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Stochastic Navier-Stokes Equations with Markov Switching

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STOCHASTIC NAVIER-STOKES EQUATIONS WITH MARKOV SWITCHING

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Dedicated to my father

I have never done anything “useful”.
—G. H. Hardy
A Mathematician’s Apology

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Abstract

This dissertation is devoted to the study of three-dimensional (regularized) stochastic Navier-Stokes equations with Markov switching. A Markov chain is introduced into the noise term to capture the transitions from laminar to turbulent flow, and vice versa. The existence of the weak solution (in the sense of stochastic analysis) is shown by studying the martingale problem posed by it. This together with the pathwise uniqueness yields existence of the unique strong solution (in the sense of stochastic analysis). The existence and uniqueness of a stationary measure is established when the noise terms are additive and autonomous. Certain exit time estimates (exponential inequalities) for solutions to such switching equations are obtained, and its connection with its counterpart in the non-switching scenario is discussed.

Chapter 1. Functional Analytic Setup

1.1. The Navier-Stokes Equations

The motion of (viscous) fluid flows is described by a system of partial differential equations known as the *Navier-Stokes equations*. If the fluid is compressible (e.g., air or gas), then the motion of such a flow is modeled by *compressible* Navier-Stokes equations; if the fluid is incompressible (e.g., water or honey), then the motion of such a flow is modeled by *incompressible* Navier-Stokes equations. In this article, we focus on the study of incompressible flows and refer the reader interested in the mathematical theory of compressible flows to, for instance, [21] and [46].

Let G be a bounded domain in $\mathbb{R}^d$. Let $\mathbf{u}(x,t)$ and $p(x,t)$ denote the velocity and pressure of the fluid at each $x \in G$ and time $t \in [0,T]$. The motion of viscous incompressible flow on G is described by the Navier-Stokes system:

$$\partial_t \mathbf{u} - \nu \triangle \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0 \qquad \text{in} \quad G \times [0,T], \tag{1.1a}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in} \quad G \times [0,T], \tag{1.1b}$$

$$\mathbf{u}(x,t) = 0 \qquad \text{on} \quad \partial G \times [0,T] \tag{1.1c}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x) \qquad \text{on} \quad G \times \{t=0\}, \tag{1.1d}$$

where $\nu > 0$ is the viscosity coefficient of the fluid. The condition (1.1b) indicates incompressiblity of the fluid. There are several well-known books on Navier-Stokes equations such as [54, 58, 59]. It is worth mentioning that if $G \subseteq \mathbb{R}^3$, then the uniqueness of the global weak solution of such a system (1.1) is an open problem, and is one among the Millennium Prize problems.

If randomness is introduced into (1.1a), then the resulting model is called *stochastic Navier-Stokes equations.* The study of stochastic Navier-Stokes equations has been an important and active area of research, and has received considerable attention in recent years. The introduction of randomness in Navier-Stokes equations arises from a need to understand (i) the velocity fluctuations observed in wind tunnels under identical experimental conditions, and (ii) the onset of turbulence. Random body forces also arise as control terms, or from random disturbances such as structural vibrations that act on the fluid. It was originally the idea of Kolmogorov (see e.g., [61]) to introduce a white noise in the Navier-Stokes system in order to obtain an invariant measure for the system. In fact, Kolmogorov's point of view to the theory of turbulence states that "the ultimate goal is to find an invariant measure of turbulence (see e.g., [7, 43, 44])."

On the other hand, randomness may occur naturally. As a matter of fact, it can be shown that (see, e.g., [8]) additive noise has to be added into the Navier-Stokes equation for a proper description of fully developed turbulence; in addition, if there are "jumps" in point vorticities (velocity gradient), then multiplicative noises should be added into the deterministic equation to adjust (see, e.g., [8]), i.e., equation (1.1a) becomes

$$\partial_t \mathbf{u} - \nu \triangle \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} + \sigma(t, \mathbf{u}) dW(t) + \int_Z \mathbf{G}(t-, \mathbf{u}, z) \tilde{N}(dz, dt),$$

where $W(t)$ is a Wiener process and $\tilde{N}(dz, ds)$ is a compensated Poisson random measure.

As mentioned in previous paragraph, the well-posedness of a (global-in-time strong) solution to a three-dimensional Navier-Stokes equation is an open problem. In order to resolve such an issue, we follow Leray's idea [36] to modify the original Navier-Stokes equation: for a mollifier k,

$$\partial_t \mathbf{u} - \nu \triangle \mathbf{u} + ((k * \mathbf{u}) \cdot \nabla)\mathbf{u} + \nabla p = 0. \tag{1.2}$$

Equation (1.2) is often called Leray regularization of Navier-Stokes equation (see, e.g., [48]). Therefore, the corresponding (regularized) stochastic Navier-Stokes equation appears as

$$\partial_t \mathbf{u} - \nu \triangle \mathbf{u} + ((k * \mathbf{u}) \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} + \sigma(t, \mathbf{u}) dW(t) + \int_Z \mathbf{G}(t-, \mathbf{u}, z)\tilde{N}(dz, dt).$$

If the noise coefficient is subject to random changes, then we capture such changes in the noise coefficient in a class of equations. We introduce a Markov chain $\{\mathfrak{r}(t)\}$ to such a class of equations appears as follows:

$$\begin{aligned} &\partial_t \mathbf{u} - \nu \triangle \mathbf{u} + ((k * \mathbf{u}) \cdot \nabla)\mathbf{u} + \nabla p \\ &= \mathbf{f} + \sigma(t, \mathbf{u}, \mathfrak{r}(t)) dW(t) + \int_Z \mathbf{G}(t-, \mathbf{u}, \mathfrak{r}(t-), z)\tilde{N}(dz, dt). \end{aligned} \tag{1.3}$$

We shall call equation (1.3) as *three-dimensional regularized stochastic Navier-Stokes equation with Markov switching*. Moreover, equation (1.3) can be transformed into the following formulation:

$$\begin{aligned} &\mathbf{du}(t) + [\nu \mathbf{A}\mathbf{u}(t) + \mathbf{B}_k(\mathbf{u}(t))]dt \\ &= \mathbf{f}(t)dt + \sigma(t, \mathbf{u}(t), \mathfrak{r}(t)) dW(t) + \int_Z \mathbf{G}(t-, \mathbf{u}(t-), \mathfrak{r}(t-), z)\tilde{N}(dz, dt). \end{aligned} \tag{1.4}$$

The study of equation (1.4) is the *main theme* of this article. In particular, we also study the *non-switching* equations which refers to the following:

$$\begin{aligned} &\mathbf{du}(t) + [\nu \mathbf{A}\mathbf{u}(t) + \mathbf{B}_k(\mathbf{u}(t))]dt \\ &= \mathbf{f}(t)dt + \sigma(t, \mathbf{u}(t)) dW(t) + \int_Z \mathbf{G}(t-, \mathbf{u}(t-), z)\tilde{N}(dz, dt). \end{aligned} \tag{1.5}$$

This thesis is organized as follows. Preliminaries, functional analytic settings, and auxiliary results will be introduced in Section 1.2. A priori estimate will be introduced in Chapter 2. Chapter 3 is devoted to the study of three-dimensional regularized stochastic Navier-Stokes equations with Markov switching (1.4). The martingale problem and existences and uniqueness of stationary measures for the equations are investigated. The exit time estimates (exponential inequalities) for solutions to switching equations are established, and its relation with the Freidlin-Wentzell type large deviation principle are studied.

1.2. Preliminaries

In this section, we review facts that will be used repeatedly throughout this work.

1.2.1. Facts from probability

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the underline probability space. Suppose $X \in L^1(\mathcal{P})$. By Markov inequality, we mean the following estimate:

$$\mathcal{P}(|X| \geq \lambda) \leq \lambda^{-1}\mathbb{E}(|X|)$$

for any $\lambda > 0$.

Definition 1 (Uniform integrability)**.** *A family of random variables $\mathfrak{X}$ in $L^1(\mathcal{P})$ is said to be uniformly integrable if*

$$\lim_{c\to\infty} \sup_{X\in\mathfrak{X}} \int_{\{|X|>c\}} |X| d\mathcal{P} = 0.$$

We introduce some properties regarding uniform integrability below and refer the interested reader to [52, Sec. 6.5.1] and [55, Lem. 5.3] for more details.

Lemma 1.2.1 (Crystal Ball Condition). *For $p > 0$, the family $\{|X_n|^p\}$ is uniformly integrable if* $\sup_n \mathbb{E}(|X_n|^{p+\delta}) < \infty$ *for some* $\delta > 0$

Lemma 1.2.2. *Let $f \in C(\Omega)$ and* $\sup_n \mathbb{E}^{\mathcal{P}_n}[|f|^{1+\delta}] \leq C$ *for some $\delta > 0$. Let $\{\mathcal{P}_n\}$ be a sequence of probability measures on Ω with $\mathcal{P}_n \Rightarrow \mathcal{P}$, as $n \to \infty$. Then we have $\mathbb{E}^{\mathcal{P}_n}(|f|) \to \mathbb{E}^{\mathcal{P}}(|f|)$.*

1.2.2. Facts from analysis

For $1 < p < \infty$, we set q to be the number such that $\frac{1}{p} + \frac{1}{q} = 1$. By basic Young inequality, we mean

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where a, b are positive numbers. If $1 \leq p < \infty$ and $a, b \geq 0$, then

$$(a+b)^p \leq 2^{p-1}(a^p + b^p). \tag{1.6}$$

Let f and g be two measurable functions in $\mathbb{R}^d$. Then the convolution of f and g is defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(t)g(x-t)dt.$$

Let p, q, and r satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, then by Young's convolution inequality, we mean

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Now we introduce the uniform integrability and tightness for general measure space and then the Vitali Convergence Theorem.

Definition 2. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ a sequence of functions on X, each of which is integrable over X. The sequence $\{f_n\}$ is said to be uniformly integrable over X provided for each $\epsilon > 0$, there is a $\delta > 0$ such that for every natural number n and measurable subset E of X,*

$$\text{if } \mu(E) < \delta, \text{then} \int_E |f_n| d\mu < \epsilon.$$

The sequence $\{f_n\}$ is said to be tight over X provided for each $\epsilon > 0$, there is a subset X_0 of X that has finite measure and, for any natural number n,

$$\int_{X \setminus X_0} |f_n| d\mu < \epsilon.$$

Theorem 1.2.3 (The Vitali Convergence Theorem)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ a sequence of functions on X that is both uniformly integrable and tight over X. Assume $f_n \to f$ pointwise almost everywhere on X and the function f is integrable over X. Then*

$$\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu.$$

In particular, if X is of finite measure, then the tightness condition on $\{f_n\}$ is removable.

Theorem 1.2.4 (Gronwall inequality)**.** *Let $T > 0$ and $\alpha(\cdot)$ an integrable function. Let $u(\cdot)$ be a Borel measurable bounded nonnegative function on $[0, T]$ and $\beta(\cdot)$ a nonnegative integrable function on $[0, T]$. If*

$$u(t) \leq \alpha(t) + \int_0^t \beta(s) u(s) ds$$

for all $0 \le t \le T$, *then*

$$u(t) \le \alpha(t) + \int_0^t \alpha(s)\beta(s)\exp\Big(\int_s^t \beta(r)dr\Big)ds$$

for all $0 \le t \le T$. *In addition, if* $\alpha(t) \ge 0$ *for* $0 \le t \le T$, *then*

$$u(t) \le \int_0^t \alpha(s)\beta(s)\exp\Big(\int_s^t \beta(r)dr\Big)ds$$

for $0 \le t \le T$. *In particular, if* $\alpha(t) = 0$ *for all* $0 \le t \le T$, *then we conclude that* $u(t) = 0$ *for all* $0 \le t \le T$.

In what follows, we shall recall some facts regarding the Sobolev spaces. The interested reader is referred to [3, 19] for more details.

By domain, we mean an open connected set in $\mathbb{R}^d$. Let $G \subset \mathbb{R}^d$ be a bounded domain, and ∂G denote the boundary of G.

Definition 3 (Geometric properties of boundary [19, App. C.1]). *We say the boundary* ∂G *is* C^k *if for each point* $x_0 \in \partial G$, *there exist* $r > 0$ *and a* C^k*-function* $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ *such that (upon relabeling and reorienting the coordinates if necessary) we have*

$$G \cap B(x_0, r) = \{x \in B(x_0, r) : x_d > \phi(x_1, \cdots, x_{d-1})\}$$

Likewise, ∂G *is* C^∞ *if* ∂G *is* C^k *for all* $k \in \mathbb{N}$, *and* ∂G *is analytic if* ϕ *is analytic.*

Definition 4 (Outer normal [19, App. C.1]).

(i) If ∂G *is* C^1, *then along* ∂G *is defined the outward pointing unit normal vector field* $\mathbf{n} = (n_1, \cdots, n_d)$. *The unit normal at any point* $x_0 \in \partial G$ *is* $\mathbf{n}(x_0) = n = (n_1, \cdots, n_d)$

(ii) Let $u \in C^1(\bar{G})$. *We call*

$$\frac{\partial u}{\partial n} := \mathbf{n} \cdot Du$$

the (outward) normal derivative of u.

Throughout this thesis, we assume that G is a bounded domain with smooth boundary, i.e., ∂G is C^∞.

Let $C_c^\infty(G)$ denote the space of infinitely differentiable functions $\phi : G \to \mathbb{R}$, with compact support in G. For a multiindex $\alpha = (\alpha_1, \cdots, \alpha_n)$ of order $|\alpha| = \alpha_1 + \cdots + \alpha_n = k$, we write

$$D^\alpha \phi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_1^{\alpha_n}} \phi.$$

Now we are in a position to introduce the definitions of weak derivative and Sobolev spaces.

Definition 5 (Weak Derivative)**.** *Suppose $u, v \in L^1_{loc}(G)$ and α is a multiindex. We say that $v \in L^1_{loc}(G)$ is the α^{th}-weak partial derivative of u, written*

$$D^\alpha u = v,$$

provided

$$\int_G u D^\alpha \phi dx = (-1)^{|\alpha|} \int_G v \phi dx$$

for all $\phi \in C_c^\infty(G)$.

Definition 6 (The Sobolev space)**.** *Fix $1 \leq p \leq \infty$ and let k be a nonnegative integer. The Sobolev space*

$$W^{k,p}(G)$$

consists of all locally summable function $u : G \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense[i] *and belongs to $L^p(G)$.*

[i]in the sense of Definition 5

The norm of $W^{k,p}(G)$ is defined by

$$\|u\|_{W^{k,p}(G)} := \begin{cases} \left(\sum_{|\alpha|\le k} \int_G |D^\alpha u|^p dx \right)^{\frac{1}{p}} & (1 \le p < \infty) \\ \sum_{|\alpha|\le k} \operatorname{ess\,sup}_G |D^\alpha u| & (p = \infty). \end{cases}$$

If $u \in C(G)$, then it is clear that u has certain values on ∂G. Therefore, it is intuitive for defining $C_0(G)$ as the space of continuous functions defined on G that vanish on ∂G. After introducing the Sobolev spaces $W^{1,p}(G)$, one may analogously define $W_0^{1,p}(G)$ as the space of $W^{1,p}$-functions that vanish on ∂G. This definition is plausible. However, notice that a function $u \in W^{1,p}(G)$ is only defined almost everywhere on G, and the d-dimensional Lebesgue measure of ∂G is zero. Therefore, the assertion "u vanishes on ∂G" is (at least, at this stage) meaningless.

To resolve this problem, we shall introduce the notion of trace operator.

Theorem 1.2.5 (Trace theorem [19, Sec. 5.5])**.** *Assume G is bounded and ∂G is C^1. Then there exists a bounded linear operator*

$$T : W^{1,p}(G) \to L^p(\partial G)$$

such that

(i) $Tu = u|_{\partial G}$ if $u \in W^{1,p}(G) \cap C(\bar{G})$, and

(ii) $\|Tu\|_{L^p(\partial G)} \le C\|u\|_{W^{1,p}(G)}$ for each $u \in W^{1,p}(G)$, with the constant C depending only on p and G.

Moreover, we call Tu the trace of u on ∂G.

With the concept of trace, we may introduce the space $W_0^{1,p}(G)$ formally.

Theorem 1.2.6 (Trace-zero functions in $W^{1,p}$ [19, Sec. 5.5])**.** *Assume G is bounded and ∂G*

is C^1. Suppose furthermore that $u \in W^{1,p}(G)$. Then

$$u \in W_0^{1,p}(G) \quad \textit{if and only if} \quad Tu = 0 \quad \textit{on} \quad \partial G.$$

The following theorems are the Sobolev inequality (see, e.g., [3, Thm. 4.12]) and the Gagiardo-Nirenberg-Sobolev inequality (see, e.g., [3, Ch. 5]). We shall remind the reader that the Sobolev inequalities may be proved under different conditions of ∂G. The interest reader may consult [3, Ch. 4] for a detailed discussion.

Theorem 1.2.7 (Sobolev Embedding/ Sobolev Inequality)**.** *Let $G \subset \mathbb{R}^n$ be a bounded domain and C a constant.*

(i) If $n > kp$, then $W^{k,p}(G) \hookrightarrow L^{\frac{np}{n-kp}}(G)$. In other words,

$$\|u\|_{L^{\frac{np}{n-kp}}}(G) \leq C\|u\|_{W^{k,p}(G)}.$$

(ii) If $n < kp$, then $W^{k,p}(G) \hookrightarrow C^{k-1-\lfloor \frac{n}{p} \rfloor, 1+\lfloor \frac{n}{p} \rfloor - \frac{n}{p}}(G)$.

In particular, if $n = 2$, then

$$W^{\frac{1}{2},2}(G) \hookrightarrow L^4(G); \tag{1.7}$$

if $n = 3$, then

$$W^{\frac{1}{2},2}(G) \hookrightarrow L^3(G); \tag{1.8}$$

Theorem 1.2.8 (The interpolation inequality/ Gagiardo-Nirenberg-Sobolev inequality)**.** *Let p, q, r, j, k, ℓ and α satisfy the following relation*

$$j \leq k \lneqq \ell, \ \frac{1}{p} - \frac{k}{n} = \alpha\Big(\frac{1}{q} - \frac{\ell}{n}\Big) + (1-\alpha)\Big(\frac{1}{r} - \frac{j}{n}\Big),$$
$$p, q, r \geq 1, \ 0 < \alpha \leq 1, \ \frac{1}{n} > \frac{1}{q} - \frac{1}{p} \geq 0.$$

Then we have

$$\|u\|_{W^{k,p}(G)} \le C\|u\|^{\alpha}_{W^{\ell,q}(G)}\|u\|^{1-\alpha}_{W^{j,r}(G)}$$

for all $u \in W^{\ell,q}(G) \cap W^{j,r}(G)$*, where* C *is a constant depending on* G*.*

In particular, when $n = 3$, we have

$$\|u\|_{W^{\frac{1}{2},2}(G)} \le C\|u\|^{\frac{1}{2}}_{W^{1,2}(G)}\|u\|^{\frac{1}{2}}_{L^2(G)}. \tag{1.9}$$

Next, we review a few essential concepts and results pertaining to operators on Hilbert spaces. The interested reader is referred to [11, 50] for more details.

Let $(U, (\cdot,\cdot)_U)$ and $(H, (\cdot,\cdot)_H)$ be two real separable Hilbert spaces. We denote by $\mathcal{L}(U,H)$ the space of all bounded linear operators from U to H. For simplicity, we use $\mathcal{L}(U)$ to denote $\mathcal{L}(U,U)$. For an operator $T \in \mathcal{L}(U,H)$, we denote by $T^* \in \mathcal{L}(H,U)$ its adjoint operator. An element $T \in \mathcal{L}(U)$ is called symmetric if $(Tu,v)_U = (u,Tv)_U$ for all $u, v \in U$. In addition, $T \in \mathcal{L}(U)$ is called nonnegative if $(Tu,u)_U \ge 0$ for all $u \in U$. Let $T \in \mathcal{L}(U)$ and $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of U. Then the trace of T is defined by

$$trT := \sum_{k\in\mathbb{N}} (Te_k, e_k)_U$$

if the series is convergent. In addition, the trace of T takes the same value for any orthonomal basis of U, and thus it is well-defined (see, e.g., [50, Rmk. B. 0.4.]).

We shall start by introducing some important operators on real separable Hilbert spaces.

Definition 7 (Nuclear operator)**.** *An element* $T \in \mathcal{L}(U,H)$ *is said to be a nuclear operator* *if there exists a sequence* $\{a_j\}_{j=1}^{\infty}$ *in* H *and a sequence* $\{b_j\}_{j=1}^{\infty}$ *in* U *such that*

$$Tu = \sum_{j=1}^{\infty} a_j (b_j, u)_U$$

for all $u \in U$ *and*

$$\sum_{j=1}^{\infty} \|a_j\|_H \|b_j\|_U < \infty.$$

Denote by $\mathcal{L}_1(U, H)$ *the collection of all nuclear operators from* U *to* H*. For* $T \in \mathcal{L}_1(U, H)$*, define*

$$\|T\|_{\mathcal{L}_1(U,H)} := \inf \Big\{ \sum_{j=1}^{\infty} \|a_j\|_H \|b_j\|_U : Tu = \sum_{j=1}^{\infty} a_j (b_j, u)_U, \ u \in U \Big\}.$$

Then $\|\cdot\|_{\mathcal{L}_1(U,H)}$ *is a norm, and* $(\mathcal{L}_1(U, H), \|\cdot\|_{\mathcal{L}_1(U,H)})$ *is a Banach space, which is called a Nuclear space.*

In particular, if $U = H$ and $T \in \mathcal{L}_1(U, H)$ is nonnegative and symmetric, then T is called trace-class operator.

Definition 8 (Hilbert-Schmidt operator)**.** *A bounded linear operator* $T : U \to H$ *is called Hilbert-Schmidt if*

$$\|T\|^2_{\mathcal{L}_2(U,H)} := \sum_{k=1}^{\infty} \|Te_k\|^2_H < \infty,$$

where $\{e_k\}_{k=1}^{\infty}$ *is an orthonormal basis of* U*. The number* $\|T\|^2_{\mathcal{L}_2(U,H)}$ *is independent of the choice of the orthonormal basis* $\{e_k\}_{k=1}^{\infty}$ *and thus is well-defined (see, e.g., [50, Rmk. B. 0.6]). Denote by* $\mathcal{L}_2(U, H)$ *the collection of all Hilbert-Schmidt operators from* U *to* H*. Then* $(\mathcal{L}_2(U, H), \|\cdot\|_{\mathcal{L}_2(U,H)})$ *is a Banach space. Furthermore, for* $S, T \in \mathcal{L}_2(U, H)$*, define*

$$(T, S)_{\mathcal{L}_2} := \sum_{k=1}^{\infty} (Se_k, Te_k)_H,$$

then $(\mathcal{L}_2(U, H), (\cdot, \cdot)_{\mathcal{L}_2})$ *is a separable Hilbert space.*

The following lemma gives useful relationships between $\mathcal{L}_1(U,H)$, $\mathcal{L}_2(U,H)$, and $\mathcal{L}(U,H)$.

Lemma 1.2.9. *Let $(K,(\cdot,\cdot)_K)$ be a further real separable Hilbert space.*

(i) If $T \in \mathcal{L}_2(U,H)$ and $S \in \mathcal{L}_2(H,K)$, then $ST \in \mathcal{L}_1(U,K)$ and

$$\|ST\|_{\mathcal{L}_1(U,K)} \le \|S\|_{\mathcal{L}_2(H,K)}\|T\|_{\mathcal{L}_2(U,H)}.$$

(ii) Let $S_1 \in \mathcal{L}(H,K)$, $S_2 \in \mathcal{L}(K,U)$, and $T \in \mathcal{L}_2(U,H)$. Then $S_1T \in \mathcal{L}_2(U,K)$, $TS_2 \in \mathcal{L}_2(K,H)$, and

$$\|S_1T\|_{\mathcal{L}_2(U,K)} \le \|S_1\|_{\mathcal{L}(H,K)}\|T\|_{\mathcal{L}_2(U,H)}$$
$$\|TS_2\|_{\mathcal{L}_2(K,H)} \le \|S_2\|_{\mathcal{L}(K,U)}\|T\|_{\mathcal{L}_2(U,H)}.$$

Recall that a bounded operator $T : U \to U$ is called compact operator if the image under T of a bounded sequence has a convergent subsequence. It is worth mentioning that both nuclear operators and Hilbert-Schmidt operators are compact operators.

Let $G \subset \mathbb{R}^3$ be a bounded domain with smooth boundary.[ii] Throughout this thesis, the notation $L^2(G)$, $H^1(G)$ etc. would mean three-dimensional vector-valued functions defined on G whose components are in $L^2(G)$, $H^1(G)$ etc.

Let $\mathcal{V}$ denote the divergence-free members in $C_0^\infty(G)$. Define the spaces H and V as the completion of $\mathcal{V}$ in $L^2(G)$ and $H^1(G)$ norms respectively. Recalling Definition 4, Theorem 1.2.5, and Theorem 1.2.6, we are in a position to introduce the characterization of spaces H and V (see, e.g., [59, Sec. 1.4, Ch. I]):

$$H = \{\mathbf{u} \in L^2(G) : \nabla\cdot\mathbf{u} = 0,\ \mathbf{u}\cdot\mathbf{n}\ _{\partial G} = 0\},$$

$$V = \{\mathbf{u} \in W_0^{1,2}(G) : \nabla\cdot\mathbf{u} = 0\},$$

[ii]Definition 3

where the divergence $\nabla \cdot \mathbf{u}$ is understood in the weak sense[iii] and $\mathbf{n}$ is the outward normal of G.

Invoking the Helmholtz decomposition, one may decompose the space $L^2(G)$ as the direct sum of H and its complement, i.e.,

$$L^2(G) = H \oplus H^{\perp}. \tag{1.10}$$

Having such a decomposition, one may further define the orthogonal projector

$$\prod_H : L^2(G) \to H, \tag{1.11}$$

which is known as the Leray-Helmholtz projector. For the detailed discussion regarding the space $H^{\perp}$ and the Leray-Helmholtz projector, we refer the interested reader to, e.g., [54, Sec. 2.5, Ch. II], [58, Sec. 1.6], and [59, Sec. 1.4, Ch. I].

Let V' denote the dual space of V. Define the Stokes operator $\mathbf{A} : V \to V'$ by $\mathbf{Au} = -\prod_H \Delta\mathbf{u}$ for $\mathbf{u} \in D(\mathbf{A})$, where $\mathcal{D}(\mathbf{A})$ is defined as

$$D(\mathbf{A}) := \left\{\mathbf{u} \in H : -\prod_H \Delta\mathbf{u} \in H\right\} = W^{2,2}(G) \cap V,$$

and the second equal sign above is guaranteed by the Cattabriga-Solonnikov Regularity Theorem (see, e.g., [54, Lem. 2.3.2, Ch. III]).

It can be shown that $\mathbf{A} : \mathcal{D}(\mathbf{A}) \to H$ is bijective if G is bounded, which implies that $\mathbf{A}^{-1} : H \to \mathcal{D}(\mathbf{A})$ is well-defined and onto (see, e.g., [24, Eq. (6.2)] or [54, Thm. 2.1.1, Ch. III] and reference therein). In addition, it follows from the Rellich theorem that the natural inclusion $i : \mathcal{D}(\mathbf{A}) \hookrightarrow H$ is compact (see, e.g., [3, Thm. 6.3], [19, Sec. 5.7]).

[iii] in the sense of Definition 5

Therefore, if we view $\mathbf{A}^{-1}$ as an operator from H to H, then $\mathbf{A}^{-1} : H \to H$ is a compact operator.

It can be shown, by using integration by parts, that $\mathbf{A}$ is symmetric: $\big(\mathbf{Au}, \mathbf{v}\big)_H = \big(\mathbf{u}, \mathbf{Av}\big)_H$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{D}(\mathbf{A})$; it turns out that $\mathbf{A}^{-1}$ is also symmetric.

Thus, we infer from the spectral theory of compact symmetric operator in Hilbert space that there exists an orthonormal basis in H, which consists of the eigenfunctions of $\mathbf{A}^{-1}$ (see, e.g., [2, Ch. V], [11, Thm. 6.11], [19, Thm. 7, App. D]). We shall state such statements formally as the following theorem and refer the interested reader to, e.g., [10, Thm. IV. 5.5] or [59, Sec. 2.6, Ch. I].

Theorem 1.2.10. *Let $G \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary.*[iv] *There exists an increasing sequence of positive real numbers $\{\lambda_k\}_{k=1}^{\infty}$, which tends to infinity, and a sequence of functions $\{e_k\}_{k=1}^{\infty}$ which is orthonormal in H, orthogonal in V and in $\mathcal{D}(\mathbf{A})$, forming a (complete) basis in H, in V, and in $\mathcal{D}(\mathbf{A})$, and a sequence of functions $\{p_k\}_{k=1}^{\infty}$ in $L_0^2(G)$*[v] *satisfying*

$$-\Delta e_k + \nabla p_k = \lambda_k e_k \quad \text{in} \quad G, \tag{1.12a}$$

$$\nabla \cdot e_k = 0 \qquad \text{in} \quad G, \tag{1.12b}$$

$$e_k = 0 \qquad \text{on} \quad \partial G. \tag{1.12c}$$

Utilizing the special basis $\{e_k\}_{k=1}^{\infty}$ obtained from Theorem 1.2.10, one obtains $\mathcal{D}(\mathbf{A}^{\frac{1}{2}}) = V$ and $\mathcal{D}(\mathbf{A}^{-\frac{1}{2}}) = V'$. In addition, $\|\mathbf{u}\|_V = \|\nabla \mathbf{u}\|_H = \|\mathbf{A}^{\frac{1}{2}}\mathbf{u}\|_H$ for $\mathbf{u} \in \mathcal{D}(\mathbf{A}^{\frac{1}{2}})$. Therefore, it is not hard to see that $(\mathbf{Au}, \mathbf{u})_H = \|\mathbf{u}\|_V^2 > 0$ for all $\mathbf{u} \neq 0$ in $\mathcal{D}(\mathbf{A})$, which

[iv] ϕ is Lipschitz continuous in Definition 3.

[v] $L_0^2(G) := L^2(G)/\mathbb{R}$ is a quotient space. See [10, Def. IV. 1. 8].

concludes that $\mathbf{A}$ is positive definite. For a detailed discussion, we indicate the reader to, e.g., [24, Sec. 6, Ch. II], [54, Sec. 2, Ch. III].

The asymptotic behavior of λ_k has been derived in [40]: $\lambda_k \sim C_{(d,G)} \cdot k^{\frac{2}{d}}$, as $k \to \infty$, where d is the dimension of the domain G and $C_{d,G}$ is a constant depending on d and G. The interested reader may also consult [29] for more details.

The Poincaré inequality in the context of this thesis appears to be

$$\|\mathbf{u}\|_H^2 \leq \frac{1}{\lambda_1} \|\mathbf{u}\|_V^2 \tag{1.13}$$

for all $\mathbf{u} \in V$, where λ_1 is the first eigenvalue of the Stokes operator $\mathbf{A}$ and is the best constant for which inequality (1.13) holds. The reader may consult [24, Eq. (5.11), Ch. II] and references therein. For Poicaré inequality in general contexts, we indicate reader to, e.g.,[19, Sec. 5.8], [24, Sec. 4, Ch. I], or [54, Sec. 1, Ch. II].

Next, we recall the basic results on weak and weak-star topologies. The contents are adapted from [11, Ch. 3].

Let E be a Banach space and let $f \in E'$, where E' is the dual space of E. We denote by $\varphi_f(x) : E \to \mathbb{R}$ the linear function $\varphi_f(x) = \langle f, x \rangle$. Then the weak topology on E, denoted $\tau(E, E')$, refers to the coarsest topology on E under which the map φ_f for each $f \in E'$ is continuous. For every $x \in E$ consider the linear functional $\varphi_x : E' \to \mathbb{R}$ defined by $f \mapsto \varphi_x(f) = \langle f, x \rangle$. The weak-star topology $\tau(E', E)$ is the coarsest topology on E' with respect to which the collection $\{\varphi_x\}_{x \in E}$ is continuous.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in E. By $x_n \to x$ weakly in E, we mean $\langle f, x_n \rangle \to \langle f, x \rangle$ for any $f \in E'$. The next theorem is a special case of Banach-Alaoglu Theorem, which characterizes an important property of the weak-star topology. The interested reader is referred to, e.g., [11, Ch. 3] or [19, App. D.4].

Theorem 1.2.11 (Banach-Alaoglu Theorem)**.** *Let E be a reflexive Banach space. Suppose the sequence $\{x_n\}_{n=1}^{\infty} \subset E$ is bounded. Then there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ and $x \in E$ such that*

$$x_{n_j} \to x$$

weakly in E.

Recall that a Polish space is a complete separable metrizable topological space (see, e.g., [34, Sec. 33, Ch. 3]). Then we introduce Lusin space (see, e.g., [41, Ch. IV]) and one theorem regarding it (see, e.g., [41, Thm. 1, Ch. IV]).

Definition 9 (Lusin space)**.** *A topological space Y that is the image by a one-to-one continuous mapping $f : X \to Y$ of a Polish space X is called a Lusin space.*

Theorem 1.2.12. *A topological space Y is a Lusin space if and only if it is homeomorphic to a topological space which is a Borel set B of a Polish space.*

1.2.3. Facts from stochastic analysis

We shall collect some concepts and facts pertaining to the infinite-dimensional stochastic analysis. The interested reader is referred to, e.g., [17], [26], [45], and [50] for details.

Throughout this subsection, $(U, (\cdot, \cdot)_U)$ is a real separable Hilbert space, and $\mathcal{L}(U)$ denotes the set of all bounded linear operator on U. We use $\mathcal{B}(X)$ to denote the Borel σ-algebra of a topological space X.

Definition 10 (Gaussian measure)**.** *Fix an element $v \in U$, define the bounded linear mapping $\phi_v : U \to \mathbb{R}$ by $\phi_v(u) := (u, v)_U$ for all $u \in U$. A probability measure μ on $(U, \mathcal{B}(U))$ is called Gaussian if for all $v \in U$, the mapping ϕ_v has a Gaussian law, i.e., for all $v \in U$ there exists $m := m(v) \in \mathbb{R}$ and $\sigma := \sigma(v) \in [0, \infty)$ such that, if $\sigma(v) > 0$,*

$$(\mu \circ \phi_v^{-1})(A) = \mu(\phi_v \in A) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_A e^{-\frac{(x-m^2)}{2\sigma^2}} dx$$

for all $A \in \mathcal{B}(\mathbb{R})$ and, if $\sigma(v) = 0$,

$$\mu \circ \phi_v^{-1} = \delta_{m(v)},$$

which is a Dirac delta function centered at $m(v)$.

Let $N(\mathbf{m}, Q)$ be an $\mathbb{R}^d$-valued Gaussian random vector with mean vector $\mathbf{m} = (m_1, \cdots, m_d)$ and covariance matrix $Q = (q_{ij})_{d\times d}$. Then the characteristic function (or Fourier transform) of $N(\mathbf{m}, Q)$ is given by

$$F_N(\mathbf{u}) = e^{i\mathbf{m}^T\mathbf{u} - \frac{1}{2}\mathbf{u}^T Q\mathbf{u}}, \tag{1.14}$$

where $\mathbf{u} \in \mathbb{R}^d$ is a vector and $\mathbf{u}^T$ denote the transpose of $\mathbf{u}$, and it is known that the characteristic function uniquely determines a random variable. We will see in next theorem that the same properties hold for the infinite-dimensional setting, i.e., the Fourier transform of a Gaussian measure takes form similar to (1.14), and the Fourier transform also uniquely determines such a measure. In addition, in the finite-dimensional case, the covariance matrix Q is a nonnegative symmetric matrix, and we will see that those properties are inherited by the "covariance operator" in the infinite-dimensional setting.

Denote by $\hat{\mu}$ the Fourier transform of μ.

Theorem 1.2.13. *A measure μ on $(U, \mathcal{B}(U))$ is Gaussian if and only if*

$$\hat{\mu}(u) := \int_U e^{i(u,v)_U} \mu(dv) = e^{i(m,u)_U - \frac{1}{2}(Qu,u)_U},$$

where $u \in U$, $m \in U$, and $Q \in \mathcal{L}(U)$ is nonnegative, symmetric with finite trace (trace-class[vi]*). In this case μ will be denoted by $N(m, Q)$, where m is called mean and Q is called covariance operator.The measure μ is uniquely determined by m and Q. Furthermore, for all $h, g \in U$,*

$$\int (x,h)_U \mu(dx) = (m,h)_U,$$
$$\int ((x,h)_U - (m,h)_U)((x,g)_U - (m,g)_U)\mu(dx) = (Qh,g)_U$$
$$\int \|x-m\|_U^2 \mu(dx) = trQ.$$

Proof. We refer the reader to [50, Thm. 2.1.2]. □

Let $Q \in \mathcal{L}(U)$ be a nonnegative symmetric trace-class operator. Then Q is a compact operator, hence, there exists an orthonormal basis $\{\mathfrak{o}_k\}_{k=1}^{\infty}$ of U consisting of eigenvectors of Q with corresponding eigenvalues $\{\zeta_k\}_{k=1}^{\infty}$. Then the following proposition connects the Gaussian measure in infinite dimension to the Gaussian random variable in one dimension.

Proposition 1.2.14 (Representation of a Gaussian random variable)**.** *Let $m \in U$ and $Q \in \mathcal{L}(U)$ be nennegative, symmetric, with finite trace. Then a U-valued random variable X on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is Gaussian with $\mathcal{P} \circ X^{-1} = N(m, Q)$ if and only if*

$$X = \sum_{k \geq 1} \sqrt{\zeta_k} \beta_k \mathfrak{o}_k + m,$$

[vi]see the paragraph after Definition 7

where β_k, $k \geq 1$, are independent real-valued random variables with $\mathcal{P} \circ \beta_k^{-1} = N(0,1)$ for all $k \geq 1$ with $\zeta_k > 0$. The series converges in $L^2(\Omega, \mathcal{F}, \mathcal{P}; U)$.

Now we are in a position to introduce one the key ingredients in the infinite-dimensional stochastic analysis: Q-Wiener process. To this end, we let $Q \in \mathcal{L}(U)$ be nonnegative, symmetric, and of trace-class. Let $T > 0$ be a fixed real number.

Definition 11 (Q-Wiener process). *A U-valued stochastic process $W(t)$, $t \in [0,T]$, on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is called a Q-Wiener process if*

(i) $W(0) = 0$,

(ii) W has continuous trajectories $\mathcal{P}$-almost surely,

(iii) W had independent increments, i.e., the random variables

$$W(t_1), W(t_2) - W(t_1), \cdots, W(t_n) - W(t_{n_1})$$

are independent for all $0 \leq t_1 < \cdots < t_n \leq T$, $n \in \mathbb{N}$,

(iv) the increments have the following Gaussian laws:

$$\mathcal{P} \circ (W(t) - W(s))^{-1} = N(0, (t-s)Q)$$

for all $0 \leq s \leq t \leq T$.

Similar to the infinite-dimensional Gaussian random variable, the Q-Wiener process admits the following representation formula which connects it to the real-valued Brownian motion.

Proposition 1.2.15 (Representation of the Q-Wiener process). *Let $\{\mathfrak{o}_k\}_{k=1}^{\infty}$ be the orthonormal basis of U consisting of the eigenvectors of Q with corresponding eigenvalues $\{\zeta_k\}_{k=1}^{\infty}$. Then a U-valued stochastic process $W(t)$, $t \in [0,T]$, is a Q-Wiener process if and* *only if*

$$W(t) = \sum_{k \geq 1} \sqrt{\zeta_k} \beta_k(t) \mathfrak{o}_k,$$

where $t \in [0, T]$ and β_k, $k \in \{n \in \mathbb{N} : \zeta_n > 0\}$, are independent real-valued Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The series converges in $L^2(\Omega, \mathcal{F}, \mathcal{P}; C([0, T]; U))$, and thus always has a $\mathcal{P}$-almost surely continuous modification. In particular, for any Q as above, there exists a Q-Wiener process on U.

Let $(H, (\cdot, \cdot)_H)$ be another real separable Hilbert space and $Q \in \mathcal{L}(U)$ be nonnegative, symmetric, and of trace-class. Define the subspace $U_0 := Q^{\frac{1}{2}}(U)$ with the inner product given by

$$(u, v)_0 := (Q^{-\frac{1}{2}} u, Q^{-\frac{1}{2}} v)_U$$

for $u, v \in U_0$, where $Q^{-\frac{1}{2}}$ is the pseudo inverse of $Q^{\frac{1}{2}}$ in the case that Q is not one-to-one (see, e.g., [50, App. C]). Then it follows from [50, Prop. C.0.3 (i)] that $(U_0, (\cdot, \cdot)_0)$ is again a separable Hilbert space.

Let $\mathcal{L}_2(U_0, H)$ denote the separable Hilbert space of the Hilbert-Schmidt operators[vii] from U_0 to H. Then it can be shown that (see, e.g., [50, p. 27])

$$\|L\|_{\mathcal{L}_2(U_0,H)} = \|L \circ Q^{\frac{1}{2}}\|_{\mathcal{L}_2(U,H)} \tag{1.15}$$

for each $L \in \mathcal{L}_2(U_0, H)$.

Let $T > 0$ be a fixed real number, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$ be a filtered probability space, and $\sigma : [0, T] \times \Omega \to \mathcal{L}_2(U_0, H)$ be predictable. If $\mathbb{E} \int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2(U_0,H)} ds < \infty$, then for

[vii] Definition 8

$t \in [0,T]$, the stochastic integral

$$\int_0^t \sigma(s)dW(s)$$

is well-defined and is a H-valued continuous square integrable martingale with the Itô isometry (see, e.g., [50, Sec. 2.3])

$$\mathbb{E}\left\|\int_0^T \sigma(s)dW(s)\right\|_H^2 = \mathbb{E}\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2(U_0,H)}ds.$$

Define

$$\langle M\rangle_t := \int_0^t \|\sigma(s)\|^2_{\mathcal{L}_2(U_0,H)}ds$$

for $t \in [0,T]$. Then it follows from [50, Lem. 2.4.3] that $\langle M\rangle_t$ is the unique continuous increasing $(\mathcal{F}_t)$-adapted process starting at zero such that $\|M(t)\|_H^2 - \langle M\rangle_t$, $t \in [0,T]$, is a local martingale.

Let f be an $(\mathcal{F}_t)$-adapted continuous H-valued process. Define

$$\int_0^T \langle f(s), \sigma(s)dW(s)\rangle := \int_0^T \tilde{\sigma}_f(s)dW(s), \tag{1.16}$$

where

$$\tilde{\sigma}_f(s)(u) := \big(f(s), \sigma(s)u\big)_H,$$

$u \in U_0$. Then it follows from [50, Lem. 2.4.2] that (1.16) is a well-defined continuous real-valued stochastic process. More precisely, $\tilde{\sigma}_f$ is a $\mathcal{P}_T/\mathcal{B}(\mathcal{L}_2(U_0,\mathbb{R}))$-measurable map[viii] from $\Omega\times[0,T]$ to $\mathcal{L}_2(U_0,\mathbb{R})$ with

$$\|\tilde{\sigma}_f(\omega,t)\|_{\mathcal{L}_2(U_0,\mathbb{R})} = \|\sigma^*(\omega,t)f(\omega,t)\|_{U_0}$$

[viii] $\mathcal{P}_T = dx \otimes \mathcal{P}$, where dx is the Lebesgue measure.

for all $(\omega, t) \in \Omega \times [0, T]$ and, for $\mathcal{P}$-almost surely,

$$\int_0^T \|\tilde{\sigma}_f(s)\|^2_{\mathcal{L}_2(U_0,\mathbb{R})} ds \leq \sup_{t\in[0,T]} \|f\|_H \int_0^T \|\sigma(s)\|_{\mathcal{L}_2(U_0,H)} ds < \infty.$$

Next, we recall some facts regarding the Poisson random measure, Point processes, and the stochastic integrals with respect to point processes. The contents are adapted from [30, Sec. 8, 9, Ch. I and Sec. 3, Ch II.].

Let $(Z, \mathcal{B}(Z))$ be a measurable space, $\mathbf{M}$ be the collection of all of nonnegative integer-valued measure on $(Z, \mathcal{B}(Z))$, and $\mathcal{B}(\mathbf{M})$ be the smallest σ-field on $\mathbf{M}$ with respect to which all $N \mapsto N(B)$ are measurable, where $N \in \mathbf{M}$, $N(B) \in \mathbb{Z}^+ \cup \{\infty\}$, and $B \in \mathcal{B}(Z)$.

Definition 12 (Poisson random measure)**.** *Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the underline probability space. An $(\mathbf{M}, \mathcal{B}(\mathbf{M}))$-valued random variable $N : \Omega \to \mathbf{M}$ is called a Poisson random measure if*

(i) for each $B \in \mathcal{B}(Z)$, $N(B)$ is Poisson distributed, i.e.,

$$\mathcal{P}(N(B) = n) = \nu(B)^n \frac{e^{-\nu(B)}}{n!}$$

for $n = 0, 1, 2, \cdots$, where $\nu(B) = \mathbb{E}(N(B))$, and

(ii) if $B_1, B_2, \cdots, B_n \in \mathcal{B}(Z)$ are disjoint, then $N(B_1), N(B_2), \cdots, N(B_n)$ are mutually independent.

The measure ν in the above definition is called the intensity measure, the mean measure, or the Lévy measure of the Poisson random measure N.

A point function p on Z is a mapping from $\mathcal{D}(p)$ to Z, where $\mathcal{D}(p)$ is the domain of p and is a countable subset of $(0, \infty)$. A point function p defines a counting measure $N_p(dz, dt)$ on $Z \times (0, \infty)$ by

$$N_p(E \times (0, t)) := \#\{s \in \mathcal{D}(p) : s \leq t,\ p(s) \in E\},$$

where $t > 0$ and $E \in \mathcal{B}(Z)$.

Let Π_Z be the collection of all point functions on Z and $\mathcal{B}(\Pi_Z)$ be the smallest σ-field on Π_Z with respect to which all $p \mapsto N_p(E \times (0,t])$, $t > 0, E \in \mathcal{B}(Z)$, are measurable.

Definition 13. *Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the underline probability space. A point process p on Z is a $(\Pi_Z, \mathcal{B}(\Pi_Z))$-valued random variable, i.e., a mapping $p : \Omega \to \Pi_Z$ which is $\mathcal{F}/\mathcal{B}(\Pi_Z)$-measurable.*

Let p be a point process with domain $\mathcal{D}(p)$. For $t > 0$, define $(\theta_t p)(s) := p(s+t)$, where the domain of $\theta_t p$ is $\mathcal{D}(\theta_t p) = \{s \in (0,\infty) : s + t \in \mathcal{D}(p)\}$. Then p is called stationary if p and $\theta_t p$ have the same probability law for every $t > 0$. A point process p is called Poisson if $N_p(dz, dt)$ is a Poisson random measure on $Z \times (0, \infty)$. A Poisson point process is stationary if and only if its intensity measure $\nu_p(dz, dt) = \mathbb{E}(N_p(dz, dt))$ is of the form

$$\nu_p(dz, dt) = \nu(dz)dt$$

for some measure $\nu(dz)$ on $(Z, \mathcal{B}(Z))$. The measure $\nu(dz)$ is called the characteristic measure of p.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a filtered probability space. A point process $p = p(t)$ on Z defined on Ω is called $(\mathcal{F}_t)$-adapted if for every $t > 0$ and $E \in \mathcal{B}(Z)$,

$$N_p(E, t) = \sum_{\substack{s \in \mathcal{D}(p) \\ s \leq t}} 1_E(p(s))$$

is $\mathcal{F}_t$-measurable. A point process p is called σ-finite if there exist $E_n \in \mathcal{B}(Z)$, $n \in \mathbb{N}$, such that $E_n \uparrow Z$ and $\mathbb{E}(N_p(E_n, t)) < \infty$ for all $t > 0$ and $n \in \mathbb{N}$.

For a given $(\mathcal{F}_t)$-adapted, σ-finite point process p, let

$$\Gamma_p = \{E \in \mathcal{B}(Z) : \mathbb{E}(N_p(E,t)) < \infty \quad \text{for all} \quad t > 0\}.$$

If $E \in \Gamma_p$, then $t \mapsto N_p(E,t)$ is an adapted, integrable increasing process, hence, there exists a natural integrable increasing process $\hat{N}_p(E,t)$ such that $\tilde{N}_p : t \mapsto \tilde{N}_p(E,t) = N_p(E,t) - \hat{N}_p(E,t)$ is a martingale.

Definition 14. *An $(\mathcal{F}_t)$-adapted point process p on $(\Omega, \mathcal{F}, \mathcal{P})$ is said to be of the class (QL)*[ix] *with respect to $(\mathcal{F}_t)$ if it is σ-finite and there exists $\hat{N}_p(E,t)$ such that*

(i) for $E \in \Gamma_p, t \mapsto \hat{N}_p(E,t)$ is a continuous $(\mathcal{F}_t)$-adapted increasing process,

(ii) for each t and almost all $\omega \in \Omega$, $E \mapsto \hat{N}_p(E,t)$ is a σ-finite measure on $(Z, \mathcal{B}(Z))$,

(iii) for $E \in \Gamma_p, t \mapsto \tilde{N}_p(E,t) = N_p(E,t) - \hat{N}_p(E,t)$ is an $(\mathcal{F}_t)$-martingale.

The random measure $\{\hat{N}_p(E,t)\}$ is called the compensator of the point process p (or $\{N_p(E,t)\}$).

Definition 15. *A point process p is called an $(\mathcal{F}_t)$-Poisson point process if it is an $(\mathcal{F}_t)$-adapted, σ-finite Poisson point process such that $\{N_p(E,t+h) - N_p(E,t)\}_{h>0, E\in\mathcal{B}(Z)}$ is independent of $\mathcal{F}_t$.*

An $(\mathcal{F}_t)$-Poisson point process is of class (QL) if and only if $t \mapsto \mathbb{E}(N_p(E,t))$ is continuous for $E \in \Gamma_p$. The compensator in such a case is given by $\hat{N}_p(E,t) = \mathbb{E}(N_p(E,t))$. In particular, a stationary $(\mathcal{F}_t)$-Poisson point process is of the class (QL) with compensator $\hat{N}_p(E,t) = \nu(E)t$, where $\nu(dz)$ is the characteristic measure of p.

Let p be a point process. Then it follows from [30, Thm. 3.1] that $\tilde{N}_p(\cdot, E)$ is a

[ix] Quasi left-continuous

square integrable martingale for $E \in \Gamma_p$, and we have

$$\langle \tilde{N}_p(\cdot, E_1), \tilde{N}_p(\cdot, E_2) \rangle = \hat{N}_p(\cdot, E_1 \cap E_2).$$

Before proceeding to the discussion regarding the stochastic integral with respect to the Poisson random measure, we shall introduce the $(\mathcal{F}_t)$-predictable process. Let H be a real separable Hilbert space.

Definition 16. *A H-valued function $f(\omega, z, t)$ defined on $\Omega \times Z \times [0, \infty)$ is called $(\mathcal{F}_t)$-predictable if the mapping $(\omega, z, t) \to f(\omega, z, t)$ is $\mathsf{P}/\mathcal{B}(H)$-measurable where P is the smallest σ-field on $\Omega \times Z \times [0, \infty)$ with respect to which all g having the following properties are measurable:*

(i) for each $t > 0, (\omega, z) \mapsto g(\omega, z, t)$ is $\mathcal{B}(Z) \times \mathcal{F}_t$-measurable;

(ii) for each (ω, z), $t \mapsto g(\omega, z, t)$ is left-continuous.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$ be the underline filtered probability space. Let p be a stationary $(\mathcal{F}_t)$-Poisson point process of class (QL). Then $N_p(dz, dt)$ is a Poisson random measure with compensator $\hat{N}_p(dz, dt) = \nu(dz)dt$.

Suppose that $f(\omega, z, t)$ is an $(\mathcal{F}_t)$-predictable process. Then

$$\int_0^{t+} \int_Z f(\cdot, z, s) N_p(dz, ds) := \sum_{\substack{s \in \mathcal{D}(p) \\ s \leq t}} f(\cdot, p(s), s).$$

The integral is well-defined whenever the right hand series is absolute convergent. In addition, if f is such that

$$\mathbb{E} \int_0^{t+} \int_Z |f(\cdot, z, s)| \nu(dz) ds < \infty \tag{1.17}$$

for every $t > 0$, then it can be shown that

$$\mathbb{E}\int_0^{t+}\int_Z |f(\cdot,z,s)|N_p(dz,ds) = \mathbb{E}\int_0^{t+}\int_Z |f(\cdot,z,s)|\nu(dz)ds. \tag{1.18}$$

Set

$$\int_0^{t+}\int_Z f(\cdot,z,s)\tilde{N}_p(dz,ds) = \int_0^{t+}\int_Z f(\cdot,z,s)N_p(dz,ds) - \int_0^{t}\int_Z f(\cdot,z,s)\nu(dz)ds.$$

Then $t \mapsto \int_0^{t+} f(\cdot,z,s)\tilde{N}_p(dz,ds)$ is an $\mathcal{F}_t$-martingale.

If f satisfies (1.17), and further,

$$\mathbb{E}\int_0^{t+}\int_Z |f(\cdot,z,s)|^2\nu(dz)ds < \infty,$$

then $t \mapsto \int_0^{t+}\int_Z f(\cdot,z,s)\tilde{N}_p(dz,ds)$ is a square integrable martingale and

$$\langle\int_0^{t+}\int_Z f(\cdot,z,s)\tilde{N}_p(dz,ds)\rangle = \int_0^{t+}\int_Z f^2(\cdot,z,s)\nu(dz)ds \tag{1.19}$$

Let $M(t)$ be a right-continuous submartingale and $T > 0$ be a real number. By basic submartingale inequality (see, e.g., [31, Ch. 3]), we mean

$$\lambda\mathcal{P}\{\sup_{0\le t\le T}|M(t)| \ge \lambda\} \le \mathbb{E}|M(T)|$$

for any $\lambda > 0$.

Let $M(t)$ be a càdlàg (right continuous with left limits) process. Denote by $[M]_t = [M,M]_t$ the quadratic variational process of $M(t)$. The following well-known theorem gives us lower and upper L^p-bound estimates for the expectation of the supremum of a martingale. The case for $1 < p < \infty$ was established by Burkholder [5], and the case of $0 < p < 1$ was established by Burkholder and Gundy [6]. Finally, the case of $p = 1$ was established by Davis [14]; we therefore refer the *Davis inequality* to the case of $p = 1$.

If $M(t)$ is of $\mathbb{R}^d$-valued, the interested reader may consult, e.g., [30, Sec. 3, Ch. III], [31, Sec. 5.6], [32, Ch. 17] or, [49, Sec. 5, Ch. IV] for the detailed discussion and proof; if $M(t)$ is of infinite-dimensional Hilbert spaced valued, the interested reader may consult, e.g., [12, Ch. 6], [38], or [51, Sec. 3.9] the detailed discussion and proof.

Theorem 1.2.16 (Burkholder-Davis-Gundy inequality)**.** *Let $T > 0$. For any $0 < p < \infty$, there exist universal constants $c_{p/2}$ and $C_{p/2}$ such that*

$$c_{p/2}\mathbb{E}\Big([M]_T^{p/2}\Big) \le \mathbb{E}\Big(\sup_{0\le t\le T}|M(t)|^p\Big) \le C_{p/2}\mathbb{E}\Big([M]_T^{p/2}\Big).$$

It may be of interest to study the optimality of the constants. The interested reader may consult, e.g., [4], [15, Sec. 3, Ch. VII], [25], [47] and references therein. For the generalizations of Burkholder-Davis-Gundy inequality, we indicate the reader to, e.g., [35].

In this thesis, we take $C_{1/2} = \sqrt{2}$ for continuous martingales starting from 0 (see, e.g., [47]); for general càdlàg martingales, we take $C_{1/2} = \sqrt{10}$ (see, [25]). For general $1 < p < \infty$ and càdlàg martingales, we take $C_{p/2} = \left(\frac{p}{p-1}\right)^p$ (see, [31, Thm. 5.6.3]).

Let K be a complete separable metric space with distance d. For $T > 0$, we denote by $\mathcal{D}([0,T];K)$ the K-valued càdlàg functions defined on $[0,T]$. Let Λ_T be the set of increasing homeomorphisms of $[0,T]$. Define the metric

$$\delta_T(x,y) = \inf_{\lambda\in\Lambda_T}\Big\{\sup_{t\in[0,T]} d(x(t), x\circ\lambda(t)) + \sup_{t\in[0,T]}|t-\lambda(t)| + \sup_{s\neq t}\ \log\frac{\lambda(t)-\lambda(s)}{t-s}\Big\}. \quad (1.20)$$

Then it is known that $(\mathcal{D}([0,T];K), \delta_T)$ is a complete separable metric space (see, e.g., [20]). The topology induced by δ_T is called the (Skorohod) J-topology and the space $\mathcal{D}([0,T];K)$ endowed with J-topology is called the Skorohod space. For more information regarding the Skorohod space, we refer the interested reader to [9, 41].

Let $m \in \mathbb{N}$. Let $\{\mathfrak{r}(t) : t \in \mathbb{R}^+\}$ be a right continuous ergodic [x] Markov chain with generator $\Gamma = (\gamma_{ij})_{m\times m}$ taking values in $\mathcal{S} := \{1, 2, 3, ...m\}$ such that

$$\mathcal{R}_t(i,j) = \mathcal{R}(\mathfrak{r}(t+h) = j | \mathfrak{r}(t) = i) = \begin{cases} \gamma_{ij}h + o(h) & \text{if} \quad i \neq j, \\ 1 + \gamma_{ii}h + o(h) & \text{if} \quad i = j, \end{cases}$$

and

$$\gamma_{ii} = -\sum_{i\neq j} \gamma_{ij}.$$

The transition probability $\mathcal{R}_t(i,j)$ satisfies the Chapman-Kolmogorov equation:

$$\mathcal{R}_{t+s}(i,k) = \sum_{j=1}^{m} \mathcal{R}_s(i,j)\mathcal{R}_t(j,k).$$

There exists a stationary distribution $\pi = (\pi_1, \cdots, \pi_m)$ for this Markov chain $\mathfrak{r}(t)$, where π_j satisfies

$$\lim_{t\to\infty} \mathcal{R}_t(i,j) = \pi_j$$

In addition, $\mathfrak{r}(t)$ admits a stochastic integral representation (see, e.g, [42, Sec. 1.7], [53, Sec. 2.1, Ch. 2], or [64, Ch. 2]): Let Δ_{ij} be consecutive, left closed, right open intervals of

[x] irreducible aperiodic positive recurrent, see [33].

the real line each having length γ_{ij} such that

$$\begin{aligned}
\Delta_{12} &= [0, \gamma_{12}), \\
\Delta_{13} &= [\gamma_{12}, \gamma_{12} + \gamma_{13}), \\
&\vdots \\
\Delta_{1m} &= \Big[\sum_{j=2}^{m-1} \gamma_{1j}, \sum_{j=2}^{m} \gamma_{1j} \Big), \\
&\vdots \\
\Delta_{2m} &= \Big[\sum_{j=2}^{m} \gamma_{1j} + \sum_{j=1, j\neq 2}^{m-1} \gamma_{2j}, \sum_{j=2}^{m} \gamma_{1j} + \sum_{j=1, j\neq 2}^{m} \gamma_{2j} \Big)
\end{aligned}$$

and so on. Define a function

$$h : \mathcal{S} \times \mathbb{R} \to \mathbb{R}$$

by

$$h(i, y) = \begin{cases} j - i & \text{if} \quad y \in \Delta_{ij}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.21}$$

Then

$$d\mathfrak{r}(t) = \int_{\mathbb{R}} h(\mathfrak{r}(t-), y) N_2(dt, dy), \tag{1.22}$$

with initial condition $\mathfrak{r}(0) = \mathfrak{r}_0$, where $N_2(dt, dy)$ is a Poisson random measure with intensity measure $dt \times \mathfrak{L}(dy)$, in which $\mathfrak{L}$ is the Lebesgue measure on $\mathbb{R}$.

We assume that such a Markov chain, Wiener process, and the Poisson random measure are independent.

1.2.4. Facts from the theory of large deviations

Here, we recall the basic definitions and properties of the theory of large deviations. The interested reader is referred to, e.g., [16, 18, 60].

Let $(\mathcal{X}, \mathcal{B})$ be a topological space, and $\{\mu_\epsilon\}$ is a sequence of probability measures defined on it.

Definition 17 (Rate function)**.** *A rate function I is a lower semicontinuous mapping $I : \mathcal{X} \to [0, \infty]$ (such that the level set $\Psi_I(\alpha) := \{x : I(x) \leq \alpha\}$ is a closed subset of $\mathcal{X}$ for all $\alpha \in [0, \infty)$). A good rate function is a rate function for which all the level sets $\Psi_I(\alpha)$ are compact subsets of $\mathcal{X}$.*

For any set Γ, $\bar{\Gamma}$ denotes its closure and Γ° denotes its interior.We adapt the convention that the infimum over an empty set is ∞.

Definition 18 (Large deviation principle)**.** *The sequence of probability measures $\{\mu_\epsilon\}$ is said to satisfy the large deviation principle with a rate function I if*

$$- \inf_{x \in \Gamma^\circ} I(x) \leq \liminf_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma) \leq \limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma) \leq - \inf_{x \in \bar{\Gamma}} I(x)$$

for all $\Gamma \in \mathcal{B}$.

One of the properties of the large deviation principle is that it is preserved under a continuous mapping.

Theorem 1.2.17 (Contraction principle)**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hausdorff topological spaces and $f : \mathcal{X} \to \mathcal{Y}$ a continuous function. Consider a good rate function $I : \mathcal{X} \to [0, \infty]$.*

(i) For each $y \in \mathcal{Y}$, define

$$J(y) := \inf\{I(x) : x \in \mathcal{X}, y = f(x)\}.$$

Then J is a good rate function on $\mathcal{Y}$.

(ii) If I controls the large deviation principle associated with a family of probability measures $\{\mu_\epsilon\}$ on $\mathcal{X}$, then J controls the large deviation principle associated with the family of probability measures $\{\mu_\epsilon \circ f^{-1}\}$ on $\mathcal{Y}$.

In the context of this article, the probability measures under consideration are (mostly) defined on a complete metric space (Polish space). In such a case, the large deviation principle is known to be equivalent the "Laplace–Varadhan principle." In what next, we will collect the definition of the Laplace-Varadhan principle, and a theorem that shows the equivalence between the large deviation principle and the Laplace-Varadhan principle. The interested reader is referred to, e.g., [31, Ch. 12]. Let E be a Polish space.

Definition 19 (Laplace-Varadhan principle)**.** *A sequence of E-valued random variables $\{X_n\}$ is said to satisfy the Laplace-Varadhan principle with a good rate function I if*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\Big(e^{-nh(X_n)}\Big) = -\inf_{x\in E}\{h(x) + I(x)\}$$

for all $h \in C_b(E)$. The inequality

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}\Big(e^{-nh(X_n)}\Big) \leq -\inf_{x\in E}\{h(x) + I(x)\}$$

is known as the upper bound for the Laplace-Varadhan principle, and the inequality

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E}\Big(e^{-nh(X_n)}\Big) \geq -\inf_{x\in E}\{h(x) + I(x)\}$$

is known as the lower bound for the Laplace-Varadhan principle.

The following theorem gives the equivalence between the large deviation principle and the Laplace-Varadhan principle. The interested reader may consult [31, Thm. 12.2.1 and 12.2.2] for the detailed discussion and proofs.

Theorem 1.2.18. *Let E be a Polish space.*

(i) Let $\{X_n\}$ be a sequence of random variables taking values in E and satisfy large deviation principle on E with a good rate function I. Then

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\Big(e^{-nh(X_n)}\Big) = -\inf_{x\in E}\{h(x) + I(x)\}$$

for all $h \in C_b(E)$.

(ii) If I is a rate function on E and the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\Big(e^{-nh(X_n)}\Big) = -\inf_{x\in E}\{h(x) + I(x)\}$$

holds for all $h \in C_b(E)$, then $\{X_n\}$ satisfies the large deviation principle with rate function I.

1.2.5. Functional analytic setup

From now on, H and V is reserved for the following specific spaces that have been introduced in the previous section:

$$H = \{\mathbf{u} \in L^2(G) : \nabla \cdot \mathbf{u} = 0,\ \mathbf{u} \cdot \mathbf{n}_{\ \partial G} = 0\},$$

$$V = \{\mathbf{u} \in W_0^{1,2}(G) : \nabla \cdot \mathbf{u} = 0\},$$

and we denote the H-norm (V-norm, resp.) by $|\cdot|$ ($\|\cdot\|$, resp.) and the inner product on H (duality pairing on V, resp.) by $(\cdot,\cdot)$ ($\langle\cdot,\cdot\rangle_V$, resp.) when there is no ambiguity. In addition, we have the following inclusion between the spaces:

$$V \hookrightarrow H \hookrightarrow V',$$

and both of the inclusions $V \hookrightarrow H$ and $H \hookrightarrow V'$ are compact embeddings (see, e.g., [54, Lem. 1.5.1 and 1.5.2, Ch. II]).

The set $\{e_i\}_{i=1}^{\infty}$ is reserved for the orthonormal basis obtained from Theorem 1.2.10 and $H_n := \text{span}\{e_i\}_{i=1}^{n}$; Π_n is the orthogonal projection of H on H_n.

Let k be a mollifier with $\|k\|_{\frac{6}{5}} = 1$. Define $b(k * \cdot, \cdot, \cdot) : V \times V \times V \to \mathbb{R}$ by

$$b(k * \mathbf{u}, \mathbf{v}, \mathbf{w}) := \sum_{i,j=1}^{3} \int_G (k * u)_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

which induces a bilinear form $\mathbf{B}_k(\mathbf{u}, \mathbf{v})$ by $b(k * \mathbf{u}, \mathbf{v}, \mathbf{w}) = \langle \mathbf{B}_k(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle_V$. A combination of (generalized) Hölder inequality and the Young convolution inequality yields

$$|b(k * \mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \|k * \mathbf{u}\|_6 \|\nabla \mathbf{v}\|_2 \|\mathbf{w}\|_3 \leq \|k\|_{\frac{6}{5}} \|\mathbf{u}\|_3 \|\nabla \mathbf{v}\|_2 \|\mathbf{w}\|_3, \tag{1.23}$$

which together with (1.8) and (1.9) further implies

$$|b(k * \mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C \|\mathbf{u}\|^{\frac{1}{2}} |\mathbf{u}|^{\frac{1}{2}} \|\mathbf{v}\| \|\mathbf{w}\|^{\frac{1}{2}} |\mathbf{w}|^{\frac{1}{2}} \tag{1.24}$$

In particular, when $\mathbf{u} = \mathbf{w}$, we have

$$|b(k * \mathbf{u}, \mathbf{v}, \mathbf{u})| \leq C \|\mathbf{u}\| \cdot |\mathbf{u}| \cdot \|\mathbf{v}\|. \tag{1.25}$$

For the sake of simplicity, we shall assume that $C = 1$ in this article.

For the (U-valued) Q-Wiener process $W(s)$ introduced in Section 1.2.3, we take $U = H$ and write

$$\| \cdot \|_{L_Q} = \| \cdot \|_{\mathcal{L}_2(H_0, H)},$$

where $\| \cdot \|_{\mathcal{L}_2(H_0,H)}$ is defined in (1.15) with $H_0 = U_0$.

Now we introduce hypotheses of the noise coefficients.

The functions $\sigma : [0, T] \times H \times \mathcal{S} \to \mathcal{L}_2(H_0, H)$ and $\mathbf{G} : [0, T] \times H \times \mathcal{S} \times Z \to H$ are continuous and satisfy the following Hypotheses **H**:

H1. For all $t \in (0, T)$ and $i \in \mathcal{S}$, there exists a constant $K > 0$ such that

$$\|\sigma(t, \mathbf{u}, i)\|_{L_Q}^p \leq K(1 + |\mathbf{u}|^p + |i|^p)$$

for $p = 2, 3$ (growth condition on σ).

H2. For all $t \in (0,T)$, there exists a constant $L > 0$ such that for all $\mathbf{u}, \mathbf{v} \in H$ and $i, j \in \mathcal{S}$,

$$\|\sigma(t,\mathbf{u},i) - \sigma(t,\mathbf{v},j)\|_{L_Q}^2 \leq L(|\mathbf{u}-\mathbf{v}|^2 + |i-j|^2)$$

(Lipschitz condition on σ).

H3. For all $t \in (0,T)$ and $i \in \mathcal{S}$, there exists a constant $K > 0$ such that

$$\int_Z |\mathbf{G}(t,\mathbf{u},i,z)|^p \nu(dz) \leq K(1 + |\mathbf{u}|^p + |i|^p)$$

for $p = 1, 2$, and 3 (growth condition on G).

H4. For all $t \in (0,T)$, there exists a constant $L > 0$ such that for all $\mathbf{u}, \mathbf{v} \in H$ and $i, j \in \mathcal{S}$,

$$\int_Z |\mathbf{G}(t,\mathbf{u},i,z) - \mathbf{G}(t,\mathbf{v},j,z)|^2 \nu(dz) \leq L(|\mathbf{u}-\mathbf{v}|^2 + |i-j|^2)$$

(Lipschitz condition on $\mathbf{G}$).

Without further notice, we will assume the noises coefficients satisfy Hypotheses **H**. Any change that we made on the noise coefficients will be clearly specified at the beginning of each chapter or section.

The transformation from (1.3) to (1.4) is sketched as follows. By applying the Leray-Helmholtz projector (1.11) to each term of (1.3), and utilizing the Helmholtz decomposition (1.10), one transforms (1.3) into (1.4). For more details, we refer the interested reader to [37, 54]. In addition, we shall remind the reader that (1.4) is understood in the following integro-variational sense:

$$\begin{aligned} &\langle \mathbf{u}(t), \rho\rangle_V + \int_0^t \langle \nu \mathbf{A}\mathbf{u}(s) + \mathbf{B}_k(\mathbf{u}(s)), \rho\rangle_V ds \\ &= \langle \mathbf{u}(0), \rho\rangle_V + \int_0^t \langle \mathbf{f}(s), \rho\rangle_V ds + \langle \int_0^t \sigma(t, \mathbf{u}(s), \mathfrak{r}(s)) dW(s), \rho\rangle_V \\ &\quad + \langle \int_0^t \int_Z \mathbf{G}(s-, \mathbf{u}(s-), \mathfrak{r}(s-), z) \tilde{N}_1(dz, ds), \rho\rangle_V \end{aligned} \tag{1.26}$$

for all $\rho \in V$.

Now we are in a position to introduce the Itô formula. We remind the reader that the Itô formula introduced here is in the context of $\mathbf{u}(t)$, which is the solution to equation (1.4). For full generality, we refer the reader to, e.g., [12, 30, 31, 42, 45, 51, 53].

Let $F : [0,T] \times V \times \mathcal{S} \to \mathbb{R}^+$ be a continuous function with its Fréchet derivatives F_t, F_v, and F_{vv} are bounded and continuous. Define the operator

$$\begin{aligned}\mathcal{L}F(t,\mathbf{v},i) := &\, F_t(t,\mathbf{v},i) + \langle -\nu\mathbf{A}\mathbf{v} - \mathbf{B}_k(\mathbf{v}) + \mathbf{f}(t), F_v(t,\mathbf{v},i)\rangle_V \\ &+ \sum_{j=1}^{m} \gamma_{ij}F(t,\mathbf{v},j) + \frac{1}{2}tr\Big(F_{vv}(t,\mathbf{v},i)\sigma(t,\mathbf{v},i)Q\sigma^*(t,\mathbf{v},i)\Big) \\ &+ \int_Z \Big(F(t,\mathbf{v}+\mathbf{G}(t,\mathbf{v},i,z),i) - F(t,\mathbf{v},i) \\ &\qquad - \Big(F_v(t,\mathbf{v},i),\mathbf{G}(t,\mathbf{v},i,z)\Big)_H\Big)\nu_1(dz).\end{aligned}$$

Then we have the following change of variables formula due to Itô (see, e.g., [42, Thm. 1.45], [53, Lem. 3 in Sec. 2.1, Ch. 2]):

$$\begin{aligned}F(t,\mathbf{u}(t),\mathfrak{r}(t)) = &\, F(0,\mathbf{u}(0),\mathfrak{r}(0)) + \int_0^t \mathcal{L}F(s,\mathbf{u}(s),\mathfrak{r}(s))ds \\ &+ \int_0^t \langle F_x(s,\mathbf{u}(s),\mathfrak{r}(s)), \sigma(s,\mathbf{u}(s),\mathfrak{r}(s))dW(s)\rangle \\ &+ \int_0^t \int_Z \Big(F(s,\mathbf{u}(s-)+\mathbf{G}(s-,\mathbf{u}(s-),\mathfrak{r}(s-),z),\mathfrak{r}(s-)) \\ &\qquad - F(s,\mathbf{u}(s-),\mathfrak{r}(s-))\Big)\tilde{N}_1(dz,ds) \\ &+ \int_0^t \int_{\mathbb{R}} \Big(F(s,\mathbf{u}(s-),\mathfrak{r}(s-)+h(\mathfrak{r}(s-),y)) \\ &\qquad - F(s,\mathbf{u}(s-),\mathfrak{r}(s-))\Big)\tilde{N}_2(ds,dy),\end{aligned}$$

$\tilde{N}_1(dz,ds)$ is the compensated Poisson random measure introduced in Section 1.2.3, $\tilde{N}_2(ds,dy) = N_2(ds,dy) - \mathfrak{L}(dy)ds$ and $N_2(ds,dy) = N(ds,dy)$ in (1.22), and $\mathfrak{L}(dy)ds$ is as in (1.22); $h(s,y)$ is defined as in (1.21). In particular, if $F(t,\mathbf{u}(t),i) = |\mathbf{u}(t)|^2$, then the term $\sum_{j=1}^{m} \gamma_{ij} |\mathbf{u}(t)|^2 = 0$, therefore, we have

$$
\begin{aligned}
|\mathbf{u}(t)|^2 = |\mathbf{u}(0)|^2 &+ 2\int_0^t \langle -\nu \mathbf{A}\mathbf{u}(s) - \mathbf{B}_k(\mathbf{u}(s)) + \mathbf{f}(s), \mathbf{u}(s)\rangle_V ds \\
&+ \int_0^t \|\sigma(s, \mathbf{u}(s), \mathfrak{r}(s))\|_{L_Q}^2 ds + 2\int_0^t \langle \mathbf{u}(s), \sigma(s, \mathbf{u}(s), \mathfrak{r}(s)) dW(s)\rangle \\
&+ \int_0^t \int_Z \Big(|\mathbf{u}(s-) + \mathbf{G}(s-, \mathbf{u}(s-), \mathfrak{r}(s-), z)|^2 - |\mathbf{u}(s-)|^2 \Big) \tilde{N}_1(dz, ds) \\
&+ \int_0^t \int_Z \Big(|\mathbf{u}(s) + \mathbf{G}(s, \mathbf{u}(s), \mathfrak{r}(s), z)|^2 - |\mathbf{u}(s)|^2 \\
&\qquad - 2\Big(\mathbf{u}(s), \mathbf{G}(s, \mathbf{u}(s), \mathfrak{r}(s), z)\Big)_H \Big) \nu_1(dz) ds.
\end{aligned}
$$

Denoted by $\{\tau_i\}_{i=1}^4$ the topologies

$$
\begin{aligned}
&\tau_1 = J\text{-topology} && \text{on} \quad \mathcal{D}([0,T]; V'), \\
&\tau_2 = \text{weak topology} && \text{on} \quad L^2(0,T;V), \\
&\tau_3 = \text{weak-star topology} && \text{on} \quad L^\infty(0,T;H), \\
&\tau_4 = \text{strong topology} && \text{on} \quad L^2(0,T;H),
\end{aligned}
$$

and Ω_i the spaces

$$
\begin{aligned}
\Omega_1 &= \mathcal{D}([0,T]; V'), \\
\Omega_2 &= L^2(0,T;V), \\
\Omega_3 &= L^\infty(0,T;H), \\
\Omega_4 &= L^2(0,T;H).
\end{aligned}
$$

Then $\{(\Omega_i, \tau_i)\}_{i=1}^4$ are all Lusin spaces[xi].

[xi] Definition 9.

Definition 20. *Define the space Ω^* by*

$$\Omega^* = \cap_{i=1}^4 \Omega_i.$$

Let τ be the supremum of the topologies[xii] *induced on Ω^* by all τ_i. Then it follows from [41, Prop. 1, Ch. IV] that*[xiii]

(i) (Ω^*, τ) is a Lusin space.

(ii) Let $\{\mu_k\}_{k\in\mathbb{N}}$ be a sequence of Borel probability laws on Ω^* (on the Borel σ-algebra $\mathcal{B}(\tau)$) such that their images $\{\mu_k^i\}_{k\in\mathbb{N}}$ on $(\Omega_i, \mathcal{B}(\tau_i))$ are tight for τ_i for all i. Then $\{\mu_k\}_{k\in\mathbb{N}}$ is tight for τ.

As shall be seen, the space (Ω^*, τ) is the path space of the "solution" $\mathbf{u}(t)$ of equation (1.4). Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a (complete) probability space on which the following are defined:

(i) $W = \{W(t) : 0 \leq t \leq T\}$, an H-valued Q-Wiener process.

(ii) $N = \{N(z,t) : 0 \leq t \leq T \quad \text{and} \quad z \in Z\}$, the Poisson random measure.

(iii) $\mathfrak{r} = \{\mathfrak{r}(t) : 0 \leq t \leq T\}$, the Markov chain.

(iv) ξ, an H-valued random variable.

Assume that ξ, W, N, and $\mathfrak{r}(t)$ are mutually independent. For each t, define the σ-field

$$\mathcal{F}_t := \sigma(\xi, \mathfrak{r}(t), W(s), N(z,s) : z \in Z, 0 \leq s \leq t) \ \cup \ \{\text{all } \mathcal{P}\text{-null sets in } \mathcal{F}\}.$$

Then it is clear that $(\mathcal{F}_t)$ satisfies the usual conditions, and both $W(t)$ and $N(z,t)$ are $\mathcal{F}_t$-adapted processes.

[xii] The coarest topology that is finer than each τ_i. See, e.g., [27, Sec. 5.2]

[xiii] Note that all the natural inclusion $\Omega_i \hookrightarrow \Omega_1$, $i = 2, 3, 4$, are continuous.

Denoting by $\mathcal{J}$ the J-topology in the space $\mathcal{D}([0,T];S)$, we define

$$\begin{aligned}\Omega^\dagger &:= \Omega^* \times \mathcal{D}([0,T];\mathcal{S}),\\ \tau^\dagger &:= \tau \times \mathcal{J}.\end{aligned} \tag{1.27}$$

We are in the position to introduce the concept of solutions of equation (1.4). The following contents regarding the concepts of solutions are adapted from [31, Ch. 6].

Definition 21 (Strong solution)**.** *The $H \times \mathcal{S}$-valued process $(\mathbf{u}(t), \mathfrak{r}(t)), t \in [0,T]$, defined on $(\Omega, \mathcal{F}, \mathcal{P})$, is called a strong solution of the stochastic Navier-Stokes equation with Markov switching* (1.4) *with initial condition $(\mathbf{u}(0), \mathfrak{r}(0)) = (\xi, r)$ if it satisfies*

(i) $(\mathbf{u}(t), \mathfrak{r}(t))$ *is $\mathcal{F}_t$-adapted with sample path in $(\Omega^\dagger, \tau^\dagger)$.*

(ii) For all $\rho \in V$, $\int_0^T \langle \nu \mathbf{A}\mathbf{u}(s) + \mathbf{B}_k(\mathbf{u}(s)), \rho\rangle_V ds < \infty$ $\mathcal{P}$-almost surely.

(iii) $\mathbb{E}\int_0^T \|\sigma(s, \mathbf{u}(s), \mathfrak{r}(s))\|_{L_Q}^2 ds < \infty$.

(iv) $\mathbb{E}\int_0^T \int_Z |\mathbf{G}(s, \mathbf{u}(s), \mathfrak{r}(s), z)|^2 \nu_1(dz) ds < \infty$.

(v) For each $t \in [0,T]$, the equation (1.26) *is satisfied $\mathcal{P}$-almost surely.*

Definition 22 (Strong uniqueness)**.** *The stochastic Navier-Stokes equation with Markov switching* (1.4) *with initial condition $(\mathbf{u}(0), \mathfrak{r}(0)) = (\xi, r)$ has a unique strong solution if, for any two strong solutions $(\mathbf{u}_1(t), \mathfrak{r}_1(t))$ and $(\mathbf{u}_2(t), \mathfrak{r}_2(t))$ on $(\Omega, \mathcal{F}, \mathcal{P})$, one has*

$$\mathcal{P}\{\omega : (\mathbf{u}_1(\omega,t), \mathfrak{r}_1(\omega,t)) = (\mathbf{u}_2(\omega,t), \mathfrak{r}(\omega,t)) \quad \forall t \in [0,T]\} = 1.$$

Similar to the study of the Navier-Stokes equations (NSEs), the study of stochastic Navier-Stokes equations (SNSEs) allows one the defines "weak solutions." However, unlike the study of NSEs, the "weak" here does not mean "less differentiable" but mean "finding solutions in another probability space."

Definition 23 (Weak solution)**.** *Suppose that, on some probability space* $(\Omega, \mathcal{F}, \mathcal{P})$*, there exists an increasing family* $(\mathcal{G}_t)$ *of sub* σ*-field of* $\mathcal{F}$*, an* $H \times \mathcal{S}$*-valued random vector* (ξ, r) *with given distribution* μ*, and* $\mathcal{G}_t$*-adapted processes* $W(t)$*,* $\tilde{N}(z,t)$*,* $\mathbf{u}(t)$*, and* $\mathfrak{r}(t)$ *such that*

(i) $(W(t), \mathcal{G}_t, \mathcal{P})$ *is an* H*-valued* Q*-Wiener martingale.*

(ii) $\tilde{N}(z,t)$ *is an square integrable martingale with respect to* $(\mathcal{G}_t)$*.*

(iii) $W(t)$*,* $N(z,t)$*, and* (ξ, r) *are mutually independent.*

(iv) For all $\rho \in V$*,* $\mathcal{P}\Big\{\omega : \int_0^T \langle \nu \mathbf{A}\mathbf{u}(\omega,s) + \mathbf{B}_k(\mathbf{u}(\omega,s)), \rho\rangle_V ds < \infty\Big\} = 1.$

(v) For all t *and* $\rho \in V$*,*

$$\begin{aligned}
&\langle \mathbf{u}(\omega,t), \rho\rangle_V + \int_0^t \langle \nu \mathbf{A}\mathbf{u}(\omega,s) + \mathbf{B}_k(\mathbf{u}(\omega,s)), \rho\rangle_V ds \\
&= \langle \xi, \rho\rangle_V + \int_0^t \langle \mathbf{f}(s), \rho\rangle_V ds + \langle \int_0^t \sigma(t, \mathbf{u}(s), \mathfrak{r}(s)) dW(\omega,s), \rho\rangle_V \\
&\quad + \langle \int_0^t \int_Z \mathbf{G}(s-, \mathbf{u}(s-), \mathfrak{r}(s-), z) \tilde{N}_1(\omega, dz, ds), \rho\rangle_V
\end{aligned}$$

$\mathcal{P}$*-almost surely.*

Then the family $(\Omega, \mathcal{F}, (\mathcal{G}_t), \mathcal{P}, \xi, r, \{W(t)\}, \{N(z,t)\}, \{\mathbf{u}(t)\}, \{\mathfrak{r}(t)\})$ *is called a weak solution of the stochastic Navier-Stokes equation* (1.4)*.*

Similar to the NSEs case, a strong solution of an SNSEs is also a weak solution. Next, we define the "uniqueness" that is suitable for the weak solutions of SNSEs.

Definition 24 (Pathwise uniqueness)**.** *A weak solution of the stochastic Navier-Stokes equation* (1.4) *is said to be pathwise unique if, for any two weak solutions give by* $(\Omega, \mathcal{F}, (\mathcal{G}_t^i), \mathcal{P}, \xi^i, r^i, \{W(t)\}, \{N(z,t)\}, \{\mathbf{u}^i(t)\}, \{\mathfrak{r}^i(t)\})$*,* $i = 1, 2$*, the following holds:*

$$\mathcal{P}\Big\{(\mathbf{u}^1(t), \mathfrak{r}^1(t)) = (\mathbf{u}^2(t), \mathfrak{r}^2(t)) \quad \forall t \geq 0\Big\} = 1.$$

1.2.6. Auxiliary results

In this thesis, we will need some compactness of functions or measures for extracting a suitable convergent (sub)sequence, and we collect them in this subsection for the benefit of the reader. The first lemma can be viewed as a desired formulation of the Aubin-Lions Lemma (see, e.g., [58, Sec. 3.4], [59, Sec. 2.1, Ch. III]) in the context of stochastic differential equations. The statement of which is adapted from [41, Lem. 3, Ch. VI].

Lemma 1.2.19. *Consider the continuous dense embeddings $V \hookrightarrow H \hookrightarrow V'$ with $V \hookrightarrow H$ and $H \hookrightarrow V'$ being compact. Suppose that a set B in $L^q(0,T;H) \cap \mathcal{D}([0,T];V')$ is relatively compact in $\mathcal{D}([0,T];V')$ and bounded in $L^q(0,T;V)$. Then B is relatively compact in $L^q(0,T;H)$.*

Next we introduce a lemma due to Aldous that gives a criterion for the tightness (compactness) of a sequence of probability measures. For the details, we refer the reader to, e.g., [1, 9, 31, 41, 63].

Lemma 1.2.20 (Aldous' criterion)**.** *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of processes with paths in the space $\mathcal{D}([0,T];V')$. Suppose that for each rationals $t \in [0,T]$, we have*

$$\lim_{N\to\infty} \limsup_{n} \mathcal{P}\Big(\|X_n(t)\|_{V'} > N\Big) = 0. \tag{1.28}$$

Then $\{X_n\}_{n=1}^{\infty}$ is tight in $\mathcal{D}([0,T];V')$ if the following condition is satisfied:

For every sequence (T_n, δ_n) where each T_n is a stopping time such that $T_n + \delta_n \le T$, and $\delta_n > 0$, $\delta_n \to 0$, we have $\|X_n(T_n+\delta_n) - X_n(T_n))\|_{V'} \to 0$ in probability as $n \to \infty$.

Proof. The interested reader may consult [31, Thm. 10.4.1], [63, p. 354, Th. 6.8] □

The next lemma characterizes a property of càdlàg functions (see, e.g., [9, Lem.

1, Ch. 3]): there can be at most finitely many points at which the jump exceeds a given positive number .

Lemma 1.2.21. *Let K be a separable Hilbert space. For each $x \in \mathcal{D}([0,T];K)$ and each positive ϵ, there exists points $t_0, t_1, \ldots, t_v$ such that*

$$0 = t_0 < t_1 < \cdots < t_v = T$$

and

$$\sup_{s,t \in [t_i, t_{i+1})} \|x(s) - x(t)\|_K < \epsilon,$$

$i = 1, 2, \ldots, v$.

Chapter 2. A Priori Estimates

In the chapter, we establish a priori estimates to the approximation of (1.4). Recalling the definitions of $\{e_i\}_{i=1}^{\infty}$, H_n, and Π_n from Section 1.2.5, we define $W_n := \Pi_n W$, $N_n := \Pi_n N$, $\sigma_n := \Pi_n \sigma$, and $\mathbf{G}_n := \Pi_n \mathbf{G}$.

Define $\mathbf{u}_n$ as the solution to following equation: for each $\mathbf{v} \in H_n$,

$$\begin{aligned}\mathbf{d}(\mathbf{u}_n(t), \mathbf{v}) =& \big[\big(-\nu \mathbf{A}\mathbf{u}_n(t) - \mathbf{B}_k(\mathbf{u}_n(t)), \mathbf{v}\big)\big]dt + \langle \mathbf{f}(t), \mathbf{v}\rangle_V dt \\ &+ \big(\sigma_n(t, \mathbf{u}_n(t), \mathfrak{r}(t))dW_n(t), \mathbf{v}\big) \\ &+ \Big(\int_Z \mathbf{G}_n(t-, \mathbf{u}_n(t-), \mathfrak{r}(t-), z)\tilde{N}_{1n}(dz, dt), \mathbf{v}\Big)\end{aligned} \tag{2.1}$$

with $\mathbf{u}_n(0) = \Pi_n \mathbf{u}(0)$, where $\tilde{N}_{1n} = \Pi_n \tilde{N}_1$ and $\tilde{N}_1(dz, ds) = N_1(dz, ds) - \nu_1(dz)ds$.

Proposition 2.0.1 (A priori estimates)**.** *Let $T > 0$ be fixed. Suppose that $\mathbb{E}|\mathbf{u}(0)|^2 < \infty$ and $\mathbf{f} \in L^2(0, T; V')$. Then, under Hypotheses $\mathbf{H}$, we have*

$$\mathbb{E}|\mathbf{u}_n(t)|^2 + \nu \mathbb{E}\int_0^t \|\mathbf{u}_n(s)\|^2 ds \le C(\mathbb{E}|\mathbf{u}(0)|^2, \mathbb{E}\int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds, \nu, K, m, T), \tag{2.2}$$

for any $t \in [0, T]$ and

$$\mathbb{E}\sup_{0\le t\le T} |\mathbf{u}_n(t)|^2 + \nu \mathbb{E}\int_0^T \|\mathbf{u}_n(s)\|^2 ds \le C(\mathbb{E}|\mathbf{u}(0)|^2, \mathbb{E}\int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds, \nu, K, m, T). \tag{2.3}$$

Suppose further that $\mathbb{E}|\mathbf{u}(0)|^3 < \infty$ and $\mathbf{f} \in L^3(0, T; V')$. Then, under Hypotheses $\mathbf{H}$, we have

$$\mathbb{E}\sup_{0\le t\le T} |\mathbf{u}_n(t)|^3 + 2\nu \mathbb{E}\int_0^T |\mathbf{u}_n(s)|\|\mathbf{u}_n(s)\|^2 ds \le C(\mathbb{E}|\mathbf{u}(0)|^3, \mathbb{E}\int_0^T \|\mathbf{f}(s)\|_{V'}^3 ds, \nu, K, m, T) \tag{2.4}$$

Proof. Let $N > 0$. Define

$$\tau_N := \inf\{t \in [0,T] : |\mathbf{u}_n(t)|^2 + \int_0^t \|\mathbf{u}_n(s)\|^2 ds > N \\ \text{or } |\mathbf{u}_n(t-)|^2 + \int_0^t \|\mathbf{u}_n(s)\|^2 ds > N\}.$$

It follows from the Itô formula that

$$\begin{aligned} |\mathbf{u}_n(t\wedge\tau_N)|^2 = |\mathbf{u}(0)|^2 &+ 2\int_0^{t\wedge\tau_N} \langle -\nu\mathbf{A}\mathbf{u}_n(s) - \mathbf{B}_k(\mathbf{u}_n(s)) + \mathbf{f}(s), \mathbf{u}_n(s)\rangle_V ds \\ &+ \int_0^{t\wedge\tau_N} \|\sigma_n(s,\mathbf{u}_n(s),\mathfrak{r}(s))\|_{L_Q}^2 ds + 2\int_0^{t\wedge\tau_N} \langle \mathbf{u}_n(s), \sigma_n(s,\mathbf{u}_n(s),\mathfrak{r}(s))dW_n(s)\rangle \\ &+ \int_0^{t\wedge\tau_N}\int_Z \Big(|\mathbf{u}_n(s-) + \mathbf{G}_n(s-,\mathbf{u}_n(s-),\mathfrak{r}(s-),z)|^2 - |\mathbf{u}_n(s-)|^2\Big)\tilde{N}_{1n}(dz,ds) \\ &+ \int_0^{t\wedge\tau_N}\int_Z \Big(|\mathbf{u}_n(s) + \mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)|^2 - |\mathbf{u}_n(s)|^2 \\ &\qquad - 2\big(\mathbf{u}_n(s), \mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)\big)_H\Big)\nu_1(dz)ds. \end{aligned} \tag{2.5}$$

Note that $\mathbf{u}_n \in H_n \subset \mathcal{D}(\mathbf{A})$, therefore, it follows from the property of $\mathbf{A}$ that[i]

$$\langle \mathbf{A}\mathbf{u}_n(s), \mathbf{u}_n(s)\rangle_V = \big(\mathbf{A}\mathbf{u}_n(s), \mathbf{u}_n(s)\big)_H = \|\mathbf{u}_n(s)\|^2,$$

which implies that

$$2\int_0^{t\wedge\tau_N} -\nu\langle \mathbf{A}\mathbf{u}_n(s), \mathbf{u}_n(s)\rangle_V ds = -2\nu\int_0^{t\wedge\tau_N} \|\mathbf{u}_n(s)\|^2 ds.$$

By the definition of $\mathbf{B}_k$, $\langle \mathbf{B}_k(\mathbf{u}_n(s)), \mathbf{u}_n(s)\rangle_V = 0$. For the external force term $\mathbf{f}$, one deduces from the basic Young inequality that

$$\begin{aligned} &2\int_0^{t\wedge\tau_N} \langle \mathbf{f}(s), \mathbf{u}_n(s)\rangle_V ds \\ &\le 2\int_0^{t\wedge\tau_N} \|\mathbf{f}(s)\|_{V'}\|\mathbf{u}_n(s)\| ds \le \frac{1}{\nu}\int_0^t \|\mathbf{f}(s)\|_{V'}^2 ds + \nu\int_0^{t\wedge\tau_N} \|\mathbf{u}_n(s)\|^2 ds. \end{aligned}$$

[i] Theorem 1.2.10 and references therein.

A simplification of the last term in (2.5) gives

$$\int_0^{t\wedge\tau_N}\int_Z\Big(|\mathbf{u}_n(s)+\mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)|^2-|\mathbf{u}_n(s)|^2 \\ -2\big(\mathbf{u}_n(s),\mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)\big)_H\Big)\nu_1(dz)ds \\ =\int_0^{t\wedge\tau_N}\int_Z|\mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)|^2\nu_1(dz)ds.$$

Now, taking expectation on the both side of (2.5), using Hypotheses **H1** and **H3**, and then putting everything together, we obtain

$$\mathbb{E}|\mathbf{u}_n(t\wedge\tau_N)|^2+\nu\mathbb{E}\int_0^{t\wedge\tau_N}\|\mathbf{u}_n(s)\|^2ds \\ \leq\mathbb{E}|\mathbf{u}(0)|^2+\frac{1}{\nu}\mathbb{E}\int_0^T\|\mathbf{f}(s)\|_{V'}^2ds+2K\mathbb{E}\int_0^t|\mathbf{u}_n(s\wedge\tau_N)|^2ds+2KT(1+m^2). \tag{2.6}$$

Denoting $C_T:=\mathbb{E}|\mathbf{u}(0)|^2+\frac{1}{\nu}\mathbb{E}\int_0^T\|\mathbf{f}(s)\|_{V'}^2ds+2KT(1+m^2)$, we utilize the Gronwall's inequality to obtain

$$\mathbb{E}|\mathbf{u}_n(t\wedge\tau_N)|^2\leq C_Te^{2KT}. \tag{2.7}$$

A combination of (2.6) and (2.7) yield

$$\mathbb{E}|\mathbf{u}_n(t\wedge\tau_N)|^2+\nu\mathbb{E}\int_0^{t\wedge\tau_N}\|\mathbf{u}_n(s)\|^2ds\leq C_T(1+2KTe^{2KT}). \tag{2.8}$$

In particular,

$$\mathbb{E}\int_0^{t\wedge\tau_N}\|\mathbf{u}_n(s)\|^2ds\leq\frac{1}{\nu}C_T(1+2KTe^{2KT}). \tag{2.9}$$

An application of Davis inequality yields

$$
\begin{aligned}
&2\mathbb{E}\sup_{0\le t\le T\wedge\tau_N}\int_0^t \langle \mathbf{u}_n(s),\sigma(s,\mathbf{u}_n(s),\mathfrak{r}(s))dW(s)\rangle\\
&\le 2\sqrt{2}\mathbb{E}\Big\{\Big(\int_0^{T\wedge\tau_N}|\mathbf{u}_n(s)|^2\|\sigma(s,\mathbf{u}_n(s),\mathfrak{r}(s))\|_{L_Q}^2\Big)^{\frac{1}{2}}\Big\}\\
&\le 2\sqrt{2}\epsilon_1\mathbb{E}\sup_{0\le t\le T\wedge\tau_N}|\mathbf{u}_n(t)|^2+2\sqrt{2}C_{\epsilon_1}\mathbb{E}\int_0^{T\wedge\tau_N}\|\sigma(s,\mathbf{u}_n(s),\mathfrak{r}(s))\|_{L_Q}^2ds
\end{aligned}
$$

and

$$
\begin{aligned}
&2\mathbb{E}\sup_{0\le t\le T\wedge\tau_N}\int_0^t\int_Z\big(\mathbf{u}_n(s-),\mathbf{G}(s-,\mathbf{u}_n(s-),\mathfrak{r}(s-),z)\big)_H\tilde{N}_1(dz,ds)\\
&\le 2\sqrt{10}\epsilon_2\mathbb{E}\sup_{0\le t\le T\wedge\tau_N}|\mathbf{u}_n(t)|^2+2\sqrt{10}C_{\epsilon_2}\mathbb{E}\int_0^{T\wedge\tau_N}\int_Z|\mathbf{G}(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)|^2\nu(dz)ds.
\end{aligned}
$$

Therefore, Itô formula implies

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T\wedge\tau_N}|\mathbf{u}_n(t)|^2+\nu\mathbb{E}\int_0^{T\wedge\tau_N}\|\mathbf{u}_n(s)\|^2ds\\
&\le \mathbb{E}|\mathbf{u}(0)|^2+\frac{1}{\nu}\mathbb{E}\int_0^T\|\mathbf{f}(s)\|_{V'}^2ds+2(\sqrt{2}\epsilon_1+\sqrt{10}\epsilon_2)\mathbb{E}\sup_{0\le t\le T\wedge\tau_N}|\mathbf{u}_n(t)|^2\\
&\quad+(2\sqrt{2}C_{\epsilon_1}+1)\mathbb{E}\int_0^{T\wedge\tau_N}\|\sigma(s,\mathbf{u}_n(s),\mathfrak{r}(s))\|_{L_Q}^2ds\\
&\quad+(2\sqrt{10}C_{\epsilon_2}+1)\mathbb{E}\int_0^{T\wedge\tau_N}\int_Z|\mathbf{G}(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)|^2\nu(dz)ds.
\end{aligned}
$$

Take $\epsilon_1=\frac{1}{8\sqrt{2}}$ and $\epsilon_2=\frac{1}{8\sqrt{10}}$. Then $C_{\epsilon_1}=2\sqrt{2}$ and $C_{\epsilon_2}=2\sqrt{10}$. One obtains from above and (2.9) that

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\sup_{0\le t\le T\wedge\tau_N}|\mathbf{u}_n(t)|^2+\nu\mathbb{E}\int_0^{T\wedge\tau_N}\|\mathbf{u}_n(s)\|^2ds\\
&\le \mathbb{E}|\mathbf{u}(0)|^2+\frac{1}{\nu}\mathbb{E}\int_0^T\|\mathbf{f}(s)\|_{V'}^2ds+50K\mathbb{E}\int_0^{T\wedge\tau_N}(1+|\mathbf{u}_n(s)|^2+m^2)ds\\
&\le \mathbb{E}|\mathbf{u}(0)|^2+\frac{1}{\nu}\mathbb{E}\int_0^T\|\mathbf{f}(s)\|_{V'}^2ds+\frac{50K}{\nu}C_T(1+2KTe^{2KT})+50KT(1+m^2)\\
&:=C_2(T). \qquad (2.10)
\end{aligned}
$$

Now consider the event $\{\tau_N < T\}$. By the definition of τ_N, we have

$$\{\tau_N < T\} = \Big\{|\mathbf{u}_n(t)|^2 + \int_0^t \|\mathbf{u}_n(s)\|^2 ds > N\Big\} \cup \Big\{|\mathbf{u}_n(t-)|^2 + \int_0^t \|\mathbf{u}_n(s)\|^2 ds > N\Big\}$$

for some $t \in [0, T]$, therefore,

$$\mathcal{P}\{\tau_N < T\} \leq \mathcal{P}\Big\{|\mathbf{u}_n(t)|^2 + \int_0^t \|\mathbf{u}_n(s)\|^2 ds > N\Big\} + \mathcal{P}\Big\{|\mathbf{u}_n(t-)|^2 + \int_0^t \|\mathbf{u}_n(s)\|^2 ds > N\Big\},$$

which together with the Markov inequality imply

$$\mathcal{P}\{\tau_N < T\} \leq \frac{\mathbb{E}\Big(|\mathbf{u}_n(t)|^2 + \int_0^t \|\mathbf{u}_n(s)\|^2 ds\Big)}{N} + \frac{\mathbb{E}\Big(|\mathbf{u}_n(t-)|^2 + \int_0^t \|\mathbf{u}_n(s)\|^2 ds\Big)}{N}.$$

Using (2.10) in above, we obtain

$$\mathcal{P}\{\tau_N < T\} \leq \frac{2}{N}\Big(\mathbb{E} \sup_{0 \leq t \leq T\wedge\tau_N} |\mathbf{u}_n(t)|^2 + \mathbb{E}\int_0^{T\wedge\tau_N} \|\mathbf{u}_n(s)\|^2 ds\Big) = \frac{2}{N} C_2(T).$$

Note that $C_2(T)$ is a constant independent of N, therefore, $\mathcal{P}\{\tau_N < T\} \to 0$ as $N \to \infty$, which implies that $\tau_N \to \infty$ almost surely as $N \to \infty$. This together with the fact that τ_N is increasing in N further imply that $T \wedge \tau_N \to T$ almost surely as $N \to \infty$. Letting $N \to \infty$ in (2.10) and (2.8), we obtain (2.3) and (2.2).

Define

$$\tau'_N := \inf\{t \in [0,T] : |\mathbf{u}_n(t)|^3 + \int_0^t |\mathbf{u}_n(s)| \|\mathbf{u}_n(s)\|^2 ds > N$$
$$\text{or } |\mathbf{u}_n(t-)|^3 + \int_0^t |\mathbf{u}_n(s)| \|\mathbf{u}_n(s)\|^2 ds > N\}.$$

Denote

$$\begin{aligned}
M(t) &= 2\int_0^t \langle \mathbf{u}_n(s), \sigma(s, \mathbf{u}_n(s), \mathfrak{r}(s)) dW_n(s)\rangle,\\
A(t) &= 2\int^t \langle -\nu \mathbf{A}\mathbf{u}_n(s) - \mathbf{B}_k(\mathbf{u}_n(s)) + \mathbf{f}(s), \mathbf{u}_n(s)\rangle_V ds + \int_0^t \|\sigma^*(s, \mathbf{u}_n(s), \mathfrak{r}(s))\|_{L_Q}^2 ds\\
&\quad + \int_0^{t+}\int_Z \Big(|\mathbf{u}_n(s) + \mathbf{G}_n(s, \mathbf{u}_n(s), \mathfrak{r}(s), z)|^2 - |\mathbf{u}_n(s)|^2\\
&\qquad - 2\big(\mathbf{u}_n(s), \mathbf{G}_n(s, \mathbf{u}_n(s), \mathfrak{r}(s), z)\big)_H \Big) \nu_1(dz) ds,\\
g(z,t) &= |\mathbf{u}_n(t-) + \mathbf{G}(t-, \mathbf{u}_n(t-), \mathfrak{r}(t-), z)|^2 - |\mathbf{u}_n(t-)|^2,
\end{aligned}$$

temporarily. Then the process $|\mathbf{u}_n(t)|^2$ defined in (2.5) is a real-valued process and can be expressed in the following formulation

$$|\mathbf{u}_n(t)|^2 = |\mathbf{u}(0)|^2 + M(t) + A(t) + \int_0^{t+}\int_Z g(z,s)\tilde{N}_{1n}(dz, ds).$$

Let $F(x) = x^{\frac{3}{2}}$. Then $F'(x) = \frac{3}{2}\sqrt{x}$ and $F''(x) = \frac{3}{4}\frac{1}{\sqrt{x}}$. Utilizing the Itô formula in Ikeda and Watanabe [30, Thm. 5.1, Ch. II] to $|\mathbf{u}_n(t)|^2$ with $F(x) = x^{\frac{3}{2}}$, we have

$$
\begin{aligned}
&(|\mathbf{u}_n(t)|^2)^{\frac{3}{2}} \\
&= (|\mathbf{u}(0)|^2)^{\frac{3}{2}} + \frac{3}{2}\int_0^t \sqrt{|\mathbf{u}_n(s)|^2}\cdot 2\langle \mathbf{u}_n(s), \sigma(s,\mathbf{u}_n(s),\mathfrak{r}(s))dW_n(s)\rangle \\
&+ \frac{3}{2}\int_0^t \sqrt{|\mathbf{u}_n(s)|^2}\Big\{2\langle -\nu\mathbf{A}\mathbf{u}_n(s) - \mathbf{B}_k(\mathbf{u}_n(s)) + \mathbf{f}(s), \mathbf{u}_n(s)\rangle_V + \|\sigma^*(s,\mathbf{u}_n(s),\mathfrak{r}(s))\|^2_{L_Q} \\
&\qquad\qquad + \int_Z \Big(|\mathbf{u}_n(s) + \mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)|^2 - |\mathbf{u}_n(s)|^2 \\
&\qquad\qquad\qquad - 2\big(\mathbf{u}_n(s), \mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)\big)_H\Big)\nu_1(dz)ds\Big\}ds \\
&+ \frac{1}{2}\int_0^t \frac{3}{4}\frac{1}{\sqrt{|\mathbf{u}_n(s)|^2}}\cdot 4\|\sigma^*(s,\mathbf{u}_n(s),\mathfrak{r}(s))\mathbf{u}_n(s)\|^2_{L_Q}ds \\
&+ \int_0^{t+}\int_Z \Big\{\Big(|\mathbf{u}_n(s-)|^2 + (|\mathbf{u}_n(s-) + \mathbf{G}_n(s-,\mathbf{u}_n(s-),\mathfrak{r}(s-),z)|^2 - |\mathbf{u}_n(s-)|^2)\Big)^{\frac{3}{2}} \\
&\qquad - (|\mathbf{u}_n(s-)|^2)^{\frac{3}{2}}\Big\}\tilde{N}_{1n}(dz,ds) \\
&+ \int_0^{t+}\int_Z \Big\{\Big(|\mathbf{u}_n(s)|^2 + (|\mathbf{u}_n(s) + \mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)|^2 - |\mathbf{u}_n(s)|^2)\Big)^{\frac{3}{2}} - (|\mathbf{u}_n(s)|^2)^{\frac{3}{2}} \\
&\qquad - (|\mathbf{u}_n(s) + \mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)|^2 - |\mathbf{u}_n(s)|^2)\cdot\frac{3}{2}\sqrt{|\mathbf{u}_n(s)|^2}\Big\}\nu_1(dz)ds.
\end{aligned}
$$

A simplification of the second term on the right of above gives

$$
3\int_0^t |\mathbf{u}_n(s)|\langle \mathbf{u}_n(s), \sigma(s,\mathbf{u}_n(s),\mathfrak{r}(s))dW_n(s)\rangle;
$$

a simplification of the third term gives

$$
\begin{aligned}
&3\int_0^t |\mathbf{u}_n(s)|\langle -\nu\mathbf{A}\mathbf{u}_n(s) - \mathbf{B}_k(\mathbf{u}_n(s)) + \mathbf{f}(s), \mathbf{u}_n(s)\rangle_V ds \\
&+ \frac{3}{2}\int_0^t |\mathbf{u}_n(s)|\|\sigma^*(s,\mathbf{u}_n,\mathfrak{r}(s))\|^2_{L_Q}ds \\
&+ \frac{3}{2}\int_0^t\int_Z |\mathbf{u}_n(s)||\mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)|^2\nu_1(dz)ds;
\end{aligned}
$$

a simplification of the forth term gives

$$\frac{3}{2}\int_0^t \frac{\|\sigma^*(s,\mathbf{u}_n(s),\mathfrak{r}(s))\mathbf{u}_n(s)\|_{L_Q}^2}{|\mathbf{u}_n(s)|}ds;$$

a simplification of the fifth term gives

$$\int_0^{t+}\int_Z \Big((|\mathbf{u}_n(s-)|+|\mathbf{G}_n(s-,\mathbf{u}_n(s-),\mathfrak{r}(s-),z)|)^3-|\mathbf{u}_n(s-)|^3\Big)\tilde{N}_{1n}(dz,ds);$$

a simplification of the sixth (last) term gives

$$\int_0^{t+}\int_Z \Big\{|\mathbf{u}_n(s)+\mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)|^3-|\mathbf{u}_n(s)|^3$$
$$-\frac{3}{2}|\mathbf{u}_n(s)|\Big((2\big(\mathbf{u}_n(s),\mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)\big)_H+|\mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)|^2\Big)\Big\}\nu_1(dz)ds.$$

Hence, we conclude

$$\begin{aligned}
|\mathbf{u}_n(t)|^3 &= |\mathbf{u}(0)|^3 \\
&+3\int_0^t |\mathbf{u}_n(s)|\langle -\nu\mathbf{A}\mathbf{u}_n(s)-\mathbf{B}_k(\mathbf{u}_n(s))+\mathbf{f}(s),\mathbf{u}_n(s),\rangle_V ds \\
&+3\int_0^t |\mathbf{u}_n(s)|\langle \mathbf{u}_n(s),\sigma_n(s,\mathbf{u}_n(s),\mathfrak{r}(s))dW_n(s)\rangle \\
&+\frac{3}{2}\int_0^t |\mathbf{u}_n(s)|\|\sigma_n^*(s,\mathbf{u}_n(s),\mathfrak{r}(s))\|_{L_Q}^2 ds+\frac{3}{2}\int_0^t \frac{\|\sigma^*(s,\mathbf{u}_n(s),\mathfrak{r}(s))\mathbf{u}_n(s)\|_{L_Q}^2}{|\mathbf{u}_n(s)|}ds \\
&+\int_0^t\int_Z \Big(|\mathbf{u}_n(s-)+\mathbf{G}_n(s,\mathbf{u}_n(s-),\mathfrak{r}(s),z)|^3-|\mathbf{u}_n(s-)|^3\Big)\tilde{N}_1(dz,ds) \\
&+\int_0^t\int_Z \Big(|\mathbf{u}_n(s)+\mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)|^3-|\mathbf{u}_n(s)|^3 \\
&\qquad -3|\mathbf{u}_n(s)|\big(\mathbf{u}_n(s),\mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)\big)_H\Big)\nu(dz)ds.
\end{aligned} \tag{2.11}$$

Taking integration up to $t \wedge \tau'_N$ and then expectation in (2.11), we have

$$\begin{aligned}
&\mathbb{E}|\mathbf{u}_n(t \wedge \tau'_N)|^3 + 3\nu \mathbb{E}\int_0^{t\wedge\tau'_N} |\mathbf{u}_n(s)|\|\mathbf{u}_n(s)\|^2 ds \\
&\leq \mathbb{E}|\mathbf{u}(0)|^3 + 3\mathbb{E}\int_0^{t\wedge\tau'_N} |\mathbf{u}_n(s)|\|\mathbf{f}(s)\|_{V'}\|\mathbf{u}_n(s)\| ds \\
&\quad + 3\mathbb{E}\int_0^{t\wedge\tau'_N} |\mathbf{u}_n(s)|\|\sigma(s, \mathbf{u}_n(s), \mathfrak{r}(s))\|^2_{L_Q} ds \\
&\quad + \mathbb{E}\int_0^{t\wedge\tau'_N}\int_Z \Big(|\mathbf{u}_n(s) + \mathbf{G}_n(s, \mathbf{u}_n(s), \mathfrak{r}(s), z)|^3 - |\mathbf{u}_n(s)|^3 \\
&\qquad\qquad - 3|\mathbf{u}_n(s)|\big(\mathbf{u}_n(s), \mathbf{G}_n(s, \mathbf{u}_n(s), \mathfrak{r}(s), z)\big)_H\Big)\nu(dz)ds.
\end{aligned} \tag{2.12}$$

An application of triangle inequality and Hypothesis **H3** yields

$$\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\tau'_N}\int_Z \Big(|\mathbf{u}_n(s) + \mathbf{G}_n(s, \mathbf{u}_n(s), \mathfrak{r}(s), z)|^3 - |\mathbf{u}_n(s)|^3 \\
&\qquad\qquad - 3|\mathbf{u}_n(s)|\big(\mathbf{u}_n(s), \mathbf{G}_n(s, \mathbf{u}_n(s), \mathfrak{r}(s), z)\big)_H\Big)\nu(dz)ds \\
&\leq \mathbb{E}\int_0^{t\wedge\tau'_N}\int_Z 6|\mathbf{u}_n(s)|^2|\mathbf{G}_n(s, \mathbf{u}_n(s), \mathfrak{r}(s), z)|\nu(dz)ds \\
&\quad + \mathbb{E}\int_0^{t\wedge\tau'_N}\int_Z 3|\mathbf{u}_n(s)||\mathbf{G}_n(s, \mathbf{u}_n(s), \mathfrak{r}(s), z)|^2\nu(dz)ds \\
&\quad + \mathbb{E}\int_0^{t\wedge\tau'_N}\int_Z |\mathbf{G}_n(s, \mathbf{u}_n(s), \mathfrak{r}(s), z)|^3\nu(dz)ds \\
&\leq 10K\mathbb{E}\int_0^{t\wedge\tau'_N} |\mathbf{u}_n(s)|^3 ds + 6K(1+m)\mathbb{E}\int_0^{t\wedge\tau'_N} |\mathbf{u}_n(s)|^2 ds \\
&\quad + 3K(1+m^2)\mathbb{E}\int_0^{t\wedge\tau'_N} |\mathbf{u}_n(s)| ds + KT(1+m^3).
\end{aligned} \tag{2.13}$$

It follows from the basic Young inequality and the property $|\cdot| \leq \|\cdot\|$ that

$$\begin{aligned}
&3\|\mathbf{f}(s)\|_{V'}|\mathbf{u}_n(s)|\|\mathbf{u}_n(s)\| \\
&\leq \frac{1}{\nu^2}\|\mathbf{f}(s)\|^3_{V'} + 2\nu(|\mathbf{u}_n(s)|\|\mathbf{u}_n(s)\|)^{\frac{3}{2}} \leq \frac{1}{\nu^2}\|\mathbf{f}(s)\|^3_{V'} + 2\nu|\mathbf{u}_n(s)|\|\mathbf{u}_n(s)\|^2.
\end{aligned} \tag{2.14}$$

Using Hypothesis **H1**, (2.13), and (2.14) in (2.12), one has

$$\begin{aligned}
&\mathbb{E}|\mathbf{u}_n(t\wedge\tau_N')|^3+\nu\mathbb{E}\int_0^{t\wedge\tau_N'}|\mathbf{u}_n(s)|\|\mathbf{u}_n(s)\|^2ds \qquad (2.15)\\
&\le \mathbb{E}|\mathbf{u}(0)|^3+\frac{1}{\nu^2}\mathbb{E}\int_0^{t\wedge\tau_N'}\|\mathbf{f}(s)\|_{V'}^3ds+6K(1+m)\mathbb{E}\int_0^{t\wedge\tau_N'}|\mathbf{u}_n(s)|^2ds\\
&\quad+4K(1+m^2)\mathbb{E}\int_0^{t\wedge\tau_N'}|\mathbf{u}_n(s)|ds+KT(1+m^3)+11k\mathbb{E}\int_0^{t\wedge\tau_N'}|\mathbf{u}_n(s)|^3ds.
\end{aligned}$$

Notice that

$$\int_0^{t\wedge\tau_N'}|\mathbf{u}_n(s)|ds\le\int_0^t|\mathbf{u}_n(s)|ds$$

since $t\wedge\tau_N'\le t$. Thus, by the Schwarz inequality, the Jesen inequality (for concave functions), the property that $|\cdot|\le\|\cdot\|$, and (2.2), we have

$$\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\tau_N'}|\mathbf{u}_n(s)|ds\le\mathbb{E}\int_0^t|\mathbf{u}_n(s)|ds \qquad (2.16)\\
&\le\mathbb{E}\Big(\int_0^t|\mathbf{u}_n(s)|^2ds\Big)^{\frac{1}{2}}\sqrt{T}\le\sqrt{T}\Big(\mathbb{E}\int_0^t\|\mathbf{u}_n(s)\|^2ds\Big)^{\frac{1}{2}};
\end{aligned}$$

we also have

$$\mathbb{E}\int_0^{t\wedge\tau_N'}|\mathbf{u}_n(s)|^2ds\le\mathbb{E}\int_0^t\|\mathbf{u}_n(s)\|^2ds\le C. \qquad (2.17)$$

Making use of the above two estimates in (2.15), we then use the Gronwall inequality to obtain

$$\mathbb{E}|\mathbf{u}_n(t\wedge\tau_N')|^3\le C(\mathbb{E}|\mathbf{u}(0)|^3,\mathbb{E}\int_0^T\|\mathbf{f}(s)\|_{V'}^3ds,\nu,K,m,T). \qquad (2.18)$$

Utilizing above bound on the last term on the right of (2.15), we conclude

$$\begin{aligned}
&\mathbb{E}|\mathbf{u}_n(t\wedge\tau_N')|^3+\nu\mathbb{E}\int_0^{t\wedge\tau_N}|\mathbf{u}_n(s)|\|\mathbf{u}_n(s)\|^2ds\\
&\le C(\mathbb{E}|\mathbf{u}(0)|^3,\mathbb{E}\int_0^T\|\mathbf{f}(s)\|_{V'}^3ds,\nu,K,m,T).
\end{aligned}$$

It follows from the Itô formula that

$$
\begin{aligned}
|\mathbf{u}_n(t)|^3 = |\mathbf{u}(0)|^3 &+ 3\int_0^t |\mathbf{u}_n(s)|\langle -\nu\mathbf{A}\mathbf{u}_n(s) - \mathbf{B}_k(\mathbf{u}_n(s)) + \mathbf{f}(s), \mathbf{u}_n(s), \rangle_V ds \\
&+ 3\int_0^t |\mathbf{u}_n(s)|\langle \mathbf{u}_n(s), \sigma_n(s, \mathbf{u}_n(s), \mathfrak{r}(s))dW_n(s)\rangle \\
&+ 3\int_0^t |\mathbf{u}_n(s)|\|\sigma_n(s, \mathbf{u}_n(s), \mathfrak{r}(s))\|_{L_Q}^2 ds \\
&+ \int_0^t\int_Z \Big(|\mathbf{u}_n(s-) + \mathbf{G}_n(s, \mathbf{u}_n(s-), \mathfrak{r}(s), z)|^3 - |\mathbf{u}_n(s-)|^3\Big) N_1(dz, ds) \\
&- 3\int_0^t\int_Z |\mathbf{u}_n(s)|\big(\mathbf{u}_n(s), \mathbf{G}_n(s, \mathbf{u}_n(s), \mathfrak{r}(s), z)\big)_H \nu(dz)ds.
\end{aligned}
\tag{2.19}
$$

Taking supremum over $T\wedge\tau_N'$ and then expectation on (2.19), we have

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T\wedge\tau_N'} |\mathbf{u}_n(t)|^3 + 3\nu\mathbb{E}\int_0^{T\wedge\tau_N'} |\mathbf{u}_n(s)|\|\mathbf{u}_n(s)\|^2 ds \qquad (2.20)\\
&\le \mathbb{E}|\mathbf{u}(0)|^3 + 3\mathbb{E}\int_0^{T\wedge\tau_N'} |\mathbf{u}_n(s)|\langle \mathbf{f}(s), \mathbf{u}_n(s)\rangle_V ds \\
&\quad + \mathbb{E}\sup_{0\le t\le T\wedge\tau_N'} 3\int_0^t |\mathbf{u}_n(s)|\langle \mathbf{u}_n(s), \sigma(s, \mathbf{u}_n(s), \mathfrak{r}(s))dW_s(s)\rangle \\
&\quad + 3\mathbb{E}\int_0^{T\wedge\tau_N'} |\mathbf{u}_n(s)|\|\sigma_n(s, \mathbf{u}_n(s), \mathfrak{r}(s))\|_{L_Q}^2 ds \\
&\quad + \mathbb{E}\int_0^{T\wedge\tau_N'}\int_Z \Big(|\mathbf{u}_n(s-) + \mathbf{G}_n(s, \mathbf{u}_n(s-), \mathfrak{r}(s), z)|^3 - |\mathbf{u}_n(s-)|^3\Big) N_1(dz, ds) \\
&\quad + 3\mathbb{E}\int_0^{T\wedge\tau_N'}\int_Z |\mathbf{u}_n(s)|\ \big(\mathbf{u}_n(s), \mathbf{G}_n(s, \mathbf{u}_n(s), \mathfrak{r}(s), z)\big)_H\ \nu(dz)ds.
\end{aligned}
$$

By the Davis inequality, we have

$$
\begin{aligned}
&3\mathbb{E}\sup_{0\le t\le T\wedge\tau_N'}\int_0^t |\mathbf{u}_n(s)|\langle \mathbf{u}_n(s), \sigma(s, \mathbf{u}_n(s), \mathfrak{r}(s))dW_n(s)\rangle \\
&\le 3\sqrt{2}\mathbb{E}\Big\{\Big(\int_0^{T\wedge\tau_N'} \|\sigma(s, \mathbf{u}_n(s), \mathfrak{r}(s))(|\mathbf{u}_n(s)|\mathbf{u}_n(s))\|_{L_Q}^2\Big)^{\frac{1}{2}}\Big\} \\
&\le 3\sqrt{2}\mathbb{E}\Big\{\sup_{0\le t\le T\wedge\tau_N'} |\mathbf{u}_n(t)|^2\Big(\int_0^{T\wedge\tau_N'} \|\sigma(s, \mathbf{u}_n(s), \mathfrak{r}(s))\|_{L_Q}^2 ds\Big)^{\frac{1}{2}}\Big\};
\end{aligned}
$$

invoking the basic Young inequality and Hypothesis **H1** and continuing,

$$
\begin{aligned}
&\leq 3\sqrt{2}\mathbb{E}\Big\{\frac{2}{3}\epsilon \sup_{0\leq t\leq T\wedge\tau'_N} |\mathbf{u}_n(t)|^3 + \frac{1}{3}C_\epsilon\Big(\int_0^{T\wedge\tau'_N} \|\sigma(s,\mathbf{u}_n(s),\mathfrak{r}(s))\|^2_{L_Q} ds\Big)^{\frac{3}{2}}\Big\}\\
&\leq 3\sqrt{2}\mathbb{E}\Big\{\frac{2}{3}\epsilon \sup_{0\leq t\leq T\wedge\tau'_N} |\mathbf{u}_n(t)|^3 + \frac{1}{3}C_\epsilon\sqrt{T}\int_0^{T\wedge\tau'_N} \|\sigma(s,\mathbf{u}_n(s),\mathfrak{r}(s))\|^3_{L_Q} ds\Big\}\\
&\leq 2\sqrt{2}\epsilon\mathbb{E}\sup_{0\leq t\leq T\wedge\tau'_N} |\mathbf{u}_n(s)|^3 + \sqrt{2}\sqrt{T}KC_\epsilon\mathbb{E}\int_0^{T\wedge\tau'_N} |\mathbf{u}_n(s)|^3 ds\\
&\quad + \sqrt{2}KT^{\frac{3}{2}}(1+m^3)C_\epsilon. \qquad (2.21)
\end{aligned}
$$

An application of triangle inequality and expanding the cubic power yields

$$
\begin{aligned}
&\mathbb{E}\int_0^{T\wedge\tau'_N}\int_Z \Big(|\mathbf{u}_n(s-)+\mathbf{G}_n(s-,\mathbf{u}_n(s-),\mathfrak{r}(s-),z)|^3 - |\mathbf{u}_n(s-)|^3\Big) N_1(dz,ds)\\
&\leq 3\mathbb{E}\int_0^{T\wedge\tau'_N}\int_Z |\mathbf{u}_n(s-)|^2 |\mathbf{G}_n(s-,\mathbf{u}_n(s-),\mathfrak{r}(s-),z)| N_1(dz,ds)\\
&\quad + 3\mathbb{E}\int_0^{T\wedge\tau'_N}\int_Z |\mathbf{u}_n(s-)||\mathbf{G}_n(s-,\mathbf{u}_n(s-),\mathfrak{r}(s-),z)|^2 N_1(dz,ds)\\
&\quad + \mathbb{E}\int_0^{T\wedge\tau'_N}\int_Z |\mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)|^3 N_1(dz,ds);
\end{aligned}
$$

invoking Hypothesis **H3** and continuing,

$$
\begin{aligned}
&\leq 3\mathbb{E}\int_0^{T\wedge\tau'_N}\int_Z |\mathbf{u}_n(s)|^2 |\mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)| \nu(dz) ds\\
&\quad + 3\mathbb{E}\int_0^{T\wedge\tau'_N}\int_Z |\mathbf{u}_n(s)||\mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)|^2 \nu(dz) ds\\
&\quad + \mathbb{E}\int_0^{T\wedge\tau'_N}\int_Z |\mathbf{G}_n(s,\mathbf{u}_n(s),\mathfrak{r}(s),z)|^3 \nu(dz) ds\\
&\leq 7K\mathbb{E}\int_0^{T\wedge\tau'_N} |\mathbf{u}_n(s)|^3 ds + 3K(1+m)\mathbb{E}\int_0^{T\wedge\tau'_N} |\mathbf{u}_n(s)|^2 ds\\
&\quad + 3K(1+m^2)\mathbb{E}\int_0^{T\wedge\tau'_N} |\mathbf{u}_n(s)| ds + K(1+m^3)T. \qquad (2.22)
\end{aligned}
$$

Employing (2.21) and (2.22) in (2.20) and then using Hypotheses **H1** and **H3** and the basic Young inequality, we have

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T\wedge\tau'_N}|\mathbf{u}_n(t)|^3+\nu\mathbb{E}\int_0^{T\wedge\tau'_N}|\mathbf{u}_n(s)|\|\mathbf{u}_n(s)\|^2ds\\
&\le \mathbb{E}|\mathbf{u}(0)|^3+\frac{1}{\nu^2}\mathbb{E}\int_0^{T\wedge\tau'_N}\|\mathbf{f}(s)\|_{V'}^3ds\\
&\quad+2\sqrt{2}\epsilon\mathbb{E}\sup_{0\le t\le T\wedge\tau'_N}|\mathbf{u}_n(s)|^3+\sqrt{2}\sqrt{T}KC_\epsilon\mathbb{E}\int_0^{T\wedge\tau'_N}|\mathbf{u}_n(s)|^3ds+\sqrt{2}KT^{\frac{3}{2}}(1+m^3)C_\epsilon\\
&\quad+3K(1+m^2)\mathbb{E}\int_0^{T\wedge\tau'_N}|\mathbf{u}_n(s)|ds+3K\mathbb{E}\int_0^{T\wedge\tau_N}|\mathbf{u}_n(s)|^3ds\\
&\quad+7K\mathbb{E}\int_0^{T\wedge\tau'_N}|\mathbf{u}_n(s)|^3ds+3K(1+m)\mathbb{E}\int_0^{T\wedge\tau'_N}|\mathbf{u}_n(s)|^2ds\\
&\quad+3K(1+m^2)\mathbb{E}\int_0^{T\wedge\tau'_N}|\mathbf{u}_n(s)|ds+K(1+m^3)T\\
&\quad+3K(1+m)\mathbb{E}\int_0^{T\wedge\tau'_N}|\mathbf{u}_n(s)|^2ds+3K\mathbb{E}\int_0^{T\wedge\tau'_N}|\mathbf{u}_n(s)|^3ds
\end{aligned}
$$

Choose $\epsilon=\frac{1}{4\sqrt{2}}$. Then $C_\epsilon=4\sqrt{2}$. The inequality above can be simplified as

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\sup_{0\le t\le T\wedge\tau'_N}|\mathbf{u}_n(t)|^3+\nu\mathbb{E}\int_0^{T\wedge\tau'_N}|\mathbf{u}_n(s)|\|\mathbf{u}_n(s)\|^2ds\\
&\le(13+8\sqrt{T})K\mathbb{E}\int_0^{T\wedge\tau'_N}|\mathbf{u}_n(s)|^3ds+6(1+m)K\mathbb{E}\int_0^{T\wedge\tau'_N}|\mathbf{u}_n(s)|^2ds\\
&\quad+6(1+m^2)K\mathbb{E}\int_0^{T\wedge\tau'_N}|\mathbf{u}_n(s)|ds+(8\sqrt{T}+1)(1+m^3)KT.
\end{aligned}
$$

Using (2.16), (2.17), and (2.18) in above, we conclude, upon a simplification, that

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T\wedge\tau'_N}|\mathbf{u}_n(t)|^3+2\nu\mathbb{E}\int_0^{T\wedge\tau'_N}|\mathbf{u}_n(s)|\|\mathbf{u}_n(s)\|^2ds\\
&\le C(\mathbb{E}|\mathbf{u}(0)|^3,\mathbb{E}\int_0^T\|\mathbf{f}(s)\|_{V'}^3ds,\nu,K,m,T),
\end{aligned}
$$

which leads $T\wedge\tau'_N\to T$ as $N\to\infty$. Therefore, (2.4) is proved. □

Chapter 3. Stochastic Navier-Stokes Equations with Markov Switching

This chapter is devoted to the study stochastic Navier-Stokes equations with Markov switching (1.4). We first establish the existence of a weak solution (in the sense of stochastic analysis) by studying the martingale problem posed by it. Then we show that the weak solutions are pathwise unique so that by a well-known result of Yamada and Watanabe [62], the existence of a unique strong solution (in the sense of stochastic analysis) is obtained.

Having the solution to equation (1.4), we turn our attention to study certain properties of the solution. We establish the existence and uniqueness of the stationary measure of the system (1.4), and obtain exit time estimates as well. These estimates will be compared to the corresponding ones obtained for the non-switching case. The relation between the latter and the Freidlin-Wentzell type large deviations are also discussed

3.1. Martingale Problem

Suppose that $\omega^\dagger = (\mathbf{u}, \mathfrak{r})$ is a solution to equation (1.4). Then it is not hard to see from the Itô formula that

$$M^{\omega^\dagger}(t) := F(t, \mathbf{u}(t), \mathfrak{r}(t)) - F(0, \mathbf{u}(0), \mathfrak{r}(0)) - \int_0^t \mathcal{L}F(s, \mathbf{u}(s), \mathfrak{r}(s))ds \tag{3.1}$$

is a μ-martingale, where $\mu := \mathcal{P} \circ {\omega^\dagger}^{-1}$ is the distribution of $\omega^\dagger$, and $\mathcal{L}$ is the operator introduced in Section 1.2.5.

Recalling (1.27), we have defined $\Omega^\dagger := \Omega^* \times \mathcal{D}([0, T]; \mathcal{S})$. Now let $\omega = (u, i)$ be a generic element in $\Omega^\dagger$. Substituting $(\mathbf{u}, \mathfrak{r})$ by (u, i) in (3.1), we obtain a canonical expression:

$$M^{\omega}(t) := F(t, u(t), i(t)) - F(0, u(0), i(0)) - \int_0^t \mathcal{L}F(s, u(s), i(s))ds. \tag{3.2}$$

The aim of this section is to identify a measure μ on the path space $\Omega^{\dagger}$ under which $M(\cdot)$ in (3.2) is a martingale, and this is called *the martingale problem posed by the stochastic Navier-Stokes equation with Markov switching* (1.4).

Recalling the definition of path space $(\Omega^{\dagger}, \tau^{\dagger})$ from (1.27), we let $\mathcal{B}$ denote the Borel σ-field of the topology $\tau^{\dagger}$. Define

$$\mathcal{F}_t := \sigma(\omega(s) : 0 \leq s \leq t,\ \omega \in \Omega^{\dagger}).$$

Recall that $\mathcal{L}$ is the operator introduced in Section 1.2.5. We are in the position to introduce the definition of a solution to a martingale problem.

Definition 25. *A probability measure μ on $(\Omega^{\dagger}, \mathcal{B})$ is called a solution of the martingale problem with the initial distribution μ_0 and operator $\mathcal{L}$ if the following hold:*

(i) The time marginal of μ at $t = 0$ is μ_0, i.e., $\mu|_{t=0} = \mu_0$

(ii) The canonical expression $M(t)$ defined in (3.2) is an $\mathcal{F}_t$-martingale.

Let $X_t(\omega) = \omega(t)$ for all $\omega \in \Omega^{\dagger}$ be the canonical process on $\Omega^{\dagger}$. Therefore, in terms of the canonical process, the definition becomes:

Definition 26. *A process $X = \{X_t\}$ with path in $(\Omega^{\dagger}, \tau^{\dagger})$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is called a solution to the martingale problem for the initial distribution μ_0 and operator $\mathcal{L}$ if the following hold:*

(i) The distribution of X_0 is μ_0.

(ii) *For any $F \in \mathcal{D}(\mathcal{L})$, the process* (3.2) *is a $\mathcal{F}_t^X$-martingale.*

The study of martingale problem was introduced by Stroock and Varadhan. Moreover, it is shown that the existence of such a measure is equivalent to the existence of a weak solution to a stochastic differential equation.

Theorem 3.1.1. *Assume that $\mathbb{E}|\mathbf{u}(0)|^3 < \infty$ and $\mathbf{f} \in L^3(0,T;V')$. Then, under Hypotheses* **H**, *there exists a unique strong solution to* (1.4).

3.1.1. The proof of existence

As mentioned in the beginning of Section 3.1, the existence of the solution is established by studying its martingale problem. There are several equivalent formulations of a solution to a martingale problem (see, e.g., [55]), and we introduce one of them in the following lemma. The interested reader is referred to [31, Prop. 7.1.2] for more details.

Lemma 3.1.2. *The following statements are equivalent:*

(i) *X is a solution to the martingale problem for the operator $\mathcal{L}$.*

(ii) *For all $f \in D(\mathcal{L})$, $0 \le t_1 < t_2 < \cdots < t_{n+1}$, $h_1, h_2, \cdots h_n \in C_b$, and $n \ge 1$, we have*

$$\mathbb{E}\Big\{\Big(f(X_{t_{n+1}}) - f(X_{t_n}) - \int_{t_n}^{t_{n+1}} \mathcal{L}f(X_s)ds\Big)\prod_{j=1}^{n} h_j(X_{t_j})\Big\} = 0$$

Let $\phi(t,i)$ be a real-valued bounded smooth function with compact support (in each variables). For $\rho \in \mathcal{D}(\mathbf{A}) \subseteq V$, $0 \le s \le t$, and each generic element $\omega = (u,i) \in \Omega^\dagger$, define

$$
\begin{aligned}
&M^{\phi}(t) - M^{\phi}(s) \\
&:= \phi(\langle u(t), \rho\rangle_V, i(t)) - \phi(\langle u(s), \rho\rangle_V, i(s)) - \int_s^t \sum_{j=1}^m \gamma_{i(r-)j}\phi(\langle u(r), \rho\rangle_V, j)dr \\
&- \int_s^t \Big(\phi^{'}(\langle u(r), \rho\rangle_V, i(r)) \cdot \langle -\nu \mathbf{A} u(r) - \mathbf{B}_k(u(r)) + \mathbf{f}(r), \rho\rangle_V\Big)dr \\
&- \frac{1}{2}\int_s^t \phi^{''}(\langle u(r), \rho\rangle_V, i(r)) \cdot \big(\rho, \sigma(r, u(r), i(r)) Q \sigma^*(r, u(r), i(r))\rho\big)_H dr \\
&- \int_s^t \int_Z \Big(\phi(\langle u(r) + \mathbf{G}(r, u(r), i(r), z), \rho\rangle_V, i(r)) - \phi(\langle u(r), \rho\rangle_V, i(r)) \\
&\qquad - \phi^{'}(\langle u(r), \rho\rangle_V, i(r)) \cdot \langle \mathbf{G}(r, u(r), i(r), z), \rho\rangle_V\Big)\nu(dz)dr,
\end{aligned} \tag{3.3}
$$

According to Lemma 3.1.2, to show $M^{\phi}(t)$ is a solution to the martingale problem, it suffices to find a Radon measure μ such that

$$
\mathbb{E}^{\mu}\Big(\prod_{j=1}^m \psi_j(s_j)(M^{\phi}(t) - M^{\phi}(s))\Big) = 0, \ \forall s < s_1 < \cdots < s_m < t,
$$

where $\psi_j \in C_b(\Omega)$ and $\mathcal{F}_s$-measurable.

Define the truncation of $M^{\phi}(t)$ as follows:

$$
\begin{aligned}
&M_n^{\phi}(t) - M_n^{\phi}(s) \\
&:= \phi(\langle u(t), \rho\rangle_V, i(t)) - \phi(\langle u(s), \rho\rangle_V, i(s)) - \int_s^t \sum_{j=1}^m \gamma_{i(r-)j}\phi(\langle u(r), \rho\rangle_V, j)dr \\
&- \int_s^t \Big(\phi^{'}(\langle u(r), \rho\rangle_V, i(r)) \cdot \langle -\nu \mathbf{A}_n u(r) - \mathbf{B}_k^n(u(r)) + \mathbf{f}(r), \rho\rangle_V\Big)dr \\
&- \frac{1}{2}\int_s^t \phi^{''}(\langle u(r), \rho\rangle_V, i(r)) \cdot \big(\rho, \sigma_n(r, u(r), i(r)) Q \sigma_n^*(r, u(r), i(r))\rho\big)_H dr \\
&- \int_s^t \int_Z \Big(\phi(\langle u(r) + \mathbf{G}_n(r, u(r), i(r), z), \rho\rangle_V, i(r)) - \phi(\langle u(r), \rho\rangle_V, i(r)) \\
&\qquad - \phi^{'}(\langle u(r), \rho\rangle_V, i(r)) \cdot \langle \mathbf{G}_n(r, u(r), i(r), z), \rho\rangle_V\Big)\nu(dz)dr,
\end{aligned} \tag{3.4}
$$

where $\mathbf{A}_n = \Pi_n \mathbf{A}$, $\mathbf{B}_k^n = \Pi_n \mathbf{B}_k$, $\sigma_n = \Pi_n \sigma$, $\mathbf{G}_n = \Pi_n \mathbf{G}$, and $\omega = (u, i) \in \Omega^\dagger$ is a generic element.

Let $(\mathbf{u}_n, \mathfrak{r})$ be the solution to equation (2.1) and denote by μ_n the (joint) distribution of $(\mathbf{u}_n, \mathfrak{r})$. Then it follows from the (finite dimensional) Itô formula that $M_n^\phi(t)$ is a μ_n-martingale, therefore, for all n,

$$\mathbb{E}^{\mu_n} \prod_{j=i}^{m} \psi_j(s_j) M_n^\phi(t) = 0, \ \forall s < s_1 < \cdots < s_m < t,$$

for $\psi_j \in C_b(\Omega)$ and $\mathcal{F}_s$-measurable. Hence,

$$\lim_{n\to\infty} \mathbb{E}^{\mu_n} \prod_{j=i}^{m} \psi_j(s_j) M_n^\phi(t) = 0, \ \forall s < s_1 < \cdots < s_m < t,$$

for $\psi_j \in C_b(\Omega)$ and $\mathcal{F}_s$-measurable.

If we can show that

M1. there exists a probability measure μ such that μ_n "converges" to μ,

M2. $\lim_{n\to\infty} M_n^\phi(t) = M^\phi(t)$ μ-almost surely, and

M3. $\lim_{n\to\infty} \mathbb{E}^{\mu_n} M^\phi(t) = \mathbb{E}^\mu M^\phi(t)$,

then we conclude that $M^\phi(t)$ is a μ-martingale.

Now we prove **M1**. Recall that $\mathbf{u}_n$ is the solution to (2.1) for each n.

Lemma 3.1.3. *The sequence* $\{\mathbf{u}_n\}$ *forms a relative compact set in* $\mathcal{D}([0,T]; V')$.

Proof. It is clear that $\{\mathbf{u}_n\}$ is a subset of $\mathcal{D}([0,T]; V')$.

Let $N > 0$. By the Markov inequality, the property that $\|\cdot\|_{V'} \le |\cdot|$, and (2.2), we have

$$\mathcal{P}\Big(\|\mathbf{u}_n(t)\|_{V'} > N\Big) \le \frac{1}{N^2}\mathbb{E}\|\mathbf{u}_n(t)\|_{V'}^2 \le \frac{1}{N^2}\mathbb{E}|\mathbf{u}_n(t)|^2 \le \frac{C}{N^2}.$$

Therefore,

$$\lim_{N\to\infty}\limsup_{n}\mathcal{P}\Big(\|\mathbf{u}_n(t)\|_{V'}>N\Big)=0.$$

Let (T_n,δ_n) be a sequence, where T_n is a stopping time with $T_n+\delta_n\le T$ and $\delta_n>0$ with $\delta_n\to 0$. For each $\epsilon>0$, the Chebyshev's inequality implies

$$\begin{aligned}&\mathcal{P}(\|\mathbf{u}_n(T_n+\delta_n)-\mathbf{u}_n(T_n)\|_{V'}>\epsilon)\\&\le\frac{1}{\epsilon^2}\mathbb{E}\|\mathbf{u}_n(T_n+\delta_n)-\mathbf{u}_n(T_n)\|_{V'}^2\le\frac{1}{\epsilon^2}\mathbb{E}|\mathbf{u}_n(T_n+\delta_n)-\mathbf{u}_n(T_n)|^2.\end{aligned}$$

It follows from the Itô formula and the Gronwall inequality that

$$\mathbb{E}|\mathbf{u}_n(T_n+\delta_n)-\mathbf{u}_n(T_n)|^2\le\Big(\frac{1}{\nu}\mathbb{E}\int_0^{\delta_n}\|\mathbf{f}(s)\|_{V'}^2ds+2K\delta_n\Big)e^{2K\delta_n}\to 0$$

as $n\to\infty$. Therefore, $\|\mathbf{u}_n(T_n+\delta_n)-\mathbf{u}_n(T_n)\|_{V'}\to 0$ in probability as $n\to\infty$. By Aldous' criterion, we conclude that $\{\mathbf{u}_n\}$ is tight in $\mathcal{D}([0,T];V')$.

Hence, $\{\mathbf{u}_n\}$ is relatively compact in $\mathcal{D}([0,T];V')$. □

We have a even stronger convergence which is proved in the following proposition.

Proposition 3.1.4. *The sequence $\{\mathbf{u}_n\}$ forms a relative compact set in $L^2(0,T;H)$.*

Proof. It follows from (2.3) that $\{\mathbf{u}_n\}$ is bounded in $L^2(0,T;V)$; also, we have

$$\mathbb{E}\int_0^T|\mathbf{u}_n(t)|^2dt\le\mathbb{E}\int_0^T\|\mathbf{u}_n(t)\|^2dt\le C,$$

which implies that $\{\mathbf{u}_n\}\subset L^2(0,T;H)\cap\mathcal{D}([0,T];V')$.

In addition, by Lemma 3.1.3, $\{\mathbf{u}_n\}$ is relatively compact in $\mathcal{D}([0,T];V')$.

Hence, the proposition follows from Lemma 1.2.19. □

Recalling Definition 20 and $\mathbf{u}_n$ being the solution to (2.1), we deduce from a priori estimates and Banach-Alaoglu theorem that $\{\mathbf{u}_n\}$ is relatively compact in (Ω_2, τ_2) and (Ω_3, τ_3). In addition, Lammata 3.1.3 and 3.1.4 imply that $\{\mathbf{u}_n\}$ is compact in (Ω_1, τ_1) and (Ω_4, τ_4), respectively. Therefore, by Prohorov's theorem, the induced distribution $\{\mu_n^*\}$[i] is tight on each space (Ω_j, τ_j) for $j = 1, 2, 3, 4$. Hence, by (ii) in Definition 20, $\{\mu_n^*\}$ is tight on (Ω^*, τ).

Let μ_n be the joint distribution of $(\mathbf{u}_n, \mathfrak{r})$. Then $\{\mu_n\}$ is tight on the space $(\Omega^\dagger, \tau^\dagger)$, hence, there exist a subsequence $\{\mu_{n_\ell}\}_\ell$ and a measure μ such that $\mu_{n_\ell} \Rightarrow \mu$.

Next, we consider **M2**. Recall from Section1.2.5 that $H_n = \text{span}\{e_j\}_{j=1}^n$ and Π_n is a projection operator from H onto H_n. Denote by $\{n_\ell\}_{\ell=1}^\infty$ the indices such that $\mu_{n_\ell} \Rightarrow \mu$.

Lemma 3.1.5. *For each $\rho \in D(\mathbf{A})$, $\Pi_{n_\ell}\rho \to \rho$ in V, as $\ell \to \infty$.*

Proof. Defining $f_j := \frac{e_j}{\sqrt{\lambda_j}}$, one sees $\|f_j\| = \frac{\|e_j\|^2}{\lambda_j} = \frac{\lambda_j}{\lambda_j} = 1$. This implies that $\{f_j\}$ is a complete orthonormal basis in V. Thus,

$$\rho = \sum_{j=1}^{\infty} (\rho, f_j)_V f_j; \quad \Pi_{n_\ell}\rho := \rho_{n_\ell} = \sum_{j=1}^{n_\ell} (\rho, f_j)_V f_j.$$

As a consequence, $\|\rho - \Pi_{n_\ell}\rho\| = \sum_{j=n_\ell+1}^\infty (\rho, f_j)_V f_j \to 0$, as $\ell \to \infty$. □

Lemma 3.1.6. *For each $\rho \in \mathcal{D}(\mathbf{A})$, we have*

$$\lim_{\ell\to\infty} \int_s^t \phi'(\langle u(r), \rho\rangle_V, i(r))\langle -\nu \mathbf{A}_{n_\ell} u(r), \rho\rangle_V dr = \int_s^t \phi'(\langle u(r), \rho\rangle_V, i(r))\langle -\nu \mathbf{A} u(r), \rho\rangle_V dr,$$

Proof. A direct computation gives, for almost all $r \in [t, s]$,

$$\langle -\nu \mathbf{A}_{n_\ell} u(r), \rho\rangle_V = -\nu\langle \mathbf{A} u(r), \rho_{n_\ell}\rangle \to -\nu\langle \mathbf{A} u(r), \rho\rangle$$

[i] $\mu_n^* := \mathcal{P} \circ \mathbf{u}_n^{-1}$

as $\ell \to \infty$, by Lemma 3.1.5. In addition,

$$|\langle -\nu \mathbf{A}_{n_\ell} u(r), \rho \rangle_V| = | -\nu \langle \mathbf{A} u(r), \rho_{n_\ell} \rangle| \leq \nu \|\rho\|_{V'} \|u(r)\|.$$

Notice that $u \in \Omega^*$, therefore, $u \in L^2(0,T;V) \subset L(0,T;V)$. Hence, the lemma follows from the Lebesgue Dominated Convergence Theorem. □

Lemma 3.1.7. *For each $\rho \in \mathcal{D}(\mathbf{A})$, we have*

$$\lim_{\ell \to \infty} \int_s^t \phi^{'}(\langle u(r), \rho \rangle_V, i(r)) \langle \mathbf{B}_k^{n_\ell}(u(r)), \rho \rangle_V dr = \int_s^t \phi^{'}(\langle u(r), \rho \rangle_V, i(r)) \langle \mathbf{B}_k(u(r)), \rho \rangle_V dr$$

Proof. A similar argument as in Lemma 3.1.6 shows that

$$\langle \mathbf{B}_k^{n_\ell}(u(r)), \rho \rangle_V \to \langle \mathbf{B}_k(u(r)), \rho \rangle_V$$

as $\ell \to \infty$ for almost all $r \in [s,t]$. In addition

$$|\langle \mathbf{B}_k^{n_\ell}(u(r)), \rho \rangle_V| = |\langle \mathbf{B}_k(u(r)), \rho_{n_\ell} \rangle_V| \leq \|u(r)\| \|u(r)\| \|\rho_{n_\ell}\|$$

by (1.25). Since $u \in \Omega^*$, $u \in L^2(0,T;V)$. Thus,

$$|\phi^{'}(\langle u(r), \rho \rangle_V, i) \langle \mathbf{B}_k(u(r)), \rho \rangle_V| \leq \|\phi^{'}\|_\infty \|\rho\| \|u(r)\|^2,$$

which is an L^1-function. Therefore, the lemma follows from the Lebesgue Dominated Convergence Theorem. □

Lemma 3.1.8. *For each $\rho \in \mathcal{D}(\mathbf{A})$, we have*

$$\lim_{\ell \to \infty} \int_s^t \phi^{''}(\langle u(r), \rho \rangle_V, i(r)) \big(\rho, \sigma_{n_\ell}(r, u(r), i(r)) Q \sigma_{n_\ell}^*(r, u(r), i(r)) \rho\big)_H dr$$
$$= \int_s^t \phi^{''}(\langle u(r), \rho \rangle_V, i(r)) \big(\rho, \sigma(r, u(r), i(r)) Q \sigma^*(r, u(r), i(r)) \rho\big)_H dr,$$

Proof. For the notation simplicity, we write $\sigma(r) = \sigma(r, u(r), i(r))$ when there is no ambiguity. A direct computation shows that

$$\begin{aligned}&\big(\rho, \sigma_{n_\ell}(r)Q\sigma^*_{n_\ell}(r)\rho\big)_H = \big(\rho_{n_\ell}, \sigma(r)Q\sigma^*_{n_\ell}(r)\rho\big)_H\\&= \big(\sigma_{n_\ell}(r)Q\sigma^*(r)\rho_{n_\ell}, \rho\big)_H = \big(\sigma(r)Q\sigma^*(r)\rho_{n_\ell}, \rho_{n_\ell}\big)_H,\end{aligned}$$

therefore,

$$\big(\rho, \sigma_{n_\ell}(r)Q\sigma^*_{n_\ell}(r)\rho\big)_H - \big(\rho, \sigma(r)Q\sigma^*(r)\rho\big)_H = \big(\rho_{n_\ell}, \sigma(r)Q\sigma^*(r)\rho_{n_\ell}\big)_H - \big(\rho, \sigma(r)Q\sigma^*(r)\rho\big)_H,$$

which implies

$$\begin{aligned}&|\big(\rho, \sigma_{n_\ell}(r)Q\sigma^*_{n_\ell}(r)\rho\big)_H - \big(\rho, \sigma(r)Q\sigma^*(r)\rho\big)_H|\\&\leq |\big(\rho_{n_\ell}, \sigma(r)Q\sigma^*(r)\rho_{n_\ell} - \sigma(r)Q\sigma^*(r)\rho\big)_H| + |\big(\rho_{n_\ell} - \rho, \sigma(r)Q\sigma^*(r)\rho\big)_H|\\&\leq |\rho_{n_\ell}||\sigma(r)Q\sigma^*(r)||\rho_{n_\ell} - \rho| + |\rho_{n_\ell} - \rho||\sigma(r)Q\sigma^*(r)||\rho|\\&= 2|\rho||\rho_{n_\ell} - \rho|\|\sigma(r)\|_{L_Q} \leq 2\|\rho\| \cdot \|\rho_{n_\ell} - \rho\| \cdot \|\sigma(r)\|_{L_Q}\end{aligned}$$

Thus, by Lemma 3.1.5,

$$\big(\rho, \sigma_{n_\ell}(r)Q\sigma^*_{n_\ell}(r)\rho\big)_H \to \big(\rho, \sigma(r)Q\sigma^*(r)\rho\big)_H$$

as $\ell \to \infty$ for all $r \in [s, t]$. In addition,

$$\begin{aligned}&|\phi''(\langle u(r), \rho\rangle_V, i(r))\big(\rho, \sigma_{n_\ell}(r, u(r), i(r))Q\sigma^*_{n_\ell}(r, u(r), i(r))\rho\big)_H|\\&\leq \|\phi''\|_\infty|\rho_{n_\ell}|\|\sigma(r)\|_{L_Q} \leq \|\phi''\|_\infty \cdot \|\rho\| \cdot \|\sigma(r, u(r), i(r))\|_{L_Q}.\end{aligned}$$

Consider

$$\int_0^T \|\sigma(r, u(r), i(r))\|_{L_Q} dr \leq \sqrt{T}\Big(\int_0^T \|\sigma(r, u(r), i(r))\|^2_{L_Q} dr\Big)^{\frac{1}{2}}.$$

Recall that $u \in \Omega^*$, therefore, $u \in L^2(0,T;H)$; Hypothesis **H1** implies

$$\int_0^T \|\sigma(r,u(r),i(r))\|_{L_Q}^2 dr \leq \int_0^T K(1+|u(r)|^2+i^2)dr < C$$

for a constant C. Therefore, we conclude that the function

$$|\phi^{''}(\langle u(r),\rho\rangle_V, i(r))\big(\rho, \sigma_{n_\ell}(r,u(r),i(r))Q\sigma_{n_\ell}^*(r,u(r),i(r))\rho\big)_H|$$

is bounded by an L^1-function, hence, the lemma follows from the Lebesgue Dominated Convergence Theorem. □

Lemma 3.1.9. *For each $\rho \in \mathcal{D}(\mathbf{A})$, we have*

$$\begin{aligned}
&\lim_{\ell\to\infty}\int_s^t\int_Z \Big(\phi(\langle u(r)+\mathbf{G}_{n_\ell}(r,u(r),i(r),z),\rho\rangle_V, i(r)) - \phi(\langle u(r),\rho\rangle_V, i(r)) \\
&\qquad - \phi^{'}(\langle u(r),\rho\rangle_V, i(r))\cdot\langle \mathbf{G}_{n_\ell}(r,u(r),i(r),z),\rho\rangle_V\Big)\nu(dz)dr \\
&= \int_s^t\int_Z \Big(\phi(\langle u(r)+\mathbf{G}(r,u(r),i(r),z),\rho\rangle_V, i) - \phi(\langle u(r),\rho\rangle_V, i(r)) \\
&\qquad - \phi^{'}(\langle u(r),\rho\rangle_V, i(r))\cdot\langle \mathbf{G}(r,u(r),i(r),z),\rho\rangle_V\Big)\nu(dz)dr,
\end{aligned}$$

Proof. It follows from Lemma 3.1.5 that

$$\langle \mathbf{G}_{n_\ell}(r,u(r),i(r),z),\rho\rangle_V = \langle \mathbf{G}(r,u(r),i(r),z),\rho_{n_\ell}\rangle_V \to \langle \mathbf{G}(r,u(r),i(r),z),\rho\rangle_V$$

as $\ell\to\infty$ for all $r\in[s,t]$. Therefore, the convergence of the integrand is shown.

For a fixed ℓ, writing $a = \langle u(r),\rho\rangle_V$ and $b = \langle u(r)+\mathbf{G}_{n_\ell}(r,u(r),i(r),z),\rho\rangle_V$, we deduce from the Mean Value Theorem that

$$\begin{aligned}
&\phi(\langle u(r)+\mathbf{G}_{n_\ell}(r,u(r),i(r),z),\rho\rangle_V, i) - \phi(\langle u(r),\rho\rangle_V, i(r)) \\
&= \phi'(c,i(r))\langle \mathbf{G}_{n_\ell}(r,u(r),i(r),z),\rho\rangle_V,
\end{aligned}$$

where $c \in (a,b)$. Therefore,

$$\begin{aligned}
&|\phi(\langle u(r) + \mathbf{G}_{n_\ell}(r,u(r),i(r),z),\rho\rangle_V, i(r)) - \phi(\langle u(r),\rho\rangle_V, i(r))| \\
&\leq \|\phi^{'}\|_\infty |\rho_{n_\ell}||\mathbf{G}(r,u(r),i(r),z)|,
\end{aligned}$$

which implies

$$\begin{aligned}
&\Big(\phi(\langle u(r) + \mathbf{G}_{n_\ell}(r,u(r),i(r),z),\rho\rangle_V, i(r)) - \phi(\langle u(r),\rho\rangle_V, i(r)) \\
&\qquad\qquad - \phi^{'}(\langle u(r),\rho\rangle_V, i(r)) \cdot \langle \mathbf{G}_{n_\ell}(r,u(r),i(r),z),\rho\rangle_V\Big) \\
&\leq |\phi(\langle u(r) + \mathbf{G}_{n_\ell}(r,u(r),i(r),z),\rho\rangle_V, i(r)) - \phi(\langle u(r),\rho\rangle_V, i(r))| \\
&\quad + |\phi^{'}(\langle u(r),\rho\rangle_V, i(r)) \cdot \langle \mathbf{G}_{n_\ell}(r,u(r),i(r),z),\rho\rangle_V| \\
&\leq 2\|\phi^{'}\|_\infty \|\rho\||\mathbf{G}(r,u(r),i(r),z)|.
\end{aligned}$$

By Hypothesis **H3**, $\int_Z |\mathbf{G}(r,u(r),i(r),z)|\nu(dz)$ is an L^1-function. Thus, this lemma follows from the Lebesgue Dominated Convergence Theorem. □

In light of Lemmata 3.1.5 to 3.1.9, **M2** has been proved. Moreover, as shown in the proofs of Lemmata 3.1.5 to 3.1.9, the expectation and limit is exchangeable, i.e.,

$$\lim_{n\to\infty} \mathbb{E} M_n^\phi(t) = \mathbb{E} \lim_{n\to\infty} M_n^\phi(t).$$

Lastly, we consider **M3**. Clearly, if the assumption of Lemma 1.2.2 is fulfilled, then **M3** is obtained. So far, we have a sequence of measures $\{\mu_n\}$, and there exists a measure μ such that $\mu_{n_\ell} \Rightarrow \mu$ as $\ell \to \infty$. Therefore, it remains to prove that

(i) $M^\phi(t)$ is continuous on $(\Omega^\dagger, \tau^\dagger)$, and

(ii) for some $\delta > 0$, $\sup_n \mathbb{E}^{\mu_n}\Big(|M^\phi(t)|^{1+\delta}\Big) < C$, where C is a constant.

We begin the proof of continuity of $M^\phi(t)$ with the following auxiliary lemma.

Lemma 3.1.10. *Let $\{u_n\}$ and u be members of (Ω^*, τ) with $u_n \to u$ as $n \to \infty$ in τ-topology. For almost all $t \in [0, T]$, $k = 0, 1, 2$, and each $i \in \mathcal{S}$, we have*

$$\frac{d^k}{dx^k}\phi(\langle u_n(t), \rho\rangle_V, i) \to \frac{d^k}{dx^k}\phi(\langle u(t), \rho\rangle_V, i),$$

as $n \to \infty$.

Proof. Define $C(u) := \{t \in [0, T]; \mathcal{P}\big(u(t) = u(t-)\big) = 1\}$. Then it follows from Lemma 1.2.21 that the complement of $C(u)$ is at most countable (see, e.g., [9, Lem. 1]). Therefore, for almost all $t \in [0, T]$, one has $u_n(t) \to u(t)$, as $n \to \infty$. This further implies that $\langle u_n(t), \rho\rangle_V \to \langle u(t), \rho\rangle_V$, as $n \to \infty$ for any $\rho \in \mathcal{D}(\mathbf{A})$. Therefore, the lemma follows from the smoothness of ϕ. □

Lemma 3.1.11. *$M^\phi(t)$ is continuous in the $(\tau^\dagger)$-topology.*

Proof. It suffices to prove that $M^\phi(t)$ is continuous in the τ-topology since there is no convergence issue in $\mathfrak{r}$.

Let $\{u_n\}$ and u be members of (Ω^*, τ) with $u_n \to u$ as $n \to \infty$ in τ-topology. Let $M^\phi(u_n(t))$ be the function where u_n is in place of u in (3.3). Given $u_n \to u$. We need to show that $\lim_{n\to\infty} M^\phi(u_n(t)) = M^\phi(u(t))$, and we prove it by taking the term-by-term limit.

The first three terms follows from Lemma 3.1.10 and the Bounded Convergence Theorem.

For the $\mathbf{A}$ term, it is not hard to see that

$$\langle \mathbf{A}u_n(r), \rho\rangle_V = \langle u_n(r), \mathbf{A}\rho\rangle_V \to \langle u(r), \mathbf{A}\rho\rangle_V$$

as $n \to \infty$ for almost all $r \in [s, t]$ by Lemma 3.1.10. Consider

$$|\langle \mathbf{A}u_n(r), \rho\rangle_V|^2 \le \|u_n(r)\|^2 \|\rho\|_{V'}^2,$$

and

$$\int_0^T \|u_n(r)\| \|\rho\|_{V'} dr = \|\rho\|_{V'} \int_0^T \|u_n(r)\| dr \le \|\rho\|_{V'} \sqrt{T} \Big(\int_0^T \|u_n(r)\|^2 dr \Big)^{\frac{1}{2}} < C$$

for all n and a constant C since $u_n \to u$ in τ-topology and thus in τ_2. This implies that

$$\sup_n \int_0^T |\langle \mathbf{A}u_n(r), \rho\rangle_V|^2 dr < \infty.$$

Thus, by Lemma 1.2.1, $\{\langle \mathbf{A}u_n(r), \rho\rangle_V\}$ is uniformly integrable, and thus by Theorem 1.2.3, we have

$$\lim_{n\to\infty} \int_s^t \langle -\nu \mathbf{A}u_n(r), \rho\rangle_V = \int_s^t \langle -\nu \mathbf{A}u(r), \rho\rangle_V.$$

For the $\mathbf{B}$ term, using the definition of $\mathbf{B}$, we have

$$\begin{aligned}
&|\langle \mathbf{B}_k(u_n(r)), \rho\rangle_V - \langle \mathbf{B}_k(u(r)), \rho\rangle_V| \\
&\le |b(k * u_n(r), u_n(r) - u(r), \rho)| + |b(k * (u_n(r) - u(r)), u(r), \rho)|,
\end{aligned}$$

which together with (1.24) further imply

$$\begin{aligned}
&|\langle \mathbf{B}_k(u_n(r)), \rho\rangle_V - \langle \mathbf{B}_k(u(r)), \rho\rangle_V| \\
&\le 2\|\rho\| \|u_n(r)\| \|u_n(r) - u(r)\|^{\frac{1}{2}} |u_n(r) - u(r)|^{\frac{1}{2}}.
\end{aligned}$$

Therefore, the Schwarz inequality implies

$$
\begin{aligned}
&\int_0^T |\langle \mathbf{B}_k(u_n(r)), \rho\rangle_V - \langle \mathbf{B}_k(u(r)), \rho\rangle_V| dr \\
&\leq 2\|\rho\| \Big(\int_0^T \|u_n(r)\|^2 dr \Big)^{\frac{1}{2}} \Big(\int_0^T \|u_n(r) - u(r)\| |u_n(r) - u(r)| dr \Big)^{\frac{1}{2}} \\
&\leq 2\|\rho\| \Big(\int_0^T \|u_n(r)\|^2 dr \Big)^{\frac{1}{2}} \Big(\int_0^T \|u_n(r)\| |u_n(r) - u(r)| dr + \int_0^T \|u(r)\| |u_n(r) - u(r)| dr \Big)^{\frac{1}{2}} \\
&\leq 2\|\rho\| \Big(\int_0^T \|u_n(r)\|^2 dr \Big)^{\frac{1}{2}} \\
&\quad \cdot \Big\{ \Big(\int_0^T \|u_n(r)\|^2 dr \Big)^{\frac{1}{2}} \Big(\int_0^T |u_n(r) - u(r)|^2 dr \Big)^{\frac{1}{2}} \\
&\qquad + \Big(\int_0^T \|u(r)\|^2 dr \Big)^{\frac{1}{2}} \Big(\int_0^T |u_n(r) - u(r)|^2 dr \Big)^{\frac{1}{2}} \Big\}^{\frac{1}{2}}.
\end{aligned}
$$

Since u_n and u are members of Ω^*, the $L^2(0,T;V)$-norms are finite. In addition, $u_n \to u$ in τ-topology implies that $u_n \to u$ in τ_4 (the strong topology in $L^2(0,T;H)$). Hence, we conclude that

$$\lim_{n\to\infty} \int_0^T |\langle \mathbf{B}_k(u_n(r)), \rho\rangle_V - \langle \mathbf{B}_k(u(r)), \rho\rangle_V| dr = 0,$$

which implies

$$\lim_{n\to\infty} \int_s^t \langle \mathbf{B}_k(u_n(r)), \rho\rangle_V dr = \int_s^t \langle \mathbf{B}_k(u(r)), \rho\rangle_V dr.$$

For the term represents the continuous noise, consider

$$
\begin{aligned}
&\big(\rho, \sigma(r, u_n(r), i(r)) Q \sigma^*(r, u_n(r), i(r)) \rho\big)_H - \big(\rho, \sigma(r, u(r), i(r)) Q \sigma^*(r, u(r), i(r)) \rho\big)_H \\
&\leq \ \rho \ \cdot \ \sigma(r, u_n(r), i(r)) Q \sigma^*(r, u_n(r), i(r)) \rho - \sigma(r, u(r), i(r)) Q \sigma^*(r, u(r), i(r)) \rho \\
&\leq |\rho|^2 \ \sigma(r, u_n(r), i(r)) Q \sigma^*(r, u_n(r), i(r)) - \sigma(r, u(r), i) Q \sigma^*(r, u(r), i(r)) \ .
\end{aligned}
$$

Recalling the definition of L_Q-norm, we see that

$$\sigma(r, u_n(r), i(r))Q\sigma^*(r, u_n(r), i(r)) - \sigma(r, u(r), i(r))Q\sigma^*(r, u(r), i(r))$$

$$= \|\sigma(r, u_n(r), i(r)) - \sigma(r, u(r), i(r))\|_{L_Q}.$$

Therefore, by Hypothesis **H2**, we have

$$\int_0^T \left(\rho, \sigma(r, u_n(r), i(r))Q\sigma^*(r, u_n(r), i(r))\rho\right)_H - \left(\rho, \sigma(r, u(r), i)Q\sigma^*(r, u(r), i(r))\rho\right)_H{}^2 dr$$
$$\leq \|\rho\|^4 \int_0^T \|\sigma(r, u_n(r), i(r)) - \sigma(r, u(r), i(r))\|_{L_Q}^2 dr \leq L\|\rho\|^4 \int_0^T |u_n(r) - u(r)|^2 dr,$$

which approaches to 0 as $n \to \infty$ since $u_n \to u$ in τ means that $u_n \to u$ in τ_4 (the strong topology in $L^2(0, T; H)$). Thus, we have

$$\lim_{n\to\infty} \int_s^t \left(\rho, \sigma(r, u_n(r), i(r))Q\sigma^*(r, u_n(r), i(r))\rho\right)_H dr$$
$$= \int_s^t \left(\rho, \sigma(r, u(r), i(r))Q\sigma^*(r, u(r), i(r))\rho\right)_H dr.$$

For the jump noise term, notice that $\mathbf{G}$ is continuous in all of its components, therefore, by Lemma 3.1.10

$$\lim_{n\to\infty} \mathbf{G}(r, u_n(r), i(r), z) = \mathbf{G}(r, u(r), i(r), z)$$

for almost all $r \in [s, t]$ and all fixed z. This implies that

$$\phi(\langle u_n(r) + \mathbf{G}(r, u_n(r), i(r), z), \rho\rangle_V, i) - \phi(\langle u_n(r), \rho\rangle_V, i(r))$$

$$- \phi'(\langle u_n(r), \rho\rangle_V, i(r)) \cdot \langle \mathbf{G}(r, u_n(r), i(r), z), \rho\rangle_V$$

converges to

$$\phi(\langle u(r) + \mathbf{G}(r, u(r), i(r), z), \rho\rangle_V, i(r)) - \phi(\langle u(r), \rho\rangle_V, i(r))$$

$$- \phi'(\langle u(r), \rho\rangle_V, i(r)) \cdot \langle \mathbf{G}(r, u(r), i(r), z), \rho\rangle_V$$

almost surely in $[s,t] \times Z$. Writing $a_n = \langle u_n(r), \rho\rangle_V$ and $b_n = \langle u_n(r) + \mathbf{G}(r, u_n(r), i(r), z)\rangle_V$, one infers from the Mean Value Theorem that

$$\phi(\langle u_n(r) + \mathbf{G}(r, u_n(r), i(r), z), \rho\rangle_V, i(r)) - \phi(\langle u_n(r), \rho\rangle_V, i(r))$$
$$= \phi^{'}(c_n, i(r))\langle \mathbf{G}(r, u_n(r), i(r), z), \rho\rangle_V,$$

where $c_n \in (a_n, b_n)$. Therefore,

$$\begin{aligned}
&\phi(\langle u_n(r) + \mathbf{G}(r, u_n(r), i(r), z), \rho\rangle_V, i) - \phi(\langle u_n(r), \rho\rangle_V, i(r)) \\
&\quad - \phi^{'}(\langle u_n(r), \rho\rangle_V, i(r)) \cdot \langle \mathbf{G}(r, u_n(r), i(r), z), \rho\rangle_V \\
&\leq \ \phi(\langle u_n(r) + \mathbf{G}(r, u_n(r), i(r), z), \rho\rangle_V, i(r)) - \phi(\langle u_n(r), \rho\rangle_V, i(r)) \\
&\quad + \ \phi^{'}(\langle u_n(r), \rho\rangle_V, i(r)) \cdot \langle \mathbf{G}(r, u_n(r), i(r), z), \rho\rangle_V \\
&= \ \phi^{'}(c_n)\langle \mathbf{G}(r, u_n(r), i(r), z), \rho\rangle_V \\
&\quad + \ \phi^{'}(\langle u_n(r), \rho\rangle_V, i(r)) \cdot \langle \mathbf{G}(r, u_n(r), i(r), z), \rho\rangle_V \\
&\leq 2\|\phi^{'}\|_\infty |\rho| |\mathbf{G}(r, u_n(r), i(r), z)|,
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_0^T \int_Z \ \phi(\langle u_n(r) + \mathbf{G}(r, u_n(r), i(r), z), \rho\rangle_V, i(r)) - \phi(\langle u_n(r), \rho\rangle_V, i(r)) \\
&\qquad - \phi^{'}(\langle u_n(r), \rho\rangle_V, i(r)) \cdot \langle \mathbf{G}(r, u_n(r), i(r), z), \rho\rangle_V \ ^2 dr \\
&\leq 4\|\rho\|^2 \int_0^T \int_Z |\mathbf{G}(r, u_n(r), i(r), z)|^2 dr \leq 4K\|\rho\|^2 \int_0^T (1 + |u_n(r)|^2) dr,
\end{aligned}$$

where the last inequality follows from Hypothesis **H3**. Since $u_n \to u$ in τ-topology, $u_n \to u$ in τ_4, which implies that

$$\sup_n \int_0^T |u_n(r)|^2 dr < C$$

for a constant C. Therefore, we have

$$\sup_n \int_0^T \int_Z \; \phi(\langle u_n(r) + \mathbf{G}(r, u_n(r), i(r), z), \rho\rangle_V, i(r)) - \phi(\langle u_n(r), \rho\rangle_V, i(r))$$
$$- \phi^{'}(\langle u_n(r), \rho\rangle_V, i(r)) \cdot \langle \mathbf{G}(r, u_n(r), i(r), z), \rho\rangle_V \; ^2 dr < C,$$

which shows the validity of Lemma 1.2.1. Hence, we conclude

$$\lim_{n\to\infty} \int_s^t \int_Z \Big(\phi(\langle u_n(r) + \mathbf{G}(r, u_n(r), i(r), z), \rho\rangle_V, i(r)) - \phi(\langle u_n(r), \rho\rangle_V, i(r))$$
$$- \phi^{'}(\langle u_n(r), \rho\rangle_V, i(r)) \cdot \langle \mathbf{G}(r, u_n(r), i(r), z), \rho\rangle_V\Big)\nu(dz)dr$$
$$= \int_s^t \int_Z \Big(\phi(\langle u(r) + \mathbf{G}(r, u(r), i(r), z), \rho\rangle_V, i(r)) - \phi(\langle u(r), \rho\rangle_V, i(r))$$
$$- \phi^{'}(\langle u(r), \rho\rangle_V, i(r)) \cdot \langle \mathbf{G}(r, u(r), i(r), z), \rho\rangle_V\Big)\nu(dz)dr,$$

which completes the proof. □

The following lemma is the final piece of the required argument. It is the only place where we require $\mathbb{E}|\mathbf{u}(0)|^3 < 0$ and $\mathbf{f} \in L^3(0, T; V')$.

Lemma 3.1.12. *Suppose that Hypotheses* $\mathbf{H}$ *is fulfilled,* $\mathbb{E}|\mathbf{u}(0)|^3 < \infty$*, and* $\mathbf{f} \in L^3(0, T; V')$*. There exist some* $\delta > 0$ *such that*

$$\sup_\ell \mathbb{E}^{\mu_{n_\ell}}\left[|M^\phi|^{1+\delta}\right] \leq C,$$

where C *is an appropriate constant.*

Proof. Recalling from (3.3) the definition of M^ϕ, we employ inequality (1.6) and the Mean

Value Theorem (on $\mathbf{G}$) to deduce

$$
\begin{aligned}
|M^{\phi}(t)|^{1+\delta} \leq 5^{\delta}|\phi(\langle u(t),\rho\rangle_V, i(r))|^{1+\delta} &+ 5^{\delta}|\phi(\langle u(s)\rangle_V, i(r))|^{1+\delta} \\
&+ 5^{\delta}\|\phi'\|_{\infty}\int_s^t |\langle \nu\mathbf{A}u(r) + \mathbf{B}_k(u(r)) + \mathbf{f}(s), \rho\rangle_V|^{1+\delta} dr \\
&+ 5^{\delta}\|\phi''\|_{\infty}\int_s^t |(\rho, \sigma(r,u(r),i(r))Q\sigma^*(r,u(r),i(r)))_H|^{1+\delta} dr \\
&+ 5^{\delta}\int_s^t \|\phi'\|_{\infty} \quad \int_Z \langle \mathbf{G}(r,u(r),i(r),z),\rho\rangle_V \nu(dz) \quad ^{1+\delta} dr
\end{aligned}
$$

since ϕ is bounded smooth function.

For the $\mathbf{A}$ term,

$$
\int_s^t |\nu\langle \mathbf{A}u(r),\rho\rangle_V|^{1+\delta} dr \leq \nu^{1+\delta}\int_s^t \big(\|u(r)\|_V \|\mathbf{A}\rho\|_{V'}\big)^{1+\delta} dr \leq C_1 \int_s^t \|u(r)\|_V^{1+\delta} dr,
$$

where $C_1 = \nu^{1+\delta}\|\mathbf{A}\rho\|_{V'}$. This implies that

$$
\mathbb{E}^{\mu_{n_\ell}}\int_s^t |\nu\langle \mathbf{A}u(r),\rho\rangle_V|^{1+\delta} dr \leq C_1 \mathbb{E}^{\mu_{n_\ell}}\int_s^t \|u(r)\|_V^{1+\delta} dr = \mathbb{E}\int_s^t \|\mathbf{u}_{n_\ell}(r)\|_V^{1+\delta} dr,
$$

hence,

$$
\sup_{\ell} \mathbb{E}^{n_\ell}\int_s^t \quad \langle \nu\mathbf{A}u(r),\rho\rangle_V \quad ^{1+\delta} dr \leq C_{\mathbf{A}}
$$

if $\delta < 1$.

For the nonlinear term, (1.24) and Hölder inequality imply

$$
\begin{aligned}
\mathbb{E}^{\mu_{n_\ell}}\Big\{ \quad \int_s^t \langle \mathbf{B}_k(u(r)),\rho\rangle_V dr \quad ^{1+\delta}\Big\} &\leq \|\rho\|_V^{1+\delta}\mathbb{E}^{\mu_{n_\ell}}\Big\{\Big(\int_s^t \|u(r)\|_V |u(r)|_H dr\Big)^{1+\delta}\Big\} \\
\leq \|\rho\|_V^{1+\delta}\Big\{\mathbb{E}^{\mu_{n_\ell}}\big(\sup_{0\leq t\leq T}|u(t)|_H^{1+\delta}\big)^p\Big\}^{\frac{1}{p}} &\Big\{\mathbb{E}^{\mu_{n_\ell}}\big(\int_s^t \|u(r)\|_V^{1+\delta} dr\big)^q\Big\}^{\frac{1}{q}},
\end{aligned}
$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Choosing q such that $(1+\delta)q = 2$, we have

$$\begin{aligned}
&\mathbb{E}^{\mu_{n_\ell}}\Big\{ \int_s^t \langle \mathbf{B}_k(u(r)), \rho\rangle_V{}^{1+\delta}\Big\} \qquad (3.5)\\
&\le \|\rho\|_V^{1+\delta}\Big\{\mathbb{E}^{\mu_{n_\ell}} \sup_{0\le t\le T} |u(t)|_H^{2(\frac{1+\delta}{1-\delta})}\Big\}^{\frac{1-\delta}{2}} \Big\{\mathbb{E}^{\mu_{n_\ell}} \int_s^t \|u(r)\|_V^2 dr\Big\}^{\frac{1+\delta}{2}}.\\
&= \|\rho\|_V^{1+\delta}\Big\{\mathbb{E} \sup_{0\le t\le T} |\mathbf{u}_{n_\ell}(t)|_H^{2(\frac{1+\delta}{1-\delta})}\Big\}^{\frac{1-\delta}{2}} \Big\{\mathbb{E} \int_s^t \|\mathbf{u}_{n_\ell}(r)\|_V^2 dr\Big\}^{\frac{1+\delta}{2}}
\end{aligned}$$

Taking $\delta = \frac{1}{5}$, we have $2(\frac{1+\delta}{1-\delta}) = 3$. The first expectation on the right of (3.5) will have a uniform bound by (2.4) if we further assume that $\mathbb{E}|\mathbf{u}(0)|^3 < \infty$. The boundedness of $\mathbb{E}\int_s^t \|\mathbf{u}_{n_\ell}(r)\|_V^2 dr$ is followed from (2.2). Therefore, (3.5) implies that

$$\sup_{\ell} \mathbb{E}^{\mu_{n_\ell}}\Big\{ \int_s^t \langle \mathbf{B}_k(u(r)), \rho\rangle_V{}^{1+\delta} dr\Big\} \le C_{\mathbf{B}}$$

if $\delta \le \frac{1}{5}$.

For martingale terms, we have

$$\begin{aligned}
&\int_s^t \big(\rho, \sigma(r, u(r), i(r))Q\sigma^*(r, u(r), i(r))\rho\big)_H{}^{1+\delta} dr\\
&\le |\rho|_H^{2(1+\delta)} \int_s^t \|\sigma(r, u(r), i(r))\|_{L_Q}^{1+\delta} dr\\
&\le |\rho|_H^{2(1+\delta)} K^{\frac{1+\delta}{2}} \int_s^t (1 + m^2 + |u(r)|_H^2)^{\frac{1+\delta}{2}} dr\\
&\le |\rho|_H^{2(1+\delta)} K^{\frac{1+\delta}{2}} T^{\frac{1+\delta}{2}} \Big((1+m^2)T + \int_s^t |u(r)|_H^2 dr\Big)^{\frac{1+\delta}{2}},
\end{aligned}$$

where the second inequality follows from Hypothesis **H1** with $p = 2$, and the last inequality follows from the concavity of the power $\frac{1+\delta}{2}$. Using Hypothesis **H3** and inequality (1.6), we have

$$\begin{aligned}
&\int_s^t \int_Z \langle \mathbf{G}(r, u(r), i(r), z), \rho\rangle_V \nu(dz){}^{1+\delta} dr\\
&\le 2^\delta |\rho|_H^{1+\delta} K^{1+\delta}\Big((1+m)^{1+\delta}T + \int_s^t |u(r)|_H^{1+\delta} dr\Big).
\end{aligned}$$

Therefore, taking expectation and then supremum over ℓ on martingale terms, the above estimates imply

$$\sup_{\ell} \mathbb{E}^{n_\ell} \int_s^t \left(\rho, \sigma(r, u(r), i(r)) Q \sigma^*(r, u(r), i(r)) \rho\right)_H^{1+\delta} dr \leq C_Q(T)$$

and

$$\sup_{\ell} \mathbb{E}^{n_\ell} \int_s^t \int_Z \langle \mathbf{G}(r, u(r), i(r), z), \rho \rangle_V \nu(dz)\, dr \leq C_{\mathbf{G}}(T)$$

since, by (2.2),

$$\begin{aligned} &\mathbb{E}^{\mu_{n_\ell}} \int_s^t \|u(r)\|_V^{1+\delta} dr = \mathbb{E} \int_s^t \|\mathbf{u}_{n_\ell}(r)\|^{1+\delta} dr \\ &\leq \mathbb{E} \int_s^t \|\mathbf{u}_{n_\ell}(r)\|^2 dr = \mathbb{E}^{\mu_{n_\ell}} \int_s^t \|u(r)\|_V^2 dr \leq C, \end{aligned}$$

if $\delta < 1$.

In conclusion, the argument above shows that for $0 < \delta \leq \frac{1}{5}$, there is a constant C such that $\sup_\ell \mathbb{E}^{n_\ell}[|M^\phi|^{1+\delta}] \leq C$ provided that $\mathbb{E}|\mathbf{u}(0)|^3$ is finite. Hence, we complete the proof. □

As **M1**, **M2**, and **M3** are shown, the existence theorem follows:

Theorem 3.1.13. *Suppose that $\mathbb{E}|\mathbf{u}(0)|^3 < \infty$ and $\mathbf{f} \in L^3(0, T; V')$. Then, under Hypotheses* **H**, *$M^\phi(t)$ is a μ-martingale, i.e., μ is a solution to the martingale problem posed by* (1.4).

3.1.2. The proof of uniqueness

In this subsection, we prove that the (weak) solution obtained in Theorem 3.1.13 is pathwise unique.

Theorem 3.1.14. *Let $\mathbb{E}|\mathbf{u}(0)|^3 < \infty$ and $\mathbf{f} \in L^3(0, T; V')$. Then, under Hypotheses* **H**, *the solution obtained in Theorem 3.1.13 is pathwise unique.*

Proof. Let $\mathbf{w} = \mathbf{u} - \mathbf{v}$, where $\mathbf{u}$, $\mathbf{v}$ are solutions with same initial data. Let $F(t, x, i) := e^{-\rho(t)}x$, where $\rho(t)$ is a function that will be determined later. Then the Itô formula implies

$$
\begin{aligned}
& e^{-\rho(t)}|\mathbf{w}(t)|^2 + 2\nu \int_s^t e^{-\rho(s)}\|\mathbf{w}(s)\|^2 ds \\
&= \int_0^t -\rho^{'}(s)e^{-\rho(s)}|\mathbf{w}(s)|^2 ds - \int_0^t e^{-\rho(s)}\langle \mathbf{B}_k(\mathbf{u}(s)) - \mathbf{B}_k(\mathbf{v}(s)), \mathbf{u}(s) - \mathbf{v}(s)\rangle_V ds \\
&\quad + \int_0^t e^{-\rho(s)}\|\sigma(s, \mathbf{u}(s), i(s)) - \sigma(s, \mathbf{v}(s), j(s))\|_{L_Q}^2 ds \\
&\quad + 2\int_0^t e^{-\rho(s)}\langle \mathbf{u}(s) - \mathbf{v}(s), [\sigma(s, \mathbf{u}(s), i(s)) - \sigma(s, \mathbf{v}(s), j(s))]dW(s)\rangle \\
&\quad + 2\int_0^t \int_Z e^{-\rho(s-)} \\
&\qquad \cdot \big(\mathbf{u}(s-) - \mathbf{v}(s-), \mathbf{G}(s-, \mathbf{u}(s-), i(s-), z) - \mathbf{G}(s-, \mathbf{v}(s-), j(s-), z)\big)_H \tilde{N}_1(dz, ds) \\
&\quad + \int_0^t \int_Z e^{-\rho(s-)}|\mathbf{G}(s-, \mathbf{u}(s-), i(s-), z) - \mathbf{G}(s-, \mathbf{v}(s-), j(s-), z)|^2 N_1(dz, ds).
\end{aligned}
\tag{3.6}
$$

Applying the basic Young inequality to the nonlinear term, we see that

$$
\begin{aligned}
& \int_0^t \int_Z e^{-\rho(s)}\langle \mathbf{B}_k(\mathbf{u}(s)) - \mathbf{B}_k(\mathbf{v}(s)), \mathbf{u}(s) - \mathbf{v}(s)\rangle_V ds \\
& \le \int_0^t e^{-\rho(s)}(\|\mathbf{w}(s)\| \cdot |\mathbf{w}(s)| \cdot \|\mathbf{u}(s)\|)ds \\
& \le \nu \int_0^t e^{-\rho(s)}\|\mathbf{w}(s)\|^2 ds + \frac{1}{4\nu}\int_0^t e^{-\rho(s)}|\mathbf{w}(s)|^2\|\mathbf{u}(s)\|^2 ds.
\end{aligned}
$$

Therefore, choosing $\rho(t) := \frac{1}{4\nu}\int_0^t \|\mathbf{u}(s)\|^2 ds$, we deduce form (3.6) that

$$
\begin{aligned}
& e^{-\rho(t)}|\mathbf{w}(t)|^2 + \nu \int_0^t e^{-\rho(s)}\|\mathbf{w}(s)\|^2 ds \qquad (3.7) \\
& \leq \int_0^t e^{-\rho(s)}\|\sigma(s,\mathbf{u}(s),i(s)) - \sigma(s,\mathbf{v}(s),j(s))\|_{L_Q}^2 ds \\
& \quad + 2\int_0^t e^{-\rho(s)}\langle \mathbf{u}(s)-\mathbf{v}(s), [\sigma(s,\mathbf{u}(s),i(s)) - \sigma(s,\mathbf{v}(s),j(s))]dW(s)\rangle \\
& \quad + 2\int_0^t \int_Z e^{-\rho(s-)} \\
& \qquad \cdot \big(\mathbf{u}(s-)-\mathbf{v}(s-), \mathbf{G}(s-,\mathbf{u}(s-),i(s-),z) - \mathbf{G}(s-,\mathbf{v}(s-),j(s-),z)\big)_H \tilde{N}_1(dz,ds) \\
& \quad + \int_0^t \int_Z e^{-\rho(s-)}|\mathbf{G}(s-,\mathbf{u}(s-),i(s-),z) - \mathbf{G}(s-,\mathbf{v}(s-),j(s-),z)|^2 N_1(dz,ds).
\end{aligned}
$$

Moreover, by the Davis and the basic Young inequalities and Hypotheses **H2** and **H4**, the martingale terms in (3.7) have the following estimates.

$$
\begin{aligned}
& \mathbb{E}\sup_{0\leq t\leq T}\int_0^t 2e^{-\rho(s)}\langle \mathbf{u}(s)-\mathbf{v}(s), [\sigma(s,\mathbf{u}(s),i(s)) - \sigma(s,\mathbf{v}(s),j(s))]dW(s)\rangle \\
& \leq 2\sqrt{L}C_{\frac{1}{2}}\Big\{\epsilon \sup_{0\leq t\leq T} e^{-\rho(t)}|\mathbf{w}(t)|^2 + C_\epsilon \int_0^T e^{-\rho(s)}|\mathbf{w}(s)|^2 ds\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathbb{E}\sup_{0\leq t\leq T}\int_0^t \int_Z 2e^{-\rho(s-)} \\
& \quad \cdot \big(\mathbf{u}(s-)-\mathbf{v}(s-), \mathbf{G}(s-,\mathbf{u}(s-),i(s-),z) - \mathbf{G}(s-,\mathbf{v}(s-),j(s-),z)\big)_H \tilde{N}_1(dz,ds) \\
& \leq 2\sqrt{L}C_{\frac{1}{2}}\mathbb{E}\Big\{\epsilon \sup_{0\leq t\leq T} e^{-\rho(t)}|\mathbf{w}(t)|^2 + C_\epsilon \int_0^T e^{-\rho(s)}|\mathbf{w}(s)|^2 ds\Big\}.
\end{aligned}
$$

As a consequence, taking supremum over $[0,T]$ and then expectation, one obtains from

(3.7) the following.

$$\mathbb{E}\sup_{0\le t\le T} e^{-\rho(t)}|\mathbf{w}(t)|^2 + \nu\mathbb{E}\int_0^T \|\mathbf{w}(s)\|^2 ds$$
$$\le 2L\mathbb{E}\int_0^T e^{\rho(s)}|\mathbf{w}(s)|^2 ds + \mathfrak{C}\epsilon\mathbb{E}\sup_{0\le t\le T} e^{-\rho(t)}|\mathbf{w}(t)|^2 + \mathfrak{C}C_\epsilon\mathbb{E}\int_0^T e^{-\rho(s)}|\mathbf{w}(s)|^2 ds,$$

where $\mathfrak{C} = 4\sqrt{L}C_{\frac{1}{2}}$. Choosing ϵ small enough so that $\mathfrak{C}\epsilon < \frac{1}{2}$, one obtains from above that

$$\mathbb{E}\sup_{0\le t\le T} e^{-\rho(t)}|\mathbf{w}(t)|^2 \le C\mathbb{E}\int_0^T e^{-\rho(s)}|\mathbf{w}(s)|^2 ds \le C\mathbb{E}\int_0^T \sup_{0\le r\le s} e^{-\rho(r)}|\mathbf{w}(r)|^2 ds,$$

where C stands for a generic constant. Furthermore, we employ the Gronwall inequality to obtain $\mathbb{E}\sup_{0\le t\le T} e^{-\rho(t)}|\mathbf{w}(t)|^2 \le 0$, which implies the pathwise uniqueness. Hence, we complete the proof. □

3.1.3. Martingale problem for non-switching case

It is not hard to see that (1.5) is a special case of (1.4), and thus the solution is guaranteed by Theorem 3.1.1. More precisely, suppose now that the noise coefficients $\sigma : [0,T]\times H \to \mathcal{L}_2(H_0,H)$ and $\mathbf{G} : [0,T]\times H\times Z \to H$ are continuous and satisfy the following Hypotheses $\mathbf{H}'$:

H1'. For all $t\in(0,T)$, there exists a constant $K>0$ such that
$$\|\sigma(t,\mathbf{u})\|_{L_Q}^p \le K(1+|\mathbf{u}|^p)$$
for $p=2,3$ (growth condition on σ).

H2'. For all $t\in(0,T)$, there exists a constant $L>0$ such that for all $\mathbf{u},\mathbf{v}\in H$,
$$\|\sigma(t,\mathbf{u})-\sigma(t,\mathbf{v})\|_{L_Q}^2 \le L(|\mathbf{u}-\mathbf{v}|^2)$$
(Lipschitz condition on σ).

H3'. For all $t\in(0,T)$, there is a constant $K>0$ such that
$$\int_Z |\mathbf{G}(t,\mathbf{u},z)|^p\nu(dz) \le K(1+|\mathbf{u}|^p)$$
for $p=1,2,$ and 3 (growth condition on $\mathbf{G}$).

H4'. For all $t \in (0, T)$, there exists a constant $L > 0$ such that for all $\mathbf{u}, \mathbf{v} \in H$,

$$\int_Z |\mathbf{G}(t, \mathbf{u}, z) - \mathbf{G}(t, \mathbf{v}, z)|^2 \nu(dz) \leq L(|\mathbf{u} - \mathbf{v}|^2)$$

(Lipschitz condition on $\mathbf{G}$).

It is clear that the Hypotheses $\mathbf{H}'$ is a subclass of the Hypotheses $\mathbf{H}$, therefore, the existence and uniqueness of the solution to equation (1.5) is guaranteed by Theorem 3.1.1.

Corollary 3.1.15. *Assume that* $\mathbb{E}|\mathbf{u}(0)|^3 < \infty$, *and* $\mathbf{f} \in L^3(0, T; V')$. *Then, under Hypotheses* $\mathbf{H}'$, *there exists a unique strong solution to* (1.5).

3.2. Stationary Measures

In this section, we study the stationary measures of the system (1.4). The study of the invariant measures of the two-dimensional Navier-Stokes equations has been addressed by several authors (see, e.g., [22, 23, 39, 56]) under a variety of conditions. Here, the noise coefficients σ and $\mathbf{G}$ are assumed to be *additive* and *autonomous*, i.e., the equation under study is

$$\mathbf{du}(t) + [\nu \mathbf{A}\mathbf{u}(t) + \mathbf{B}_k(\mathbf{u}(t))]dt = \mathbf{f}(t)dt + \sigma(\mathfrak{r}(t))dW(t) + \int_Z \mathbf{G}(\mathfrak{r}(t-), z)\tilde{N}_1(dz, dt), \quad (3.9)$$

where $\mathbf{u}(0) = \mathbf{u}_0 \in H$.

Instead of Hypotheses $\mathbf{H}$, we assume that the noise coefficients σ and $\mathbf{G}$ satisfy Hypotheses $\mathbf{A}$ (throughout this section). Though these hypotheses can be written more simply, we state it as below since it would be more useful for our future work when the Markov chain has more general state space. Suppose that the functions $\sigma : \mathcal{S} \to \mathcal{L}_2(H_0, H)$ and $\mathbf{G} : \mathcal{S} \times Z \to H$ are continuous functions and satisfy

A1. For any $i \in \mathcal{S}$, there exist a constant $K > 0$ such that

$$\|\sigma(i)\|_{L_Q}^2 \leq K(1 + |i|^2),$$

(growth condition on σ).

A2. For any $i, j \in \mathcal{S}$, there exist a constant $L > 0$ such that

$$\|\sigma(i) - \sigma(j)\|_{L_Q}^2 \leq L(|i - j|^2),$$

(Lipschitz condition on σ).

A3. For any $i \in \mathcal{S}$, there exists a constant $K > 0$ such that

$$\int_Z |\mathbf{G}(i, z)|^p \nu(dz) \leq K(1 + |i|^p),$$

for $p = 1, 2$, and 4 (growth condition on $\mathbf{G}$).

A4. For any $i, j \in \mathcal{S}$, there exist a constant $L > 0$ such that

$$\int_Z |\mathbf{G}(i, z) - \mathbf{G}(j, z)|^p \nu(dz) \leq L(|i - j|^p),$$

$p = 1, 2$, and 4 (Lipschitz condition on $\mathbf{G}$).

It is clear that Hypotheses **A** is a subclass of the Hypotheses **H**, therefore, the existence and uniqueness of equation (3.9) follows from Theorem 3.1.1. In addition, under a new hypothesis, the solution $\mathbf{u}$ of equation (3.9) satisfy the following modified estimates.

Proposition 3.2.1. *Let $T > 0$ be fixed. Assume that $\mathbb{E}|\mathbf{u}(0)|^2 < \infty$ and $\mathbf{f} \in L^2(0, T; V')$. Then, under Hypotheses* **A**, *the solution* $\mathbf{u}$ *of equation* (3.9) *satisfies the following estimates.*

$$\begin{aligned} &\mathbb{E}|\mathbf{u}(t)|^2 + \nu \mathbb{E} \int_0^t \|\mathbf{u}(s)\|^2 ds \\ &\leq \mathbb{E}|\mathbf{u}(0)|^2 + \frac{1}{\nu} \mathbb{E} \int_0^t \|\mathbf{f}(s)\|_{V'}^2 ds + 2(1 + m^2)Kt \\ &= C(\mathbb{E}|\mathbf{u}(0)|^2, \mathbb{E} \int_0^t \|\mathbf{f}(s)\|_{V'}^2 ds, \nu, m, K, t) \end{aligned} \tag{3.10}$$

for each $t \in (0,T]$, and

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}|\mathbf{u}(t)|^2 + \nu\mathbb{E}\int_0^T \|\mathbf{u}(s)\|^2 ds \\
&\le 2\mathbb{E}|\mathbf{u}(0)|^2 + \frac{4}{3\nu}\mathbb{E}\int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds + 100(1+m^2)KT \\
&= C(\mathbb{E}|\mathbf{u}(0)|^2, \mathbb{E}\int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds, \nu, m, K, T).
\end{aligned} \tag{3.11}$$

Proof. The proof is skipped since it follows a similar argument as the proof of Proposition 2.0.1. □

The next step is to establish auxiliary results for proving exponential stability.

$$M_1(t) = \int_0^t \langle \mathbf{u}(s), \sigma(\mathfrak{r}(s))dW(s)\rangle \tag{3.12}$$

$$M_2(t) = \int_0^t \int_Z \Big(|\mathbf{u}(s) + \mathbf{G}(\mathfrak{r}(s), z)|^2 - |\mathbf{u}(s)|^2\Big)\tilde{N}_1(dz, ds). \tag{3.13}$$

Let us denote $M^*(T) := \sup_{0\le t\le T}|M(t)|$ for a martingale $M(t)$.

Lemma 3.2.2. *Assume that $\mathbb{E}|\mathbf{u}(0)|^2 < \infty$ and $\mathbf{f} \in L^2(0,T;V')$. In addition, if $\lim_{T\to\infty}\frac{1}{T}\int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds = F > 0$, then there exist a sequence $\{T_n\}$ with $T_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} M_i^*(T_n)/T_n = 0$ almost surely for $i = 1, 2$.*

Proof. For $M_1(t)$, utilizing the Davis inequality, we have

$$\begin{aligned}
\mathbb{E}M_1^*(T) &= \mathbb{E}\sup_{t\in[0,T]}\int_0^t \langle \mathbf{u}(s), \sigma(\mathfrak{r}(s))dW(s)\rangle \\
&\le \sqrt{2}\mathbb{E}\Big\{\Big(\int_0^T \|\sigma^*(\mathfrak{r}(s))\mathbf{u}(s)\|_{L_Q}^2 ds\Big)^{\frac{1}{2}}\Big\} \le \sqrt{2}\mathbb{E}\Big\{\Big(\int_0^T \|\sigma^*(\mathfrak{r}(t))\|_{L_Q}^2 |\mathbf{u}(s)|^2 ds\Big)^{\frac{1}{2}}\Big\};
\end{aligned}$$

invoking Hypothesis **A1**, the property $|\cdot| \le \|\cdot\|$, the Schwarz inequality, and continuing

$$\le \sqrt{2}K(1+m^2)\mathbb{E}\Big\{\Big(\int_0^T \|\mathbf{u}(s)\|^2 ds\Big)^{\frac{1}{2}}\Big\} \le \sqrt{2}K(1+m^2)\Big(\mathbb{E}\int_0^T \|\mathbf{u}(s)\|^2 ds\Big)^{\frac{1}{2}}.$$

In addition, it follows from (3.11) that

$$\Big(\mathbb{E}\int_0^T \|\mathbf{u}(s)\|^2 ds\Big)^{\frac{1}{2}} \le \Big(\frac{2}{\nu}\mathbb{E}|\mathbf{u}(0)|^2 + \frac{4}{3\nu^2}\mathbb{E}\int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds + \frac{100(1+m^2)KT}{\nu}\Big)^{\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}\frac{\mathbb{E}M_1^*(T)}{T} &\le \frac{\sqrt{2}K(1+m^2)}{T}\Big(\frac{2}{\nu}\mathbb{E}|\mathbf{u}(0)|^2 + \frac{4}{3\nu^2}\mathbb{E}\int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds + \frac{100(1+m^2)KT}{\nu}\Big)^{\frac{1}{2}} \\ &= \frac{\sqrt{2}K(1+m^2)}{\sqrt{T}}\Big(\frac{2}{\nu}\frac{\mathbb{E}|\mathbf{u}(0)|^2}{T} + \frac{4}{3\nu^2}\frac{\mathbb{E}\int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds}{T} + \frac{100(1+m^2)K}{\nu}\Big)^{\frac{1}{2}}.\end{aligned}$$

Moreover, it follows from the assumption of $\mathbf{f}$ that $\mathbb{E}M_1^*(T)/T \to 0$ as $T \to \infty$. Thus, there exists a subsequence $\{T_{1,n}\}$ such that $M_1^*(T_{1,n})/T_{1,n} \to 0$, as $n \to \infty$, almost surely.

For $M_2(t)$, we have

$$\begin{aligned}M_2(t) &= \int_0^t \int_Z \Big(|\mathbf{u}(s) + \mathbf{G}(\mathfrak{r}(s),z)|^2 - |\mathbf{u}(s)|^2\Big)\tilde{N}_1(dz,ds) \\ &= \int_0^t \int_Z 2\big(\mathbf{u}(s), \mathbf{G}(\mathfrak{r}(s),z)\big)\tilde{N}_1(dz,ds) + \int_0^t \int_Z |\mathbf{G}(\mathfrak{r}(s),z)|^2 \tilde{N}_1(dz,ds),\end{aligned}$$

therefore, the Davis inequality, Hypothesis **A3**, and the property $|\cdot| \le \|\cdot\|$, imply

$$\begin{aligned}\mathbb{E}M_2^*(T) &\le \sqrt{10}\mathbb{E}\Big\{\Big(\int_0^T \int_Z \big(\mathbf{u}(s), G(\mathfrak{r}(s),z)\big)^2 \nu(dz)ds\Big)^{\frac{1}{2}}\Big\} \\ &\quad + \sqrt{10}\mathbb{E}\Big\{\Big(\int_0^T \int_Z |\mathbf{G}(\mathfrak{r}(s),z)|^4 \nu(dz)ds\Big)^{\frac{1}{2}}\Big\} \\ &\le \sqrt{10}\mathbb{E}\Big\{\Big(\int_0^T \int_Z |\mathbf{u}(s)|^2 |\mathbf{G}(\mathfrak{r}(s),z)|^2 ds\Big)^{\frac{1}{2}}\Big\} + \sqrt{10KT(1+m^4)} \\ &\le \sqrt{10K(1+m^2)}\mathbb{E}\Big\{\Big(\int_0^T \|\mathbf{u}(s)\|^2 ds\Big)^{\frac{1}{2}}\Big\} + \sqrt{10KT(1+m^4)};\end{aligned}$$

invoking the Schwarz inequality and (3.11), continuing

$$\leq \sqrt{10K(1+m^2)}\Big(\mathbb{E}\int_0^T \|\mathbf{u}(s)\|^2 ds\Big)^{\frac{1}{2}} + \sqrt{10KT(1+m^4)}$$
$$\leq \sqrt{10K(1+m^2)}\Big(\frac{2}{\nu}\mathbb{E}|\mathbf{u}(0)|^2 + \frac{4}{3\nu^2}\mathbb{E}\int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds + \frac{100(1+m^2)KT}{\nu}\Big)^{\frac{1}{2}}$$
$$+ \sqrt{10KT(1+m^4)}.$$

This implies that $\mathbb{E}M_2^*(T)/T \to 0$ as $T \to \infty$. Therefore, by an analogous argument as for $M_1(t)$, there exist a sequence $\{T_{2,n}\}$ such that $M_2^*(T_{2,n})/T_{2,n} \to 0$ almost surely as $n \to \infty$.

Let $\{T_n\}$ be a common subsequence of $\{T_{1,n}\}$ and $\{T_{2,n}\}$. Then

$$\lim_{n\to\infty} \frac{M_i^*(T_n)}{T_n} = 0$$

almost surely for $i = 1, 2$. □

Remark. *The argument employed in Lemma 3.2.2 is in the context of stochastic Navier-Stokes equations. One may employ other method to deduce the such a limit. For instance, one may utilize [57, Lem. 2.1] to obtain the almost surely limits of M_1 and M_2 of the original sequence instead of a subsequence (cf. [56, Eq. (3.17)]).*

Recall that λ_1 is the first eigenvalue of the Stokes operator $\mathbf{A}$ and K is the constant in Hypotheses **A**.

Lemma 3.2.3. *Assume that $\mathbb{E}|\mathbf{u}(0)|^2 < \infty$ and $\mathbf{f} \in L^2(0,T;V')$. In addition, if $\lim_{T\to\infty} \frac{1}{T}\int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds = F > 0$ and*

$$K < \frac{\nu^3\lambda_1 - F/\nu}{2(1+m^2)},$$

then

$$\lim_{T\to\infty}\Big(\nu\lambda_1-\frac{1}{\nu T}\int_0^T\|\mathbf{u}(s)\|^2ds\Big)>0$$

almost surely.

Proof. It follows from the Itô formula and the basic Young inequality that

$$\begin{aligned}&\sup_{0\le t\le T}|\mathbf{u}(t)|^2+\nu\int_0^T\|\mathbf{u}(s)\|^2ds\\&\le|\mathbf{u}(0)|^2+\frac{1}{\nu}\int_0^T\|\mathbf{f}(s)\|_{V'}^2ds+\int_0^T\|\sigma(\mathfrak{r}(s))\|_{L_Q}^2ds+2M_1^*(T)+M_2^*(T)\\&\quad+\int_0^T\int_Z|\mathbf{G}(\mathfrak{r}(s),z)|^2\nu(dz)ds,\end{aligned}$$

where $M_1(T)$ and $M_2(T)$ are defined as in (3.12) and (3.13), respectively, and $M_i^*(T)$, $i=1,2$, are introduced before Lemma 3.2.2. Using Hypotheses **A1**, **A3**, and dropping $\sup_{0\le t\le T}|\mathbf{u}(t)|^2$, we have

$$\begin{aligned}&\frac{1}{\nu T}\int_0^T\|\mathbf{u}(s)\|^2ds\\&\le\frac{|\mathbf{u}(0)|^2}{\nu^2T}+\frac{1}{\nu^3}\frac{\int_0^T\|\mathbf{f}(s)\|_{V'}^2ds}{T}+\frac{2(1+m^2)KT}{\nu^2T}+\frac{2M_1^*(T)}{\nu^2T}+\frac{M_2^*(T)}{\nu^2T},\end{aligned}$$

which implies

$$\begin{aligned}&\nu\lambda_1-\frac{1}{\nu T}\int_0^T\|\mathbf{u}(s)\|^2ds\\&\ge\nu\lambda_1-\Big(\frac{|\mathbf{u}(0)|^2}{\nu^2T}+\frac{1}{\nu^3}\frac{\int_0^T\|\mathbf{f}(s)\|_{V'}^2ds}{T}+\frac{2(1+m^2)K}{\nu^2}+\frac{2M_1^*(T)}{\nu^2T}+\frac{M_2^*(T)}{\nu^2T}\Big).\end{aligned}$$

By Lemma 3.2.2, the assumption of $\mathbf{f}$, and the requirement of K, we conclude

$$\lim_{n\to\infty}\Big(\nu\lambda_1-\frac{1}{\nu T_n}\int_0^{T_n}\|\mathbf{u}(s)\|^2ds\Big)\ge\nu\lambda_1-\frac{F}{\nu^3}-\frac{2(1+m^2)K}{\nu^2}>0$$

almost surely. ∎

Let $\mathbf{u}_i(t)$ be the solution of (3.9) with initial conditions $\mathbf{u}_i(0) = \mathbf{u}_i$ and $\mathfrak{r}_i(0) = \mathfrak{r}_i$. Write $\mathbf{w}(t) = \mathbf{u}_1(t) - \mathbf{u}_2(t)$, $\sigma_{12}(t) = \sigma(\mathfrak{r}_1(t)) - \sigma(\mathfrak{r}_2(t))$, and $\mathbf{G}_{12}(t,z) = \mathbf{G}(\mathfrak{r}_1(t), z) - \mathbf{G}(\mathfrak{r}_2(t), z)$. Now we are in a position to introduce the exponential stability.

Theorem 3.2.4 (Exponential stability)**.** *Assume that* $\mathbb{E}|\mathbf{u}(0)|^2 < \infty$ *and* $\mathbf{f} \in L^2(0,T;V')$. *Suppose that* $\lim_{T\to\infty} \frac{1}{T}\int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds = F > 0$, $K < \frac{\nu^3\lambda_1 - F/\nu}{2(1+m^2)}$, *and* $L < \frac{\nu^3\lambda_1 - F/\nu}{2(1+m^2)}$. *Then*

$$\lim_{t\to\infty} |\mathbf{w}(t)|^2 = 0$$

for almost all $\omega \in \Omega$.

Proof. It follows from the Itô formula that

$$\begin{aligned}
&|\mathbf{w}(t)|^2 + 2\nu \int_0^t \|\mathbf{w}(s)\|^2 ds + \int_0^t \langle \mathbf{B}_k(\mathbf{u}_1(s)) - \mathbf{B}_k(\mathbf{u}_2(s)), \mathbf{u}_1(s) - \mathbf{u}_2(s)\rangle_V ds \\
&= |\mathbf{w}(0)|^2 + \int_0^t \|\sigma_{12}(s)\|_{L_Q}^2 ds + 2\int_0^t \langle \mathbf{w}(s), \sigma_{12}(s) dW(s)\rangle \\
&\quad + \int_0^t \int_Z \Big(|\mathbf{w}(s) + \mathbf{G}_{12}(s,z)|^2 - |\mathbf{w}(s)|^2\Big)\tilde{N}_1(dz, ds) \\
&\quad + \int_0^t \int_Z \Big(|\mathbf{w}(s) + \mathbf{G}_{12}(s,z)|^2 - |\mathbf{w}(s)|^2 - 2(\mathbf{w}(s), \mathbf{G}_{12}(s,z))\Big)\nu(dz)ds.
\end{aligned} \tag{3.14}$$

For the non-linear term, we have

$$|\langle \mathbf{B}_k(\mathbf{u}_1(s)) - \mathbf{B}_k(\mathbf{u}_2(s)), \mathbf{u}_1(s) - \mathbf{u}_2(s)\rangle_V| \leq \|\mathbf{w}(s)\| |\mathbf{w}(s)| \|\mathbf{u}_1(s)\|,$$

therefore, one infers from the basic Young inequality that

$$\begin{aligned}
&2\int_0^t |\langle \mathbf{B}_k(\mathbf{u}_1(s)) - \mathbf{B}_k(\mathbf{u}_2(s)), \mathbf{u}_1(s) - \mathbf{u}_2(s)\rangle_V| ds \\
&\leq \nu \int_0^t \|\mathbf{w}(s)\|^2 ds + \frac{1}{\nu}\int_0^t |\mathbf{w}(s)|^2 \|\mathbf{u}_1(s)\|^2 ds.
\end{aligned}$$

Let

$$\tilde{M}_1(t) = \int_0^t \langle \mathbf{w}(s), \sigma_{12}(s) dW(s) \rangle$$

$$\tilde{M}_2(t) = \int_0^t \int_Z \Big(|\mathbf{w}(s) + \mathbf{G}_{12}(s,z)|^2 - |\mathbf{w}(s)|^2 \Big) \tilde{N}_1(dz, ds).$$

Using the notation $\tilde{M}_i(t)$, $i = 1, 2$, the estimate for nonlinear term, and Hypotheses **A2** and **A4** in (3.14), we obtain

$$\begin{aligned} &|\mathbf{w}(t)|^2 + \nu \int_0^t \|\mathbf{w}(s)\|^2 ds \\ &\le |\mathbf{w}(0)|^2 + \frac{1}{\nu} \int_0^t |\mathbf{w}(s)|^2 \|\mathbf{u}_1(s)\|^2 ds + \int_0^t L |\mathfrak{r}_1(s) - \mathfrak{r}_2(s)|^2 ds + 2\tilde{M}_1^*(T) + \tilde{M}_2^*(T) \\ &\quad + \int_0^t L |\mathfrak{r}_1(s) - \mathfrak{r}_2(s)|^2 ds. \end{aligned}$$

Using the Poincaré inequality (1.13) and the fact that $|\mathfrak{r}_1(s) - \mathfrak{r}_2(s)| \le m$, we have

$$|\mathbf{w}(t)|^2 + \int_0^t (\nu\lambda_1 - \frac{1}{\nu}\|\mathbf{u}_1(s)\|^2) |\mathbf{w}(s)|^2 ds \le |\mathbf{w}(0)|^2 + 2\tilde{M}_1^*(T) + \tilde{M}_2^*(T) + 2m^2 LT.$$

Writing $C_T = |\mathbf{w}(0)|^2 + 2\tilde{M}_1^*(T) + \tilde{M}_2^*(T) + 2m^2 LT$, one infers from the Gronwall inequality that

$$|\mathbf{w}(t)|^2 \le C_T e^{-\int_0^T (\nu\lambda_1 - \frac{1}{\nu}\|\mathbf{u}_1(s)\|^2) ds} = C_T e^{-(\nu\lambda_1 - \frac{1}{\nu T}\int_0^T \|\mathbf{u}(s)\|^2 ds) T}.$$

The theorem follows from Lemma 3.2.3 and the fact that C_T is growing as a polynomial in T. □

Now we study the existence of the stationary measure induced by $\mathbf{u}(t)$, the solution to (3.9). Denote by $\mathcal{Q}_t(x, i; B, j)$ the transition probability function of $(\mathbf{u}, \mathfrak{r})$:

$$\mathcal{Q}_t(x, i; B, j) = \mathcal{P}((\mathbf{u}(t), \mathfrak{r}(t)) \in (B, j) | (\mathbf{u}(0), \mathfrak{r}(0)) = (x, i)),$$

where $x \in H$, $B \in \mathcal{B}(H)$, and $i, j \in \mathcal{S}$.

Let $\phi(x,i) : H \times \mathcal{S} \to \mathbb{R}$ be a bounded continuous function. Let $\mathbf{u}(t)$ be a solution to (3.9) with initial conditions $\mathbf{u}_0 = x$ and $\mathfrak{r}_0 = i$. Define

$$(\mathcal{Q}_t\phi)(x,i) := \sum_{k=1}^{m} \int_H \phi(z,k)\mathcal{Q}_t(x,i;dz,k) = \mathbb{E}^{x,i}(\phi(\mathbf{u}(t),\mathfrak{r}(t))). \tag{3.15}$$

We say $\lambda(y,j)$ is a stationary measure if

$$\sum_{j=1}^{m} \int_H (\mathcal{Q}_t\phi)(y,j)\lambda(dy,j) = \sum_{j=1}^{m} \int_H \phi(y,j)\lambda(dy,j)$$

for all $t \geq 0$ and $\phi \in C_b(H \times \mathcal{S})$.

Theorem 3.2.5. *Assume that* $\mathbb{E}|\mathbf{u}(0)|^2 < \infty$ *and* $\|\mathbf{f}(x)\|_{V'}^2 = F > 0$. *If*

$$K < \frac{\nu^3\lambda_1 - F/\nu}{2(1+m^2)} \quad \text{and} \quad L < \frac{\nu^3\lambda_1 - F/\nu}{2(1+m^2)},$$

then there exists a unique stationary measure, with support in $V \times \mathcal{S}$ *and a finite second moment, for the solution* $\mathbf{u}$ *of the equation* (3.9).

Proof. We begin the proof by showing the existence. It follows from (3.10) that

$$\frac{\nu}{t}\mathbb{E}\int_0^t \|\mathbf{u}(s)\|^2 ds \leq C \tag{3.16}$$

for $t > 1$ and C is an appropriate constant independent of t. Hence by the Chebyshev inequality

$$\lim_{N\to\infty} \sup_{t>1} \frac{1}{t}\int_0^t \mathcal{P}(\|\mathbf{u}(s)\| > N)ds = 0 \tag{3.17}$$

follows.

Let $\{t_n\}$ be any increasing sequence of positive numbers with $\lim_{n\to\infty} t_n = \infty$. Define probability measures λ_n as follows:

$$\lambda_n(B,j) = \frac{1}{t_n}\int_0^{t_n} \mathcal{Q}_s(x,i;B,j)ds \tag{3.18}$$

for all $B \in \mathcal{B}(H)$ and $j \in \mathcal{S}$.

Let N be an positive integer and $A := \{v \in V : \|v\| \leq N\}$, which is a bounded set in V. Then by the compact embedding $i : V \hookrightarrow H$, A is a relative compact set in H. Consider

$$\begin{aligned}
\lambda_n(A^c,j) &= \frac{1}{t_n}\int_0^{t_n} \mathcal{Q}_s(x,i;A^c,j)ds \\
&= \frac{1}{t_n}\int_0^{t_n} \mathcal{P}((\mathbf{u}(t),\mathfrak{r}(t)) \in (A^c,j)|(\mathbf{u}(0),\mathfrak{r}(0)) = (x,i)) \\
&= \frac{1}{t_n}\int_0^{t_n} \mathcal{P}(\|\mathbf{u}(t)\| > N \text{ and } \mathfrak{r}(t) = j|\mathbf{u}(0) = x \text{ and } \mathfrak{r}(0) = i),
\end{aligned}$$

which together with (3.17) further imply

$$\lambda_n(A^c,j) = \frac{1}{t_n}\int_0^{t_n} \mathcal{P}(\|\mathbf{u}(t)\| > N \text{ and } \mathfrak{r}(t) = j|\mathbf{u}(0) = x \text{ and } \mathfrak{r}(0) = i) \to 0$$

as $N \to \infty$. Hence $\{\lambda_n\}$ is tight in the space of probability measures on $(H \times \mathcal{S}, \mathcal{B}(H \times \mathcal{S}))$ equipped with the weak topology. Therefore, the Prokhorov theorem implies that there exists a subsequence $\{\lambda_{n_\ell}\}$ such that $\lambda_{n_\ell} \to \lambda$.

For $\phi \in C_b(H \times \mathcal{S})$, we have

$$\begin{aligned}
&\sum_{j=1}^{m}\int_H (\mathcal{Q}_t\phi)(y,j)\lambda(dy,j) = \lim_{\ell\to\infty}\sum_{j=1}^{m}\int_H (\mathcal{Q}_t\phi)(y,j)\lambda_{n_\ell}(dy,j) \\
&= \lim_{\ell\to\infty}\sum_{j=1}^{m}\int_H (\mathcal{Q}_t\phi)(y,j)\frac{1}{t_{n_\ell}}\int_0^{t_{n_\ell}} \mathcal{Q}_s(x,i;dy,j)ds.
\end{aligned}$$

Using (3.15) in above, we further have

$$
\begin{aligned}
&\sum_{j=1}^{m}\int_H(\mathcal{Q}_t\phi)(y,j)\lambda(dy,j) \qquad (3.19)\\
&=\lim_{\ell\to\infty}\sum_{j=1}^{m}\sum_{k=1}^{m}\int_H\int_H\phi(z,k)\mathcal{Q}_t(y,j;dz,k)\frac{1}{t_{n_\ell}}\int_0^{t_{n_\ell}}\mathcal{Q}_s(x,i;dy,j)ds.
\end{aligned}
$$

The Chapman-Kolmogorov equation gives

$$
\mathcal{Q}_{s+t}(x,i;dz,k)=\sum_{j=1}^{m}\int_H\mathcal{Q}_s(x,i;dy,j)\mathcal{Q}_t(y,j;dz,k).
$$

Plugging the above formula into (3.19), we have

$$
\sum_{j=1}^{m}\int_H(\mathcal{Q}_t\phi)(y,j)\lambda(dy,j)=\lim_{\ell\to\infty}\sum_{k=1}^{m}\int_H\phi(z,k)\frac{1}{t_{n_\ell}}\int_0^{t_{n_\ell}}\mathcal{Q}_{s+t}(x,i;dz,k)ds.
$$

Note that

$$
\begin{aligned}
&\lim_{\ell\to\infty}\frac{1}{t_{n_\ell}}\int_0^{t_{n_\ell}}\mathcal{Q}_{s+t}(x,i;dz,k)ds=\lim_{\ell\to\infty}\frac{1}{t_{n_\ell}}\int_t^{t_{n_\ell}+t}\mathcal{Q}_u(x,i;dz,k)du\\
&=\lim_{\ell\to\infty}\frac{1}{t_{n_\ell}}\Big(\int_0^{t_{n_\ell}}\mathcal{Q}_u(x,i;dz,k)du+\int_{t_{n_\ell}}^{t_{n_\ell}+t}\mathcal{Q}_u(x,i;dz,k)du-\int_0^{t}\mathcal{Q}_u(x,i;dz,k)du\Big)\\
&=\lim_{\ell\to\infty}\frac{1}{t_{n_\ell}}\int_0^{t_{n_\ell}}\mathcal{Q}_u(x,i;dz,k)du
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\sum_{j=1}^{m}\int_H(\mathcal{Q}_t\phi)(y,j)\lambda(dy,j)&=\lim_{\ell\to\infty}\sum_{k=1}^{m}\int_H\phi(z,k)\frac{1}{t_{n_\ell}}\int_0^{t_{n_\ell}}\mathcal{Q}_{s+t}(x,i;dz,k)ds\\
&=\lim_{\ell\to\infty}\sum_{k=1}^{m}\int_H\phi(z,k)\frac{1}{t_{n_\ell}}\int_0^{t_{n_\ell}}\mathcal{Q}_u(x,i;dz,k)du=\sum_{k=1}^{m}\int_H\phi(z,k)\lambda(dz,k).
\end{aligned}
$$

Hence, we conclude that λ is a stationary measure.

For the second moment of λ, one employs the lower semi-continuity of the H-norm to deduce

$$
\sum_{j=1}^{m}\int_H(|y|^2+j^2)\lambda(dy,j)\le\liminf_{\ell\to\infty}\sum_{j=1}^{m}\int_H(|y|^2+j^2)\lambda_{n_\ell}(dy,j).
$$

Then using (3.15) and (3.18) with $\phi(y,j) = |y|^2 + j^2$, one further deduce

$$\sum_{j=1}^{m} \int_H (|y|^2 + j^2)\lambda_{n_\ell}(dx,i) = \sum_{j=1}^{m} \int_H \phi(y,j) \frac{1}{n_\ell} \int_0^{n_\ell} \mathcal{Q}_t(x,i;dy,j)dt$$
$$= \frac{1}{n_\ell} \int_0^{n_\ell} \sum_{j=1}^{m} \int_H \phi(y,j)\mathcal{Q}_t(x,i;dy,j)dt = \frac{1}{n_\ell} \int_0^{n_\ell} \mathbb{E}(|\mathbf{u}^y(t)|^2 + i^2)dt < \infty$$

by (3.11). Therefore, the second moment for the measure λ exists.

For the support of λ, let $\mathbf{u}$ denote the solution of (3.9) started at $\mathbf{u}_0$ and $\mathfrak{r}_0$, where the (joint) distribution of $(\mathbf{u}_0, \mathfrak{r}_0)$ is given by λ. By (3.16), it follows that $\mathbf{u}(s)$ is almost surely V-valued for almost all s. In particular, λ has support in $V \times \mathcal{S}$.

Let λ_1 and λ_2 be two stationary measures. To show the uniqueness, it suffices to show that

$$\sum_{i=1}^{m} \int_H \phi(x,i)\lambda_1(dx,i) = \sum_{i=1}^{m} \int_H \phi(x,i)\lambda_2(dx,i)$$

for all $\phi \in C_b(H \times \mathcal{S})$.

Let $\mathbf{u}^x$ denote the solution of (3.9) with $\mathbf{u}_0 = x$ and $\mathfrak{r}^i(t)$ the Markov chain $\mathfrak{r}(t)$ with $\mathfrak{r}_0 = i$. Define

$$\lambda_T^{x,i}(B,j) = \frac{1}{T} \int_0^T \mathcal{Q}_t(x,i;B,j)dt$$

for all $B \in \mathcal{B}(H)$ and $j \in \mathcal{S}$. Let $\lambda(dx,i)$ denote a stationary measure. Then by stationarity,

$$\sum_{i=1}^{m} \int_H \phi(x,i)\lambda(dx,i) = \sum_{i=1}^{m} \sum_{j=1}^{m} \int_H \int_H \phi(y,j)\lambda_T^{x,i}(dy,j)\lambda(dx,i)$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \int_H \int_H \phi(y,j)\lambda_T^{x,i}(dy,j)\lambda(dx|i)\pi_i,$$

where $\pi = (\pi_1, \cdots, \pi_m)$ is the unique stationary distribution of the Markov chain $\mathfrak{r}(t)$.

Furthermore, one obtains

$$\sum_{i=1}^{m} \int_H \phi(x,i)\lambda(dx,i) = \sum_{i=1}^{m}\sum_{j=1}^{m} \int_H \int_H \phi(y,j)\lambda_T^{x,i}(dy,j)\lambda(dx|i)\pi_i$$
$$= \sum_{i=1}^{m} \int_H \frac{1}{T}\int_0^T \mathbb{E}^{x,i}(\phi(\mathbf{u}(t),\mathfrak{r}(t)))dt\lambda(dx|i)\pi_i = \sum_{i=1}^{m} \int_H \frac{1}{T}\int_0^T \mathbb{E}(\phi(\mathbf{u}^x(t),\mathfrak{r}^i(t)))dt\lambda(dx|i)\pi_i.$$

Hence,

$$\begin{aligned}
&\sum_{i=1}^{m} \int_H \phi(x,i)\lambda_1(dx,i) - \sum_{i=1}^{m} \int_H \phi(w,i)\lambda_2(dw,i) \\
= \;& \sum_{i=1}^{m} \int_H \frac{1}{T}\int_0^T \mathbb{E}(\phi(\mathbf{u}^x(t),\mathfrak{r}^i(t)))dt\lambda_1(dx|i)\pi_i \\
&\quad - \sum_{i=1}^{m} \int_H \frac{1}{T}\int_0^T \mathbb{E}(\phi(\mathbf{u}^w(t),\mathfrak{r}^i(t)))dt\lambda_2(dw|i)\pi_i \\
\leq \;& \sum_{i=1}^{m} \int_H \int_H \frac{1}{T}\int_0^T \mathbb{E}\; \phi(\mathbf{u}^x(t),\mathfrak{r}^i(t)) - \phi(\mathbf{u}^w(t),\mathfrak{r}^i(t)) \; dt\lambda_1(dx|i)\lambda_2(dw|i)\pi_i.
\end{aligned} \tag{3.20}$$

By Theorem 3.2.4 and the continuity of ϕ, we have

$$\phi(\mathbf{u}^x(t),\mathfrak{r}^i(t)) - \phi(\mathbf{u}^w(t),\mathfrak{r}^i(t)) \;\to 0$$

as $t \to \infty$ for almost all $\omega \in \Omega$. Therefore,

$$\frac{1}{T}\int_0^T \; \phi(\mathbf{u}^x(t),\mathfrak{r}^i(t)) - \phi(\mathbf{u}^w(t),\mathfrak{r}^i(t)) \; dt \to 0$$

as $T \to \infty$. Finally, it follows from Lebesgue Dominated Convergence Theorem that (3.20)$\to 0$ as $T \to \infty$. This implies

$$\int_{H\times\mathcal{S}} \phi(x,i)d\lambda_1 = \int_{H\times\mathcal{S}} \phi(x,i)d\lambda_2$$

for all $\phi \in C_b(H \times \mathcal{S})$. So that $\lambda_1 = \lambda_2$. $\square$

3.2.1. Stationary measures for non-switching case

Consider the following equation:

$$\mathbf{du}(t) + [\nu \mathbf{Au}(t) + \mathbf{B}_k(\mathbf{u}(t))]dt = \mathbf{f}(x)dt + \sigma(x)dW(t) + \int_Z \mathbf{G}(x,z)\tilde{N}_1(dz,dt) \tag{3.21}$$

with $\mathbf{u}(0) = \mathbf{u}_0 \in H$, where the noise coefficients satisfy the following Hypotheses $\mathbf{A}'$:

A1'. There exist a constant $K > 0$ such that

$$\|\sigma(x)\|_{L_Q}^2 \leq K,$$

(growth condition on σ).

A2'. There exists a constant $K > 0$ such that

$$\int_Z |\mathbf{G}(x,z)|^p \nu(dz) \leq K$$

for $p = 1, 2$, and 4 (growth condition on $\mathbf{G}$).

It is clear that Hypotheses $\mathbf{A}'$ is a subclass of Hypotheses $\mathbf{H}$, therefore, the solution $\mathbf{u}$ to equation (3.21) exists and is unique. Moreover, the system (3.21) admits a unique stationary measure since Hypotheses $\mathbf{A}'$ is a subclass of Hypotheses $\mathbf{A}$:

Corollary 3.2.6. *Assume that* $\mathbb{E}|\mathbf{u}(0)|^2 < \infty$ *and* $\|\mathbf{f}(x)\|_{V'}^2 = F > 0$. *If*

$$K < \frac{\nu^3\lambda_1 - F/\nu}{2},$$

then there exists a unique stationary measure, with support in V *and a finite second moment, for the solution* $\mathbf{u}$ *of the equation* (3.21).

3.3. Exponential Inequalities

In this section, we focus our attention on a completely different probabilistic behavior of solutions on a class of stochastic Navier-Stokes equations with Markov switching.

Let $W(t)$ be a one-dimensional Wiener process defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$. Let $r > 0$ be fixed, define $\tau_r := \inf\{t \in [0,T] : |W(t)| > r\}$, the first time (before time T) that the process $W(t)$ exists the interval $(-r, r)$. In addition, it can be shown that

$$\mathcal{P}(\tau_r < T) = \mathcal{P}(\sup_{0\leq t\leq T} |W(t)| > r) \leq 2e^{-\frac{r^2}{2t}}.$$

It can be seen from above that the exist time estimate for the (one-dimensional) Wiener process decays exponentially, therefore, we usually refer it *exponential inequality*. In general, exponential estimates for exit times for a class of stochastic evolution equations were obtained systematically by Chow and Menaldi [13]. Inspired by their work, we consider those exit time estimates (exponential inequalities) for the solution to equation (3.9) and start the investigation from a less-complicated case:

$$\mathbf{du}(t) + [\nu\mathbf{A}\mathbf{u}(t) + \mathbf{B}_k(\mathbf{u}(t))]dt = \mathbf{f}(t)dt + \sigma(\mathfrak{r}(t))dW(t) \tag{3.23}$$

with $\mathbf{u}(0) = \mathbf{u}_0 \in H$, where σ satisfy Hypotheses $\mathbf{A}$.

Proposition 3.3.1. *Assume that there exists a constant $F > 0$ such that*

$$\int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds \leq F.$$

Then, for any given $r > 0$, the solution $\mathbf{u}$ of (3.23) satisfies

$$\mathcal{P}\{\sup_{0\leq t\leq T} |\mathbf{u}(t)| > r\} \leq C_1 \exp\Big(-r^2 e^{-2K(1+m^2)T}\Big),$$

where $C_1 = \exp\Big(|\mathbf{u}(0)|^2 + \frac{F}{\nu} + K(1+m^2)T\Big)$.

Proof. It follows from the Itô formula and the basic Young inequality that

$$
\begin{aligned}
&|\mathbf{u}(t)|^2 + 2\nu \int_0^t \|\mathbf{u}(s)\|^2 ds \\
&= |\mathbf{u}(0)|^2 + 2\int_0^t \langle \mathbf{f}(s), \mathbf{u}(s)\rangle_V ds + 2\int_0^t \langle \mathbf{u}(s), \sigma(\mathfrak{r}(s))dW(s)\rangle + \int_0^t \|\sigma^*(\mathfrak{r}(s))\|_{L_Q}^2 ds \\
&\le |\mathbf{u}(0)|^2 + \frac{1}{\nu}\int_0^t \|\mathbf{f}(s)\|_{V'}^2 ds + \nu \int_0^t \|\mathbf{u}(s)\|^2 ds + \eta(t) + 2\int_0^t \|\sigma^*(\mathfrak{r}(s))\mathbf{u}(s)\|_{L_Q}^2 ds \qquad (3.24)\\
&\quad + \int_0^t \|\sigma^*(\mathfrak{r}(s))\|_{L_Q}^2 ds,
\end{aligned}
$$

where

$$\eta(t) := 2\int_0^t \langle \mathbf{u}(s), \sigma(\mathfrak{r}(s))dW(s)\rangle - 2\int_0^t \|\sigma^*(\mathfrak{r}(s))\mathbf{u}(s)\|_{L_Q}^2 ds.$$

By Hypothesis **A1**, one deduces

$$\int_0^t \|\sigma^*(\mathfrak{r}(s))\mathbf{u}(s)\|_{L_Q}^2 ds \le \int_0^t K(1+i^2)|\mathbf{u}(s)|^2 ds \le K(1+m^2)\int_0^t |\mathbf{u}(s)|^2 ds.$$

Using above estimate and the assumption on $\mathbf{f}$ in (3.24), we have

$$
\begin{aligned}
&|\mathbf{u}(t)|^2 + \nu \int_0^t \|\mathbf{u}(s)\|^2 ds \\
&\le |\mathbf{u}(0)|^2 + \frac{F}{\nu} + \eta(t) + 2K(1+m^2)\int_0^t |\mathbf{u}(s)|^2 ds + K(1+m^2)t.
\end{aligned}
$$

Dropping the second term on the left and using the Grownwall inequality, we have

$$
\begin{aligned}
|\mathbf{u}(t)|^2 &\le \Big(|\mathbf{u}(0)|^2 + \frac{F}{\nu} + \eta(t) + K(1+m^2)t\Big) \\
&\quad + \int_0^t \Big(|\mathbf{u}(0)|^2 + \frac{F}{\nu} + \eta(s) + K(1+m^2)s\Big) 2K(1+m^2)e^{2K(1+m^2)(t-s)} ds \\
&\le \Big(|\mathbf{u}(0)|^2 + \frac{F}{\nu} + \sup_{0\le s\le t} \eta(s) + K(1+m^2)t\Big)e^{2K(1+m^2)t}.
\end{aligned}
$$

Hence, for fixed $r > 0$, we obtain,

$$\begin{aligned}
&\mathcal{P}\Big\{\sup_{0\le t\le T}|\mathbf{u}(t)| > r\Big\}\\
&\le \mathcal{P}\Big\{\Big(|\mathbf{u}(0)|^2 + \frac{F}{\nu} + \sup_{0\le t\le T}\eta(t) + K(1+m^2)T\Big)\Big)e^{2K(1+m^2)T} > r^2\Big\}\\
&= \mathcal{P}\Big\{\sup_{0\le t\le T}\eta(t) > r^2 e^{-2K(1+m^2)T} - |\mathbf{u}(0)|^2 - \frac{F}{\nu} - K(1+m^2)T\Big\}\\
&= \mathcal{P}\Big\{\sup_{0\le t\le T} e^{\eta(t)} > \exp\Big(r^2 e^{-2K(1+m^2)T} - |\mathbf{u}(0)|^2 - \frac{F}{\nu} - K(1+m^2)T\Big)\Big\}\\
&\le \exp\Big(-r^2 e^{-2K(1+m^2)T} + |\mathbf{u}(0)|^2 + \frac{F}{\nu} + K(1+m^2)T\Big)
\end{aligned}$$

by the basic submartingale inequality, and the proof is complete. □

Next we consider the solution $\mathbf{u}$ to equation (3.9), where σ and $\mathbf{G}$ satisfy Hypotheses **A**.

Proposition 3.3.2. *Assume that there exists a constant $F > 0$ such that*

$$\int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds \le F.$$

Then, for any given $r > 0$, the solution $\mathbf{u}$ of (3.9) satisfies

$$\mathcal{P}\{\sup_{0\le t\le T}|\mathbf{u}(t)| > r\} \le C_2 e^{-\frac{r^2}{2}}$$

where $C_2 = 2\exp\Big(\frac{|\mathbf{u}(0)|^2}{2} + \frac{F}{2\nu} + K(1+m^2)T\Big)$.

Proof. It follows from the Itô formula that

$$\begin{aligned}
|\mathbf{u}(t)|^2 &= |\mathbf{u}(0)|^2\\
&+ \int_0^t \Big((\|\sigma^*(\mathfrak{r}(s))\|_{L_Q}^2 + 2\langle \mathbf{f}(s), \mathbf{u}(s)\rangle - 2\nu\|\mathbf{u}(s)\|^2 + \int_Z |\mathbf{G}(\mathfrak{r}(s), z)|^2\Big) ds \qquad (3.25)\\
&+ 2\int_0^t \langle \mathbf{u}(s), \sigma(\mathfrak{r}(s))dW(s)\rangle + \int_0^t \Big(2(\mathbf{u}(s), \mathbf{G}(\mathfrak{r}(s), z)) + |\mathbf{G}(\mathfrak{r}(s), z)|^2\Big)\tilde{N}(dz, ds).
\end{aligned}$$

For $\alpha > 0$, let $F_\alpha(x) = \log(1+\alpha x)^{\frac{1}{\alpha}}$. Applying Itô formula to (3.25) with $F_\alpha(x)$, we have

$$\begin{aligned}
&\frac{1}{\alpha}\log(1+\alpha|\mathbf{u}(t)|^2)\\
&= \frac{1}{\alpha}\log(1+\alpha|\mathbf{u}(0)|^2) + \eta_\alpha(t) + \theta_\alpha(t)\\
&\quad + \int_0^t \frac{1}{1+\alpha|\mathbf{u}(s)|^2}\Big(\|\sigma^*(s)\|_{L_Q}^2 + 2\langle \mathbf{f}(s), \mathbf{u}(s)\rangle - 2\nu\|\mathbf{u}(s)\|^2\\
&\qquad\qquad + \int_Z |\mathbf{G}(s,z)|^2\nu(dz)\Big)ds,
\end{aligned} \tag{3.26}$$

where

$$\eta_\alpha(t) := \int_0^t \frac{1}{1+\alpha|\mathbf{u}(s)|^2}\langle 2\mathbf{u}(s), \sigma(\mathfrak{r}(s))dW(s)\rangle - 2\int_0^t \frac{\|\sigma^*(\mathfrak{r}(s))\mathbf{u}(s)\|_{L_Q}^2}{(1+\alpha|\mathbf{u}(s)|^2)^2}ds$$

and

$$\begin{aligned}
\theta_\alpha(t) :&= \frac{1}{\alpha}\int_0^t\int_Z \log\Big(1+\frac{\alpha(2(\mathbf{u}(s), \mathbf{G}(\mathfrak{r}(s),z)) + |\mathbf{G}(\mathfrak{r}(s),z)|^2)}{1+\alpha|\mathbf{u}(s)|^2}\Big)\tilde{N}(dz,ds)\\
&\quad + \frac{1}{\alpha}\int_0^t\int_Z\Big\{\log\Big(1+\frac{\alpha(2(\mathbf{u}(s), \mathbf{G}(\mathfrak{r}(s),z)) + |\mathbf{G}(\mathfrak{r}(s),z)|^2)}{1+\alpha|\mathbf{u}(s)|^2}\Big)\\
&\qquad\qquad - \frac{\alpha((\mathbf{u}(s), \mathbf{G}(\mathfrak{r}(s),z)) + |\mathbf{G}(\mathfrak{r}(s),z)|^2)}{1+\alpha|\mathbf{u}(s)|^2}\Big\}\nu(dz)ds.
\end{aligned}$$

Since $1/(1+\alpha|\mathbf{u}(s)|^2) \leq 1$, we have

$$\int_0^t \frac{\|\sigma^*(\mathfrak{r}(s))\|_{L_Q}^2}{1+\alpha|\mathbf{u}(s)|^2}ds \leq K(1+m^2)T, \quad \int_0^t\int_Z \frac{|\mathbf{G}(\mathfrak{r}(s),z)|^2}{1+\alpha|\mathbf{u}(s)|^2}\nu(dz)ds \leq K(1+m^2)T, \tag{3.27}$$

and from the basic Young inequality that

$$\int_0^t \frac{2\langle \mathbf{f}(s), \mathbf{u}(s)\rangle}{1+\alpha|\mathbf{u}(s)|^2}ds \leq \frac{F}{\nu} + \nu\int_0^t \frac{\|\mathbf{u}(s)\|^2}{1+\alpha|\mathbf{u}(s)|^2}ds.$$

Utilizing above estimates in (3.26), we obtain

$$\begin{aligned}
&\frac{1}{\alpha}\log(1+\alpha|\mathbf{u}(t)|^2) + \nu\int_0^t \frac{\|\mathbf{u}(s)\|^2}{1+\alpha|\mathbf{u}(s)|^2}ds\\
&\leq \frac{1}{\alpha}\log(1+\alpha|\mathbf{u}(0)|^2) + 2K(1+m^2)T + \frac{F}{\nu} + \sup_{0\leq t\leq T}\eta_\alpha(t) + \sup_{0\leq t\leq T}\theta_\alpha(t).
\end{aligned}$$

Thus, for given $r > 0$,

$$
\begin{aligned}
&\mathcal{P}\Big\{\sup_{0\le t\le T}|\mathbf{u}(t)| > r\Big\}\\
&= \mathcal{P}\Big\{\log(1+\alpha|\mathbf{u}(t)|^2)^{\frac{1}{\alpha}} > \log(1+\alpha r^2)^{\frac{1}{\alpha}}\Big\}\\
&\le \mathcal{P}\Big\{\log(1+\alpha|\mathbf{u}(0)|^2)^{\frac{1}{\alpha}} + 2K(1+m^2)T + \frac{F}{\nu} + \sup_{0\le t\le T}\eta_\alpha(t) + \sup_{0\le t\le T}\theta_\alpha(t) > \log(1+\alpha r^2)^{\frac{1}{\alpha}}\Big\}\\
&= \mathcal{P}\Big\{\sup_{0\le t\le T}\eta_\alpha(t) + \sup_{0\le t\le T}\theta_\alpha(t) > \frac{1}{\alpha}\log\Big(\frac{1+\alpha r^2}{1+\alpha|\mathbf{u}(0)|^2}\Big) - \frac{F}{\nu} - 2K(1+m^2)T\Big\}\\
&\le \mathcal{P}\Big\{\sup_{0\le t\le T}\eta_\alpha(t) > \frac{1}{2}\Big(\frac{1}{\alpha}\log\Big(\frac{1+\alpha r^2}{1+\alpha|\mathbf{u}(0)|^2}\Big) - \frac{F}{\nu} - 2K(1+m^2)T\Big)\Big\}\\
&\quad + \mathcal{P}\Big\{\sup_{0\le t\le T}\theta_\alpha(t) > \frac{1}{2}\Big(\frac{1}{\alpha}\log\Big(\frac{1+\alpha r^2}{1+\alpha|\mathbf{u}(0)|^2}\Big) - \frac{F}{\nu} - 2K(1+m^2)T\Big)\Big\}\\
&= \mathcal{P}\Big\{\sup_{0\le t\le T}e^{\eta_\alpha(t)} > \exp\Big(\frac{1}{2}\Big(\frac{1}{\alpha}\log\Big(\frac{1+\alpha r^2}{1+\alpha|\mathbf{u}(0)|^2}\Big) - \frac{F}{\nu} - 2K(1+m^2)T\Big)\Big)\Big\}\\
&\quad + \mathcal{P}\Big\{\sup_{0\le t\le T}e^{\theta_\alpha(t)} > \exp\Big(\frac{1}{2}\Big(\frac{1}{\alpha}\log\Big(\frac{1+\alpha r^2}{1+\alpha|\mathbf{u}(0)|^2}\Big) - \frac{F}{\nu} - 2K(1+m^2)T\Big)\Big)\Big\}.
\end{aligned}
$$

Notice that both $e^{\eta_\alpha(t)}$ and $e^{\theta_\alpha(t)}$ are martingales, therefore, we conclude from the basic submartingale inequality that

$$
\mathcal{P}\{\sup_{0\le t\le T}|\mathbf{u}(t)| > r\} \le 2\exp\Big(K(1+m^2)T + \frac{F}{2\nu} - \log\Big(\frac{1+\alpha r^2}{1+\alpha|\mathbf{u}(0)|^2}\Big)^{\frac{1}{2\alpha}}\Big).
$$

Taking $\alpha \to 0$, we obtain the desired estimate. □

3.3.1. Exponential inequality and the large deviation principle

In this subsection, we study the exit time estimate for the solution to the following (non-switching) equation

$$
\mathbf{du}(t) + [\nu\mathbf{A}\mathbf{u}(t) + \mathbf{B}_k(\mathbf{u}(t))]dt = \mathbf{f}(t)dt + \sigma(t)dW(t) \tag{3.28}
$$

with initial condition $\mathbf{u}(0) = \mathbf{u}_0$, and $\sigma(t)$ satisfies the following growing condition: For all $t \in [0, T]$, there exists a constant $K > 0$ such that

$$\|\sigma(t)\|_{L_Q} \leq K. \tag{3.29}$$

It is clear that equation (3.28) is a special case of equation (3.9), therefore, the exit time estimates that we obtained in Propositions 3.3.1 and 3.3.2 shall be able to apply to the solution to equation (3.28). In what next, we will study the exit time estimate by using small noise asymptotics provided by large deviations theory. It is worthwhile to point out that the analysis is carried out despite the fact that the stochastic equations do not have a small parameter in the noise term.

Remark. *The content of this subsection is adapted from the author's earlier work [28].*

Consider the unique solution $\mathbf{z}(t)$ of

$$\mathbf{dz} + \mathbf{Az}dt = \sigma(t)dW(t). \tag{3.30}$$

with $\mathbf{z}(0) = 0$. Define $\mathbf{v} := \mathbf{u} - \mathbf{z}$, and notice that

$$\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} &= \frac{\partial \mathbf{u}}{\partial t} - \frac{\partial \mathbf{z}}{\partial t} \\
&= (-\mathbf{Au} - \mathbf{B}_k(\mathbf{u}) + \mathbf{f}(t) + \sigma(t)\frac{dW}{dt}) - (-\mathbf{Az} + \sigma(t)\frac{dW}{dt}) \\
&= -\mathbf{A}(\mathbf{u} - \mathbf{z}) - \mathbf{B}_k(\mathbf{u}) + \mathbf{f}(t) = -\mathbf{Av} - \mathbf{B}_k(\mathbf{v} + \mathbf{z}) + \mathbf{f}
\end{aligned}$$

Therefore, with $\mathbf{z}$ given, solving for $\mathbf{u} - \mathbf{z}$ would be equivalent to solving for $\mathbf{v}$ in

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{Av} + \mathbf{B}_k(\mathbf{v} + \mathbf{z}) + \mathbf{f} = 0 \tag{3.31}$$

with initial data $\mathbf{v}(0) = \mathbf{u}_0 \in H$. Note that equation (3.31) is a non-random, nonlinear partial differential equation and is solved for each ω, where ω enters the equation through $\mathbf{z}(\omega)$.

From a priori bounds, one can easily show that (similar to Proposition 2.3 in [**?**]),

$$\mathbb{E}\left(\sup_{0\le t\le T} |\mathbf{z}(t)|^2\right) + \mathbb{E}\left(\int_0^T \|\mathbf{z}(t)\|^2 dt\right) \le C(\nu, T, K)$$

where $C(\nu, T, K)$ is a finite constant that depends on ν, T, and K that appear in the (3.29). Hence, one obtains that almost surely, $\mathbf{z} \in C_0([0,T]; H) \cap L^2(0,T;V)$.

Lemma 3.3.3. *Given a function $\varphi \in C_0([0,T]; H) \cap L^2(0,T;V)$, let map $\Lambda : \varphi \mapsto \mathbf{v}_\varphi$ be defined by*

$$\frac{\partial \mathbf{v}_\varphi}{\partial t} + \mathbf{A}\mathbf{v}_\varphi + \mathbf{B}_k(\mathbf{v}_\varphi + \varphi) + \mathbf{f} = 0 \tag{3.32}$$

for $t \in [0,T]$, with $\mathbf{v}_\varphi(0) = \mathbf{u}(0)$. Then Λ is a continuous map from $C_0([0,T]; H) \cap L^2(0,T;V)$ to the space $C([0,T]; H) \cap L^2(0,T;V)$.

Proof. Consider functions φ_1 and φ_2 in $C_0([0,T]; H) \cap L^2(0,T;V)$, and denote the corresponding solutions of equation (3.32) as $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Let

$$\mathbf{w}_i := \mathbf{v}_i + \varphi_i \ \text{ for } \ i = 1, 2.$$

Then, by the energy equality,

$$\begin{aligned} &|\mathbf{v}_1(t) - \mathbf{v}_2(t)|^2 + 2\nu \int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|^2 ds \\ &= 2\int_0^t \langle \mathbf{B}_k(\mathbf{w}_1(s)) - \mathbf{B}_k(\mathbf{w}_2(s)), \mathbf{v}_1(s) - \mathbf{v}_2(s)\rangle_V ds. \end{aligned} \tag{3.33}$$

By the basic properties of the bilinear operator $\mathbf{B}_k$, we have,

$$\langle \mathbf{B}_k(\mathbf{w}_1(s)), \mathbf{v}_1(s) - \mathbf{v}_2(s)\rangle_V$$
$$= \langle \mathbf{B}_k(\mathbf{w}_1(s), \mathbf{w}_2(s)), \mathbf{v}_1(s) - \mathbf{v}_2(s)\rangle_V + \langle \mathbf{B}_k(\mathbf{w}_1(s), \varphi_1 - \varphi_2), \mathbf{v}_1(s) - \mathbf{v}_2(s)\rangle_V$$

which enables us to write the integrand on the right side of (3.33) (suppressing the time parameter s) as

$$\begin{aligned}
&\langle \mathbf{B}_k(\mathbf{w}_1) - \mathbf{B}_k(\mathbf{w}_2), \mathbf{v}_1 - \mathbf{v}_2\rangle_V \\
&= \langle \mathbf{B}_k(\mathbf{w}_1 - \mathbf{w}_2, \mathbf{w}_2), \mathbf{v}_1 - \mathbf{v}_2\rangle_V + \langle \mathbf{B}_k(\mathbf{w}_1, \varphi_1 - \varphi_2), \mathbf{v}_1 - \mathbf{v}_2\rangle_V \\
&= \langle \mathbf{B}_k(\mathbf{v}_1 - \mathbf{v}_2, \mathbf{w}_2), \mathbf{v}_1 - \mathbf{v}_2\rangle + \langle \mathbf{B}_k(\varphi_1 - \varphi_2, \mathbf{w}_2), \mathbf{v}_1 - \mathbf{v}_2\rangle_V \\
&\quad + \langle \mathbf{B}_k(\mathbf{w}_1, \varphi_1 - \varphi_2), \mathbf{v}_1 - \mathbf{v}_2\rangle_V. \qquad (3.34)
\end{aligned}$$

Thus the integral on the right side of (3.33) can be split into three integrals, each of which is bounded as follows: First, consider

$$\begin{aligned}
&\int_0^t \langle \mathbf{B}_k(\mathbf{v}_1(s) - \mathbf{v}_2(s), \mathbf{w}_2(s)), \mathbf{v}_1(s) - \mathbf{v}_2(s)\rangle_V ds \\
&\leq \frac{\nu}{6}\int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|^2 ds + \frac{3}{\nu}\int_0^t |\mathbf{v}_1(s) - \mathbf{v}_2(s)|^2 \|\mathbf{w}_2(s)\|^2 ds \qquad (3.35)
\end{aligned}$$

by applying (1.25) and the basic Young inequality. Next, consider

$$
\begin{aligned}
&\int_0^t \langle \mathbf{B}_k(\varphi_1(s)-\varphi_2(s),\mathbf{w}_2(s)),\mathbf{v}_1(s)-\mathbf{v}_2(s)\rangle_V ds \\
&\leq \frac{\nu}{6}\int_0^t \|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|^2 ds + \frac{3}{\nu}\int_0^t \|\varphi_1(s)-\varphi_2(s)\|^2_{L^4(G)}\|\mathbf{w}_2(s)\|^2_{L^4(G)} ds \\
&\leq \frac{\nu}{6}\int_0^t \|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|^2 ds + \frac{3}{2\nu}\int_0^t |\varphi_1(s)-\varphi_2(s)|^2\|\mathbf{w}_2(s)\|^2 ds \\
&\quad + \frac{3}{2\nu}\int_0^t \|\varphi_1(s)-\varphi_2(s)\|^2|\mathbf{w}_2(s)|^2 ds \\
&\leq \frac{\nu}{6}\int_0^t \|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|^2 ds + \frac{3}{2\nu}\Big[\sup_{0\leq s\leq T}|\varphi_1(s)-\varphi_2(s)|^2\int_0^t \|\mathbf{w}_2(s)\|^2 ds \\
&\quad + \sup_{0\leq s\leq T}|\mathbf{w}_2(s)|^2\int_0^t \|\varphi_1(s)-\varphi_2(s)\|^2 ds\Big], \qquad (3.36)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\int_0^t \langle \mathbf{B}_k(\mathbf{w}_1(s),\varphi_1(s)-\varphi_2(s)),\mathbf{v}_1(s)-\mathbf{v}_2(s)\rangle_V ds \\
&\leq \frac{\nu}{6}\int_0^t \|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|^2 ds + \frac{3}{2\nu}\Big[\sup_{0\leq s\leq T}|\varphi_1(s)-\varphi_2(s)|^2\int_0^t \|\mathbf{w}_1(s)\|^2 ds \\
&\quad + \sup_{0\leq s\leq T}|\mathbf{w}_1(s)|^2\int_0^t \|\varphi_1(s)-\varphi_2(s)\|^2 ds\Big] \qquad (3.37)
\end{aligned}
$$

by the same reasoning employed in obtaining (3.36). Using bounds (3.35), (3.36) and (3.37) in equation (3.33), we obtain upon simplification,

$$
\begin{aligned}
&|\mathbf{v}_1(t)-\mathbf{v}_2(t)|^2 + \nu\int_0^t \|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|^2 ds \\
&\leq \frac{6}{\nu}\int_0^t |\mathbf{v}_1(s)-\mathbf{v}_2(s)|^2\|\mathbf{w}_2(s)\|^2 ds \\
&\quad + \frac{3}{\nu}\big(\sup_{0\leq s\leq T}|\varphi_1(s)-\varphi_2(s)|^2\big)\int_0^t (\|\mathbf{w}_1(s)\|^2+\|\mathbf{w}_2(s)\|^2) ds \\
&\quad + \frac{3}{\nu}\big(\sup_{0\leq s\leq T}|\mathbf{w}_1(s)|^2 + \sup_{0\leq s\leq T}|\mathbf{w}_2(s)|^2\big)\int_0^t \|\varphi_1(s)-\varphi_2(s)\|^2 ds. \qquad (3.38)
\end{aligned}
$$

Dropping the second term on the left, and applying the Gronwall inequality, we obtain

$$\begin{aligned}
&|\mathbf{v}_1(t)-\mathbf{v}_2(t)|^2 \\
&\leq \frac{3}{\nu}\big(\sup_{0\leq s\leq T}|\varphi_1(s)-\varphi_2(s)|^2\big)\int_0^t(\|\mathbf{w}_1(s)\|^2+\|\mathbf{w}_2(s)\|^2)\exp\{\frac{6}{\nu}\int_s^t\|\mathbf{w}_2(r)\|^2dr\}ds \\
&\quad+\frac{3}{\nu}\big(\sup_{0\leq s\leq T}|\mathbf{w}_1(s)|^2+\sup_{0\leq s\leq T}|\mathbf{w}_2(s)|^2\big) \\
&\quad\cdot\int_0^t\|\varphi_1(s)-\varphi_2(s)\|^2\exp\{\frac{6}{\nu}\int_s^t\|\mathbf{w}_2(r)\|^2dr\}ds. \qquad (3.39)
\end{aligned}$$

If $\varphi_n\to\varphi$ in $C_0([0,T];H)\cap L^2(0,T;V)$, as $n\to\infty$, it is simple to obtain an upper bound uniform in n for $\sup_{0\leq t\leq T}|\mathbf{w}_n(t)|$ and $\int_0^T\|\mathbf{w}_n(s)\|ds$, where $\mathbf{w}_n:=\mathbf{v}_n+\varphi_n$. Hence, (3.39) allows us to conclude that $\mathbf{v}_n-\mathbf{v}\to 0$ in $C_0([0,T];H)$, and we use this result to estimate (3.38) to justify that $\mathbf{v}_n-\mathbf{v}\to 0$ in $L^2(0,T;V)$ as well. The continuity of the map Λ has thus been proven. □

For each $h\in L^2(0,T;H_0)$, we will use the notation $\mathcal{G}^0(\int_0^\cdot h(s)ds)$ to denote the set of all solutions of the equation

$$dx(t)+\mathbf{A}x(t)dt=\sigma(t)h(t)dt$$

with $x(0)=0$.

For each $\epsilon>0$, let $\mathbf{z}^\epsilon$ denote the solution of

$$\mathbf{dz}^\epsilon(t)+\mathbf{A}\mathbf{z}^\epsilon(t)dt=\sqrt{\epsilon}\sigma(t)dW(t)$$

for $0\leq t\leq T$ with $\mathbf{z}^\epsilon(0)=0$. Then $\mathbf{z}^\epsilon(t)=\sqrt{\epsilon}\int_0^t S_{t-s}\sigma(s)dW(s)$ where S is the semigroup generated by $\mathbf{A}$. It is well-known (cf. [**?**]) that the large deviations rate function for the family $\{\mathbf{z}^\epsilon\}$ is given by,

$$I(x) = \inf_{\{h \in L^2(0,T;H_0): x \in \mathcal{G}_0(\int_0^\cdot h(s)ds)\}} \frac{1}{2}\int_0^T |h(s)|_0^2 ds.$$

Define the map Γ from $C_0([0,T];H) \cap L^2(0,T;V)$ to $C([0,T];H) \cap L^2(0,T;V)$ by

$$\Gamma(\mathbf{z}) = \mathbf{z} + \Lambda(\mathbf{z}).$$

Then Γ is continuous by Lemma 3.3.3, and $\mathbf{u}^\epsilon = \Gamma(\mathbf{z}^\epsilon)$ for all $\epsilon > 0$. Hence, by Theorem 1.2.17 (the contraction principle), $\{\mathbf{u}^\epsilon\}$ satisfies the large deviation principle large deviations principle with rate function

$$J(A) = \inf_{x\, \in\, \Gamma^{-1}(A)} I(x)$$

for any Borel set A in $C([0,T];H) \cap L^2(0,T;V)$, and in particular,

$$\limsup_{\epsilon \to 0} \epsilon \log \mathcal{P}\{\mathbf{u}^\epsilon \in B_r^c\} \leq -J(B_r^c). \tag{3.40}$$

Thus, for any given $\delta > 0$, there exists an $\epsilon_1 > 0$ such that for all $0 < \epsilon \leq \epsilon_1$,

$$\mathcal{P}\{\mathbf{u}^\epsilon \in B_r^c\} \leq \exp\Big(-\frac{1}{\epsilon}(J(B_r^c) - \delta)\Big).$$

That is,

$$\mathcal{P}\Big\{\mathbf{z} \in \frac{1}{\sqrt{\epsilon}}\Gamma^{-1}(B_r^c)\Big\} \leq \exp\Big(-\frac{1}{\epsilon}(J(B_r^c) - \delta)\Big). \tag{3.41}$$

Let A denote the set $\Gamma(\frac{1}{\sqrt{\epsilon}}\Gamma^{-1}(B_r^c))$. Then (3.41) can be written as

$$\mathcal{P}\{\mathbf{u} \in A\} \leq \exp\Big(-\frac{1}{\epsilon}(J(B_r^c) - \delta)\Big).$$

We have thus proved the following theorem:

Theorem 3.3.4. *For any given $r > 0$ and $\delta > 0$, there exists a large positive constant ρ_0, such that for all $\rho \geq \rho_0$ if we define the set $A_\rho := \Gamma(\rho\Gamma^{-1}(B_r^c))$, then solution $\mathbf{u}$ of equation (3.28) satisfies*

$$\mathcal{P}\{\mathbf{u}(t) \in A_\rho\} \leq \exp(-\rho(J(B_r^c) - \delta)) \tag{3.42}$$

where $B_r = \{h \in C([0,T]; H) : \sup_{0 \leq t \leq T} |h(t)|^2 < r\}$,

$$J(B_r^c) = \inf_{x \in \Gamma^{-1}(B_r^c)} I(x),$$

and

$$I(x) = \inf_{\{h \in L^2(0,T;U_0): x \in \mathcal{G}^0(\int_0^\cdot h(s)ds)\}} \frac{1}{2} \int_0^T |h(s)|_0^2 ds.$$

Remark.

(i) In case $\rho_0 = 1$, A_1 coincides with B_r^c, and the theorem gives the rate of decay as $J(B_r^c)$. Also, if we can ascertain the existence of an R such that $B_R^c \subseteq A_{\rho_0}$, the above result leads to a simpler inequality.

(ii) Since we know, by Proposition 3.3.1 that the rate of decay is of the order of r^2, we can follow the above procedure by considering the set

$$F_r = \{x : J(x) \leq r^2\}$$

for $r > 0$ and define the set G_r as any open neighborhood of F_r. Then given any $\delta > 0$, there exists an $\epsilon_1 > 0$ such that for all $\epsilon < \epsilon_1$, we have

$$\begin{aligned}\mathcal{P}\{\mathbf{u}^\epsilon \in G_r^c\} &\leq \exp\Big(-\frac{1}{\epsilon}(J(G_r^c) - \delta)\Big) \\ &\leq \exp\Big(-\frac{1}{\epsilon}(r^2 - \delta)\Big)\end{aligned}$$

by the definition of G_r. Thus, we can conclude that

$$\mathcal{P}\Big\{\mathbf{u} \in \Gamma(\frac{1}{\sqrt{\epsilon}}\Gamma^{-1}(G_r^c))\Big\} \leq \exp\Big(-\frac{1}{\epsilon}(r^2 - \delta)\Big). \tag{3.43}$$

3.3.2. Discussion

Let $\mathbf{u}$ be the solution to equation (3.28), which is a special case of equation (3.23). Thus, it follows from Proposition 3.3.1 that $\mathbf{u}$ satisfies

$$\mathcal{P}\{\sup_{0\leq t\leq T}|\mathbf{u}(t)|>r\}\leq C_1\exp\Big(-r^2e^{-2K(1+m^2)T}\Big), \tag{3.44}$$

where $C_1=\exp\Big(|\mathbf{u}(0)|^2+\frac{F}{\nu}+K(1+m^2)T\Big)$. In addition, processing the argument of the proof of Proposition 3.3.2, one deduces that $\mathbf{u}$ satisfies

$$\mathcal{P}\Big\{\sup_{0\leq t\leq T}|\mathbf{u}(t)|>r\Big\}\leq C_2e^{-r^2}, \tag{3.45}$$

where $C_2=\exp\Big(|\mathbf{u}(0)|^2+\frac{F}{\nu}+K(1+m^2)T\Big)$.

Comparing the inequalities (3.44) and (3.45), we see that the latter is sharper than the former. Moreover, we see from the two inequalities that there are many constants that will effect the estimates, but the constants K, m, and T are the factor for making (3.44) looser[ii].

On the other hand, (3.43) gives a different story. (3.43) reveals that the exponential inequality may be obtained independent of all the factors in both (3.44) and (3.45); it only depends ϵ, the smallness of the noise, which is arbitrary according to Theorem 3.3.4. However, there is still a price to pay if one employs Theorem 3.3.4 to obtain the exponential inequality. Notice that ϵ appears on the both side of (3.43). Therefore, if one chooses a small ϵ to obtain a sharper estimate, one enlarges the set (whose diameter depends on r and ϵ) that $\mathbf{u}$ escapes from while in (3.44), one only need to enlarge K, m, or T (without effecting the size of the r-ball).

[ii]F is the constant relates the external forcing $\mathbf{f}$, K is the constant that appears in the growing condition of σ, and m is the largest element of the state space $\mathcal{S}$ of the Markov chain $\mathfrak{r}(t)$.

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