1-10-2021

A Conjecture on the Irregularity Function for Local Geometric Langlands Parameters and the Formal Frenkel-Gross Connection

Andrew Alaniz

Louisiana State University and Agricultural and Mechanical College

A CONJECTURE ON THE IRREGULARITY FUNCTION FOR LOCAL GEOMETRIC LANGLANDS PARAMETERS AND THE FORMAL FRENKEL-GROSS CONNECTION

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Andrew Alaniz
B.S., University of Texas Pan American, 2008
M.S., University of Texas Pan American, 2013
May 2021

This thesis is dedicated to my ever-loving parents and wife.

Acknowledgments

I would like to thank my advisor Dan Sage for the opportunity to work on such an interesting problem, and for countless hours of instruction and insightful conversation on the wonderful world of mathematics. I will always be indebted to my parents and brothers who always provided loving encouraged and unrelenting support during my time of study. To my wonderful wife, thank you for constant love and constant companionship throughout this all.

Table of Contents

Abstract

For a simple complex algebraic group G, M. Kamgarpour and D. Sage have shown that the adjoint irregularity of an irregular singular flat G-bundle on the formal punctured disc is bounded from below by the rank of G, moreover the rank is realized by the formal Frenkel-Gross connection. This is a geometric analog of a conjecture of Gross and Reeder on the swan conductor of arithmetic local Langlands parameters. In this work, we explore an interesting combinatorial problem which arises when trying to consider the minimal value of the irregularity function with respect to an arbitrary representation of G.

Chapter 1. Introduction and Motivations

This thesis is an attempt to establish and explore an interesting combinatorial problem which arises when trying to consider the minimal value of a certain integer-valued local invariant in the geometric Langlands program—in the style of Arinkin, Frenkel and Gaitsgory [5], [1]—with wild ramification. This invariant, namely the *irregularity*, can be understood to measure how wildly ramified a local geometric Langlands parameter is. These considerations in turn stem from a conjecture concerning ramification data of Langlands parameters in, what is more contemporarily referred to as, the *classical* or *arithmetic* local Langlands conjectures, which like much of the Langlands program with wild ramification, is for the most part still rather mysterious.

Very roughly, in the arithmetic setting of the local Langlands correspondence, the main objects of interest are, on one side of the correspondence, Galois representations, or rather representations of the Weil-Deligne group of a local field, and on the other side, so-called irreducible admissible representations. In this study, we will primarily be concerned with the objects on the Galois side of local Langlands correspondences. In [7], Gross and Reeder formulate, and prove in some instances, a conjecture concerning a lower bound on the (adjoint) *Swan conductor* of a wildly ramified Langlands parameter. In the geometric setting, formal flat G-bundles on algebraic curves replace Galois representations to serve as Langlands parameters. In this setting, the analog of the Swan conductor is the irregularity of a formal flat G-bundle. An analogy between these two types of Langlands parameters was noted by Sage and Kamgarpour in [12], where the authors prove an analogous lower bound on the irregularity under lighter restrictions as the aforementioned conjecture of Gross and Reeder. The existence of such an analogy between these two different settings of the Langlands program is just one example in a well-known collection of such analogies which arise when switching between arithmetic and geometry in the Langlands program. Therefore, this work is towards a better understanding of wild ramification in the general framework of the Langlands program.

1.1. The Langlands Program

The Langlands program, in perhaps broadest of terms, aims to build dictionaries or *correspondences* between different areas of mathematics. In its original inception, this was formulated in terms of relating Galois groups in algebraic number theory to more analytic flavored objects called automorphic forms and the representation theory of algebraic groups over local and global fields. These ideas of R.P. Langlands followed earlier work by Harish-Chandra, Selberg and Gelfand concerning trace formulas for semisimple Lie groups while attempting to also incorporate new connections to number theory via categorical constructions. While more recent statements can be formulated for G an arbitrary reductive algebraic group, for our purposes, it suffices to consider $G = GL_n$, in fact, all relevant constructions in this thesis will be concerned with the situation where G is a simple algebraic group. All algebraic number theoretic objects in what follows can be found in any introductory text on the subject.

In number theory, the Langlands correspondence is a conjectural correspondence between n-dimensional complex linear representations of $Gal_F := Gal(\overline{F}/F)$, where F is a finite field extension of $\mathbb{Q}$, i.e. a number field or a function field of a curve over $\mathbb{F}_q$, and automorphic representations of the n-dimensional general linear group $GL_n(\mathbb{A}_F)$, where $\mathbb{A}_F$ is the ring of adeles of F. The latter objects can be realized in representations given by functions on the double coset space $GL_n(F) \setminus GL_n(\mathbb{A}_F)/GL_n(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of all formal completions of F. Such a correspondence should also exhibit *reciprocity* and *functoriality*. We outline these concepts. Firstly, objects on both sides of the correspondence should have associated L-functions. Reciprocity can then be understood as the conjecture that the L-functions coming from the two sides are equivalent. Furthermore, such constructions should behave well with respect to precomposing with analytic homomorphisms between dual groups so that the associated L-function remains invariant with respect to such a change. This was Langlands' original notion of functoriality.

As a quick illustrative example of such a correspondence, we roughly state the now famous theorem for $G = GL_n$ as proved by Harris, Lan, Taylor and Thorne in 2013 and then by Scholze in 2017. Let E be a totally real field or a number field with complex multiplication. Let π be a cuspidal automorphic representation of $GL_n(\mathbb{A}_E)$. Modulo some further restrictive algebraicity conditions on π, there exists a Galois representation

$$\rho_\pi : Gal_E \to GL_n(\overline{\mathbb{Q}_\ell})$$

which is *canonically* attached to π.

The local Langlands conjectures can then be considered as a refinement of these conjectures by way of considering the above constructions over local fields rather than global fields. For G a reductive algebraic group over K a local field, the local Langlands conjectures then predict that the irreducible complex representations of the locally compact group $G(K)$ should correspond to homomorphisms ϕ from the Weil-Deligne group of K to the complex Langlands dual group of G, together with an irreducible representation ρ of the component group of the centralizer of ϕ.

1.2. Gross-Reeder Conjecture

We recall the setup of [10] along with some useful observations from [12]. Let G be a simple complex algebraic group. If the Langlands parameter $\phi : W \times SL_2(\mathbb{C}) \to G$ is discrete and inertially discrete, then the Swan conductor of $Ad(\phi)$ is greater than or equal to the rank of G, i.e.

$$\mathrm{sw}(Ad\phi) \geq \mathrm{rk}(G).$$

While this conjecture remains open in full generality, many important cases have been shown. Moreover, by analyzing the case when equality was attained, this led Gross and Reeder to their construction of *simple wild parameters*. Using the Langlands correspondence they then went on also to construct *simple supercuspidal representations* of p-adic groups with dual group G which correspond under the local Langlands program to simple wild parameters. Simple supercuspidal representations are in turn the simplest examples of what are known as *epipelagic* representations. Epipelagic representations were constructed by Reeder and Yu in [20]. As we can see, these inquiries and constructions by Gross and Reeder initiated an important new direction in the local Langlands program. This theory also has important applications to the global Langlands program.

Recently, Heinloth, Ngo and Yun used these results to construct Kloosterman sheaves-ℓ-adic local systems on $\mathbb{P}^1 \setminus \{0, \infty\}$ whose single wildly ramified singularity corresponds to a simple wild parameter. This then can be seen to provide an example of a wildly ramified Langlands correspondence between ℓ-adic local systems and Hecke eigensheaves. See [11] for more details into this aspect of study.

One can use "Weil's Rosetta Stone", see [6], to translate from number theory to geometry, and even more startlingly, to physics and quantum field theories. For the latter vein of study we refer the interested reader to [19], [21] and [14] for fascinating expositions into this aspect of the Langlands program. These connections relate algebraic properties of fields to geometric properties of algebraic curves defined over $\mathbb{C}$. In the geometric world, as noted above, it is now well known that formal flat G-bundles play the role of Langlands parameters, see for example [15]. Let $K = \mathbb{C}((t))$ be the field of formal Laurent series and we denote by $D^\times = Spec(K)$ the formal punctured disk. A formal flat G-bundle (E, ∇) is then a principal G-bundle E on $D^\times$ endowed with a connection ∇ (which is automatically flat). Interestingly, switching to the geometric setting affords the ability to fully prove the geometric analog of the Gross-Reeder conjecture.

1.3. A Geometric Analog of the Gross-Reeder Conjecture

Observe the following theorems of Sage and Kamgarpour [12].

Theorem 1 (Sage, Kamgarpour). Let G be a simple group and let (E, ∇) be an irregular singular formal flat G-bundle. Then,

$$irr(Ad(\nabla)) \geq rk(G). \tag{1.3.1}$$

Where $(Ad(E), Ad(\nabla))$ is the associated adjoint bundle of (E, ∇).

Theorem 2 (Sage, Kamgarpour). For G and (E, ∇) as above, then the following are equivalent:

1. $irr(Ad(\nabla)) = rk(G)$
2. $s(\nabla) = 1/h$
3. ∇ is a formal Frenkel-Gross connection.

This explicit result, and characterization of the formal Frenkel-Gross connection which we will denote by ∇_{FG}, strongly suggests that the Frenkel-Gross connection should be viewed as the geometric analog of the simple wild parameters of Gross and Reeder. Some natural questions arise. How special is the Frenkel-Gross connection? And, why is the adjoint representation needed for these statements?

1.4. A Conjecture on the Minimal Irregularity

To address the inquiries stated above, let $\mathfrak{g} = Lie(G)$ be a simple Lie algebra and let $V \in Rep(\mathfrak{g})$ be a finite dimensional representation of $\mathfrak{g}$. Denote by $\mathfrak{L}^{irr} \subset \mathfrak{L} = \Omega^1(\mathfrak{g}(K))/G(K)$ the space of irregular formal flat connections. For a formal flat G-bundle (E, ∇), we will denote by $irr_V : \mathfrak{L} \to \mathbb{Z}$ the irregularity of the associated flat vector bundle (V_E, V_∇), and we shall refer to this local invariant as the λ-irregularity (λ will denote the highest weight of an irreducible V) of a formal flat connection $\nabla \in \mathfrak{L}$. Once properly defined, this invariant will provide us the technology to tackle the following conjecture.

The minimal irregularity Conjecture: In the above set-up, and for all $V \in Rep(\mathfrak{g})$,

$$irr_V(\nabla^{FG}) = \min_{\nabla \in \mathfrak{L}^{irr}} \{irr_V(\nabla)\}. \tag{1.4.1}$$

Note, for this conjecture to be completely affirmed with respect to a specific simple Lie algebra $\mathfrak{g}$, it must hold for all $V \in Rep(\mathfrak{g})$. An important observation to also consider is that an arbitrary representation V of a simple Lie algebra $\mathfrak{g}$ will be completely reducible. This means that V will decompose into a direct sum of irreducible representations, $V \cong \bigoplus V_i$. Also, the irregularity is additive i.e.

$$irr_V(\nabla) = \sum irr_{V_i}(\nabla).$$

Therefore, rather than considering $V \in Rep(\mathfrak{g})$ we can restrict to only considering $V \in$

$Irr(\mathfrak{g})$. As we will see, while this conjecture seems rather innocuous, there will arise combinatorial complexities, already even for $\mathfrak{g} = A_2$, for which we will need to provide some novel constructions in order to derive formulae for the λ-irregularity of certain minimal classes of connections in $\mathfrak{L}^{irr}$. In what follows, we show that the inequality

$$irr_V(\nabla_{FG}) < irr_V(\nabla)$$

holds for all "generic" $\nabla \in \mathfrak{L}^{irr}$, and furthermore we will prove the minimal irregularity conjecture in full generality for the cases A_1 and A_2.

Chapter 2. Preliminaries

Throughout this work we will for the most part only be considering constructions for type A simple Lie algebras. Unless otherwise stated $\mathfrak{g} = Lie(G) = \mathfrak{sl}_n$, i.e. $G = SL_n$. Most of the representation theory we recall here can be found in standard textbooks on representation theory, for example see [9] or [4]. For the relevant bundle theory and results on irregular connections I will be using as references [3], [12], [13].

2.1. Irreducible Representations

Let $M_n(k)$ be the associative algebra of all $n \times n$ matrices over the field k and we write Lie$(M_n(k))$ for the corresponding Lie algebra, i.e. $\mathfrak{gl}_n(k) = Lie(M_n(k))$ with dim $\mathfrak{gl}_n(k) = n^2$. A *representation* of a Lie algebra L over k is then a homomorphism of Lie algebras $\rho : L \to \mathfrak{gl}_n(k)$, or $\rho \in \mathrm{Hom}(L, \mathfrak{gl}_n(k))$ for short, for some n, and ρ is called a representation of degree or *dimension* n. Two representations ρ, ρ' of the same degree are said to be equivalent if there exists a non-singular matrix T such that

$$\rho'(x) = T^{-1}\rho(x)T$$

for all $x \in L$.

A *left L-module* is a vector space V over k together with a multiplication (or action)

$$L \times V \to V$$

$$(x, v) \mapsto xv$$

which is bilinear, and for all $x, y \in L$ and $v \in V$ we have

$$[xy]v = x(yv) - y(xv).$$

Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra with Cartan subalgebra $\mathfrak{h}$. Let R be the associated root system. Then we say that an element $\lambda \in \mathfrak{h}^*$ is integral if $2(\lambda, \alpha)/(\alpha, \alpha)$ is an integer for every root. Choosing a set of positive roots R^+, λ is dominant if $(\lambda, \alpha) \geq 0$ for all positive roots. λ is dominant integral if it is both dominant and integral. For λ and μ we say the λ is higher than μ if you can express $\lambda - \mu$ as a linear combination of positive roots with non-negative coefficients, we denote this by $\mu \preceq \lambda$. A weight of a representation V of $\mathfrak{g}$ is called a highest weight if λ is higher than all other weights of V. The theorem of the highest weight tells us that if V is a finite dimensional irreducible representation of $\mathfrak{g}$ then V has a unique highest weight, and this highest weight is dominant integral. We will denote the finite dimensional irreducible representations of $\mathfrak{g}$ by $Irr(\mathfrak{g})$.

Let $\mathfrak{g} = \mathfrak{sl_n} = Lie(SL_n)$ and let $\mathfrak{h}$ be the maximal Cartan subalgebra consisting of the diagonal matrices, we can write $\mathfrak{h} = \{diag(\theta_1, \ldots, \theta_n) \mid \theta_i \in \mathbb{C}, \sum_{i=1}^n \theta_i = 0\}$. For $1 \leq i \leq n$ set $\varepsilon_i(diag(\theta_1, \ldots, \theta_n) = \theta_i$. We have

$$\mathfrak{h}^* = \bigoplus_{i=1}^n \mathbb{C}\varepsilon_i / \langle \sum_{i=1}^n \varepsilon_i = 0 \rangle.$$

The set of positive roots is $R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}$. The weight lattice is then

$$P = \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i / \langle \sum_{i=1}^n \varepsilon_i = 0 \rangle$$

The set of irreducible representations $Irr(\mathfrak{sl}_n)$ are parameterized by $n-1$-tuples $(k_1, \ldots, k_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$, they will lie in the following cone of P

$$P^{++} = \left\{ \sum_{i=1}^{n-1} k_i \varepsilon_i \in P \mid k_1 \geq \cdots \geq k_{n-1} \geq 0 \right\}$$

with highest weights of the form

$$\begin{aligned}\lambda =& k_1\varepsilon_1 + k_2(\varepsilon_1+\varepsilon_2) + \cdots + k_{n-1}(\varepsilon_1 + \cdots + \varepsilon_{n-1}) \\ =& (k_1 + \cdots + k_{n-1})\varepsilon_1 + (k_2 + \cdots + k_{n-1})\varepsilon_2 + \cdots + k_{n-1}\varepsilon_{n-1}\end{aligned}$$

with $\varepsilon_i \in \mathfrak{h}^*$. With respect to such an $\mathfrak{h}$, each $V_\lambda \in Irr(\mathfrak{sl}_n)$ has a weight space decomposition

$$V_\lambda = \bigoplus_{\mu \preceq \lambda} V_\lambda^\mu$$

where $\preceq$ is the partial order on weights. Since we are dealing with finite dimensional representations, $dim V_\lambda^\mu < \infty$ for all $\mu \preceq \lambda$. For our purposes it will be convenient to define the following function

$$\begin{aligned}\chi : Irr(\mathfrak{sl}_n) &\to \mathbb{Z}_{\geq 0} \\ V_\lambda &\mapsto \sum_{\mu \neq 0} dim V_\lambda^\mu,\end{aligned}$$

which is well defined since $\sum dim V_\lambda^\mu < \infty$. In the literature $dim V_\lambda^\mu$ is called a Kostka number and is usually denoted $K_{\lambda\mu}$. In general $K_{\lambda\mu}$ counts the number of semistandard Young tableaux of shape λ and content μ, both considered as partitions. We refer the interested reader to [8] for the details on the combinatorics of Young tableaux and their importance in representation theory. For our purposes, the relevant combinatorics of V_λ can be encapsulated by considering the convex hull of the set of weights appearing in V_λ, following the constructions of [18], we call this the *weight polytope* of λ, and denote it by $\mathcal{P}(V_\lambda) \subset \mathbb{Z}^{n-1} \subset \mathbb{R}^{n-1}$. We will sometimes write $\mathcal{P}_\lambda$ for $\mathcal{P}(V_\lambda)$.

2.2. Formal Flat Connections

Let G be a topological group. A *principal G-bundle* is a fiber bundle (P, X, π, F), with a continuous right action of G, $P \times G \to P$, that preserves the fibers of P, i.e. for $y \in P_x$, then $yg \in P_x$ for every $g \in G$, and that acts freely and transitively such that for each $x \in X$, and $y \in P_x$, $G \to P_x$ is a homeomorhpism, via $g \mapsto yg$, i.e. each fiber of the bundle is homeomorphic to G.

Let $K = \mathbb{C}(\!(t)\!)$. For G a simple complex algebraic group of finite rank, a formal flat G-bundle (E, ∇) is a principal G-bundle E on the formal punctured disk $\mathcal{D}^\times \cong Spec(K)$ with a connection ∇, which is automatically flat. After choosing a trivialization the connection can be written in terms of its matrix $[\nabla] \in \mathfrak{g}(K)$, we can write $\nabla = d + [\nabla]\frac{dt}{t} \in \Omega^1(\mathfrak{g}(K))$. Changing the trivialization by an element of the loop group $g \in G(K)$, changes the matrix by the so-called *gauge action*

$$g.[\nabla] = Ad(g)[\nabla] - (dg)g^{-1}.$$

Accordingly, the set of isomorphism classes of flat G-bundles on $D^\times$, which we denote by $Bun_G(D^\times)$ is isomorphic to the quotient $\Omega^1(\mathfrak{g}(K))/G(K)$ where the loop group $G(K)$ acts by the gauge action.

A flat G-bundle (E, ∇) on $D^\times$ is called *regular* singular if the connection matrix has only simple poles with respect to some trivialization, otherwise it is called *irregular*. Irregular formal flat G-bundles are wildly ramified geometric Langlands parameters. Following cues from the arithmetic Langlands correspondence, one can then ask "how irregular" an irregular singular flat G-bundle is. This is measured by two invariants: the slope—a rational number—and the irregularity with respect to a representation of G, an integer-valued invariant.

2.3. Slope

Let $b \geq 1$ be an integer then there is a ramified cover $D_b^\times = Spec(\mathbb{C}((u)))$ with $u = t^{1/b}$ and a trivialization of the pullback bundle such that the pullback connection is of the form

$$d + (X_{-a}u^{-a} + X_{1-a}u^{1-a} + \dots)\frac{du}{u}$$

$X_i \in \mathfrak{g}$, X_{-a} non-nilpotent and $a \geq 0$. The quotient a/b is independent of the choice of such an expression, one calls it the *slope* of ∇, which we denote by $s(\nabla)$. By the contravariance of *Spec*, we can lift the order of the pole, $\mathrm{ord} : \sum_{n \geq n_0} a_n t^n \mapsto -n_0$ in the induced diagram

$$\begin{tikzcd} \mathbb{C}((u)) \arrow[r, "\mathrm{ord}_b"] & \frac{1}{b}\mathbb{Z} \subset \mathbb{Q} \\ \mathbb{C}((t)) \arrow[u] \arrow[r, "\mathrm{ord}"] & \mathbb{Z} \arrow[u] \end{tikzcd}$$

where $u = t^{1/b}$, so that $\mathrm{ord}_b = \frac{ord}{b} \in \mathbb{Q}$ in the diagram. The slope is positive if and only if the flat G-bundle is irregular, and the smallest possible slope is $1/h$ where h is the Coxeter number of G, these results and other results related to the slope of irregular connections can be found in [2], [12].

2.4. Irregularity

We recall the constructions in [12]. Let $G = GL_n$, although the following results will hold for more general G. In this case a flat G-bundle is equivalent to a pair (E, ∇) where E is a vector bundle on $D^\times$ endowed with a connection ∇. After passing to a ramified cover $D_b^\times$ the pullback connection $\pi_b^* E$ has a Jordan decomposition, which is called a

Levelt-Turrittin (LT) decomposition, into a finite sum

$$\bigoplus (L_i \otimes M_i, \nabla_{L_i} \otimes \nabla_{M_i}),$$

where (L_i, ∇_{L_i}) is rank one and (M_i, ∇_{M_i}) is regular singular. For (E, ∇) a formal flat G-bundle there will always exist a positive integer b and a trivialization of $\pi_b^* E$ such that

$$\pi_b^* \nabla = d + (h+n)\frac{du}{u}$$

with $h \in \mathfrak{h}[u^{-1}]$ and $n \in \mathfrak{n}(\mathbb{C})$ such that h and n commute and any such pair of h and n is unique. Let s_i be the slope of the flat connection $(L_i \otimes M_i, \nabla_{L_i} \otimes \nabla_{M_i})$, then the irregularity, $irr(\nabla)$ is

$$\sum s_i \cdot dim(M_i) \in \mathbb{Z}_{\geq 0}.$$

Moreover, it can be shown that $irr(\nabla) = 0$ if and only if ∇ regular singular. The Levelt-Turrittin form essentially gives an explicit computational tool to compute the irregularity. Let $G = GL_n$ with B the upper triangular matrices and H the diagonal matrices. Let (E, ∇) be a formal flat G-bundle with Levelt-Turrittin form $d + (h+n)\frac{du}{u}$ with respect to B and H. Then since h is diagonal we can write it as

$$h = diag(h_1, \ldots, h_n)$$

with $h_i \in \mathbb{C}[u^{-1}]$. From here we can equivalently define the irregularity as

$$irr(\nabla) = \sum_{i=1}^{n} \{\max\{0, \frac{ord(h_i)}{b}\}\}$$

where ord is the order of the pole of h_i.

We consider an example to illustrate. For $G = SL_5$ consider the connection with

Levelt-Turrittin form

$$\begin{pmatrix} t^{-1/3} & & & & \\ & \zeta t^{-1/3} & & & \\ & & \zeta^2 t^{-1/3} & & \\ & & & t^{-1/2} & \\ & & & & -t^{-1/2} \end{pmatrix}$$

with ζ a primitive cube root of unity. In this case from the notation above $n = 0$. This is an example of a non-generic irregular formal flat G-bundle, by which we mean a connection that has mixed Galois orbits for the valuations. Its irregularity is $1/3 + 1/3 + 1/3 + 1/2 + 1/2 = 2$.

Recall, by the theorems of Sage and Kamgarpour listed in section 1, the formal irregular flat connection with minimal possible slope $1/h$ is called the Frenkel-Gross connection denoted ∇_{FG}, its connection matrix is of the form

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ t & & & 0 \end{pmatrix} \in \mathfrak{sl}_n((t)), \tag{2.4.1}$$

and we will work with the inverse form

$$\begin{pmatrix} 0 & & & t^{-1} \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}, \tag{2.4.2}$$

which changes the wildly ramified point from ∞ to 0.

The Frenkel-Gross connection is one in, what we shall call, a larger *combinatorial class* of irregular connections called formal *Coxeter* connections introduced by Sage and Karmgarpour in [13]. For example, for $\mathfrak{sl}_n$, let N_r be the matrix with 1's on the rth sub-diagonal and 0's everywhere else, and let E_r be the matrix with 1's on the $(n-r)$th superdiagonal and 0's everywhere else. Then, the formal Coxeter connection can be written as

$$\nabla^q_{r,m} = d + t^{-m}q(N_r + t^{-1}E_r)\frac{dt}{t}$$

with for $q \in \mathbb{C}^\times$ and $m, r \in \mathbb{Z}_{\geq 0}$ such that $gcd(r,h) = 1$ and $1 \leq r \leq h$. Note, if one restricts ∇_{FG} to the formal neighborhood at the irregular singular point we get $\nabla^{-1}_{1,0}$. ∇_{FG} is also an example of a *cohomologically rigid* irregular flat connection, which are connections that can be determined by their monodromy data around the singular points. This is equivalent to having no infinitesimal deformations. The Frenkel-Gross connection was defined by Frenkel and Gross in [7] and has been exploited by Heinloth, Yun, Ngo, Kamgarpour-Sage, Lam, Templier and others for its many interesting properties, including relations to mirror symmetry, for example see [17].

In a specified combinatorial class, we can also *parameterize* a formal connection by multiplying each of its non-zero entries by a non-zero scalar. In some sense these are "smooth" parameters. For instance, by parameterizing the Frenkel-Gross connection we can arrive at the following form

$$\begin{pmatrix} 0 & & & a_1 t^{-1} \\ a_2 & 0 & & \\ & \ddots & \ddots & \\ & & a_n & 0 \end{pmatrix} \tag{2.4.3}$$

with $a_i \neq 0$ for all i. Its Levelt-Turrittin form will be

$$Ct^{-1/n} \begin{pmatrix} 1 & & & \\ & \zeta & & \\ & & \ddots & \\ & & & \zeta^{n-1} \end{pmatrix} \in \mathfrak{sl}_n(\!(u)\!) \tag{2.4.4}$$

where $u = t^{1/n}$, $\zeta = e^{2\pi i/n}$ and $C = (\prod a_i)^{1/n}$. We denote by $\mathbf{p}(\nabla)$ the number of non-zero scalars needed to parameterize a formal flat irregular connection ∇. Observe that by taking the LT form of ∇_{FG}, we reduce the number of parameters needed to parameterize ∇_{FG}. We can see that $\mathbf{p}(\nabla_{FG}) = 1$, and in fact it follows that $\mathbf{p}(\nabla^q_{r,m}) = 1$ for all q, r, m as above. Thus, the Frenkel-Gross connection has the minimal possible slope and number of parameters.

Another relevant combinatorial class of irregular formal flat connection is what we shall call a *diagonalizable* connection, and denote by ∇_D. This type of connection will have connection matrix as follows,

$$\begin{pmatrix} a_1(t^{-1}) & & & \\ & \ddots & & \\ & & a_{n-1}(t^{-1}) & \\ & & & a_n(t^{-1}) \end{pmatrix} \in \mathfrak{sl}_n(\!(t)\!) \tag{2.4.5}$$

where the $a_i(t^{-1})$ are polynomials in t^{-1} such that $\sum_i a_i(t^{-1}) = 0$. As we can see, contrary to the Coxeter connections, there is no need to lift to a field extension for the LT form. It follows that $s(\nabla_D) = \max_i(\deg(a_i))$. We note also that the irregularity of any ∇_D will be minimal when $s(\nabla_D) = \max(\deg(a_i)) = 1$. For example, a diagonalizable irregular formal flat connection with connection matrix

$$t^{-1}\begin{pmatrix} a_1 & & & \\ & \ddots & & \\ & & a_{n-1} & \\ & & & -\sum_{i=1}^{n-1} a_i \end{pmatrix} \in \mathfrak{sl}_n((t)) \tag{2.4.6}$$

has slope 1 and $\mathbf{p} = n-1$. We denote the diagonal connection with connection matrix (2.6) $\nabla(a_1, \ldots, a_{n-1})$.

Remark 0.1. *In general, specifying a combinatorial class in $\mathfrak{L}^{irr}$ amounts to specifying the Galois structure along with choosing specific eigenvalues.*

Now, for a fixed irreducible representation V_λ and a formal G-bundle (E, ∇), there is an associated flat vector bundle for V_λ, $(V_\lambda E, V_\lambda \nabla)$. Let ∇ be a formal flat connection of a specific combinatorial type, then the irregularity with respect to V_λ will have the following form,

$$irr_{V_\lambda}(\nabla) = \sum_\mu{}' dim(V_\lambda^\mu)(-val([\nabla]\cdot\mu)) \tag{2.4.7}$$

where $\sum'$ denotes the sum over μ appearing in V_λ such that the valuation of $[\nabla]\cdot\mu$ is strictly negative. We have denoted by $[\nabla]$ the LT form of ∇. We denote by

$$irr_\lambda(\nabla) := irr_{V_\lambda}(\nabla) \tag{2.4.8}$$

the λ-irregularity of a $\nabla \in \mathfrak{L}^{irr}$. For $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ we will give explicit formulae for irr_λ to resolve the minimal irregularity conjecture. For $\mathfrak{g} = \mathfrak{sl}_n$, fix a $V_\lambda \in Irr(\mathfrak{g})$ in these cases and consider its weight polytope $\mathcal{P}_\lambda \subset \mathbb{Z}^{n-1} \subset \mathbb{R}^{n-1}$. For $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$, the points of $\mathcal{P}_\lambda$ are arranged in unbroken strings of weights. The multiplicity of a weight $\mu \preceq \lambda$ in a string of weights is constant so one may consider the multiplicity of a string of weights in $\mathcal{P}_\lambda$. The multiplicity of a string of weights increases from outer to inner. Recall the function $\chi(V_\lambda) = dimV_\lambda - dimV_\lambda^0$. We will define a non-negative integer $\mathbf{c}$ which we shall call the *cancellation parameter* of ∇. The cancellation parameter counts lattice points, with multiplicity, lying on the intersection of the weight polytope and a lower dimensional subspace determined by ∇, which we denote by $\ell(\nabla) \subset \mathbb{R}^{n-1}$. For $\mathfrak{sl}_2$, $\mathfrak{sl}_3$ these subspaces will be either the origin, or lines through the origin, in the ambient real euclidean space containing $\mathcal{P}_\lambda$, for all irreducible representations. It will follow that

$$\mathbf{c}(\nabla) = \sum_{\mu \neq 0 \in \ell(\nabla) \cap \mathcal{P}_\lambda} K_{\lambda\mu}. \tag{2.4.9}$$

2.5. Minimal Irregularity

We can now discuss an ambiguity which arises when trying to find irregular formal flat connections which are minimal irregular with respect to a fixed irreducible representation. Given $V_\lambda \in Irr(\mathfrak{g})$ we want to find connections $\nabla' \in \mathfrak{L}^{irr}$ such that

$$irr_\lambda(\nabla') \leq irr_\lambda(\nabla) \tag{2.5.1}$$

for all $\nabla \in \mathfrak{L}^{irr}$. An obvious candidate is the formal Frenkel-Gross connection since, as noted above, it has the minimal possible slope and number of smooth parameters. The Frenkel-Gross connection also has a known formula with respect to the adjoint representation. In our notation this formula can be written

$$irr_{Ad}(\nabla_{FG}) = rk(\mathfrak{g}). \tag{2.5.2}$$

For example, let $\mathfrak{g} = \mathfrak{sl}_2$, then by the above formula we have that $irr_{Ad}(\nabla_{FG}) = 1$, the minimal possible irregularity. Now note that here is also another irreducible representation, the standard representation std, for which ∇_{FG} also yields the minimal possible irregularity, in other words, for $\mathfrak{sl}_2$ we will have

$$irr_{std}(\nabla_{FG}) = irr_{Ad}(\nabla_{FG}) = 1. \tag{2.5.3}$$

The following question then arises. Are there other irreducible representations for which the minimal possible irregularity is realized by connections which are not the Frenkel-Gross connection? As we will see, settling this will settle the minimal irregularity conjecture for $\mathfrak{sl}_2$.

2.6. Generic Formal Flat Connections

We consider $G = SL_n$. Given a connection of a fixed combinatorial class we have an associated partition of n. If b is a part (so $1 \leq b \leq n$), then the associated diagonal entries all have valuation $-a/b$ for some positive a relatively prime to b, unless the part has size 1 with a 0 eigenvalue. Now, suppose that the n diagonal entries have no $\mathbb{Q}$-linear dependence relations other than the obvious trace 0 condition. It is then the case that any nonzero integral weight vector μ evaluated on this diagonal matrix is nonzero and will contribute $dim(V^{\mu}) \cdot \frac{a}{b}$ for one of the valuations appearing among the entries. Now, If the partition is not the one associated to Coxeter connections, in other words, not the partition with the single part n, then $b < n$ for each part, therefore $a/b > 1/n$. Thus, in this case

$$\begin{aligned}
irr_{V_\lambda}(\nabla) &= \sum_{\mu \neq 0 \in V_\lambda} dim(V_\lambda^\mu) a/b \\
&> \sum_{\mu \neq 0 \in V_\lambda} dim(V_\lambda^\mu) 1/n \\
&= irr_{V_\lambda}(\nabla_{FG}).
\end{aligned}$$

Therefore, the conjecture is true generically. We have the following theorem.

Theorem 1. *For generic SL_n-connections ∇ and any representation V which has an irreducible component that is not the trivial representation, we have the following strict inequality*

$$irr_V(\nabla) > irr_V(\nabla_{FG}). \tag{2.6.1}$$

We should note that this is a considerably more complicated theorem for non-generic ∇ where a strict inequality would fail. For $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ we find that there are only two and three types, respectively, of Galois orbits to consider. For the various choice of partition in these cases, it is clear that you get the smallest possible irregularity if you take the slope $1/2$ connection for $\mathfrak{sl}_2$ and the slope $1/3$ connection for $\mathfrak{sl}_3$.

Let us now turn to the task of proving the minimal irregularity conjecture for $\mathfrak{sl}_2$.

Chapter 3. $\mathfrak{sl}_2$

For $\mathfrak{sl}_2$, the minimal irregularity conjecture can be verified rather straightforwardly via our methods. In this case the complexity of all the relevant data is at a minimum. Moreover, we have a simplification for the formula of the λ-irregularity. In this case, the relevant formal irregular connections are what are called *toral* connections see [13] for results and constructions of toral connections. Consequently, irr_λ takes the simple form

$$irr_\lambda(\nabla) = s(\nabla)(\chi(V_\lambda) - \mathbf{c}(\nabla)).$$

Observe, $Irr(\mathfrak{sl}_2)$ is parameterized simply by $\mathbb{Z}_{\geq 0}$, and strings of weights, with respect to a given irreducible, are just decreasing sequences of integers whose pairwise differences are congruent modulo 2. The weight lattice has rank 1. For a connection $\nabla \in \mathfrak{L}^{irr}$, the slope can take values in $\mathbb{Q} \cap [1/2, \infty)$. For $\mathfrak{g} = \mathfrak{sl}_n$ in general, types of ∇ such that $s(\nabla) \in \mathbb{Q} \cap [1/n, 1]$ can be broken into two classes, ∇ which have fractional slope less than one, and ∇ with slope equal to one. Returning to the $\mathfrak{sl}_2$, we have that there are only two such type of connections to consider, ∇_{FG} with slope $1/2$, and $\nabla(a_1)$ with slope 1, where $a_1 \in \mathbb{R}^\times$. Respectively, we have

$$\begin{pmatrix} 0 & a_1 t^{-1} \\ a_2 & 0 \end{pmatrix}, \text{ and } t^{-1}\begin{pmatrix} a_1 & 0 \\ 0 & -a_1 \end{pmatrix}.$$

As noted above because we compute the irregularity from a connection's LT form, for the Frenkel-Gross connection, we have

$$Ct^{-1/2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with $C = \sqrt{a_1 a_2}$. We note in this case $\mathbf{p}(\nabla_{FG}) = \mathbf{p}(\nabla(a_1)) = 1$. Moreover, there can be no non-trivial cancellation. Thus, $\mathbf{c}(\nabla_{FG}) = 0$.

Let us explicitly derive the formula in this case. We will use the pullback of the Frenkel-Gross to define $irr_\lambda(\nabla_{FG})$ for a given $V_\lambda \in Irr(\mathfrak{sl}_2)$. All the irreducibles in this case are symmetric tensor products of the standard 2 dimensional representation $St \in Irr(\mathfrak{sl}_2)$ with respect to $[\nabla_{FG}]$. Choosing a basis e_0, e_1 we have

$$[\nabla_{FG}].e_0 = t^{-1/2}e_0$$

$$[\nabla_{FG}].e_1 = -t^{-1/2}e_1$$

$$\begin{aligned} irr_{St}(\nabla_{FG}) : &= \operatorname{ord}_2(t^{-1/2})\dim(V_{-1}) + \operatorname{ord}_2(\zeta t^{-1/2})\dim(V_1) \\ &= \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

where the V^i are the weight spaces for St, i.e. $St \simeq V^{-1} \oplus V^1$. Let $a \in \mathbb{Z}_{\geq 0}$ then $\left\{e_0^{a-i} \otimes e_1^i\right\}$ for $i = 0, \ldots, a$ is a basis for $Sym^a(St) \in Irr(\mathfrak{sl}_2)$ with respect to a maximal Cartan $\mathfrak{h}$. We can establish via the Leibniz rule that for $(i, j) \in \mathbb{Z}^2_{\geq 0}$ with $0 \leq i, j \leq a$ such that $i + j = a \in \mathbb{Z}_{\geq 0}$

$$[\nabla_{FG}]e_0^i \otimes e_1^j = (i + j\zeta)t^{-1/2}e_0^i \otimes e_1^j. \tag{3.0.1}$$

Note that $i = a - j$ and in this case $\zeta = \zeta_2 = -1$ so we can rewrite the above as

$$\begin{aligned} [\nabla_{FG}^{LT}]e_0^{a-j} \otimes e_1^j &= (a - j + j\zeta)t^{-1/2}e_0^{a-j} \otimes e_1^j \\ &= (a - 2j)t^{-1/2}e_0^{a-j} \otimes e_1^j \end{aligned}$$

for $0 \leq j \leq a$. Let $h_j(t) := (a - 2j)t^{-1/2}$ and $e^j := e_0^{a-j} \otimes e_1^j$, then we can clean up the previous lines as

$$[\nabla_{FG}]e^j = h_j(t)e^j \tag{3.0.2}$$

for $j = 0, 1, \dots, a$. So via the definition of the standard irregularity we define

$$\begin{aligned} irr_{Sym^a(St)}(\nabla_{FG}) &:= \sum_{j=0}^{a} \operatorname{ord}(h_j(t)) \cdot \dim(V_j) \\ &= \begin{cases} \frac{a+1}{2} & \text{for } a \not\equiv 0 \mod 2 \\ \\ \frac{a}{2} & \text{for } a \equiv 0 \mod 2 \end{cases} \\ &= \left\lfloor \frac{a+1}{2} \right\rfloor . \end{aligned}$$

We can repeat the same argument except with diagonal type connection $[\nabla(a_1)] = t^{-1} diag(a_1, -a_1)$ instead of ∇_{FG} to arrive at

$$irr_{Sym^a(St)}(\nabla(a_1)) = \begin{cases} a+1 & \text{for } a \not\equiv 0 \mod 2 \\ \\ a & \text{for } a \equiv 0 \mod 2 \end{cases} .$$

Writing in our formalism, where $V_\lambda = Sym^a(St)$ with highest weight a, we have shown

$$irr_\lambda(\nabla_{FG}) = \frac{1}{2}\chi(V_\lambda). \tag{3.0.3}$$

Let us describe $\ell(\nabla(a_1))$. Since $a_1 \neq 0$ we can define $\ell(\nabla(a_1))$ as the line (in $\mathbb{R}$) through 0 and a_1. Since we are in one dimension $\ell(a_1) \cap \mathcal{P}_\lambda = \mathcal{P}_\lambda$, therefore we cannot cancel without trivializing so we get $\mathbf{c}(\nabla(a_1)) = 0$. Therefore, we have

$$irr_\lambda(\nabla(a_1)) = \chi(V_\lambda). \tag{3.0.4}$$

Moreover, we see that for all $V_\lambda \in Irr(\mathfrak{sl}_2)$, we have the strict inequality

$$irr_\lambda(\nabla_{FG}) < irr_\lambda(\nabla(a_1)). \tag{3.0.5}$$

This confirms the minimal irregularity conjecture for formal flat $\mathfrak{sl}_2(\!(t)\!)$-connections, and we have shown the following.

Lemma 1.1. *Let $k \in \mathbb{Z}_{>0}$, for $V_\lambda \in Irr(\mathfrak{sl}_2)$ with dominant weight $\lambda = k\varepsilon_1$, and $a_1 \in \mathbb{R}^\times$ then we have the following:*

i.) $irr_\lambda \nabla(a_1) = \chi(V_\lambda)$,

ii.) $irr_\lambda \nabla_{FG} = \frac{1}{2}\chi(V_\lambda)$,

where ∇_{FG} is the slope 1/2 Frenkel-Gross connection, $\nabla(a_1)$ is the slope 1 minimal diagonalizable connection, and $\chi(V_\lambda) = \sum_{\mu \neq 0} dim V_\lambda^\mu$.

Corollary 1.1. *For SL_2, the Frenkel-Gross connection is the unique minimal irregular singular formal flat connection, by which we mean*

$$irr_\lambda \nabla_{FG} = \min_{\nabla \in \mathcal{L}} \{irr_\lambda \nabla\}$$

for all $V_\lambda \in Irr(\mathfrak{sl}_2)$.

For SL_n and $n \geq 3$ we will see that the λ-irregularity is, in some sense, not as algebraic, as it is for SL_2. By this we mean, as we see in the above result, $irr_\lambda \nabla$ can be given entirely in terms of data coming from the representation V_λ and the Lie algebra via its Coxeter number.

We continue with the $\mathfrak{sl}_3$ case.

Chapter 4. $\mathfrak{sl}_3$

In this case, as was the case for $\mathfrak{sl}_2$, for the relevant classes of connections, namely those with slopes 1, 1/2 and 1/3, irr_λ takes the form

$$irr_\lambda(\nabla) = s(\nabla)(\chi(V_\lambda) - \mathbf{c}(\nabla)).$$

Let $V_\lambda \in Irr(\mathfrak{sl}_3)$, then the highest weight will be of the form $\lambda = (k_1 + k_2)\varepsilon_1 + k_2\varepsilon_2 = k_1\varepsilon_1 - k_2\varepsilon_3$ and the weight polytopes are in two dimensional weight lattices $\mathcal{P}_\lambda \subset \mathbb{Z}^2 \subset \mathbb{R}^2$. The multiplicity of strings of weights are well known in this case, for instance see [9] . We recall the structure. For an arbitrary weight polytope $\mathcal{P}_\lambda$ the strings of weights are arranged in hexagons and triangles with their multiplicities potentially decreasing from outer to inner strings. There are special cases. If $k_2 = 0$ then the strings of weights will be arranged as a sequence of only regular triangles and the multiplicity of the strings are always one, in other words $K_{\lambda\mu} = 1$ for all $\mu \preceq \lambda = k_1\varepsilon_1$ with $k_1 \geq 1$. If $k_2 = k_1$ then the polytope will consist of a sequence of only regular hexagons and the multiplicities of the strings will increase by one from outer to inner, with the outer most hexagonal string having multiplicity 1. In general, the strings in a weight polytope will be arranged in a sequence of outer non-regular hexagons in which the multiplicities decrease by one from outer to inner until you reach an inner sequence of regular triangles where the multiplicities will be constant. A useful numerical property of $\chi(V_\lambda)$ is that it is dual invariant. This implies dual invariance of irr_λ, therefore without loss of generality we can consider only the irreducibles with $\lambda = k_1\varepsilon_1 - k_2\varepsilon_3$ and $k_1 \geq k_2$ in the cone of dominant weights in the weight lattice. We note now an important difference in the $\mathfrak{sl}_3$ case, which did not occur in rank one, but will persist in higher rank. In this case we must consider the lower rank toral connections along with the toral connections of rank equal to the rank of the Lie algebra. Let us first consider the irregular toral connections with fractional slope less than one. We denote the three parameter Frenkel-Gross connection with slope 1/3 as $\nabla_{1/3}$ and the two parameter slope 1/2 connection as $\nabla_{1/2}$. As $\mathfrak{sl}_3((t))$-connections we have the following connection matrices

$$[\nabla_{1/3}] = \begin{pmatrix} 0 & 0 & a_1 t^{-1} \\ a_2 & 0 & 0 \\ 0 & a_3 & 0 \end{pmatrix}$$

and

$$diag([\nabla_{1/2}], 0) = \begin{pmatrix} 0 & a_1 t^{-1} & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For the slope one connections we have the two parameter connection $\nabla(a_1, a_2)$ and one parameter connection $\nabla(a_1)$, considered as $\mathfrak{sl}_3((t))$-connections, with connection matrices

$$[\nabla(a_1, a_2)] = t^{-1} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & -a_1 - a_2 \end{pmatrix}$$

and

$$diag([\nabla(a_1)], 0) = t^{-1} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & -a_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As before, to arrive at explicit formulae for irr_λ, we describe the $\ell(\nabla)$ subspaces of the polytopes to determine $\mathbf{c}(\nabla)$. As we will see, in rank two, there can occur non-trivial cancellation of parameters.

Let $V_\lambda \in Irr(\mathfrak{sl}_3)$, then $\mathcal{P}_\lambda \subset \mathbb{Z}^2 \subset \mathbb{R}^2$. To address the minimal irregularity conjecture for $\mathfrak{sl}_3$, it follows to systematically understand where the function $\mathbf{c}(\nabla)$ is maximized for a given irreducible representation. Our parameter space is then $\mathbb{R}^2$. Choosing a point $(a_1, a_2) \neq \mathbf{0} \in \mathbb{R}^2$ we let $\ell(\nabla(a_1, a_2))$ be the line through $\mathbf{0}$ and the point (a_1, a_2). For the two parameter slope one irregular toral connection $\nabla(a_1, a_2)$ we note that $\mathbf{p} = rk(\mathfrak{sl}_3)$. For the one parameter slope one connection $\nabla(a_1)$ we have that $\mathbf{p} < rk(\mathfrak{sl}_3)$ which should indicate that $\mathbf{c}(\nabla(a_1))$ will not always be 0. Indeed, observe that for $\nabla(a_1)$ there is a cancellation that will automatically occur in the weight polytope $\mathcal{P}_\lambda$. This happens because its $\mathfrak{sl}_3$ matrix has a zero in the third diagonal entry, therefore it follows that

$$\mathbf{c}(\nabla(a_1)) = \sum_{\mu \neq 0 \in \ell(\varepsilon_3) \cap \mathcal{P}_\lambda} K_{\lambda\mu} \tag{4.0.1}$$

for all $a_1 \neq 0$ and V_λ. For $\mathbf{c}(\nabla(a_1, a_2))$, we first note that if a_1 and a_2 are $\mathbb{Q}$-linearly independent

$$\mathbf{c}(\nabla(a_1, a_2)) = 0.$$

for all $\mathcal{P}_\lambda$. A further analysis of the weight polytopes $\mathcal{P}_\lambda$ for $\lambda = k_1\varepsilon_1 - k_2\varepsilon_3$ and $k_1 \geq k_2$ gives us that

$$\max(\mathbf{c}(\nabla(a_1, a_2))) = \begin{cases} \sum_{\mu \neq 0 \in \ell(\varepsilon_1) \cap \mathcal{P}_\lambda} K_{\lambda\mu} & \text{if } k_1 - k_2 \not\equiv 0 \mod 3 \\ \sum_{\mu \neq 0 \in \ell(\varepsilon_1 - \varepsilon_3) \cap \mathcal{P}_\lambda} K_{\lambda\mu} & \text{if } k_1 - k_2 \equiv 0 \mod 3 \end{cases}. \tag{4.0.2}$$

In the above formulas we have written $\ell(\varepsilon_i)$ to be the line through the weight ε_i and the zero weight i.e. the origin. Let us denote by $r(\varepsilon_i)$ the ray extending from the zero weight through the weight ε_i. Since $\ell(\varepsilon_i) = -r(\varepsilon_i) \cup r(\varepsilon_i)$,

$$\sum_{\mu \neq 0 \in \ell(\varepsilon_i) \cap \mathcal{P}_\lambda} K_{\lambda\mu} = \sum_{\mu \neq 0 \in -r(\varepsilon_i) \cap \mathcal{P}_\lambda} K_{\lambda\mu} + \sum_{\mu \neq 0 \in r(\varepsilon_i) \cap \mathcal{P}_\lambda} K_{\lambda\mu}. \tag{4.0.3}$$

For certain irreducible representations, the relevant lines will be symmetric about the origin. Let us turn our attention to the irregular formal flat connections with slope less than one.

The parameter restriction condition for $\nabla_{1/3}$ is the same as before namely, $a_1, a_2, a_3 \neq 0$. Since $\mathbf{p}(\nabla_{1/3}) = 1$, we have that $\mathbf{c}(\nabla_{1/3}) = 0$, and we arrive at the formula

$$irr_\lambda \nabla_{1/3} = \frac{\chi(V_\lambda)}{3}.$$

Considering $\nabla_{1/2}$ as a $\mathfrak{sl}_3(\!(t)\!)$-connection we see that while $\mathbf{p}(\nabla_{1/2}) = 1$, $\mathbf{c}(\nabla_{1/2})$ will not always be zero. Indeed, by simple observation we see that for an arbitrary V_λ, $\chi(V_\lambda)$ will not always be an even number so, by the integrality of the irregularity, $\mathbf{c}(\nabla_{1/2})$ will surely not always be zero. By the same reasoning employed for the one parameter connection $\nabla(a_1)$ of slope one, we see that there is a zero in the last diagonal entry of $\nabla_{1/2}$ considered as an $\mathfrak{sl}_3(\!(t)\!)$-connection, therefore we deduce that

$$\mathbf{c}(\nabla_{1/2}) = \sum_{\mu \neq 0 \in \ell(\varepsilon_3) \cap \mathcal{P}_\lambda} K_{\lambda\mu}. \tag{4.0.4}$$

As noted above, there is an induced dual invariance of irr_λ so that the relevant alcove we must consider is made up of the points in between the boundaries of dominant weights of the form $\lambda = k_1\varepsilon_1$ and $\lambda = k_2(\varepsilon_1 - \varepsilon_3)$ with $k_1, k_2 \geq 1$. Let us examine our constructions for the irreducible representations with highest weights on the boundaries of our alcove. There is a well-known formula for the dimension of the zero weight space, for instance see

[16], we have

$$dimV_\lambda^0 = \begin{cases} 1 + min\{k_1, k_2\} & \text{if } k_1 - k_2 \equiv 0 \mod 3 \\ \\ 0 & \text{if } k_1 - k_2 \not\equiv 0 \mod 3 \end{cases}. \tag{4.0.5}$$

It follows that for all V_λ that when a_1 and a_2 are $\mathbb{Q}$-linearly independent

$$\sum_{\mu \neq 0 \in \ell \cap \mathcal{P}_\lambda} dimV_\lambda^\mu = 0. \tag{4.0.6}$$

The first examples of such irreducible representations are the so-called fundamental representations. They are the standard representation, with highest weight $\lambda = \varepsilon_1$, and the adjoint representation, with highest weight $\lambda = \varepsilon_1 - \varepsilon_3$.

Let $std \cong \mathbb{C}^3 \in Irr(\mathfrak{sl}_3)$ be the standard representation. Firstly, we note that

$$\chi(std) = dim(std) - dim(std^0) = 3.$$

So, for $\nabla_{1/3}$ we then have

$$irr_{std}(\nabla_{1/3}) = \frac{3}{3} = 1.$$

For $\nabla_{1/2}$, in the standard representation $K_{\lambda\mu} = 1$ for all $\mu \preceq \lambda$ therefore

$$\mathbf{c}(\nabla_{1/2}) = \sum_{\mu \neq 0 \in \ell(\varepsilon_3) \cap \mathcal{P}_\lambda} K_{\lambda\mu} = \sum_{\mu \neq 0 \in \ell(\varepsilon_3) \cap \mathcal{P}_\lambda} 1 = 1,$$

thus

$$irr_{std}(\nabla_{1/2}) = \frac{3-1}{2} = 1.$$

We proceed with the slope one connections $\nabla(a_1, a_2)$. For $\ell(\nabla(a_1, a_2))$ we choose a point in the parameter space and take $\ell(\nabla(a_1, a_2))$ to be the line from that point to $\mathbf{0} \in \mathbb{R}^2$ which will correspond to the connection $\nabla(a_1, a_2)$ with parameters $(a_1, a_2) \neq 0 \in \mathbb{R}^2$. For $\nabla(a_1, a_2)$ we find that the domain of $irr_{St}\nabla(a_1, a_2)$ is broken into six sectors separated by three lines. Along said lines the function $irr_{St} : \nabla(a_1, a_2) \to \mathbb{Z}_{\geq 0}$ is minimized. This happens since if you choose (a_1, a_2) lying on one of these lines you can cancel the parameter and the resulting connection will still be irregular singular. Therefore this produces three kinds of irregular singular connections with slope one, corresponding to connections with coordinate lying on one of the three lines in the domain of $irr_{St}\nabla(a_1, a_2)$, they are of the form $\nabla(a_1, 0), \nabla(0, a_2)$ and $\nabla(a_1, -a_1)$.

We compute the irregularities of $\nabla_{1/i}$ for $i = 2, 3$, $\nabla(a_1, a_2)$ and $\nabla(a_1)$ in the standard representation $std \in Irr(\mathfrak{sl}_3)$.For the one parameter connection $\nabla(a_1)$ we have

$$irr_{std}(\nabla(a_1)) = 3 - \sum_{\mu \neq 0 \in \ell(\varepsilon_3) \cap \mathcal{P}_\lambda} 1 = 3 - 1 = 2.$$

For the two parameter connection $\nabla(a_1, a_2)$, since $k_1 = 1 \not\equiv 0 \mod 3$, we have that

$$\max(\mathbf{c}(\nabla(a_1, a_2))) = \sum_{\mu \neq 0 \in \ell(\varepsilon_1) \cap \mathcal{P}_\lambda} 1 = 1.$$

Therefore

$$\min\{irr_{std}(\nabla(a_1, a_2))\} = 3 - 1 = 2.$$

In conclusion we see that for the standard representation, the minimal irregularity problem is solved by the two fractional slope connections. In other words

$$irr_{std}(\nabla_{1/i}) = \min_{\nabla \in \mathcal{L}}\{irr_{std}(\nabla)\} \tag{4.0.7}$$

for $i = 2$ and 3.

We now consider the eight dimensional adjoint representation $ad \in Irr(\mathfrak{sl}_3)$ with highest weight $\lambda = \varepsilon_1 - \varepsilon_3$. For the slope one diagonalizable type connection $\nabla(a_1, a_2)$ the domain of $irr_{Ad}\nabla(a_1, a_2)$ again breaks into six sectors separated by three lines. It follows that

$$irr_{Ad}\nabla(a_1,-a_1) = irr_{Ad}\nabla(a_1,-2a_1) = irr_{Ad}\nabla(a_1,-\frac{1}{2}a_1) = 4$$

and $irr_{Ad}\nabla(a_1,a_2) = 6$ otherwise. In this case we have

$$\chi(ad) = dim(ad) - dim(ad^0) = 8 - 2 = 6,$$

and, as for the standard representation, $K_{\lambda\mu} = 1$ for all $\mu \preceq \lambda$. Observe that in the adjoint representation, $\ell(\varepsilon_3) \cap \mathcal{P}_\lambda = \varnothing$, therefore

$$\sum_{\mu \neq 0 \in \ell(\varepsilon_3)\cap\mathcal{P}_\lambda} K_{\lambda\mu} = 0. \tag{4.0.8}$$

It follows that

$$irr_{ad}(\nabla_{1/3}) = \frac{6}{3} = 2.$$

This is in agreement with the results of [KS19]. For $\nabla_{1/2}$ we see that via (4.6), $\mathbf{c}(\nabla_{1/2}) = 0$, thus

$$irr_{ad}(\nabla_{1/2}) = \frac{6}{2} = 3.$$

Note, in contrast to the case for the standard representation we have

$$irr_{ad}(\nabla_{1/3}) < irr_{ad}(\nabla_{1/2}). \tag{4.0.9}$$

For the one parameter connection $\nabla(a_1)$, since $\mathbf{c}(\nabla(a_1)) = 0$,

$$irr_{ad}(\nabla(a_1)) = 6.$$

Lastly, for the two parameter, slope one connection $\nabla(a_1,a_2)$, since $k_1 - k_2 = k_1 - k_1 = 0 \equiv 0 \mod 3$ and by the above discussion

$$\max(\mathbf{c}(\nabla(a_1,a_2))) = \sum_{\mu\neq 0\in\ell(\varepsilon_1-\varepsilon_3)\cap\mathcal{P}_\lambda} 1 = 2.$$

Thus

$$\min\{irr_{ad}(\nabla(a_1, a_2))\} = 6 - 2 = 4.$$

We conclude that

$$irr_{ad}(\nabla_{1/3}) = \min_{\nabla \in \mathcal{L}}\{irr_{ad}(\nabla)\}, \tag{4.0.10}$$

and we note that in the adjoint representation, the minimal irregularity problem is solved by the unique minimal slope connection $\nabla_{1/3}$.

Let us consider the six dimensional irreducible representation $Sym^2(std) \in Irr(\mathfrak{sl}_3)$ with highest weight $2\varepsilon_1$. In this case it follows that

$$\begin{aligned} irr_{2\varepsilon_1}(\nabla_{1/3}) &= 6/3 \\ &= 2 \\ &= 4/2 = irr_{2\varepsilon_1}(\nabla_{1/2}) \end{aligned}$$

and

$$irr_{2\varepsilon_1}(\nabla(a_1)) = 4 = \min\{irr_{2\varepsilon_1}(\nabla(a_1, a_2))\}.$$

Therefore as with the standard representation

$$irr_{2\varepsilon_1}(\nabla_{1/i}) = \min_{\nabla \in \mathcal{L}}\{irr_{2\varepsilon_1}(\nabla)\} \tag{4.0.11}$$

for $i = 2$ and 3.

Let us now consider the irreducible representations along the boundaries of our reduced alcove in the cone of dominant weights P^{++}.

We first consider the irreducible representations with highest weights $k_1\varepsilon_1$ with $k_1 \in \mathbb{Z}_{\geq 1}$. These correspond to the symmetric powers of the standard representation. The structure of the weight polytope $\mathcal{P}_{k_1\varepsilon_1}$ particularly nice. As noted above, in this case, the strings of weights are arranged purely as concentric regular triangles. Another useful fact in this case is that the multiplicity of all the weights are one, i.e. $K_{\lambda\mu} = 1$ throughout the polytope. Therefore, our approach essentially turns into a counting problem. For $k_1 \geq 1$

$$\begin{aligned}\chi(V_{k_1\varepsilon_1}) &= dim(V_{k_1\varepsilon_1}) - dim(V^0_{k_1\varepsilon_1}) \\ &= \frac{(k_1+1)(k_1+2)}{2} - \begin{cases} 1 & \text{if } k_1 \equiv 0 \mod 3 \\ 0 & \text{if } k_1 \not\equiv 0 \mod 3 \end{cases}.\end{aligned}$$

As we have seen throughout, the relevant data coming from the connection is the parameter cancellation number. Note that in general

$$\mathbf{c}(\nabla_{1/2}) = \mathbf{c}(\nabla(a_1)) = \sum_{\mu\neq 0\in\ell(\varepsilon_3)\cap\mathcal{P}_\lambda} K_{\lambda\mu}.$$

Another useful fact is that in $\mathcal{P}_{k_1\varepsilon_1}$

$$\sum_{\mu\neq 0\in\ell(\varepsilon_3)\cap\mathcal{P}_\lambda} K_{\lambda\mu} = \sum_{\mu\neq 0\in\ell(\varepsilon_1)\cap\mathcal{P}_\lambda} K_{\lambda\mu}. \tag{4.0.12}$$

Therefore by (3.10) and (3.2), to compute all the parameter cancellation numbers, it follows to evaluate the sums over the two intersections, $\ell(\varepsilon_1) \cap \mathcal{P}_\lambda$ and $\ell(\varepsilon_1 - \varepsilon_3) \cap \mathcal{P}_\lambda$. Since $K_{\lambda\mu} = 1$ we have

$$\begin{aligned}\sum_{\mu\neq 0\in\ell(\varepsilon_1)\cap\mathcal{P}_\lambda} K_{\lambda\mu} &= \sum_{\mu\neq 0\in\ell(\varepsilon_1)\cap\mathcal{P}_\lambda} 1 \\ &= \sum_{\mu\neq 0\in -r(\varepsilon_1)\cap\mathcal{P}_\lambda} 1 + \sum_{\mu\neq 0\in r(\varepsilon_1)\cap\mathcal{P}_\lambda} 1.\end{aligned}$$

We observe that $\sum_{\mu\neq 0\in r(\varepsilon_1)\cap\mathcal{P}_\lambda} 1$ counts the number of triangles, therefore

$$\sum_{\mu\neq 0\in r(\varepsilon_1)\cap\mathcal{P}_\lambda} 1 = \left\lfloor \frac{k_1+2}{3} \right\rfloor.$$

For $\sum_{\mu\neq 0\in -r(\varepsilon_1)\cap\mathcal{P}_\lambda} 1$, by computational inspection we find the following curious identity

$$\sum_{\mu\neq 0\in -r(\varepsilon_1)\cap\mathcal{P}_\lambda} 1 = \left\lfloor \frac{k_1}{2} \right\rfloor - \left\lfloor \frac{k_1}{3} \right\rfloor. \tag{4.0.13}$$

Putting this together arrive at the following formula

$$\sum_{\mu\neq 0\in \ell(\varepsilon_1)\cap\mathcal{P}_\lambda} 1 = \left\lfloor \frac{k_1}{2} \right\rfloor - \left\lfloor \frac{k_1}{3} \right\rfloor + \left\lfloor \frac{k_1+2}{3} \right\rfloor. \tag{4.0.14}$$

We note that it follows to sum over $\mu \neq 0 \in \ell(\varepsilon_1 - \varepsilon_3) \cap \mathcal{P}_\lambda$ only when $k_1 \equiv 0 \mod 3$. And, in such cases the line $\ell(\varepsilon_1 - \varepsilon_3)$ intersects the polytope the same number of times in the two directions, therefore for $k_1 \equiv 0 \mod 3$

$$\sum_{\mu\neq 0\in -r(\varepsilon_1-\varepsilon_3)\cap\mathcal{P}_\lambda} 1 = \sum_{\mu\neq 0\in r(\varepsilon_1-\varepsilon)\cap\mathcal{P}_\lambda} 1$$

so that

$$\sum_{\mu\neq 0\in \ell(\varepsilon_1-\varepsilon_3)\cap\mathcal{P}_\lambda} 1 = 2 \sum_{\mu\neq 0\in r(\varepsilon_1-\varepsilon_3)\cap\mathcal{P}_\lambda} 1.$$

We observe that for $k_1 \equiv 0 \mod 3$ the sum over the intersection $r(\varepsilon_1 - \varepsilon_3) \cap \mathcal{P}_\lambda$ again just counts the number of triangles, therefore

$$\begin{aligned}\sum_{\mu\neq 0\in \ell(\varepsilon_1-\varepsilon_3)\cap\mathcal{P}_\lambda} 1 &= 2 \sum_{\mu\neq 0\in r(\varepsilon_1-\varepsilon_3)\cap\mathcal{P}_\lambda} 1 \\ &= 2 \cdot \frac{k_1}{3}.\end{aligned}$$

With these formulae we can now give explicit formulas for $irr_{k_1\varepsilon_1}$. In other words, have shown the following lemma.

Lemma 1.2. *Let $k_1 \in \mathbb{Z}_{\geq 1}$, then for $V_{k_1\varepsilon_1} \in Irr(\mathfrak{sl}_3)$ we have*

$$irr_{k_1\varepsilon_1}(\nabla_{1/3}) = \frac{\chi(V_{k_1\varepsilon_1})}{3}, \tag{4.0.15}$$

$$irr_{k_1\varepsilon_1}(\nabla_{1/2}) = \frac{1}{2}\left[\chi(V_{k_1\varepsilon_1}) - \left[\left\lfloor\frac{k_1}{2}\right\rfloor - \left\lfloor\frac{k_1}{3}\right\rfloor + \left\lfloor\frac{k_1+2}{3}\right\rfloor\right]\right], \tag{4.0.16}$$

$$irr_{k_1\varepsilon_1}(\nabla(a_1)) = \chi(V_{k_1\varepsilon_1}) - \left[\left\lfloor\frac{k_1}{2}\right\rfloor - \left\lfloor\frac{k_1}{3}\right\rfloor + \left\lfloor\frac{k_1+2}{3}\right\rfloor\right], \tag{4.0.17}$$

$$\min\{irr_{k_1\varepsilon_1}(\nabla(a_1,a_2))\} = \chi(V_{k_1\varepsilon_1}) - \begin{cases} 2\cdot\frac{k_1}{3} & \textit{if } k_1 \equiv 0 \mod 3 \\ \\ \lfloor\frac{k_1}{2}\rfloor - \lfloor\frac{k_1}{3}\rfloor + \lfloor\frac{k_1+2}{3}\rfloor & \textit{if } k_1 \not\equiv 0 \mod 3 \end{cases}, \tag{4.0.18}$$

where $\chi(V_{k_1\varepsilon_1}) = \frac{(k_1+1)(k_1+2)}{2} - \begin{cases} 1 & \textit{if } k_1 \equiv 0 \mod 3 \\ \\ 0 & \textit{if } k_1 \not\equiv 0 \mod 3 \end{cases}.$

We proceed to derive the analogous formulae for the irreducible representations on the other boundary component. Without loss of generality, these irreducible representations will have highest weights $\lambda = k_2(\varepsilon_1 - \varepsilon_3)$ with $k_2 \in \mathbb{Z}_{\geq 1}$. For such irreducible representations, the weight polytopes will no longer have constant Kostka numbers. However, the strings of weights will be arranged in sequences of concentric regular hexagons. The strings of weights will have multiplicities, from inner to outer, $k_2 - i$ with $i = 0, 1, \ldots, k_2 - 1$. It follows that for $k_2 \neq 0$,

$$\begin{aligned}\chi(V_{k_2(\varepsilon_1-\varepsilon_3)}) &= (k_2+1)^3 - (k_2+1) \\ &= (k_2+1)((k_2+1)^2 - k_2 - 1) \\ &= (k_2+1)(k_2^2 + k_2) \\ &= k_2(k_2+1)^2.\end{aligned}$$

Another simplicity here is that all the intersections $\ell \cap \mathcal{P}_\lambda$ enjoy the property

$$\sum_{\mu \neq 0 \in -r \cap \mathcal{P}_\lambda} K_{\lambda\mu} = \sum_{\mu \neq 0 \in r \cap \mathcal{P}_\lambda} K_{\lambda\mu},$$

so that it will always be the case that

$$\sum_{\mu \neq 0 \in \ell \cap \mathcal{P}_\lambda} K_{\lambda\mu} = 2 \sum_{\mu \neq 0 \in r \cap \mathcal{P}_\lambda} K_{\lambda\mu}. \tag{4.0.19}$$

In determining the parameter cancellation numbers we find that, as before, we need only consider the sums over the two intersections $\ell(\varepsilon_3) \cap \mathcal{P}_\lambda$ and $\ell(\varepsilon_1 - \varepsilon_3) \cap \mathcal{P}_\lambda$. We find that

$$\begin{aligned}
\sum_{\mu \neq 0 \in \ell(\varepsilon_1 - \varepsilon_3) \cap \mathcal{P}_\lambda} K_{\lambda\mu} &= 2 \sum_{\mu \neq 0 \in r(\varepsilon_1 - \varepsilon_3) \cap \mathcal{P}_\lambda} K_{\lambda\mu} \\
&= 2 \sum_{i=0}^{k_2 - 1} (k_2 - i) \\
&= 2(k_2 + \sum_{i=1}^{k_2-1} (k_2 - i)) \\
&= 2(k_2 + \sum_{i=1}^{k_2-1} k_2 + \sum_{i=1}^{k_2-1} i \\
&= 2 \left(k_2 + k_2(k_2 - 1) + \frac{(k_2 - 1)k_2}{2} \right) \\
&= k_2(3(k_2 - 1) + 2).
\end{aligned}$$

For $\ell(\varepsilon_3) \cap \mathcal{P}_\lambda$ we find that

$$\begin{aligned}
\sum_{\mu \neq 0 \in \ell(\varepsilon_3) \cap \mathcal{P}_\lambda} K_{\lambda\mu} &= 2 \sum_{\mu \neq 0 \in r(\varepsilon_3) \cap \mathcal{P}_\lambda} K_{\lambda\mu} \\
&= 2 \sum_{\substack{i=0 \\ i \text{ odd}}}^{k_2 - 1} (k_2 - i).
\end{aligned}$$

For the two parameter slope one connection $\nabla(a_1, a_2)$ we find that

$$\max(\mathbf{c}(\nabla(a_1, a_2))) = \sum_{\mu \neq 0 \in \ell(\varepsilon_1 - \varepsilon_3) \cap \mathcal{P}_\lambda} K_{\lambda\mu}. \tag{4.0.20}$$

Therefore we have shown the following identities

Lemma 1.3. *Let $k_2 \in \mathbb{Z}_{\geq 1}$, then for $V_{k_2\varepsilon_1} \in Irr(\mathfrak{sl}_3)$ we have*

$$irr_{k_2(\varepsilon_1-\varepsilon_3)}(\nabla_{1/3}) = \frac{\chi(V_{k_2(\varepsilon_1-\varepsilon_3)})}{3}, \tag{4.0.21}$$

$$irr_{k_2(\varepsilon_1-\varepsilon_3)}(\nabla_{1/2}) = \frac{1}{2}\left[\chi(V_{k_2(\varepsilon_1-\varepsilon_3)}) - 2\sum_{\substack{i=0 \\ i \text{ odd}}}^{k_2-1}(k_2-i)\right], \tag{4.0.22}$$

$$irr_{k_2(\varepsilon_1-\varepsilon_3)}(\nabla(a_1)) = \chi(V_{k_2(\varepsilon_1-\varepsilon_3)}) - 2\sum_{\substack{i=0 \\ i \text{ odd}}}^{k_2-1}(k_2-i), \tag{4.0.23}$$

$$\min\{irr_{k_2(\varepsilon_1-\varepsilon_3)}(\nabla(a_1,a_2))\} = \chi(V_{k_2(\varepsilon_1-\varepsilon_3)}) - k_2(3(k_2-1)+2), \tag{4.0.24}$$

with $\chi(V_{k_2(\varepsilon_1-\varepsilon_3)}) = k_2(k_2+1)^2$.

For the remaining cases, we consider $V_\lambda \in Irr(\mathfrak{sl}_3)$ with highest weight $\lambda = k_1\varepsilon_1 - k_2\varepsilon_3$ and $k_1 > k_2 \neq 0$. We note that the corresponding formulas in this case will generalize the previous two lemmas. Thus, we will use the last resulting lemma to prove the minimal irregularity conjecture for $\mathfrak{sl}_3$.

In this case the strings of weights in the corresponding polytope will be arranged in an outer sequence of non-regular hexagons and an inner sequence of regular triangles. For the outer sequence of hexagons the Kostka numbers will again increase by one as we move from outer to inner until we reach the sequence of triangles, where they will remain constant. Let us denote by $\mathcal{T}$ the data coming from the sequence of triangles and by $\mathcal{H}$ the data coming from the sequence of hexagons. As we will see, in this case, the intersection sums will break into a $\mathcal{T}$-sum and an $\mathcal{H}$-sum. We note that since $k_1 > k_2$, $k_1 - k_2 > 0$. As expected, we find that the relevant intersections to sum over will be $\ell(\varepsilon_1) \cap \mathcal{P}_\lambda$ and $\ell(\varepsilon_1 - \varepsilon_3) \cap \mathcal{P}_\lambda$.

We determine the Kostka numbers for the strings of weights. On the sequence of hexagons, from inner to outer, they will be, as before, $k_2 - i$ for $i = 0, 1, \dots, k_2 - 1$. For the sequence of triangles, where they are constant, they will be $k_2 + 1$. Let us consider the intersection $\ell(\varepsilon_3) \cap \mathcal{P}_\lambda$. We first compute the $\mathcal{H}$ and $\mathcal{T}$ data individually. The $\mathcal{H}$-data will be

$$\begin{aligned}\sum_{\mu \neq 0 \in \ell(\varepsilon_3) \cap \mathcal{P}_\lambda} K_{\lambda\mu} &= \sum_{\mu \neq 0 \in -r(\varepsilon_3) \cap \mathcal{P}_\lambda} K_{\lambda\mu} + \sum_{\mu \neq 0 \in r(\varepsilon_3) \cap \mathcal{P}_\lambda} K_{\lambda\mu} \\ &= \sum_{\substack{i=0 \\ i \text{ even}}}^{k_2-1} (k_2 - i) + \sum_{\substack{i=0 \\ i \text{ odd}}}^{k_2-1} (k_2 - i) \\ &= \sum_{i=0}^{k_2-1} (k_2 - i) \\ &= \frac{k_2(3(k_2-1)+2)}{2}.\end{aligned}$$

For the $\mathcal{T}$-data we can modify the identity (3.12) by using $k_1 - k_2$ rather than k_1 and instead of 1 for the Kostka number, we have $k_2 + 1$. Therefore, we find that the $\mathcal{T}$-data is

$$\begin{aligned}\sum_{\mu \neq 0 \in \ell(\varepsilon_1) \cap \mathcal{P}_\lambda} (k_2+1) &= (k_2+1) \sum_{\mu \neq 0 \in \ell(\varepsilon_1) \cap \mathcal{P}_\lambda} 1 \\ &= (k_2+1)\left(\left\lfloor \frac{k_1-k_2}{2} \right\rfloor - \left\lfloor \frac{k_1-k_2}{3} \right\rfloor + \left\lfloor \frac{k_1-k_2+2}{3} \right\rfloor\right).\end{aligned}$$

Putting this together we conclude

$$\begin{aligned}\sum_{\mu \neq 0 \in \ell(\varepsilon_3) \cap \mathcal{P}_\lambda} K_{\lambda\mu} =& \frac{k_2(3(k_2-1)+2)}{2} \\ &+ (k_2+1)\left(\left\lfloor \frac{k_1-k_2}{2} \right\rfloor - \left\lfloor \frac{k_1-k_2}{3} \right\rfloor + \left\lfloor \frac{k_1-k_2+2}{3} \right\rfloor\right).\end{aligned} \tag{4.0.25}$$

We proceed with the intersection $\ell(\varepsilon_1 - \varepsilon_3) \cap \mathcal{P}_\lambda$. As usual, we only consider this case when $\lambda = k_1\varepsilon_1 - k_2\varepsilon_3$ and $k_1 - k_2 \equiv 0 \mod 3$. As we've seen throughout, the intersection sum is the same for r and $-r$. For the $\mathcal{H}$-data we have

$$\sum_{\mu\neq 0\in\ell(\varepsilon_1-\varepsilon_3)\cap\mathcal{P}_\lambda} K_{\lambda\mu} = 2\sum_{i=0}^{k_2-1}(k_2-i)$$

$$= k_2(3(k_2-1)+2).$$

And for the $\mathcal{T}$-data we can deduce

$$\sum_{\mu\neq 0\in\ell(\varepsilon_1-\varepsilon_3)\cap\mathcal{P}_\lambda} K_{\lambda\mu} = \frac{2(k_2+1)(k_1-k_2)}{3}.$$

Combining, we have

$$\sum_{\mu\neq 0\in\ell(\varepsilon_1-\varepsilon_3)\cap\mathcal{P}_\lambda} K_{\lambda\mu} = k_2(3(k_2-1)+2)+\frac{2(k_2+1)(k_1-k_2)}{3}. \tag{4.0.26}$$

Once again, we compose these results into explicit formulae for $irr_{k_1\varepsilon_1-k_2\varepsilon_3}$. And, we have shown the following lemma.

Lemma 1.4. *Let $V_\lambda \in Irr(\mathfrak{sl}_3)$ with highest weight $\lambda = k_1\varepsilon_1 - k_2\varepsilon_3$ and $k_1 > k_2 \neq 0$. It follows that*

$$irr_{k_1\varepsilon_1-k_2\varepsilon_3}(\nabla_{1/3}) = \frac{\chi(V_{k_1\varepsilon_1-k_2\varepsilon_3})}{3} \tag{4.0.27}$$

$$irr_{k_1\varepsilon_1-k_2\varepsilon_3}(\nabla_{1/2}) = \frac{1}{2}\Big[\chi(V_{k_1\varepsilon_1-k_2\varepsilon_3}) - (\textit{R.H.S. of (4.25)})\Big] \tag{4.0.28}$$

$$irr_{k_1\varepsilon_1-k_2\varepsilon_3}(\nabla(a_1)) = \chi(V_{k_1\varepsilon_1-k_2\varepsilon_3}) - (\textit{R.H.S. of (4.25)}) \tag{4.0.29}$$

$$\min\{irr_{k_1\varepsilon_1-k_2\varepsilon_3}(\nabla(a_1,a_2))\} = \chi(V_{k_1\varepsilon_1-k_2\varepsilon_3}) \tag{4.0.30}$$

$$-\begin{cases} k_2(3(k_2-1)+2)+\frac{2(k_2+1)(k_1-k_2)}{3} & \textit{if } k_1-k_2\equiv 0 \mod 3 \\ (\textit{R.H.S. of (4.25)}) & \textit{if } k_1-k_2\not\equiv 0 \mod 3 \end{cases}$$

with $\chi(V_{k_1\varepsilon_1-k_2\varepsilon_3}) = \frac{(k_1+1)(k_2+1)(k_1+k_2+2)}{2} - \begin{cases} 1+min\{k_1,k_2\} & \textit{if } k_1-k_2\equiv 0 \mod 3 \\ 0 & \textit{if } k_1-k_2\not\equiv 0 \mod 3 \end{cases}.$

To prove the conjecture for all $Irr(\mathfrak{sl}_3)$ we must show inequalities among the above identities. We first note that (4.28) < (4.29) will always hold. Also, in the case $k_1 - k_2 \not\equiv 0 \mod 3$ we see that (4.29) = (4.30). We first want to show (4.27) < (4.28).

Note (4.27) < (4.28) if and only if

$$\frac{\chi}{3} < \frac{\chi - R.H.S.}{2} \Leftrightarrow 6(R.H.S.) < 2\chi,$$

where we multiply by 2 to get rid of denominators in the expressions. Expanding and simplifying we have that the last inequality is the same as

$$6(k_2+1)\mathcal{T}(k_1-k_2) < k_1^2k_2 + k_2^2k_1 + k_1^2 - 9k_2^2 + 3k_1k_2 + 3k_1 + k_2 + 3 \tag{4.0.31}$$

where we've written $\mathcal{T}(k_1-k_2) = \left\lfloor \frac{k_1-k_2}{2} \right\rfloor - \left\lfloor \frac{k_1-k_2}{3} \right\rfloor + \left\lfloor \frac{k_1-k_2+2}{3} \right\rfloor$. Let $k_1 = k_2 + a$ where $a \geq 0$ is an integer. Then the R.H.S. of (4.31) becomes

$$\begin{aligned}(k_2+a)^2k_2 + k_2^2(k_2+a) + (k_2+a)^2 - 9k_2^2 + 3(k_2+a)k_2 + 3(k_2+a) + k_2 + 3 = \\ = 2k_2^3 + (3a-5)k_2^2 + (5a+4)k_2 + a^2 + 3a + 3.\end{aligned}$$

Note also that

$$\begin{aligned}6(k_2+1)(\frac{a}{2} - \frac{a}{3} + 1 + \frac{a+2}{3}) &= 6(k_2+1)(a+\frac{5}{3}) \\ &= \frac{6(k_2+1)(3a+10)}{6} \\ &= (k_2+1)(3a+10).\end{aligned} \tag{4.0.32}$$

This allows us to approximate the LHS of (4.31). Therefore, we have

$$\begin{aligned}(\text{RHS of } (4.31)) - (\text{LHS of } (4.31)) &\geq (\text{RHS of } (4.31)) - (4.32) \\ &= 2k_2^3 + (3a-5)k_2^2 + (2-6)k_2 + (a^2-7).\end{aligned} \tag{4.0.33}$$

We see that every term in (4.33) is positive except the last three when $a = 1$ or the last two when $a = 2$. Therefore, if $a \geq 3$ we have that

$$\text{(RHS of (4.31))} - \text{(LHS of (4.31))} \geq 0 \tag{4.0.34}$$

for all $k_2 \geq 0$. For $a = 2, 1$ we can use (4.33). Note, If $a = 2$, (4.33) becomes $2k_2^3 + k_2^2 - 2k_2 - 3$, therefore if $k_2 = 0$, $2k_2^3 + k_2^2 - 2k_2 - 3 = -3$ and similarly if $k_2 = 1$ we have -2. So we must check $(k_1, k_2) = (2, 0), (3, 1)$. Also, if $a = 1$ we then when $k_2 = 0, 1, 2$ (4.33) gives -6,-10, and -6, respectively. Thus, in total, we must check when $(k_1, k_2) = (2, 0), (3, 1), (1, 0), (2, 1), (3, 2)$. For $(k_1, k_2) = (2, 0)$ we have $12 < 13$. For $(k_1, k_2) = (3, 1)$ we have $12 < 34$. For $(k_1, k_2) = (1, 0)$ we have $6 < 7$. For $(k_1, k_2) = (2, 1)$ we have $12 < 17$. And, for $(k_1, k_2) = (3, 2)$ we have $18 < 35$. Therefore we have shown the following theorem.

Theorem 2. *Let $\nabla_{1/2}$ and $\nabla_{1/3}$ be as above, then for all $V_\lambda \in Irr(\mathfrak{sl}_3)$*

$$irr_\lambda(\nabla_{1/3}) \leq irr_\lambda(\nabla_{1/2}), \tag{4.0.35}$$

with equality holding only for V_λ with $\lambda = \varepsilon_1$ and $2\varepsilon_1$.

Proof. □

We now consider the main ingredient to prove the minimal irregularity conjecture, which is to compare (4.27) and (4.30) when $k_1 - k_2 \equiv 0 \mod 3$. For the conjecture to hold it should follow that (4.27) < (4.30). This is the same as

$$\frac{\chi}{3} < \chi - \left(k_2(3(k_2 - 1) + 2) + \frac{2(k_2 + 1)(k_1 - k_2)}{3}\right).$$

Simplifying and expanding we find that this is equivalent to whether

$$0 < k_1^2 k_2 + k_2^2 k_1 + 2k_1 k_2 - 6k_2^2 + k_1^2 - 4k_2 + k_1 \tag{4.0.36}$$

is always true. Now since $k_1 - k_2 > 0$ and $k_1 - k_2 \equiv 0 \mod 3$ we can write $k_1 = k_2 + 3b$ with $b \geq 1$. Rewriting the RHS we have

$$2k_2^3 + (9b-3)k_2^2 + (9b^2 + 12b - 3)k_2 + 9b^2 + 3b.$$

We see that $9b - 3$ and $9b^2 + 12b - 3$ will always be positive for $b \geq 1$, therefore (4.36) will always hold. This shows the conjecture for $\mathfrak{sl}_3$. We have the following theorem.

Theorem 3. *For all $V_\lambda \in Irr(\mathfrak{sl}_3)$ not the trivial representation,*

$$irr_\lambda(\nabla_{1/3}) = \min_{\nabla \in \mathfrak{L}^{irr}} \{irr_\lambda(\nabla)\}.$$

In other words, the minimal irregularity conjecture holds true for SL_3.

Proof. $\square$

Chapter 5. Concluding Remarks

In summary, the main directive of this thesis was to describe how for $G = SL_2$ and SL_3, the Frenkel-Gross connection realizes the strict lower bound of the irregularity function irr_λ of a formal flat G-bundle, with respect to an arbitrary element of $V_\lambda \in Irr(G)$. In closing, we examine some examples in order to illustrate what occurs for other simple G.

Consider $\mathfrak{g} = \mathfrak{sp}_4$, with Coxeter number 4. In this case, the eigenvalues must come in pairs. We list the relevant combinatorial classes of connections to be considered. We have the Coxeter connections $\nabla_{1/4}$ and $\nabla_{1/2}$ with LT forms

$$Ct^{-1/4}\begin{pmatrix} 1 & & & \\ & \zeta_4 & & \\ & & \zeta_4^2 & \\ & & & \zeta_4^3 \end{pmatrix} \text{ and } Ct^{-1/2}\begin{pmatrix} 1 & & & \\ & \zeta_2 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

respectively, where ζ_i is an ith root of unity. The diagonalizable connection $\nabla(a_1, a_2)$ will have connection matrix

$$t^{-1}\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & -a_1 & \\ & & & -a_2 \end{pmatrix}.$$

And, in this case there is also a non-generic connection with two different slopes, which we denote by $\nabla_{1/2,1}$, its LT form can be written as follows

$$\begin{pmatrix} [\nabla_{1/2}] & \\ & [\nabla(a_1)] \end{pmatrix} = \begin{pmatrix} Ct^{-1/2} & & & \\ & C\zeta_2 t^{-1/2} & & \\ & & a_1 t^{-1} & \\ & & & -a_1 t^{-1} \end{pmatrix}.$$

Since there does exist explicit formulae for the dimensions of arbitrary weight spaces for $V_\lambda \in Irr(\mathfrak{sp}_4)$, the techniques applied in the proofs presented above should also be able to be applied to address the minimal irregularity conjecture for Sp_4.

For groups of higher rank it seems that one can only find explicit formulas for dimensions of arbitrary weight spaces of certain special irreducible representations. Therefore, new or modifications of our techniques might need to be considered to prove the the minimal irregularity conjecture in full generality for a given simple Lie algebra of a specified rank.

Bibliography

[1] Dima Arinkin and Dennis Gaitsgory. Singular support of coherent sheaves and the geometric langlands conjecture. *Selecta Mathematica*, 21(1):1–199, 2015.

[2] Christopher L Bremer and Daniel S Sage. Moduli spaces of irregular singular connections. *International Mathematics Research Notices*, 2013(8):1800–1872, 2013.

[3] Christopher L Bremer and Daniel S Sage. A Theory of Minimal K-Types for Flat G-Bundles. *International Mathematics Research Notices*, 2018(11):3507–3555, 02 2017.

[4] Roger Carter and Roger William Carter. *Lie algebras of finite and affine type*, volume 96. Cambridge University Press, 2005.

[5] Edward Frenkel. Lectures on the langlands program and conformal field theory. In *Frontiers in number theory, physics, and geometry II*, pages 387–533. Springer, 2007.

[6] Edward Frenkel. Gauge theory and langlands duality. In *Séminaire Bourbaki : volume 2008/2009 exposés 997-1011 - Avec table par noms d'auteurs de 1848/49 à 2008/09*, number 332 in Astérisque. Société mathématique de France, 2010. talk:1010.

[7] Edward Frenkel and Benedict Gross. A rigid irregular connection on the projective line. *Annals of mathematics*, pages 1469–1512, 2009.

[8] Mr William Fulton and William Fulton. *Young tableaux: with applications to representation theory and geometry*, volume 35. Cambridge University Press, 1997.

[9] W. Fulton, W.F.J. Harris, and J. Harris. *Representation Theory: A First Course*. Graduate Texts in Mathematics. Springer New York, 1991.

[10] Benedict H Gross, Mark Reeder, et al. Arithmetic invariants of discrete langlands parameters. *Duke Mathematical Journal*, 154(3):431–508, 2010.

[11] Jochen Heinloth, Bao-Châu Ngô, and Zhiwei Yun. Kloosterman sheaves for reductive groups. *Annals of mathematics*, pages 241–310, 2013.

[12] Masoud Kamgarpour and Daniel S. Sage. A geometric analogue of a conjecture of gross and reeder. *American Journal of Mathematics*, 141(5):14571476, 2019.

[13] Masoud Kamgarpour and Daniel S. Sage. Rigid connections on p1 via the bruhattits building. *Proceedings of the London Mathematical Society*, May 2020.

[14] Anton Kapustin and Edward Witten. Electric-magnetic duality and the geometric langlands program. *arXiv preprint hep-th/0604151*, 2006.

[15] Nicholas M Katz. On the calculation of some differential galois groups. *Inventiones mathematicae*, 87(1):13–61, 1987.

[16] Shrawan Kumar and Dipendra Prasad. Dimension of zero weight space: An algebro-geometric approach. *Journal of Algebra*, 403:324–344, 2014.

[17] Thomas Lam and Nicolas Templier. The mirror conjecture for minuscule flag varieties. *arXiv preprint arXiv:1705.00758*, 2017.

[18] Zhuo Li, Y Cao, and Zhenheng Li. Cross-section lattices of j-irreducible monoids and orbit structures of weight polytopes. *arXiv preprint arXiv:1411.6140*, 2014.

[19] Martin Luu. Local langlands duality and a duality of conformal field theories. *arXiv preprint arXiv:1506.00663*, 2015.

[20] Mark Reeder and Jiu-Kang Yu. Epipelagic representations and invariant theory. *Journal of the American Mathematical Society*, 27(2):437–477, 2014.

[21] Edward Witten. Gauge theory and wild ramification. *Analysis and Applications*, 6(04):429–501, 2008.

Vita

Andrew Alaniz

377 Lockett Hall aalani1@lsu.edu

Department of Mathematics

Louisiana State University

Baton Rouge, Louisiana

Areas of specialization

Representation Theory; The Local Langlands Correspondence

Education

2021, PhD in Mathematics, Louisiana State University, Baton Rouge, LA

2013, MSc in Mathematics, University of Texas Pan American, Edinburg, TX

2008, BA in Political Science, University of Texas Pan American, Edinburg, TX

Honors and Awards

2015-2018, GAANN Fellow, Louisiana State University, Baton Rouge, LA

2014-2015, VIGRE Outreach Traneeship, Louisiana State University, Baton Rouge, LA

2013-2014, Dept. of Mathematics Tuition Award, Louisiana State University, Baton Rouge, LA

2012-2013, GAANN Fellow, University of Texas Pan American, Edinburg, TX

2013, HESTEC Graduate Poster 1st Place, University of Texas Pan American, Edinburg, TX

Publications and Papers

A. Alaniz, L. Huang, E. Kennedy, L. Smolinsky, *The Social and Work Structure of an After-school Math Club*, in press Middle Grades Research Journal, 2019

A. Alaniz, T. Huber, *On Cubic Multisections of Eisenstein Series*, The Ramanujan Journal, October 2013

A. Alaniz, *On Cubic Multisections*, Masters Thesis, University of Texas Pan American, August 2013