (a) both subharmonics are of equal strength, in case (c) the three-dimensional subharmonic is stronger, almost by an order of magnitude, than the two-dimensional one, and in case (d) the opposite is true. It is seen from Figure 5.68 that, as far as the shear layer growth is concerned, the re-distribution of energy between the two- and three-dimensional subharmonics
Figure 5.65: Effect of the initial energy densities $E_{mn}$: Development of the modal energy densities $E_{mn}$ and the growth of the shear layer $\delta$. $\phi_{21i} = \phi_{22i} = 0$. Long spanwise wave length ($\gamma = 0.1$), cases (a), (b)
Terms have been multiplied by a factor of $10^3$.

Figure 5.66: Effect of the initial energy densities $E_{nn}$: Energy interaction with the mean flow. $\phi_{21} = \phi_{22} = 0$. Long spanwise wave length ($\gamma = 0.1$), cases (a), (b)
Terms have been multiplied by a factor $10^3$.

Figure 5.67: Effect of the initial energy densities $E_{mn}$: Net energy interaction between modes. $\phi_{21i} = \phi_{22i} = 0$. Long spanwise wave length ($\gamma = 0.1$), cases (a), (b).
Table 5.3: Initial conditions for the amplitudes ($E_{mn}$) used in examining the effect of energy distribution between modes

<table>
<thead>
<tr>
<th>case</th>
<th>$E_{10i}$</th>
<th>$E_{20i}$</th>
<th>$E_{11i}$</th>
<th>$E_{21i}$</th>
<th>$E_{22i}$</th>
<th>$\gamma$</th>
</tr>
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<tr>
<td>a</td>
<td>$10^{-04}$</td>
<td>$10^{-02}$</td>
<td>$10^{-04}$</td>
<td>$10^{-02}$</td>
<td>$10^{-02}$</td>
<td>0.1</td>
</tr>
<tr>
<td>b</td>
<td>$10^{-06}$</td>
<td>$10^{-02}$</td>
<td>$10^{-06}$</td>
<td>$10^{-02}$</td>
<td>$10^{-02}$</td>
<td>0.1</td>
</tr>
<tr>
<td>c</td>
<td>$0.1 \times 10^{-04}$</td>
<td>$10^{-02}$</td>
<td>$1.9 \times 10^{-04}$</td>
<td>$10^{-02}$</td>
<td>$10^{-02}$</td>
<td>0.1</td>
</tr>
<tr>
<td>d</td>
<td>$1.9 \times 10^{-04}$</td>
<td>$10^{-02}$</td>
<td>$0.1 \times 10^{-04}$</td>
<td>$10^{-02}$</td>
<td>$10^{-02}$</td>
<td>0.1</td>
</tr>
<tr>
<td>e</td>
<td>$10^{-06}$</td>
<td>$10^{-03}$</td>
<td>$10^{-06}$</td>
<td>$10^{-03}$</td>
<td>$10^{-03}$</td>
<td>0.1</td>
</tr>
<tr>
<td>f</td>
<td>$10^{-06}$</td>
<td>$1.4 \times 10^{-03}$</td>
<td>$10^{-06}$</td>
<td>$0.2 \times 10^{-03}$</td>
<td>$1.4 \times 10^{-03}$</td>
<td>0.1</td>
</tr>
<tr>
<td>g</td>
<td>$10^{-06}$</td>
<td>$1.4 \times 10^{-03}$</td>
<td>$10^{-06}$</td>
<td>$1.4 \times 10^{-03}$</td>
<td>$0.2 \times 10^{-03}$</td>
<td>0.1</td>
</tr>
<tr>
<td>h</td>
<td>$10^{-06}$</td>
<td>$0.2 \times 10^{-03}$</td>
<td>$10^{-06}$</td>
<td>$1.4 \times 10^{-03}$</td>
<td>$1.4 \times 10^{-03}$</td>
<td>0.1</td>
</tr>
</tbody>
</table>

to within our order of magnitude does not severely influence the shear layer development. The observed trend tends to show that a stronger three-dimensional subharmonic leads to a slightly slower growth of the shear layer, while the opposite is true when the two-dimensional one is stronger. Far downstream after the subharmonics have saturated and the shear layer growth is weak this situation is reversed mainly because of the higher viscous dissipation associated with the three-dimensional subharmonic. The fundamentals, particularly 20 and 22 are considerably modified far downstream due to the mode-mode interactions. They survive the longest in the case of an initially stronger three-dimensional subharmonic primarily because of their nonlinear interactions with it (Figures 5.69(b) and (e)).

Overall the conclusion from the two cases examined thus far is that the main factor in the flow evolution far downstream is the total energy of the subharmonics.
Additional tests have shown that the initial energy re-distribution between two- and three-dimensional subharmonics to within an order of magnitude does not have a considerable influence on the flow when the total subharmonic energy is initially the same.

The influence of the initial strength of the fundamentals is shown in Figures 5.70 ~ 5.72. Since the fundamentals are the dominant modes initially, the shear layer growth is severely affected and reduced when their net initial energy content is reduced. As a consequence of the slower initial growth of the shear layer the fundamentals saturate further downstream due to the modification of the local Strouhal number, and spanwise wave number (for the three-dimensional modes) while the potential for energy extraction of the subharmonics is enhanced (see Figure 5.71) because of the same reason. This contributes to the enhanced growth of the shear layer when the subharmonics become dominant (Figure 5.72(f)). As Figure 5.72 shows, the mode-mode interactions are considerably stronger over the majority of the domain of evolution of the shear layer when the initial fundamental energy is reduced. In a short-lived initial region the reverse is true. However, this is soon reversed because downstream all the modes are stronger.

In Figure 5.73 we can see that the effect of the initial distribution of energy among the fundamentals, when their cumulative initial energy is fixed, on the shear layer growth is weak. In the case presented an initially weaker mode 21 results in reduction of shear layer growth downstream, where the subharmonics are dominant,
Figure 5.68: Effect of the initial energy densities $E_{mn}$: Development of the modal energy densities $E_{mn}$ and the growth of the shear layer $\delta$. $\phi_{21i} = \phi_{22i} = 0$. Long spanwise wave length ($\gamma = 0.1$), cases (a), (c), (d)
Figure 5.69: Effect of the initial energy densities $E_{mn}$: Net energy interaction between modes. $\phi_{21} = \phi_{22} = 0$. Long spanwise wave length ($\gamma = 0.1$), cases (a), (c), (d)
Figure 5.70: Effect of the initial energy densities $E_{mn}$: Development of the modal energy densities $E_{mn}$ and the growth of the shear layer $\delta$. $\phi_{21i} = \phi_{22i} = 0$. Long spanwise wave length ($\gamma = 0.1$), cases (b), (e)
Figure 5.71: Effect of the initial energy densities $E_{mn}$: Energy interaction with the mean flow. $\phi_{21i} = \phi_{22i} = 0$. Long spanwise wave length ($\gamma = 0.1$), cases (b), (e).

Terms have been multiplied by a factor of $10^3$. 
Figure 5.72: Effect of the initial energy densities $E_{nni}$: Net energy interaction between modes. $\phi_{21} = \phi_{22} = 0$. Long spanwise wave length ($\gamma = 0.1$), cases (b), (e).

Terms have been multiplied by a factor $10^3$. 
while an initially weak mode 20 leads to faster growth in the same region. The initial growth of the shear layer is also weakly affected. This is more visible in the interaction with the mean flow shown in Figure 5.74. The secondary peaks of the three-dimensional fundamentals, induced by mode-mode interactions, become almost as strong as the original ones, induced by the interaction with the mean flow that are still strong far downstream (see Figure 5.75).

The cases involving variation of the initial energies to within an order of magnitude that have been presented here certainly do not cover all the possible combinations of all the initial conditions. However, additional tests we have performed confirm the basic conclusions reached. Finally, although some effects of the initial energies shown here appear to be weak with respect to the overall shear layer growth, they can make a profound difference on the local variations of some of the individual modes.

5.6 Three-Dimensional Shear Layer: Effect of the Initial Frequency Parameter ($\beta_0$)

In this section we are going to examined the effect of the initial frequency parameter on the development of the shear layer.

Thus far we have examined the effects of various parameters for an initial Strouhal number of the subharmonics of $\beta_0 = 0.22$ so that the fundamentals are near the most amplified frequency ($\beta_f = 0.44$) to begin with. This case has been chosen as the most realistic one for a situation where the subharmonic forcing is
Figure 5.73: Effect of the initial energy densities $E_{mn}$: Development of the modal energy densities $E_{mn}$ and the growth of the shear layer $\delta$. $\phi_{21i} = \phi_{22i} = 0$. Long spanwise wave length ($\gamma = 0.1$), cases (e), (f), (g), (h).
Figure 5.74: Effect of the initial energy densities $E_{mn}$: Energy interaction with the mean flow. $\phi_{21i} = \phi_{22i} = 0$. Long spanwise wave length ($\gamma = 0.1$), cases (e), (f), (g), (h).

Terms have been multiplied by a factor of $10^3$. 
Figure 5.75: Effect of the initial energy densities $E_{\text{ini}}$: Net energy interaction between modes. $\phi_{21i} = \phi_{22i} = 0$. Long spanwise wave length ($\gamma = 0.1$), cases (e), (f), (g), (h).
imposed at a frequency half that of the fundamental. Under this condition the interaction between fundamental and subharmonic can produce either the subharmonic or a mode with $3/2$ the frequency of the fundamental. The subharmonic through self-interaction can produce the fundamental and the latter through self-interaction may produce frequencies twice that of itself. The generation of these additional modes can quickly generate other modes and the situation becomes very complicated very fast.

All modes that can be generated in this way, however, are at frequencies higher than the fundamental and this means that they do not have (at least according to linear stability) a high potential for growth by interacting with the mean flow. Therefore the case examined thus far involving the fundamental and its subharmonic (the first one) is reasonable especially if the energy levels of all other modes are considerably smaller. The mode with frequency $3/2$ of the fundamental is the only one of those that can be generated that can still experience sustained growth through the interaction with the mean flow. This mode has been observed to be present in experiments (Ho & Huang 1982) and its effects have been dealt with elsewhere (Nikitopoulos & Liu 1990) for the purely two-dimensional case.

Theoretically speaking we can study under the present analysis any pair of frequencies related by a factor of two and call them "fundamental" and "subharmonic". Realistically in such a case these two modes except for the interaction between them will definitely have the most amplified frequency (the natural frequency) of the shear
layer to interact with. This has two serious consequences. First, it is obvious that in such case the two frequency model is no longer acceptable, second, the interactions between the three primary modes can definitely generate frequencies with the potential to grow through their interaction with the mean flow and that can therefore not be ignored. It is therefore understandable that examining the effect of different initial frequencies (Strouhal numbers) is rather academic. The only case that could be considered realistic is that of the most amplified (natural) frequency and its first harmonic. Keeping this in mind we will examine three initial "subharmonic" frequencies $\beta_0 = 0.1, 0.22 \text{ and } 0.4$. The last one is in the neighborhood of the most amplified one.

Figures 5.76 ~ 5.78 show the results of the nonlinear analysis with initial conditions those of case 2 (see Table 5.1). According to Figure 5.76 the position of the peaks of the energy densities of the large-scale structures move upstream with increasing initial frequency, and their magnitude decreases. Lower initial frequencies have a longer amplification region from the interaction with the mean flow as linear stability indicates. Consequently the large-scale coherent structures overall extract more energy from the mean flow and the nonlinear wave mode-mode interactions as shown in Figures 5.77 and 5.78 tend to be more vigorous. As a result, the shear layer thickness grows stronger when $\beta_0$ is low than when it is high. The shear layer thickness roughly doubles for all initial frequencies as it is well known from purely two-dimensional studies. As it is also known from the previous studies of the purely
two-dimensional shear layer the plateaus in the shear layer growth are higher for lower initial frequencies because of the longer amplification life of the wave modes and the subsequent increased energy drain of the mean flow. This trend is apparent in the results shown in Figure 5.76 and is independent of the three-dimensionality of the flow. When the initial frequency \( \beta_0 \) is 0.4, the initial frequency of the fundamental wave modes corresponds to 0.8. In this case the "fundamental" wave modes are close to the neutral stability point according to the linear stability analysis. Therefore the "fundamental" wave modes in this case become immediately damped and are overtaken by the "subharmonics" very soon. As a result of this fact, and in conjunction with what is known from the experimental observations regarding the correlation of the modal energy evolution with visible events such as the shear layer roll-up and vortex-pairing, it is possible that when \( \beta_0 = 0.4 \) the first roll-up is due to the "subharmonic" even though it starts out being much weaker than the "fundamental". This case is actually one where a frequency close to the first harmonic of the natural frequency of the shear layer is forced at a level higher than that of the natural frequency.

At low initial frequencies, the initial local growth of the "subharmonics" because of interaction with the mean flow is weaker, but increases further downstream as the Strouhal number approaches the most amplified value. The overall interaction with the mean flow of the three-dimensional subharmonic is impaired because of a local unfavorable mode-mode interaction (around \( x/\delta \approx 200 \)) as shown in Figure 5.78(c).
It is also observed that the net interactions between modes can reverse direction several times more at low initial frequencies. For instance $EWW_{11}$ and $EWW_{10}$ reverse direction four times. The vigorous variation of the mode-mode interactions are responsible for the appearance of multiple peaks of "fundamental" energies (mode 21 has three). It is also worth pointing out that very far downstream, where the "fundamentals" are sustained through favorable to them interactions with the "sub-harmonics", the three-dimensional "fundamental" 21 becomes stronger than the two-dimensional one.

5.7 Three-Dimensional Shear Layer: Effect of the Initial Reynolds Number ($Re_0$)

The experimental works by Konrad (1976), Breidenthal (1978) and Bernal & Roshko (1986) have indicated that certain aspects of the evolution of three-dimensional shear layers depend on the Reynolds number. This provides adequate motivation for us to examine the effect of the Reynolds number on the evolution of the five modes included in this study.

Results of initial Reynolds number variation are shown in Figures 5.80 ~ 5.83 for case 2 (see Table 5.1). It is seen from Figure 5.80 that the initial growth of all the modes is reduced as the Reynolds number is decreased. This is to be expected since linear stability indicates lower amplification rates at lower Reynolds numbers. The reduced growth of the modes is primarily attributed to the reduced strength of the interaction of all of them with the mean flow (Figure 5.81) and the initially higher
Figure 5.76: Effect of the initial frequency $\beta_0$: Development of the modal energy densities $E_{mn}$ and the growth of the shear layer $\delta$. Two- and three-dimensional modes are of equal initial strength. $\phi_{21i} = \phi_{22i} = 0$. Long spanwise wave length ($\gamma = 0.1$), $Re_0 = 100$
Terms have been multiplied by a factor of $10^3$.

Figure 5.77: Effect of the initial frequency $\beta_0$: Energy interaction with the mean flow. Two- and three-dimensional modes are of equal initial strength. $\phi_{21i} = \phi_{22i} = 0$. Long spanwise wave length ($\gamma = 0.1$), $Re_0 = 100$
Figure 5.78: Effect of the initial frequency $\beta_0$ : Net energy interaction between modes. Two- and three-dimensional modes are of equal initial strength. $\phi_{21i} = \phi_{22i} = 0$. Long spanwise wave length ($\gamma = 0.1$), $Re_0 = 100$. Terms have been multiplied by a factor $10^3$. 
Terms have been divided by a factor of $\pi$.

Figure 5.79: Effect of the initial frequency $\beta_0$ : Total phase angle difference $\Psi_{ijkl}^{pq}$. Two- and three-dimensional modes are of equal initial strength. $\phi_{21i} = \phi_{22i} = 0$. Long spanwise wave length ($\gamma = 0.1$)
viscous dissipation (Figures 5.82(b), (d) and (e)). The peaks of all the fundamentals increase monotonically with increasing initial Reynolds number and are also moved downstream. As a consequence, the growth of the shear layer at the early stage of its development is more pronounced as the Reynolds number increases. The strength of the subharmonic peaks is also increased with increasing Reynolds number partly because of the early favorable interaction with other modes (Figure 5.83) and mainly because of increased interaction with the mean flow (Figure 5.81). These two mechanisms are of course coupled and mutually enhance or impair each other. Overall, the mode-mode interactions are stronger with increasing Reynolds number because the participating modes are stronger.

It is observed that as the initial Reynolds number increases the sensitivity of evolution of the shear layer and the wave modes is reduced. This is expected as the Reynolds number increases towards the inviscid limit. It is also observed that at lower Reynolds numbers the shear layer sustains some noticeable growth even after the subharmonics have saturated and are in decline. This is quite visible for the $Re_0 = 50$ case and is primarily due to the reduced ability of the subharmonics and the fundamentals (especially the three-dimensional ones), to feed energy back to the mean flow and counter the effect of viscosity (see Figure 5.81(a)).

The case presented here is indicative of the effect that the initial Reynolds number may have on the development of the modes and the shear layer. Of course all other initial conditions were held constant and it is obvious that the effects of the
initial Reynolds number variation will be quantitatively, and to a certain extent qualitatively, different when all other initial conditions are varied. The purpose of this section was to show that there is some noticeable effect at low Reynolds numbers, the possible consequences of which will be put in perspective in the next section.

5.8 Interpretations of Vortex Merging, Spanwise Wave Length Appearance and Doubling

A number of experiments have been performed, particularly during recent years, focusing on the shear layer large-scale structures, both two- and three-dimensional. Quantitative measurements of large-scale, individual wave component energies and shear layer growth have been performed (e.g. Huang & Ho 1982; Zhang et al 1984; Cohen & Wygnanski 1987) assuming purely two-dimensional structure. Comparisons of previous results of the present theory (Nikitopoulos & Liu 1987, 1990) with such measurements have indicated excellent qualitative agreement and satisfactory quantitative agreement considering that initial conditions were not fully available in the experiments. The experimental work in three-dimensional shear layers is mostly of qualitative nature based on flow visualization. Some local measurements have been performed but not in a form that would allow direct comparisons with our results. More specifically, energy levels of two-dimensional and three-dimensional modes have not been measured yet, and in fact, the modal energy level measurements that have been preformed in the past for the "two-dimensional" modes may
Figure 5.80: Effect of the initial Reynolds number $Re_0$: Development of the modal energy densities $E_{mn}$ and the growth of the shear layer $\delta$. Two- and three-dimensional modes are of equal initial strength. $\phi_{21i} = \phi_{22i} = 0$. Long spanwise wave length ($\gamma = 0.1$), $\beta_0 = 0.22$
Terms have been multiplied by a factor of $10^3$.

Figure 5.81: Effect of the initial Reynolds number $Re_0$: Energy interaction with the mean flow. Two- and three-dimensional modes are of equal initial strength. $\phi_{21i} = \phi_{22i} = 0$. Long spanwise wave length ($\gamma = 0.1$), $\beta_0 = 0.22$
Terms have been multiplied by a factor $10^3$

Figure 5.82: Effect of the initial Reynolds number $Re_0$: Viscous dissipation. Two- and three-dimensional modes are of equal initial strength. $\phi_{21i} = \phi_{22i} = 0$. Long spanwise wave length ($\gamma = 0.1$), $\beta_0 = 0.22$
Figure 5.83: Effect of the initial Reynolds number $Re_0$: Net energy interaction between modes. Two- and three-dimensional modes are of equal initial strength. $\phi_{21i} = \phi_{22i} = 0$. Long spanwise wave length ($\gamma = 0.1$), $\beta_0 = 0.22$.
actually represent energies of two- and three-dimensional modes combined since no attempt to discriminate between them was ever made. In addition, the multitude of initial conditions relevant to the multimode problem make such measurements a very challenging undertaking.

Given this lack of quantitative experimental data appropriate for direct comparisons, we will limit ourselves to a qualitative interpretation of some of the observed experimental trends in three-dimensional shear layers on the basis of the present theory. In our effort to do this we have selected a few representative cases shown in Figures 5.84 and 5.85. Figure 5.84 shows in turn the evolution of the five individual mode energies (Figure 5.84(a)), the evolution of the energies of the fundamentals combined and the subharmonics combined (Figure 5.84(b)), and the evolution of the energies of the three-dimensional short wave-length mode and the three-dimensional long wave-length modes combined (Figure 5.84(c)). The growth of the shear layer is shown as well in Figure 5.84(d). It has been established from previous experimental and theoretical studies (Ho & Huang 1982; Nikitopoulos & Liu 1987) that

- the peak of the two-dimensional fundamental energy is associated with the completion of the first roll-up of the shear layer,

- the region where the two-dimensional subharmonic becomes comparable in strength to the fundamental marks the beginning of vortex merging, and

- the peak of the two-dimensional subharmonic marks the completion of vortex merging.
In parallel to the energy evolution the initial fast growth of the shear layer is caused by the fundamental and the associated first roll-up, and the renewed step-like growth of the shear layer is brought about by the growth and dominance of the subharmonic and the ensuing vortex merging. The same interpretation can obviously be rendered, on the basis of combined fundamental and subharmonic energies, in the presence of three-dimensional modes. In such case the location of the relevant energy peaks and the parity of subharmonic and fundamental strength are modified depending of course on initial conditions. In any case the essence of the interpretation does not change.

Regarding the three-dimensional structure, experimental observations indicate (Jimenez et al 1985; Bell & Mehta 1992) that the spanwise wave-length often doubles downstream and this doubling occurs near the location where vortex merging begins. Also observations indicate that a strong three-dimensional structure manifested in the force of streamwise vorticity streaks first appears with the first roll-up of the shear layer in the braids connecting successive vortices. If we observe Figure 5.84(c) we can see that the peak of the short spanwise wave-length fundamental is in the same neighborhood as the peak of the combined fundamentals. If we interpret the peak of 22 mode the location where the streamwise vorticity becomes most pronounced agreement with experimental observation becomes apparent. Regarding the doubling of the spanwise wave-length, if we associate its first appearance with the neighborhood in which parity of the strength of 22 and 21+11 is effected, and
compare with the corresponding neighborhood of vortex merging, we again find consistency with experimental observation. This is particularly so considering that it has been pointed out by Corcos & Lin (1984) and Bell & Mehta (1992), among others, that the identification of the exact location of these three-dimensional events in experiments often depends on the resolution of the visual or quantitative observation. Although this fact may cast a shadow of doubt on some experimental observations it has been argued by Lasherás et al (1986) that the location of the first appearance of streamwise vorticity and other three-dimensional events downstream is significantly dependent on initial conditions. The results we have presented in previous sections indicate that the general evolution of the modes depends on a variety initial conditions. In principle, this is in agreement with Lasherás' observation. The first appearance of streamwise vorticity is most likely dependent on the initial spanwise wave number and the initial modal energy contents since these are more likely to influence the peak of 22, according to our results presented in previous sections. The initial Reynolds number has been argued by Konrad (1976), Breidenthal (1978) and Bernal & Roshko (1986) to be a potential factor in the appearance of the streamwise vortex streaks. In Figure 5.85 we show results for a lower Reynolds number showing that the 22 peak associated with the emergence of the streamwise vorticity has been moved upstream. Bearing in mind that the Reynolds number sensitivity diminishes as this parameter increases, it is unlikely that it can be a considerable factor at higher Reynolds numbers. Again it can be argued that the
existence of a multitude of initial conditions that can not be measured nor controlled in experiments, can obscure the true factors in the appearance and behavior of the three-dimensional structures.

Finally, Figure 5.86 presents a case where no doubling of spanwise wave-length would be observed because the short wave-length fundamental is initially weaker. The experiments of Lasheras et al (1986) and Lasheras & Choi (1988) show no change in spanwise wave-length with downstream distance. They always used forcing of a single wave-length of the three-dimensional structure in their experiments and according to our interpretation it is possible that the level of forcing was strong enough to favor only the wave-length forced just like the case presented in Figure 5.86.

In this section we have given a qualitative interpretation of experimental observations tied to our quasi-steady, quasi-parallel, nonlinear theory. By no means is this interpretation unique especially since the presence of additional modes, such as a short wave subharmonic (12) could modify it. However, this interpretation can explain the qualitative observations in a satisfactory manner employing only the five modes considered here.
Figure 5.84: Interpretations of vortex merging, spanwise wave length appearance and doubling. Long spanwise wave length ($\gamma = 0.1$), $\beta_0 = 0.22$, $Re_0 = 200$
Figure 5.85: Interpretations of vortex merging, spanwise wave length appearance and doubling. Long spanwise wave length ($\gamma = 0.1$), $\beta_0 = 0.22$, $Re_0 = 50$
Figure 5.86: Interpretations of vortex merging, spanwise wave length appearance and doubling. Long spanwise wave length ($\gamma = 0.1$), $\beta_0 = 0.22$, $Re_0 = 200$ when $E_{21i} > E_{22i}$
Chapter 6
Conclusions and Further Research

6.1 Conclusions

We have formulated here the nonlinear interaction problem for 2 two-dimensional wave modes and 3 three-dimensional wave modes in a developing shear layer. To solve the nonlinear problem, we first formulated and solved the local linear, viscous, stability problem for two- and three-dimensional disturbances. The nonlinear formulation used, is based on an energy method which leads to a set of nonlinearly-coupled, ordinary differential equations governing the evolution of the modal energies and phases. The nonlinear equations involve integral coefficients that are calculated from the eigenfunctions resulting from the solution of the local viscous linear stability problem. The linear problem is an eigenvalue problem and was solved by using a traditional shooting method with orthonormalization, to maintain linear independence of the solutions. Bi-directional integration with matching at an intermediate point within the doubly infinite domain was used to ensure the convergence and the
accuracy in satisfying the boundary conditions at ±∞. The boundary conditions are given by the asymptotic solutions of the linear equations at infinity. Based on the linear stability results and the calculated integral coefficients of the nonlinear problem, general resonance conditions for the interactions between the fundamentals and their subharmonics were formulated.

Solutions of a nonlinear problem were obtained for two-dimensional, subharmonic-fundamental interaction problem which was treated as a test case. The full three-dimensional five-mode problem involved two three-dimensional fundamentals, one three-dimensional subharmonic, and a two-dimensional fundamental-subharmonic pair. This problem was solved for a variety of combinations of the initial parameters. These parameters were identified as the initial Strouhal number, spanwise wave number and Reynolds number, the initial energy levels for the five interacting modes, their initial phase angles relative to the two-dimensional subharmonic and the spanwise phases relative to the three-dimensional subharmonic. Given the multitude of initial conditions a limited parametric analysis was carried out.

In all cases, except for that examining the initial Strouhal number effect, the fundamental was taken to be near the most amplified frequency (the natural response frequency of the shear layer). The initial energy level of the fundamentals was always taken to be at least an order of magnitude higher than that of the subharmonics, in accordance with existing quantitative measurements in both unperturbed and forced shear layers.
The most general conclusion reached from this work is that, independently of any of the initial conditions, nonlinear interactions between the fundamental and subharmonic wave modes will clearly favor the fundamentals in the far downstream region of the shear layer where the subharmonics are dominant. The fundamentals are thus preserved much further downstream than their interaction with the mean flow could afford them in the absence of these nonlinear interactions with the subharmonics. This implies that the mechanism exists that will produce harmonics of the most amplified frequency far downstream. On the other hand, the nonlinear mode-mode interactions tend to favor the subharmonics during the early stages of development when the fundamental is dominant. However, the direction of this early interaction does not have the universality of the one favoring the fundamentals far downstream and is sensitive to initial conditions. Depending on the initial conditions this interaction can be forced to be initially unfavorable to the subharmonics and contributes to their considerable weakening. It was shown that depending on the initial modal phase angles, particularly those of the long wavelength fundamental and subharmonic, the downstream growth of the shear layer and the associated local entrainment can be strongly affected by the presence of these nonlinear interactions. This high sensitivity of the shear layer growth is limited to a narrow band of initial phase angle values. The fundamentals and the growth of the shear layer are unaffected by the choice of initial phase angles in the initial region where the fundamentals dominate. The three-dimensional subharmonic 11 is
much more sensitive to the nonlinear interactions with other modes compared to the two-dimensional one, because (a) 11 is less capable to draw energy from the mean flow, (b) is subject to stronger viscous dissipation and (c) has the ability to interact with all other modes.

The effect of the spanwise phase angle of the long-wave-length fundamental on the evolution of the wave modes and the shear layer far downstream, where the subharmonics become dominant, can be considerable. This effect is dependent on the initial modal energies and the initial spanwise wave number. Cases examined, where the long wave-length fundamental was decoupled from all others and all interactions that require its participation were turned off, revealed that the nonlinear, mode-mode, three-wave interactions can be very important in determining the flow evolution. Comparisons with purely two-dimensional cases of identical net energy content of the fundamentals and subharmonics have shown that the presence of the three-dimensional modes may contribute to increases or decreases of the shear layer growth depending on the setting of the parameters that control the nonlinear mode interactions and consequently can either enhance or attenuate entrainment and mixing. The effect of the spanwise phase of the short wave-length fundamental was shown to be weak.

The growth of the three-dimensional shear layer in general, displays the same trends observed for the two-dimensional one. Namely the shear layer grows in a step-wise manner, and two successive plateaus are observed in its growth. The first
one is associated with the growth and saturation of the fundamentals, while the second one is associated with the growth and saturation of the subharmonics. The primary mechanism driving the growth of the shear layer is the interaction of the wave modes with the mean flow. The shear layer thickness roughly doubles as a result of the presence of the subharmonics. The level of the plateaus is dependent on the initial frequency parameter and is more or less invariant to initial conditions if the fundamental is initially stronger than the subharmonic. This quasi invariance of the plateaus to initial conditions, other than the initial frequency parameter, reflects that the integral entrainment, up to each plateau which is due to the presence of the large-scale structure, is unaffected. Local variations of the entrainment rate, however, can be significant, and the downstream distance from the onset of the shear layer where this almost invariant integral entrainment has been realized, can be significantly altered.

The evolution of the shear layer is strongly influenced by the net energy content of all the fundamentals combined, as well as the net energy of the combined subharmonics. When the subharmonic energy is reduced the step-wise growth is moved downstream and the growth rate is reduced. A variation of energy distribution between the two- and three-dimensional modes to within an order of magnitude and at constant total subharmonic and fundamental energies, is shown to have weak effects. Shear layer growth is also shown to be sensitive to the initial spanwise wave number. The sensitivity depends on the initial strength of the three-dimensional wave modes
relative to the two-dimensional ones. The sensitivity of the shear layer on the spanwise wave number is more evident when the three-dimensional modes are set to be initially much stronger than the two-dimensional ones. In general, three-dimensional modes with initially higher wave numbers are less effective in influencing the shear layer evolution directly, or indirectly through nonlinear, mode-mode interactions.

Some of our results based on the energies of the five modes included in this analysis, provide a reasonable interpretation for experimental observations regarding the appearance of the three-dimensional structure, the spanwise wave-length doubling downstream, and the three-dimensional vortex-merging event. The appearance of three dimensionality is associated with the peaking of the short wave-length fundamental, which occurs in the neighborhood of the first roll-up. The beginning of the process leading to wave-length doubling is associated with the location where the energy of the short wave-length fundamental becomes comparable in strength with the long wave-length mode energy. This location is in the neighborhood of the initiation of vortex-merging. As speculated by other previous investigations a weak dependence of the location of these events on Reynolds number has been shown to exist. A much stronger dependence on other initial conditions has also been demonstrated in agreement with experimental evidence. A possible explanation for the absence of spanwise wave-length doubling has been given on the basis of the influence of initial energy levels on the mode evolution in the shear layer.
During the last decade increasing recognition of the importance of the role of large-scale structures in transitional shear flows has motivated a number of studies aiming at the enhancement of mixing and transport by forcing large-scale structure modes. This can be achieved both by active and passive forcing. Most work in this area tied to practical applications in combustors, ribbed heat exchangers, electronic cooling systems and more recently in gas turbine blade ducts, involves forcing of the subharmonic of the naturally dominant large-scale structure mode with satisfactory results. In order to achieve optimum mixing and transport augmentation in the early stages of transition using large-scale structure forcing, it is necessary to understand the mechanisms and parameters that govern the evolution of several two- and three-dimensional large-scale structure modes in this flow and their mutual interactions with other scales of motion. In this context the results of the present theoretical work can provide guidelines for the choice of appropriate forcing parameters for practical applications utilizing multi-large-scale structure forcing. It can also aid in the interpretation of observed trends in mixing and transport augmentation experiments and provide the basis for the successful modeling and prediction of transitional shear flows incorporating large-scale structure effects.

Furthermore the results of our investigation provide guidelines for the development of quantitative experiments that can clarify the influence of the initial conditions identified herein, on the evolution of forced three-dimensional shear layers and the role of the three-dimensional large-scale structure in the transition process. Such
experiments are not available to date and are necessary in order to quantitatively assess the validity of the theoretical conclusions and results of this study.

6.2 Further Research

As we mentioned in chapter 1, we do not include the small-scale turbulence in the present research because the focus of the present study is on the mutual interaction of the multiple large-scale structure wave modes with the mean flow and with each other. Therefore, our analysis is limited to the initial stages of development of the shear layer before the breakdown of the large-scale structure into small-scale turbulence which dictates the growth of the shear flow thereafter. In general the energy transfer mechanisms among the mean flow, large-scale coherent structure and small-scale turbulence are that the large-scale structure initially amplifies by gaining energy from the mean flow, and is damped by losing energy to the small-scale turbulence. The small-scale turbulence gains energy from both the mean flow and the large-scale structure and loses energy through viscous dissipation. Thus, the small-scale turbulence can be enhanced by obtaining energy directly from the mean flow and indirectly through the large-scale wave modes. The growth of the mixing layer is due to the energy transfer from the mean flow to both the large-scale structure and the small-scale turbulence. If initially the energy level of the small-scale turbulence is high enough compared with that of the large-scale structure, we must include the small-scale turbulence. As we increase the initial turbulence energy, the growth of the instability wave modes will be reduced because of increased
energy drain caused by the small-scale turbulence. Therefore, some large-scale wave modes can be reduced or possibly eliminated by introducing high initial energy levels of small-scale turbulence. Including small-scale turbulence effects to the three-dimensional multi-mode problem requires phase averaging and either a closure or shape assumptions for the small-scale related stresses, and is something that can be pursued in the future.

In a natural flow there are usually many wave modes. To study the nonlinear interactions between the two- and three-dimensional fundamental and subharmonic wave modes in a spatially developing shear layer, the selected five wave modes represent the minimum number that needs to be considered. However, we realize that the three-dimensional subharmonic 12 wave mode interacts with everybody directly or indirectly as shown in Figure G.1. For instance, the subharmonic 12 wave mode can interact directly with all modes except the subharmonic 10 wave mode while the subharmonic 10 wave mode can interact indirectly with the fundamental 22 wave mode by the agency of 12 the wave mode. Therefore we have included the formulated governing equations for 12 wave mode in Appendix G as a prelude to future understanding. It will be very interesting to see how the three-dimensional subharmonic 12 wave mode will affect the development of the three-dimensional shear layer.
Bibliography


Appendix A

Comparison with Experiments

Comparison of predictions obtained by the application of the energy method (Nikitopoulos & Liu 1990) to a purely two-dimensional shear layer, with the measurements of [10] (Ho & Huang 1982). Two (a, b) and three (c) two-dimensional large-scale structure modes were taken into account in these investigations.
Small-scale turbulence dominates.

Subharmonic peaks.

Fundamental peaks

Present

Ref. [10]

Fund. (Calc.)

Sub. (Calc.)

Fund. [10]

Sub. [10]

Figure A.1: Mean flow (a) and modal energy density evolution of a shear layer perturbed by forcing the first subharmonic.
Appendix B

Wave Mode Terms

According to equations (2.13) and (2.14) in chapter 2, \( \tilde{u} \), \( \tilde{v} \) and \( \tilde{w} \) are represented by following:

\[
\tilde{u} = (u_{10} e^{-i\beta t} + u_{10}^* e^{i\beta t}) + (u_{20} e^{-2i\beta t} + u_{20}^* e^{2i\beta t})
+ (u_{11} e^{-i\beta t} + u_{11}^* e^{i\beta t}) \cos(\gamma z) + (u_{21} e^{-2i\beta t} + u_{21}^* e^{2i\beta t}) \cos(\gamma z)
+ (u_{22} e^{-2i\beta t} + u_{22}^* e^{2i\beta t}) \cos(2\gamma z) \quad (B.1)
\]

\[
\tilde{v} = (v_{10} e^{-i\beta t} + v_{10}^* e^{i\beta t}) + (v_{20} e^{-2i\beta t} + v_{20}^* e^{2i\beta t})
+ (v_{11} e^{-i\beta t} + v_{11}^* e^{i\beta t}) \cos(\gamma z) + (v_{21} e^{-2i\beta t} + v_{21}^* e^{2i\beta t}) \cos(\gamma z)
+ (v_{22} e^{-2i\beta t} + v_{22}^* e^{2i\beta t}) \cos(2\gamma z) \quad (B.2)
\]

\[
\tilde{w} = (w_{11} e^{-i\beta t} + w_{11}^* e^{i\beta t}) \sin(\gamma z) + (w_{21} e^{-2i\beta t} + w_{21}^* e^{2i\beta t}) \sin(\gamma z)
+ (w_{22} e^{-2i\beta t} + w_{22}^* e^{2i\beta t}) \sin(2\gamma z) \quad (B.3)
\]

The complex amplitudes of terms (\( \tilde{u}_i \tilde{u}_j - \tilde{u}_i \tilde{u}_j^* \)) in equation (2.8) denoted as (\( \tilde{u}_i \tilde{u}_j - \tilde{u}_i \tilde{u}_j^* \))\(_{mn} \) for each mode \( mn \) are shown below for the simple case where \( \phi_{21} \), \( \phi_{22} \)
are equal to 0. Thus every term containing modes $2n$ contains $\cos(n\gamma z + \phi_{2n})$ or $\sin(n\gamma z + \phi_{2n})$ rather than $\cos(n\gamma z)$ or $\sin(n\gamma z)$ as indicated.

1. Two-dimensional Subharmonic 10 Mode: $e^{-i\beta t}$

Here, we seek the terms with $e^{-i\beta t}$.

\[
(\ddot{u}^2 - \ddot{u}^2)_{10} = 2u_{10}^{\prime}u_{20}^{\prime} + u_{11}^{\prime}u_{21}^{\prime}
\]  \hspace{1cm} (B.4)

\[
(\ddot{u}\ddot{v} - \ddot{u}\ddot{v})_{10} = u_{10}^{\prime}v_{20}^{\prime} + u_{20}^{\prime}v_{10}^{\prime} + \frac{u_{11}^{\prime}v_{21}^{\prime}}{2} + \frac{u_{21}^{\prime}v_{11}^{\prime}}{2}
\]  \hspace{1cm} (B.5)

\[
(\ddot{u}\ddot{w} - \ddot{u}\ddot{w})_{10} = 0
\]  \hspace{1cm} (B.6)

\[
(\ddot{v}^2 - \ddot{v}^2)_{10} = 2v_{10}^{\prime}v_{20}^{\prime} + v_{11}^{\prime}v_{21}^{\prime}
\]  \hspace{1cm} (B.7)

\[
(\ddot{v}\ddot{w} - \ddot{v}\ddot{w})_{10} = 0
\]  \hspace{1cm} (B.8)

\[
(\ddot{w}^2 - \ddot{w}^2)_{10} = w_{11}^{\prime}w_{21}^{\prime}
\]  \hspace{1cm} (B.9)

2. Two-dimensional Fundamental 20 Mode: $e^{-2i\beta t}$

\[
(\ddot{u}^2 - \ddot{u}^2)_{20} = u_{10}^{\prime2} + \frac{u_{11}^{\prime2}}{2}
\]  \hspace{1cm} (B.10)

\[
(\ddot{u}\ddot{v} - \ddot{u}\ddot{v})_{20} = u_{10}^{\prime}v_{10}^{\prime} + \frac{u_{11}^{\prime}v_{11}^{\prime}}{2}
\]  \hspace{1cm} (B.11)

\[
(\ddot{u}\ddot{w} - \ddot{u}\ddot{w})_{20} = 0
\]  \hspace{1cm} (B.12)

\[
(\ddot{v}^2 - \ddot{v}^2)_{20} = v_{10}^{\prime2} + \frac{v_{11}^{\prime2}}{2}
\]  \hspace{1cm} (B.13)

\[
(\ddot{v}\ddot{w} - \ddot{v}\ddot{w})_{20} = 0
\]  \hspace{1cm} (B.14)

\[
(\ddot{w}^2 - \ddot{w}^2)_{20} = \frac{w_{11}^{\prime2}}{2}
\]  \hspace{1cm} (B.15)
3. Three-dimensional Subharmonic 11 Mode: $e^{-i\beta t}(\cos(\gamma z) \text{or} \sin(\gamma z))$

\[
\begin{align*}
(\ddot{u}^2 - \ddot{u}^2)_{11} &= 2 \, w_{10}^* u_{21} + 2 \, u_{11}^* u_{20} + u_{11}^* u_{22} \\
(\ddot{u} \ddot{v} - \ddot{u} \ddot{v})_{11} &= u_{10}^* v_{21} + u_{11}^* v_{20} + v_{10}^* u_{21} + v_{11}^* u_{20} \\
&\quad + \frac{u_{11}^* v_{22}}{2} + \frac{v_{11}^* u_{22}}{2} \\
(\ddot{u} \ddot{w} - \ddot{u} \ddot{w})_{11} &= u_{10}^* w_{21} + w_{11}^* u_{20} + \frac{u_{11}^* w_{22}}{2} - \frac{w_{11}^* u_{22}}{2} \\ 
(\ddot{v}^2 - \ddot{v}^2)_{11} &= 2 \, v_{10}^* v_{21} + 2 \, v_{11}^* v_{20} + v_{11}^* v_{22} \\ 
(\ddot{v} \ddot{w} - \ddot{v} \ddot{w})_{11} &= v_{10}^* w_{21} + w_{11}^* v_{20} + \frac{v_{11}^* w_{22}}{2} - \frac{w_{11}^* v_{22}}{2} \\
(\ddot{w}^2 - \ddot{w}^2)_{11} &= w_{11}^* w_{22} 
\end{align*}
\]

4. Three-dimensional Fundamental 21 Mode: $e^{-2i\beta t}(\cos(\gamma z) \text{or} \sin(\gamma z))$

\[
\begin{align*}
(\ddot{u}^2 - \ddot{u}^2)_{21} &= 2 \, w_{10}^* u_{11} \\
(\ddot{u} \ddot{v} - \ddot{u} \ddot{v})_{21} &= u_{10}^* v_{11} + u_{11}^* v_{10} \\
(\ddot{u} \ddot{w} - \ddot{u} \ddot{w})_{21} &= u_{10}^* w_{11} \\
(\ddot{v}^2 - \ddot{v}^2)_{21} &= 2 \, v_{10}^* v_{11} \\
(\ddot{v} \ddot{w} - \ddot{v} \ddot{w})_{21} &= v_{10}^* w_{11} \\
(\ddot{w}^2 - \ddot{w}^2)_{21} &= 0
\end{align*}
\]
5. Three-dimensional Fundamental 22 Mode: $e^{-2i\beta t}(\cos(2\gamma z) or \sin(2\gamma z))$

\[
(\ddot{u}^2 - \dddot{u}^2)_{22} = \frac{u_{11}^2}{2} \tag{B.28}
\]

\[
(\ddot{u}\ddot{v} - \dddot{u}\dddot{v})_{22} = \frac{u_{11}v_{11}}{2} \tag{B.29}
\]

\[
(\ddot{u}\ddot{w} - \dddot{u}\dddot{w})_{22} = \frac{u_{11}w_{11}}{2} \tag{B.30}
\]

\[
(\ddot{v}^2 - \dddot{v}^2)_{22} = \frac{v_{11}^2}{2} \tag{B.31}
\]

\[
(\ddot{v}\ddot{w} - \dddot{v}\dddot{w})_{22} = \frac{v_{11}w_{11}}{2} \tag{B.32}
\]

\[
(\ddot{w}^2 - \dddot{w}^2)_{22} = -\frac{w_{11}^2}{2} \tag{B.33}
\]
Appendix C

Momentum Equations for Fundamental Wave Modes

In here, we represent the momentum equations for wave modes 20, 21 and 22. The continuity equation of 20 mode is

$$\frac{\partial u_{20}}{\partial x} + \frac{\partial v_{20}}{\partial y} = 0 \tag{C.1}$$

The x-momentum equation of 20 mode is

$$-i\beta_{20}u_{20} + U\frac{\partial u_{20}}{\partial x} + v_{20}\frac{\partial U}{\partial y} = -\frac{\partial p_{20}}{\partial x}$$

$$+ \frac{1}{Re_0}\left(\frac{\partial^2 u_{20}}{\partial x^2} + \frac{\partial^2 u_{20}}{\partial y^2}\right) - \frac{\partial}{\partial x}(\bar{u}^2 - \overline{\bar{u}}^2)_{20} - \frac{\partial}{\partial y}(\bar{u}\bar{v} - \overline{\bar{u}\bar{v}})_{20} \tag{C.2}$$

The y-momentum equation of 20 mode is

$$-i\beta_{20}v_{20} + U\frac{\partial v_{20}}{\partial x} = -\frac{\partial p_{20}}{\partial y} + \frac{1}{Re_0}\left(\frac{\partial^2 v_{20}}{\partial x^2} + \frac{\partial^2 v_{20}}{\partial y^2}\right)$$

$$- \frac{\partial}{\partial x}(\bar{v}\bar{u} - \overline{\bar{v}\bar{u}})_{20} - \frac{\partial}{\partial y}(\bar{v}^2 - \overline{\bar{v}}^2)_{20} \tag{C.3}$$

The continuity equation of 21 mode is

$$\frac{\partial u_{21}}{\partial x} + \frac{\partial v_{21}}{\partial y} + \gamma w_{21} = 0 \tag{C.4}$$
The x-momentum equation of 21 mode is

\[-i\beta_{21} u_{t21} + U \frac{\partial u_{t21}}{\partial x} + v_{t21} \frac{\partial U}{\partial y} = -\frac{\partial p_{t21}}{\partial x}\]

\[+ \frac{1}{Re_0} \left( \frac{\partial^2 u_{t21}}{\partial x^2} + \frac{\partial^2 u_{t21}}{\partial y^2} - \gamma^2 u_{t21} \right) - \frac{\partial}{\partial x} \left( \ddot{u}^2 - \dddot{u}^2 \right)_{21} \]

\[= -\frac{\partial}{\partial y} \left( \dddot{u} \dddot{v} - \dddot{v} \dddot{u} \right)_{21} - \gamma u_{t10} w_{t11} \]  \hspace{1cm} (C.5)

The y-momentum equation of 21 mode is

\[-i\beta_{21} v_{t21} + U \frac{\partial v_{t21}}{\partial x} = -\frac{\partial p_{t21}}{\partial y} + \frac{1}{Re_0} \left( \frac{\partial^2 v_{t21}}{\partial x^2} + \frac{\partial^2 v_{t21}}{\partial y^2} - \gamma^2 v_{t21} \right) \]

\[= -\frac{\partial}{\partial x} \left( \ddot{u} \dddot{v} - \dddot{u} \dddot{v} \right)_{21} - \frac{\partial}{\partial y} \left( \dddot{v} \dddot{u} - \dddot{v} \dddot{u} \right)_{21} \]  \hspace{1cm} (C.6)

The z-momentum equation of 21 mode is

\[-i\beta_{21} w_{t21} + U \frac{\partial w_{t21}}{\partial x} = \gamma p_{t21} + \frac{1}{Re_0} \left( \frac{\partial^2 w_{t21}}{\partial x^2} + \frac{\partial^2 w_{t21}}{\partial y^2} - \gamma^2 w_{t21} \right) \]

\[= -\frac{\partial}{\partial x} \left( \ddot{u} \dddot{w} - \dddot{u} \dddot{w} \right)_{21} - \frac{\partial}{\partial y} \left( \dddot{v} \dddot{w} - \dddot{v} \dddot{w} \right)_{21} \]  \hspace{1cm} (C.7)

The continuity equation of 22 mode is

\[\frac{\partial u_{t22}}{\partial x} + \frac{\partial v_{t22}}{\partial y} + 2\gamma w_{t22} = 0 \]  \hspace{1cm} (C.8)

The x-momentum equation of 22 mode is

\[-i\beta_{22} u_{t22} + U \frac{\partial u_{t22}}{\partial x} + v_{t22} \frac{\partial U}{\partial y} = -\frac{\partial p_{t22}}{\partial x}\]

\[+ \frac{1}{Re_0} \left( \frac{\partial^2 u_{t22}}{\partial x^2} + \frac{\partial^2 u_{t22}}{\partial y^2} - 4\gamma^2 u_{t22} \right) - \frac{\partial}{\partial x} \left( \ddot{u}^2 - \dddot{u}^2 \right)_{22} \]

\[= -\frac{\partial}{\partial y} \left( \dddot{u} \dddot{v} - \dddot{v} \dddot{u} \right)_{22} - \gamma u_{t11} w_{t11} \]  \hspace{1cm} (C.9)

The y-momentum equation of 22 mode is

\[-i\beta_{22} v_{t22} + U \frac{\partial v_{t22}}{\partial x} = -\frac{\partial p_{t22}}{\partial y} + \frac{1}{Re_0} \left( \frac{\partial^2 v_{t22}}{\partial x^2} + \frac{\partial^2 v_{t22}}{\partial y^2} - 4\gamma^2 v_{t22} \right) \]
\[- \frac{\partial}{\partial x} (\tilde{u}\tilde{v} - \tilde{u}\tilde{v})_{22} - \frac{\partial}{\partial y} (\tilde{v}^2 - \tilde{v}^2)_{22} - \gamma \nu l_{11} \nu l_{11} \] (C.10)

The z-momentum equation of 22 mode is

\[-i\beta_{22} \nu l_{22} + U \frac{\partial \nu l_{22}}{\partial x} = 2\gamma p l_{22} + \frac{1}{Re_0} (\frac{\partial^2 \nu l_{22}}{\partial x^2} + \frac{\partial^2 \nu l_{22}}{\partial y^2} - 4\gamma^2 \nu l_{22}) \]

\[-\frac{\partial}{\partial x} (\tilde{u}\tilde{v} - \tilde{u}\tilde{v})_{22} - \frac{\partial}{\partial y} (\tilde{v}\tilde{w} - \tilde{v}\tilde{w})_{22} - \nu l_{11}^2 \] (C.11)

where \((\ )^*\) is the complex conjugate of \((\ )\), and \((\tilde{u}^2 - \tilde{u}^2)_{mn}, (\tilde{u}\tilde{v} - \tilde{u}\tilde{v})_{mn}, (\tilde{u}\tilde{v} - \tilde{u}\tilde{v})_{mn}, (\tilde{v}^2 - \tilde{v}^2)_{mn}, (\tilde{v}\tilde{w} - \tilde{v}\tilde{w})_{mn}\) and \((\tilde{w}^2 - \tilde{w}^2)_{mn}\) are given in Appendix B.
The mechanical energy equation for 20 wave mode is

\[ U \frac{d}{dx} [ |u_{20}|^2 + |v_{20}|^2 ] = \]

Rate of change of modal energy

\[-2 \text{Re}(u_{20}v_{20}^*) \frac{\partial U}{\partial y} \]

Energy exchange with the mean flow

\[-\frac{\partial}{\partial x} [2 \text{Re}(u_{20}v_{20}^*)] - \frac{\partial}{\partial y} [2 \text{Re}(v_{20}p_{20}^*)] \]

\[+ \frac{1}{Re_0} \left( \frac{\partial^2}{\partial x^2} (|u_{20}|^2 + |v_{20}|^2) + \frac{\partial^2}{\partial y^2} (|u_{20}|^2 + |v_{20}|^2) \right) \]

\[- \frac{2}{Re_0} \left( |\frac{\partial u_{20}}{\partial x}|^2 + |\frac{\partial v_{20}}{\partial x}|^2 + |\frac{\partial u_{20}}{\partial y}|^2 + |\frac{\partial v_{20}}{\partial y}|^2 \right) \]

\[- \frac{\partial}{\partial x} \left[2 \text{Re}(u_{10}^2 u_{20} + \frac{u_{11}^2 u_{20}}{2} + u_{10}^* v_{10} v_{20} + \frac{u_{11}^* v_{11} v_{20}}{2}) \right] \]

\[- \frac{\partial}{\partial y} \left[2 \text{Re}(u_{10}^* v_{10}^* u_{20} + \frac{u_{11}^* v_{11}^* u_{20}}{2} + v_{10}^* v_{20} + \frac{v_{11}^* v_{20}}{2}) \right] \]

Transport term
\[ +2\text{Re}l\{u_{10}^2 \frac{\partial u_{20}}{\partial x} + u_{10}^* v_{10}^* \frac{\partial v_{20}}{\partial x} + (u_{10}^* v_{10} \frac{\partial u_{20}}{\partial y} + v_{10}^2 \frac{\partial v_{20}}{\partial y})\} \]

Energy interaction between 10 and 20

\[ +2\text{Re}l\{\frac{1}{2}(u_{11}^2 \frac{\partial u_{20}}{\partial x} + u_{11}^* v_{11}^* \frac{\partial v_{20}}{\partial x}) + \frac{1}{2}(u_{11}^* v_{11} \frac{\partial u_{20}}{\partial y} + v_{11}^2 \frac{\partial v_{20}}{\partial y})\} \]

Energy interaction between 20 and 11,

(D.1)

and for wave mode 21:

\[
\frac{U}{2} \frac{d}{dx} [u_{21}^2 + |v_{21}|^2 + |w_{21}|^2] = \]

Rate of change of modal energy

\[-2\text{Re}l(u_{21}v_{21}/2) \frac{\partial U}{\partial y}\]

Energy exchange with the mean flow

\[-\frac{\partial}{\partial x}\{2\text{Re}l(u_{21}^2/2)\} - \frac{\partial}{\partial y}\{2\text{Re}l(v_{21}^2/2)\} \]

\[+ \frac{1}{2\text{Re}_0} \left[ \frac{\partial^2}{\partial x^2}(|u_{21}|^2 + |v_{21}|^2 + |w_{21}|^2) + \frac{\partial^2}{\partial y^2}(|u_{21}|^2 + |v_{21}|^2 + |w_{21}|^2) \right] \]

\[-\frac{1}{\text{Re}_0} \left[ |\frac{\partial u_{21}}{\partial x}|^2 + |\frac{\partial v_{21}}{\partial x}|^2 + |\frac{\partial w_{21}}{\partial x}|^2 + |\frac{\partial u_{21}}{\partial y}|^2 + |\frac{\partial v_{21}}{\partial y}|^2 + |\frac{\partial w_{21}}{\partial y}|^2 \right] \]

\[-\gamma^2 (|u_{21}|^2 + |v_{21}|^2 + |w_{21}|^2) \]

\[-\frac{\partial}{\partial x}\{2\text{Re}l(u_{10}^* u_{11}^* u_{21} + u_{10}^* v_{11}^* v_{21} + u_{10}^* v_{11}^* w_{21}/2 + w_{10}^* v_{11}^* w_{21}/2)\} \cos(\phi_{21}) \]

\[-\frac{\partial}{\partial y}\{2\text{Re}l(u_{10}^* v_{11}^* u_{21} + u_{11}^* v_{10}^* v_{21} + u_{11}^* v_{10}^* w_{21}/2 + v_{10}^* v_{11}^* w_{21}/2)\} \cos(\phi_{21}) \]

Transport term

\[+2\text{Re}l\{\frac{1}{2}(u_{10}^* u_{11} \frac{\partial u_{21}}{\partial x} + v_{10}^* u_{11} \frac{\partial v_{21}}{\partial x}) + \frac{1}{2}(u_{10}^* v_{11} \frac{\partial u_{21}}{\partial y} + v_{10}^* v_{11} \frac{\partial v_{21}}{\partial y}) \]

\[-\gamma \frac{1}{2}(u_{10}^* v_{11}^* u_{21} + v_{10}^* v_{11}^* v_{21}) \cos(\phi_{21}) \]
Energy interaction between 10 and 21

\[ +2\text{Re}l\left[ \frac{1}{2}(u_{\tau_1}u_{\tau_0} \frac{\partial u_{\tau_1}}{\partial x} + v_{\tau_1}u_{\tau_0} \frac{\partial v_{\tau_1}}{\partial x} + w_{\tau_1}u_{\tau_0} \frac{\partial w_{\tau_1}}{\partial x}) \right] \\
+ \frac{1}{2}(u_{\tau_1}u_{\tau_0} \frac{\partial u_{\tau_1}}{\partial y} + v_{\tau_1}u_{\tau_0} \frac{\partial v_{\tau_1}}{\partial y} + w_{\tau_1}u_{\tau_0} \frac{\partial w_{\tau_1}}{\partial y}) \cos(\phi_{\tau_1}) \]

Energy interaction between 11 and 21,

\[ (D.2) \]

and for wave mode 22:

\[ \frac{U}{2} \frac{d}{dx} [|u_{\tau_2}|^2 + |v_{\tau_2}|^2 + |w_{\tau_2}|^2] = \]

Rate of change of modal energy

\[-2\text{Re}l(u_{\tau_2}v_{\tau_2}) \frac{\partial U}{\partial y} \]

Energy exchange with the mean flow

\[-\frac{\partial}{\partial x} \left[ 2\text{Re}l\left( \frac{u_{\tau_2}v_{\tau_2}}{2} \right) \right] - \frac{\partial}{\partial y} \left[ 2\text{Re}l\left( \frac{v_{\tau_2}w_{\tau_2}}{2} \right) \right] \]

\[ + \frac{1}{2\text{Re}_0} \left[ \frac{\partial^2}{\partial x^2} (|u_{\tau_2}|^2 + |v_{\tau_2}|^2 + |w_{\tau_2}|^2) + \frac{\partial^2}{\partial y^2} (|u_{\tau_2}|^2 + |v_{\tau_2}|^2 + |w_{\tau_2}|^2) \right] \]

\[-\frac{1}{\text{Re}_0} \left[ \frac{\partial u_{\tau_2}}{\partial x} |u_{\tau_2}|^2 + \frac{\partial v_{\tau_2}}{\partial x} |v_{\tau_2}|^2 + \frac{\partial w_{\tau_2}}{\partial x} |w_{\tau_2}|^2 + \frac{\partial u_{\tau_2}}{\partial y} |u_{\tau_2}|^2 + \frac{\partial v_{\tau_2}}{\partial y} |v_{\tau_2}|^2 + \frac{\partial w_{\tau_2}}{\partial y} |w_{\tau_2}|^2 \right] \]

\[-\gamma^2 (|u_{\tau_2}|^2 + |v_{\tau_2}|^2 + |w_{\tau_2}|^2) \]

\[-\frac{\partial}{\partial x} \left[ 2\text{Re}l\left( \frac{u_{\tau_1}^2v_{\tau_2}}{4} + \frac{u_{\tau_1}v_{\tau_1}v_{\tau_2}}{4} + \frac{w_{\tau_1}^2u_{\tau_1}w_{\tau_2}}{4} \right) \right] \cos(\phi_{\tau_2}) \]

\[-\frac{\partial}{\partial y} \left[ 2\text{Re}l\left( \frac{v_{\tau_1}^2u_{\tau_2}}{4} + \frac{w_{\tau_1}v_{\tau_1}v_{\tau_2}}{4} + \frac{w_{\tau_1}^2u_{\tau_1}w_{\tau_2}}{4} \right) \right] \cos(\phi_{\tau_2}) \]

Transport terms

\[ +2\text{Re}l\left[ \frac{1}{4} (u_{\tau_1}^2 \frac{\partial u_{\tau_2}}{\partial x} + v_{\tau_1}u_{\tau_1} \frac{\partial v_{\tau_2}}{\partial x} + w_{\tau_1}u_{\tau_1} \frac{\partial w_{\tau_2}}{\partial x}) \right] \\
+ \frac{1}{4} (u_{\tau_1}v_{\tau_1} \frac{\partial u_{\tau_2}}{\partial y} + v_{\tau_1}^2 \frac{\partial v_{\tau_2}}{\partial y} + w_{\tau_1}v_{\tau_1} \frac{\partial w_{\tau_2}}{\partial y}) \]
where \( \text{Rel}( ) \) is the real part of the complex quantity ( ).
Appendix E

The Phase Angle Equations for Fundamental Wave Modes

The phase angle equation for 20 wave mode is

\[ 2\beta_{20}(|u_{20}|^2 + |v_{20}|^2) \]

Temporal phase

\[ + U[2Im(u_{20} \frac{\partial u_{20}}{\partial x} + v_{20} \frac{\partial v_{20}}{\partial x})] \]

Rate of change of phase

\[ +2Im(u_{20}v_{20}) \frac{\partial U}{\partial y} = \]

Phase shift from interacting with the mean flow

\[ - \frac{\partial}{\partial x}[2Im(u_{20}p_{20}')] - \frac{\partial}{\partial y}[2Im(v_{20}p_{20}')] \]

Pressure field induced phase shift

\[ + \frac{2}{Re_0}[2Im \frac{\partial}{\partial x}(u_{20} \frac{\partial u_{20}^*}{\partial x} + v_{20} \frac{\partial v_{20}^*}{\partial x}) + 2Im \frac{\partial}{\partial y}(u_{20} \frac{\partial u_{20}^*}{\partial y} + v_{20} \frac{\partial v_{20}^*}{\partial y})] \]

Viscosity induced phase shift

\[ - \frac{\partial}{\partial x}[2Im(u_{10}^2u_{20} + \frac{u_{11}^2u_{20}}{2} + u_{10}v_{10}^*v_{20}^* + \frac{u_{11}^*v_{11}^*u_{20}}{2})] \]
\[- \frac{\partial}{\partial y} [2 \text{Im}(u_{10}^{*} v_{10}^{*} u_{20} + \frac{u_{11}^{*} v_{11}^{*} u_{20}}{2} + v_{10}^{*} v_{10}^{*} u_{20} + \frac{v_{11}^{*} v_{11}^{*} u_{20}}{2})]\]

Phase shift induced by convective transport

\[+2 \text{Im}\left[\frac{1}{2}(u_{11}^{*} \frac{\partial u_{20}}{\partial x} + u_{11}^{*} \frac{\partial v_{20}}{\partial x}) + \frac{1}{2}(u_{11}^{*} v_{11}^{*} \frac{\partial u_{20}}{\partial y} + v_{11}^{*} v_{11}^{*} \frac{\partial v_{20}}{\partial y})\right]\]

Phase shift due to interaction between 10 and 20

\[+2 \text{Im}\left[\frac{1}{2}(u_{10}^{*} \frac{\partial u_{20}}{\partial x} + u_{10}^{*} \frac{\partial v_{20}}{\partial x}) + (u_{10}^{*} v_{10}^{*} \frac{\partial u_{20}}{\partial y} + v_{10}^{*} v_{10}^{*} \frac{\partial v_{20}}{\partial y})\right]\]

Phase shift due to interaction between 20 and 11, \text{(E.1)}

where \(\text{Im}(\ )\) is the imaginary part of the complex quantity ( ), and for wave mode 21:

\[2 \beta_{21}[|u_{21}|^2 + |v_{21}|^2 + |w_{21}|^2]/2]\]

Temporal phase

\[+ \frac{U}{2} [2 \text{Im}(u_{21} \frac{\partial u_{21}}{\partial x} + v_{21} \frac{\partial v_{21}}{\partial x} + w_{21} \frac{\partial w_{21}}{\partial x})]\]

Rate of change of phase

\[+2 \text{Im}(u_{21} v_{21}^{*}/2) \frac{\partial U}{\partial y} =\]

Phase shift from interacting with the mean flow

\[- \frac{\partial}{\partial x} [2 \text{Im}(\frac{u_{21} p_{21}^{*}}{2})] - \frac{\partial}{\partial y} [2 \text{Im}(\frac{v_{21} p_{21}^{*}}{2})]\]

Phase angle of pressure work

\[+ \frac{1}{Re_{0}} [\text{Im}(u_{21} \frac{\partial u_{21}}{\partial x} + v_{21} \frac{\partial v_{21}}{\partial x} + w_{21} \frac{\partial w_{21}}{\partial x}) + \text{Im}(u_{21} \frac{\partial u_{21}}{\partial y} + v_{21} \frac{\partial v_{21}}{\partial y} + w_{21} \frac{\partial w_{21}}{\partial y})]\]
Phase angle of viscous terms

\[-\frac{\partial}{\partial x}[2\text{Im}(u_{10}^*v_{11}^*u_{21} + \frac{u_{10}^*v_{11}^*v_{121}}{2} + \frac{u_{11}^*v_{10}^*v_{121}}{2} + \frac{u_{10}^*u_{11}^*w_{121}}{2})] \cos(\phi_{21})\]

\[-\frac{\partial}{\partial y}[2\text{Im}(\frac{u_{10}^*v_{11}^*u_{21}}{2} + \frac{u_{11}^*v_{10}^*u_{21}}{2} + v_{10}^*v_{11}^*v_{121} + \frac{v_{10}^*w_{11}^*w_{121}}{2})] \cos(\phi_{21})\]

Phase angle of convective transport term

\[+2\text{Im}(\frac{1}{2}(w_{10}^*u_{11}^* \frac{\partial u_{21}}{\partial x} + v_{10}^*u_{11}^* \frac{\partial v_{21}}{\partial x})) + \frac{1}{2}(w_{10}^*v_{11}^* \frac{\partial u_{21}}{\partial y} + v_{10}^*v_{11}^* \frac{\partial v_{21}}{\partial y})\]

\[-\frac{\gamma}{2}(w_{10}^*w_{11}^*u_{21} + v_{10}^*w_{11}^*v_{21})] \cos(\phi_{21})\]

Phase shift due to interaction between 10 and 21

\[+2\text{Im}(\frac{1}{2}(w_{11}^*u_{10}^* \frac{\partial u_{21}}{\partial x} + v_{11}^*u_{10}^* \frac{\partial v_{21}}{\partial x} + w_{11}^*u_{10}^* \frac{\partial w_{21}}{\partial x})) \]

\[+\frac{1}{2}(w_{11}^*v_{10}^* \frac{\partial u_{21}}{\partial y} + v_{11}^*v_{10}^* \frac{\partial v_{21}}{\partial y} + w_{11}^*v_{10}^* \frac{\partial w_{21}}{\partial y})] \cos(\phi_{21})\]

Phase shift due to interaction between 11 and 21, (E.2)

for wave mode 22:

\[2\beta_{22}[(|u_{22}|^2 + |v_{22}|^2 + |w_{22}|^2)/2]\]

Temporal phase

\[+\frac{U}{2}[2\text{Im}(u_{22}^* \frac{\partial u_{22}}{\partial x} + v_{22}^* \frac{\partial v_{22}}{\partial x} + w_{22}^* \frac{\partial w_{22}}{\partial x})]\]

Rate of change of phase

\[+2\text{Im}(u_{22}^*v_{22}^*/2) \frac{\partial U}{\partial y} =\]

Phase shift from interacting with the mean flow

\[-\frac{\partial}{\partial x}[2\text{Im}(\frac{u_{22}^*p_{22}^*}{2})] - \frac{\partial}{\partial y}[2\text{Im}(\frac{v_{22}^*p_{22}^*}{2})]\]
Phase angle of pressure work

\[ + \frac{1}{Re_0} \left[ \text{Im} \frac{\partial}{\partial x} (u_{22}' \frac{\partial u_{22}''}{\partial x} + v_{22}' \frac{\partial v_{22}''}{\partial x} + w_{22}' \frac{\partial w_{22}''}{\partial x}) ight] + \text{Im} \frac{\partial}{\partial y} (u_{22}' \frac{\partial u_{22}''}{\partial y} + v_{22}' \frac{\partial v_{22}''}{\partial y} + w_{22}' \frac{\partial w_{22}''}{\partial y}) \]

Phase angle of viscous terms

\[ - \frac{\partial}{\partial x} \left[ 2\text{Im} \left( \frac{u_{11}' u_{22}''}{4} + \frac{u_{11}' v_{11}' v_{22}''}{4} + \frac{u_{11}' w_{11}' w_{22}''}{4} \right) \right] \cos(\phi_{22}) \]
\[ - \frac{\partial}{\partial y} \left[ 2\text{Im} \left( \frac{v_{11}' v_{22}''}{4} + \frac{v_{11}' v_{11}' u_{22}''}{4} + \frac{v_{11}' w_{11}' w_{22}''}{4} \right) \right] \cos(\phi_{22}) \]

Phase angle of convective transport term

\[ + 2\text{Im} \left[ \frac{1}{4} \left( u_{11}' \frac{\partial u_{22}''}{\partial x} + v_{11}' u_{11}' \frac{\partial v_{22}''}{\partial x} + w_{11}' u_{11}' \frac{\partial w_{22}''}{\partial x} \right) \right] + \frac{1}{4} \left( u_{11}' v_{11}' \frac{\partial u_{22}''}{\partial y} + v_{11}' v_{11}' \frac{\partial v_{22}''}{\partial y} + w_{11}' v_{11}' \frac{\partial w_{22}''}{\partial y} \right) \]
\[ - \frac{\gamma}{2} (u_{11}' w_{11}' u_{22}'' + v_{11}' w_{11}' v_{22}'' + w_{11}' w_{11}' w_{22}'') \cos(\phi_{22}) \]

Phase shift due to interaction between 11 and 22

(E.3)
Appendix F

Integral Coefficients $\Sigma_{ij}^{pq} \Delta_{kl}$

\[
\Sigma_{10}^{10} \Delta_{20} = \int_{-\infty}^{\infty} (-i\alpha_{20} \dot{u}_{10} \dot{u}_{20} - i\alpha_{20} \dot{v}_{10} \dot{v}_{20}) \, dy \\
+ \int_{-\infty}^{\infty} (\dot{u}_{10} \dot{v}_{20} \frac{\partial \dot{u}_{20}}{\partial y} + \dot{v}_{10} \dot{v}_{20} \frac{\partial \dot{v}_{20}}{\partial y}) \, dy, \quad (F.1)
\]

\[
\Sigma_{11}^{11} \Delta_{20} = \int_{-\infty}^{\infty} -\frac{i\alpha_{20}}{2} (\ddot{u}_{11} \ddot{u}_{20} + \ddot{v}_{11} \ddot{v}_{20}) \, dy \\
+ \frac{1}{2} \int_{-\infty}^{\infty} (\ddot{u}_{11} \ddot{v}_{20} \frac{\partial \ddot{u}_{20}}{\partial y} + \ddot{v}_{11} \ddot{v}_{20} \frac{\partial \ddot{v}_{20}}{\partial y}) \, dy, \quad (F.2)
\]

\[
\Sigma_{10}^{21} \Delta_{11} = \frac{1}{2} \int_{-\infty}^{\infty} i\alpha_{11} (\dddot{u}_{10} \dddot{u}_{21} \dddot{u}_{11} + \dddot{v}_{10} \dddot{v}_{21} \dddot{v}_{11}) \, dy \\
+ \frac{1}{2} \int_{-\infty}^{\infty} (\dddot{u}_{10} \dddot{v}_{21} \frac{\partial \dddot{u}_{21}}{\partial y} + \dddot{v}_{10} \dddot{v}_{21} \frac{\partial \dddot{v}_{21}}{\partial y}) \, dy \\
- \frac{1}{2} \int_{-\infty}^{\infty} \gamma (\dddot{u}_{10} \dddot{u}_{21} \dddot{u}_{11} + \dddot{v}_{10} \dddot{v}_{21} \dddot{v}_{11}) \, dy, \quad (F.3)
\]

\[
\Sigma_{11}^{11} \Delta_{21} = \frac{1}{2} \int_{-\infty}^{\infty} -i\alpha_{21} (\dddot{u}_{10} \dddot{u}_{11} \dddot{u}_{21} + \dddot{v}_{10} \dddot{v}_{11} \dddot{v}_{21}) \, dy \\
+ \frac{1}{2} \int_{-\infty}^{\infty} (\dddot{u}_{10} \dddot{v}_{21} \frac{\partial \dddot{u}_{21}}{\partial y} + \dddot{v}_{10} \dddot{v}_{21} \frac{\partial \dddot{v}_{21}}{\partial y}) \, dy \\
- \frac{1}{2} \int_{-\infty}^{\infty} \gamma (\dddot{u}_{10} \dddot{u}_{11} \dddot{u}_{21} + \dddot{v}_{10} \dddot{v}_{11} \dddot{v}_{21}) \, dy, \quad (F.4)
\]

\[
\Sigma_{11}^{10} \Delta_{21} = -\frac{1}{2} \int_{-\infty}^{\infty} i\alpha_{21} (\dddot{u}_{11} \dddot{u}_{10} \dddot{u}_{21} + \dddot{v}_{11} \dddot{v}_{10} \dddot{v}_{21} + \dddot{v}_{11} \dddot{v}_{10} \dddot{v}_{21}) \, dy \\
+ \frac{1}{2} \int_{-\infty}^{\infty} (\dddot{u}_{11} \dddot{v}_{10} \frac{\partial \dddot{u}_{21}}{\partial y} + \dddot{v}_{11} \dddot{v}_{10} \frac{\partial \dddot{v}_{21}}{\partial y} + \dddot{v}_{11} \dddot{v}_{10} \frac{\partial \dddot{v}_{21}}{\partial y}) \, dy, \quad (F.5)
\]
\[ \Sigma_{11}^{11} \Delta_{22} = -\frac{1}{4} \int_{-\infty}^{\infty} i\alpha_{22} (\dot{u}_{11}^2 \ddot{u}_{22}^2 + \dot{v}_{11} \ddot{u}_{22} + \ddot{w}_{11} \dot{u}_{11} \ddot{w}_{22}) dy \\
+ \frac{1}{4} \int_{-\infty}^{\infty} (\dot{u}_{11} \ddot{v}_{11} + \dot{v}_{11} \ddot{v}_{22} + \ddot{w}_{11} \dot{v}_{11} \ddot{w}_{22}) dy \\
- \frac{1}{2} \int_{-\infty}^{\infty} \gamma (\dot{u}_{11} \ddot{w}_{11} \dot{u}_{22} + \dot{v}_{11} \ddot{w}_{11} \dot{v}_{22} + \ddot{w}_{11}^2 \dot{w}_{22}) dy \quad (F.6) \]
Appendix G

The Six Wave Mode Problem

G.1 Wave Mode Terms

Three-dimensional Subharmonic 12 Mode: $e^{-i\beta t} (\cos(2\gamma z)\text{or } \sin(2\gamma z))$

\[(\ddot{u}^2 - \ddot{u}^2)_{12} = 2 u_{12}^* u_{20} + 2 u_{11}^* u_{21} + u_{10}^* u_{22} \]  \hspace{1cm} (G.1)

\[(\ddot{w} - \ddot{u}^2)_{12} = u_{12}^* v_{10} + u_{10}^* v_{22} + v_{12}^* u_{20} + v_{10}^* u_{22} \]  \hspace{1cm} (G.2)

\[(\ddot{u}w - \ddot{u}w)_{12} = u_{10}^* w_{12} + u_{12}^* w_{20} + \frac{u_{11}^* w_{21}}{2} + \frac{w_{11}^* u_{21}}{2} \]  \hspace{1cm} (G.3)

\[(\ddot{v}^2 - \ddot{v}^2)_{12} = 2 v_{12}^* v_{10} + 2 v_{10}^* v_{22} + v_{11}^* v_{21} \]  \hspace{1cm} (G.4)

\[(\ddot{w} - \ddot{w})_{12} = v_{10}^* w_{12} + w_{12}^* v_{10} + \frac{v_{11}^* w_{21}}{2} + \frac{w_{11}^* v_{21}}{2} \]  \hspace{1cm} (G.5)

\[(\ddot{w}^2 - \ddot{w}^2)_{12} = -\frac{w_{11}^* w_{21}}{2} \]  \hspace{1cm} (G.6)
G.2 The Mechanical Energy Equations for 12 Wave Modes

In here we present one equation which contains both the energy and the phase angle equations. To get the energy equation, we can take the real part of the equation G.7 and take the imaginary part of the equation G.7 for the phase angle equation.

\[
\frac{U}{2} \frac{d}{dx} [ |u_{12}|^2 + |v_{12}|^2 + |w_{12}|^2 ] = -2(t_{12}v_{12}^*/2) \frac{dU}{dy} \\
- \frac{\partial}{\partial x} \left[ 2\left( \frac{u_{12}v_{12}^*}{2} \right) \right] - \frac{\partial}{\partial y} \left[ 2\left( \frac{v_{12}w_{12}^*}{2} \right) \right] \\
+ \frac{1}{2Re_0} \left[ \frac{\partial^2}{\partial x^2} \left( |u_{12}|^2 + |v_{12}|^2 + |w_{12}|^2 \right) + \frac{\partial^2}{\partial y^2} \left( |u_{12}|^2 + |v_{12}|^2 + |w_{12}|^2 \right) \right] \\
- \frac{1}{Re_0} \left[ \frac{\partial u_{12}}{\partial x} \right]^2 + \frac{\partial v_{12}}{\partial x} \right]^2 + \frac{\partial w_{12}}{\partial x} \right]^2 + \frac{\partial u_{12}}{\partial y} \right]^2 + \frac{\partial v_{12}}{\partial y} \right]^2 + \frac{\partial w_{12}}{\partial y} \right]^2 \\
- \gamma^2 (|u_{12}|^2 + |v_{12}|^2 + |w_{12}|^2) \\
- \left[ \frac{\partial}{\partial x} \left( \frac{u_{12}^2u_{20}^*}{2} + \frac{v_{12}^2v_{20}^*}{2} + \frac{w_{12}^2w_{20}^*}{2} \right) \right] \\
+ \frac{\partial}{\partial y} \left( \frac{u_{12}u_{11}v_{21}^*}{2} + \frac{v_{12}u_{11}v_{21}^*}{2} + \frac{w_{12}u_{11}w_{21}^*}{2} \right) \\
+ \left[ \frac{\partial}{\partial x} \left( \frac{u_{12}^2u_{20}^*}{2} + \frac{v_{12}^2v_{20}^*}{2} + \frac{w_{12}^2w_{20}^*}{2} \right) \right] \\
- \left[ \frac{\partial}{\partial y} \left( \frac{u_{12}u_{11}v_{21}^*}{2} + \frac{v_{12}u_{11}v_{21}^*}{2} + \frac{w_{12}u_{11}w_{21}^*}{2} \right) \right] \\
\left[ \frac{\partial u_{12}}{\partial x} \right]^2 + \frac{\partial v_{12}}{\partial x} \right]^2 + \frac{\partial w_{12}}{\partial x} \right]^2 + \frac{\partial u_{12}}{\partial y} \right]^2 + \frac{\partial v_{12}}{\partial y} \right]^2 + \frac{\partial w_{12}}{\partial y} \right]^2 \\
+ \gamma \left[ \frac{u_{12}^2u_{21}w_{11}^* + v_{12}^2v_{21}w_{11}^* + w_{12}^2w_{21}w_{11}^*}{2} \right] \\
\left[ \frac{\partial u_{12}}{\partial x} \right]^2 + \frac{\partial v_{12}}{\partial x} \right]^2 + \frac{\partial w_{12}}{\partial x} \right]^2 + \frac{\partial u_{12}}{\partial y} \right]^2 + \frac{\partial v_{12}}{\partial y} \right]^2 + \frac{\partial w_{12}}{\partial y} \right]^2 \\
+ \gamma \left[ \frac{u_{12}^2u_{21}w_{11}^* + v_{12}^2v_{21}w_{11}^* + w_{12}^2w_{21}w_{11}^*}{2} \right] \\
\left[ \frac{\partial u_{12}}{\partial x} \right]^2 + \frac{\partial v_{12}}{\partial x} \right]^2 + \frac{\partial w_{12}}{\partial x} \right]^2 + \frac{\partial u_{12}}{\partial y} \right]^2 + \frac{\partial v_{12}}{\partial y} \right]^2 + \frac{\partial w_{12}}{\partial y} \right]^2 \\
+ \gamma \left[ \frac{u_{12}^2u_{21}w_{11}^* + v_{12}^2v_{21}w_{11}^* + w_{12}^2w_{21}w_{11}^*}{2} \right] \]
\[-\frac{1}{2}[(u_{11}u_{21})_x \frac{\partial u_{12}}{\partial x} + (v_{11}v_{21})_x \frac{\partial v_{12}}{\partial x} + (u_{11}w_{21})_x \frac{\partial w_{12}}{\partial x}]
\]
\[+ (u_{11}u_{21})_y \frac{\partial u_{12}}{\partial y} + (v_{11}v_{21})_y \frac{\partial v_{12}}{\partial y} + (u_{11}w_{21})_y \frac{\partial w_{12}}{\partial y}]
\[-(u_{12}u_{10})_x \frac{\partial u_{22}}{\partial x} + (v_{12}v_{10})_x \frac{\partial v_{22}}{\partial x} + (w_{12}w_{10})_x \frac{\partial w_{22}}{\partial x}]
\[+(u_{12}v_{10})_y \frac{\partial u_{22}}{\partial y} + (v_{12}v_{10})_y \frac{\partial v_{22}}{\partial y} + (w_{12}w_{10})_y \frac{\partial w_{22}}{\partial y}\]

(G.7)

**G.3 Integral Coefficients \( \Sigma_{ij}^{pq} \Delta_{kl} \)**

\[\Sigma_{12}^{12} \Delta_{20} = \int_{-\infty}^{\infty} (-i \alpha^2_{20} \hat{u}_{12}^2 \hat{u}_{20} - i \alpha^2_{20} \hat{v}_{12} \hat{v}_{20})dy\]
\[+ \int_{-\infty}^{\infty} (\hat{u}_{12} \hat{v}_{20} \frac{\partial \hat{u}_{20}}{\partial y} + \hat{v}_{12} \hat{v}_{20} \frac{\partial \hat{v}_{20}}{\partial y})dy, \quad (G.8)\]

\[\Sigma_{12}^{22} \Delta_{10} = \int_{-\infty}^{\infty} i \alpha_{10} (\hat{u}_{12} \hat{u}_{22} \hat{u}_{10} + \hat{v}_{10} \hat{v}_{22} \hat{v}_{12})dy\]
\[+ \int_{-\infty}^{\infty} (\hat{u}_{10} \hat{v}_{22} \frac{\partial \hat{u}_{12}}{\partial y} + \hat{v}_{10} \hat{v}_{22} \frac{\partial \hat{v}_{12}}{\partial y})dy, \quad (G.9)\]

\[\Sigma_{12}^{21} \Delta_{11} = \frac{1}{2} \int_{-\infty}^{\infty} i \alpha_{11} (\hat{u}_{12} \hat{u}_{11} \hat{u}_{21} + \hat{v}_{12} \hat{v}_{11} \hat{v}_{21} + \hat{w}_{12} \hat{w}_{11} \hat{w}_{21})dy\]
\[+ \frac{1}{2} \int_{-\infty}^{\infty} (\hat{u}_{12} \hat{v}_{21} \frac{\partial \hat{u}_{11}}{\partial y} + \hat{v}_{12} \hat{v}_{21} \frac{\partial \hat{v}_{11}}{\partial y} + \hat{w}_{12} \hat{v}_{21} \frac{\partial \hat{w}_{11}}{\partial y})dy\]
\[+ \frac{1}{2} \int_{-\infty}^{\infty} \gamma (\hat{u}_{12} \hat{w}_{11} \hat{u}_{21} + \hat{v}_{12} \hat{w}_{11} \hat{v}_{21} + \hat{w}_{12} \hat{w}_{11} \hat{w}_{21})dy, \quad (G.10)\]

\[\Sigma_{12}^{10} \Delta_{22} = - \int_{-\infty}^{\infty} i \alpha_{22} (\hat{u}_{12} \hat{u}_{10} \hat{u}_{22} + \hat{v}_{12} \hat{v}_{10} \hat{v}_{22} + \hat{w}_{12} \hat{w}_{10} \hat{w}_{22})dy\]
\[+ \int_{-\infty}^{\infty} (\hat{u}_{12} \hat{v}_{10} \frac{\partial \hat{u}_{22}}{\partial y} + \hat{v}_{12} \hat{v}_{10} \frac{\partial \hat{v}_{22}}{\partial y} + \hat{w}_{12} \hat{v}_{10} \frac{\partial \hat{w}_{22}}{\partial y})dy, \quad (G.11)\]

\[\Sigma_{11}^{21} \Delta_{12} = \frac{1}{2} \int_{-\infty}^{\infty} i \alpha_{12} (\hat{u}_{11} \hat{u}_{12} \hat{u}_{21} + \hat{v}_{11} \hat{v}_{12} \hat{v}_{21} + \hat{w}_{11} \hat{w}_{12} \hat{w}_{21})dy\]
\[+ \frac{1}{2} \int_{-\infty}^{\infty} (\hat{v}_{11} \hat{u}_{21} \frac{\partial \hat{u}_{12}}{\partial y} + \hat{v}_{11} \hat{v}_{21} \frac{\partial \hat{v}_{12}}{\partial y} + \hat{v}_{11} \hat{w}_{21} \frac{\partial \hat{w}_{12}}{\partial y})dy \quad (G.12)\]

Figure G.1 shows the nonlinear energy interactions among the wave modes.
Figure G.1: Diagram of the nonlinear energy exchanges among the wave modes for the six-mode case

A: $S_{12}^{12} D_{22}$
B: $S_{11}^{21} D_{12}$
C: $S_{12}^{12} D_{20}$
D: $S_{11}^{10} D_{22}$
E: $S_{21}^{11} D_{12}$
Vita

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