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Nonlinear Potential Analysis on Sobolev Multiplier Spaces

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NONLINEAR POTENTIAL ANALYSIS ON SOBOLEV MULTIPLIER SPACES

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Abstract

We characterize preduals and Köthe duals to a class of Sobolev multiplier type spaces. Our results fit in well with the modern theory of function spaces of harmonic analysis and are also applicable to nonlinear partial differential equations. As a maneuver, we make use of several tools from nonlinear potential theory, weighted norm inequalities, and the theory of Banach function spaces to obtain our results. After characterizing the preduals, we establish a capacitary strong type inequality which resolves a special case of a conjecture by David R. Adams. As a consequence, we obtain several equivalent norms for Choquet integrals associated to Bessel or Riesz capacities. This enables us to obtain bounds for the Hardy-Littlewood maximal function on Choquet spaces associated to Bessel or Riesz capacities in a sublinear setting. Finally, we extend those maximal function bounds to full range of exponents, which allow us to deduce Sobolev type embeddings on certain Choquet spaces.

Chapter 1. Introduction

For $\alpha > 0$ and $s > 1$, let $H^{\alpha,s} = \{u = G_\alpha * f : f \in L^s(\mathbb{R}^n)\}$ be the space of Bessel potentials of L^s functions in $\mathbb{R}^n$, $n \geq 1$. Here G_α is Bessel kernel of order α and $*$ denotes the convolution operator in $\mathbb{R}^n$. Associated to $H^{\alpha,s}$ is the Bessel capacity $\mathrm{Cap}_{\alpha,s}(\cdot)$ defined for any set $E \subseteq \mathbb{R}^n$ by

$$\mathrm{Cap}_{\alpha,s}(E) = \inf\{\|f\|^s_{L^s(\mathbb{R}^n)} : G_\alpha * f \geq 1 \text{ on } E\}.$$

For $p \geq 1$, we let $M_p^{\alpha,s}$ be the space of functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{M_p^{\alpha,s}} = \sup_K \left(\frac{\int_K |f(x)|^p dx}{\mathrm{Cap}_{\alpha,s}(K)}\right)^{1/p} < +\infty,$$

where the supremum is taken over all compact sets $K \subseteq \mathbb{R}^n$ such that $\mathrm{Cap}_{\alpha,s}(K) \neq 0$. It is by now known (see Chapter 2) that a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to $M_p^{\alpha,s}$ if and only if $|f|^{\frac{p}{s}} u \in L^s$ for any $u \in H^{\alpha,s}$ and the following inequality holds:

$$\left(\int_{\mathbb{R}^n} (G_\alpha * h)^s |f|^p dx\right)^{\frac{1}{p}} \leq C\|h\|^{\frac{s}{p}}_{L^s(\mathbb{R}^n)}, \qquad \forall h \geq 0.$$

Thus $M_p^{\alpha,s}$ can be viewed as a Sobolev multiplier type space in $\mathbb{R}^n$ (see [MS1, MS2]). We note that such spaces or their homogeneous counterparts appear naturally and play an important role in many super-critical nonlinear PDEs including the Navier-Stokes system (see, e.g., [VW, KV, HMV, Ph2, Ph3, PhV, NP, AP2, PhPh, L-R, Ger]).

For $0 < \alpha s \leq n$, and $p > 1$, a major part of this thesis is to investigate preduals X of $M_p^{\alpha,s}$. Those are spaces X such that $X^* \approx M_p^{\alpha,s}$ in the sense that for every linear functional $L \in X^*$, there is a unique $f \in M_p^{\alpha,s}$ such that

$$L(g) = \int_{\mathbb{R}^n} f(x)g(x)dx, \qquad \forall g \in X,$$

and that $A^{-1}\|L\| \leq \|f\|_{M_p^{\alpha,s}} \leq A\|L\|$ for some constant $A > 0$.

It will be shown in Section 3.2 that X could be any of the spaces $(M_p^{\alpha,s})'$, $\mathcal{N}_{p'}^{\alpha,s}$, $N_{p'}^{\alpha,s}$, $\tilde{N}_{p'}^{\alpha,s}$. and $B_{p'}^{\alpha,s}$, where $p' = p/(p-1)$. The readers are referred to Section 3.2 for the precise definitions of such predual spaces. Here we mention that $(M_p^{\alpha,s})'$ is the Köthe dual of $M_p^{\alpha,s}$, and $N_q^{\alpha,s}$ is the space of all measurable functions g such that there exists a local A_1 weight $w \geq 0$ such that $\int_0^\infty \mathrm{Cap}_{\alpha,s}(\{w > t\})dt \leq 1$ and

$$\left(\int_{\mathbb{R}^n} |g(x)|^{p'} w(x)^{1-p'} dx\right)^{1/p'} < +\infty.$$

The 'norm' of a function $g \in N_{p'}^{\alpha,s}$ is the defined as the infimum of the above quantity over all such weights w.

As one of the main results, we obtain in Theorem 3.2.8 that these preduals are isomorphic, i.e.,

$$\mathcal{N}_{p'}^{\alpha,s} \approx N_{p'}^{\alpha,s} \approx (M_p^{\alpha,s})' \approx \widetilde{N}_{p'}^{\alpha,s} \approx B_{p'}^{\alpha,s}. \tag{1.1}$$

Moreover, in the case when $\mathrm{Cap}_{\alpha,s}$ is strongly subadditive, which is known for $s = 2$ and $0 < \alpha \leq 1$ (see Chapter 2), we show in Theorem 3.2.7 that all of the above spaces are Banach function spaces in the sense of [Lux] and the following isometric isomorphisms hold:

$$(M_p^{\alpha,s})' = \widetilde{N}_{p'}^{\alpha,s} = B_{p'}^{\alpha,s}.$$

One can think of the isomorphism $(M_p^{\alpha,s})' \approx N_{p'}^{\alpha,s}$ obtained in (1.1) as a concrete description for the abstract space $(M_p^{\alpha,s})'$. This enables us to obtain localized bounds for the Hardy-Littlewood maximal function and standard Calderón-Zygmund operators on $(M_p^{\alpha,s})'$ (or on any of the spaces in (1.1)), see Theorem 3.3.2.

The above results as well as other results obtained in Chapter 3 are based on the paper [OP1]. The methods used here are based mainly on tools from nonlinear potential theory, weighted norm inequalities, the theory of Banach function spaces, and a lemma of Komlós [Kom].

In Chapter 4, we present the results obtained in the papers [OP2, OP3]. Here an application related to the preduals of $M_p^{\alpha,s}$ will be considered. Our motivation is the following conjecture made by David R. Adams in [Ad4]: There exists a constant $A > 0$ such that

$$\int_{\mathbb{R}^n} (G_\alpha * f) dC \leq A \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} dx, \qquad \forall f \geq 0. \tag{1.2}$$

Here we write, for any function φ,

$$\int_{\mathbb{R}^n} |\varphi| dC = \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{x \in \mathbb{R}^n : |\varphi(x)| > t\}) dt,$$

which is the Choquet integral of $|\varphi|$ with respect to the capacity $\mathrm{Cap}_{\alpha,s}$.

More precisely, Adams [Ad4] obtained (1.2) in the context of Riesz potentials and capacities for *integers* $\alpha \in (0, n)$ and suggested that it should hold for all *real* $\alpha \in (0, n)$. Here, as another main result, we answer this question positively for both Riesz and Bessel potentials. Our resolution of (1.2) enables us to find new characterizations of the L^1 Choquet integral defined above. Indeed, for any function φ we have

$$\int_{\mathbb{R}^n} |\varphi| dC \simeq \lambda_{\alpha,s}(\varphi) \simeq \beta_{\alpha,s}(\varphi), \tag{1.3}$$

where

$$\lambda_{\alpha,s}(\varphi) = \inf \left\{ \|f\|_{(M_p^{\alpha,s})'} : 0 \le f \in (M_p^{\alpha,s})', G_\alpha * f \ge |\varphi| \right\},$$
$$\beta_{\alpha,s}(\varphi) = \inf \left\{ \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} dx : f \ge 0, G_\alpha * f \ge |\varphi| \right\}.$$

Here $A \simeq B$ stands for the two-sided bound $c^{-1}A \le B \le cA$ for some constant $c > 0$.

In turn, the characterization (1.3) allows us to obtain the boundedness of the local Hardy-Littlewood maximal function in a sublinear setting:

$$\int_{\mathbb{R}^n} (\mathbf{M}^{\text{loc}} f)^q dC \le A \int_{\mathbb{R}^n} |f|^q dC, \tag{1.4}$$

where $\mathbf{M}^{\text{loc}} f$ is the local Hardy-Littlewood maximal function of f defined by (3.5) below, and $q > (n-\alpha)/n$ (see Theorem 4.2.5). Note here that the exponent q could be smaller than 1.

Finally, we extend the range of q in (1.4) to the optimal one $q > (n - \alpha s)/n$ (see Theorem 4.3.2) by showing that at the end-point $q_0 = (n - \alpha s)/n$ one has the following capacitary "weak type" bound

$$\text{Cap}_{\alpha,p}(\{\mathbf{M}^{\text{loc}} f > t\}) \le C\, t^{-q_0} \int_{\mathbb{R}^n} |f|^{q_0} dC, \qquad \forall t > 0, \tag{1.5}$$

(see Theorem 4.3.1). Our approach to (1.5) relies heavily on nonlinear potential theory in which the so-called Wolff's potentials will play an important role. We also prove and use certain convexity property of a weak L^{q_0} Choquet space. With the bound (1.4) for $q > (n - \alpha s)/n$ and the bound (1.5) at hand, we are able to deduce various Sobolev type inequalities on Choquet spaces (see, e.g., Theorem 4.4.1).

Chapter 2. Preliminaries

2.1. Capacities and the Space $L^1(C)$

Let α be a real number and $s > 1$. We define the space of Bessel potentials $H^{\alpha,s} = H^{\alpha,s}(\mathbb{R}^n)$, $n \geq 1$, as the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{H^{\alpha,s}} = \|(1-\Delta)^{\frac{\alpha}{2}} u\|_{L^s(\mathbb{R}^n)}.$$

Here the operator $(1-\Delta)^{\frac{\alpha}{2}}$ is understood as $(1-\Delta)^{\frac{\alpha}{2}} := \mathcal{F}^{-1}(1+|\xi|^2)^{\frac{\alpha}{2}}\mathcal{F}$, where $\mathcal{F}$ is the Fourier transform in $\mathbb{R}^n$. In the case $\alpha > 0$, it follows that (see, e.g., [MH]) a function u belongs to $H^{\alpha,s}$ if and only if

$$u = G_\alpha * f$$

for some $f \in L^s$, and moreover $\|u\|_{H^{\alpha,s}} = \|f\|_{L^s}$. Here G_α is the Bessel kernel of order α defined by $G_\alpha(x) := \mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{-\alpha}{2}}](x)$.

The Bessel potential space $H^{\alpha,s}$, $\alpha > 0, s > 1$, can be viewed as a fractional generalization of the standard Sobolev space $W^{k,s} = W^{k,s}(\mathbb{R}^n)$, $k \in \mathbb{N}, s > 1$. The latter, by definition, consists of functions in L^s whose distributional derivatives up to order k also belong to L^s. The norm of a function $u \in W^{k,s}$ is given by $\|u\|_{W^{k,s}} = \sum_{|\beta|=k} \|D^\beta u\|_{L^s} + \|u\|_{L^s}$. Indeed, it follows from the theory of singular integrals that for any $k \in \mathbb{N}$ and $s > 1$ we have $H^{k,s} \approx W^{k,s}$, i.e., there exists a constant $A > 0$ such that

$$A^{-1}\|u\|_{H^{k,s}} \leq \|u\|_{W^{k,s}} \leq A\|u\|_{H^{k,s}}. \tag{2.1}$$

Recall that the Bessel capacity $\mathrm{Cap}_{\alpha,s}(\cdot)$, $\alpha > 0, s > 1$, is defined for every subset E of $\mathbb{R}^n$ by

$$\mathrm{Cap}_{\alpha,\,s}(E) := \inf\Big\{\|f\|_{L^s}^s : f \geq 0, G_\alpha * f \geq 1 \text{ on } E\Big\}.$$

It is an outer capacity, i.e., for any set $E \subset \mathbb{R}^n$,

$$\mathrm{Cap}_{\alpha,s}(E) = \inf\{\mathrm{Cap}_{\alpha,s}(G) : G \supset E, G \text{ open}\},$$

and is countably subadditive in the sense that for $E_i \subset \mathbb{R}^n$, $i = 1, 2, \ldots,$

$$\mathrm{Cap}_{\alpha,s}\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leq \sum_{i=1}^{\infty} \mathrm{Cap}_{\alpha,s}(E_i).$$

Moreover, it has the following basic properties of a Choquet capacity (see [AH]):

(i) $\mathrm{Cap}_{\alpha,s}(\emptyset) = 0$;

(ii) if $E_1 \subset E_2$, then $\mathrm{Cap}_{\alpha,s}(E_1) \leq \mathrm{Cap}_{\alpha,s}(E_2)$;

(iii) if $K_1 \supset K_2 \supset \ldots$ is a decreasing sequence of compact sets of $\mathbb{R}^n$, then

$$\mathrm{Cap}_{\alpha,s}\Big(\bigcap_{i=1}^{\infty} K_i\Big) = \lim_{i\to\infty} \mathrm{Cap}_{\alpha,s}(K_i);$$

(iv) if $E_1 \subset E_2 \subset \ldots$ is an increasing sequence of subsets of $\mathbb{R}^n$, then

$$\mathrm{Cap}_{\alpha,s}\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \lim_{i\to\infty} \mathrm{Cap}_{\alpha,s}(E_i). \tag{2.2}$$

Thus by the Capacitability Theorem (see [Cho, Mey]), for any Borel (or more generally Suslin) set $E \subset \mathbb{R}^n$ we have

$$\mathrm{Cap}_{\alpha,s}(E) = \sup\{\mathrm{Cap}_{\alpha,s}(K) : K \subset E, K \text{ compact}\}.$$

By (2.1) we see that if α is a positive integer, then $\mathrm{Cap}_{\alpha,s}(E) \simeq C_{\alpha,s}(E)$ for any set $E \subset \mathbb{R}^n$ (see also [AH]). Here for a compact set $K \subset \mathbb{R}^n$ and $\alpha \in \mathbb{N}$, we define

$$C_{\alpha,s}(K) = \inf\{\|\varphi\|_{W^{\alpha,s}} : \varphi \in C_c^\infty, \varphi \geq 1 \text{ on } K\},$$

and $C_{\alpha,s}(\cdot)$ is extended to any set E of $\mathbb{R}^n$ by letting

$$C_{\alpha,s}(E) := \inf_{\substack{G \supset E \\ G \text{ open}}} \left\{ \sup_{\substack{K \subset G \\ K \text{ compact}}} C_{\alpha,s}(K) \right\}. \tag{2.3}$$

The notion of Choquet integral will be important in this work. Let $w : \mathbb{R}^n \to [0, \infty]$ be defined $\mathrm{Cap}_{\alpha,s}$-quasieverywhere, i.e., defined except for only a set of zero capacity $\mathrm{Cap}_{\alpha,s}$. The Choquet integral of w is defined by

$$\int_{\mathbb{R}^n} w dC := \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{x \in \mathbb{R}^n : w(x) > t\}) dt. \tag{2.4}$$

A function $\tilde{f}$ is said to be quasicontinuous with respect to $\mathrm{Cap}_{\alpha,s}$ if for any $\epsilon > 0$ there exists an open set G such that $\mathrm{Cap}_{\alpha,s}(G) < \epsilon$ and $\tilde{f}$ is continuous in $G^c := \mathbb{R}^n \setminus G$ (see [AH]). We let $L^1(C)$ be the space of quasicontinuous (hence quasieverywhere defined) functions f in $\mathbb{R}^n$ such that

$$\|f\|_{L^1(C)} := \int |f| dC < +\infty. \tag{2.5}$$

In general, $L^1(C)$ is only a quasi-Banach space (see Proposition 2.1.2 below) as $\|\cdot\|_{L^1(C)}$ may not satisfy the triangle inequality. However, by a theorem of Choquet (see [Cho, Den]), $\|\cdot\|_{L^1(C)}$ satisfies the triangle inequality (hence $L^1(C)$ is a Banach space) if and only if the associated capacity $\mathrm{Cap}_{\alpha,s}$ is strongly subadditive. By definition, the capacity $\mathrm{Cap}_{\alpha,s}$ is strongly subadditive if for any two sets $E_1, E_2 \subset \mathbb{R}^n$,

$$\mathrm{Cap}_{\alpha,s}(E_1 \cup E_2) + \mathrm{Cap}_{\alpha,s}(E_1 \cap E_2) \leq \mathrm{Cap}_{\alpha,s}(E_1) + \mathrm{Cap}_{\alpha,s}(E_2). \tag{2.6}$$

It is known that $\mathrm{Cap}_{\alpha,2}$, $0 < \alpha \leq 1$ is strongly subadditive, and $\mathrm{Cap}_{1,s}$, $s > 1$, is equivalent to one that is strongly subadditive.

For $s = 2$ and $\alpha \in (0, 1]$, it is known that $\mathrm{Cap}_{\alpha,s}(\cdot)$ is strongly subadditive in the sense of (2.6) (see [Lan, pp. 141–145]). We note that the book [Lan] considers only Riesz capacities, i.e., homogeneous versions of $\mathrm{Cap}_{\alpha,2}(\cdot)$. However, the argument there also applies to Bessel capacities since for any $\alpha \in (0, 1]$ the Bessel kernel $G_{2\alpha}$ is continuous and subharmonic in $\mathbb{R}^n \setminus \{0\}$ (hence the First Maximum Principle in the sense of [Lan, Theorem 1.10] holds).

On the other hand, for $\alpha = 1$, the capacity $C_{1,s}(\cdot)$ is strongly subadditive for any $s > 1$. Indeed, this can be proved by adapting the proof of [HKM, Theorem 2.2] to our nonhomogeneous setting.

We shall need the following metric properties of $\mathrm{Cap}_{\alpha,s}(\cdot)$ (see [AH]): For any $0 < r \leq 1$,

$$\mathrm{Cap}_{\alpha,s}(B_r) \simeq r^{n-\alpha s} \quad \text{if } \alpha s < n \tag{2.7}$$

and

$$\mathrm{Cap}_{\alpha,s}(B_r) \simeq [\log(\tfrac{2}{r})]^{1-s} \quad \text{if } \alpha s = n.$$

For $r \geq 1$ and $\alpha s \leq n$ we have

$$\mathrm{Cap}_{\alpha,s}(B_r) \simeq r^n. \tag{2.8}$$

On the other hand, we have for any non-empty set E with $\mathrm{diam}(E) \leq 1$,

$$\mathrm{Cap}_{\alpha,s}(E) \simeq 1 \quad \text{if } \alpha s > n. \tag{2.9}$$

By Sobolev Embedding Theorem for any Lebesgue measurable set E,

$$|E|^{1-\alpha s/n} \leq C\, \mathrm{Cap}_{\alpha,s}(E) \quad \text{if } \alpha s < n. \tag{2.10}$$

Moreover, by Young's inequality for convolution we have, for $s = n/\alpha > 1$,

$$\|G_\alpha * f\|_{L^q} \leq \|G_\alpha\|_{L^r} \|f\|_{L^{\frac{n}{\alpha}}}$$

for any $n/\alpha \leq q < +\infty$ and $r = nq/(n + q(n-\alpha))$. Thus for any $\epsilon \in (0, 1]$ we find

$$|E|^{\epsilon} \leq C(\epsilon)\, \mathrm{Cap}_{\alpha,s}(E) \quad \text{if } \alpha s = n.$$

Note that using the bound $\|G_\alpha * f\|_{L^s} \le \|G_\alpha\|_{L^1}\|f\|_{L^s}$, we also find that

$$|E| \le C\,\mathrm{Cap}_{\alpha,s}(E) \quad \text{for all } \alpha > 0, s > 1.$$

It follows that if $\mathrm{Cap}_{\alpha,s}(E) = 0$ then the Lebesgue measure of E is zero.

The Choquet integral of a $\mathrm{Cap}_{\alpha,s}$-quasieverywhere defined function $w : \mathbb{R}^n \to [0, \infty]$ was defined by (2.4). We also let $L^1(C)$ be the space of quasicontinuous functions f in $\mathbb{R}^n$ such that (2.5) holds. Perhaps, a better notation for $L^1(C)$ should be $L^1(\mathrm{Cap}_{\alpha,s})$ to indicate its dependence on $\mathrm{Cap}_{\alpha,s}$. But we shall use the notation $L^1(C)$ for simplicity and implicitly understand that $C = \mathrm{Cap}_{\alpha,s}$.

In general, the 'norm' of $L^1(C)$ is only a quasinorm, i.e., we only have

$$\|f + g\|_{L^1(C)} \le 2\|g\|_{L^1(C)} + 2\|g\|_{L^1(C)}.$$

However, if $\mathrm{Cap}_{\alpha,s}$ is strongly subadditive then it is actually a norm by a theorem of Choquet (see [Cho, Den]).

In [Ad3, Theorem 4] the following quasiadditivity result was obtained for $\mathrm{Cap}_{\alpha,s}$:

$$\sum_{j=1}^{\infty} \mathrm{Cap}_{\alpha,s}(E \cap \{j-1 \le |x| < j\}) \le C\,\mathrm{Cap}_{\alpha,s}(E) \tag{2.11}$$

for all $E \subset \mathbb{R}^n$, where $C = C(n, \alpha, s) > 0$. We now use (2.11) to obtain the following density result for the space $L^1(C)$.

Proposition 2.1.1. *$C_c(\mathbb{R}^n)$ is dense in $L^1(C)$, where $C_c(\mathbb{R}^n)$ is the linear space of continuous functions with compact support in $\mathbb{R}^n$.*

Proof. We first show that the set of all bounded continuous functions is dense in $L^1(C)$. Let $f \in L^1(C)$ be given. For $M > 0$, we define $f_M(x) = f(x)$ if $|f(x)| \le M$, $f_M(x) = M$ if

$f(x) > M$, and $f_M(x) = -M$ if $f(x) < -M$. Note that

$$\begin{aligned}\|f_M - f\|_{L^1(C)} &= \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{|f_M - f| > t\})dt \\ &= \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{|f| > M + t\})dt \\ &= \int_M^\infty \mathrm{Cap}_{\alpha,s}(\{|f| > t\})dt \to 0,\end{aligned}$$

as $M \to \infty$. For any $\epsilon > 0$, choose an $M > 0$ such that $\|f_M - f\|_{L^1(C)} < \epsilon$. As f_M is quasicontinuous (since f is quasicontinuous), there exists an open set G such that $\mathrm{Cap}_{\alpha,s}(G) < \epsilon$ and $f_M\big|_{G^c}$ is continuous.

By Tietze Extension Theorem, we can find a continuous function v such that $|v| \leq M$ and $v = f_M$ on G^c. Then

$$\begin{aligned}\|v - f_M\|_{L^1(C)} &= \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{|v - f_M| > t\})dt \\ &= \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{x \in G : |v(x) - f_M(x)| > t\})dt \\ &= \int_0^{2M} \mathrm{Cap}_{\alpha,s}(\{x \in G : |v(x) - f_M(x)| > t\})dt \\ &\leq 2M\mathrm{Cap}_{\alpha,s}(G) < 2M\,\epsilon.\end{aligned}$$

As a result,

$$\|f - v\|_{L^1(C)} \leq 2\|f - f_M\|_{L^1(C)} + 2\|f_M - v\|_{L^1(C)} < 2\epsilon + 4M\,\epsilon,$$

which yields the claim.

Now we claim that C_c is dense in $L^1(C)$. All we need to do is to approximate bounded continuous functions by functions in C_c. To this end, let v be a bounded continuous function, say, $|v| \leq M$ for some $M > 0$. For each $N = 1, 2, ...$, let $O_N = \{|v| > 1/N\}$, then O_N is open and

$$\mathrm{Cap}_{\alpha,s}(O_N) \leq N\|v\|_{L^1(C)} < +\infty.$$

We observe that

$$\begin{aligned}
\|v\chi_{O_N^c}\|_{L^1(C)} &= \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{x \in O_N^c : |v(x)| > t\})dt \\
&= \int_0^{1/N} \mathrm{Cap}_{\alpha,s}(\{x \in O_N^c : |v(x)| > t\})dt \\
&\leq \int_0^{1/N} \mathrm{Cap}_{\alpha,s}(\{|v| > t\})dt \to 0,
\end{aligned}$$

as $N \to \infty$. Thus for any $\epsilon > 0$, there is an open set O such that $\mathrm{Cap}_{\alpha,s}(O) < \infty$ and $\|v\chi_{O^c}\|_{L^1(C)} < \epsilon$. Since $\mathrm{Cap}_{\alpha,s}(O) < \infty$, by (2.11) we have

$$\sum_{j=0}^\infty \mathrm{Cap}_{\alpha,s}(O \cap \{j \leq |x| < j+1\}) < \infty,$$

and so there is a positive integer j_0 such that

$$\mathrm{Cap}_{\alpha,s}(O \cap \{|x| \geq j_0\}) \leq \sum_{j=j_0}^\infty \mathrm{Cap}_{\alpha,s}(O \cap \{j \leq |x| < j+1\}) < \epsilon.$$

Let $O_1 = O \cap \{|x| < j_0\}$ and $O_2 = O \cap \{|x| \geq j_0\}$, then $O = O_1 \cup O_2$, O_1 is bounded, and $\mathrm{Cap}_{\alpha,s}(O_2) < \epsilon$. Let η be a continuous function with compact support such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on O_1. We have

$$\begin{aligned}
\|\eta v - v\|_{L^1(C)} &\leq 2\|(\eta v - v)\chi_{O^c}\|_{L^1(C)} \\
&\qquad + 4\|(\eta v - v)\chi_{O_1}\|_{L^1(C)} + 4\|(\eta v - v)\chi_{O_2}\|_{L^1(C)} \\
&= 2\|(\eta v - v)\chi_{O^c}\|_{L^1(C)} + 4\|(\eta v - v)\chi_{O_2}\|_{L^1(C)},
\end{aligned}$$

since $\eta \equiv 1$ on O_1.

On the other hand, note that

$$\begin{aligned}
\|(\eta v - v)\chi_{O^c}\|_{L^1(C)} &= \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{x \in O^c : |(\eta v)(x) - v(x)| > t\})dt \\
&\leq \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{x \in O^c : 2|v(x)| > t\})dt \\
&\leq 2\|v\chi_{O^c}\|_{L^1(C)} \\
&< 2\epsilon.
\end{aligned}$$

Also,

$$\begin{aligned}
\|(\eta v - v)\chi_{O_2}\|_{L^1(C)} &\leq \int_0^{2M} \mathrm{Cap}_{\alpha,s}(\{x \in O_2 : |(\eta v)(x) - v(x)| > t\})dt \\
&\leq 2M\mathrm{Cap}_{\alpha,s}(O_2) \\
&< 2M\,\epsilon.
\end{aligned}$$

Thus, we conclude that

$$\|\eta v - v\|_{L^1(C)} < 4\epsilon + 8M\,\epsilon,$$

and since ηv has compact support, the proof is then complete. □

We are now ready to establish the completeness of $L^1(C)$.

Proposition 2.1.2. *The quasinorm space $L^1(C)$ is complete for any $\alpha > 0$ and $s > 1$.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in $L^1(C)$. We need to show that $u_n \to u$ in $L^1(C)$ for some $u \in L^1(C)$. Since C_c is dense in $L^1(C)$, we may assume that $\{u_n\} \subset C_c$.

As $\{u_n\}$ is a Cauchy sequence, we can find positive integers $n_1 < n_2 < \cdots$ such that

$$\int_0^\infty \mathrm{Cap}_{\alpha,s}(\{|u_m - u_n| > t\})dt < 4^{-j} \tag{2.12}$$

for all $m, n \geq n_j$, $j = 1, 2, \ldots$ In particular,

$$\int_0^{2^{-j}} \mathrm{Cap}_{\alpha,s}(|u_{n_{j+1}} - u_{n_j}| > 2^{-j})dt < 4^{-j}$$

and hence

$$\mathrm{Cap}_{\alpha,s}(|u_{n_{j+1}} - u_{n_j}| > 2^{-j}) < 2^{-j}.$$

Let $G_j = \{|u_{n_{j+1}} - u_{n_j}| > 2^{-j}\}$. Then G_j is open and $\mathrm{Cap}_{\alpha,s}(G_j) < 2^{-j}$. We now set

$$H_m = \bigcup_{j \geq m} G_j.$$

Then we have

$$\mathrm{Cap}_{\alpha,s}(H_m) \leq \sum_{j \geq m} \mathrm{Cap}_{\alpha,s}(G_j) < \sum_{j \geq m} 2^{-j} \to 0 \tag{2.13}$$

as $m \to \infty$.

Observe that for any $x \in H_m^c$, we have

$$\sum_{j \geq m} |u_{n_{j+1}}(x) - u_{n_j}(x)| \leq \sum_{j \geq m} 2^{-j} < +\infty.$$

Thus if we let $u : H_m^c \to \mathbb{R}$ be defined by

$$u(x) = \lim_{k \to \infty} u_{n_k}(x) = u_{n_m}(x) + \lim_{k \to \infty} \sum_{j=m+1}^{k} (u_{n_j}(x) - u_{n_{j-1}}(x)),$$

then by the Weierstrass M-Test we see that u is continuous in H_m^c.

As the set H_m^c is increasing, the function u can be extended to define in the union $\bigcup_{m \geq 1} H_m^c$. It is now easy to see from (2.13) that u is quasicontinuous.

Now by (2.2) and the Monotone Convergence Theorem we have for each $n \geq 1$,

$$\begin{aligned}
\|u_n - u\|_{L^1(C)} &= \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{|u_n - u| > t\})dt \\
&\leq \int_0^\infty \mathrm{Cap}_{\alpha,s}\left(\bigcup_{N\geq 1}\bigcap_{k\geq N}\{|u_n - u_{n_k}| > t\}\right) dt \\
&= \int_0^\infty \lim_{N\to\infty} \mathrm{Cap}_{\alpha,s}\left(\bigcap_{k\geq N}\{|u_n - u_{n_k}| > t\}\right) dt \\
&= \lim_{N\to\infty} \int_0^\infty \mathrm{Cap}_{\alpha,s}\left(\bigcap_{k\geq N}\{|u_n - u_{n_k}| > t\}\right) dt.
\end{aligned}$$

Thus by (2.12) for each $j = 1, 2, \ldots,$ and $n \geq n_j$, we have

$$\|u_n - u\|_{L^1(C)} \leq 4^{-j}.$$

This completes the proof of the proposition. □

We also define $\mathfrak{M}^{\alpha,s} = \mathfrak{M}^{\alpha,s}(\mathbb{R}^n)$ as the space of locally finite signed measure μ in $\mathbb{R}^n$ such that the norm $\|\mu\|_{\mathfrak{M}^{\alpha,s}} < +\infty$, where

$$\|\mu\|_{\mathfrak{M}^{\alpha,s}} := \sup_K \frac{|\mu|(K)}{\mathrm{Cap}_{\alpha,s}(K)}$$

with the supremum being taken over all compact sets $K \subset \mathbb{R}^n$ such that $\mathrm{Cap}_{\alpha,s}(K) \neq 0$. The following duality relation was stated without proof in [Ad4]. Indeed, it can be proved using Proposition 2.1.1 and the formula

$$\int_{\mathbb{R}^n} ud|\mu| = \sup\left\{\int_{\mathbb{R}^n} vd\mu : v \in C_c(\mathbb{R}^n), |v| \leq u\right\},$$

which holds for all $u \in C_c(\mathbb{R}^n)$ and $u \geq 0$.

Theorem 2.1.3. *Let $\alpha > 0$ and $s > 1$. We have $(L^1(C))^* = \mathfrak{M}^{\alpha,s}$ in the sense that each bounded linear functional $L \in (L^1(C))^*$ corresponds to a unique measure $\nu \in \mathfrak{M}^{\alpha,s}$ in such* *a way that*

$$L(f) = \int_{\mathbb{R}^n} f(x)d\nu(x) \tag{2.14}$$

for all $f \in L^1(C)$*. Moreover,* $\|L\| = \|\nu\|_{\mathfrak{M}^{\alpha,s}}$*.*

Remark 2.1.4. *The right-hand side of* (2.14) *makes sense since for* $f \in L^1(C)$ *and* $t \in \mathbb{R}$*, we have that the set* $\{f > t\} = F \setminus N$ *for a* G_δ *set* F *and a set* N *with* $\mu(N) = \mathrm{Cap}_{\alpha,s}(N) = 0$*. Here* μ *should be understood as the completion of* μ*, and note that if* $\mathrm{Cap}_{\alpha,s}(N) = 0$ *then* $N \subset \widetilde{N}$*, where* $\widetilde{N}$ *is a* G_δ *set with* $\mathrm{Cap}_{\alpha,s}(\widetilde{N}) = 0$*.*

2.2. The Spaces $L^p(C)$, $L^{p,\infty}(C)$, and Capacitary Strong Type Inequality

For $0 < p < \infty$, we denote by $L^p(C)$ the space of all q.e. defined functions u in $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} |u|^p dC < +\infty$, with quasi-norm

$$\|u\|_{L^p(C)} := \left(\int_{\mathbb{R}^n} |u|^p dC\right)^{\frac{1}{p}}.$$

The 'weak' version of $L^p(C)$ is denoted by $L^{p,\infty}(C)$ which consists of all q.e. defined functions u in $\mathbb{R}^n$ such that $\|u\|_{L^{p,\infty}(C)} < +\infty$, where

$$\|u\|_{L^{p,\infty}(C)} := \sup_{\lambda>0} \lambda \, \mathrm{Cap}_{\alpha,s}(\{|u| > \lambda\})^{\frac{1}{p}}.$$

One of the fundamental results of nonlinear potential theory is the following Maz'ya's capacitary inequality, originally obtained by Maz'ya, and subsequently extended by Adams, Dahlberg, and Hansson:

$$\int_{\mathbb{R}^n} (G_\alpha * f)^s dC \leq A \int_{\mathbb{R}^n} f^s dx, \tag{2.15}$$

which holds for any nonnegative Lebesgue measurable function f. See, e.g., [Maz, MS2, AH], and in particular, see Section 2.3.1 and the historical comments in Section 2.3.13 of

[Maz]. This kind of capacitary inequalities and their many applications are discussed in Chapters 2,3, and 11 of [Maz]. Let us call (2.15) the Capacitary Strong Type inequality which we restate it as the following theorem.

Theorem 2.2.1. *Let $\alpha > 0$, $1 < s < \infty$. There is a constant A, depending only on n and s such that*

$$\int_0^\infty \mathrm{Cap}_{\alpha,s}(\{x \in \mathbb{R}^n : G_\alpha * f(x) \geq \lambda\})d\lambda^s \leq A\|f\|^s_{L^s(\mathbb{R}^n)}$$

for all $f \in L^s_+(\mathbb{R}^n)$.

Proposition 2.2.2. *$L^1(C)$ is normable.*

Proof. We will need the following functional which is defined by

$$\gamma_{\alpha,s}(u) := \inf \left\{ \int f^s dx : 0 \leq f \in L^s(\mathbb{R}^n) \text{ and } G_\alpha * f \geq |u|^{\frac{1}{s}} \text{ q.e.} \right\}$$

for each q.e. defined function u in $\mathbb{R}^n$. Note that $\gamma_{\alpha,s}(tu) = |t|\gamma_{\alpha,s}(u)$ for all $t \in \mathbb{R}$, and moreover,

$$\gamma_{\alpha,s}(u) := \inf \{t > 0 : |u| \in t\, H\},$$

where H is the set of all nonnegative and q.e. defined functions g in $\mathbb{R}^n$ such that $g^{\frac{1}{s}} \leq G_\alpha * f$ q.e. for a function $f \in L^s(\mathbb{R}^n)$, $f \geq 0$, such that $\|f\|_{L^s} \leq 1$. Note that if we define a nonlinear operator T by

$$T(h) := \left(G_\alpha * |h|^{\frac{1}{s}}\right)^s, \qquad \forall h \in L^1(\mathbb{R}^n),$$

then by reverse Minkowski's inequality we see that T is superadditive on $L^1_+(\mathbb{R}^n)$. This yields that the set H is convex (see [KV, Lemma 2.4]) and thus the functional $\gamma_{\alpha,s}(\cdot)$ is subadditive.

On the other hand, we can deduce from the Capacitary Strong Type inequality and [AH, Proposition 7.4.1] the following equivalence

$$\int_0^\infty \mathrm{Cap}_{\alpha,s}(\{x : |u(x)| > t\})dt \simeq \gamma_{\alpha,s}(u), \tag{2.16}$$

which holds for all q.e. defined functions u in $\mathbb{R}^n$. In particular, we find that the space $L^1(C)$ is normable for all $\alpha > 0, s > 1$ and $\alpha s \leq n$. □

Corollary 2.2.3. *$L^p(C)$ is normable for $p \geq 1$.*

The following important result is a direct consequence of Capacitary Strong Type inequality of Theorem 2.2.1 above.

Theorem 2.2.4 (Maz'ya-Adams-Dahlberg-Hasson)**.** *Let $\alpha > 0$, $s > 1$ and suppose that ν is a nonnegative locally finite measure in $\mathbb{R}^n$. Then the following properties are equivalent:*

(i) *The inequality*

$$\int_{\mathbb{R}^n} (G_\alpha * f)^s d\nu \leq A_1 \int_{\mathbb{R}^n} f^s dx$$

holds for all functions $f \in L^s(\mathbb{R}^n), f \geq 0$.

(ii) *The inequality*

$$\nu(K) \leq A_2 \, \mathrm{Cap}_{\alpha,s}(K)$$

holds for all compact sets $K \subset \mathbb{R}^n$.

(iii) *The weak-type inequality*

$$\sup_{t>0} t^s \nu(\{x \in \mathbb{R}^n : G_\alpha * f(x) > t\}) \leq A_3 \int_{\mathbb{R}^n} f^s dx$$

holds for all functions $f \in L^s(\mathbb{R}^n), f \geq 0$.

Moreover, the least possible values of A_i, $i = 1, 2, 3$, are equivalent.

2.3. Banach Function Spaces

Most of the spaces under our consideration fit well in the context of Banach function spaces in the sense of [Lux]. In the setting of $\mathbb{R}^n$ with Lebesgue measure as the underlying measure, a Banach function space X on $\mathbb{R}^n$ is the set of all Lebesgue measurable functions f such that $\|f\|_X := \rho(|f|)$ is finite. Here $\rho(f)$, $f \geq 0$, is a given metric function $(0 \leq \rho(f) \leq \infty)$ that obeys the following properties:

(P1) $\rho(f) = 0$ if and only if $f(x) = 0$ a.e. in $\mathbb{R}^n$; $\rho(f_1 + f_2) \leq \rho(f_1) + \rho(f_2)$; and $\rho(\lambda f) = \lambda \rho(f)$ for any constant $\lambda \geq 0$.

(P2) If $\{f_j\}$, $j = 1, 2, \ldots$, is a sequence of nonnegative measurable functions and $f_j \uparrow f$ a.e. in $\mathbb{R}^n$, then $\rho(f_j) \uparrow \rho(f)$.

(P3) If E is any bounded and measurable subset of $\mathbb{R}^n$, and χ_E is its characteristic function, then $\rho(\chi_E) < +\infty$.

(P4) For every bounded and measurable subset E of $\mathbb{R}^n$, there exists a finite constant $A_E \geq 0$ (depending only on the set E) such that $\int_E f dx \leq A_E \rho(f)$ for any nonnegative measurable function f in $\mathbb{R}^n$.

It follows from property (P2) that any Banach function space X is complete (see [Lux]). We also have that, for measurable functions f_1 and f_2, if $|f_1| \leq |f_2|$ a.e. in $\mathbb{R}^n$ and $f_2 \in X$, then it follows that $f_1 \in X$ and $\|f_1\|_X \leq \|f_2\|_X$.

Given a Banach function space X, the Köthe dual space (or the associate space) to X, denoted by X', is the set of all measurable functions f such that $fg \in L^1(\mathbb{R}^n)$ for all $g \in X$. It turns out that X' is also a Banach function space with the associate metric function $\rho'(f)$, $f \geq 0$, defined by

$$\rho'(f) := \sup\left\{ \int |fg|dx : g \in X,\ \|g\|_X \leq 1 \right\}.$$

By definition, the second associate space X'' to X is given by $X'' = (X')'$, i.e., X'' is the Köthe dual space to X'. The following theorems are important in the theory of Banach function spaces (see [Lux]).

Theorem 2.3.1. *Every Banach function space X coincides with its second associate space X'', i.e., $X = X''$ with equality of norms.*

Theorem 2.3.2. *$X^* = X'$ (isometrically) if and only if the space X has an absolutely continuous norm.*

Here we say that X has an absolutely continuous norm if the following properties are satisfied for any $f \in X$:

(a) If E is a bounded set of $\mathbb{R}^n$ and E_j are measurable subsets of E such that $|E_j| \to 0$ as $j \to \infty$, the $\|f\chi_{E_j}\|_X \to 0$ as $j \to \infty$.

(b) $\|f\chi_{\mathbb{R}^n \setminus B_j(0)}\|_X \to 0$ as $j \to \infty$.

It is known that X has an absolutely continuous norm if and only if any sequence $f_j \in X$ such that $|f_j| \downarrow 0$ a.e. in $\mathbb{R}^n$ has the property that $\|f_j\|_X \downarrow 0$ (see [Lux, page 14]).

Chapter 3. Sobolev Multiplier Type Spaces

3.1. Basic Properties

Let $M_p^{\alpha,s} = M_p^{\alpha,s}(\mathbb{R}^n)$, $\alpha > 0, s > 1, p \geq 1$ be the *Sobolev Multiplier Type Space* defined as the set of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that the trace inequality

$$\left(\int_{\mathbb{R}^n} |u|^s |f|^p dx\right)^{\frac{1}{p}} \leq C\|u\|_{H^{\alpha,s}}^{\frac{s}{p}} \tag{3.1}$$

holds for all $u \in C_c^\infty(\mathbb{R}^n)$. A norm of a function $f \in M_p^{\alpha,s}$ is defined to be the least possible constant C in (3.1). By Theorem 2.2.4, the norm is equivalent to the quantity

$$\sup_K \left(\frac{\int_K |f(x)|^p dx}{\mathrm{Cap}_{\alpha,s}(K)}\right)^{1/p}, \tag{3.2}$$

where the supremum is taken over all compact sets $K \subset \mathbb{R}^n$ with non-zero capacity. In what follows, we shall tacitly use (3.2) for functions $f \in M_p^{\alpha,s}$, that is, we redefine

$$\|f\|_{M_p^{\alpha,s}} := \sup_K \left(\frac{\int_K |f(x)|^p dx}{\mathrm{Cap}_{\alpha,s}(K)}\right)^{1/p}.$$

It is worth mentioning that one also has (see [MS2, Remark 3.1.1]):

$$\|f\|_{M_p^{\alpha,s}} \simeq \sup_{K:\,\mathrm{dian}(K)\leq 1} \left(\frac{\int_K |f(x)|^p dx}{\mathrm{Cap}_{\alpha,s}(K)}\right)^{1/p}.$$

Thus in view of (2.9), we have $\|f\|_{M_p^{\alpha,s}} \simeq \sup_{x\in\mathbb{R}^n} \|f\|_{L^p(B_1(x))}$ provided $\alpha s > n$. That is, when $\alpha s > n$, $M_p^{\alpha,s}$ can be identified with the space of uniformly local L^p functions in $\mathbb{R}^n$. For this reason, we shall be mainly interested in the case $\alpha s \leq n$. On the other hand, for $\alpha s < n$ by (2.7) below we see that $M_p^{\alpha,s}$ is continuously embedded into a local Morrey space. If $f \in L^\infty(\mathbb{R}^n)$ then $f \in M_p^{\alpha,s}$ for any $p \geq 1$, and

$$\|f\|_{M_p^{\alpha,s}} \leq C\|f\|_{L^\infty(\mathbb{R}^n)}. \tag{3.3}$$

On the other hand, when $\alpha s < n$ by (2.10) we have

$$\|f\|_{M_p^{\alpha,s}} \le C\|f\|_{L^{\frac{np}{\alpha s},\infty}(\mathbb{R}^n)},$$

where $L^{\frac{np}{\alpha s},\infty}(\mathbb{R}^n)$ is the weak $L^{\frac{np}{\alpha s}}$ space.

Our first result provides another equivalent norm for the space $M_p^{\alpha,s}$, $p > 1$.

Theorem 3.1.1. *For $p > 1$ and $\alpha > 0, s > 1$, with $\alpha s \le n$, we have*

$$\|f\|_{M_p^{\alpha,s}} \simeq \sup_w \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p},$$

where the supremum is taken over all nonnegative $w \in L^1(C) \cap A_1^{\rm loc}$ with $\|w\|_{L^1(C)} \le 1$ and $[w]_{A_1^{\rm loc}} \le \overline{\mathbf{c}}(n,\alpha,s)$ for a constant $\overline{\mathbf{c}}(n,\alpha,s) \ge 1$ that depends only on n, α and s.

Here $A_1^{\rm loc}$ is the class of local A_1 weights which consists of nonnegative locally integrable functions w in $\mathbb{R}^n$ such that

$$\mathbf{M}^{\rm loc} w(x) \le C w(x) \tag{3.4}$$

for a.e. $x \in \mathbb{R}^n$. The $A_1^{\rm loc}$ characteristic constant of w, $[w]_{A_1^{\rm loc}}$, is defined as the least possible constant C in the above inequality. The operator $\mathbf{M}^{\rm loc}$ stands for the (center) local Hardy-Littlewood maximal function defined for each $f \in L^1_{\rm loc}(\mathbb{R}^n)$ by

$$\mathbf{M}^{\rm loc} f(x) = \sup_{0<r\le 1} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy. \tag{3.5}$$

We recall that the (center) Hardy-Littlewood maximal function $\mathbf{M}f$ of f is defined similarly except that the supremum is now taken over all $r > 0$. If (3.4) holds a.e. with $\mathbf{M}$ in place of $\mathbf{M}^{\rm loc}$, then we say that w belongs to the class A_1.

One should relate Theorem 3.1.1 to [AX1, Theorem 2.2] and [AX2, Lemma 11] in the context of (homogeneous) Morrey spaces. Here we mention that our approach to Theorem 3.1.1 actually provides a new proof of [AX2, Lemma 11] in which the result of [OV] can be completely avoided.

To prove Theorem 3.1.1 we need the following preliminary results. A homogeneous version of the next theorem can be found in [MS1]. But our approach here is different from that of [MS1] at least in the case $s > 2 - \alpha/n$.

Theorem 3.1.2. *Let $s > 1$, $\alpha > 0$, and $\alpha s \leq n$. If $t \in (1, n/(n-\alpha))$ for $s \leq 2 - \alpha/n$ and $t \in (1, n(s-1)/(n-\alpha s))$ for $s > 2 - \alpha/n$, then for any nonnegative measure μ and $V = G_\alpha * (G_\alpha * \mu)^{\frac{1}{s-1}}$ we have*

$$\mathbf{M}^{\text{loc}}(V^t)(x_0) \leq A V^t(x_0), \qquad \forall x_0 \in \mathbb{R}^n, \tag{3.6}$$

where A is a constant independent of μ.

Proof. We shall use the following properties of G_α (see [AH, Sect. 1.2.4]):

$$G_\alpha(x) \simeq |x|^{\alpha - n}, \qquad \forall |x| \leq 4, \qquad 0 < \alpha < n, \tag{3.7}$$

and

$$G_\alpha(x) \leq c\, G_\alpha(x+y), \qquad \forall |x| \geq 2, \qquad |y| \leq 1. \tag{3.8}$$

Note that (3.7) and (3.8) yield that for any $t \in (1, n/(n-\alpha))$ we have

$$\mathbf{M}^{\text{loc}}(G_\alpha^t(\cdot - z))(x) \leq C\, G_\alpha^t(x - z), \qquad \forall x, z \in \mathbb{R}^n, \tag{3.9}$$

where C is independent of x and z. This can be verified by inspecting the case $x \in B_3(z)$ and the case $x \notin B_3(z)$ separately.

First we consider the case $s \leq 2 - \alpha/n$ and $t \in (1, n/(n-\alpha))$. By Minkowski's inequality and (3.9) we have, for $x_0 \in \mathbb{R}^n$ and $r \in (0,1]$,

$$\begin{aligned}
\fint_{B_r(x_0)} V^t(y)dy &= \fint_{B_r(x_0)} \left[\int_{\mathbb{R}^n} G_\alpha(y-z)(G_\alpha * \mu)(z)^{\frac{1}{s-1}} dz\right]^t dy \\
&\leq \left[\int_{\mathbb{R}^n} (G_\alpha * \mu)(z)^{\frac{1}{s-1}} \left(\fint_{B_r(x_0)} G_\alpha^t(y-z)dy\right)^{\frac{1}{t}} dz\right]^t \\
&\leq C \left[\int_{\mathbb{R}^n} (G_\alpha * \mu)(z)^{\frac{1}{s-1}} G_\alpha(x_0 - z)dz\right]^t \\
&= C\, V^t(x_0).
\end{aligned}$$

Thus we get (3.6) when $t \in (1, n/(n-\alpha))$ and $s \leq 2 - \alpha/n$. In fact, the proof is valid for all $s > 1$.

We now consider the case $s > 2 - \alpha/n$ and $t \in (1, n(s-1)/(n - \alpha s))$. By Hölder's inequality we may assume that $t > s - 1$. Let $x_0 \in \mathbb{R}^n$ and $r \in (0,1]$. We write

$$V(x) = V_1(x) + V_2(x),$$

where

$$\begin{aligned}
V_1(x) &= \int_{|y-x_0|>3} G_\alpha(x-y)\varphi(y)dy, \\
V_2(x) &= \int_{|y-x_0|\leq 3} G_\alpha(x-y)\varphi(y)dy,
\end{aligned}$$

with

$$\varphi(y) = (G_\alpha * \mu(y))^{\frac{1}{s-1}}.$$

Observe that for $|x - x_0| \leq 1$ and $|y - x_0| > 3$, it holds that

$$G_\alpha(x-y) \leq A G_\alpha(x_0 - y). \tag{3.10}$$

Indeed, since $|x-y| \geq |y-x_0| - |x-x_0| \geq 3 - 1 = 2$ and $|x - x_0| \leq 1$, by (3.8) we find

$$G_\alpha(x-y) \leq A G_\alpha(x - y + x_0 - x) = A G_\alpha(x_0 - y).$$

Now by (3.10) we have

$$V_1(x) \le A \int_{\mathbb{R}^n} G_\alpha(x_0 - y)\varphi(y)dy = AV(x_0)$$

for all $|x - x_0| \le 1$. This yields

$$⨍_{B_r(x_0)} V_1^t(x)dx \le AV^t(x_0). \tag{3.11}$$

As for V_2 we write

$$V_2(x) = \int_{|x-u-x_0|\le 3} G_\alpha(u)\varphi(x-u)du \le c[V_{21}(x) + V_{22}(x)],$$

where

$$V_{21}(x) = \int_{|x-u-x_0|\le 3} G_\alpha(u)\varphi_1(x-u)du,$$
$$V_{22}(x) = \int_{|x-u-x_0|\le 3} G_\alpha(u)\varphi_2(x-u)du,$$

with

$$\varphi_1(x-u) = \left(\int_{|z-x_0|>5} G_\alpha(x-u-z)d\mu(z)\right)^{\frac{1}{s-1}},$$
$$\varphi_2(x-u) = \left(\int_{|z-x_0|\le 5} G_\alpha(x-u-z)d\mu(z)\right)^{\frac{1}{s-1}}.$$

Using (3.8), for $|z - x_0| > 5$, $|x - u - x_0| \le 3$, and $|x - x_0| \le 1$, we have

$$G_\alpha(x-u-z) \le AG_\alpha(x-u-z-x+x_0) = AG_\alpha(x_0-u-z).$$

Thus for such x and u it follows that

$$\varphi_1(x-u) \le A\left[\int_{\mathbb{R}^n} G_\alpha(x_0-u-z)d\mu(z)\right]^{\frac{1}{s-1}} = A\varphi(x_0-u).$$

Hence,

$$
\begin{aligned}
V_{21}(x) &\leq A \int_{|x-u-x_0|\leq 3} G_\alpha(u)\varphi(x_0-u)du \\
&\leq A \int_{\mathbb{R}^n} G_\alpha(x_0-y)\varphi(y)dy \\
&= AV(x_0).
\end{aligned}
$$

As this holds for all $|x-x_0|\leq 1$ we deduce that

$$
⨍_{B_r(x_0)} V_{21}^t(x)dx \leq AV^t(x_0). \tag{3.12}
$$

It is now left to estimate V_{22}. Note that for $|x-x_0|\leq 1$,

$$
V_{22}(x) = \int_{|y-x_0|\leq 3} G_\alpha(x-y)\varphi_2(y)dy \leq \int_{|y-x|\leq 4} G_\alpha(x-y)\varphi_2(y)dy,
$$

and thus by (3.7) we have

$$
V_{22}(x) \leq C\int_0^5 \frac{\int_{B_\rho(x)}\varphi_2(y)dy}{\rho^{n-\alpha}}\frac{d\rho}{\rho}. \tag{3.13}
$$

We next claim that for $|x-x_0|\leq 1$,

$$
V_{22}(x) \leq C\int_0^{100} \left(\frac{\mu(B_\rho(x))}{\rho^{n-\alpha s}}\right)^{\frac{1}{s-1}}\frac{d\rho}{\rho}. \tag{3.14}
$$

Assuming (3.14), we have

$$
⨍_{B_r(x_0)} V_{22}^t(x)dx \leq C(Q_1+Q_2), \tag{3.15}
$$

where

$$
\begin{aligned}
Q_1 &= ⨍_{B_r(x_0)} \left(\int_0^r \left(\frac{\mu(B_\rho(x))}{\rho^{n-\alpha s}}\right)^{\frac{1}{s-1}}\frac{d\rho}{\rho}\right)^t dx, \\
Q_2 &= ⨍_{B_r(x_0)} \left(\int_r^{100} \left(\frac{\mu(B_\rho(x))}{\rho^{n-\alpha s}}\right)^{\frac{1}{s-1}}\frac{d\rho}{\rho}\right)^t dx.
\end{aligned}
$$

For Q_2 we observe that if $x \in B_r(x_0)$ and $\rho \geq r$ then $B_\rho(x) \subset B_{2\rho}(x_0)$, and so

$$Q_2 \leq \fint_{B_r(x_0)} \left(\int_r^{100} \left(\frac{\mu(B_{2\rho}(x_0))}{\rho^{n-\alpha s}} \right)^{\frac{1}{s-1}} \frac{d\rho}{\rho} \right)^t dx \leq C V^t(x_0),$$

where we used [Ad1, Theorem 2] in the last inequality.

For Q_1, we first bound, with $x \in B_r(x_0)$ and $\epsilon > 0$,

$$\begin{aligned} \left(\int_0^r \left(\frac{\mu(B_\rho(x))}{\rho^{n-\alpha s}} \right)^{\frac{1}{s-1}} \frac{d\rho}{\rho} \right)^t &\leq c\, r^{\frac{\epsilon t}{s-1}} \sup_{0<\rho<r} \left(\frac{\mu(B_\rho(x))}{\rho^{n-\alpha s+\epsilon}} \right)^{\frac{t}{s-1}} \\ &\leq c\, r^{\frac{\epsilon t}{s-1}} \sup_{0<\rho<r} \left(\int_{|x-z|<\rho} \frac{d\mu(z)}{|x-z|^{n-\alpha s+\epsilon}} \right)^{\frac{t}{s-1}} \\ &\leq c\, r^{\frac{\epsilon t}{s-1}} \left(\int_{|x_0-z|<2r} \frac{d\mu(z)}{|x-z|^{n-\alpha s+\epsilon}} \right)^{\frac{t}{s-1}}. \end{aligned}$$

This and Minkowski's inequality (recall that $t > s-1$) yield

$$\begin{aligned} Q_1 &\leq c\, r^{\frac{\epsilon t}{s-1}} \fint_{B_r(x_0)} \left(\int_{|x_0-z|<2r} \frac{d\mu(z)}{|x-z|^{n-\alpha s+\epsilon}} \right)^{\frac{t}{s-1}} dx \\ &\leq c\, r^{\frac{\epsilon t}{s-1}} \left[\int_{|x_0-z|<2r} d\mu(z) \left(\fint_{B_r(x_0)} \frac{dx}{|x-z|^{(n-\alpha s+\epsilon)\frac{t}{s-1}}} \right)^{\frac{s-1}{t}} \right]^{\frac{t}{s-1}}. \end{aligned}$$

We now choose an $\epsilon > 0$ such that $(n-\alpha s+\epsilon)\frac{t}{s-1} < n$, which is possible since

$$\frac{n(s-1)}{t} > n - \alpha s.$$

Then by simple calculations and [Ad1, Theorem 2] we arrive at

$$Q_1 \leq c \left(\frac{\mu(B_{2r}(x_0))}{r^{n-\alpha s}} \right)^{\frac{t}{s-1}} \leq c \left(\int_0^4 \left(\frac{\mu(B_\rho(x_0))}{\rho^{n-\alpha s}} \right)^{\frac{1}{s-1}} \frac{d\rho}{\rho} \right)^t dx \leq C V^t(x_0).$$

Thus in view of (3.15) and the above estimates for Q_1 and Q_2 we get

$$\fint_{B_r(x_0)} V_{22}^t(x) dx \leq A V^t(x_0). \tag{3.16}$$

Estimates (3.11), (3.12), and (3.16) yield the bound (3.6) as desired.

Therefore, what's left now is to verify inequality (3.14). In view of (3.13) we need to estimate $\int_{B_\rho(x)} \varphi_2(y)dy$. To this end, we first observe that for $y \in B_\rho(x)$, $\rho \in (0,5]$, by (3.7) it holds that

$$\begin{aligned}
\varphi_2(y) &= \left(\int_{|z-x_0|\le 5} G_\alpha(y-z)d\mu(z) \right)^{\frac{1}{s-1}} \\
&\le \left(\int_{|z-y|\le 11} G_\alpha(y-z)d\mu(z) \right)^{\frac{1}{s-1}} \\
&\le C \left(\int_{|z-y|\le 11} \frac{1}{|y-z|^{n-\alpha}} d\mu(z) \right)^{\frac{1}{s-1}}.
\end{aligned}$$

Thus for $0 < \rho \le 5$ we have

$$\int_{B_\rho(x)} \varphi_2(y)dy \le C(I_1 + I_2), \tag{3.17}$$

where

$$I_1 = \int_{B_\rho(x)} \left(\int_{|z-y|<\rho} \frac{1}{|y-z|^{n-\alpha}} d\mu(z) \right)^{\frac{1}{s-1}} dy,$$

and

$$\begin{aligned}
I_2 &= \int_{B_\rho(x)} \left(\int_{\rho\le|z-y|\le 11} \frac{1}{|y-z|^{n-\alpha}} d\mu(z) \right)^{\frac{1}{s-1}} dy \\
&\le \int_{B_\rho(x)} \left(\int_\rho^{12} \frac{\mu(B_t(y))}{t^{n-\alpha}} \frac{dt}{t} \right)^{\frac{1}{s-1}} dy \\
&\le c\,\rho^n \left(\int_\rho^{12} \frac{\mu(B_{2t}(x))}{t^{n-\alpha}} \frac{dt}{t} \right)^{\frac{1}{s-1}}.
\end{aligned}$$

We now claim that

$$I_1 \le c\,\mu(B_{2\rho}(x))^{\frac{1}{s-1}} \rho^{n+\frac{\alpha-n}{s-1}}. \tag{3.18}$$

Indeed, we have

$$I_1 \le \int_{|x-y|\le\rho} \left(\int_{|z-x|<2\rho} \frac{d\mu(z)}{|z-y|^{n-\alpha}} \right)^{\frac{1}{s-1}} dy,$$

and thus when $s > 2$ by Hölder's inequality with exponents $s-1$ and $\frac{s-1}{s-2}$ and Fubini's theorem we obtain

$$I_1 \le \left(\int_{|z-x|<2\rho} d\mu(z) \int_{|x-y|\le\rho} \frac{dy}{|z-y|^{n-\alpha}} \right)^{\frac{1}{s-1}} |B_\rho(x)|^{\frac{s-2}{s-1}}$$
$$\le c\,\mu(B_{2\rho}(x))^{\frac{1}{s-1}} \rho^{\frac{\alpha}{s-1}} \rho^{\frac{n(s-2)}{s-1}} = c\,\mu(B_{2\rho}(x))^{\frac{1}{s-1}} \rho^{n+\frac{\alpha-n}{s-1}}.$$

On the other hand, when $2-\alpha/n < s \le 2$ we use Minkowski's inequality to get

$$I_1 \le \left[\int_{|z-x|<2\rho} \left(\int_{|x-y|\le\rho} \frac{dy}{|z-y|^{\frac{n-\alpha}{s-1}}} \right)^{s-1} d\mu(z) \right]^{\frac{1}{s-1}}$$
$$\le c\,\mu(B_{2\rho}(x))^{\frac{1}{s-1}} \rho^{n+\frac{\alpha-n}{s-1}}.$$

Thus the claim (3.18) follows. At this point combining estimates (3.13), (3.17)-(3.18) we obtain

$$V_{22}(x) \le C \int_0^5 \left(\frac{\mu(B_\rho(x))}{\rho^{n-\alpha s}} \right)^{\frac{1}{s-1}} \frac{d\rho}{\rho} + CJ(x), \tag{3.19}$$

where

$$J(x) = \int_0^5 \rho^\alpha \left(\int_\rho^{12} \frac{\mu(B_{2t}(x))}{t^{n-\alpha}} \frac{dt}{t} \right)^{\frac{1}{s-1}} \frac{d\rho}{\rho}.$$

Using Hardy's inequality of the form

$$\left\{ \int_0^\infty \left(\int_\rho^\infty f(t)dt \right)^q \rho^\alpha \frac{d\rho}{\rho} \right\}^{\frac{1}{q}} \le \frac{q}{\alpha} \left\{ \int_0^\infty (tf(t))^q t^\alpha \frac{dt}{t} \right\}^{\frac{1}{q}},$$

$1 \le q < \infty$, $\alpha > 0$, $f \ge 0$, when $s \le 2$ we find

$$J(x) \le C \int_0^{12} \left(\frac{\mu(B_{2t}(x))}{t^{n-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dt}{t} \le C \int_0^{24} \left(\frac{\mu(B_\rho(x))}{\rho^{n-\alpha s}} \right)^{\frac{1}{s-1}} \frac{d\rho}{\rho}.$$

When $s > 2$ we have

$$\left(\int_\rho^{12} \frac{\mu(B_{2t}(x))}{t^{n-\alpha}} \frac{dt}{t}\right)^{\frac{1}{s-1}} \leq \left(\sum_{k=0}^{k_0} \int_{2^k\rho}^{2^{k+1}\rho} \frac{\mu(B_{2t}(x))}{t^{n-\alpha}} \frac{dt}{t}\right)^{\frac{1}{s-1}}$$
$$\leq C \left(\sum_{k=0}^{k_0} \frac{\mu(B_{2^{k+2}\rho}(x))}{(2^k\rho)^{n-\alpha}}\right)^{\frac{1}{s-1}}$$
$$\leq C \sum_{k=0}^{k_0} \left(\frac{\mu(B_{2^{k+2}\rho}(x))}{(2^k\rho)^{n-\alpha}}\right)^{\frac{1}{s-1}},$$

where $k_0 = k_0(\rho)$ is an integer such that $2^{k_0+1}\rho \geq 12$ and $2^{k_0+1}\rho < 24$. This yields

$$\left(\int_\rho^{12} \frac{\mu(B_{2t}(x))}{t^{n-\alpha}} \frac{dt}{t}\right)^{\frac{1}{s-1}} \leq C \int_{2\rho}^{48} \left(\frac{\mu(B_{2t}(x))}{t^{n-\alpha}}\right)^{\frac{1}{s-1}} \frac{dt}{t}.$$

Thus by Fubini's theorem we get

$$J(x) \leq C \int_0^5 \rho^\alpha \int_{2\rho}^{48} \left(\frac{\mu(B_{2t}(x))}{t^{n-\alpha}}\right)^{\frac{1}{s-1}} \frac{dt}{t} \frac{d\rho}{\rho}$$
$$\leq C \int_0^{48} \left(\frac{\mu(B_{2t}(x))}{t^{n-\alpha}}\right)^{\frac{1}{s-1}} \int_0^{t/2} \rho^\alpha \frac{d\rho}{\rho} \frac{dt}{t}$$
$$= C \int_0^{48} \left(\frac{\mu(B_{2t}(x))}{t^{n-\alpha s}}\right)^{\frac{1}{s-1}} \frac{dt}{t}$$
$$\leq C \int_0^{96} \left(\frac{\mu(B_\rho(x))}{\rho^{n-\alpha s}}\right)^{\frac{1}{s-1}} \frac{dt}{t}.$$

Now combining (3.19) with the above estimates for $J(x)$ we arrive at (3.14) as desired.

The proof of the theorem is complete. □

For any set $E \subset \mathbb{R}^n$ with $0 < \mathrm{Cap}_{\alpha,s}(E) < \infty$, by [AH, Theorems 2.5.6 and 2.6.3] one can find a nonnegative measure $\mu = \mu^E$ with $\mathrm{supp}(\mu) \subset \overline{E}$ (called capacitary measure for E) such that the function $V^E = G_\alpha * ((G_\alpha * \mu)^{\frac{1}{s-1}})$ satisfies the following properties:

$$\mu^E(\overline{E}) = \mathrm{Cap}_{\alpha,s}(E) = \int_{\mathbb{R}^n} V^E d\mu^E = \int_{\mathbb{R}^n} (G_\alpha * \mu^E)^{\frac{s}{s-1}} dx, \tag{3.20}$$

$$V^E \geq 1 \quad \text{quasieverywhere on } E,$$

and

$$V^E \leq A \quad \text{on } \mathbb{R}^n. \tag{3.21}$$

Lemma 3.1.3. *Let E, $\mu = \mu^E$, and V^E be as above and let $0 < \alpha s \leq n$. If*

$$\delta \in (1, n/(n-\alpha))$$

for $s < 2$ and $\delta \in (s-1, n(s-1)/(n-\alpha s))$ for $s \geq 2$, then the function $(V^E)^\delta \in A_1^{\mathrm{loc}}$ with $[(V^E)^\delta]_{A_1^{\mathrm{loc}}} \leq c(n,\alpha,s,\delta)$. Moreover, $(V^E)^\delta \in L^1(C)$ with $\|(V^E)^\delta\|_{L^1(C)} \leq C\,\mathrm{Cap}_{\alpha,s}(E)$.

Proof. Thanks to Theorem 3.1.2, we just meed to prove the last statement of the lemma. By [AH, Proposition 6.1.2] we see that V^E and hence $(V^E)^\delta$ are quasicontinuous. We have

$$\|(V^E)^\delta\|_{L^1(C)} = \delta \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{V^E > \rho\})\rho^{\delta-1}d\rho. \tag{3.22}$$

For $s \geq 2$, by [AM, Proposition 4.4] and (3.20) it holds that

$$\mathrm{Cap}_{\alpha,s}(\{V^E > \rho\}) \leq C\,\mu^E(\mathbb{R}^n)\rho^{1-s} = C\,\mathrm{Cap}_{\alpha,s}(E)\rho^{1-s}.$$

For $1 < s < 2$, let ν be the capacitary measure for the set $\{V^E > \rho\}$. By Fubini's theorem we have

$$\begin{aligned}\mathrm{Cap}_{\alpha,s}(\{V^E > \rho\}) &= \int_{\mathbb{R}^n} d\nu \leq \rho^{-1}\int_{\mathbb{R}^n} V^E d\nu \\ &= \rho^{-1}\int_{\mathbb{R}^n} (G_\alpha * \mu^E(y))^{\frac{1}{s-1}}(G_\alpha * \nu(y))dy.\end{aligned}$$

Thus by Hölder's inequality it follows that

$$\begin{aligned}\mathrm{Cap}_{\alpha,s}(\{V^E > \rho\}) &\leq \rho^{-1}\left\{\int_{\mathbb{R}^n}(G_\alpha * \mu^E(y))^{\frac{s}{s-1}}dy\right\}^{2-s} \times \\ &\qquad \times \left\{\int_{\mathbb{R}^n}(G_\alpha * \mu^E(y))(G_\alpha * \nu(y))^{\frac{1}{s-1}}dy\right\}^{s-1} \\ &= \rho^{-1}\mathrm{Cap}_{\alpha,s}(E)^{2-s}\int_{\mathbb{R}^n} G_\alpha * ((G_\alpha * \nu)^{\frac{1}{s-1}})d\mu^E \\ &\leq C\,\rho^{-1}\mathrm{Cap}_{\alpha,s}(E)^{2-s}\mu^E(\mathbb{R}^n)^{s-1} = C\,\rho^{-1}\mathrm{Cap}_{\alpha,s}(E).\end{aligned}$$

Using (3.21)-(3.22) and the above estimates for $\mathrm{Cap}_{\alpha,s}(\{V^E > \rho\})$, we get

$$\|(V^E)^\delta\|_{L^1(C)} \leq C\,\mathrm{Cap}_{\alpha,s}(E)\int_0^A \rho^{-\max\{s-1,1\}}\rho^{\delta-1}d\rho \leq C\,\mathrm{Cap}_{\alpha,s}(E),$$

as desired. □

We are now ready to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. By Theorem 2.1.3, given $w \in L^1(C)$ with $\|w\|_{L^1(C)} \leq 1$, one has

$$\begin{aligned}\int |f(x)|^p w(x)dx &\leq \||f|^p\|_{M_1^{\alpha,s}}\int_0^\infty \mathrm{Cap}_{\alpha,s}(\{w > t\})dt \\ &= \|f\|^p_{M_p^{\alpha,s}}\|w\|_{L^1(C)},\end{aligned}$$

which yields

$$\sup\left\{\int |f(x)|^p w(x)dx : w \in L^1(C), \|w\|_{L^1(C)} \leq 1\right\} \leq \|f\|^p_{M_p^{\alpha,s}}.$$

On the other hand, fix a constant δ such that $\delta \in (1, n/(n-\alpha))$ if $s < 2$ and $\delta \in (s-1, n(s-1)/(n-\alpha s))$ if $s \geq 2$, and let E be a compact subset of $\mathbb{R}^n$ with $\mathrm{Cap}_{\alpha,s}(E) > 0$. Then, with V^E as in Lemma 3.1.3, we can find a constant $\overline{\mathbf{c}} = \overline{\mathbf{c}}(n, \alpha, s) \geq 1$ such that

$[V^E/\mathrm{Cap}_{\alpha,s}(E)]_{A_1^{\mathrm{loc}}} \leq \overline{\mathbf{c}}$. Thus we have

$$\frac{\int_E |f(x)|^p dx}{\mathrm{Cap}_{\alpha,s}(E)} \leq \frac{\int_E |f(x)|^p (V^E)^\delta dx}{\mathrm{Cap}_{\alpha,s}(E)} \leq \int_{\mathbb{R}^n} |f(x)|^p \left(\frac{(V^E)^\delta}{\mathrm{Cap}_{\alpha,s}(E)}\right) dx$$
$$\leq C \sup\left\{\int |f(x)|^p w(x) dx : w \in L^1(C), \|w\|_{L^1(C)} \leq 1, [w]_{A_1^{\mathrm{loc}}} \leq \overline{\mathbf{c}}\right\}.$$

This finishes the proof of the theorem. □

We notice that by [MV, Theorem 1.2] it holds that

$$\|f\|_{M_p^{\alpha,s}} \simeq \left\|\frac{G_\alpha * [G_\alpha * (|f|^p)]^{s'}}{G_\alpha * (|f|^p)}\right\|_{L^\infty(\{G_\alpha * (|f|^p) > 0\})}^{\frac{1}{p(s'-1)}}.$$

Thus it follows from [KV, Proposition 2.9] and [KV, Theorem 2.10] that both $|f|^{\frac{p}{s'}}$ and $G_\alpha * (|f|^p)$ belong to $\mathcal{Z}$. Here $\mathcal{Z} = \mathcal{Z}_{s'}$ is the space of measurable functions h such that the integral equation

$$u = G_\alpha * (u^{s'}) + \epsilon |h| \qquad \text{a.e.}$$

has a nonnegative solution $u \in L^{s'}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\epsilon > 0$. A norm for $\mathcal{Z}$ can be defined by

$$\|h\|_{\mathcal{Z}} = \inf\{t > 0 : G_\alpha * |h|^{s'} \leq t^{s'-1}|h| \quad \text{a.e.}\} = \left\|\frac{G_\alpha * |h|^{s'}}{|h|}\right\|_{L^\infty(\{|h|>0\})}^{\frac{1}{s'-1}}$$

(see [KV, page 3455]). Moreover, by [KV, Theorem 2.10] we have

$$\||f|^{\frac{p}{s'}}\|_{\mathcal{Z}} \simeq \|G_\alpha * (|f|^p)\|_{\mathcal{Z}}^{\frac{1}{s'}}.$$

With these observations, we see that a function $f \in M_p^{\alpha,s}$ if and only if $|f|^{\frac{p}{s'}} \in \mathcal{Z}$ and

$$\left\||f|^{\frac{p}{s'}}\right\|_{\mathcal{Z}}^{\frac{s'}{p}} \simeq \|f\|_{M_p^{\alpha,s}}.$$

3.2. Preduals

The main goal of this section is to find 'good' predual spaces to the space $M_p^{\alpha,s}$, $p > 1$. By a good predual space in this context we mean one that fits in well with the theory of function spaces of harmonic analysis and partial differential equations. For example, one should be able to demonstrate the behavior of basic operators such as the Hardy-Littlewood maximal function and Calderón-Zygmund operators on such a space. A natural candidate for a predual of $M_p^{\alpha,s}$ is its Köthe dual space $(M_p^{\alpha,s})'$ defined by

$$(M_p^{\alpha,s})' = \left\{ \text{measurable functions } f : \sup \int |fg| dx < +\infty \right\}, \tag{3.23}$$

where the supremum is taken over all functions g in the unit ball of $M_p^{\alpha,s}$. The norm of $f \in (M_p^{\alpha,s})'$ is defined as the above supremum. Indeed, as in [KV], using the p-convexity of $M_p^{\alpha,s}$ we find that this is the case , i.e.,

$$[(M_p^{\alpha,s})']^* = M_p^{\alpha,s},$$

(see Proposition 3.2.5).

We observe however that the space $(M_p^{\alpha,s})'$ is quite abstract and thus it is desirable to find a more concrete space that is isomorphic to it. In this paper, inspired from the work [AX1, AX2], several other predual spaces to $M_p^{\alpha,s}$ will be constructed. In particular, we find a Banach function space (in the sense of [Lux]) $\mathcal{N}_{p'}^{\alpha,s}$, $p' = p/(p-1)$, such that $(M_p^{\alpha,s})' \approx \mathcal{N}_{p'}^{\alpha,s}$ for all $p > 1, \alpha > 0$, and $\alpha p \leq n$. More importantly, the space $\mathcal{N}_{p'}^{\alpha,s}$ and other predual spaces that we construct have a nice structure that the Hardy-Littlewood maximal function and standard Calderón-Zygmund type operators behave well on them in a reasonable sense. As a result, the *local* Hardy-Littlewood maximal function $\mathbf{M}^{\rm loc}$ (see (3.5)) is shown to be bounded on $(M_p^{\alpha,s})'$. We remark that whereas the Hardy-Littlewood maximal function $\mathbf{M}$ is bounded on $M_p^{\alpha,s}$ for any $p > 1$ and $\alpha s \leq n$ (see [MS1]), it fails to be bounded on $(M_p^{\alpha,s})'$. This is because $L^\infty(\mathbb{R}^n) \hookrightarrow M_p^{\alpha,s}$ and thus $(M_p^{\alpha,s})' \hookrightarrow L^1(\mathbb{R}^n)$. This phenomenon happens simply because of the inhomogeneity of the Sobolev space under consideration. In the homogeneous case, where the space $H^{\alpha,s}$ in (3.1) is replaced with its homogeneous counterpart $\dot{H}^{\alpha,s}$ (the space of Riesz potentials), such a phenomenon does not exist.

3.2.1. $(N_{p'}^{\alpha,s})^* \approx M_p^{\alpha,s}$

Inspired by Theorem 3.1.1 we define the following space. For $q > 1$ and $\alpha > 0, s > 1$, let $N_q^{\alpha,s} = N_q^{\alpha,s}(\mathbb{R}^n)$ be the space of all measurable functions g such that there exists a weight $w \in L^1(C) \cap A_1^{\rm loc}$ with $\|w\|_{L^1(C)} \leq 1$ and $[w]_{A_1^{\rm loc}} \leq \overline{\mathbf{c}}(n,\alpha,s)$ such that

$$\left(\int_{\mathbb{R}^n} |g(x)|^q w(x)^{1-q} dx\right)^{1/q} < +\infty.$$

This implies that, for such w, $g = 0$ a.e. on the set $\{w = 0\}$. The 'norm' of a function $g \in N_q^{\alpha,s}$ is the defined as

$$\|g\|_{N_q^{\alpha,s}} = \inf_w \left(\int_{\mathbb{R}^n} |g(x)|^q w(x)^{1-q} dx\right)^{1/q},$$

where the infimum is taken over all $w \in L^1(C) \cap A_1^{\rm loc}$ with $\|w\|_{L^1(C)} \leq 1$ and $[w]_{A_1^{\rm loc}} \leq \overline{\mathbf{c}}(n,\alpha,s)$.

Our first duality result can now be stated.

Theorem 3.2.1. *Let $p > 1$, $\alpha > 0, s > 1$, with $\alpha s \leq n$, and $p' = p/(p-1)$. We have*

$$\left(N_{p'}^{\alpha,s}\right)^* \approx M_p^{\alpha,s}$$

in the sense that each bounded linear functional $L \in \left(N_{p'}^{\alpha,s}\right)^*$ corresponds to a unique $f \in$

$M_p^{\alpha,s}$ such that $L(g) = L_f(g)$ for all $g \in N_{p'}^{\alpha,s}$, where

$$L_f(g) = \int_{\mathbb{R}^n} f(x)g(x)dx, \qquad g \in N_{p'}^{\alpha,s}.$$

Moreover, we have

$$\|f\|_{M_p^{\alpha,s}} \simeq \|L_f\|_{\left(N_{p'}^{\alpha,s}\right)^*}.$$

In Theorems 3.1.1 and 3.2.1, we can also drop the A_1^{loc} and the quasicontinuity conditions on the weights w and obtain the following similar results with equality of norms.

Theorem 3.2.2. *For $p > 1$ and $\alpha > 0, s > 1$, with $\alpha s \leq n$, we have*

$$\|f\|_{M_p^{\alpha,s}} = \sup_w \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{1/p},$$

where the supremum is taken over all weights w such that w is defined $\text{Cap}_{\alpha,s}$- quasieverywhere and $\int_{\mathbb{R}^n} wdC \leq 1$.

Moreover, we have $\left(\widetilde{N}_{p'}^{\alpha,s}\right)^* = M_p^{\alpha,s}$, where $\widetilde{N}_q^{\alpha,s} = \widetilde{N}_q^{\alpha,s}(\mathbb{R}^n)$, $q > 1$, is the space of all measurable functions g such that

$$\|g\|_{\widetilde{N}_q^{\alpha,s}} := \inf_w \left(\int_{\mathbb{R}^n} |g(x)|^q w(x)^{1-q}dx\right)^{1/q} < +\infty,$$

Here the infimum is taken over all nonnegative q.e. defined function $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} wdC \leq 1$.

The spaces $N_q^{\alpha,s}$ and $\widetilde{N}_q^{\alpha,s}$ are obviously quasinormed spaces. However, at this point it is not clear if they are normable or complete for all $\alpha > 0, s > 1$ with $\alpha s \leq n$ and $q > 1$. We now introduce two Banach spaces are also preduals of $M_p^{\alpha,s}$. The first one is of course the Köthe dual space $(M_p^{\alpha,s})'$ defined earlier in (3.23). The second one is a block type space in the spirit of [BRV], which we call $B_q^{\alpha,s}$, $q > 1$.

3.2.2. $(B_{p'}^{\alpha,s})^* \approx M_p^{\alpha,s}$

Definition 3.2.3. *Let $q > 1$, $\alpha > 0$, and $s > 1$. We define $B_q^{\alpha,s} = B_q^{\alpha,s}(\mathbb{R}^n)$ to be the space of all functions f of the form*

$$f = \sum_j c_j a_j,$$

where the convergence is in pointwise a.e. sense. Here $\{c_j\} \in l^1$ and each $a_j \in L^q(\mathbb{R}^n)$ is such that there exists a bounded set $A_j \subset \mathbb{R}^n$ for which $a_j = 0$ a.e. in $\mathbb{R}^n \setminus A_j$ and $\|a_j\|_{L^q} \le \mathrm{Cap}_{\alpha,s}(A_j)^{\frac{1-q}{q}}$. The norm of a function $f \in B_q^{\alpha,s}$ is defined as

$$\|f\|_{B_q^{\alpha,s}} = \inf\Big\{ \sum_j |c_j| : f = \sum_j c_j a_j \text{ a.e.}\Big\}.$$

It is now easy to see from the definition that $B_q^{\alpha,s}$ is a Banach space. Both $(M_p^{\alpha,s})'$ and $B_{p'}^{\alpha,s}$ are also preduals of $M_p^{\alpha,s}$.

Theorem 3.2.4. *Let $p > 1$, $\alpha > 0$, and $s > 1$. We have*

$$[(M_p^{\alpha,s})']^* = \left(B_{p'}^{\alpha,s}\right)^* = M_p^{\alpha,s},$$

with equalities of norms.

We will now follow an idea in [KV, Proposition 2.11] and use the p-convexity of $M_p^{\alpha,s}$ to show that $(M_p^{\alpha,s})'$ is actually a predual space of $M_p^{\alpha,s}$.

Proposition 3.2.5. *We have $[(M_p^{\alpha,s})']^* = M_p^{\alpha,s}$ (isometrically) for any $\alpha > 0, s > 1$, $p > 1$.*

Proof. It is obvious that $M_p^{\alpha,s}$ is p-convex with p-convexity constant 1, i.e., for every choice of m functions $\{f_i\}_{i=1}^m$ in $M_p^{\alpha,s}$, we have

$$\left\| \Big(\sum_{i=1}^m |f_i|^p \Big)^{\frac{1}{p}} \right\|_{M_p^{\alpha,s}} \le \Big(\sum_{i=1}^m \|f_i\|_{M_p^{\alpha,s}}^p \Big)^{\frac{1}{p}}. \tag{3.24}$$

Now using the fact that $\ell^{p'}((M_p^{\alpha,s})^*) = \ell^p(M_p^{\alpha,s})^*$ we have for any choice of m functions $\{g_i\}_{i=1}^m$ in $(M_p^{\alpha,s})'$,

$$\begin{aligned}
\Big(\sum_{i=1}^m \|g_i\|_{(M_p^{\alpha,s})'}^{p'}\Big)^{\frac{1}{p'}} &= \Big(\sum_{i=1}^m \|g_i\|_{(M_p^{\alpha,s})^*}^{p'}\Big)^{\frac{1}{p'}} \\
&= \sup_{\|\{f_i\}\|_{\ell^p(M_p^{\alpha,s})}\le 1} \sum_{i=1}^m \int f_i(x)g_i(x)dx \\
&\le \sup_{\|\{f_i\}\|_{\ell^p(M_p^{\alpha,s})}\le 1} \int \Big(\sum_{i=1}^m |f_i(x)|^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^m |g_i(x)|^{p'}\Big)^{\frac{1}{p'}} dx \\
&\le \sup_{\|\{f_i\}\|_{\ell^p(M_p^{\alpha,s})}\le 1} \Big\|\Big(\sum_{i=1}^m |f_i|^p\Big)^{\frac{1}{p}}\Big\|_{M_p^{\alpha,s}} \Big\|\Big(\sum_{i=1}^m |g_i|^{p'}\Big)^{\frac{1}{p'}}\Big\|_{(M_p^{\alpha,s})'}.
\end{aligned}$$

Thus in view of (3.24) we see that $(M_p^{\alpha,s})'$ is p'-concave with p'-concavity constant 1, i.e.,

$$\Big(\sum_{i=1}^m \|g_i\|_{(M_p^{\alpha,s})'}^{p'}\Big)^{\frac{1}{p'}} \le \Big\|\Big(\sum_{i=1}^m |g_i|^{p'}\Big)^{\frac{1}{p'}}\Big\|_{(M_p^{\alpha,s})'}.$$

Then by [LT, Proposition 1.a.7] the space $(M_p^{\alpha,s})'$ must have an absolutely continuous norm. Hence, Theorems 2.3.1 and 2.3.2 yield that

$$[(M_p^{\alpha,s})']^* = (M_p^{\alpha,s})'' = M_p^{\alpha,s},$$

as desired. □

Remark 3.2.6. *The proof shows that $(M_p^{\alpha,s})'$ has an absolutely continuous norm and thus it is a separable Banach space (see [Lux]).*

Having introduced several predual spaces to $M_p^{\alpha,s}$, a natural question to us now is whether they are isometrically isomorphic or at least isomorphic. We will show eventually that they are all indeed isomorphic. In the case the capacity $\mathrm{Cap}_{\alpha,s}$ is strongly subadditive we claim that

$$N_{p'}^{\alpha,s} \approx (M_p^{\alpha,s})' = \widetilde{N}_{p'}^{\alpha,s} = B_{p'}^{\alpha,s}, \tag{3.25}$$

provided $\mathrm{Cap}_{\alpha,s}$ is strongly subadditive.

The first relation in (3.25) provides us with a new concrete description for the abstract space $(M_p^{\alpha,s})'$ and enables us 'to do harmonic analysis' on it when $\mathrm{Cap}_{\alpha,s}$ is strongly subadditive. In order to deal with all capacities, we now introduce another space which we call $\mathcal{N}_q^{\alpha,s}$, $q > 1$. Eventually, we show that $\mathcal{N}_{p'}^{\alpha,s} \approx (M_p^{\alpha,s})'$ for all $p > 1$ and $\alpha s \leq n$. To this end, we first modify the space $L^1(C)$, which in general is only a quasinormed space. Let $\mathcal{L}^1(C)$ be the space of measurable functions w such that

$$\sup_g \int |g(x)||w(x)|dx < +\infty,$$

where the supremum is taken over all $g \in M_1^{\alpha,s}$ such that $\|g\|_{M_1^{\alpha,s}} \leq 1$. In other words, $\mathcal{L}^1(C)$ is the Köthe dual of $M_1^{\alpha,s}$ with the norm $\|w\|_{\mathcal{L}^1(C)}$ being defined as the above supremum. It is easy to see that $L^1(C) \hookrightarrow \mathcal{L}^1(C)$.

3.2.3. $(\mathcal{N}_{p'}^{\alpha,s})^* \approx M_p^{\alpha,s}$ and the Equivalence of Preduals

For $q > 1$, we now define $\mathcal{N}_q^{\alpha,s} = \mathcal{N}_q^{\alpha,s}(\mathbb{R}^n)$ as the space of all measurable functions g such that there exists a weight $w \in \mathcal{L}^1(C) \cap A_1^{\mathrm{loc}}$ with $\|w\|_{\mathcal{L}^1(C)} \leq 1$ and $[w]_{A_1^{\mathrm{loc}}} \leq \overline{\mathbf{c}}(n, \alpha, s)$ such that

$$\left(\int_{\mathbb{R}^n} |g(x)|^q w(x)^{1-q} dx\right)^{1/q} < +\infty.$$

As in the case of $N_q^{\alpha,s}$, the norm of a function $g \in \mathcal{N}_q^{\alpha,s}$ is the defined as the infimum of the left-hand side above over all $w \in \mathcal{L}^1(C) \cap A_1^{\mathrm{loc}}$ with $\|w\|_{\mathcal{L}^1(C)} \leq 1$ and $[w]_{A_1^{\mathrm{loc}}} \leq \overline{\mathbf{c}}(n, \alpha, s)$.

Theorem 3.2.7. *Let $p > 1$, $\alpha > 0$, $s > 1$, $\alpha s \leq n$. Then $\mathcal{N}_{p'}^{\alpha,s}$ and $(M_p^{\alpha,s})'$ are Banach function spaces, and $\mathcal{N}_{p'}^{\alpha,s} \approx (M_p^{\alpha,s})'$ (thus $(\mathcal{N}_{p'}^{\alpha,s})^* \approx M_p^{\alpha,s}$). Moreover, if* $\mathrm{Cap}_{\alpha,s}$ *is strongly* *subadditive then* $N^{\alpha,s}_{p'}$, $\widetilde{N}^{\alpha,s}_{p'}$, *and* $B^{\alpha,s}_{p'}$ *are also Banach function spaces, and*

$$\mathcal{N}^{\alpha,s}_{p'} \approx N^{\alpha,s}_{p'} \approx (M^{\alpha,s}_{p})' = \widetilde{N}^{\alpha,s}_{p'} = B^{\alpha,s}_{p'}.$$

Finally, we have the following isomorphism result which applies to all capacities.

Theorem 3.2.8. *Let* $p > 1$, $\alpha > 0$, $s > 1$, $\alpha s \leq n$. *We have*

$$\mathcal{N}^{\alpha,s}_{p'} \approx N^{\alpha,s}_{p'} \approx (M^{\alpha,s}_{p})' \approx \widetilde{N}^{\alpha,s}_{p'} \approx B^{\alpha,s}_{p'}. \tag{3.26}$$

In general, the space of continuous functions with compact support C_c is not dense in $M^{\alpha,s}_{p}$. We shall let $\mathring{M}^{\alpha,s}_{p}$ denote the closure of C_c in $M^{\alpha,s}_{p}$. As it turns out, we have that $\mathring{M}^{\alpha,s}_{p}$ is a predual of $\mathcal{N}^{\alpha,s}_{p'}$.

Theorem 3.2.9. *Let* $p > 1$, $\alpha > 0$, $s > 1$, *with* $\alpha s \leq n$. *We have*

$$(\mathring{M}^{\alpha,s}_{p})^{*} \approx \mathcal{N}^{\alpha,s}_{p'}$$

in the sense that each bounded linear functional $L \in (\mathring{M}^{\alpha,s}_{p})^{*}$ *corresponds to a unique* $g \in \mathcal{N}^{\alpha,s}_{p'}$ *such that* $L(v) = \int_{\mathbb{R}^n} v(x)g(x)dx$ *for all* $v \in \mathring{M}^{\alpha,s}_{p}$, *and* $\|g\|_{\mathcal{N}^{\alpha,s}_{p'}} \simeq \|L\|_{(\mathring{M}^{\alpha,s}_{p})^{*}}$.

As a consequence of Theorem 3.2.9, we obtain a triplet duality relation

$$\mathring{M}^{\alpha,s}_{p}\text{–}\mathcal{N}^{\alpha,s}_{p'}\text{–}M^{\alpha,s}_{p},$$

which is analogous to the famous triplet VMO–H^1–BMO of harmonic analysis (see [CW]). See also [AX2] where a similar triplet was claimed without proof in the context of Morrey spaces. We mention that our proof of Theorem 3.2.9 is completely different from the VMO–H^1 duality proof of [CW]. It is based on the relation $\mathcal{N}^{\alpha,s}_{p'} \approx (M^{\alpha,s}_{p})'$, Radon-Nikodym Theorem, and Hahn-Banach Theorem. Moreover, it can also be easily modified to provide a proof the claimed triplet in [AX2]. For other related results in the Morrey space setting, see [ST, ISY].

Proof of Theorem 3.2.1. If $f \in M_p^{\alpha,s}$, then for any $g \in N_{p'}^{\alpha,s}$ and $w \in L^1(C)$, $\|w\|_{L^1(C)} \le 1$, such that $\int |g|^{p'} w^{1-p'} dx < +\infty$, one has

$$\left|\int f(x)g(x)dx\right| \le \left(\int |f|^p w dx\right)^{\frac{1}{p}} \left(\int |g|^{p'} w^{1-p'} dx\right)^{\frac{1}{p'}}$$

by Hölder's inequality. Thus it follows from the proof of Theorem 3.1.1 that

$$\left|\int f(x)g(x)dx\right| \le \|f\|_{M_p^{\alpha,s}} \|g\|_{N_{p'}^{\alpha,s}},$$

and so $L_f \in (N_{p'}^{\alpha,s})^*$.

Conversely, let $L \in (N_{p'}^{\alpha,s})^*$ be given. If $g \in L^{p'}$ with $\mathrm{supp}(g) \subset E$ for a bounded set E with positive capacity, then with V^E and δ as in Lemma 3.1.3 we have

$$\int_{\mathbb{R}^n} |g|^{p'} \left[(V^E)^\delta / \mathrm{Cap}_{\alpha,s}(E)\right]^{1-p'} dx \le \mathrm{Cap}_{\alpha,s}(E)^{p'-1} \|g\|_{L^{p'}}^{p'}.$$

Thus $g \in N_{p'}^{\alpha,s}$ with

$$\|g\|_{N_{p'}^{\alpha,s}} \le C \mathrm{Cap}_{\alpha,s}(E)^{\frac{1}{p}} \|g\|_{L^{p'}},$$

and so

$$|L(g)| \le C\|L\| \mathrm{Cap}_{\alpha,s}(E)^{\frac{1}{p}} \|g\|_{L^{p'}}.$$

By Riesz's representation theorem there is an $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$L(g) = \int f(x)g(x)dx \tag{3.27}$$

for all $g \in L^{p'}$ with compact support. In particular, if $g = \mathrm{sgn}(f)|f|^{p-1}\chi_K$ for any compact set K, then we deduce

$$\begin{aligned} |L(g)| = \int_K |f|^p &\le C\|L\| \mathrm{Cap}_{\alpha,s}(K)^{\frac{1}{p}} \|g\|_{L^{p'}} \\ &= C\|L\| \mathrm{Cap}_{\alpha,s}(K)^{\frac{1}{p}} \left(\int_K |f|^p\right)^{\frac{1}{p'}}. \end{aligned}$$

This implies $f \in M_p^{\alpha,s}$ and

$$\|f\|_{M_p^{\alpha,s}} \leq C\|L\|.$$

Note that for any $g \in N_{p'}^{\alpha,s}$, the functions

$$g_k := \max\{\min\{g, k\}, -k\}\chi_{B_k(0)}, \quad k \geq 1,$$

converge to g in $N_{p'}^{\alpha,s}$ as $k \to \infty$. Also, for any $g \in N_{p'}^{\alpha,s}$ and $k \geq 1$ we have

$$\int |f||g|_k dx \leq C\|f\|_{M_p^{\alpha,s}}\|g_k\|_{N_{p'}^{\alpha,s}} \leq C\|f\|_{M_p^{\alpha,s}}\|g\|_{N_{p'}^{\alpha,s}},$$

and thus by Fatou's lemma we get $fg \in L^1(\mathbb{R}^n)$. Then by continuity, (3.27), and Lebesgue Dominated Convergence Theorem we arrive at

$$L(g) = \lim_{k\to\infty} L(g_k) = \lim_{k\to\infty}\int f(x)g_k(x)dx = \int f(x)g(x)dx$$

for all $g \in N_{p'}^{\alpha,s}$.

This completes the proof of the theorem. □

Remark 3.2.10. *The above proof shows that bounded functions with compact support f are dense in $N_q^{\alpha,s}$. For such f, we define $\rho_\epsilon * f = \epsilon^{-n}\rho(\epsilon^{-1}\cdot) * f$, where $\epsilon \in (0,1)$ and $\rho \in C_c^\infty(B_1(0))$. Let B be a ball such that $\mathrm{supp}(f) \subset B$ and $\mathrm{supp}(\rho_\epsilon * f) \subset B$ for any $\epsilon \in (0,1)$. Then take a weight $w \in L^1(C) \cap A_1^{\mathrm{loc}}$ such that $w \geq 1$ on B. We have*

$$\|\rho_\epsilon * f - f\|_{N_q^{\alpha,s}} \leq C\left(\int_{\mathbb{R}^n} |\rho_\epsilon * f - f|^q w^{1-q}dx\right)^{\frac{1}{q}} \leq C\|\rho_\epsilon * f - f\|_{L^q}.$$

Thus we see that $C_c^\infty(\mathbb{R}^n)$ is dense in $N_q^{\alpha,s}$. Likewise, we also have that $C_c^\infty(\mathbb{R}^n)$ is dense in the space $\mathcal{N}_q^{\alpha,s}$.

Proof of Theorem 3.2.2. One just needs to follow the proofs of Theorems 3.1.1 and 3.2.1 and replace the function $(V^E)^\delta$ with the characteristic function χ_E. $\square$

Remark 3.2.11. *In general, functions in $N_{p'}^{\alpha,s}$ (hence $\widetilde{N}_{p'}^{\alpha,s}$) do not belong to $L^{p'}_{\text{loc}}(\mathbb{R}^n)$. To see this, consider the case $p = 2, \alpha = 1/4, s = 2,$ and $n \geq 3$. Let $g(x) = |x|^{-n+1}$ for $|x| < 1$ and for $g(x) = |x|^{-n-1}$ for $|x| \geq 1$. Also, let $w(x) = g(x)$ for any $x \in \mathbb{R}^n$. Then using* (2.7) *and* (2.8) *it can be shown that $w \in A_1^{\text{loc}} \cap L^1(C)$. Moreover, we have $g^2 w^{-1} \in L^1(\mathbb{R}^n)$. Thus $g \in N_{p'}^{\alpha,s}$ (by enlarging $\overline{\mathbf{c}}(n, \alpha, s)$ if necessary) but $g \notin L^2(B_1(0))$.*

On the other hand, if in the definition of $N_{p'}^{\alpha,s}$ we consider only weights w such that $w \in L^\infty(\mathbb{R}^n) \cap L^1(C) \cap A_1^{\text{loc}}$ with $\|w\|_{L^1(C)} \leq 1$ and $[w]_{A_1^{\text{loc}}} \leq \overline{\mathbf{c}}(n, \alpha, s)$ (or only weights w such that $w \in L^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} w dC \leq 1$ for $\widetilde{N}_{p'}^{\alpha,s}$), then Theorems 3.2.1 and 3.2.2 still remain valid for those versions of $N_{p'}^{\alpha,s}$ and $\widetilde{N}_{p'}^{\alpha,s}$. Moreover, functions in such spaces belong to $L^{p'}(\mathbb{R}^n)$.

Proof of Theorem 3.2.4. By Proposition 3.2.5, we just need to show $(B_{p'}^{\alpha,s})^* = M_p^{\alpha,s}$. Let $f \in M_p^{\alpha,s}$ and $g \in B_{p'}^{\alpha,s}$. Suppose that $g = \sum_j c_j a_j$ where $a_j = 0$ in $\mathbb{R}^n \setminus A_j$ and $\|a_j\|_{L^{p'}} \leq \text{Cap}_{\alpha,s}(A_j)^{-1/p}$. We have

$$
\begin{aligned}
\left| \int f(x) g(x) dx \right| &\leq \sum_j |c_j| \int_{A_j} |f a_j| dx \\
&\leq \sum_j |c_j| \|f\|_{L^p(A_j)} \|a_j\|_{L^{p'}} \\
&\leq \sum_j |c_j| \|f\|_{L^p(A_j)} \text{Cap}_{\alpha,s}(A_j)^{-1/p} \\
&\leq \sum_j |c_j| \|f\|_{M_p^{\alpha,s}}.
\end{aligned}
$$

Thus,

$$\left|\int f(x)g(x)dx\right| \leq \|f\|_{M_p^{\alpha,s}} \|g\|_{B_{p'}^{\alpha,s}}. \tag{3.28}$$

Conversely, let $L \in B_{p'}^{\alpha,s}$ be given. If $0 \neq g \in L^{p'}$ with $\text{supp}(g) \subset E$ for a bounded set E then $g \in B_{p'}^{\alpha,s}$ as we can write $g = \text{Cap}_{\alpha,s}(E)^{1/p}\|g\|_{L^{p'}}\tilde{g}$, where $\tilde{g} = g/(\text{Cap}_{\alpha,s}(E)^{1/p}\|g\|_{L^{p'}})$, and so

$$\|g\|_{B_{p'}^{\alpha,s}} \leq \text{Cap}_{\alpha,s}(E)^{\frac{1}{p}}\|g\|_{L^{p'}}.$$

This gives

$$|L(g)| \leq \|L\|\text{Cap}_{\alpha,s}(E)^{\frac{1}{p}}\|g\|_{L^{p'}}.$$

Then as in the proof of Theorem 3.2.1 we can find an $f \in M_p^{\alpha,s}$ with $\|f\|_{M_p^{\alpha,s}} \leq \|L\|$, and

$$L(g) = \int f(x)g(x)dx \tag{3.29}$$

for all $g \in L^{p'}$ with compact support.

We will now show that (3.29) holds for all $g \in B_{p'}^{\alpha,s}$. Note that for any $g \in B_{p'}^{\alpha,s}$, we have a representation $g = \sum_j c_j a_j$ where $a_j = 0$ in $\mathbb{R}^n \setminus A_j$, A_j's are bounded sets, $\|a_j\|_{L^{p'}} \leq \text{Cap}_{\alpha,s}(A_j)^{-1/p}$, and $\sum_j |c_j| < +\infty$. Thus the functions

$$g_k := \sum_{|j|\leq k} c_j a_j, \qquad k \geq 1,$$

have compact support and converge to g in $B_{p'}^{\alpha,s}$ as $k \to \infty$. Also, if $h_k = \sum_{|j|\leq k} |c_j||a_j|$, $k \geq 1$, then $h_k \in B_{p'}^{\alpha,s}$ and $\|h_k\|_{B_{p'}^{\alpha,s}} \leq \sum_j |c_j|$. Thus using (3.28) we have

$$\int |f|h_k dx \leq \|f\|_{M_p^{\alpha,s}}\|h_k\|_{B_{p'}^{\alpha,s}} \leq \|f\|_{M_p^{\alpha,s}} \sum_j |c_j|,$$

and by Fatou's lemma we get $f\sum_j |c_j||a_j|$ and $fg \in L^1(\mathbb{R}^n)$.

Now by continuity, (3.29), and Lebesgue Dominated Convergence Theorem we arrive at

$$L(g) = \lim_{k\to\infty} L(g_k) = \lim_{k\to\infty} \int f(x)g_k(x)dx = \int f(x)g(x)dx$$

for all $g \in B_{p'}^{\alpha,s}$. □

Remark 3.2.12. *The proof above shows that if $g = \sum_j c_j a_j$, where $a_j = 0$ in $\mathbb{R}^n \setminus A_j$ for a bounded set A_j, $\|a_j\|_{L^{p'}} \le \mathrm{Cap}_{\alpha,s}(A_j)^{-1/p}$, and $\sum_j |c_j| < +\infty$, then the series $\sum_j c_j a_j$ converges absolutely a.e. in $\mathbb{R}^n$.*

Recall that $\mathcal{L}^1(C)$ is defined as the Köthe dual of $M_1^{\alpha,s}$. Also, by Theorem 2.1.3 we see that $L^1(C)$ is continuously embedded into $\mathcal{L}^1(C)$. Moreover, the norm of a function $g \in \mathcal{N}_q^{\alpha,s}$ is the defined as

$$\|g\|_{\mathcal{N}_q^{\alpha,s}} = \inf_w \left(\int_{\mathbb{R}^n} |g(x)|^q w(x)^{1-q} dx \right)^{1/q}, \tag{3.30}$$

where the infimum is taken over all $w \in \mathcal{L}^1(C) \cap A_1^{\mathrm{loc}}$ with $\|w\|_{\mathcal{L}^1(C)} \le 1$ and $[w]_{A_1^{\mathrm{loc}}} \le \overline{\mathbf{c}}(n, \alpha, s)$.

We remark that if $\mathrm{Cap}_{\alpha,s}$ is strongly subadditive, then it can be shown from Hahn-Banach Theorem, Theorem 2.1.3, and an approximation argument that $\|f\|_{L^1(C)} = \|f\|_{\mathcal{L}^1(C)}$ for all $f \in L^1(C)$.

Let δ be a fixed constant such that

$$\delta \in (1, n/(n-\alpha))$$

if $s < 2$ and $\delta \in (s-1, n(s-1)/(n-\alpha s))$ if $s \ge 2$. We observe that if E is subset of $\mathbb{R}^n$ such that $0 < \mathrm{Cap}_{\alpha,s}(E) < \infty$ and V^E is as in Lemma 3.1.3, then by Theorem 2.1.3 and

Lemma 3.1.3 we have

$$\int |g(x)| \left(\frac{(V^E)^\delta}{\mathrm{Cap}_{\alpha,s}(E)} \right) dx \leq C$$

for all $g \in L^1_{\mathrm{loc}} \cap M_1^{\alpha,s}$ such that $\|g\|_{M_1^{\alpha,s}} \leq 1$. That is,

$$\left\| (V^E)^\delta / \mathrm{Cap}_{\alpha,s}(E) \right\|_{\mathcal{L}^1(C)} \leq C.$$

Moreover, $[V^E/\mathrm{Cap}_{\alpha,s}(E)]_{A_1^{\mathrm{loc}}} \leq \overline{\mathbf{c}}(n,\alpha,s)$ for some $\overline{\mathbf{c}}(n,\alpha,s) \geq 1$. Thus by a simple modification of the proofs of Theorems 3.1.1 and 3.2.1 we obtain the following duality result.

Theorem 3.2.13. *For $p > 1$ and $\alpha > 0, s > 1$, with $\alpha s \leq n$, we have*

$$\|f\|_{M_p^{\alpha,s}} \simeq \sup_w \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p},$$

where the supremum is taken over all weights $w \in \mathcal{L}^1(C) \cap A_1^{\mathrm{loc}}$ with $\|w\|_{\mathcal{L}^1(C)} \leq 1$ and $[w]_{A_1^{\mathrm{loc}}} \leq \overline{\mathbf{c}}(n,\alpha,s)$. Moreover, we have $\left(\mathcal{N}_{p'}^{\alpha,s}\right)^* \approx M_p^{\alpha,s}$.

The fact that $\|\cdot\|_{\mathcal{L}^1(C)}$ is a norm yields the following important result. A related result in the setting of Morrey spaces can be found in [MST].

Theorem 3.2.14. *The space $\mathcal{N}_q^{\alpha,s}$ with the norm given by* (3.30) *is a Banach function space.*

Proof. For any $g \in \mathcal{N}_q^{\alpha,s}$ we set

$$\|g\|_1 := \inf \Big\{ \sum_j |c_j| : g = \sum_j c_j b_j \text{ a.e.} \Big\},$$

where each b_j is a block in $\mathcal{N}_q^{\alpha,s}$, i.e., $b_j \in \mathcal{N}_q^{\alpha,s}$ and $\|b_j\|_{\mathcal{N}_q^{\alpha,s}} \leq 1$. It is easy to see that $\|\cdot\|_1$ is actually a norm and $(\mathcal{N}_q^{\alpha,s}, \|\cdot\|_1)$ is a Banach space. That $\|g\|_1 = 0$ implies

$$g = 0$$

a.e. can be checked as follows. Since $\|g\|_1 = 0$, for any $\epsilon > 0$, there exist $\{c_j\} \in \ell^1$ and blocks b_j's such that $g = \sum_j c_j b_j$ and $\sum_j |c_j| < \epsilon$. Then for any $\varphi \in C_c(\mathbb{R}^n)$ by Theorem 3.2.13 we have

$$\int |g||\varphi| dx \le \sum_j |c_j| \int |b_j||\varphi| dx \le \sum_j |c_j| \|b_j\|_{\mathcal{N}_q^{\alpha,s}} \|\varphi\|_{M_{q'}^{\alpha,s}} \le \epsilon \, \|\varphi\|_{M_{q'}^{\alpha,s}},$$

which yields that $\int |g||\varphi| dx = 0$ for all $\varphi \in C_c(\mathbb{R}^n)$ and hence $g = 0$ a.e.

We next show that

$$\|g\|_{\mathcal{N}_q^{\alpha,s}} = \|g\|_1 \tag{3.31}$$

for all $g \in \mathcal{N}_q^{\alpha,s}$ and thus property (P1) in the definition of Banach function space is fulfilled. That $\|g\|_1 \le \|g\|_{\mathcal{N}_q^{\alpha,s}}$ is obvious. To show the converse, we will show that

$$\|g\|_{\mathcal{N}_q^{\alpha,s}} \le (1+\epsilon)^2 \|g\|_1, \qquad \forall \epsilon > 0. \tag{3.32}$$

For any $g \in \mathcal{N}_q^{\alpha,s}$, $g \neq 0$, and any $\epsilon > 0$, there exist $\{c_j\} \in \ell^1$ and blocks b_j's such that $g = \sum_j c_j b_j$ and

$$\sum_j |c_j| \le (1+\epsilon)\|g\|_1.$$

Since $\|b_j\|_{\mathcal{N}_q^{\alpha,s}} \le 1$, we can find $w_j \in \mathcal{L}^1(C) \cap A_1^{\mathrm{loc}}$, with $\|w_j\|_{\mathcal{L}^1(C)} \le 1$ and

$$[w_j]_{A_1^{\mathrm{loc}}} \le \overline{\mathbf{c}}(n, \alpha, s)$$

, such that

$$\left(\int |b_j|^q w_j^{1-q} dx \right)^{\frac{1}{q}} \le 1 + \epsilon.$$

By Hölder's inequality we have

$$|g|^q \le \Big(\sum_j |c_j||b_j| \Big)^q \le \Big(\sum_j |c_j| w_j \Big)^{q-1} \Big(\sum_j c_j |b_j|^q w_j^{1-q} \Big).$$

Let $w = \sum_j |c_j| w_j$. It is easy to see that $w \in \mathcal{L}^1(C) \cap A_1^{\mathrm{loc}}$, with

$$\|w\|_{\mathcal{L}^1(C)} \leq \sum_j |c_j|$$

and $[w]_{A_1^{\mathrm{loc}}} \leq \bar{\mathbf{c}}(n, \alpha, s)$. We then have

$$\int |g|^q w^{1-q} dx \leq \sum_j |c_j| \int |b_j|^q |w_j|^{1-q} dx \leq \sum_j |c_j| (1+\epsilon)^q.$$

This gives

$$\int |g|^q \Big(\frac{w}{\sum_j |c_j|} \Big)^{1-q} dx \leq \Big(\sum_j |c_j| \Big)^q (1+\epsilon)^q,$$

and so

$$\|g\|_{\mathcal{N}_q^{\alpha,s}} \leq \sum_j |c_j| \, (1+\epsilon) \leq (1+\epsilon)^2 \|g\|_1.$$

Thus we obtain (3.32) and so (3.31) follows.

As properties (P3) and (P4) are easy to check, what's left now is to verify the Fatou property (P2). To this end, let $\{f_j\}$, $j = 1, 2, \ldots$, be a sequence of nonnegative measurable functions in $\mathcal{N}_q^{\alpha,s}$ and $f_j \uparrow f$ a.e. in $\mathbb{R}^n$. We just need to show that $\|f\|_{\mathcal{N}_q^{\alpha,s}} \leq \sup_{j \geq 1} \|f_j\|_{\mathcal{N}_q^{\alpha,s}}$. For this, we may assume that

$$\sup_{j \geq 1} \|f_j\|_{\mathcal{N}_q^{\alpha,s}} = M < +\infty.$$

Then for any $j \geq 1$ and $\epsilon > 0$ we can find $w_j \in \mathcal{L}^1(C) \cap A_1^{\mathrm{loc}}$, with $\|w_j\|_{\mathcal{L}^1(C)} \leq 1$ and $[w_j]_{A_1^{\mathrm{loc}}} \leq \bar{\mathbf{c}}(n, \alpha, s)$, such that

$$\Big(\int |f_j(x)|^q w_j(x)^{1-q} dx \Big)^{\frac{1}{q}} \leq M + \epsilon. \tag{3.33}$$

Note that if $g = 1/C$, where C is the constant in (3.3) then we have $g \in M_1^{\alpha,s}$ and $\|g\|_{M_1^{\alpha,s}} \leq 1$. This yields that

$$\frac{1}{C} \int w_j(x) dx \leq \|w_j\|_{\mathcal{L}^1(C)} \leq 1, \qquad \forall j \geq 1.$$

Thus by Komlós Theorem (see [Kom]), one can find some subsequence of $\{w_j\}$, still denoted by $\{w_j\}$, and a function w such that

$$\sigma_k(x) := \frac{1}{k}\sum_{j=1}^{k} w_j(x) \to w(x)$$

for almost everywhere x. Moreover, any subsequence of $\{w_j\}$ is also Cesàro convergent to w almost everywhere. Then for any function g such that $\|g\|_{M_1^{\alpha,s}} \le 1$, by Fatou's lemma we have

$$\begin{aligned}
\int |g(x)|w(x)dx &\le \liminf_{k\to\infty} \int |g(x)|\sigma_k(x)dx \\
&= \liminf_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} \int |g(x)|w_j(x)dx \\
&\le \liminf_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} \|w_j\|_{\mathcal{L}^1(C)} \\
&\le 1.
\end{aligned}$$

This shows that $w \in \mathcal{L}^1(C)$ and $\|w\|_{\mathcal{L}^1(C)} \le 1$. Furthermore, for each $j \ge 1$, by the convexity of the function $t \mapsto t^{1-q}$ on $(0,\infty)$, we have

$$\begin{aligned}
\int f_j(x)^q w(x)^{1-q} dx &= \int f_j(x)^q \lim_{k\to\infty} \Big[\frac{1}{k}\sum_{m=j}^{j+k-1} w_m(x)\Big]^{1-q} dx \\
&\le \liminf_{k\to\infty} \int f_j(x)^q \Big[\frac{1}{k}\sum_{m=j}^{j+k-1} w_m(x)\Big]^{1-q} dx \\
&\le \liminf_{k\to\infty} \int f_j(x)^q \,\frac{1}{k}\sum_{m=j}^{j+k-1} w_m(x)^{1-q} dx \\
&\le \liminf_{k\to\infty} \int \frac{1}{k}\sum_{m=j}^{j+k-1} f_m(x)^q\, w_m(x)^{1-q} dx,
\end{aligned}$$

where we used $0 \leq f_j \leq f_m$ for $m \geq j$ in the last bound. By (3.33), this gives

$$\int f_j(x)^q w(x)^{1-q} dx \leq (M+\epsilon)^q,$$

and letting $j \to \infty$ we get

$$\int f(x)^q w(x)^{1-q} dx \leq (M+\epsilon)^q.$$

As this holds for all $\epsilon > 0$ we arrive at

$$\|f\|_{\mathcal{N}_q^{\alpha,s}} \leq M = \sup_{j \geq 1} \|f_j\|_{\mathcal{N}_q^{\alpha,s}},$$

which completes the proof of the theorem. □

We now obtain the main result of this section.

Theorem 3.2.15. *For $\alpha > 0, s > 1$ with $\alpha s \leq n$ and $p > 1$ we have*

$$\left(\mathcal{N}_{p'}^{\alpha,s}\right)' \approx M_p^{\alpha,s}, \tag{3.34}$$

and

$$\mathcal{N}_{p'}^{\alpha,s} \approx \left(M_p^{\alpha,s}\right)'. \tag{3.35}$$

Proof. The relation (3.34) is just a consequence of Theorem 3.2.13 and in fact more precisely we have

$$\|f\|_{\left(\mathcal{N}_{p'}^{\alpha,s}\right)'} \leq \|f\|_{M_p^{\alpha,s}} \leq C\|f\|_{\left(\mathcal{N}_{p'}^{\alpha,s}\right)'}.$$

To prove (3.35) we note from (3.34) that $\left(\mathcal{N}_{p'}^{\alpha,s}\right)'' \approx \left(M_p^{\alpha,s}\right)'$. On the other hand, by Theorems 2.3.1 and 3.2.14 we find $\left(\mathcal{N}_{p'}^{\alpha,s}\right)'' = \mathcal{N}_{p'}^{\alpha,s}$. Thus we obtain (3.35) as claimed. □

Remark 3.2.16. *If we drop the A_1^{loc} condition in the definition of $\mathcal{N}_q^{\alpha,s}$, then we get another space which we call $\widetilde{\mathcal{N}}_q^{\alpha,s}$. For this space we have $\left(\widetilde{\mathcal{N}}_{p'}^{\alpha,s}\right)' = M_p^{\alpha,s}$ and $\widetilde{\mathcal{N}}_{p'}^{\alpha,s} = \left(M_p^{\alpha,s}\right)'$, $p > 1$. In particular, we have $\mathcal{N}_q^{\alpha,s} \approx \widetilde{\mathcal{N}}_q^{\alpha,s}$, $q > 1$.*

In order to prove Theorem 3.2.7 we need the following lemmas.

Lemma 3.2.17. *Suppose that* $q > 1$ *and* $\mathrm{Cap}_{\alpha,s}$ *is strongly subadditive. Then* $N_q^{\alpha,s}$ *and* $\widetilde{N}_q^{\alpha,s}$ *are Banach spaces and we have* $N_q^{\alpha,s} \approx \widetilde{N}_q^{\alpha,s} \approx B_q^{\alpha,s}$ *with*

$$\|f\|_{\widetilde{N}_q^{\alpha,s}} \le \|f\|_{B_q^{\alpha,s}} \le c_1 \|f\|_{\widetilde{N}_q^{\alpha,s}} \le c_1 \|f\|_{N_q^{\alpha,s}} \le c_2 \|f\|_{B_q^{\alpha,s}}.$$

Proof. By reasoning as in the proof of Theorem 3.2.14 we find

$$\|f\|_{N_q^{\alpha,s}} = \inf\Big\{ \sum_j |c_j| : f = \sum_j c_j b_j \text{ a.e.}\Big\}, \tag{3.36}$$

where each $b_j \in N_q^{\alpha,s}$ and $\|b_j\|_{N_q^{\alpha,s}} \le 1$. Likewise,

$$\|f\|_{\widetilde{N}_q^{\alpha,s}} = \inf\Big\{ \sum_j |c_j| : f = \sum_j c_j b_j \text{ a.e.}\Big\}, \tag{3.37}$$

where each $b_j \in \widetilde{N}_q^{\alpha,s}$ and $\|b_j\|_{\widetilde{N}_q^{\alpha,s}} \le 1$. Note here that to verify (3.36) we use the completeness of $L^1(C)$ (Theorem 2.1.2) to obtain that if $w = \sum_j |c_j| w_j$, where $\{c_j\} \in \ell^1$ and $w_j \in L^1(C)$ with $\|w_j\|_{L^1(C)} \le 1$, then w is quasicontinuous and $\|w_j\|_{L^1(C)} \le \sum_j |c_j|$. Now (3.36) and (3.37) yield that $N_q^{\alpha,s}$ and $\widetilde{N}_q^{\alpha,s}$ are Banach spaces.

Note that if $a \in L^q(\mathbb{R}^n)$ is such that there exists a bounded set $A \subset \mathbb{R}^n$ for which $a = 0$ a.e. in $\mathbb{R}^n \setminus A$ and $\|a\|_{L^q} \le \mathrm{Cap}_{\alpha,s}(A)^{\frac{1-q}{q}}$, then obviously

$$\|a\|_{\widetilde{N}_q^{\alpha,s}} \le 1$$

and by Lemma 3.1.3

$$\|a\|_{N_q^{\alpha,s}} \le C.$$

Thus it follows from (3.36) and (3.37) that

$$\|f\|_{\widetilde{N}_q^{\alpha,s}} \le \|f\|_{B_q^{\alpha,s}}, \qquad \text{and } \|f\|_{N_q^{\alpha,s}} \le C \|f\|_{B_q^{\alpha,s}}.$$

Also, it is obvious that $\|f\|_{\widetilde{N}_q^{\alpha,s}} \leq \|f\|_{N_q^{\alpha,s}}$ and so we just need to show

$$\|f\|_{B_q^{\alpha,s}} \leq C\,\|f\|_{\widetilde{N}_q^{\alpha,s}} \tag{3.38}$$

for any $f \in \widetilde{N}_q^{\alpha,s}$. Now for $f \in \widetilde{N}_q^{\alpha,s}$, we can find a nonnegative function w defined quasieverywhere such that

$$\int w\,dC = \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{w > t\})dt \leq 1$$

and

$$\left(\int |f(x)|^q w(x)^{1-q} dx\right)^{\frac{1}{q}} \leq 2\|f\|_{\widetilde{N}_q^{\alpha,s}}.$$

Note that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} 2^k \mathrm{Cap}_{\alpha,s}(\{2^{k-1} < w \leq 2^k\}) \\
&\qquad = \frac{1}{4}\sum_{k\in\mathbb{Z}} \int_{2^{k-2}}^{2^{k-1}} \mathrm{Cap}_{\alpha,s}(\{2^{k-1} < w \leq 2^k\})dt \\
&\qquad \leq \frac{1}{4}\sum_{k\in\mathbf{Z}} \int_{2^{k-2}}^{2^{k-1}} \mathrm{Cap}_{\alpha,s}(\{w > t\})dt \\
&\qquad = \frac{1}{4}\int w\,dC \leq \frac{1}{4}.
\end{aligned}$$

Let $E_k = \{2^{k-1} < w \leq 2^k\}$ for $k \in \mathbb{Z}$ and $D_l = \{l-1 \leq |x| < l\}$ for $l = 1, 2, \ldots$ Note that $w < +\infty$ quasieverywhere and hence

$$f = \sum_{k,l} f\chi_{E_k\cap D_l} = \sum_{k,l} c_{k,l}a_{k,l} \qquad \text{a.e.,}$$

where $\sum_{k,l} = \sum_{k\in\mathbb{Z}}\sum_{l\geq 1}$ and

$$c_{k,l} = \|f\|_{L^q(E_k\cap D_l)}\mathrm{Cap}_{\alpha,s}(E_k\cap D_l)^{(q-1)/q},$$

$$a_{k,l} = \|f\|^{-1}_{L^q(E_k\cap D_l)}\mathrm{Cap}_{\alpha,s}(E_k\cap D_l)^{(1-q)/q} f\chi_{E_k\cap D_l}.$$

Here we understand that $a_{k,l} = 0$ whenever $f = 0$ a.e. in $E_k \cap D_l$. It is obvious that

$$\|a_{k,l}\|_{L^q} = \mathrm{Cap}_{\alpha,s}(E_k \cap D_l)^{(1-q)/q},$$

whenever $a_{k,l} \neq 0$. Moreover, we have that

$$\begin{aligned}
\sum_{k,l} c_{k,l} &= \sum_{k,l} \left(\int_{E_k \cap D_l} |f(x)|^q w(x)^{1-q} w(x)^{q-1} dx \right)^{\frac{1}{q}} \mathrm{Cap}_{\alpha,s}(E_k \cap D_l)^{\frac{q-1}{q}} \\
&\leq \sum_{k,l} \left(\int_{E_k \cap D_l} |f(x)|^q w(x)^{1-q} dx \right)^{\frac{1}{q}} 2^{k(q-1)/q} \mathrm{Cap}_{\alpha,s}(E_k \cap D_l)^{\frac{q-1}{q}} \\
&\leq \Big(\sum_{k,l} \int_{E_k \cap D_l} |f(x)|^q w(x)^{1-q} dx \Big)^{\frac{1}{q}} \Big(\sum_{k,l} 2^k \mathrm{Cap}_{\alpha,s}(E_k \cap D_l) \Big)^{\frac{q-1}{q}},
\end{aligned}$$

where we used Hölder's inequality in the last line.

On the other hand, it follows from the quasiadditivity of $\mathrm{Cap}_{\alpha,s}$ (see (2.11)) we have

$$\sum_{l \geq 1} \mathrm{Cap}_{\alpha,s}(E_k \cap D_l) \leq C\, \mathrm{Cap}_{\alpha,s}(E_k).$$

Thus,

$$\begin{aligned}
\sum_{k,l} c_{k,l} &\leq C \left(\int |f(x)|^q w(x)^{1-q} dx \right)^{\frac{1}{q}} \left(\sum_k 2^k \mathrm{Cap}_{\alpha,s}(E_k) \right)^{\frac{q-1}{q}} \\
&\leq C \|f\|_{\widetilde{N}^{p',\alpha,s}}.
\end{aligned}$$

We have succeeded to decompose f as the sum $f = \sum_j c_j a_j$ such that $\|c_j\|_{l^1} \leq C\|f\|_{\widetilde{N}^{p',\alpha,s}}$ and $\|a_j\|_{L^q} \leq \mathrm{Cap}_{\alpha,s}(A_j)^{(1-q)/q}$ with $\{a_j \neq 0\} \subset A_j$ for a bounded set A_j. Thus by the definition of $B_q^{\alpha,s}$ we obtain $f \in B_q^{\alpha,s}$ with the bound (3.38). □

Lemma 3.2.18. *Suppose that $q > 1$ and $\mathrm{Cap}_{\alpha,s}$ is strongly subadditive. Then $N_q^{\alpha,s}$, $\widetilde{N}_q^{\alpha,s}$, and $B_q^{\alpha,s}$ are Banach function spaces.*

Proof. First we show that $\widetilde{N}_q^{\alpha,s}$ is a Banach function space and for that we just need to check the Fatou property (P2). The proof is similar to that of Theorem 3.2.14. Let $\{f_j\}$, $j = 1, 2, \ldots$, be a sequence of nonnegative measurable functions in $\widetilde{N}_q^{\alpha,s}$ and $f_j \uparrow f$ a.e. in $\mathbb{R}^n$. Suppose that $\sup_{j\geq 1} \|f_j\|_{\widetilde{N}_q^{\alpha,s}} = M < +\infty$. It is enough to show that $\|f\|_{\widetilde{N}_q^{\alpha,s}} \leq M$.

As in the proof of Theorem 3.2.14, for any $j \geq 1$ and $\epsilon > 0$ we can find a nonnegative and q.e. defined weight w_j with $\int w_j \, dC \leq 1$ such that

$$\left(\int |f_j(x)|^q w_j(x)^{1-q} dx \right)^{\frac{1}{q}} \leq M + \epsilon$$

and $\int w_j(x) dx \leq C$. Then by Komlós Theorem, one can find a subsequence of $\{w_j\}$, still denoted by $\{w_j\}$, and a function w such that $\sigma_k(x) := \dfrac{1}{k} \sum_{j=1}^k w_j(x) \to w(x)$ for almost everywhere x. Moreover, any subsequence of $\{w_j\}$ is also Cesàro convergent to w almost everywhere. By redefining $w(x)$ to be zero for all the points x such that $\sigma_k(x) \not\to w(x)$, one has

$$w(x) \leq \liminf_{k\to\infty} \sigma_k(x) \qquad \text{quasieverywhere.}$$

Hence,

$$\begin{aligned} \int_0^\infty \mathrm{Cap}_{\alpha,s}\left(\{w > t\}\right) dt &\leq \int_0^\infty \mathrm{Cap}_{\alpha,s}\left(\left\{\liminf_{k\to\infty} \sigma_k > t\right\}\right) dt \\ &\leq \int_0^\infty \liminf_{k\to\infty} \mathrm{Cap}_{\alpha,s}(\{\sigma_k > t\}) dt \\ &\leq \liminf_{k\to\infty} \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{\sigma_k > t\}) dt \\ &\leq \liminf_{k\to\infty} \frac{1}{k} \sum_{j=1}^k \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{w_j > t\}) dt, \end{aligned}$$

where we used the strong subadditivity of $\mathrm{Cap}_{\alpha,s}$ in the last inequality. This gives

$$\int w \, dC \leq \liminf_{k\to\infty} \frac{1}{k} \sum_{j=1}^k \int w_j \, dC \leq 1.$$

Moreover, as in the proof of Theorem 3.2.14 we have

$$\int |f(x)|^q w(x)^{1-q} dx \le (M+\epsilon)^q$$

and so $\|f\|_{\widetilde{N}_q^{\alpha,s}} \le M$ as desired.

Next we show that $N_q^{\alpha,s}$ is a Banach function space. Let $\{f_j\}$, $j = 1, 2, \ldots$, be a sequence of nonnegative measurable functions in $N_q^{\alpha,s}$ and $f_j \uparrow f$ a.e. in $\mathbb{R}^n$. Suppose that $\sup_{j\ge1} \|f_j\|_{N_q^{\alpha,s}} = M < +\infty$. By Lemma 3.2.17 and the Fatou property of $\widetilde{N}_q^{\alpha,s}$, we have $\|f\|_{\widetilde{N}_q^{\alpha,s}} \le M$. In particular, $f \in \widetilde{N}_q^{\alpha,s} \cap N_q^{\alpha,s}$ and thus $g_j := f - f_j \in \widetilde{N}_q^{\alpha,s} \cap N_q^{\alpha,s}$ for all $j \ge 1$ and $g_j \downarrow 0$ a.e.

On the other hand, Theorem 3.2.1 implies that $\left(\widetilde{N}_q^{\alpha,s}\right)^* = \left(\widetilde{N}_q^{\alpha,s}\right)'$ with equality of norms and hence it follows from Theorem 2.3.2 that $\widetilde{N}_q^{\alpha,s}$ has an absolutely continuous norm. This yields that $\|f - f_j\|_{\widetilde{N}_q^{\alpha,s}} = \|g_j\|_{\widetilde{N}_q^{\alpha,s}} \downarrow 0$. Thus by Lemma 3.2.17 we then obtain $\|g_j\|_{N_q^{\alpha,s}} \downarrow 0$. This yields $\|f_j\|_{N_q^{\alpha,s}} \uparrow \|f\|_{N_q^{\alpha,s}}$ and the Fatou property (P2) follows for $N_q^{\alpha,s}$. It is now easy to see that $N_q^{\alpha,s}$ is a Banach function space.

The proof that $B_q^{\alpha,s}$ is a Banach function space can be proceeded similarly, as long as we can verify the following properties of $B_q^{\alpha,s}$:

$$\|f\|_{B_q^{\alpha,s}} = \||f|\|_{B_q^{\alpha,s}}, \qquad \forall f \in B_q^{\alpha,s}, \tag{3.39}$$

and

$$0 \le f \le g \text{ a.e. } \Rightarrow \|f\|_{B_q^{\alpha,s}} \le \|g\|_{B_q^{\alpha,s}}, \qquad \forall g \in B_q^{\alpha,s}. \tag{3.40}$$

Equality (3.39) is easy to see from the identities $|f| = f\mathrm{sgn}(f)$ and $f = |f|\mathrm{sgn}(f)$. To see (3.40), suppose that $g \in B_q^{\alpha,s}$ and $g = \sum_j c_j a_j$, where $\{c_j\} \in \ell^1$ and each $a_j \in L^q(\mathbb{R}^n)$ is such that there exists a bounded set $A_j \subset \mathbb{R}^n$ for which $a_j = 0$ a.e. in $\mathbb{R}^n \setminus A_j$

and $\|a_j\|_{L^q} \le \mathrm{Cap}_{\alpha,s}(A_j)^{\frac{1-q}{q}}$. Then for $0 \le f \le g$ we can write

$$f = \sum_j c_j f g^{-1} a_j \chi_{\{g\neq 0\}} \qquad \text{a.e.}$$

Note that $\|f g^{-1} a_j \chi_{\{g\neq 0\}}\|_{L^q} \le \|a_j\|_{L^q} \le \mathrm{Cap}_{\alpha,s}(A_j)^{\frac{1-q}{q}}$, and thus $f \in B_q^{\alpha,s}$ and

$$\|f\|_{B_q^{\alpha,s}} \le \|g\|_{B_q^{\alpha,s}}.$$

This completes the proof of the lemma. □

We can now prove Theorem 3.2.7.

Proof of Theorem 3.2.7. By Theorems 3.2.2 and 3.2.4 we have

$$\left(M_p^{\alpha,s}\right)' = \left(\widetilde{N}_{p'}^{\alpha,s}\right)'' = \left(B_{p'}^{\alpha,s}\right)''.$$

Thus if $\mathrm{Cap}_{\alpha,s}$ is strongly subadditive then by Theorem 2.3.1 and Lemma 3.2.18 we find

$$(M_p^{\alpha,s})' = \widetilde{N}_{p'}^{\alpha,s} = B_{p'}^{\alpha,s}.$$

Likewise, by Theorem 3.2.1 we have $\left(N_{p'}^{\alpha,s}\right)'' \approx (M_p^{\alpha,s})'$ and so it follows from Theorem 2.3.1 and Lemma 3.2.18 that $N_{p'}^{\alpha,s} \approx (M_p^{\alpha,s})'$.

Now the theorem follows from Theorems 3.2.14, 3.2.15, and Lemma 3.2.18. □

Proof of Theorem 3.2.8. Using (2.16) and the subadditivity of $\gamma_{\alpha,s}(\cdot)$, and arguing as in the proof of Theorem 3.2.14 we find

$$\|f\|_{N_q^{\alpha,s}} \simeq \inf\Big\{\sum_j |c_j| : f = \sum_j c_j b_j \text{ a.e. where } \|b_j\|_{N_q^{\alpha,s}} \le 1\Big\},$$

and

$$\|f\|_{\widetilde{N}_q^{\alpha,s}} \simeq \inf\Big\{\sum_j |c_j| : f = \sum_j c_j b_j \text{ a.e. where } \|b_j\|_{\widetilde{N}_q^{\alpha,s}} \le 1\Big\}.$$

At this point we can repeat the argument in the proof of Lemma 3.2.17 to obtain $N_q^{\alpha,s} \approx \widetilde{N}_q^{\alpha,s} \approx B_q^{\alpha,s}$. Combining this with Theorem 3.2.15 we get the theorem. $\square$

3.3. Boundedness of Local Hardy-Littlewood Maximal Function on Preduals

The way the spaces $N_{p'}^{\alpha,s}$ and $\mathcal{N}_{p'}^{\alpha,s}$ are constructed and Theorem 3.2.8, we obtain the following important results regarding the behavior of the Hardy-Littlewood maximal functions and Calderón-Zygmund operators on those spaces.

Theorem 3.3.1. *Let $p > 1$, $\alpha > 0$, $s > 1$, and $\alpha s \leq n$. Then the local Hardy-Littlewood maximal function $\mathbf{M}^{\text{loc}}$ is bounded on S where S is any of the spaces in* (3.26)*.*

We recall the Hardy-Littlewood maximal function $\mathbf{M}$ is bounded on $M_p^{\alpha,s}$, $\alpha s \leq n$, (see [MS1]). However, unlike $\mathbf{M}$, standard singular integrals are generally unbounded on $M_p^{\alpha,s}$. Take for example the j-th Riesz transform,

$$R_j(f)(x) = c(n)\,\text{p.v.} \int \frac{x_j - y_j}{|x-y|^{n+1}} f(y)dy, \qquad j = 1, 2, \ldots, n,$$

and adapt the argument of [RT, Theorem 1.1] to our setting, using the fact that $L^\infty \hookrightarrow M_p^{\alpha,s}$.

On the other hand, $\mathbf{M}$ fails to be bounded on any of the spaces in (3.26), since they are included in L^1. Likewise, the first Riesz transform R_1, say, is also unbounded on these spaces. To see that, take a nonnegative function $f \in C_c^\infty(B_1(0))$ such that $f = 1$ on $B_{1/2}(0)$. Then for any $x = (x_1, x') = (x_1, x_2, \ldots, x_n)$ with $x_1 > 1$ we have

$$R_1(f)(x) \geq c(n) \int_{B_{1/2}(0)} \frac{x_1 - y_1}{|x-y|^{n+1}} dy \geq c\, \frac{x_1}{|x|^{n+1}}.$$

This shows that $R_1(f) \notin L^1$, since

$$\|R_1(f)\|_{L^1} \geq c \int_1^\infty \int_{|x'|<x_1} \frac{x_1}{|x|^{n+1}} dx' dx_1 \geq c \int_1^\infty \int_{|x'|<x_1} x_1^{-n} dx' dx_1$$
$$= c \int_1^\infty x_1^{-1} dx_1 = +\infty,$$

and thus it does not belong to any of the mentioned spaces.

However, the following 'localized' boundedness property is applicable to $\mathbf{M}$ and any standard Calderón-Zygmund operator.

Theorem 3.3.2. *Let $q > 1$, $\alpha > 0$, $s > 1$, and $\alpha s \leq n$. Suppose that T is an operator (not necessarily linear or sublinear) such that*

$$\int |T(f)|^q w dx \leq C_1 \int |f|^q w dx$$

holds for all $f \in L^q(w)$ and all $w \in A_1$, with a constant C_1 depending only on n, q, and the bound for the A_1 constant of w. Then for any measurable function f such that $\mathrm{supp}(f) \subset B_{R_0}(x_0)$*, $x_0 \in \mathbb{R}^n, R_0 > 0$, we have*

$$\|T(f)\chi_{B_{R_0}(x_0)}\|_S \leq C_2 \|f\|_S,$$

where $S = N_q^{\alpha,s}, \mathcal{N}_q^{\alpha,s}$, $(M_{q'}^{\alpha,s})'$, $\widetilde{N}_q^{\alpha,s}$, $B_q^{\alpha,s}$, or $M_q^{\alpha,s}$. Here the constant

$$C_2 = C_2(n, \alpha, s, q, R_0)$$

.

We mention that Theorem 3.3.2 can be applied to the so-called (nonlinear) m-harmonic transform $\mathcal{H}_m$, $m > 1$, where for each vector field $F \in L^m(\Omega, \mathbb{R}^n)$ we define $\mathcal{H}_m(F) = \nabla u$ with $u \in W_0^{1,m}(\Omega)$ being the unique solution of $\Delta_m u = \mathrm{div}(|F|^{m-2}F)$ in

Ω. Here Ω is a bounded C^1 domain in $\mathbb{R}^n$ and Δ_m is the m-Laplacian defined as $\Delta_m u = \mathrm{div}(|\nabla u|^{m-2}\nabla u)$. Indeed, this is possible since the weighted bound

$$\int_\Omega |\mathcal{H}_m(F)|^q w dx \leq C(n, m, q, \Omega, [w]_{A_1}) \int_\Omega |F|^q w dx$$

holds for all weights $w \in A_1$ and $q \geq m$ (see [Ph1, MP] for $q > m$ and [AP1] for $q = m$). For m-Laplace equations with measure data, where the exponent q can be less than the natural exponent m, see [Ph3, NP].

To prove Theorems 3.3.1 and 3.3.2, we need the following basic results about A_1^{loc} weights. We first observe that if $w \in A_1^{\mathrm{loc}}$ then for any ball B_r with radius $r \leq 1/2$ we have

$$\fint_{B_r} w(y)dy \leq 2^n [w]_{A_1^{\mathrm{loc}}} \inf_{B_r} w. \tag{3.41}$$

Let B be any ball such that the radius of B is $r(B) = 1/2$. We claim that there is a constant $c(n) > 0$ such that

$$\int_{(t+1/2)B} w(y)dy \leq c(n)[w]_{A_1^{\mathrm{loc}}} \int_{tB} w(y)dy, \qquad \forall t \geq 1. \tag{3.42}$$

Indeed, for any $t \geq 1$, let $A = (t+1/2)B \setminus tB$. We can cover A by balls B_k of radius $r(B_k) = 1/2$ such that $|B_k \cap tB| \geq c(n)$, and

$$\sum_k \chi_{B_k}(x) \leq N(n).$$

Thus using (3.41) we have

$$\begin{aligned}
\int_A w(y)dy &\leq \sum_k \int_{B_k} w(y)dy \leq c[w]_{A_1^{\mathrm{loc}}} \sum_k \inf_{B_k} w \\
&\leq c[w]_{A_1^{\mathrm{loc}}} \sum_k \inf_{B_k \cap tB} w \leq c[w]_{A_1^{\mathrm{loc}}} \sum_k \fint_{B_k \cap tB} w(y)dy \\
&\leq c[w]_{A_1^{\mathrm{loc}}} \int_{tB} w(y) \sum_k \chi_{B_k}(y)dy \leq c[w]_{A_1^{\mathrm{loc}}} \int_{tB} w(y)dy.
\end{aligned}$$

It follows that

$$\int_{(t+1/2)B} w(y)dy \le (c[w]_{A_1^{\mathrm{loc}}} + 1) \int_{tB} w(y)dy,$$

and the claim (3.42) follows.

Using (3.42) we see that if $w \in A_1^{\mathrm{loc}}$ with $[w]_{A_1^{\mathrm{loc}}} \le \overline{\mathbf{c}}$, then for any ball B_r with radius $r \le R_0$, $R_0 > 0$, we have

$$⨍_{B_r} w(y)dy \le C(n, R_0, \overline{\mathbf{c}}) \inf_{B_r} w. \tag{3.43}$$

Now using (3.43) and a minor modification of the proof of [Ryc, Lemma 1.1], we obtain the following result.

Lemma 3.3.3. *Let $w \in A_1^{\mathrm{loc}}$ with $[w]_{A_1^{\mathrm{loc}}} \le \overline{\mathbf{c}}$ and $B = B_{R_0}(x_0)$, $R_0 > 0$. Then there exists a weight $\overline{w} \in A_1$ such that $w = \overline{w}$ in B and $[\overline{w}]_{A_1} \le c(n, R_0, \overline{\mathbf{c}})$.*

As a consequence of [Ryc, Lemma 2.11] and (3.43), we also have the following weighted bound for $\mathbf{M}^{\mathrm{loc}}$.

Lemma 3.3.4. *For any $p > 1$ and $w \in A_1^{\mathrm{loc}}$ with $[w]_{A_1^{\mathrm{loc}}} \le \overline{\mathbf{c}}$, it holds that*

$$\int_{\mathbb{R}^n} \mathbf{M}^{\mathrm{loc}}(f)^p w dx \le C(n, p, \overline{\mathbf{c}}) \int_{\mathbb{R}^n} |f|^p w dx.$$

In fact, Lemma 3.3.4 also holds for w in the larger class of A_p^{loc} weights (see [Ryc, Lemma 2.11]).

We are now ready to prove Theorems 3.3.1 and 3.3.2.

Proof of Theorem 3.3.1. This theorem follows from Theorem 3.2.8 and Lemma 3.3.4. □

Proof of Theorem 3.3.2. Let $w \in A_1^{\mathrm{loc}}$ with $[w]_{A_1^{\mathrm{loc}}} \le \overline{\mathbf{c}}$ and suppose that $\mathrm{supp}(f) \subset B_{R_0}(x_0)$ for some $R_0 > 0$. By Lemma 3.3.3, there exists a weight $\overline{w} \in A_1$ with $[\overline{w}]_{A_1} \le$

$C(n, R_0, \overline{\mathbf{c}})$ such that $\overline{w} = w$ in $B_{R_0}(x_0)$. Thus it follows from the hypothesis of the theorem that

$$\begin{aligned}\int |T(f)|^q \chi_{B_{R_0}(x_0)} w dx &\leq \int |T(f)|^q \overline{w} dx \\ &\leq C(n, q, \overline{\mathbf{c}}) \int |f|^q \overline{w} dx \\ &= C(n, q, \overline{\mathbf{c}}) \int |f|^q w dx.\end{aligned}$$

Theorem 3.3.2 now follows from Theorems 3.2.8 and 3.1.1. □

3.4. The Homogeneous Case

Let $p \geq 1$, $\alpha > 0$, and $s > 1$ be such that $\alpha s < n$. The homogeneous version of $M_p^{\alpha,s}$, denoted as $\dot{M}_p^{\alpha,s}$, is the space of functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that the trace inequality

$$\left(\int_{\mathbb{R}^n} |u|^s |f|^p dx \right)^{\frac{1}{p}} \leq C \|u\|_{\dot{H}^{\alpha,s}}^{\frac{s}{p}} \tag{3.44}$$

holds for all $u \in C_c^\infty(\mathbb{R}^n)$. A norm of a function $f \in \dot{M}_p^{\alpha,s}$ is defined as the least possible constant C in the above inequality. In (3.44), $\dot{H}^{\alpha,s}$ stands for the space of Riesz potentials which consists of functions of the form $u = I_\alpha * f$ for some $f \in L^s(\mathbb{R}^n)$, and $\|u\|_{\dot{H}^{\alpha,s}} = \|f\|_{L^s}$. Here I_α, $\alpha \in (0, n)$, is the Riesz kernel defined as the inverse Fourier transform of $|\xi|^\alpha$ (in the distributional sense), and explicitly we have $I_\alpha(x) = \gamma(n, \alpha)|x|^{\alpha - n}$, where $\gamma(n, \alpha) = \Gamma(\frac{n-\alpha}{2})/[\pi^{n/2} 2^\alpha \Gamma(\frac{\alpha}{2})]$. It is known that (see [MH]) $\dot{H}^{\alpha,s}$ is the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\dot{H}^{\alpha,s}} = \|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^s(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(u))\|_{L^s(\mathbb{R}^n)}$$

In the case $\alpha = k \in \mathbb{N}$ and $s > 1$ we have $\dot{H}^{k,s} \approx \dot{W}^{k,s}$, where $\dot{W}^{k,s} = \dot{W}^{k,s}(\mathbb{R}^n)$ is the homogeneous Sobolev space with norm being defined as

$$\|u\|_{\dot{W}^{k,s}} = \sum_{|\beta|=k} \|D^\beta u\|_{L^s}.$$

The capacity associated to $\dot{H}^{\alpha,s}$ is the Riesz capacity defined for each set $E \subset \mathbb{R}^n$ by

$$\mathrm{cap}_{\alpha,\,s}(E) := \inf\Big\{\|f\|^s_{L^s} : f \geq 0, I_\alpha * f \geq 1 \text{ on } E\Big\}.$$

It is known that the norm of a function $f \in \dot{M}^{\alpha,s}_p$ is equivalent to the quantity

$$\sup_K \left(\frac{\int_K |f(x)|^p dx}{\mathrm{cap}_{\alpha,s}(K)}\right)^{1/p},$$

where the supremum is taken over all compact sets $K \subset \mathbb{R}^n$ (see [MS2, AH]). For this reason, we shall use this quantity as the norm for $\dot{M}^{\alpha,s}_p$, $p \geq 1$, in what follows.

For $s = 2$ and $\alpha \in (0, 1]$, it is known that $\mathrm{cap}_{\alpha,s}(\cdot)$ is strongly subadditive (see [Lan, pp. 141-145]). On the other hand, for $\alpha = 1$, $\mathrm{cap}_{1,s}(\cdot)$ is equivalent to the capacity $c_{1,s}(\cdot)$, which is a strongly subadditive capacity (see [HKM, Theorem 2.2]). Here for each compact set $K \subset \mathbb{R}^n$,

$$c_{1,s}(K) := \inf\left\{\int |\nabla u|^s dx : \varphi \in C^\infty_c, \varphi \geq 1 \text{ on } K\right\},$$

and $c_{1,s}(\cdot)$ is extended to all sets E as in (2.3).

Note that for all balls B_r, $r > 0$, we have

$$\mathrm{cap}_{\alpha,s}(B_r) \simeq r^{n-\alpha s},$$

and for any measurable set E it follows from the Sobolev Embedding Theorem that

$$|E|^{1-\alpha s/n} \leq C\, \mathrm{cap}_{\alpha,s}(E).$$

This lower bound implies that $L^{\frac{n}{\alpha s},\infty}(\mathbb{R}^n) \hookrightarrow \dot{M}_p^{\alpha,s}$ with the estimate

$$\|f\|_{\dot{M}_p^{\alpha,s}} \leq C\|f\|_{L^{\frac{np}{\alpha s},\infty}(\mathbb{R}^n)}. \tag{3.45}$$

The capacity $\mathrm{cap}_{\alpha,s}$ is quasiadditive in the following sense. There exists a constant $C = C(n, \alpha, s) > 0$ such that for any set E we have (see [Ad3, Eq.(7)] and [Ad2])

$$\sum_{j=-\infty}^{\infty} \mathrm{cap}_{\alpha,s}(E \cap \{2^{j-1} \leq |x| < 2^j\}) \leq C\,\mathrm{cap}_{\alpha,s}(E).$$

The homogeneous version of $L^1(C)$ is $\dot{L}^1(C)$ which is defined analogously using the Riesz capacity $\mathrm{cap}_{\alpha,s}$. Likewise, the homogeneous version of $\mathcal{L}^1(C)$ is $\dot{\mathcal{L}}^1(C)$ which consists of measurable functions w such that

$$\|w\|_{\dot{\mathcal{L}}^1(C)} := \sup_g \int |g(x)||w(x)|dx < +\infty,$$

where the supremum is taken over all $g \in \dot{M}_1^{\alpha,s}$ such that $\|g\|_{\dot{M}_1^{\alpha,s}} \leq 1$. That is, $\dot{\mathcal{L}}^1(C)$ is the Köthe dual of $\dot{M}_1^{\alpha,s}$. It is easy to see that the quasinormed space $\dot{L}^1(C)$ is continuously embedded into the Banach space $\dot{\mathcal{L}}^1(C)$.

Let E be a subset of $\mathbb{R}^n$ such that $0 < \mathrm{cap}_{\alpha,s}(E) < \infty$. By [AH, Theorems 2.5.6 and 2.6.3], the capacitary measure for E exists as a nonnegative measure μ^E with $\mathrm{supp}(\mu^E) \subset \overline{E}$ such that the function $V^E = I_\alpha * ((I_\alpha * \mu)^{\frac{1}{s-1}})$ satisfies the following properties:

$$\mu^E(\overline{E}) = \mathrm{Cap}_{\alpha,s}(E) = \int_{\mathbb{R}^n} V^E d\mu^E = \int_{\mathbb{R}^n} (I_\alpha * \mu^E)^{\frac{s}{s-1}} dx,$$

$$V^E \geq 1 \quad \text{quasieverywhere on } E, \quad \text{and} \quad V^E \leq A \quad \text{on } \mathbb{R}^n.$$

We have the following homogeneous version of Lemma 3.1.3.

Lemma 3.4.1. *Let E, μ^E, and V^E be as above with $0 < \alpha s < n$. If $\delta \in (1, n/(n-\alpha))$ for $s < 2$ and $\delta \in (s-1, n(s-1)/(n-\alpha s))$ for $s \geq 2$, then the function $(V^E)^\delta \in A_1$ with $[(V^E)^\delta]_{A_1} \leq c(n, \alpha, s, \delta)$. Moreover, $(V^E)^\delta \in \dot{L}^1(C)$ with $\|(V^E)^\delta\|_{\dot{L}^1(C)} \leq C \operatorname{cap}_{\alpha,s}(E)$.*

This lemma follows from Lemmas 2 and 3 of [MS1, Sub-section 2.6.3]. The following 'renorming' theorem for $\dot{M}_p^{\alpha,s}$ can be proved as in the inhomogeneous setting.

Theorem 3.4.2. *For $p > 1$ and $\alpha > 0, s > 1$, with $\alpha s < n$, we have*

$$\|f\|_{\dot{M}_p^{\alpha,s}} \simeq \sup_w \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p}, \tag{3.46}$$

where the supremum is taken over all nonnegative $w \in \dot{L}^1(C) \cap A_1$ with $\|w\|_{\dot{L}^1(C)} \leq 1$ and $[w]_{A_1} \leq \overline{\mathbf{c}}(n, \alpha, s)$ for a constant $\overline{\mathbf{c}}(n, \alpha, s) \geq 1$. The equivalence (3.46) *also holds if we replace $\dot{L}^1$ by $\dot{\mathcal{L}}^1$. Moreover, we have*

$$\|f\|_{\dot{M}_p^{\alpha,s}} = \sup_w \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p},$$

where the supremum is taken over all weights w such that w is defined $\operatorname{cap}_{\alpha,s}$- quasieverywhere and $\int_0^\infty \operatorname{cap}_{\alpha,s}(\{x \in \mathbb{R}^n : w(x) > t\}) dt \leq 1$.

The homogeneous versions of $N_q^{\alpha,s}, \mathcal{N}_q^{\alpha,s}, \widetilde{N}_q^{\alpha,s}$, and $B_q^{\alpha,s}$ are denoted by $\dot{N}_q^{\alpha,s}$, $\dot{\mathcal{N}}_q^{\alpha,s}, \dot{\widetilde{N}}_q^{\alpha,s}$, and $\dot{B}_q^{\alpha,s}$, respectively. They are defined similarly using $\operatorname{cap}_{\alpha,s}$ in place of $\operatorname{Cap}_{\alpha,s}$. We have the following relations.

Theorem 3.4.3. *Let $p > 1$ and $\alpha > 0, s > 1$, with $\alpha s < n$. Then*

$$\left(\dot{N}_{p'}^{\alpha,s}\right)^* \approx \left(\dot{\mathcal{N}}_{p'}^{\alpha,s}\right)^* \approx \dot{M}_p^{\alpha,s} = \left(\dot{\widetilde{N}}_{p'}^{\alpha,s}\right)^* = \left(\dot{B}_{p'}^{\alpha,s}\right)^* = \left[(\dot{M}_p^{\alpha,s})'\right]^*. \tag{3.47}$$

Moreover, the spaces $\dot{\mathcal{N}}_{p'}^{\alpha,s}$ and $(\dot{M}_p^{\alpha,s})'$ are Banach function spaces and $\dot{\mathcal{N}}_{p'}^{\alpha,s} \approx (\dot{M}_p^{\alpha,s})'$. Additionally, if $\operatorname{cap}_{\alpha,s}$ is strongly subadditive then $\dot{N}_{p'}^{\alpha,s}$, $\dot{\widetilde{N}}_{p'}^{\alpha,s}$, and $\dot{B}_{p'}^{\alpha,s}$ are also

Banach function spaces, and

$$\dot{\mathcal{N}}_{p'}^{\alpha,s} \approx \dot{N}_{p'}^{\alpha,s} \approx (\dot{M}_p^{\alpha,s})' = \dot{\tilde{N}}_{p'}^{\alpha,s} = \dot{B}_{p'}^{\alpha,s}.$$

In general, if no strong subadditivity is assumed on $\mathrm{cap}_{\alpha,s}$ *then we have*

$$\dot{\mathcal{N}}_{p'}^{\alpha,s} \approx \dot{N}_{p'}^{\alpha,s} \approx (\dot{M}_p^{\alpha,s})' \approx \dot{\tilde{N}}_{p'}^{\alpha,s} \approx \dot{B}_{p'}^{\alpha,s}. \tag{3.48}$$

Note that by (3.45) and (3.47), all spaces in (3.48) are continuously embedded into the Lorentz space $L^{\frac{np}{np-\alpha s},1}$.

The homogeneous version of Theorem 3.2.9 reads as follows.

Theorem 3.4.4. *Let* $p > 1$, $\alpha > 0$, $s > 1$, *with* $\alpha s < n$. *We have*

$$(\mathring{\dot{M}}_p^{\alpha,s})^* \approx \dot{\mathcal{N}}_{p'}^{\alpha,s},$$

where we define $\mathring{\dot{M}}_p^{\alpha,s}$ *as the closure of* C_c *in* $\dot{M}_p^{\alpha,s}$.

It is known that the Hardy-Littlewood maximal function $\mathbf{M}$ and standard Calderón-Zygmund operators are bounded on $\dot{M}_p^{\alpha,s}$ (see [MV] and [MS1]). For other spaces, we have the following results.

Theorem 3.4.5. *Let* $q > 1$, $\alpha > 0$, $s > 1$, *and* $\alpha s < n$. *Suppose that* T *is an operator (not necessarily linear or sublinear) such that*

$$\int |T(f)|^q w dx \leq C_1 \int |f|^q w dx$$

holds for all $f \in L^q(w)$ *and all* $w \in A_1$, *with a constant* C_1 *depending only on* n, q, *and the bound for the* A_1 *constant of* w. *Then* T *is bounded on* $\dot{N}_q^{\alpha,s}, \dot{\mathcal{N}}_q^{\alpha,s}$, $(\dot{M}_{q'}^{\alpha,s})'$, $\dot{\tilde{N}}_q^{\alpha,s}$, *and* $\dot{B}_q^{\alpha,s}$.

Chapter 4. Further Applications

4.1. Adams' Conjecture

In [Ad4], Adams conjectured (in the context of Riesz capacities and Riesz potentials) that another capacitary strong type inequality

$$\int_{\mathbb{R}^n} (G_\alpha * f)dC \leq A \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} dx \tag{4.1}$$

holds for any nonnegative Lebesgue measurable function f (see [Ad2, Equ. (3.11)]). (The integral $\int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} dx$ is understood as ∞ whenever $f = \infty$ on a set of positive Lebesgue measure. In the case $f \equiv 0$, it is understood as 0). Moreover, he essentially showed for the corresponding Riesz capacities and potentials that this is true provided α is an *integer* in $(0, n)$ (see [Ad2, p. 23]). However, we observe that his argument does not appear to work for Bessel capacities and Bessel potentials as in (4.1) even with integers $\alpha \in (0, n)$.

One of the main purposes of this section is to verify (4.1) for any real $\alpha > 0$.

Theorem 4.1.1. *Let $\alpha > 0$ and $s > 1$ be such that $\alpha s \leq n$. There exists some constant $A > 0$ such that* (4.1) *holds for any nonnegative Lebesgue measurable function f.*

Proof. Let $L^1(C)$ denote the space of quasicontinuous function f in $\mathbb{R}^n$ such that

$$\|f\|_{L^1(C)} := \int_{\mathbb{R}^n} |f| dC < +\infty.$$

Recall a function f is said to be quasicontinuous (with respect to $\text{Cap}_{\alpha,s}$) if for any $\epsilon > 0$ there exists an open set O such that $\text{Cap}_{\alpha,s}(O) < \epsilon$ and f is continuous in $O^c := \mathbb{R}^n \setminus O$. It is known that the dual of $L^1(C)$ can be identify with the space $\mathfrak{M}^{\alpha,s} = \mathfrak{M}^{\alpha,s}(\mathbb{R}^n)$ which consists of locally finite signed measures μ in $\mathbb{R}^n$ such that the norm $\|\mu\|_{\mathfrak{M}^{\alpha,s}} < +\infty$ (see

Theorem 2.1.3). Here we define

$$\|\mu\|_{\mathfrak{M}^{\alpha,s}} := \sup_K \frac{|\mu|(K)}{\text{Cap}_{\alpha,s}(K)},$$

where the supremum is taken over all compact sets $K \subset \mathbb{R}^n$ such that $\text{Cap}_{\alpha,s}(K) \neq 0$.

By Proposition 2.2.2, $L^1(C)$ is normable and thus it follows from Hahn-Banach Theorem that for any $u \in L^1(C)$ we have

$$\|u\|_{L^1(C)} \simeq \sup\left\{\left|\int ud\mu\right| : \|\mu\|_{\mathfrak{M}^{\alpha,s}} \leq 1\right\}. \tag{4.2}$$

Let f be a nonnegative measurable and bounded function with compact support. Applying (4.2) with $u = G_\alpha * f$ we have

$$\begin{aligned}
\int_{\mathbb{R}^n} G_\alpha * f dC &\leq A \sup_{\|\mu\|_{\mathfrak{M}^{\alpha,s}} \leq 1} \int G_\alpha * f d|\mu| \\
&= A \sup_{\|\mu\|_{\mathfrak{M}^{\alpha,s}} \leq 1} \int (G_\alpha * |\mu|) f dx \\
&\leq A \|f\|_{(M^{\alpha,s}_{s'})'} \sup_{\|\mu\|_{\mathfrak{M}^{\alpha,s}} \leq 1} \|G_\alpha * |\mu|\|_{M^{\alpha,s}_{s'}} \\
&\leq A \|f\|_{(M^{\alpha,s}_{s'})'},
\end{aligned}$$

where the last inequality follows from [MV, Theorem 1.2]. By density (see Remark 3.2.10) we see that the inequality

$$\int_{\mathbb{R}^n} G_\alpha * f dC \leq A \|f\|_{(M^{\alpha,s}_{s'})'} \tag{4.3}$$

holds for any nonnegative function $f \in (M^{\alpha,s}_{s'})'$.

In proving (4.1) we may assume that $\int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} dx < +\infty$ and hence f is finite a.e. by our convention. In this case we must have that $f \in (M^{\alpha,s}_{s'})'$. Indeed, for any $g \in M^{\alpha,s}_{s'}$ such that $\|g\|_{M^{\alpha,s}_{s'}} \leq 1$ by [KV], there exists a nonnegative function $u \in L^{s'}_{\text{loc}}(\mathbb{R}^n)$

such that

$$u = G_\alpha * (u^{s'}) + \frac{|g|}{M} \qquad \text{a.e.}$$

for a constant $M > 0$ independent of g and u. Thus, as in [BP] (see also [KV]), we have

$$\begin{aligned}
\int_{\mathbb{R}^n} f|g|dx &= M \int_{\mathbb{R}^n} f(u - G_\alpha * (u^{s'}))dx \qquad (4.4)\\
&= M \int_{\mathbb{R}^n} (fu - u^{s'} G_\alpha * f)dx \\
&= M \int_{\mathbb{R}^n} G_\alpha * f \left(u \frac{f}{G_\alpha * f} - u^{s'} \right) dx \\
&\leq M s^{-s}(s-1)^{s-1} \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} dx,
\end{aligned}$$

where we used the Young's inequality $ab - a^{s'}/s' \leq b^s/s$, $a, b \geq 0$, in the last inequality. Thus taking the supremum over $g \in M_{s'}^{\alpha,s}$ such that $\|g\|_{M_{s'}^{\alpha,s}} \leq 1$ in (4.4), we find

$$\|f\|_{(M_{s'}^{\alpha,s})'} \leq C \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} dx < +\infty. \qquad (4.5)$$

Finally, combining (4.3) with (4.5) we obtain (4.1) as desired. □

4.2. Various Characterizations of $L^1(C)$

For a q.e. defined function u in $\mathbb{R}^n$, recall that

$$\gamma_{\alpha,s}(u) := \inf \left\{ \int f^s dx : 0 \leq f \in L^s(\mathbb{R}^n) \text{ and } G_\alpha * f \geq |u|^{\frac{1}{s}} \text{ q.e.} \right\}.$$

We further define the following quantities:

$$\lambda_{\alpha,s}(u) := \inf \left\{ \|f\|_{(M_{s'}^{\alpha,s})'} : 0 \leq f \in (M_{s'}^{\alpha,s})' \text{ and } G_\alpha * f \geq |u| \text{ q.e.} \right\}$$

and

$$\beta_{\alpha,s}(u) := \inf \left\{ \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} dx : f \geq 0,\, G_\alpha * f \geq |u| \text{ q.e.} \right\}.$$

Recall the important equivalence that

$$\int_{\mathbb{R}^n} |u| dC \simeq \gamma_{\alpha,s}(u). \tag{4.6}$$

Theorem 4.2.1. *Let $\alpha > 0$ and $s > 1$ be such that $\alpha s \leq n$. For any q.e. defined function u in $\mathbb{R}^n$ it holds that*

$$\int_{\mathbb{R}^n} |u| dC \simeq \lambda_{\alpha,s}(u) \simeq \beta_{\alpha,s}(u). \tag{4.7}$$

In particular, we have

$$\mathrm{Cap}_{\alpha,s}(E) \simeq \lambda_{\alpha,s}(\chi_E) \simeq \beta_{\alpha,s}(\chi_E)$$

for any set $E \subset \mathbb{R}^n$.

In order to prove Theorem 4.2.1, we first prove the following "integration by parts" lemma.

Lemma 4.2.2. *Let $\alpha > 0, s > 1$ be such that $\alpha s \leq n$. Suppose that μ is a nonnegative measure such that the diameter of* $\mathrm{supp}(\mu)$ *is less than* 1*. Then there is a constant $C = C(n, \alpha, s) > 0$ such that, for $f = (G_\alpha * \mu)^{s'-1}$, we have*

$$(G_\alpha * f)^s \leq C G_\alpha * [f(G_\alpha * f)^{s-1}]$$

pointwise everywhere in $\mathbb{R}^n$.

Remark 4.2.3. *For Riesz potentials, this lemma has been established for all $f \geq 0$ in [VW] (see also [KV, Ver]). In our setting, which deals with Bessel potentials, it is necessary to require μ to have compact support.*

Proof of Lemma 4.2.2. Without loss of generality, we may assume that $\mathrm{supp}(\mu) \subset B_{1/2}(0)$. With $f = (G_\alpha * \mu)^{s'-1}$, we write $f = f_1 + f_2$, where

$$f_1 = f\chi_{B_3(0)} \quad \text{and} \quad f_2 = f\chi_{B_3(0)^c} \quad (B_3(0)^c = \mathbb{R}^n \setminus B_3(0)).$$

Then

$$(G_\alpha * f)^s \le C[(G_\alpha * f_1)^s + (G_\alpha * f_2)^s]. \tag{4.8}$$

We shall use the following pointwise two-sided estimates for G_α (see, e.g., [AH, Section 1.2.4]):

$$G_\alpha(x) \simeq |x|^{\alpha-n}, \quad \forall |x| \le 15, (0 < \alpha < n). \tag{4.9}$$

and

$$G_\alpha(x) \simeq G_\alpha(x+y), \quad \forall |x| \ge 3, |y| \le 1, (\alpha > 0). \tag{4.10}$$

We mention that (4.10) follows from the asymptotic behavior G_α near infinity that can be found, e.g., in [AH, Equ. 1.2.24].

We now write

$$[G_\alpha * f_1(x)]^s = \int_{|y|\le 3} G_\alpha(x-y) f(y) \left[\int_{|z|\le 3} G_\alpha(x-z) f(z) dz \right]^{s-1} dy.$$

Thus if $|x| \ge 10$, then $|x-z| \ge 7 \ge |y-z|$, which yields that

$$G_\alpha(x-z) \le G_\alpha(y-z).$$

Therefore, we get

$$[G_\alpha * f_1(x)]^s \le G_\alpha * [f(G_\alpha * f)^{s-1}](x)$$

in the case $|x| \ge 10$.

On the other hand, if $|x| < 10$, then for $|y| \le 3$ by (4.9) we have

$$G_\alpha(x-y) \simeq |x-y|^{\alpha-n}.$$

Thus applying [Ver, Lemma 2.1] we obtain

$$[G_\alpha * f_1(x)]^s \le C G_\alpha * [f_1 (G_\alpha * f_1)^{s-1}](x) \le C G_\alpha * [f(G_\alpha * f)^{s-1}](x)$$

in the case $|x| < 10$.

Combining these two estimates we get that

$$[G_\alpha * f_1(x)]^s \le C G_\alpha * [f(G_\alpha * f)^{s-1}](x), \quad \forall x \in \mathbb{R}^n. \tag{4.11}$$

To estimate $[G_\alpha * f_2(x)]^s$ we first observe the following bound

$$f_2(x) \le C G_\alpha * f(x), \quad \forall x \in \mathbb{R}^n. \tag{4.12}$$

Inequality (4.12) is trivial when $|x| < 3$. On the other hand, for $|x| \ge 3$, we have by (4.10),

$$\begin{aligned}(f_2(x))^{s-1} &= \int_{|y|<1/2} G_\alpha(x-y) d\mu(y) \le C \int_{|y|<1/2} G_\alpha(x) d\mu(y) \\ &= C \|\mu\| \, G_\alpha(x).\end{aligned}$$

Note that for $|y - x| < 1/2$ and $|x| \ge 3$, by (4.10) we have

$$f(y)^{s-1} = \int_{|z|<1/2} G_\alpha(y-z) d\mu(z) \ge c_0 \, G_\alpha(x) \, \|\mu\| \,,$$

and so, for $|x| \ge 3$,

$$\begin{aligned} G_\alpha * f(x) &\ge \int_{|y-x|<1/2} G_\alpha(x-y) f(y) dy \\ &\ge \int_{|y-x|<1/2} G_\alpha(x-y) (c_0 \, G_\alpha(x) \, \|\mu\|)^{s'-1} dy \\ &\ge c \, (\|\mu\| \, G_\alpha(x))^{s'-1} \ge c_1 f_2(x). \end{aligned}$$

Thus (4.12) is verified. Now by Hölder's inequality and (4.12) we have

$$[G_\alpha * f_2]^s \le C G_\alpha * (f_2^s) \le C G_\alpha * [f(G_\alpha * f)^{s-1}]. \tag{4.13}$$

At this point, combining (4.8), (4.11), and (4.13), we obtain the lemma. □

We are now ready to prove Theorem 4.7.

Proof of Theorem 4.7. Let u be a q.e. defined function in $\mathbb{R}^n$. Suppose that f is a nonnegative measurable function such that $G_\alpha * f \geq |u|$ quasi-everywhere. Then by (4.3) and (4.5) it follows that

$$\int_{\mathbb{R}^n} |u| dC \leq \int_{\mathbb{R}^n} G_\alpha * f dC \leq A_1 \|f\|_{(M_{s'}^{\alpha,s})'} \leq A_2 \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} ds.$$

Now taking the infimum over such f we arrive at

$$\int_{\mathbb{R}^n} |u| dC \lesssim \lambda_{\alpha,s}(u) \lesssim \beta_{\alpha,s}(u).$$

Thus to complete the proof, it is left to show that

$$\beta_{\alpha,s}(u) \lesssim \int_{\mathbb{R}^n} |u| dC. \tag{4.14}$$

To this end, we first show (4.14) for $u = \chi_E$, where E is any set such that $\mathrm{Cap}_{\alpha,s}(E) > 0$ and the diameter of E is less than 1. By [AH, Theorems 2.5.6 and 2.6.3] one can find a nonnegative measure $\mu = \mu^E$ with $\mathrm{supp}(\mu) \subset \overline{E}$ (called capacitary measure for E) such that the function $V^E = G_\alpha * ((G_\alpha * \mu)^{s'-1})$ satisfies the following properties:

$$\mu^E(\overline{E}) = \mathrm{Cap}_{\alpha,s}(E) = \int_{\mathbb{R}^n} V^E d\mu^E = \int_{\mathbb{R}^n} (G_\alpha * \mu^E)^{s'} dx,$$

and

$$V^E \geq 1 \quad \text{quasieverywhere on } E.$$

Let $f = (G_\alpha * \mu)^{s'-1}$. By Lemma 4.2.2, we have

$$\chi_E \leq (V^E)^s = (G_\alpha * f)^s \leq C G_\alpha * [f(G_\alpha * f)^{s-1}] \quad \text{q.e.}$$

Thus,

$$\begin{aligned}\beta_{\alpha,s}(\chi_E) &\le C\int_{\mathbb{R}^n} f^s(G_\alpha * f)^{(s-1)s}\left\{G_\alpha * [f(G_\alpha * f)^{s-1}]\right\}^{1-s} dx\\ &\le C\int_{\mathbb{R}^n} f^s(G_\alpha * f)^{(s-1)s}(G_\alpha * f)^{(1-s)s} dx\\ &= C\int_{\mathbb{R}^n} f^s dx = C\int_{\mathbb{R}^n}(G_\alpha * \mu)^{s'} dx = C\,\mathrm{Cap}_{\alpha,s}(E),\end{aligned}$$

as desired.

We now let $\{\mathcal{B}^j\}_{j\ge 0}$ be a covering of $\mathbb{R}^n$ by open balls with unit diameter. This covering is chosen in such a way that it has a finite multiplicity depending only on n. We shall use the following quasi-additivity of $\mathrm{Cap}_{\alpha,s}$:

$$\sum_{j\ge 0}\mathrm{Cap}_{\alpha,s}(E\cap\mathcal{B}^j) \le M\mathrm{Cap}_{\alpha,s}(E) \tag{4.15}$$

for any set $E\subset\mathbb{R}^n$. For compact sets E, a proof of (4.15) can be found in [MS2, Proposition 3.1.5]. The same proof also works for any set E provided one uses [AH, Corollary 2.6.8].

In proving (4.14) we may assume that $\int_{\mathbb{R}^n}|u|dC < +\infty$. Let $E_k = \{2^{k-1} < |u| \le 2^k\}$ and $E_{j,k} = E_k\cap\mathcal{B}^j$ for $k\in\mathbb{Z}$ and $j\ge 0$. We have

$$\beta_{\alpha,s}(u) = \beta_{\alpha,s}\left(\sum_{k\in\mathbb{Z}}|u|\chi_{E_k}\right) \le \beta_{\alpha,s}\left(\sum_{k\in\mathbb{Z}}\sum_{j\ge 0}|u|\chi_{E_{j,k}}\right). \tag{4.16}$$

For $k\in\mathbb{Z}$ and $j\ge 0$, let

$$f_{j,k} = (G_\alpha * \mu^{E_{j,k}})^{s'-1} \quad\text{and}\quad F_{j,k} = f_{j,k}(G_\alpha * f_{j,k})^{s-1}.$$

By the above argument, we have

$$G_\alpha * (2^k F_{jk}) \ge c\,|u|\chi_{E_{j,k}} \quad \text{q.e.}$$

and

$$\int_{\mathbb{R}^n} (2^k F_{jk})^s (G_\alpha * (2^k F_{j,k}))^{1-s} dx \leq C 2^k \mathrm{Cap}_{\alpha,s}(E_{jk}).$$

By (4.5), this gives

$$\left\| 2^k F_{j,k} \right\|_{(M_{s'}^{\alpha,s})'} \leq C 2^k \mathrm{Cap}_{\alpha,s}(E_{jk}). \tag{4.17}$$

Set $F = \sup_{j,k} 2^k F_{j,k}$. Then we have $(G_\alpha * F)^{1-s} \leq (G_\alpha * (2^k F_{j,k}))^{1-s}$ for any $k \in \mathbb{Z}$ and $j \geq 0$. Moreover,

$$G_\alpha * F \geq c \sum_{k \in \mathbb{Z}} |u| \chi_{E_k} \geq c_1 \sum_{k \in \mathbb{Z}} \sum_{j \geq 0} |u| \chi_{E_{j,k}} \qquad \text{q.e.}$$

due to the finite multiplicity of $\{\mathcal{B}^j\}_{j \geq 0}$. Also, it follows from (4.15) and (4.17) that

$$\begin{aligned} \|F\|_{(M_{s'}^{\alpha,s})'} &\leq C \sum_{k \in \mathbb{Z}} \sum_{j \geq 0} 2^k \mathrm{Cap}_{\alpha,s}(E_{j,k}) \leq C_1 \sum_{k \in \mathbb{Z}} 2^k \mathrm{Cap}_{\alpha,s}(E_k) \\ &\leq C \int_{\mathbb{R}^n} |u| dC < +\infty. \end{aligned}$$

In particular, F is finite a.e. and thus there is a set N such that $|N| = 0$ and

$$\mathbb{R}^n = \cup_{k \in \mathbb{Z}, j \geq 0} \{0 < F \leq 2^{k+1} F_{j,k}\} \cup \{F = 0\} \cup N.$$

Thus we find

$$\begin{aligned} \beta_{\alpha,s} \left(\sum_{k \in \mathbb{Z}} \sum_{j \geq 0} |u| \chi_{E_{j,k}} \right) &\leq C \int_{\mathbb{R}^n} F^s (G_\alpha * F)^{1-s} dx \\ &\leq C \sum_{k \in \mathbb{Z}} \sum_{j \geq 0} \int_{\{0 < F \leq 2^{k+1} F_{j,k}\}} F^s (G_\alpha * F)^{1-s} dx \\ &\leq C \sum_{k \in \mathbb{Z}} \sum_{j \geq 0} \int_{\mathbb{R}^n} (2^k F_{j,k})^s (G_\alpha * (2^k F_{j,k}))^{1-s} dx \\ &\leq C \sum_{k \in \mathbb{Z}} \sum_{j \geq 0} 2^k \mathrm{Cap}_{\alpha,s}(E_{jk}) \leq C \int_{\mathbb{R}^n} |u| dC. \end{aligned}$$

Inequality (4.14) now follows from (4.16) and the last bound, which completes the proof of the theorem. □

Remark 4.2.4. *For Riesz potentials* $I_\alpha * f$ *and Riesz capacities* $\mathrm{cap}_{\alpha,s}$, $\alpha \in (0,n), s > 1$, *the corresponding bound* (4.14) *can be obtained using* (4.6) *and the pointwise bound*

$$(I_\alpha * f)^s \leq C I_\alpha * [f(I_\alpha * f)^{s-1}], \tag{4.18}$$

which holds for all nonnegative measurable functions f *(see [VW, Ver]). Indeed, for any* $f \geq 0$ *such that* $I_\alpha * f \geq |u|^{\frac{1}{s}}$ *q.e., by* (4.18) *we have* $C I_\alpha * [f(I_\alpha * f)^{s-1}] \geq |u|$ *q.e., and thus again by* (4.18),

$$\begin{aligned} \beta_{\alpha,s}(u) &\leq C \int_{\mathbb{R}^n} f^s (I_\alpha * f)^{(s-1)s} I_\alpha * [f(I_\alpha * f)^{s-1}]^{1-s} dx \\ &\leq C \int_{\mathbb{R}^n} f^s dx. \end{aligned}$$

Minimizing over such f *and recalling* (4.6), *we get the corresponding bound* (4.14) *as desired.*

We end this section by an easy application of Theorem 4.2.1.

Theorem 4.2.5. *Let* $\alpha > 0$ *and* $s > 1$ *be such that* $\alpha s \leq n$. *For any* $q > (n-\alpha)/n$ *and any measurable and q.e. defined function* f, *we have*

$$\int_{\mathbb{R}^n} (\mathbf{M}^{\mathrm{loc}} f)^q dC \leq C(n,\alpha,s,q) \int_{\mathbb{R}^n} |f|^q dC.$$

Proof. By Theorem 4.2.1, we have

$$\int_{\mathbb{R}^n} |f|^q dC \simeq \inf \left\{ \int_{\mathbb{R}^n} h^s (G_\alpha * h)^{1-s} dx : h \geq 0, (G_\alpha * h)^{\frac{1}{q}} \geq |f| \text{ q.e.} \right\}.$$

On the other hand, for any $h \geq 0$ and $(G_\alpha * h)^{\frac{1}{q}} \geq |f|$ q.e. by Theorem 3.1.2 we have

$$\mathbf{M}^{\mathrm{loc}} f \leq \mathbf{M}^{\mathrm{loc}} [(G_\alpha * h)^{\frac{1}{q}}] \leq C (G_\alpha * h)^{\frac{1}{q}}$$

pointwise everywhere, provided $q > (n-\alpha)/n$. Thus

$$\int_{\mathbb{R}^n} |f|^q dC \geq c \inf \left\{ \int_{\mathbb{R}^n} g^s (G_\alpha * g)^{1-s} dx : g \geq 0, (G_\alpha * g)^{\frac{1}{q}} \geq \mathbf{M}^{\text{loc}} f \text{ q.e.} \right\}$$
$$\simeq \int_{\mathbb{R}^n} (\mathbf{M}^{\text{loc}} f)^q dC.$$

This completes the proof of the theorem. □

We note that the bound $q > (n-\alpha)/n$ in Theorem 4.2.5 is not sharp. A sharp version will be discussed in the next section.

4.3. Boundedness of Local Hardy-Littlewood Maximal Function on $L^p(C)$

Let us review some basic terminologies that have been introduced in Section 2.2. For $0 < p < \infty$, we denote by $L^p(C)$ the space of all q.e. defined functions u in $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} |u|^p dC < +\infty$, with quasi-norm

$$\|u\|_{L^p(C)} := \left(\int_{\mathbb{R}^n} |u|^p dC \right)^{\frac{1}{p}}.$$

The 'weak' version of $L^p(C)$ is denoted by $L^{p,\infty}(C)$ which consists of all q.e. defined functions u in $\mathbb{R}^n$ such that $\|u\|_{L^{p,\infty}(C)} < +\infty$, where

$$\|u\|_{L^{p,\infty}(C)} := \sup_{\lambda > 0} \lambda \operatorname{Cap}_{\alpha,s}(\{|u| > \lambda\})^{\frac{1}{p}}.$$

Theorem 4.3.1. *For $s > 1$ and $0 < \alpha < n/s$, let $p = \frac{n-\alpha s}{n}$. Then for any measurable function $f \in L^p(C)$, it holds that*

$$\left\| \mathbf{M}^{\text{loc}}(f) \right\|_{L^{p,\infty}(C)} \leq A \|f\|_{L^p(C)}, \tag{4.19}$$

where the Choquet integral is associated to the Bessel capacity $\operatorname{Cap}_{\alpha,s}$.

It is obvious that if $|f(x)| \leq M$ a.e. then $\mathbf{M}^{\text{loc}}(f)(x) \leq M$ everywhere. Thus by Theorem 4.3.1 and interpolating we obtain the following strong type estimate for the Hardy-Littlewood maximal function.

Theorem 4.3.2. *Let $\alpha > 0$ and $s > 1$ be such that $\alpha s < n$. For any $q > (n-\alpha s)/n$ and any measurable function $f \in L^q(C)$, we have*

$$\left\|\mathbf{M}^{\mathrm{loc}}(f)\right\|_{L^q(C)} \leq A \left\|f\right\|_{L^q(C)},$$

where the Choquet integral is associated to the Bessel capacity $\mathrm{Cap}_{\alpha,s}$.

The homogeneous versions corresponding to Theorems 4.3.1 4.3.2 are given by the following.

Theorem 4.3.3. *For $s > 1$ and $0 < \alpha < n/s$, let $p = \frac{n-\alpha s}{n}$. Then for any measurable function $f \in L^p(C)$, it holds that*

$$\left\|\mathbf{M}(f)\right\|_{L^{p,\infty}(C)} \leq A \left\|f\right\|_{L^p(C)}, \tag{4.20}$$

where the Choquet integral is associated to the Riesz capacity $\mathrm{cap}_{\alpha,s}$.

Theorem 4.3.4. *Let $\alpha > 0$ and $s > 1$ be such that $\alpha s < n$. For any $q > (n-\alpha s)/n$ and any measurable function $f \in L^q(C)$, we have*

$$\left\|\mathbf{M}(f)\right\|_{L^q(C)} \leq A \left\|f\right\|_{L^q(C)},$$

where the Choquet integral is associated to the Riesz capacity $\mathrm{cap}_{\alpha,s}$.

Remark 4.3.5. *It is worth mentioning that Theorem 4.3.2 also holds in the case $\alpha s = n$. Indeed, by adapting the proof of Theorem 4.3.1 to the case $\alpha s = n$, we can show that* (4.19) *holds for any $p = \epsilon \in (0,1)$. Then interpolation yields the result of Theorem 4.2.5 in the case $\alpha s = n$.*

Several preliminary results are needed to prove Theorem 4.3.1. We first start with a potential theoretic one. Let η be a nonnegative function in $C_c^\infty(B_1(0))$ such that $\eta(x) \leq 1$ for $|x| \leq 1$ and $\eta(x) \geq 1/2$ for $|x| \geq 1/2$. Also, let $\eta_m(x) = 2^{mn}\eta(2^n x)$ for $m \in \mathbb{Z}$.

Following [AH], for any nonnegative measure μ in $\mathbb{R}^n$, we define a nonlinear potential of μ by

$$\mathcal{V}^{\mu}_{\alpha,s}(x) := \sum_{m=0}^{\infty} 2^{-m\alpha s'} \eta_m * (\eta_m * \mu)^{s'-1}.$$

We mention that for $s > 2 - \frac{\alpha}{n}$ one can also use the Havin-Maz'ya potential $V^{\mu}_{\alpha,s}(x) := G_\alpha * (G_\alpha * \mu)^{s'-1}$ instead. But this potential does not serve our purpose well in the case $1 < s \leq 2 - \frac{\alpha}{n}$ (see the remark after [AH, Proposition 6.3.12]).

Recall from [AH, Section 4.5] that $\mathcal{V}^{\mu}_{\alpha,s}$ is comparable to a Wolff's potential in the sense that

$$c_1 \sum_{m=2}^{\infty} 2^{m(n-\alpha s)} \mu(B_{2^{-m}}(x))^{s'-1} \leq \mathcal{V}^{\mu}_{\alpha,s}(x)$$

and

$$\mathcal{V}^{\mu}_{\alpha,s}(x) \leq c_2 \sum_{m=-1}^{\infty} 2^{m(n-\alpha s)} \mu(B_{2^{-m}}(x))^{s'-1}. \tag{4.21}$$

We define the truncated Wolff's potential $W^{R}_{\alpha,s}(\mu)$, $0 < R \leq \infty$, of a nonnegative measure μ by

$$W^{R}_{\alpha,s}(\mu)(x) := \int_0^R \left(\frac{\mu(B_\rho(x))}{\rho^{n-\alpha s}} \right)^{\frac{1}{s-1}} \frac{d\rho}{\rho}, \qquad x \in \mathbb{R}^n.$$

With this, we see that (4.21) implies that

$$\mathcal{V}^{\mu}_{\alpha,s}(x) \leq C\, W^{4}_{\alpha,s}(\mu)(x). \tag{4.22}$$

We shall need the following estimate.

Lemma 4.3.6. *Let $s > 1$, $\alpha > 0$, and $\alpha s < n$. For any nonnegative measure μ and any $0 < r \leq 1$, $x_0 \in \mathbb{R}^n$, we have*

$$\left\| \mathcal{V}^{\mu}_{\alpha,s} \right\|_{L^{\frac{(s-1)n}{n-\alpha s},\infty}(B_r(x_0))} \leq A |B_r(x_0)|^{\frac{n-\alpha s}{(s-1)n}} W^{8}_{\alpha,s}(\mu)(x_0),$$

where A is a constant independent of μ, r, and x_0. Here $L^{\frac{(s-1)n}{n-\alpha s},\infty}(B_r(x_0))$ stands for the weak $L^{\frac{(s-1)n}{n-\alpha s}}$ space over the ball $B_r(x_0)$ (with respect to the Lebesgue measure).

Proof. Let $t = \frac{(s-1)n}{n-\alpha s}$. In view of (4.22), it is enough to show that

$$\left\|W^4_{\alpha,s}(\mu)(\cdot)\right\|_{L^{t,\infty}(B_r(x_0))} \leq C\,|B_r(x_0)|^{\frac{1}{t}} W^8_{\alpha,s}(\mu)(x_0) \tag{4.23}$$

for any $0 < r \leq 1$ and $x_0 \in \mathbb{R}^n$.

It is obvious that

$$\left\|W^4_{\alpha,s}(\mu)(\cdot)\right\|_{L^{t,\infty}(B_r(x_0))} \leq C(P_1 + P_2), \tag{4.24}$$

where

$$P_1 = \left\|\int_0^r \left(\frac{\mu(B_\rho(\cdot))}{\rho^{n-\alpha s}}\right)^{\frac{1}{s-1}} \frac{d\rho}{\rho}\right\|_{L^{t,\infty}(B_r(x_0))}$$

and

$$P_2 = \left\|\int_r^4 \left(\frac{\mu(B_\rho(\cdot))}{\rho^{n-\alpha s}}\right)^{\frac{1}{s-1}} \frac{d\rho}{\rho}\right\|_{L^{t,\infty}(B_r(x_0))}.$$

To bound P_2 we observe that if $x \in B_r(x_0)$ and $\rho \geq r$ then $B_\rho(x) \subset B_{2\rho}(x_0)$, and so

$$\begin{aligned} P_2 &\leq \left\|\int_r^4 \left(\frac{\mu(B_{2\rho}(x_0))}{\rho^{n-\alpha s}}\right)^{\frac{1}{s-1}} \frac{d\rho}{\rho}\right\|_{L^{t,\infty}(B_r(x_0))} \\ &\leq C\,|B_r(x_0)|^{\frac{1}{t}} W^8_{\alpha,s}(\mu)(x_0). \end{aligned}$$

To bound P_1, we let $\tilde{\mu}$ be the restriction of μ to the ball $B_{2r}(x_0)$. Observe that for $x \in B_r(x_0)$ we have

$$\begin{aligned} \int_0^r \left(\frac{\mu(B_\rho(x))}{\rho^{n-\alpha s}}\right)^{\frac{1}{s-1}} \frac{d\rho}{\rho} &= \int_0^r \left(\frac{\tilde{\mu}(B_\rho(x))}{\rho^{n-\alpha s}}\right)^{\frac{1}{s-1}} \frac{d\rho}{\rho} \\ &\leq C\tilde{\mu}(B_{2r}(x_0))^{\frac{\alpha s}{(s-1)n}} \mathbf{M}(\tilde{\mu})(x)^{\frac{1}{t}}, \end{aligned}$$

where the last bound follows as in the proof of [AH, Proposition 3.1]. By the weak type (1,1) bound for $\mathbf{M}$, this yields

$$
\begin{aligned}
P_1 &\le C\tilde{\mu}(B_{2r}(x_0))^{\frac{\alpha s}{(s-1)n}}\tilde{\mu}(B_{2r}(x_0))^{\frac{1}{t}} \\
&\le C\,|B_r(x_0)|^{\frac{1}{t}}\left(\frac{\mu(B_{2r}(x_0))}{r^{n-\alpha s}}\right)^{\frac{1}{s-1}} \\
&\le C\,|B_r(x_0)|^{\frac{1}{t}}W^{8}_{\alpha,s}(\mu)(x_0).
\end{aligned}
$$

Thus in view of (4.24) and the above estimates for P_1 and P_2 we get (4.23) as desired. The proof of the lemma is complete. □

We next prove Theorem 4.3.1 in the special case where f is the characteristic function of a measurable set.

Lemma 4.3.7. *Let E be a measurable subset of $\mathbb{R}^n$ such that $\mathrm{Cap}_{\alpha,s}(E) < +\infty$ for $s > 1$ and $0 < \alpha < n/s$. Then we have*

$$\left\|\mathbf{M}^{\mathrm{loc}}(\chi_E)\right\|_{L^{\frac{n-\alpha s}{n},\infty}(C)} \le C\,\mathrm{Cap}_{\alpha,s}(E)^{\frac{n}{n-\alpha s}}$$

for a constant C independent of E.

Proof. By [AH, Theorem 6.3.9] one can find a nonnegative measure $\mu = \mu^E$ with $\mathrm{supp}(\mu) \subset \overline{E}$ (called capacitary measure for E) such that the function $\mathcal{V}^E := \mathcal{V}^{\mu_E}_{\alpha,s} = \sum_{m=0}^{\infty} 2^{-m\alpha s'}\eta_m * (\eta_m * \mu^E)^{s'-1}$ satisfies the following properties:

$$\mu^E(\overline{E}) = \mathrm{Cap}_{\alpha,s}(E) = \int_{\mathbb{R}^n} \mathcal{V}^E d\mu^E, \tag{4.25}$$

and

$$\mathcal{V}^E \ge 1 \quad \text{quasieverywhere on } E.$$

Thus it follows that

$$\chi_E \leq (\mathcal{V}^E)^{\frac{(s-1)n}{n-\alpha s}} \quad \text{q.e.}$$

We next claim that for all $x_0 \in \mathbb{R}^n$,

$$\mathbf{M}^{\mathrm{loc}}(\chi_E)(x_0) \leq C\, \left[W^8_{\alpha,s}(\mu)(x_0)\right]^{\frac{(s-1)n}{n-\alpha s}}. \tag{4.26}$$

Indeed, for any $x_0 \in \mathbb{R}^n$ and $0 < r \leq 1$, by Lemma 4.3.6 we have

$$\begin{aligned} \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} \chi_E dy &= \frac{1}{|B_r(x_0)|}\, \|\chi_E\|_{L^{1,\infty}(B_r(x_0))} \\ &\leq \frac{1}{|B_r(x_0)|}\, \|\mathcal{V}^E\|^{\frac{(s-1)n}{n-\alpha s}}_{L^{\frac{(s-1)n}{n-\alpha s},\infty}(B_r(x_0))} \\ &\leq C\, \left[W^8_{\alpha,s}(\mu)(x_0)\right]^{\frac{(s-1)n}{n-\alpha s}}, \end{aligned}$$

which proves the claim.

Now it follows from (4.26) that, for any $\lambda > 0$,

$$\begin{aligned} \mathrm{Cap}_{\alpha,s}(\{\mathbf{M}^{\mathrm{loc}}(\chi_E) > \lambda\}) &\leq \mathrm{Cap}_{\alpha,s}\left(\left\{W^8_{\alpha,s}(\mu)(\cdot) > (\lambda/C)^{\frac{n-\alpha s}{(s-1)n}}\right\}\right) \\ &\leq C\, \frac{\mu(\mathbb{R}^n)}{\lambda^{\frac{n-\alpha s}{n}}}, \end{aligned}$$

where we used [AH, 6.3.12] in the last inequality.

By (4.25), this yields

$$\mathrm{Cap}_{\alpha,s}(\{\mathbf{M}^{\mathrm{loc}}(\chi_E) > \lambda\}) \leq C\, \frac{\mathrm{Cap}_{\alpha,s}(E)}{\lambda^{\frac{n-\alpha s}{n}}},$$

which proves the lemma. □

We will make use of the following inequality which is referred to as the (locally) p-convexity of $L^{p,\infty}(C)$, $0 < p < 1$.

Lemma 4.3.8. *Let $s > 1$, $0 < \alpha \le n/s$, and $0 < p < 1$. Then for any integer $m > 0$, we have*

$$\left\|\sum_{k=1}^{m} f_k\right\|_{L^{p,\infty}(C)} \le M \left(\sum_{k=1}^{m} \|f_k\|^p_{L^{p,\infty}(C)}\right)^{\frac{1}{p}}. \tag{4.27}$$

Remark 4.3.9. *Using the normability of $L^1(C)$, it is easy to see that inequality* (4.27) *also holds if $L^{p,\infty}(C)$ is replaced by $L^p(C)$ for $0 < p \le 1$.*

Proof of Lemma 4.3.8. The proof of (4.27) where $L^{p,\infty}(C)$ is replaced with $L^{p,\infty}(K, \lambda)$, for a compact metric measure space (K, λ), was obtained in [Kal]. For a similar result in the context of a general measure space (X, μ), see [Gra, Exercise 1.1.14].

The key to the proof of (4.27) in our setting is the normability of $L^1(C)$ mentioned above. As in [Gra, Exercise 1.1.14], we first observe that by the countable subadditivity of $\text{Cap}_{\alpha,s}$, one has

$$\left\|\max_{1\le k\le m} |f_k|\right\|^p_{L^{p,\infty}(C)} \le \sum_{k=1}^{m} \|f_k\|^p_{L^{p,\infty}(C)}. \tag{4.28}$$

Also, by the subadditivity of $\text{Cap}_{\alpha,s}$, for any $\lambda > 0$ one has

$$\begin{aligned}\text{Cap}_{\alpha,s}(\{|f_1 + \cdots f_m| > \lambda\}) &\le \text{Cap}_{\alpha,s}(\{|f_1 + \cdots f_m| > \lambda, \max_{1\le k\le m} |f_k| \le \lambda\}) \\ &\quad + \text{Cap}_{\alpha,s}(\{\max_{1\le k\le m} |f_k| > \lambda\}).\end{aligned}$$

Let

$$I = \sup_{\lambda>0} \lambda^p \text{Cap}_{\alpha,s}(\{|f_1 + \cdots f_m| > \lambda, \max_{1\le k\le m} |f_k| \le \lambda\}).$$

Then it follows from the last bound and (4.28) that

$$\left\|\sum_{k=1}^{m} f_k\right\|^p_{L^{p,\infty}(C)} \le I + \sum_{k=1}^{m} \|f_k\|^p_{L^{p,\infty}(C)}. \tag{4.29}$$

To estimate I, we notice that for any $\lambda > 0$,

$$
\begin{aligned}
&\lambda^p \mathrm{Cap}_{\alpha,s}(\{|f_1 + \cdots f_m| > \lambda, \max_{1\le k\le m} |f_k| \le \lambda\}) \\
&\quad \le \lambda^{p-1} \int_0^{\lambda} \mathrm{Cap}_{\alpha,s}(\{|f_1 + \cdots f_m| > t\} \cap \{\max_{1\le k\le m} |f_k| \le \lambda\}) dt \\
&\quad \le \lambda^{p-1} \int_{\{\max\limits_{1\le k\le m} |f_k| \le \lambda\}} |f_1 + \cdots f_m| dC \\
&\quad \le C\lambda^{p-1} \sum_{k=1}^{m} \int_{\{\max\limits_{1\le k\le m} |f_k| \le \lambda\}} |f_k| dC,
\end{aligned}
$$

where we used the normability of $L^1(C)$ in the last inequality. This gives

$$
\begin{aligned}
&\lambda^p \mathrm{Cap}_{\alpha,s}(\{|f_1 + \cdots f_m| > \lambda, \max_{1\le k\le m} |f_k| \le \lambda\}) \\
&\quad \le C\lambda^{p-1} \sum_{k=1}^{m} \int_{\{|f_k| \le \lambda\}} |f_k| dC \\
&\quad = C\lambda^{p-1} \sum_{k=1}^{m} \int_0^{\lambda} \mathrm{Cap}_{\alpha,s}(\{|f_k| > t\}) dt \\
&\quad \le \frac{C}{1-p} \sum_{k=1}^{m} \|f_k\|^p_{L^{p,\infty}(C)}.
\end{aligned} \tag{4.30}
$$

Finally, we combine (4.29) and (4.30) to obtain (4.27) as desired. □

We are now ready to prove Theorem 4.3.1.

Proof of Theorem 4.3.1. Recall that if $E_1 \subset E_2 \subset \ldots$ is an increasing sequence of subsets of $\mathbb{R}^n$, then

$$
\mathrm{Cap}_{\alpha,s}\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \lim_{i\to\infty} \mathrm{Cap}_{\alpha,s}(E_i).
$$

Thus by Lemma 4.3.8 we have

$$
\left\| \sum_{k\in\mathbb{Z}} |f_k| \right\|_{L^{p,\infty}(C)} \le M \left(\sum_{k\in\mathbb{Z}} \|f_k\|^p_{L^{p,\infty}(C)} \right)^{\frac{1}{p}}, \tag{4.31}
$$

which holds since $p = (n - \alpha s)/n \in (0, 1)$.

Now for $f \in L^p(C)$, we write

$$f = \sum_{k\in\mathbb{Z}} f\chi_{E_k} \quad \text{q.e.},$$

where E_k is the set $E_k := \{2^{k-1} < |f| \le 2^k\}$. Thus,

$$|f(x)| \le \sum_{k\in\mathbb{Z}} 2^k \chi_{E_k} \quad \text{q.e.},$$

which yields

$$\mathbf{M}^{\mathrm{loc}}(f)(x_0) \le \sum_{k\in\mathbb{Z}} 2^k\, \mathbf{M}^{\mathrm{loc}}(\chi_{E_k})(x_0), \quad \forall x_0 \in \mathbb{R}^n.$$

Hence applying (4.31) and Lemma 4.3.7, we get

$$\begin{aligned}\left\|\mathbf{M}^{\mathrm{loc}}(f)\right\|_{L^{p,\infty}(C)} &\le M \left(\sum_{k\in\mathbb{Z}} 2^{kp} \left\|\mathbf{M}^{\mathrm{loc}}(\chi_{E_k})\right\|_{L^{p,\infty}(C)}^p\right)^{\frac{1}{p}} \\ &\le M \left(\sum_{k\in\mathbb{Z}} 2^{kp}\, \mathrm{Cap}_{\alpha,s}(E_k)\right)^{\frac{1}{p}}.\end{aligned}$$

This gives

$$\begin{aligned}\left\|\mathbf{M}^{\mathrm{loc}}(f)\right\|_{L^{p,\infty}(C)} &\le A \left(\sum_{k\in\mathbb{Z}} \int_{2^{k-2}}^{2^{k-1}} t^p\, \mathrm{Cap}_{\alpha,s}(\{|f| > t\})\frac{dt}{t}\right)^{\frac{1}{p}} \\ &\le A \left(\int_0^\infty t^p\, \mathrm{Cap}_{\alpha,s}(\{|f| > t\})\frac{dt}{t}\right)^{\frac{1}{p}} \\ &= A\,\|f\|_{L^p(C)},\end{aligned}$$

as desired. □

We now have to remarks about the sharpness of Theorem 4.3.1.

Remark 4.3.10. *The exponent $p = (n - \alpha s)/n$ in Theorem 4.3.1 is sharp. To see this, suppose that $\alpha s < n$ and let $0 < p_1 < (n - \alpha s)/n$, say, $p_1 = \frac{n-\alpha s}{n(1+\delta)}$ for some $\delta > 0$. Note that for any $\epsilon \in (0, 1/2)$ and $|x| < 1/2$, we have*

$$\mathbf{M}^{\mathrm{loc}}(\chi_{B_\epsilon(0)})(x) \ge \frac{1}{|B_{|x|+\epsilon}(x)|} \int_{B_{|x|+\epsilon}(x)} \chi_{B_\epsilon(0)}(y)dy = c\left(\frac{\epsilon}{|x|+\epsilon}\right)^n,$$

where $c = |B_1(0)|$. *Thus, for* $(2\epsilon)^n \le t < 1$,

$$\begin{aligned}
&\mathrm{Cap}_{\alpha,s}(\{x \in B_{1/2}(0) : \mathbf{M}^{\mathrm{loc}}(\tfrac{1}{c}\chi_{B_\epsilon(0)})(x) > t\}) \qquad (4.32)\\
&\ge \mathrm{Cap}_{\alpha,s}(\{x \in B_{1/2}(0) : \left(\frac{\epsilon}{|x|+\epsilon}\right)^n > t\})\\
&= \mathrm{Cap}_{\alpha,s}(B_{\epsilon t^{-1/n}-\epsilon}(0)).
\end{aligned}$$

As $\mathrm{Cap}_{\alpha,s}(B_r(z)) \simeq r^{n-\alpha s}$ *for any* $0 < r \le 1$ *and* $z \in \mathbb{R}^n$, *by choosing* $t = (2\epsilon)^n$ *this gives*

$$\left\|\mathbf{M}^{\mathrm{loc}}(\chi_{B_\epsilon(0)})\right\|_{L^{p_1,\infty}(C)} \ge C(2\epsilon)^n(1/2-\epsilon)^{n(1+\delta)}.$$

On the other hand, we obviously have

$$\left\|\chi_{B_\epsilon(0)}\right\|_{L^{p_1}(C)} \simeq \epsilon^{n(1+\delta)}.$$

Thus the bound

$$\left\|\mathbf{M}^{\mathrm{loc}}(\chi_{B_\epsilon(0)})\right\|_{L^{p_1,\infty}(C)} \le A\left\|\chi_{B_\epsilon(0)}\right\|_{L^{p_1}(C)}$$

cannot hold uniformly in $\epsilon \in (0, 1/2)$.

Remark 4.3.11. *Inequality* (4.19) *fails to hold if the space* $L^{p,\infty}(C)$ *is replaced by the space* $L^p(C)$. *Indeed, with* $\alpha s < n$ *and* $\epsilon \in (0, 1/8)$, *by* (4.32) *we have*

$$\begin{aligned}
\left\|\mathbf{M}^{\mathrm{loc}}(\chi_{B_\epsilon}(0))\right\|_{L^{\frac{n-\alpha s}{n}}(C)}^{\frac{n-\alpha s}{n}} &\ge C\int_{(2\epsilon)^n}^1 t^{\frac{n-\alpha s}{n}-1}\left(\epsilon\, t^{-1/n}-\epsilon\right)^{n-\alpha s} dt\\
&= C\epsilon^{n-\alpha s}\int_{(2\epsilon)^n}^1 t^{-1}\left(1-t^{1/n}\right)^{n-\alpha s} dt\\
&\ge C\epsilon^{n-\alpha s}\int_{(2\epsilon)^n}^{1/2} t^{-1} dt\\
&= C\epsilon^{n-\alpha s}\ln(2^{-n-1}\epsilon^{-n}).
\end{aligned}$$

Thus the bound

$$\left\|\mathbf{M}^{\mathrm{loc}}(\chi_{B_\epsilon(0)})\right\|_{L^{\frac{n-\alpha s}{n}}(C)} \leq A\left\|\chi_{B_\epsilon(0)}\right\|_{L^{\frac{n-\alpha s}{n}}(C)}$$

cannot hold uniformly in $\epsilon \in (0, 1/8)$.

4.4. Sobolev Type Embeddings on $L^p(C)$

Theorem 4.4.1. *Let* $s > 1$, $0 < \alpha < n/s$, $q \geq (n - \alpha s)/n$, *and* $0 < \beta < (n - \alpha s)/q$. *With*

$$q^* = \frac{(n-\alpha s)q}{n-\alpha s-\beta q},$$

for any measurable function $f \in L^q(C)$ *it holds that*

$$\|G_\beta * f\|_{L^{q^*,\infty}(C)} \leq A\|f\|_{L^q(C)} \quad \text{provided } q = \frac{n-\alpha s}{n},$$

$$\|G_\beta * f\|_{L^{q^*}(C)} \leq A\|f\|_{L^q(C)} \quad \text{provided } q > \frac{n-\alpha s}{n},$$

and

$$\|G_\beta * f\|_{L^{q_1}(C)} \leq A\|f\|_{L^q(C)} \quad \text{provided } q \leq q_1 < q^*.$$

Here $L^q(C)$ *is associated to the Bessel capacity* $\mathrm{Cap}_{\alpha,s}$.

The homogeneous version of Theorem 4.4.1 is given as follows.

Theorem 4.4.2. *Let* $s > 1$, $0 < \alpha < n/s$, $q \geq (n - \alpha s)/n$, *and* $0 < \beta < (n - \alpha s)/q$. *With*

$$q^* = \frac{(n-\alpha s)q}{n-\alpha s-\beta q},$$

for any measurable function $f \in L^q(C)$ *it holds that*

$$\|I_\beta * f\|_{L^{q^*,\infty}(C)} \leq A\|f\|_{L^q(C)} \quad \text{provided } q = \frac{n-\alpha s}{n}$$

and

$$\|I_\beta * f\|_{L^{q^*}(C)} \leq A\|f\|_{L^q(C)} \quad \text{provided } q > \frac{n-\alpha s}{n}.$$

Here $L^q(C)$ *is associated to the Riesz capacity* $\mathrm{cap}_{\alpha,s}$.

The proof of Theorem 4.4.2 is based mainly on the pointwise inequality

$$I_\beta(f)(x) \le C \, \|f\|_{L^p(\mathbb{R}^n)}^{\beta p/n} \mathbf{M}(f)(x)^{1-\beta p/n}, \quad 1 \le p < n/\beta,$$

(see [AH, Proposition 3.1.2]), and Theorems 4.3.3, 4.3.4. It is simpler than that of Theorem 4.4.1 and thus we shall present only the proof of Theorem 4.4.1.

Proof of Theorem 4.4.1. By the pointwise behavior of Bessel kernel (see [AH, Section 1.2.4]), we have

$$\begin{aligned} G_\beta * f(x) &= \int_{|x-y|\le 1/2} G_\beta(x-y) f(y) dy + \int_{|x-y|>1/2} G_\beta(x-y) f(y) dy \\ &\le C \int_0^1 t^{\beta-n} \int_{B_t(x)} |f(y)| dy \frac{dt}{t} + C \int_{\mathbb{R}^n} e^{-|x-y|/2} |f(y)| dy \\ &=: C(J_1(f)(x) + J_2(f)(x)). \end{aligned}$$

Arguing as in the proof of [AH, Proposition 3.1.2], we find

$$J_1(f)(x) \le C \, \|f\|_{L^p(\mathbb{R}^n)}^{\beta p/n} \mathbf{M}^{\mathrm{loc}}(f)(x)^{1-\beta p/n}$$

provided $1 \le p < n/\beta$.

As $\mathrm{Cap}_{\alpha,s}(E) \ge c\,|E|^{1-\alpha s/n}$ for any measurable set E, we see that

$$\begin{aligned} \|f\|_{L^q(C)}^q &= q \int_0^\infty t^{q-1} \mathrm{Cap}_{\alpha,s}(\{|f| > t\}) dt \\ &\ge c \int_0^\infty t^{q-1} |\{|f| > t\}|^{1-\alpha s/n} dt \\ &= c\,\|f\|_{L^{nq/(n-\alpha s),q}(\mathbb{R}^n)}^q \\ &\ge c\,\|f\|_{L^{nq/(n-\alpha s)}(\mathbb{R}^n)}^q, \end{aligned} \tag{4.33}$$

where $L^{nq/(n-\alpha s),q}(\mathbb{R}^n)$ is a Lorentz space (see, e.g., [Gra]).

We remark that if we use the lower bound $\mathrm{Cap}_{\alpha,s}(E) \geq c\,|E|$, then we obtain that $\|f\|_{L^q(C)} \geq c\,\|f\|_{L^q(\mathbb{R}^n)}$, which by interpolation yields $\|f\|_{L^q(C)} \geq c\,\|f\|_{L^{q_1}(\mathbb{R}^n)}$ for all $q \leq q_1 \leq nq/(n-\alpha s)$. But this will not be needed in the paper except for $q_1 = nq/(n-\alpha s)$.

Note that $q^*(1-\beta q/(n-\alpha s)) = q$, and thus when $q > (n-\alpha s)/n$ by the above bounds and Theorem 4.3.2 it follows that

$$\begin{aligned} &\|J_1(f)\|_{L^{q^*}(C)} \qquad (4.34)\\ &\quad\leq C\,\|f\|^{\beta q/(n-\alpha s)}_{L^{nq/(n-\alpha s)}(\mathbb{R}^n)} \left\|\mathbf{M}^{\mathrm{loc}}(f)\right\|^{1-\beta q/(n-\alpha s)}_{L^{q^*(1-\beta q/(n-\alpha s))}(C)}\\ &\quad\leq C\,\|f\|^{\beta q/(n-\alpha s)}_{L^q(C)} \left\|\mathbf{M}^{\mathrm{loc}}(f)\right\|^{1-\beta q/(n-\alpha s)}_{L^q(C)}\\ &\quad\leq C\,\|f\|_{L^q(C)}\,. \end{aligned}$$

By using Theorem 4.3.1, the bound (4.34) also holds in the case $q = (n-\alpha s)/n$ provided $L^{q^*}(C)$ is replaced with $L^{q^*,\infty}(C)$.

Now suppose that the support of f is contained in a ball $B_1(x_0)$. Then the support of $\mathbf{M}^{\mathrm{loc}}(f)$ is contained in $B_2(x_0)$, and thus we get

$$\begin{aligned} &\|J_1(f)\|_{L^q(C)} \qquad (4.35)\\ &\quad\leq C\,\|f\|^{\beta q/(n-\alpha s)}_{L^{nq/(n-\alpha s)}(\mathbb{R}^n)} \left\|\mathbf{M}^{\mathrm{loc}}(f)\chi_{B_2(x_0)}\right\|^{1-\beta q/(n-\alpha s)}_{L^{q(1-\beta q/(n-\alpha s))}(C)}\\ &\quad\leq C\,\|f\|^{\beta q/(n-\alpha s)}_{L^q(C)} \left\|\mathbf{M}^{\mathrm{loc}}(f)\right\|^{1-\beta q/(n-\alpha s)}_{L^q(C)}\\ &\quad\leq C\,\|f\chi_{B_1}(x_0)\|_{L^q(C)}\,. \end{aligned}$$

Let $\{Q_j\}$ be a partition of $\mathbb{R}^n$ into a countable collection of closed cubes with diameters $1/2$ and with disjoint interiors. Then for any $(n-\alpha s)/n \leq q \leq 1$ and $f \in L^q(C)$ by

Remark 4.3.9 we have

$$\begin{aligned}
\|J_1(f)\|_{L^q(C)}^q &\le \left\|\sum_j J_1(f\chi_{Q_j})\right\|_{L^q(C)}^q \le C\sum_j \left\|J_1(f\chi_{Q_j})\right\|_{L^q(C)}^q \\
&\le C\sum_j \left\|f\chi_{Q_j}\right\|_{L^q(C)}^q \quad \text{(by (4.35))} \\
&\le C\int_0^\infty t^{q-1}\sum_j \mathrm{Cap}_{\alpha,s}(\{|f|>t\}\cap Q_j)dt \\
&\le C\,\|f\|_{L^q(C)}^q ,
\end{aligned}$$

where we used quasi-additivity of $\mathrm{Cap}_{\alpha,s}$ (see [MS2, Proposition 3.1.5]) in the last inequality. Note that if $q>1$ then by Hölder's inequality and the last bound, we also have

$$\|J_1(f)\|_{L^q(C)}^q = \|J_1(f)^q\|_{L^1(C)} \le C\,\|J_1(|f|^q)\|_{L^1(C)} \le C\,\|f\|_{L^q(C)}^q .$$

Thus we have obtained

$$\|J_1(f)\|_{L^{q^*}(C)} + \|J_1(f)\|_{L^q(C)} \le C\,\|f\|_{L^q(C)} \tag{4.36}$$

for all $q>(n-\alpha s)/n$, and this also holds for $q=(n-\alpha s)/n$ provided $L^{q^*}(C)$ is replaced with $L^{q^*,\infty}(C)$.

To bound $J_2(f)$, we first observe that by Hölder's inequality we have

$$J_2(f)(x) \le C\left(\int_{\mathbb{R}^n} e^{-|x-y|/2}|f(y)|^p dy\right)^{\frac{1}{p}}, \qquad p\ge 1.$$

With $\{Q_j\}$ being a partition of $\mathbb{R}^n$ as above, for any $p\ge 1$ we have

$$J_2(x) \le C\left(\sum_j \int_{Q_j} e^{-\mathrm{dist}(x,Q_j)/2}|f(y)|^p dy\right)^{\frac{1}{p}}.$$

Thus for any $0 < \epsilon \le p$ by Remark 4.3.9 we find

$$\|J_2(f)\|_{L^\epsilon(C)}^\epsilon \le C \left\| \sum_j \left(\int_{Q_j} |f(y)|^p dy \right) e^{-\mathrm{dist}(\cdot, Q_j)/2} \right\|_{L^{\epsilon/p}(C)}^{\epsilon/p}$$
$$\le C \sum_j \left(\int_{Q_j} |f(y)|^p dy \right)^{\epsilon/p} \left\| e^{-\mathrm{dist}(\cdot, Q_j)/2} \right\|_{L^{\epsilon/p}(C)}^{\epsilon/p}.$$

We now choose $p = nq/(n - \alpha s)$ and $\epsilon = q$ to get from (4.33), the above inequality, and quasi-additivity of $\mathrm{Cap}_{\alpha,s}$ that

$$\|J_2(f)\|_{L^q(C)}^q \le C \sum_j \left\| f \chi_{Q_j} \right\|_{L^q(C)}^q \le C \|f\|_{L^q(C)}^q. \tag{4.37}$$

Similarly, for any $p_1 > p$ we have

$$\|J_2(f)\|_{L^{p_1}(C)}^{p_1} \le C \left\| \sum_j \left(\int_{Q_j} |f(y)|^p dy \right) e^{-\mathrm{dist}(\cdot, Q_j)/2} \right\|_{L^{p_1/p}(C)}^{p_1/p}$$
$$\le C \left(\sum_j \int_{Q_j} |f(y)|^p dy \left\| e^{-\mathrm{dist}(\cdot, Q_j)/2} \right\|_{L^{p_1/p}(C)} \right)^{p_1/p}$$
$$\le C \left(\int_{\mathbb{R}^n} |f(y)|^p dy \right)^{p_1/p}.$$

Thus with $p = nq/(n - \alpha s)$ by (4.33) we find

$$\|J_2(f)\|_{L^{p_1}(C)} \le C \|f\|_{L^q(C)} \qquad \forall p_1 > nq/(n - \alpha s). \tag{4.38}$$

Now using (4.37), (4.38) and interpolation we arrive at

$$\|J_2(f)\|_{L^{q^*}(C)} + \|J_2(f)\|_{L^q(C)} \le C \|f\|_{L^q(C)}. \tag{4.39}$$

Finally, combining (4.36) with (4.39) and interpolation we obtain the theorem. □

Bibliography

[Ad1] D. R. Adams, *Traces of potentials II*, Indiana Univ. Math. J. **22** (1972/73), 907–918.

[Ad2] D. R. Adams, *Sets and functions of finite L^p-capacity*, Indiana Univ. Math. J. **27** (1978), 611–627.

[Ad3] D. R. Adams, *Quasi-additivity and sets of finite L^p-capacity*, Pacific J. Math. **79** (1978), no. 2, 283–291.

[Ad4] D. R. Adams, *Choquet integrals in potential theory*, Publ. Mat. **42** (1998), no. 1, 3–66.

[AH] D. R. Adams and L. I. Hedberg, *Function Spaces and Potential Theory*, Springer-Verlag, Berlin, 1996.

[AM] D. R. Adams and N. G. Meyers, *Thinness and Wiener criteria for non-linear potentials*, Indiana Univ. Math. J. **22** (1972), 169–197.

[AX1] D. R. Adams and J. Xiao, *Nonlinear analysis on Morrey spaces and their capacities*, Indiana Univ. Math. J. **53** (2004), 1629–1663.

[AX2] D. R. Adams and J. Xiao, *Morrey spaces in harmonic analysis*, Ark. Mat. **50** (2012), no. 2, 201–230.

[AP1] K. Adimurthi and N. C. Phuc, *An end-point global gradient weighted estimate for quasilinear equations in non-smooth domains*, Manuscripta Math. **150** (2016), 111–135.

[AP2] K. Adimurthi and C. P. Nguyen, *Quasilinear equations with natural growth in the gradients in spaces of Sobolev multipliers*, Calc. Var. Partial Differential Equations **57** (2018), no. 3, Art. 74, 23 pp.

[BP] P. Baras and M. Pierre, *Critère d'existence de solutions positives pour des équations semilinéaires non monotones*, Ann. Inst. H. Poincaré, Analyse Non Linéaire **2** (1985), 185–212.

[BRV] O. Blasco, A. Ruiz, and L. Vega, *Non-interpolation in Morrey-Campanato and block spaces*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), no. 1, 31–40.

[Cho] G. Choquet, *Theory of capacities*, Ann. Inst. Fourier (Grenoble) **5** (1953–1954), 131–295.

[CW] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), no. 4, 569–645.

[Den] D. Denneberg, *Non-additive measure and integral*, Theory and Decision Library. Series B: Mathematical and Statistical Methods **27**. Kluwer Academic Publishers Group, Dordrecht (1994), x+178 pp.

[Ger] P. Germain, *Multipliers, paramultipliers, and weak-strong uniqueness for the Navier-Stokes equations*, J. Differential Equations, **226** (2006), 373–428.

[Gra] L. Grafakos, *Classical Fourier analysis*, Third edition, Graduate Texts in Mathematics, **249**, Springer, New York, (2014), xviii+638 pp.

[HMV] K. Hansson, V. G. Maz'ya, and I. E. Verbitsky, *Criteria of solvability for multidimensional Riccati equations*, Ark. Mat. **37** (1999), no. 1, 87–120.

[HKM] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Univ. Press, Oxford, 1993.

[ISY] T. Izumi, E. Sato, and K. Yabuta, *Remarks on a subspace of Morrey spaces*, Tokyo J. Math. **37** (2014), 185–197.

[Kal] N. J. Kalton, *Linear operators on L_p for* $0 < p < 1$, Trans. Amer. Math. Soc. **259** (1980), 319–355.

[KV] N. J. Kalton and I. E. Verbitsky, *Nonlinear equations and weighted norm inequalities*, Trans. Amer. Math. Soc. **351** (1999), no. 9, 3441–3497.

[Kom] J. Komlós, *A generalization of a problem of Steinhaus*, Acta Math. Acad. Sci. Hungar. **18** (1967) 217–229.

[Lan] N. S. Landkof, *Foundations of modern potential theory*, Translated from the Russian by A. P. Doohovskoy. Die Grundlehren der mathematischen Wissenschaften, Band 180. Springer-Verlag, New York-Heidelberg (1972), x+424 pp.

[L-R] P.G. Lemarié-Rieusset, *Recent Developments in the Navier-Stokes Problem*, Chapman & Hall, CRC Press, Boca Raton (2002).

[LT] J. Lindenstrauss and L Tzafriri, *Classical Banach spaces II. Function spaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], 97. Springer-Verlag, Berlin-New York, 1979. x+243 pp.

[Lux] W. A. J. Luxemburg, *Banach function spaces*, Thesis, Technische Hogeschool te Delft (1955), 70 pp.

[Maz] V. Maz'ya, *Sobolev spaces with applications to elliptic partial differential equa-*

tions. Second, revised and augmented edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 342. Springer, Heidelberg, 2011. xxviii+866 pp.

[MST] M. Mastylo, Y. Sawano, and H. Tanaka, *Morrey-type space and its Köthe dual space*, Bull. Malays. Math. Sci. Soc. **41** (2018), no. 3, 1181–1198.

[MH] V. G. Maz'ja and V. P. Havin, *A nonlinear potential theory*, Uspehi Mat. Nauk **27** (1972), no. 6, 67–138 (in Russian). English translation: Russ. Math. Surv. **27** (1972), 71–148.

[MS1] V. G. Maz'ya and T.O. Shaposhnikova, *The Theory of Multipliers in Spaces of Differentiable Functions*, Pitman, New York (1985).

[MS2] V. G. Maz'ya and T.O. Shaposhnikova, *Theory of Sobolev Multipliers. With Applications to Differential and Integral Operators*, Grundlehren der Mathematischen Wissenschaften, vol. **337**, Springer-Verlag, Berlin (2009), p. xiv+609.

[MV] V. G. Maz'ya and I. E. Verbitsky, *Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multipliers*, Ark. Mat. **33** (1995), 81–115.

[MP] T. Mengesha and N. C. Phuc, *Global estimates for quasilinear elliptic equations on Reifenberg flat domains*, Arch. Ration. Mech. Anal. **203** (2012), 189–216.

[Mey] N. G. Meyers, *A theory of capacities for potentials of functions in Lebesgue classes*, Math. Scand. **26** (1970) 255–292.

[NP] Q.-H. Nguyen and N. C. Phuc, *Good-λ and Muckenhoupt-Wheeden type bounds in quasilinear measure datum problems, with applications*, Math. Ann. **374** (2019), 67–98.

[OP1] K. H. Ooi and N. C. Phuc, *Characterizations of predual spaces to a class of Sobolev multiplier type spaces*. Submitted for publication. Available at: http://arxiv.org/abs/2005.04349

[OP2] K. H. Ooi and N. C. Phuc, *On a capacitary strong type inequality and related capacitary estimates*. To appear in Rev. Mat. Iberoamericana. Available at https://arxiv.org/abs/2009.09291

[OP3] K. H. Ooi and N. C. Phuc, *The Hardy-Littlewood maximal function, Choquet integrals, and embeddings of Sobolev type*. Submitted for publication.

[OV] J. Orobitg and J. Verdera, *Choquet integrals, Hausdorff content and the Hardy- Littlewood maximal operator*, Bull. Lond. Math. Soc. **30** (1998), 145–150.

[Ph1] N. C. Phuc, *Weighted estimates for nonhomogeneous quasilinear equations with discontinuous coefficients* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **10** (2011), 1–17.

[Ph2] N. C. Phuc, *Quasilinear Riccati type equations with super-critical exponents*, Comm. Partial Differential Equations **35** (2010), no. 11, 1958–1981.

[Ph3] N. C. Phuc, *Nonlinear Muckenhoupt-Wheeden type bounds on Reifenberg flat domains, with applications to quasilinear Riccati type equations*, Adv. Math. **250** (2014), 387–419.

[PhPh] T. V. Phan and N. C. Phuc, *Stationary Navier-Stokes equations with critically singular external forces: existence and stability results*, Adv. Math. **241** (2013), 137–161.

[PhV] N. C. Phuc, and I. E. Verbitsky, *Quasilinear and Hessian equations of Lane-Emden type*, Ann. of Math. (2) **168** (2008), no. 3, 859–914.

[RT] M. Rosenthal and H. Triebel *Calderón-Zygmund operators in Morrey spaces*, Rev. Mat. Complut. **27** (2014), 1–11.

[Ryc] V. S. Rychkov, *Littlewood-Paley theory and function spaces with A_p^{loc} weights*, Math. Nachr. **224** (2001), 145–180.

[ST] Y. Sawano and H. Tanaka, *The Fatou property of block spaces*, J. Math. Sci. Univ. Tokyo **22** (2015), no. 3, 663–683.

[Ver] I. E. Verbitsky, *Nonlinear potentials and trace inequalities*, The Maz'ya Anniversary Collection, Eds. J. Rossmann, P. Takác, and G. Wildenhain, Operator Theory: Adv. Appl. **110** (1999), 323–343.

[VW] I. E. Verbitsky and R. L. Wheeden, *Weighted trace inequalities for fractional integrals and applications to semilinear equations*, J. Funct. Anal. **129** (1995), no. 1, 221–241.

Vita

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