1991

Some Results on Minors for Graphs and Matroids.

Lawrence Alan Wargo
Louisiana State University and Agricultural & Mechanical College

Order Number 9219584

Some results on minors for graphs and matroids

Wargo, Lawrence Alan, Ph.D.

The Louisiana State University and Agricultural and Mechanical Col., 1991

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

SOME RESULTS ON MINORS FOR GRAPHS AND MATROIDS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfilment of the
requirement of the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

Lawrence Alan Wargo
B.A., LaSalle College, 1981
M.A., SUNY at Stony Brook, 1984
December, 1991

Acknowledgements

This dissertation is dedicated to my parents, Vincent and Helen, my brothers Vinny and Bill, and my sisters Mary Ellen, Karen, Donna, Patty, Susan, and Beth. Their love and support throughout my life has been profound. If they could see what I see, they would see how beautiful each of them is.

An old Buddhist proverb reads: " When the student is ready, the teacher appears." This work would not have been completed without the guidance and encouragement of my advisor and teacher Prof. James Oxley. Through times of my best composure and times of my worst tantrums, he has been a model for me, instilling by example both the work ethic and enthusiasm necessary for doing and teaching Mathematics. (I will not mention the fact that he does not appreciate American basketball enough.) I would also like to thank his family for their friendship during this time.

I would also like to express my appreciation to the faculty of the department of Mathematics at LSU. They have shown a genuine interest, not just in me but to all graduate students during my stay in Baton Rouge. I feel grateful for having had the opportunity to study with them, and not under them. I would like to thank Papita, Jill, Karen, Asadian, and Tony for their help along the way.

I would also like to express my thanks to the staff of the department of Mathematics at LSU : Susan, Soula, Yvonne, Loc, Nell and Betty. Their help and assistance over the years has been invaluable.

Finally, I would like to send my thanks to the House of 710 : Jana, Vibhas, Andy, Nattu, Gooja, and Rangan. Without their help, the final form of this dissertation would not have been possible. Without their friendship and camaraderie my life would have been much more difficult.

Table of Contents

Abstract

This dissertation solves some problems relating to the theory of graphs. The first type of problem considered concerns the structure of various classes of graphs which arise naturally from outerplanar graphs. These problems are motivated by Chartrand and Harary's well-known characterization of outerplanar graphs. This theorem states that K_4 and $K_{2,3}$ are the only non-outerplanar graphs for which both G\e, the deletion of the edge e from the graph G, <u>and</u> G/e, the contraction of the edge e, are outerplanar for all edges e of G. Following Gubser's characterization of almost-planar graphs , we begin our study of graphs related to outerplanar graphs by characterizing the non-outerplanar graphs for which G\e <u>or</u> G/e is outerplanar. We call these graphs almost-outerplanar (or 1-outerplanar). We then consider the corresponding problem for almost-outerplanar graphs and characterize the graphs G that are not almost-outerplanar such that G\e <u>or</u> G/e is almost-outerplanar or outerplanar for every edge e of G.

We end our study of graphs arising from outerplanar graphs by relaxing Chartrand and Harary's condition characterizing outerplanarity in a different way. This time we describe the non-outerplanar graphs G for which G\e and G/e are outerplanar for <u>at least</u> one edge e.

The second problem we solve is motivated by Hartvigsen and Zemel's characterization of graphs having the property that every circuit basis is fundamental. This theorem states that a graph has every circuit basis fundamental if and only if the graph has no minor isomorphic to one of five graphs. We consider the corresponding problem for binary matroids. We show that, in general, the class of binary matroids for which every circuit basis is fundamental is not closed under the taking of minors. However, this class is closed under the taking of series-

minors. We also describe some general properties of this class of matroids. We end this chapter by extending Hartvigsen and Zemel's result to the class of regular matroids.

Chapter 1

Introduction

In this chapter we establish the notation and terminology that will be used throughout this dissertation. We begin by setting up the graph theory background which we will use in Chapters 2-5. We then give the necessary material on binary matroids which will be used in Chapter 5.

1.1 Some graph theory preliminaries.

Brad Gubser (1990) characterized a class of graphs obtained by relaxing an excluded-minor condition for planar graphs due to Wagner (1937). Wagner's result, which is a restatement of Kuratowski's famous characterization of non-planar graphs, asserts that if G is a non-planar graph such that G\e, the deletion of the edge e from G, <u>and</u> G/e, the contraction of the edge e, are planar, then G is isomorphic to K_5 or $K_{3,3}$. Gubser characterized those non-planar graphs G such that G\e <u>or</u> G/e is planar. He called these graphs almost-planar graphs. Following Gubser's work, we characterize three classes of graphs which arise when one relaxes in various ways an excluded-minor condition for outerplanar graphs. This study is motivated by Chartrand and Harary's (1967) characterization of outerplanar graphs. An outerplanar graph is a graph that has a planar embedding in which all of its vertices lie on the boundary of the infinite face. The following result is a well-known extension of Chartrand and Harary's characterization.

(1.1.1) Theorem: The following are equivalent for a graph G:

(i) G is outerplanar.

(ii) G has no minor isomorphic to K_4 or $K_{2,3}$.

(iii) G has no subgraph homeomorphic from K_4 or $K_{2,3}$.

Any unexplained graph-theoretic terminology used throughout this dissertation will follow Bondy and Murty (1976). We allow a graph to have loops and parallel edges; a <u>simple graph</u> has no loops or parallel edges. We let V(G) and E(G) denote the sets of vertices and edges of G, respectively. Let $X \subseteq E(G)$. The deletion and contraction of X from G will be denoted by $G\backslash X$ and G/X, respectively.

A <u>k-connected</u> graph is a graph in which every two vertices are joined by at least k internally-disjoint paths. A 1-connected graph is also called simply a connected graph. The <u>connectivity</u> of a graph G is the maximum value of k such that G is k-connected. A vertex v of a connected graph G is a <u>cut-vertex</u> of G if $G\backslash\{v\}$ is disconnected, where $G\backslash\{v\}$ is obtained from G by deleting the vertex v and all edges incident on v. A connected graph that has no cut vertices is called a <u>block</u>. A <u>block of a graph</u> is a subgraph that is a block and is maximal with this property. If a single edge is a block of G, and this edge is not a loop of G, then we call this edge a <u>cut edge</u> of G. The classes of graphs with connectivity 2 and 3 will play a significant role in the problems we will consider.

An edge e = uv which is not a cut edge is said to be <u>subdivided</u> or <u>series extended</u> when it is deleted and replaced by a path (of arbitrary non-zero length) connecting u and v. Any two edges of such a path are said to be <u>in series</u>. G is a <u>subdivision</u> or <u>series extension</u> of H, or equivalently, G is <u>homeomorphic</u> from H, if G can be obtained from H by a sequence of edge subdivisions. Edges e and f are <u>parallel</u> in G if {e,f} is a cycle of G. A graph G is a parallel extension of a graph H if $H = G\backslash X$ and every edge of X is parallel to an edge of G not in X. We say that G is a series-parallel extension of H if there is a sequence of graphs H_1, $H_2, \ldots, H_n$ such that $G \cong H_n$, $H \cong H_1$, and for all i in $\{2,3,\ldots,n\}$, there is an edge e_i in $E(H_i)$ such that H_i is either a parallel extension or series extension of H_{i-1}. A graph G without isolated vertices is called a series-parallel network if it can be obtained from a loop or a cut edge by a sequence of parallel extensions or series extensions. The following result of Dirac (1952) will play an extensive part in Chapters 2 - 4.

(1.1.2) Theorem: A graph G with at least one edge is a series-parallel network if and only if it is a block having no subgraph that is a subdivision of K_4.

The graph H is a minor of G if there are disjoint subsets X and Y of E(G) such that $G\backslash X/Y$ is isomorphic to H together with a (possibly empty) set of isolated vertices. H is a series-minor of G if $G\backslash X/Y$ is isomorphic to H together with a set of isolated vertices, and every element of Y is in series with an edge of $G\backslash X$ not in Y. We say the graph G has an H-minor if G has a minor isomorphic to H.

Let G_1 and G_2 be graphs with disjoint vertex sets and (u_i,v_i) be a non-loop edge of G_i for $i = 1$ and 2, where (u_i,v_i) denotes an edge directed from the vertex u_i to the vertex v_i. Then the graph $P((G_1;(u_1,v_1)),(G_2;(u_2,v_2)))$ will denote the parallel connection of G_1 and G_2 with respect to the basepoints (u_1,v_1) and (u_2,v_2). It is defined as follows: we first delete (u_i,v_i) from G_i for $i=1$ and 2. We then identify the vertices u_1 and u_2 as a new vertex u and identify the vertices v_1 and v_2 as a new vertex v. We complete the parallel connection by joining u and v by a new edge p. If $|E(G_i)| \geq 3$ for $i=1$ and 2 and (u_i,v_i) is neither a loop nor a cut edge, then $P((G_1;(u_1,v_1)),(G_2;(u_2,v_2)))\backslash p$ is called the 2-sum of G_1 and G_2, which we denote by $G_1 \oplus_2 G_2$. The following are two fundamental properties of 2-sum. Their elementary proofs are omitted.

(1.1.3) Proposition: Let G be a loopless graph with connectivity 2 having at least four edges. Then G is a 2-sum of 2-connected graphs G_1 and G_2 each having at least three edges. Moreover, G_i is isomorphic to a minor of G for $i=1$ and 2.

(1.1.4) Proposition: If H is a simple 3-connected graph which is isomorphic to a minor of the 2-sum of G_1 and G_2, then G_1 or G_2 has a minor isomorphic to H.

Returning to Chartrand and Harary's characterization of outerplanar graphs, an immediate consequence of Theorem (1.1.1) is that if G is a non-outerplanar graph without isolated vertices such that for every edge e of G, both G\e and G/e are outerplanar, then G is isomorphic to K_4 or $K_{2,3}$. A few natural questions arise from this result: Can we describe the non-outerplanar graphs G such that for every edge e of G either G\e or G/e is outerplanar? and, in an analogous fashion, can we describe the graphs G that are not almost-outerplanar such that for every edge e of G either G\e or G/e is almost-outerplanar or outerplanar? In a similar way, we can relax Chartrand and Harary's condition for outerplanarity and ask : what are the non-outerplanar graphs G such that G has at least one edge e such that both G\e and G/e are outerplanar? In Chapters 2-4 we answer these questions by giving a full description of the classes of graphs which arise from these questions. We also describe some properties of these classes. In addition, for two of these classes we also give an excluded-minor description.

1.2 Some preliminaries on binary matroids.

In this section we establish the notation and terminology which we will use in Chapter 5. Any unexplained terminology will follow Oxley (1992).

A matroid M on a ground set E is a collection **C** of non-empty subsets of E, called circuits, satisfying the following axioms:

(C1) If C_1 and C_2 are members of **C** and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C2) Let C_1 and C_2 be distinct circuits and let $e \in C_1 \cap C_2$. Then there is a circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2) \backslash e$.

Two important examples of matroids are those which are derived from graphs and those which are derived from vector spaces. We define the former as follows: Let G be a graph. To G we associate the matroid M(G), called the cycle matroid of G, in a natural way. It is defined on the set of edges E(G) and the circuits of M(G) are precisely the cycles of G.

For the latter we proceed as follows: Let A be a matrix with entries from a field F. Let E denote the set of column labels of A, where the columns of A are viewed as elements of a vector space over F. We associate with A the vector matroid M[A]. It is defined on the set E. The circuits of M[A] are the minimal linearly dependent subsets of E. We say a matroid M is representable over some field F if M is isomorphic to M[A] for some matrix A with entries from F. In Chapter 5 we will concentrate on binary matroids. M is a binary matroid if M is representable over GF(2). The following is a property of binary matroids which we will use. Its proof can be found, for example, in Lehman (1964).

(1.2.1) Proposition: The following are equivalent:

(i) M is a binary matroid.

(ii) The symmetric difference of any set of circuits of M is either empty or contains a circuit.

A particularly important class of matroids are those which are representable over every field. Such matroids are called regular.

Let M be a binary matroid having n elements and let V(n,2) be the n-dimensional vector space over GF(2). The circuit space of M is the subspace of V(n,2) generated by the incidence vectors of the circuits of M. A subset of these incidence vectors which is a basis for this subspace is called a circuit basis of M. Let P be such a circuit basis where $|P| = d$. We call P fundamental if there is an ordering of the circuits in P such that $C_j \backslash (C_1 \cup C_2 \cup \ldots \cup C_{j-1})$ is non-empty for $2 \leq j \leq d$. Hartvigsen and Zemel (1989) obtained an excluded-minor characterization for the class of graphic matroids in which every circuit basis is fundamental. In Chapter 5, we consider the corresponding problem for binary matroids. In general, we show that the class of binary matroids in which every circuit basis is fundamental is not closed under the taking of minors. However, we show this class is closed under the taking of series-minors. We also establish some properties which this class satisfies. We finish this chapter by extending Hartvigsen and Zemel's result to the class of regular matroids.

1.3 Basic notation.

A reference to "Figure (j.k)" refers to figure k of chapter j. A reference to "item (i.j.k)" refers to item k of section j of chapter i.

References are by authors' names and the year of publication.

Chapter 2

A Characterization of Almost-Outerplanar Graphs

2.1 Introduction.

The purpose of this chapter is to characterize a class of graphs which arises naturally when one relaxes Chartrand and Harary's (1967) characterization of outerplanar graphs. (Theorem (1.1.1).)

An immediate consequence of Theorem (1.1.1) is that if G is a non-outerplanar graph such that for every edge e, both G\e and G/e are outerplanar, then G is isomorphic to K_4 or $K_{2,3}$. In this chapter we characterize the non-outerplanar graphs G such that, for every edge e of G, the deletion or the contraction of e from G is outerplanar. We call such graphs almost-outerplanar (or 1-outerplanar). Our main result gives a characterization of all such graphs.

Before we begin characterizing the class of almost-outerplanar graphs, we will establish a few general lemmas about this class. We begin with the following easy lemma which we will use repeatedly in determining whether or not a given graph is almost-outerplanar.

(2.1.1) Lemma: Let G be an almost-outerplanar graph. Then G has either K_4 or $K_{2,3}$ as a minor. Moreover, for every edge e of G, at least one of G\e and G/e has neither K_4 nor $K_{2,3}$ as a minor.

Proof: By definition, G is not outerplanar but, for every edge e of G, at least one of G\e and G/e is outerplanar. The lemma now follows from Theorem (1.1.1). ∎

Let $\mathbf{C}_1$ be the union of the classes of almost-outerplanar graphs and outerplanar graphs. Clearly $\mathbf{C}_1$ is closed under isomorphisms. Moreover:

(2.1.2) Lemma: $\mathbf{C}_1$ is closed under the taking of minors.

Proof: We first observe that, by Theorem (1.1.1) the class of outerplanar graphs is closed under the taking of minors.

Now suppose that G is an almost-outerplanar graph and let H be an arbitrary minor of G. If H is outerplanar, then H is certainly in $\mathbf{C}_1$. Hence, we may assume that H is not outerplanar. If $e \in E(H)$, then $e \in E(G)$, so $G\backslash e$ or G/e is outerplanar. As $H\backslash e$ and H/e are minors of $G\backslash e$ and G/e, respectively, it follows that $H\backslash e$ or H/e is outerplanar. Thus H is almost-outerplanar and so is in $\mathbf{C}_1$. ∎

As a first step towards characterizing all almost-outerplanar graphs, we establish a few results describing the general structure of such graphs.

(2.1.3) Proposition: If G is a connected almost-outerplanar graph, then G is 2-connected.

Proof: Suppose G has connectivity 1. Let $G_1, G_2, \ldots, G_k$ be the blocks of G. As G is not outerplanar, some G_j, say G_1, is not outerplanar. Suppose $k > 1$ and let e be an edge in $E(G_i)$ for some $i \neq 1$. Both $G\backslash e$ and G/e have G_1 as a block and, therefore, neither is outerplanar. This contradicts our choice of G. Thus $k=1$ and G is a block. Since G has K_4 or $K_{2,3}$ as a minor, G is 2-connected. ∎

(2.1.4) Proposition: If G is a disconnected almost-outerplanar graph, then G is the union of a 2-connected almost-outerplanar graph and a set of isolated vertices.

Proof: Let G have as its connected components $G_1, G_2, \ldots, G_k$. Then G_j is not outerplanar for some j, say j = 1. Suppose $E(G_i) \neq \emptyset$ for some $i \neq 1$, and let $e \in E(G_i)$. Both G\e and G/e have G_1 as a component and, therefore, neither is outerplanar. This contradicts our choice of G. Thus $E(G_i)$ is empty, which implies $|V(G_i)| = 1$ for all $i \neq 1$. ∎

By the above propositions, if G is an almost-outerplanar graph without isolated vertices, then G is 2-connected. In what follows, we will assume that G is 2-connected. Using Theorem (1.1.1) we will show the following:

(2.1.5) Lemma: Let G be a 2-connected non-outerplanar graph. Then either there are subsets X and Y of E(G) such that G\X/Y is isomorphic to K_4 or $K_{2,3}$ where every element of Y is in series with some element of G\X not in Y, or G has a $K_{2,4}$-series-minor.

Proof: G has a series-minor isomorphic to K_4 or $K_{2,3}$. Hence there are subsets X and Y as described above such that G\X/Y is obtained from K_4 or $K_{2,3}$ by adding a, possibly empty, set Z of vertices. We prove the lemma by induction on $|Z|$.

If $|Z| = 0$, the lemma clearly holds. Assume it holds for $|Z| < n$ and let $|Z| = n$. Choose a vertex v in Z. Now, for some H in $\{K_4, K_{2,3}\}$, G\X is obtained from H by subdividing edges and adding the members of Z as isolated vertices. Since v is not isolated in the 2-connected graph G, there are two internally disjoint paths from v to V(H) that meet V(H) in different vertices. By analyzing the various cases that arise, it is not difficult to show that either G has a $K_{2,4}$-minor, or for some subset X' of E(G), the graph G\X'is obtained from a subdivision of K_4 or $K_{2,3}$ by adding at most $|Z| - 1$ isolated vertices. In fact, this analysis shows that if G has a K_4-minor, then G\X' is obtained from a subdivision of K_4 by adding at most $|Z| - 1$ isolated vertices, and if G has a $K_{2,3}$-minor, then G\X' is obtained from a subdivision of K_4 or $K_{2,3}$ by adding at most $|Z| - 1$ isolated vertices. The lemma now follows by induction. ∎

(2.1.6) Lemma: Let G be a 2-connected almost-outerplanar graph. Then there are subsets X and Y of E(G) such that G\X/Y is isomorphic to K_4 or $K_{2,3}$ where every element of Y is in series with an element of G\X not in Y.

Proof: Either E(G) has such subsets or G has a $K_{2,4}$- minor. It is easy to check that $K_{2,4}$ is not almost-outerplanar. Hence, by Lemma (2.1.5), G cannot have such a minor. The lemma now follows. ∎

In the next section we shall determine all 3-connected almost-outerplanar graphs.

2.2 The 3-connected case.

Throughout this section G will denote an almost-outerplanar graph.

(2.2.1) Lemma: Let G be an almost-outerplanar graph and let n be an integer exceeding two. Then for every edge e of G, one of G\e and G/e has no W_n-minor.

Proof: For every edge e of G, one of G\e and G/e is outerplanar, and so has no K_4-minor and hence has no W_3-minor. ∎

(2.2.2) Lemma: Let G be 3-connected. Then G contains no (K_5\e)-minor.

Proof: Suppose that G contains a (K_5\e)-minor. We observe that (K_5\e) has K_4 as a proper restriction. Let f be an edge of the (K_5\e)-minor not contained in this K_4-restriction. Then both

$(K_5\backslash e)\backslash f$ and $(K_5\backslash e)/f$ have K_4-restrictions. Consequently, both $G\backslash f$ and G/f have K_4 -minors. This contradicts our choice of G. ∎

The following result of Robertson and Seymour (1984) enables us to determine all simple 3-connected almost-outerplanar graphs.

(2.2.3) Theorem: Let G be a simple 3-connected graph with no $(K_5\backslash e)$-minor and at least four vertices. Then G is a wheel or G is isomorphic to $K_{3,3}$ or $(K_5\backslash e)^*$.

(2.2.4) Theorem: Let G be a simple 3-connected graph. Then G is almost-outerplanar if and only if G is a wheel.

Proof: Suppose G is almost-outerplanar. Since G has K_4 or $K_{2,3}$ as a minor, $|V(G)| \geq 4$. Also, by Lemma (2.2.2), G does not have a $(K_5\backslash e)$-minor. Thus, by Theorem (2.3), G is a wheel or G is isomorphic to $K_{3,3}$ or $(K_5\backslash e)^*$.

(2.2.5) Lemma: Neither $K_{3,3}$ nor $(K_5\backslash e)^*$ is almost-outerplanar.

Proof: Let $K_{3,3}$ and $(K_5\backslash e)^*$ be labeled as in Figure (2.1).

Figure (2.1)

It is easy to see that $(K_{3,3}\backslash f)/e,g \cong K_4$ and that the simple graph associated with $(K_{3,3}/f)/h$ is isomorphic to K_4. Consequently $K_{3,3}$ is not almost-outerplanar.

Similarly, $((K_5\backslash e)^*\backslash x)/f,g \cong K_4$ and the simple graph associated with $((K_5\backslash e)^*/g)/h$ is isomorphic to K_4. We conclude that $(K_5\backslash e)^*$ is not almost-outerplanar. ∎

Combining this lemma with Theorem (2.2.3), we conclude that G is a wheel.

Now suppose G is a wheel. Because G has a W_3-minor, G is not outerplanar. Let e be a rim edge of G and let f be a spoke of G. It is routine to check that both $G\backslash e$ and G/f have planar embeddings in which all the vertices lie on the infinite face. Thus G is almost-outerplanar. ∎

2.3 The connectivity 2 case.

In this section we will characterize the almost-outerplanar graphs with connectivity 2. Throughout this section G will denote an almost-outerplanar graph with connectivity 2. We partition the set of all such graphs G into the following classes:

(a) G has a W_4-minor.

(b) G has a W_3-minor but no larger wheel as a minor.

(c) G is a series-parallel network having a $K_{2,3}$-series-minor.

To describe the graphs G in the first two classes we will use the following:

(2.3.1) Proposition: Let G be an almost-outerplanar graph with connectivity 2 and suppose that W_n is the largest wheel occurring as a minor of G where $n \geq 3$. Then G is a series-parallel extension of W_n.

Proof: We will use induction on $|E(G)|$ to establish the proposition. By Proposition (2.1.3), G is a 2-sum of two 2-connected graphs G_1 and G_2 with respect to the basepoints (u_1,v_1) and (u_2,v_2). As G has a W_n-minor and W_n is simple and 3-connected, Proposition (2.1.4) implies that one of G_1 and G_2, say G_1, has a W_n-minor. We now describe the structure of G_2.

(2.3.2) Lemma: G_2 is a circuit or a cocircuit.

Proof: Since G_2 is 2-connected, it has a circuit C containing the basepoint p. We may assume that $G_2 \neq C$ otherwise the lemma holds. Let $q \in E(G_2\backslash C)$. Then C is a circuit of $G_2\backslash q$, and so $G\backslash q$ has a G_1-minor and therefore has a K_4-minor. Consequently $G\backslash q$ is not outerplanar. Therefore G/q, which equals $P((G_1;(u_1,v_1)), (G_2/q;(u_2,v_2)))\backslash p$ is outerplanar.

We will show that G_2/q has no circuit containing p and having at least two elements. Suppose, on the contrary, that D is a circuit of G_2/q with $p \in D$ and $|D| \geq 2$. Then deleting the elements of $D\backslash p$ produces a minor of G which has a G_1-minor. This contradicts the fact that G/q is outerplanar. Therefore, p must be a loop in G_2/q. Consequently, p and q are parallel in G_2. But q was arbitrarily chosen in $E(G_2\backslash C)$. Thus every edge of $E(G_2\backslash C)$ is parallel to p.

If $|C| > 2$ and $e \in C\backslash p$, then G_1 is a minor of both $G\backslash e$ and G/e; a contradiction. Hence, $|C| = 2$, and every edge of $G_2\backslash p$ is parallel to p. Thus G_2 is a cocircuit. This completes the proof of Lemma (2.3.2). ∎

By this lemma, we conclude that whenever we express G as a 2-sum, one of the parts of the 2-sum must be a circuit or a cocircuit. We want to show that G is a series-parallel extension of an n-wheel for some $n \geq 3$.

The graph G is almost-outerplanar. Therefore it has an almost-outerplanar subgraph of which G is a series-parallel extension. Among all such subgraphs, let G' be one for which $|E(G')|$ is minimal. If $G' \cong W_n$ for some $n \geq 3$, then the proof is complete. Suppose G' is not a wheel. By the previous lemma, G' is a 2-sum of two 2-connected graphs G_1' and G_2' where G_1' has a W_n-minor and G_2' is a circuit or a cocircuit. Thus G' is a series-parallel extension of G_1'. Therefore G is a series-parallel extension of G_1', which contradicts the minimality of $|E(G')|$. Hence $G' \cong W_n$ and this completes the proof of Proposition (2.3.1). ∎

Before we determine exactly which series-parallel extensions of wheels are in $\mathbf{C}_I$, we note a few easy lemmas.

(2.3.3) Lemma: Suppose e is an edge of an almost-outerplanar graph G such that G/e is not outerplanar. Then e is not in any 2-edge circuits in G.

Proof: Assume that there is a 2-circuit {e,e'} containing e. As G/e has a K_4- or $K_{2,3}$-minor and has e' as a loop, both G/e\e' and G/e/e' have K_4 or $K_{2,3}$ as a minor. Hence neither G\e' nor G/e' is outerplanar; a contradiction. ∎

In a very similar way we can establish the following:

(2.3.4) Lemma: Suppose e is an edge of an almost-outerplanar graph G such that G\e is not outerplanar. Then e is not in any 2-edge cocircuit in G.

We now return to the problem of characterizing the almost-outerplanar graphs G with connectivity 2 as described previously.

(a) G has a W_n-minor.

Let n be the largest integer such that G has a W_n-minor. By Proposition (2.3.1), G is a series-parallel extension of W_n. In this section we describe which such extensions are almost-outerplanar.

(2.3.5) Proposition: Let G be an almost-outerplanar graph with connectivity 2 and suppose that W_n is the largest wheel occurring as a minor of G, where $n \geq 4$. Then G is almost-outerplanar if and only if either (a) G is obtained from W_n by series extending rim edges, or (b) G is obtained from W_n by adding edges parallel to spokes.

Proof: By Proposition (2.3.1), it suffices to show that only rim edges of W_n may be series extended and only spokes of W_n may be parallel extended within the class of almost-outerplanar graphs. Let e be a spoke of W_n or an edge parallel to a spoke of W_n. Since $W_n\backslash e$ has a minor homeomorphic from W_{n-1}, the graph $W_n\backslash e$ is not outerplanar. By Lemma (2.3.4), we conclude that e cannot be in any 2-edge cocircuit in G. Thus, spokes of W_n may only be parallel extended within the class of almost-outerplanar graphs.

If f is any rim edge of W_n or any edge obtained by series extending a rim edge, then W_n/f has a W_{n-1}-minor. By Lemma (2.3.3), f cannot be in any 2-edge circuit in G. Thus, edges lying on the rim of W_n may only be series extended within the class of almost-outerplanar graphs.

Now let G be any such extension of W_n. It is easy to check that both $G\backslash f$ and G/e are outerplanar. ∎

We remark that the analysis used in Proposition (2.3.5) is not complete in the case when G contains a 3-wheel but no larger wheel. This is due to the edge-transitivity of the 3-wheel which does not allow us to distinguish between rim edges and spokes as in the case of n-wheels for $n \geq 4$. We will, however, show that at most three edges of the 3-wheel may be series-extended within the class of almost-outerplanar graphs where these three edges form a triangle of W_3.

(b) G has a W_3-minor but no larger wheel as a minor.

Let G be a graph with connectivity 2 having a W_3-minor but no larger wheel as a minor. By Proposition (2.3.1), we know G is a series-parallel extension of W_3. Suppose f is an edge added parallel to an edge of W_3. Then $G\backslash f$ has a K_4-minor. By Lemma (2.3.4), f cannot be subdivided within the class of almost-outerplanar graphs. Hence, if G is obtained from W_3 by parallel extending an edge e, then any edge in this parallel class can only be parallel extended. Similarly, Lemma (2.3.3) implies that if G is obtained by subdividing an edge of W_3, then any edge in this series class can only be subdivided. Thus, a necessary condition for a graph G obtained from W_3 to be almost-outerplanar is that G is obtained by either parallel extending an edge of W_3 or series extending an edge, but not by a combination of the two. More precisely, we have the following:

(2.3.6) Proposition: Let G be a graph with connectivity 2 such that G has a W_3-minor but no larger wheel as a minor. Let C be any 3-circuit of W_3. Then G is almost-outerplanar if and only if G is isomorphic to a graph that is obtained from W_3 in one of the following ways:

(i) Series extending at least two edges of C;

additional edges may be added parallel to any of the remaining edges of $E(W_3\backslash C)$.

(ii) Series extending at most one edge e of W_3;

additional edges may be added parallel to any of the remaining edges of $E(W_3\backslash e)$.

Remark: In the proof of Proposition (2.3.6), we will show that if no edge is series extended, then all of the edges of W_3 may be parallel extended.

Remark: We have observed that G is obtained from W_3 by series extending edges or parallel extending edges, but not by a combination of the two. We will establish the proposition in the following way. We will first show that at most three edges may be series extended. Next, we will show that if two or three edges are series extended, then these edges must either be contained in or form a 3-circuit C. In the former case, we will further show that the remaining edge of the circuit C cannot be parallel extended. Finally we will show that if at most one edge is series extended, then all of the remaining edges may be parallel extended.

We will begin with the following:

(2.3.7) Lemma: If G is obtained from W_3 by series extending at least four edges, then G has a proper minor isomorphic to the non-almost-outerplanar graph H shown in Figure (2.2).

Proof: Suppose G is obtained from W_3 by series extending at least four edges. Then G has as a minor a graph that is obtained from W_3 by subdividing four edges. It follows that G has a minor isomorphic to the graph H shown in Figure (2.2).

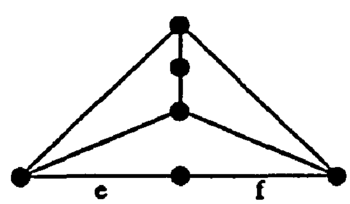


Figure (2.2)

Using the labeling in that figure, it is easy to check that H/e is homomorphic from K_4 and that (H\e)\f is the union of $K_{2,3}$ and an isolated vertex. Thus H is not almost-outerplanar and, consequently, G is not almost-outerplanar. ∎

We thus conclude that at most three edges of W_3 may be series extended within the class of almost-outerplanar graphs.

Now suppose G is obtained from W_3 by subdividing three edges. Suppose also that these edges do not form a 3-circuit. Using the edge-transitivity of W_3, we conclude G is isomorphic to a subdivision of the graph H shown in Figure (2.2) or G is isomorphic to the graph H' shown in Figure (2.3). We have already seen that H is not almost-outerplanar. Using the labeling in Figure (2.3), it is easy to see that H'\e and H'/e have subgraphs homeomorphic from $K_{2,3}$. We conclude if G is obtained from W_3 by series extending three edges, then these three edges must form a 3-circuit.

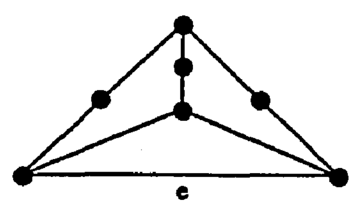


Figure (2.3)

If G is obtained by series extending exactly two edges of W_3, then the previous argument shows these edges must be contained in a 3-circuit of W_3. Let G be obtained from W_3 by series extending the edges e_1 and e_2 and let $\{e_1,e_2,e_3\}$ be a 3-circuit of W_3. We claim that no edges may be added parallel to e_3 within the class of almost-outerplanar graphs. To see this, suppose that f is added parallel to e_3. Then neither G\f nor G/f is outerplanar for G\f certainly has a subgraph homeomorphic from W_3, while it is routine to check that the simple graph associated with G/f is isomorphic to $K_{2,3}$.

Hence, if G is obtained by series extending exactly two edges of W_3, these edges must be contained in a 3-circuit of W_3 and no edge may be added parallel to the third edge of this 3-circuit. On the other hand, we claim edges may be added parallel to any of the remaining edges. It is easy to check that if f is any such edge, then G/f is outerplanar.

Using the edge-transitivity of W_3 and Lemma (2.3.3), it is now easy to see that if G is obtained from W_3 by series extending at least two edges of W_3, then G is isomorphic to an almost-outerplanar minor of the graph obtained from W_3 by series extending all rim edges and parallel extending all spokes.

Next, suppose G is obtained from W_3 by series extending exactly one edge. We claim that in this case edges may be added parallel to any of the remaining edges. To see this, let f be such an edge. Then $|V(G/f)| = 4$, and V(G/f) has a vertex of degree 2. Hence, G/f cannot have a $K_{2,3}$- or a K_4-minor. Thus G/f is outerplanar for all such f. Lastly, if G is obtained from W_3 by parallel extending edges only, then clearly G/e is outerplanar for every edge e of G. Hence, by the edge transitivity of W_3, if G is obtained from W_3 by series extending at most one edge, then G is isomorphic to an almost-outerplanar minor of the graph obtained from W_3 by parallel extending all edges except possibly one spoke which may be series extended or parallel extended.

The proof of the converse is much quicker. Suppose G is obtained from W_3 as described above. We need to check that G is almost-outerplanar. It is routine to check that both G\e and G/f are outerplanar for any edge e obtained by a series extension and any edge f obtained by a parallel extension. This completes the proof of Proposition (2.3.6). ▮

(c) G is a series-parallel network having a $K_{2,3}$-series-minor.

We now consider the case when G is a graph with connectivity 2 having no W_3-minor. Then by Theorem (2.1.2) G is a series-parallel network. Consequently, by Theorem (1.1.1) we have the following:

(2.3.8) Lemma: If G is a connected almost-outerplanar graph having no W_3-minor, then G is a series-parallel network having a $K_{2,3}$-series-minor.

In this section, we will establish the following:

(2.3.9) Proposition: Let G be an almost-outerplanar series-parallel network. Then G is obtained from $K_{2,3}$ in the following way: for each of the three non-trivial series classes of $K_{2,3}$, we can do exactly one of the following:

(i) Series extend some of the edges in the class.

(ii) Parallel extend some of the edges in the class.

Moreover, we can add any number of new edges joining the two degree-three vertices of $K_{2,3}$.

Proof: By Theorem (1.1.1) and Lemma (2.1.5), $G\backslash X/Y \cong K_{2,3}$ for some subsets X and Y of E(G), where every element of Y is in series with an element of $G\backslash X$ not in Y. We will now establish the following:

(2.3.10) Lemma: $G\backslash X$ is homeomorphic from $K_{2,3}$.

Proof: Since $G\backslash X/Y \cong K_{2,3}$ and every edge of Y is in series with some element of $G\backslash X$, we can partition Y into subsets Y_1,Y_2, and Y_3 where $G\backslash X/Y$ is as shown in Figure (2.4) and $Y_i = \{y \in Y : y \text{ is in series with } i \text{ in } G\backslash X\}$. The lemma follows easily. ∎

We now know that G\X is as shown in Figure (2.5) where each of the paths P_1,P_2, and P_3 joining u and v has length at least two.

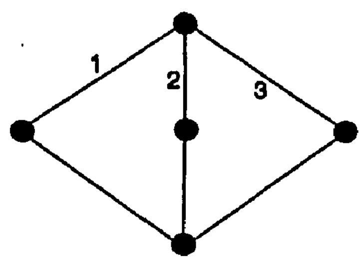


Figure (2.4)

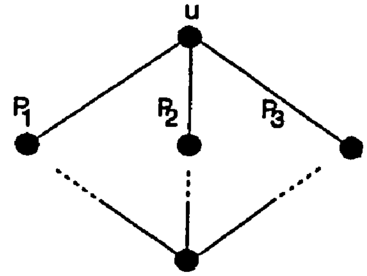


Figure (2.5)

Now every edge e of X must join two vertices of G\X otherwise both G\e and G/e have $K_{2,3}$-minors. Since G has no W_3-minor, no edge of X joins an internal vertex of one of P_1,P_2, and P_3 to an internal vertex of a different one of these paths. Thus, every edge of X must join vertices of some P_i for $i \in \{1,2,3\}$. This means the edges of X are either parallel to edges of P_i or join non-adjacent vertices of P_i. In the former case, the length of P_i must be two, otherwise G has an edge $x \in X$ such that both G\x and G/x have $K_{2,3}$-minors. (See Figure (2.6).) In the latter case, the edges $x \in X$ joining non-adjacent vertices of P_i must be of the form uv, where u and v are the degree 3 vertices of $K_{2,3}$. For if $x = u'v' \neq uv$, then it is easy to check that G has an edge e such that both G\e and G/e have $K_{2,3}$-minors. (See Figure (2.6).) On the other hand, if $x = uv$, then it is easy to check that G/x is outerplanar.

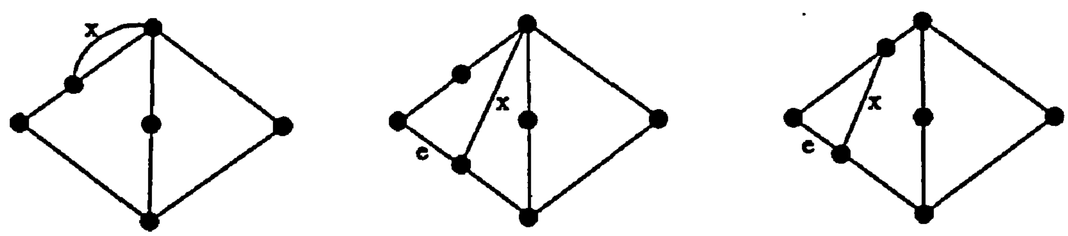


Figure (2.6)

Let $K^+_{2,3}$ be the graph shown in Figure (2.7). From the above analysis, we conclude that if S is a non-trivial series class of $K_{2,3}$ or $K^+_{2,3}$, then either both of the edges in this class can be series extended or both can be parallel extended; we cannot series extend one edge of S and parallel extend the other edge and still remain within the class of almost-outerplanar graphs.

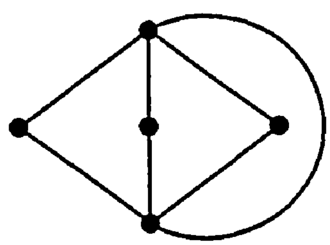


Figure (2.7)

Finally, if G is obtained from $K_{2,3}$ as described in Proposition (2.3.9), it is easy to see that G/e, G\f, and G/g are outerplanar (see Figure (2.8)).

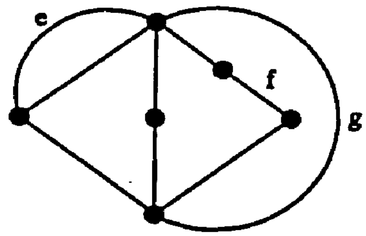


Figure (2.8)

This completes the proof of Proposition (2.3.9). ∎

This completes our characterization of almost-outerplanar graphs.

2.4 Some consequences.

Our aim in this section is to use our characterization of almost-outerplanar graphs to describe two classes of graphs closely related to such graphs, and to obtain a bound for the maximal number of edges for a simple almost-outerplanar graph on a fixed number of vertices. To begin, we will describe the class of graphs which are not outerplanar and for which the deletion of every edge is outerplanar. We note that these graphs are, in particular, almost-outerplanar. We will establish the following:

(2.4.1) Proposition: Let G be a non-outerplanar graph without isolated vertices. Then $G \backslash e$ is outerplanar for every edge e of G if and only if G is a 3-wheel or G is homeomorphic from $K_{2,3}$.

Proof: Clearly G is simple. If G is 3-connected, then using Theorem (2.2.4) it is easy to check that $G \cong W_3$. For the case when G has connectivity 2, we note the following easy lemma:

(2.4.2) Lemma: If any edge of W_3 is subdivided, then we obtain a graph G with an edge e such that G\e is not outerplanar.

By this lemma and Propositions (2.3.1), (2.3.6), and (2.3.9), we conclude that G is a series-parallel extension of $K_{2,3}$ or $K_{2,3}^+$. In fact, we have the following:

(2.4.3) Lemma: Let G have connectivity 2. Then G\e is outerplanar for every edge e of G if and only if G is homeomorphic from $K_{2,3}$.

Proof: Suppose G\e is outerplanar. We have already shown G cannot be obtained from a wheel. By our earlier description (See Proposition (2.3.9) and its proof.), G is obtained from $K_{2,3}$ by subdividing edges and, possibly, adding a new edge joining the two vertices of degree 3 of $K_{2,3}$. If such an edge is added, however, deleting this new edge does not destroy every $K_{2,3}$-minor. Hence, G is obtained from $K_{2,3}$ by subdividing edges. We need only note that if G is homeomorphic from $K_{2,3}$, then G\e is outerplanar as deleting any edge from G destroys a non-trivial path joining the two vertices of degree 3 of $K_{2,3}$. ∎

The proof of Proposition (2.4.1) is completed by combining Lemmas (2.4.2) and (2.4.3) with Proposition (2.3.9). ∎

Continuing in this direction, we now describe the non-outerplanar graphs G for which G/e is outerplanar for every edge e of G. Again, we observe that these graphs are, in particular, almost-outerplanar. We will establish the following:

(2.4.4) Proposition: Let G be a non-outerplanar graph. Then G/e is outerplanar for every edge e of G if and only if G is obtained from W_3, $K_{2,3}$, or $K_{2,3}^{+}$ by adding edges parallel to edges of these graphs.

Proof: If G has a W_4-minor, then clearly for each rim edge of this wheel G/e is non-outerplanar. Consequently, by Theorem (2.2.4) and Lemma (2.3.4), we conclude that if G is 3-connected, then $G \cong W_3$ or G is obtained from W_3 by adding edges parallel to edges of W_3.

If G has connectivity 2 and has a W_3-minor, Proposition (2.3.6) implies that G is, in particular, a non-trivial series extension of W_3. But clearly G/e has a W_3-minor if e is in a non-trivial series class of G. Thus G cannot have a W_3-minor. Therefore, by Proposition (2.3.9) G is a series-parallel extension of $K_{2,3}$ or $K_{2,3}^{+}$. As in the preceding argument, G cannot be a non-trivial series extension of $K_{2,3}$ or $K_{2,3}^{+}$. The result now follows. ∎

The final application of our characterization of almost-outerplanar graphs is to obtain a bound for the number of edges of a simple almost-outerplanar graph on a fixed number of vertices.

(2.4.5) Proposition: Let G be a simple almost-outerplanar graph on n vertices. Then $|E(G)| \leq 2n-2$. Moreover, W_{n-1} is the only graph which attains this bound.

Proof: Let H be a simple almost-outerplanar graph on n vertices such that among all such graphs H has the greatest number of edges. Since W_{n-1} is a simple almost-outerplanar graph on n vertices with $|E(W_{n-1})| = 2n-2$, we know that $|E(H)|$ is at least $2n - 2$. We shall argue by induction on n to show that H is a wheel for $n \geq 4$.

Since H is non-outerplanar, it has a K_4- or a $K_{2,3}$-minor. Hence $|V(H)| \geq 4$. Moreover, if $|V(H)| = 4$, clearly $H \cong W_3$. Hence, if $|V(H)| = 4$, then $|E(G)| = 2n-2$ and $G \cong W_3$.

Now assume the proposition is true for $|V(H)| < n$, and let $|V(H)| = n > 4$. From Proposition (2.1.4), we conclude H is 2-connected. If H is 3-connected, then because the only simple 3-connected almost-outerplanar graphs are wheels, the result holds. We can therefore assume H has connectivity 2. From Proposition (2.3.1) and Proposition (2.3.9), we conclude that H is obtained from W_n by series extending rim edges or H is obtained from $K_{2,3}$ by series extending some of the edges in any of the three non-trivial series classes of $K_{2,3}$. But series extending edges of $K_{2,3}$ or W_n lowers the average degree of the graph. It is easy to check that the average degree of $K_{2,3}$ is 14/5, and the average degree of W_n is $4 - 4/(n+1)$. Since $14/5 < 4 - 4/(n+1)$ for $n \geq 3$, the result follows immediately. ∎

2.5 The excluded minor characterization.

In this section we obtain an excluded minor characterization for the class $\mathbf{C}_1$ of graphs consisting of almost-outerplanar graphs and outerplanar graphs. We showed in Lemma (2.1.2) that $\mathbf{C}_1$ is minor closed. Moreover, we will establish the following:

(2.5.1) Proposition: A connected graph G is a member of $\mathbf{C}_1$ if and only if G has no minor isomorphic to one of the six non-isomorphic graphs obtained from K_4 and $K_{2,3}$ by adding either a loop or cut edge.

Proof: Suppose G has a minor isomorphic to one of the six graphs described above. As contracting and deleting the loop or cut edge from each of these graphs leaves a graph isomorphic to K_4 or $K_{2,3}$, we conclude $G \notin C_1$.

For the converse, suppose G has no minor isomorphic to any of the six graphs described here. Suppose also that G is not a member of C_1. Then G is, in particular, non-outerplanar. Hence some block of G has a K_4- or $K_{2,3}$-series-minor. By restricting to this block, if necessary, we may assume in what follows that G is 2-connected. Moreover, since $K_{2,4}$ has $K_{2,3}$ with a cut edge as a restriction minor, and since by hypothesis G has no such minor, by Lemma (2.1.5) there are subsets X and Y of E(G) such that $G\backslash X/Y$ is isomorphic to K_4 or $K_{2,3}$ where every element of Y is in series with some element of $G\backslash X$ not in Y. Thus $G\backslash X$ is homeomorphic from K_4 or $K_{2,3}$. We first consider the case when $G\backslash X$ is homeomorphic from K_4.

If $G\backslash X$ is obtained from K_4 by series extending at least four distinct edges, then by Lemma (2.3.7) G has a minor isomorphic to the graph shown in Figure (2.2). It is routine to check that the graph shown in that figure has $K_{2,3}$ with a cut edge as a minor. But this means G has such a minor; a contradiction. Similarly, if $G\backslash X$ is obtained from K_4 by series extending three distinct edges which do not form a 3-circuit, then G has a minor isomorphic to one of the graphs shown in Figures (2.2) and (2.3). It is routine to check that the graph shown in Figure (2.3) has $K_{2,3}$ with a loop as a minor. This again contradicts our choice of G. In any case, we conclude that $G\backslash X$ is obtained from K_4 by series extending at most three distinct edges, and that these edges must form a 3-circuit.

Now suppose $G\backslash X$ is obtained from K_4 by series extending a single edge of K_4. Since G is not almost-outerplanar, Proposition (2.3.6) implies that X cannot be empty. It is not hard to show using Proposition (2.3.6) that G has a minor isomorphic to one of the graphs shown in Figure (2.9). It is routine to check that each of the graphs shown in that figure has a minor isomorphic to K_4 with either a loop or cut edge.

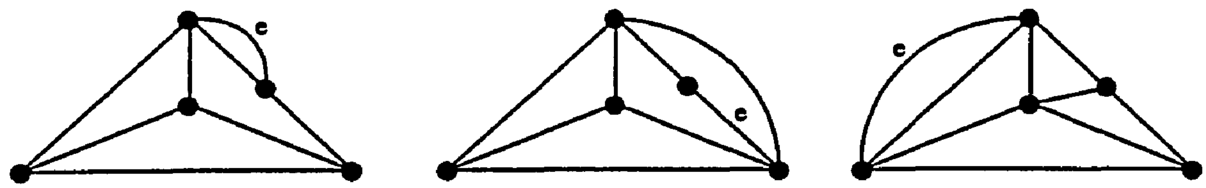


Figure (2.9)

Similarly, if G\X is obtained from K_4 by series extending at least two edges, then G has a minor isomorphic to one of the graphs shown in Figures (2.9) and (2.10). Again, it is easy to check that H_3/e, the simple graph associated with H_4/e, and H_5/e have a minor isomorphic to K_4 or $K_{2,3}$ with a loop or cut edge. This completes the case when G\X is homeomorphic from K_4.

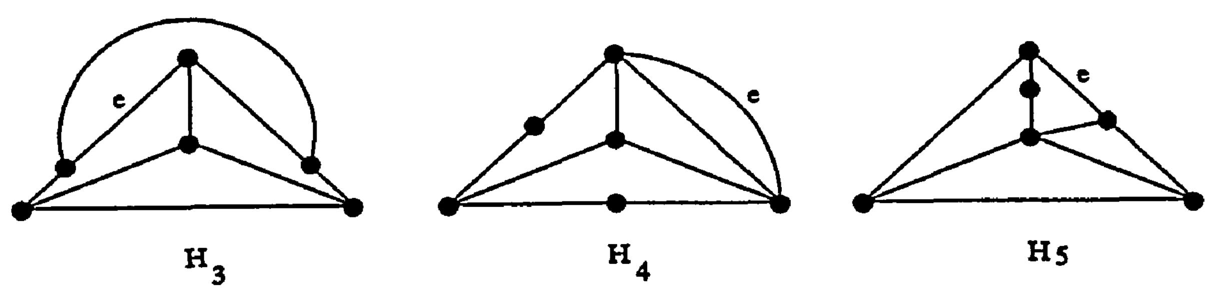


Figure (2.10)

We can now assume that G\X is homeomorphic from $K_{2,3}$ and that G does not have a K_4-minor. Since G is not almost-outerplanar and has no K_4-minor, by Proposition (2.3.9) and its proof, G has a minor isomorphic to one of the graphs shown in Figure (2.11). It is routine to check that H_6/e, H_7/e , and $H_8\backslash e$ all have minors isomorphic to $K_{2,3}$ with a loop or cut edge.

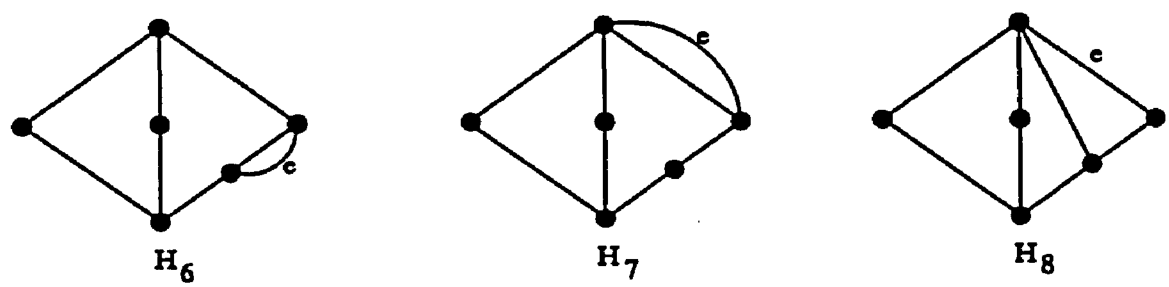


Figure (2.11)

In any case, we conclude that if G is not almost-outerplanar, then G has as a minor one of the six non-isomorphic graphs obtained from K_4 or $K_{2,3}$ by adding a loop or cut edge. This contradicts our choice of G. The result now follows. ▮

(2.5.2) Corollary: A connected graph G is a member of $\mathbf{C}_1$ if and only if G has no series-minor isomorphic to one of the six non-isomorphic graphs obtained from K_4 and $K_{2,3}$ by adding a loop or cut edge.

Chapter 3

A Characterization of 2-Outerplanar Graphs

3.1 Introduction.

In this chapter we continue investigating graphs arising from outerplanar graphs. In the previous chapter, we described the non-outerplanar graphs G such that for every edge e of G either G\e or G/e is outerplanar. We called these graphs almost-outerplanar or 1-outerplanar. In this chapter we consider an analogous problem and characterize the non-outerplanar graphs G such that, for all edges e of G, at least one of G\e and G/e is 1-outerplanar or outerplanar. We call such graphs <u>2-outerplanar</u>.

We note that because a 1-outerplanar graph has the property that the deletion or the contraction of every edge is outerplanar, and because the union of the classes of 1-outerplanar and outerplanar graphs is minor-closed, every 1-outerplanar graph is 2-outerplanar. We state this formally in the following.

(3.1.1) Lemma: If G is 1-outerplanar, then G is 2-outerplanar.

An easy consequence of our definition of 2-outerplanar graphs is the following.

(3.1.2) Lemma: If G is 2-outerplanar, then, for every pair of distinct edges e and f of G, at least one of G\e,f, G/e,f, G\e/f, or G/e\f is outerplanar.

Let C_2 be the union of the classes of 2-outerplanar graphs and outerplanar graphs.

(3.1.3) Lemma: C_2 is minor-closed.

Proof: We have already observed that the class of outerplanar graphs is closed under the taking of minors.

Now suppose G is 2-outerplanar and let H be an arbitrary minor of G. If H is outerplanar or 1-outerplanar, then H is certainly in C_2. We may therefore assume that H is neither outerplanar nor 1-outerplanar. If $e \in E(H)$, then $e \in E(G)$, so $G\backslash e$ or G/e is 1-outerplanar. But C_1 is minor-closed. Hence one of $H\backslash e$ and H/e is 1-outerplanar or outerplanar. Hence H is in C_2. ∎

As a first step towards characterizing 2-outerplanar graphs, we establish a few results describing the structures of 2-outerplanar graphs which have connectivity at most 1. If G is 1-outerplanar, then Propositions (2.3.5), (2.3.6), and (2.3.9) give a general description of such graphs. We may therefore assume G is 2-outerplanar but not 1-outerplanar.

(3.1.4) Proposition: Let G be a 2-outerplanar graph which is not 1-outerplanar and which has connectivity 1. Then G is a 1-outerplanar graph with either a cut edge or a loop.

Proof: Let $G_1, G_2, \ldots, G_k$ be the blocks of G. As G is not outerplanar, some G_j, say G_1, is not outerplanar. Suppose G has at least two edges e_1 and e_2 that are not in $E(G_1)$. Then all of $G\backslash e_1, e_2$, G/e_1e_2, $G\backslash e_1/e_2$, and $G\backslash e_2/e_1$ have G_1 as a block. Therefore none of these are outerplanar. By Lemma (3.1.2), this contradicts our choice of G. Thus, $|E(G)| - |E(G_1)| \leq 1$. Since G_1 is a block and G has connectivity 1, it follows that G consists of a block G_1 that is not outerplanar and a block G_2 that is either a cut edge or a loop.

Suppose G_1 is not 1-outerplanar. Let $e \in E(G_1)$ such that neither $G_1\backslash e$ nor G_1/e is outerplanar. Then both these graphs have one of K_4 and $K_{2,3}$ as a minor. Hence both $G\backslash e$ and G/e have as a minor K_4 with a loop or cut edge or $K_{2,3}$ with a loop or cut edge. Hence neither $G\backslash e$ nor G/e is 1-outerplanar by Proposition (2.5.1). This means G cannot be 2-outerplanar. This contradicts our choice of G. Hence G_1 is 1-outerplanar. ∎

(3.1.5) Proposition: Suppose G is a disconnected 2-outerplanar graph. Then G is the union of a connected 2-outerplanar graph and a set of isolated vertices, or G is the union of a 2-connected 1-outerplanar graph, an isolated cut edge or loop, and a set of isolated vertices.

Proof: Let G have $G_1, G_2, \ldots, G_k$ as its connected components. Then G_j is not outerplanar for some j, say G_1. Suppose $E(G_i) \neq \emptyset$ for some $i \neq 1$, and let $e \in E(G_i)$. Clearly there can be at most one edge that is not in G_1, otherwise G has two edges e and f such that all of $G\backslash e,f$, $G/e,f$, $G\backslash e/f$, and $G\backslash f/e$ have G_1 as a component and, therefore, none of them is outerplanar; a contradiction. Clearly $G\backslash e$ and G/e have G_1 as a component, so neither is outerplanar. If G_1 is not 1-outerplanar, then G_1 has as a minor K_4 with a loop or cut edge or $K_{2,3}$ with a loop or cut edge. Hence neither $G\backslash e$ nor G/e is 1-outerplanar. This again contradicts our choice of G. Thus we may assume that G_1 is 1-outerplanar. If G_1 is not 2-connected, then G has as a minor K_4 with a loop or cut edge, or $K_{2,3}$ with a loop or a cut edge. Again, neither $G\backslash e$ nor G/e is 1-outerplanar. Thus if G_1 is 1-outerplanar, it is 2-connected.

On the other hand, if $E(G_i) = \emptyset$ for all $i \neq 1$, then since G_1 is a non-outerplanar connected component of G clearly G_1 is a connected 2-outerplanar graph. ∎

3.2 The 3-connected case.

In this section we will first determine the simple 3-connected 2-outerplanar graphs. We will do this by first describing the simple 3-connected 2-outerplanar graphs with no $(K_5\backslash e)$-minor, and then by describing those with a $(K_5\backslash e)$-minor. For the former we will use Theorem (2.2.3) to establish the following.

(3.2.1) Proposition: Let G be a simple 3-connected graph with no $(K_5\backslash e)$-minor. Then G is 2-outerplanar.

Proof: By Theorem (2.2.3), G is a wheel, or G is isomorphic to $K_{3,3}$ or $(K_5\backslash e)^*$. Since wheels are, in particular, 1-outerplanar, Lemma (3.1.1) implies that wheels are also 2-outerplanar. Hence we may assume that G is isomorphic to $K_{3,3}$ or $(K_5\backslash e)^*$. The following lemma completes the proof of Proposition (3.2.1).

(3.2.2) Lemma: Both $K_{3,3}$ and $(K_5\backslash e)^*$ are 2-outerplanar.

Proof: It suffices to show that both $K_{3,3}$ and $(K_5\backslash e)^*$ have the property that the deletion or contraction of every edge is 1-outerplanar. Let e be any edge of $K_{3,3}$. We have $K_{3,3}/e \cong W_4$, which, by Theorem (2.2.4), is 1-outerplanar. Hence, $K_{3,3}$ is 2-outerplanar. On the other hand, let $G \cong (K_5\backslash e)^*$. Let S_1 be the edges of $(K_5\backslash e)^*$ that are contained in some 3-circuit, and let S_2 be the remaining edges of $(K_5\backslash e)^*$. Because the automorphism group of $E((K_5\backslash e)^*)$ is transitive on S_1 and on S_2, it suffices to show that both $(K_5\backslash e)^*\backslash f_1$ and $(K_5\backslash e)^*/f_2$ are 1-outerplanar for $f_1 \in S_1$ and $f_2 \in S_2$. We illustrate the last two graphs in Figure (3.1) (ii) and (iii). By Theorem (2.2.4) and Proposition (2.3.6), each of these graphs is 1-outerplanar. We conclude that $(K_5\backslash e)^*$ is 2-outerplanar. Hence $K_{3,3}$ and $(K_5\backslash e)^*$ are both 2-outerplanar. ∎

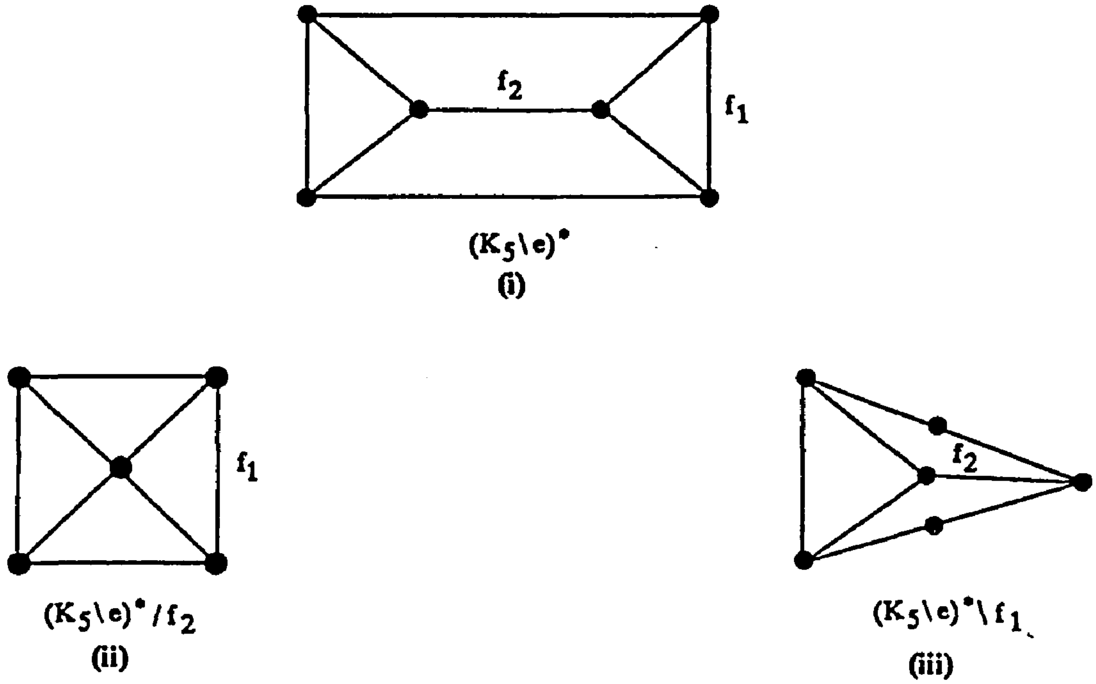


Figure (3.1)

We will now consider the case when G has a $(K_5\backslash e)$-minor. By a series of lemmas we will show that the only non-trivial extension or coextension of $(K_5\backslash e)$ within the class of 2-outerplanar graphs is K_5.

(3.2.3) Lemma: Let G be a single-element coextension of $(K_5\backslash e)$. Then G is not 2-outerplanar.

Proof: Let $(K_5\backslash e)$ be as shown in Figure (3.2)(i). If G is a non-trivial coextension of $(K_5\backslash e)$, then G is obtained from $(K_5\backslash e)$ by splitting a vertex of degree 4. Thus G is isomorphic to one of the graphs shown in Figure (3.2) (ii)-(iii). We will show that G has an edge f such that neither G\f nor G/f is in C_1. This will establish the lemma.

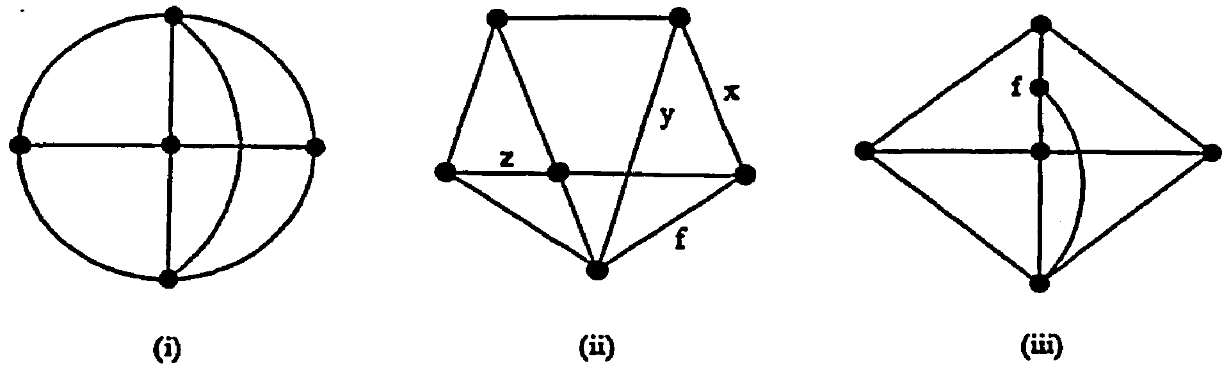


Figure (3.2)

To this end, suppose first that f is the edge labelled in (ii) of the above figure. It is easy to check that $(G\backslash f)\backslash x,z$ is $K_{2,3}$ with a cut edge, and $(G/f)/y$ has a minor isomorphic to K_4 with a loop. Similarly, for the edge labelled f in Figure (3.2)(iii), it is easy to check that both $G\backslash f$ and G/f have as a minor K_4 with a cut edge. We conclude that G is not 2-outerplanar. Hence there are no non-trivial 3-connected coextensions of $(K_5\backslash e)$ within the class of 2-outerplanar graphs.

Now suppose that G is a 2-connected trivial coextension of $(K_5\backslash e)$. Then G is obtained by subdividing an edge f of $(K_5\backslash e)$. Let f_1 and f_2 be the new edges obtained from this subdivision. Since $G/f_1 \cong K_5\backslash e$, and since $(K_5\backslash e)$ is not in C_1, it suffices to show $G\backslash f_1$ is not in C_1. But it is routine to check that $G\backslash f_1$ has as a minor K_4 with a cut edge. Consequently, G cannot be 2-outerplanar. This completes the proof of Lemma (3.2.3). ∎

(3.2.4) Lemma: K_5 is 2-outerplanar.

Proof: By the edge-transitivity of K_5, it suffices to show K_5/f is almost-outerplanar for any edge f of K_5. But K_5/f is isomorphic to K_4 with all the edges incident on a fixed vertex parallel extended. By Proposition (2.3.6), K_5/f is 1-outerplanar. Hence, K_5 is 2-outerplanar. ∎

(3.2.5) Corollary: $K_5\backslash e$ is 2-outerplanar.

Proof: Since $K_5\backslash e$ is a non-outerplanar minor of K_5, it is 2-outerplanar by Lemma (3.1.3). ▮

(3.2.6) Lemma: There are no 2-connected extensions or coextensions of K_5 within the class of 2-outerplanar graphs.

Proof: Clearly there is no non-trivial extension of K_5. On the other hand, if G is obtained from K_5 by adding an edge f parallel to a given edge of K_5, then $G\backslash f$ has as a minor K_4 with a cut edge, while G/f has as a minor K_4 with a loop. Consequently, neither $G\backslash f$ nor G/f is 1-outerplanar or outerplanar. Hence G cannot be 2-outerplanar.

Now suppose G is obtained from K_5 by splitting the vertex v. Let v_1 and v_2 be the new vertices joined by the edge f. Then it is easy to check that $G\backslash f$ and G/f both have as a minor K_4 with a cut edge. Thus G contains an edge f such that neither $G\backslash f$ nor G/f is 1-outerplanar or outerplanar. Consequently, G cannot be 2-outerplanar. This completes the proof of the lemma. ▮

Combining Propositions (3.2.1) and Lemmas (3.2.2)-(3.2.6), we have thus established the following.

(3.2.7) Proposition: Let G be a simple 3-connected graph. Then G is 2-outerplanar if and only if G is an n-wheel for $n \geq 3$, or G is isomorphic to one of $K_{3,3}$, K_5, $K_5\backslash e$ and $(K_5\backslash e)^*$. (See Figure (3.3).)

It still remains for us to describe the non-simple 3-connected 2-outerplanar graphs. In Lemma (3.2.6) we showed that K_5 has no extensions within the class of 2-outerplanar graphs.

We will show, in fact, that only wheels and $K_5\backslash e$ admit trivial extensions within the class of 2-outerplanar graphs, with these extensions themselves being of a very restricted nature.

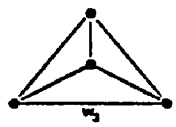
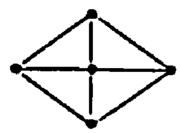
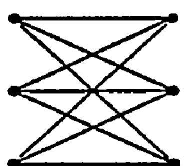
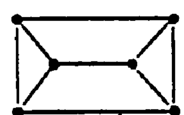
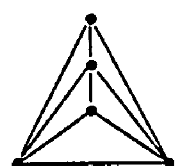
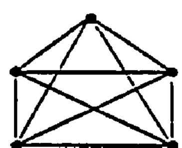
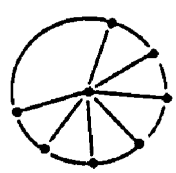
Along the lines of Lemma (3.2.6) we have the following more general result about $K_{3,3}$.

(3.2.8) Lemma: There are no 2-connected extensions or coextensions of $K_{3,3}$ within the class of 2-outerplanar graphs.

Proof: It is routine to check that if G is a non-trivial extension of $K_{3,3}$, then G has an edge f such that both $G\backslash f$ and G/f have as a minor K_4 with a cut edge. Thus neither $G\backslash f$ nor G/f is 1-outerplanar or outerplanar, so G cannot be 2-outerplanar.

Suppose G is obtained by adding an edge f parallel to some edge of $K_{3,3}$. Since $K_{3,3}$ is not in $\mathbf{C}_1$ by Lemma (2.2.5), and since $G\backslash f \cong K_{3,3}$, it suffices to show that G/f is not in $\mathbf{C}_1$. It is easy to check that G/f has as a minor $K_{2,3}$ with a loop. By Proposition (2.5.1) we conclude that G/f is not in $\mathbf{C}_1$. Consequently, G cannot be 2-outerplanar.

Now suppose G is a 2-connected single-element coextension of $K_{3,3}$. Since the degree of every vertex of $K_{3,3}$ is 3, we only need to check the case when G is obtained from $K_{3,3}$ by subdividing an edge f. Let f_1 and f_2 be the new edges obtained by this subdivision. Because $G/f_1 \cong K_{3,3}$ is not in $\mathbf{C}_1$ by Lemma (2.2.5), it suffices to show $G\backslash f_1$ is not in $\mathbf{C}_1$. Let g be an edge of $K_{3,3}$ that is adjacent to f. It is clear that $(G\backslash f_1)/g$ has as a restriction $K_{2,3}$ with a cut edge. Thus $G\backslash f_1$ is not in $\mathbf{C}_1$. Consequently, G cannot be 2-outerplanar. ∎

Number of edges	Graphs		
6		 W_3	
8		 W_4	
9	 $K_{3,3}$		
10		 K_5	
2n		 W_n	

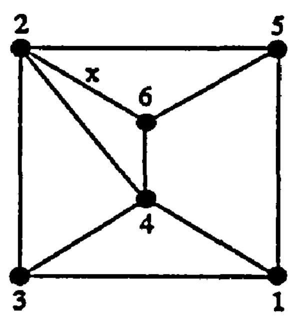
The simple 3-connected 2-outerplanar graphs

Figure (3.3)

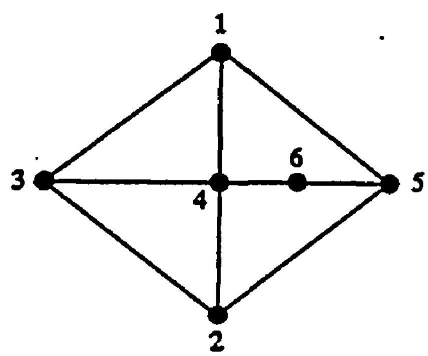
(3.2.9) Lemma: There are no extensions of $(K_5\backslash e)^*$ within the class of 2-outerplanar graphs.

Proof: Suppose G is obtained from $(K_5\backslash e)^*$ by parallel extending an edge f. We need to consider two cases here, depending on whether or not the edge f is in a 3-circuit of $(K_5\backslash e)^*$. Because $(K_5\backslash e)^*$ is not in C_1 by Lemma (2.2.5), and because $G\backslash f \cong (K_5\backslash e)^*$, it suffices to show that G/f is not in C_1 in both cases. If f is not contained in a 3-circuit, then it is easy to check that G/f is isomorphic to W_4 with a loop. Thus by Proposition (2.5.1), G/f is not in C_1 if f is not contained in a 3-circuit. On the other hand, if f is in a 3-circut, then again it is not hard to check that G/f has W_3 with a loop as a minor. Again by Proposition (2.5.1) this means G/f is not in C_1.

Suppose H is a 2-outerplanar single-element extension of $(K_5\backslash e)^*$. By the preceding argument, H must be a non-trivial extension. Consequently H is obtained from $(K_5\backslash e)^*$ by joining non-adjacent vertices of $(K_5\backslash e)^*$. By the symmetry of $(K_5\backslash e)^*$, we may assume H is the graph shown in Figure (3.4)(a).



H
(a)



H\x
(b)

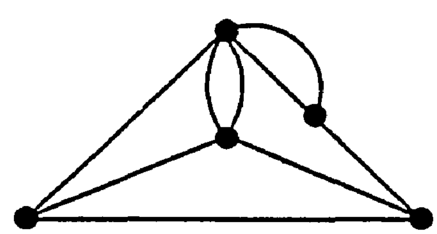

H/x
(c)

Figure (3.4)

Figure (3.4) shows explicitly an edge x for which neither $H\backslash x$ nor H/x is in $\mathbf{C}_1$. It is routine to check that the graph shown in Figure (3.4)(b) has as a minor $K_{2,3}$ with a cut edge, and that the graph shown in Figure (3.4)(c) has as a minor K_4 with a loop. We conclude that H cannot be 2-outerplanar. ∎

(3.2.10) Lemma: Let G be a 3-connected trivial extension of $K_5\backslash e$. Then G is 2-outerplanar if and only if G is obtained from $K_5\backslash e$ by parallel extending only those edges incident on two vertices of degree 4.

Proof: Suppose G is 2-outerplanar and is obtained by adding an edge f in parallel to some edge of $K_5\backslash e$. Let S_1 be the subset of $E(K_5\backslash e)$ consisting of those (three) edges whose ends have degree 4, and $S_2 = E(K_5\backslash e)\backslash S_1$. Because the automorphism group of $K_5\backslash e$ is transitive on S_1 and on S_2, we only need to check two cases here, depending on whether f is parallel to an edge of S_1 or S_2.

In the first case, G/f is clearly outerplanar and so G itself is 2-outerplanar.

In the second case, G/f has as a minor K_4 with a loop, so G/f is not in $\mathbf{C}_1$. As $G\backslash f \cong K_5\backslash e$, it follows that in the second case, G is not 2-outerplanar. ∎

We now want to describe the 2-outerplanar parallel extensions of wheels. First, by Lemma (3.1.1) and Proposition (2.3.6), if G has a W_3-minor but no larger wheel as a minor, then any subset of the edges of G may be parallel extended. We can therefore assume G has a W_4-minor. In Proposition (2.3.5) we showed that parallel extending spokes yields 1-outerplanar, hence, 2-outerplanar graphs. The next lemma describes all the 2-outerplanar parallel extensions of wheels.

(3.2.11) Proposition: Let G be a parallel extension of W_n. If $n \geq 4$, then G is 2-outerplanar if and only if G is obtained from W_n by parallel extending spokes and, possibly, adding exactly one edge parallel to a rim edge. If $n = 3$, then G is 2-outerplanar if and only if G is a parallel extension of W_3.

Proof: Suppose G is a 2-outerplanar parallel extension of W_n, where $n \geq 4$. By the preceding remarks, we only need to consider the case when G is obtained from a wheel by parallel extending rim edges. We first observe that if f is parallel to a rim edge, then G/f has as a minor K_4 with a loop. Hence, by Proposition (2.5.1), G/f is not in $\mathbf{C}_1$. Consequently, it suffices to show G\f is 1-outerplanar. If G is obtained by parallel extending at least two rim edges, or if G is obtained by adding at least two edges parallel to a given rim edge, then G\f is clearly isomorphic to a wheel with at least one rim edge parallel extended. By Proposition (2.3.5), this means G\f cannot be 1-outerplanar. We conclude at most one edge may be added parallel to a rim edge.

Conversely, if one edge is added parallel to a rim edge then clearly G\f is 1-outerplanar by Proposition (2.3.5). This completes the proof of the proposition when $n \geq 4$. For $n = 3$, the result follows by Proposition (2.3.5) and Lemma (3.1.1). ∎

On combining Propositions (3.2.1) and (3.2.11) and Lemma (3.2.10), we can summarize our results in the following.

(3.2.12) Proposition: Let G be a 3-connected graph. If G is simple, then G is 2-outerplanar if and only if G is a wheel, or G is isomorphic to one of $K_{3,3}$, K_5, $K_5 \backslash e$ and $(K_5 \backslash e)^*$. If G is not simple, then G is 2-outerplanar if and only if G is a parallel extension of W_n or $K_5 \backslash e$ described as follows:

(i) If $n \geq 4$, then G is obtained from W_n by parallel extending spokes and, possibly, adding exactly one edge parallel to a rim edge.

(ii) If $n = 3$, then G is any parallel extension of W_3.

(iii) If G is a parallel extension of $K_5\backslash e$, then G is obtained from $K_5\backslash e$ by parallel extending only those edges of $K_5\backslash e$ incident on two vertices of degree 4.

3.3 The connectivity-2 case.

We next consider the case when G is a 2-outerplanar graph with connectivity 2. By Proposition (1.1.3) we can write G as a 2-sum of two 2-connected graphs G_1 and G_2 with respect to the basepoints (u_1,v_1) and (u_2,v_2). Because G is not outerplanar, G has a W_3- or a $K_{2,3}$-minor. We will first consider the case when G has a W_3-minor.

(a) G has a W_3-minor.

By Proposition (1.1.4) we may assume that G_1 has a W_3-minor. By a result of Seymour (1977), the basepoints (u_1,v_1) must be an element of a W_3-minor of G_1. The next lemma describes the structure of G_2.

(3.3.1) Proposition: Let G be a 2-outerplanar graph with connectivity 2, where G is a 2-sum of G_1 and G_2 with respect to the basepoints (u_1,v_1) and (u_2,v_2) and where G_1 has a W_3-minor. If (u_2,v_2) is in a 2-circuit of G_2, then G_2 is a cocircuit or G_2 is isomorphic to K_3 with exactly one edge in a parallel class of size at least two. If (u_2,v_2) is in no 2-circuit of G_2, then G_2 is a circuit with, possibly, exactly one edge in a parallel class of size two.

Proof: First suppose (u_2,v_2) is in a 2-circuit of G_2. If G_2 is a cocircuit, then the proposition holds. Hence we may assume G_2 is not a cocircuit. Let f be an edge of G_2 such that f is not parallel with (u_2,v_2). Let C be a circuit of G_2 containing f and (u_2,v_2). Then $|C| \geq 3$. If $|C| > 3$, then it follows that G_2 has a minor isomorphic to the graph whose edges consist of a circuit C' of size 4 and those edges parallel with (u_2,v_2), where C' contains f and (u_2,v_2). Let $C'=\{(u_2,v_2),f,f_1,f_2\}$. Because (u_2,v_2) is in a 2-circuit of G_2 and because $|C'| = 4$, both $G\backslash f_1$ and G/f_1 have a minor isomorphic to K_4 with a cut edge. By Proposition (2.5.1), this means neither $G\backslash f_1$ nor G/f_1 is in $\mathbf{C}_1$. Consequently, G cannot be 2-outerplanar if $|C| > 3$. Hence, $|C| = 3$. Let C_p denote the parallel class of G_2 containing (u_2,v_2). We will show $E(G_2) = E(C) \cup E(C_p)$. Suppose that there is an edge $g \in E(G_2) \backslash (E(C) \cup E(C_p))$. Let C' be a circuit of G_2 containing g and an edge f of C. If $|C'| = 2$, then the edge f becomes a loop in G/g. This means G/g has as a minor K_4 with a loop. Clearly $G\backslash g$ has as a minor K_4 with a cut edge. Thus neither $G\backslash g$ nor G/g is in $\mathbf{C}_1$, so G cannot be 2-outerplanar. Hence $|C'| \geq 3$. Consequently C is a circuit of both $G_2\backslash g$ and G_2/g. Thus both $G\backslash g$ and G/g have as a minor K_4 with a cut edge. We conclude G cannot be 2-outerplanar. Since g was an arbitrary edge of $E(G_2) \backslash (E(C) \cup E(C_p))$, we must have $E(G_2) = E(C) \cup E(C_p)$.

Next suppose (u_2,v_2) is not in a 2-circuit of G_2. Let C be a minimum-sized circuit of G_2 containing (u_2,v_2). If $C = G_2$, then the result holds. Hence we can assume $G_2 \neq C$. Let $f \notin C$. If f is parallel to an edge of $C\backslash(u_2,v_2)$, we will show that f is the only such edge of G_2. For suppose there are at least two such edges. Because $|C| \geq 3$, both $G\backslash f$ and G/f have as a minor K_4 with a loop. Consequently, G is not 2-outerplanar. We conclude G_2 has at most one edge parallel to an edge of $C\backslash(u_2,v_2)$.

Now suppose f is not parallel to an edge of C. Let C' be a circuit of G_2 containing f and (u_2,v_2). First, suppose that $E(C') \subseteq E(C) \cup \{f\}$. Then f is a chord of C. Thus by the choice of C, we must have that f is parallel to an edge of C, contradicting our choice of f. We conclude $E(C') \not\subset E(C) \cup \{f\}$. Let $g \in E(C')\backslash(E(C)\cup\{f\})$. Because $|C| \geq 3$, and because (u_2,v_2) is in no 2-element circuit, $G\backslash g$ has as a minor K_4 with a cut edge and G/g has as a minor K_4 with a loop. Thus $G\backslash g$ and G/g are not 1-outerplanar. Hence G cannot be 2-outerplanar. We therefore conclude that f must be parallel to an edge of C. Hence G_2 is a circuit with, possibly, exactly one edge in a parallel class of size two. This completes the proof of Proposition (3.3.1). ∎

We have already established that K_5, $K_5\backslash e$, and $K_{3,3}$ have no coextensions in C_2. (Lemmas (3.2.6), (3.2.3), and (3.2.8), respectively.) Furthermore, there are no extensions of $(K_5\backslash e)^*$ in C_2. (Lemma (3.2.9).) Combining these results with Propositions (3.3.1) we have the following.

(3.3.2) Corollary: Let G be a 2-outerplanar graph with connectivity 2 having a W_n-minor. Then G is obtained from W_n or $(K_5\backslash e)^*$ in one of the following ways:

(i) series extending W_n or $(K_5\backslash e)^*$; in the case when G is obtained from W_n, exactly one edge may be added parallel to an edge obtained through this extension.

(ii) parallel extending edges of W_n.

(iii) taking the parallel connection of W_n and triangles where distinct edges of W_n are used as basepoints of these connections.

Remark: We observe that the construction described in (iii) amounts to joining adjacent vertices of W_n by a new path of length 2.

It still remains for us to determine which series of operations described in (i)-(iii) yield 2-outerplanar graphs when applied to W_n and $(K_5\backslash e)^*$. In order to do this, we will partition all such graphs into the following classes:

(a) Those graphs G which have a W_4-minor.

(b) Those graphs G which have a W_3-minor but no larger wheel as a minor.

(c) Those graphs G which are series-parallel networks having a $K_{2,3}$-series-minor.

(a) G has a W_4-minor.

Let G be a graph with connectivity 2 and let $n \geq 4$ be the largest integer such that G has a W_n-minor. Then we can write G as a 2-sum of 2-connected graphs G_1 and G_2 where G_1 has a W_n-minor for some $n \geq 4$. We have the following variant of Corollary (3.3.2).

(3.3.3) Lemma: If G is 2-outerplanar, then G_2 is a circuit with, possibly, one edge in a parallel class of size two, or G_2 is a cocircuit.

Proof: By the preceding remarks, it suffices to show that if G is a parallel connection of W_n and a triangle, then G is not 2-outerplanar. Since W_n has a W_4-minor for every $n \geq 4$, it suffices to show that if G is obtained from W_4 by adding a new path of length two joining two adjacent vertices of W_4, then G is not 2-outerplanar. By the symmetry of the automorphism group of W_4, this entails considering two cases, depending on whether the basepoint of the parallel connection is a rim edge or a spoke (See Figure (3.5)). With the labeling in Figure (3.5), it is routine to check that, in the first case, $G\backslash x$ has as a minor K_4 with a cut edge and G/x has as a minor K_4 with a loop. In the second case, it is again routine to check that $G\backslash x$ has as a minor K_4 with a cut edge while G/x has as a minor $K_{2,3}$ with a cut edge. In either case we conclude by Proposition (2.5.1) that G has an edge for which neither $G\backslash x$ nor G/x is in C_1.

Consequently, G cannot be 2-outerplanar. Combining this with Corollary (3.3.2) completes the proof. ▮

1. 2.

Figure (3.5)

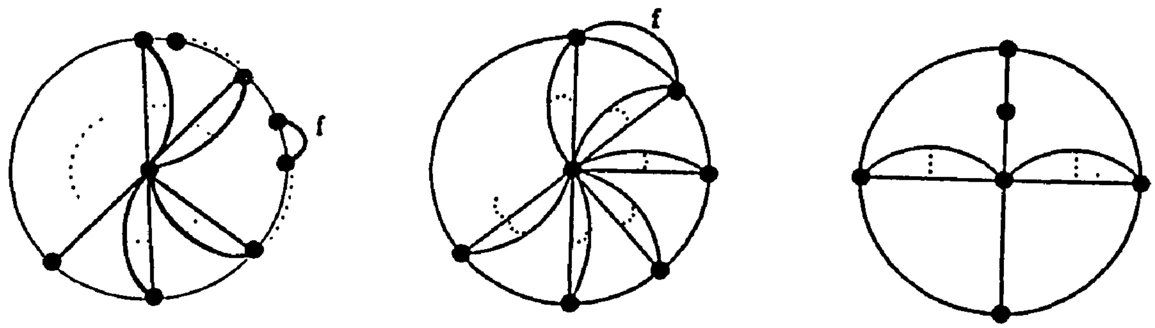
We will describe the 2-outerplanar graphs arising from n-wheels for $n \geq 4$ in the following proposition. Following the proposition, we will describe those arising from $(K_5 \backslash e)^*$.

(3.3.4) Proposition: Let G be a graph with connectivity 2 and suppose that W_n is the largest wheel occurring as a minor of G, where $n \geq 4$. Suppose G does not have a $(K_5 \backslash e)^*$-minor. Then G is 2-outerplanar if and only if G is obtained from W_n in one of the following ways:

(i) series-extending rim edges with, possibly, exactly one edge f added parallel to a new edge obtained through this extension; additional edges may only be added parallel to spokes (Figure (3.6) (i)).

(ii) series-extending rim edges and adding exactly one edge f parallel to a rim edge; additional edges may be added parallel to spokes (Figure (3.6) (ii)).

(iii) in addition, when $n = 4$ subdividing a spoke of W_4; in that case, only neighboring spokes to the subdivided spoke may be parallel extended (Figure (3.6)(iii)).



(i) (ii) (iii)

Figure (3.6)

Proof: Suppose G is 2-outerplanar. Lemma (3.3.3) implies, in particular, that G may be obtained from W_n by replacing an edge of W_n with a path of length at least two and then, possibly, adding an edge parallel to an edge of this path. First, suppose this path has length at least three. Then we claim the edge replaced by this path must be a rim edge. To see this, suppose instead that this edge is a spoke. Since W_n contains a W_4-minor for all $n \geq 4$, it suffices to show that if a spoke of W_4 is replaced by a path of length at least three, then the graph G obtained is not 2-outerplanar. Let x be any edge of this new path. It is easily checked that $G\backslash x$ has as a minor K_4 with a cut edge. Since G/x is isomorphic to W_4 with a spoke subdivided, we conclude from Proposition (2.3.5) that neither $G\backslash x$ nor G/x is in C_1. Hence G is not 2-outerplanar.

Next, suppose a rim edge is replaced by a path of length at least 2, and exactly one edge x is added parallel to an edge of this path. We claim we can only series extend rim edges of this graph or parallel extend spokes within the class of 2-outerplanar graphs. For suppose a second series class containing a rim element has an edge y added in parallel to it. It is easy to check in this case that both $G\backslash y$ and G/y have as a minor K_4 with a loop.

On the other hand, suppose any of the remaining rim edges of G are series extended and any of the spokes are parallel extended. We observe that contracting any spoke or deleting any edge on the rim in a trivial parallel class yields an outerplanar graph. If the rim edge is in a parallel class of size at least two, then its deletion yields a 1-outerplanar graph by Proposition (2.3.5). We have shown that if G is obtained from a wheel by replacing a rim edge by a path of length at least two and then adding an edge parallel to an edge of this path, then the only further series-parallel extensions of G within the class of 2-outerplanar graphs are series extensions of rim edges and parallel extensions of spokes.

Now suppose we subdivide a spoke of W_n. We will show that the graph obtained in this way is 2-outerplanar if and only if $n = 4$. If $n \geq 5$, it suffices to show by taking a W_5-minor that subdividing a spoke of W_5 yields a graph that is not 2-outerplanar. Using the labeling in Figure (3.7), it is routine to check that G/x has as a minor $K_{2,3}$ with a cut edge. Clearly G\x is homeomorphic from W_4 with a spoke subdivided. Thus by Proposition (2.3.5) neither G\x nor G/x is in C_1. Hence G is not 2-outerplanar if $n \geq 5$.

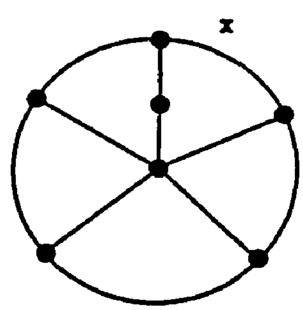


Figure (3.7)

On the other hand, it is routine to check that subdividing a spoke of W_4 yields a 2-outerplanar graph. Let G be the graph obtained from W_4 by subdividing a spoke. We claim no further series extensions of W_4 are possible within the class of 2-outerplanar graphs, and that only the neighboring spokes may be parallel extended. To see this, first suppose another edge of W_4 is subdivided. Then G is isomorphic to one of the graphs shown in Figure (3.8).

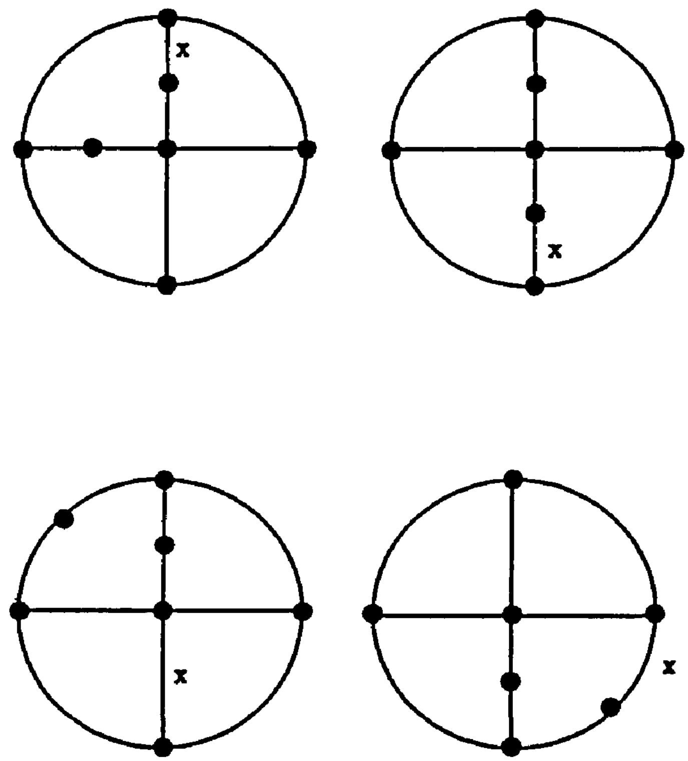


Figure (3.8)

Using the labeling of this figure, it is routine to check that neither G\x nor G/x is in C_1. We conclude that no further series extensions are possible within the class of 2-outerplanar graphs. Finally, for the remaining cases here it is routine to check that only the neighboring spokes of the spoke subdivided may be parallel extended otherwise G has an edge f such that neither G\f nor G/f is in C_1. (See Figure (3.9).)

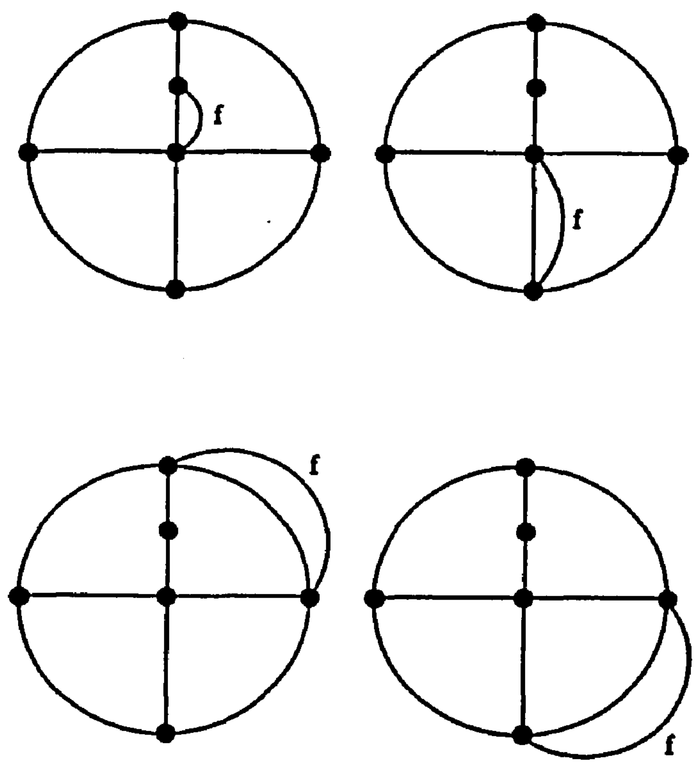


Figure (3.9)

We have now shown that if G is obtained from W_n by series extending a spoke, then G is 2-outerplanar if and only if G is, in fact, obtained from W_4 by subdividing a spoke and, possibly, parallel extending the neighboring spokes.

Now suppose that G is obtained from W_n by adding an edge f parallel to a rim edge. We have already established that no spokes may be subdivided within the class of 2-outerplanar graphs. In fact, using arguments similar to those above, it is routine to check that G is 2-outerplanar if and only if either the remaining rim edges are subdivided or edges are added parallel to spokes. This completes the proof of Proposition (3.3.4). ∎

Now let G be a graph with connectivity 2 having a $(K_5\backslash e)^*$-minor. In Lemma (3.2.9) we showed there are no extensions of $(K_5\backslash e)^*$ within the class $\mathbf{C}_2$. Since $(K_5\backslash e)^*$ has a W_4-minor, we can apply Lemma (3.3.2) to conclude that if G is 2-outerplanar, then G is homeomorphic from $(K_5\backslash e)^*$.

Let S be the three edges of $(K_5\backslash e)^*$ that are contained in no 3-circuit of $(K_5\backslash e)^*$. We will establish the following.

(3.3.5) Proposition: Let G be a graph with connectivity 2 having a $(K_5\backslash e)^*$-minor. Then G is 2-outerplanar if and only if G is obtained from $(K_5\backslash e)^*$ by series extending some subset of the edges of S.

Proof: Suppose G is 2-outerplanar. We have observed that G is a series extension of $(K_5\backslash e)^*$. Suppose G is obtained by subdividing an edge e not in S. If x is an edge obtained through this subdivision, it is routine to check that neither $G\backslash x$ nor G/x is not in $\mathbf{C}_1$. Hence G cannot be 2-outerplanar.

On the other hand, suppose G is obtained from $(K_5\backslash e)^*$ by series extending any member of S. Then G is as shown in Figure (3.10) where the path P has length at least two. It is routine to check that $G\backslash x$ is outerplanar for any edge $x \in P \cup \{x_1,x_2\}$ and $G\backslash y$ is 1-outerplanar for any other edge y. Hence G is 2-outerplanar. This completes the proof of Proposition (3.3.5). ∎

This completes our characterization of the 2-outerplanar graphs having a W_4-minor.

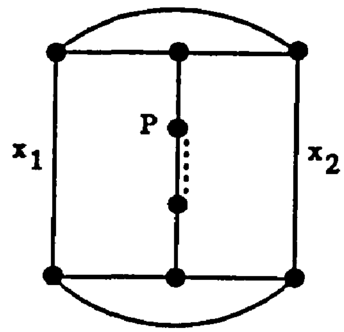


Figure (3.10)

(b) G has a W_3-minor but no larger wheel as a minor.

In this section we describe the 2-outerplanar graphs with connectivity 2 having a W_3-minor but no larger wheel as a minor. By Proposition (3.3.1), if G is 2-outerplanar then G must be obtained from W_3 by a sequence of the following operations:

(i) replacing an edge of W_3 with a path of length at least two and then, possibly, adding an edge parallel to an edge of this path;

(ii) adding edges parallel to edges of W_3;

(iii) joining two vertices of W_3 by a new path of length 2.

In what follows, we will consider which combination of these operations yield 2-outerplanar graphs. As many of our arguments are similar to those used in the previous section, we will omit the more routine proofs of this section.

(3.3.6) Lemma: Let G be a graph obtained from W_3 by replacing an edge e of W_3 by a path of length at least 2 and then adding an edge parallel to an edge of this path. Then the only further operations which can be performed on G within the class of 2-outerplanar graphs are the following:

(i) series extending any member of {f,g}, where {e,f,g} is a triangle of W_3;

(ii) parallel extending any other edge y not in {e,f,g}.

Proof: The proof is a routine check of which sequence of the operations (i)-(iii) as described above yield 2-outerplanar graphs when performed on G. The details are omitted. ∎

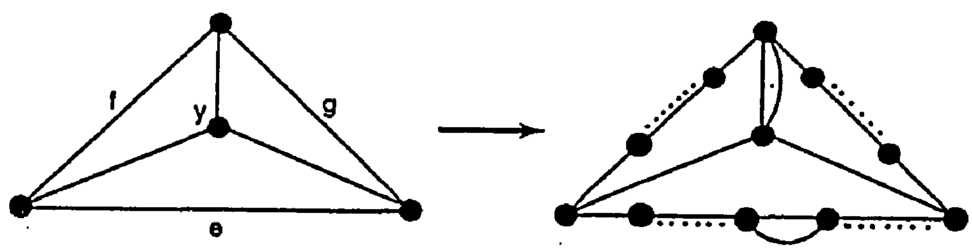


Figure (3.11)

(3.3.7) Lemma: Let G be a graph obtained from W_3 by joining two vertices of W_3 by a new path P of length 2. Then the only further operations which can be performed on G within the class of 2-outerplanar graphs are those illustrated in Figures (3.12) and (3.13).

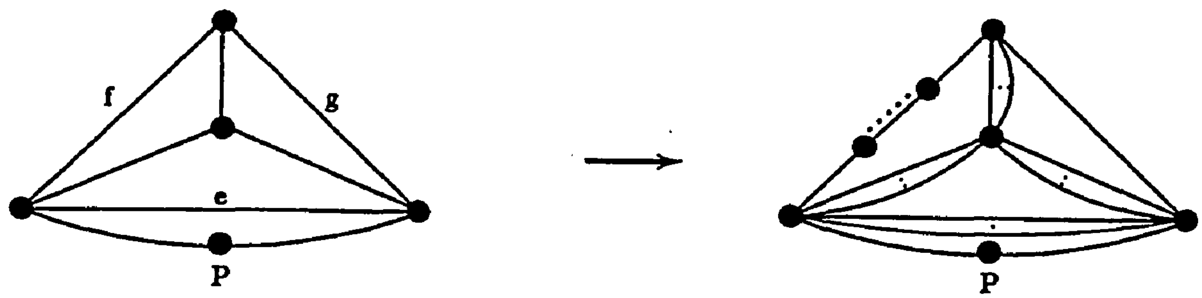


Figure (3.12)

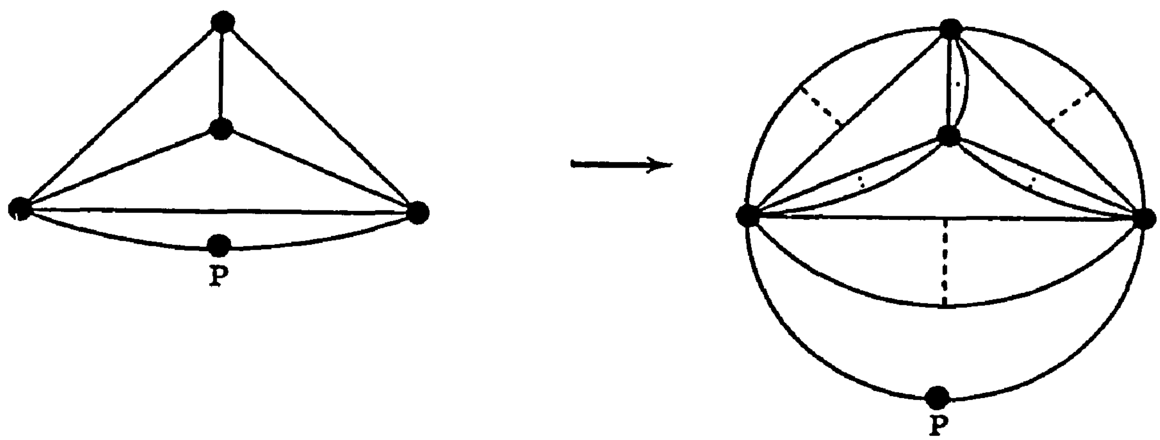


Figure (3.13)

Proof: The following steps outline a sketch of the proof. The details of these steps are routine and are omitted.

(a) No new path of length 2 joining vertices of G may be added to G.

(b) One and only one edge f of G may be subdivided where e and f are in a 3-circuit, say {e,f,g}, of G. Only edges parallel to those edges not on P or in {e,f,g} may be added to G. (See Figure (3.12).)

(c) Any edge not in P may be parallel extended in the case when we consider only parallel extensions of G. (See Figure (3.13).)

(3.3.8) Lemma: Let G be a graph obtained from W_3 by either series extending edges of W_3 or by parallel extending edges of W_3, but not by a combination of both. Then G is 2-outerplanar if and only if G is one of the graphs shown in Figure (3.14).

Proof: Again, the details of the proof amount to checking which sequence of series extensions and parallel extensions yield 2-outerplanar graphs when applied to W_3. We omit these details. We observe that the graphs shown in Figure (3.14) (v) are 1-outerplanar, and, therefore, they are 2-outerplanar by Lemma (3.1.1). In Figure (3.14) (i), $\{x_1,x_2\}$ denotes a parallel class of G. ∎

Combining Lemmas (3.3.6)-(3.3.8) gives us a full description of the 2-outerplanar graphs with connectivity 2 having a W_3-minor but no larger wheel as a minor.

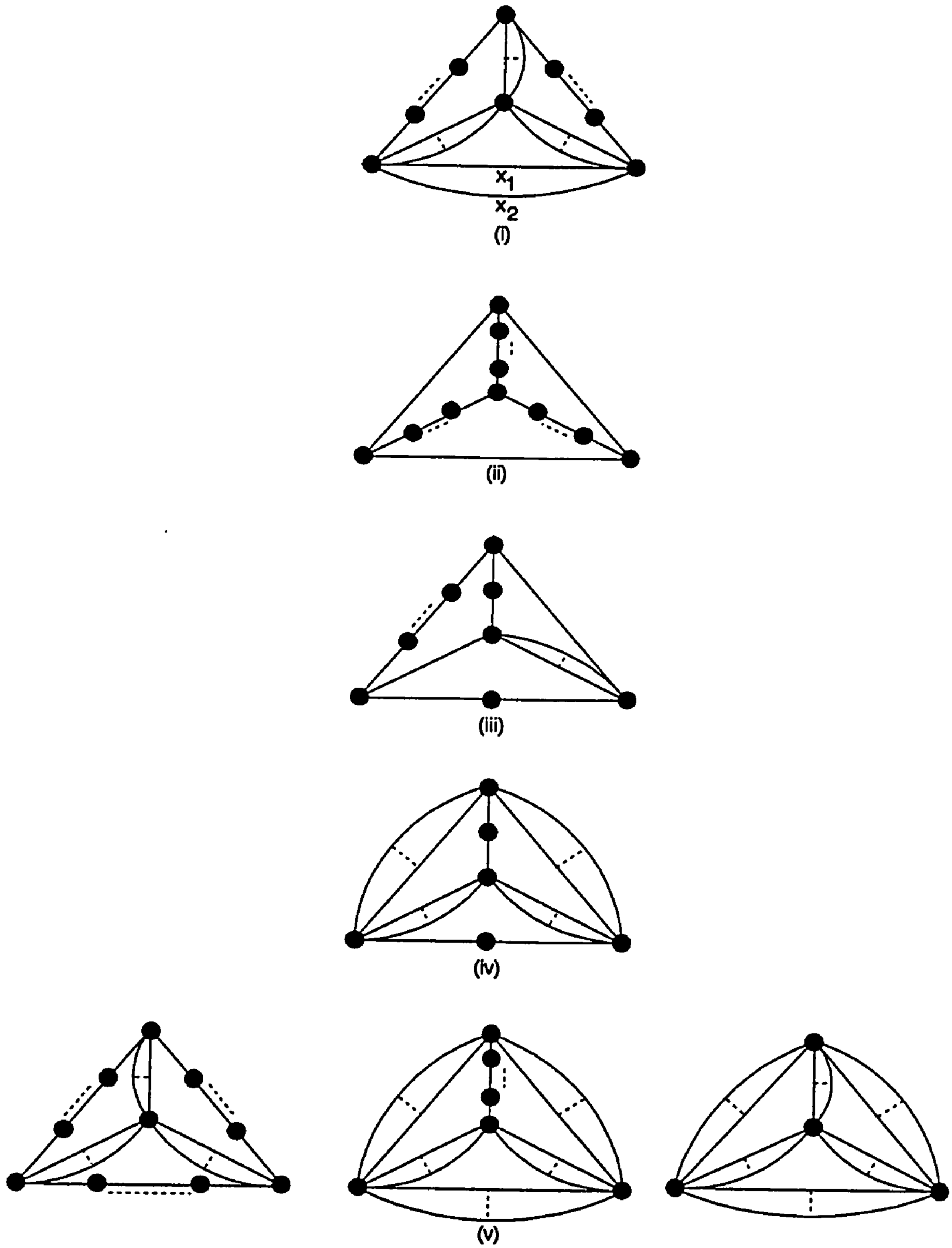


Figure (3.14)

(c) G is a series-parallel network having a $K_{2,3}$-series-minor.

In this section we describe the 2-outerplanar graphs which are series-parallel networks having a $K_{2,3}$-series-minor. An immediate consequence of Lemma (2.1.5) is the following.

(3.3.9) Lemma: Let G be a 2-connected 2-outerplanar graph. Then there exists subsets X and Y of E(G) such that G\X/Y is isomorphic to either K_4 or $K_{2,3}$ where every element of Y is in series with some element of G\X not in Y, or G has a $K_{2,4}$-series-minor.

(3.3.10) Proposition: Let G be a series-parallel network. G is 2-outerplanar if and only if G is isomorphic to one of the following:

(i) a 1-outerplanar graph as described in Proposition (2.3.9).

(ii) a 1-outerplanar graph obtained from $K_{2,3}$ or $K_{2,3}^{+}$ by adding a loop.

(iii) a graph obtained from $K_{2,3}$ in the following way: replacing one of the paths (or non-trivial series classes) of $K_{2,3}$ joining the two degree three vertices by a new path whose length is at least three and then adding exactly one edge parallel to an edge of this new path; from each of the remaining two non-trivial series classes of $K_{2,3}$ we can do exactly one of the following: series extend some of the edges in the class, or parallel extend some of the edges in the class. Moreover, we can add any number of new edges joining the two degree three vertices of $K_{2,3}$.

(iv) a graph obtained from $K_{2,3}$ as follows: join a degree three vertex to a degree two vertex by a new path of length two. For the remaining edges of $K_{2,3}$ we can do exactly one of the following: series extend some of the edges in a non-trivial series class, or parallel extend some of the edges in such a class.

Moreover, we can add any number of edges joining the two degree three vertices of $K_{2,3}$.

(v) $K_{2,4}$ with any number of new edges joining the two degree four vertices of $K_{2,4}$.

Proof: Suppose G is 2-outerplanar. We will establish Proposition (3.3.10) by a series of lemmas. To begin, if G is 1-outerplanar, then (i) follows immediately from Lemma (3.1.1) and Proposition (2.3.9). So we can assume G is 2-outerplanar but not 1-outerplanar. Consequently by Lemma (3.3.9), G\X/Y is isomorphic to $K_{2,3}$ for some subsets X and Y of E(G), where every element of Y is in series with an element of G\X not in Y, or G has a $K_{2,4}$-series-minor. We will first consider the case when X has a loop or cut edge, that is, the case when G\X is homeomorphic from $K_{2,3}$ with a loop or cut edge.

(3.3.11) Lemma: If G\X is homeomorphic from $K_{2,3}^+$ with a loop, then G is isomorphic to a 1-outerplanar graph obtained from $K_{2,3}$ or $K_{2,3}$ by adding a loop.

Proof: By Lemma (3.3.9) we know that G\X is isomorphic to one of the graphs shown in Figure (3.15), where each of the paths P_1,P_2, and P_3 joining u and v has length at least two and where P_4 is a closed path beginning at u or w, where w is an internal vertex of P_3, say.

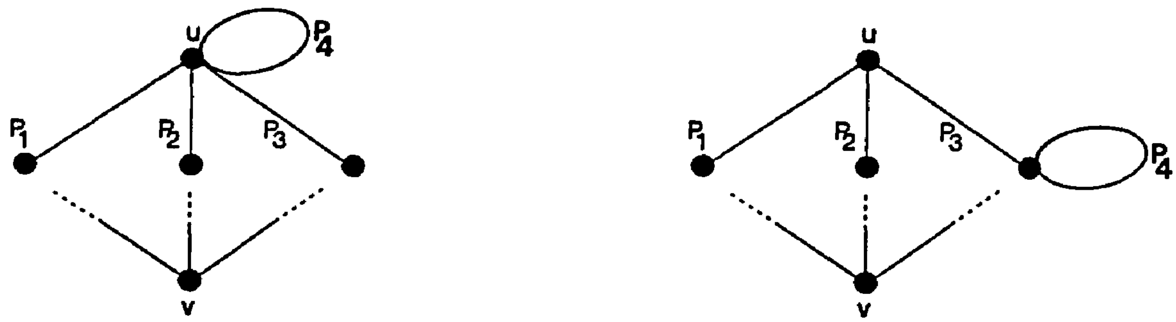


Figure (3.15)

It is obvious that if P_4 is not a loop, that is, if P_4 has length at least two, then the deletion of any edge of P_4 from G has as a minor $K_{2,3}$ with a cut edge while the contraction of any such edge has as a minor $K_{2,3}$ with a loop. Hence P_4 is a loop. Furthermore, every edge e of X must join two vertices of G\X otherwise both G\e and G/e have as a minor $K_{2,3}$ with a loop. Since G has no W_3-minor, e does not join an internal vertex of one of P_1,P_2, and P_3 to an internal vertex of a different one of these paths. Thus, every edge of X must join vertices of some P_i for $i \in \{1,2,3\}$. This means the edges of X are either parallel to edges of P_i or join non-adjacent vertices of P_i. The proof of Lemma (3.3.11) now follows similar lines to the proof of Proposition (2.3.9). ∎

We will now assume G\X is homeomorphic from $K_{2,3}$ with a cut edge. We will establish the following.

(3.3.12) Lemma: If G\X is homeomorphic from $K_{2,3}$ with a cut edge, then G is isomorphic to either $K_{2,4}$ with any number of edges joining the two degree four vertices of $K_{2,4}$, or G is obtained from $K_{2,3}$ as described in (iii) or (iv) above.

Proof: Let e = uw be the cut edge of G\X. Since G has connectivity 2 and has no W_3-minor, it is not hard to show that we can assume in what follows that u has degree three. Since G has connectivity 2, there is a path joining w to some vertex of G\X. First, suppose w is joined to v where v has degree 3. If w and v are joined by a path whose length is at least two, then it is clear that both the graphs obtained by deleting and contracting an edge of this path have as a minor $K_{2,3}$ with a cut edge. Hence w and v are incident in G. Thus u and v are joined by four paths whose lengths are at least two. As above, it is easy to check that these paths must have length 2. The proof that G can be obtained from $K_{2,4}$ by adding any number of new edges joining the two degree four vertices of $K_{2,4}$ is similar to the proof of Proposition (2.3.9).

Now suppose w is joined to an internal vertex of one of the paths P_i. If w is not joined to such a vertex by an edge, then G is not 2-outerplanar. We will show that this path P_i must have length two. For suppose that $i = 1$, say, and that P_1 has length at least three. It is routine to check that G\e has as minor $K_{2,3}$ with a cut edge and that G/e has as a minor $K_{2,3}$ with a loop or cut edge. (See Figure (3.16).) Hence P_1 has length two. Let $P_1=uu_1v$. It is routine to show that only edges parallel to uu_1 may be added parallel to edges of P_1. The remainder of the proof is similar to the proof of Proposition (2.3.9). ∎

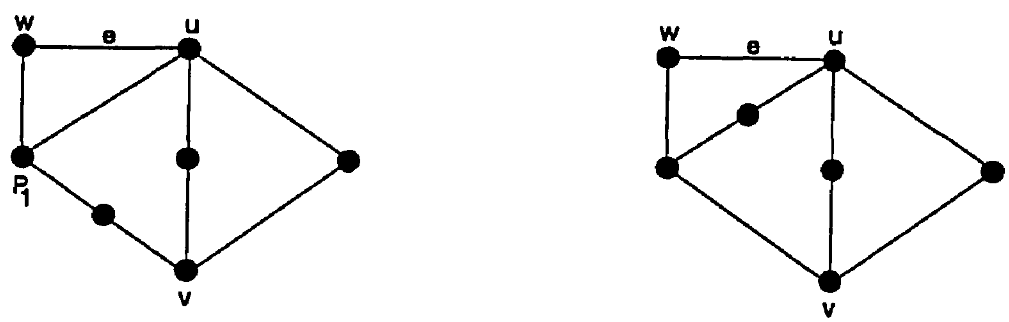


Figure (3.16)

In a similar way, we can show the following.

(3.3.13) Lemma: If G\X is homeomorphic from $K_{2,3}$ with a cut edge with one end an initial vertex of one of the paths P_i, then G is isomorphic to a graph obtained from $K_{2,3}$ as described above in (iv).

It remains for us to consider the case when G\X is homeomorphic from $K_{2,3}$. We have already considered the case when G\X is homeomorphic from $K_{2,3}$ with a loop or cut edge. We can therefore assume that $G \backslash X$ is as shown in Figure (3.17) where each of the paths P_1, P_2 and P_3 joining u and v has length at least two.

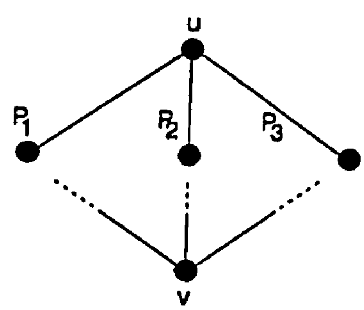


Figure (3.17)

Again, it is routine to show using the technique of Proposition (2.3.9) that G is isomorphic to a graph obtained from $K_{2,3}$ as described in (ii). Combining this with Lemmas (3.3.11)-(3.3.13) establishes the necessity of Proposition (3.3.10).

The proof of the converse is much quicker. Suppose G is obtained from $K_{2,3}$ as described above. Because every 1-outerplanar graph is 2-outerplanar, we may assume G is a graph obtained from $K_{2,3}$ as described in (ii)-(v). Clearly if G is as described in (ii), then deleting the loop leaves a 1-outerplanar graph while the deletion or the contraction of any remaining edge yields an outerplanar graph. If G is a graph as described in (v), then again it is easy to check by Proposition (2.3.9) that the contraction of any edge is either 1-outerplanar or outerplanar. Suppose, then, that G is obtained from $K_{2,3}$ as described in (iii) or (iv). (See Figure (3.18).)

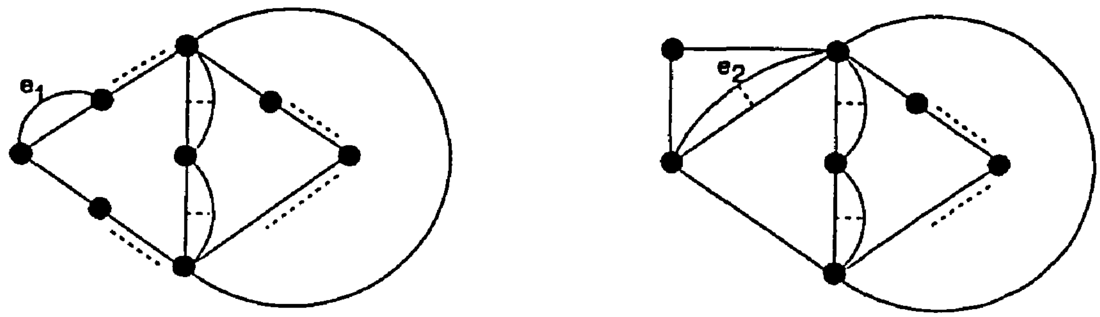


Figure (3.18)

Using the labeling of that figure, it is routine to check that G\x is 1-outerplanar for any edge x in the same parallel class as e_1, and that G/y is 1-outerplanar for any edge y in the non-trivial series class containing e_2. Clearly the deletion or contraction of any of the remaining edges yields an outerplanar graph. This completes the proof of Proposition (3.3.10). ∎

This completes our characterization of 2-outerplanar graphs with connectivity 2. Combining the results of Sections 2 and 3 with Proposition (3.1.4) and (3.1.5) completes our characterization of 2-outerplanar graphs.

Chapter 4

A Characterization of α-Outerplanar Graphs

4.1 Introduction.

In this chapter we consider another class of graphs arising in a natural way from outerplanar graphs. Chartrand and Harary's characterization of outerplanar graphs (Theorem (1.1.1)) tells us that there are just two non-outerplanar graphs, K_4 and $K_{2,3}$, for which every proper minor is outerplanar. We now ask the question: what are the non-outerplanar graphs G such that, for some edge α of G, both $G\backslash\alpha$ and G/α are outerplanar? We call such graphs α-outerplanar.

Throughout this chapter, whenever we use α to denote an edge of a graph G, it will be implicit that both the deletion and contraction of α from G are outerplanar.

From Theorem (1.1.1), it is immediate that K_4 and $K_{2,3}$ are α-outerplanar. In this chapter we characterize all α-outerplanar graphs and describe some properties of the members of this class.

To begin, let C_α denote the class of α-outerplanar graphs together with the class of outerplanar graphs. We have the following:

(4.1.1) Lemma: $\mathbf{C}_\alpha$ is closed under the taking of minors.

Proof: Suppose that both $G\backslash\alpha$ and G/α are outerplanar, and let H be a proper minor of G. If $\alpha \in E(H)$, then it is immediate that both $H\backslash\alpha$ and H/α are outerplanar. Hence $H \in \mathbf{C}_\alpha$ . . If $\alpha \notin E(H)$, then H is a minor of at least one of $G\backslash\alpha$ and G/α. Hence H is an outerplanar minor of G. Consequently, H is a member of $\mathbf{C}_\alpha$. ∎

The next two propositions describe some of the general structure of those α-outerplanar graphs that are not 2-connected. The elementary proofs of both are omitted.

(4.1.2) Proposition: Let G be an α-outerplanar graph with connectivity 1. Then G is a union of outerplanar blocks and exactly one α-outerplanar block.

(4.1.3) Proposition: Let G be a disconnected α-outerplanar graph. Then G is a disjoint union of connected outerplanar graphs and exactly one connected α-outerplanar graph.

4.2 The 3-connected case.

In this section we will characterize the 3-connected α-outerplanar graphs.

(4.2.1) Lemma: Let G be a connected graph which has a W_m-minor for some $m \geq 4$. Then G is not α-outerplanar.

Proof: Let W_n be the largest wheel which is a minor of G. Let e be a rim edge and f be a spoke of this W_n-minor. Because both W_n/e and $W_n\backslash f$ have K_4-minors, both G/e and $G\backslash f$ have K_4-minors. If g is an edge of G which is not an edge of this W_n-minor, then clearly $G\backslash g$ or G/g

has a W_n-minor. Thus G does not have an edge whose deletion and contraction are both outerplanar. Hence G is not α-outerplanar. ∎

The following proposition is a straightforward consequence of the last lemma.

(4.2.2) Proposition: Let G be a simple 3-connected graph. Then G is α-outerplanar if and only if $G \cong W_3$.

Proof: Suppose G is α-outerplanar. By Lemma (4.2.1), G has no W_4-minor. Since W_3 is the only simple 3-connected graph with no W_4-minor, we conclude $G \cong W_3$.

Conversely, if $G \cong W_3$, then clearly both $G\backslash x$ and G/x are outerplanar for every edge x of G. Hence G is α-outerplanar. ∎

By Proposition (4.2.2), if G is a 3-connected α-outerplanar graph, then G is obtained from W_3 by adding loops or by parallel extending edges of W_3. We observe that if G is α-outerplanar, then adding loops to G will not affect this property. Moreover we have the following.

(4.2.3) Lemma: Let H be an α-outerplanar graph. Then any edge of H not in the parallel class containing α may be parallel extended within the class of α-outerplanar graphs.

Proof: Let e be an edge of H not in parallel with α. Then e is an edge in $H\backslash\alpha$ and H/α, neither of which has a W_3- or $K_{2,3}$-minor. Let f be added to H in parallel with e, the resulting graph being H'. Clearly neither $H'\backslash\alpha$ nor H'/α has a W_3- or $K_{2,3}$-minor. The lemma now follows. ∎

We remark that parallel extending α may create a K_4- or $K_{2,3}$-minor in $H\backslash\alpha$ or H/α. For instance, if $H \cong W_3$ and every edge of H is parallel extended, then $H\backslash e$ has a W_3-minor for every edge e of H.

Now fix an edge of W_3. By the edge transitivity of W_3 we can choose any edge of W_3 here. Combining Proposition (4.2.2) and Lemma (4.2.3) we have the following.

(4.2.4) Proposition: Let G be a 3-connected non-outerplanar graph. Then G is α-outerplanar if and only if G is obtained from W_3 by adding loops or by parallel extending any set of edges of $W_3\backslash\alpha$, where α is any edge of W_3.

4.3 The connectivity 2 case.

In this section we will characterize the α-outerplanar graphs with connectivity 2. If G is such a graph, then since G is, in particular, non-outerplanar, it has a W_3- or $K_{2,3}$-series-minor. We will partition the set of all such α-outerplanar graphs into the following classes:

(a) Those graphs G which have a W_3-series-minor.

(b) Those graphs G which are series-parallel networks having a $K_{2,3}$-series-minor.

(a) G has a W_3-series-minor.

Before we begin describing the α-outerplanar graphs having a W_3-series-minor, we establish the following.

(4.3.1) Lemma: Let G be a 2-connected α-outerplanar graph. Let H be isomorphic to W_3 or $K_{2,3}$, and suppose G has H as a series-minor. Then there are subsets X and Y of E(G) such that

$G \backslash X/Y$ is isomorphic to H where every element of Y is in series with an element of $G \backslash X$ not in Y.

Proof: The proof follows immediately from the proof of Lemma (2.1.5). ∎

Now let G be an α-outerplanar graph with connectivity 2 having a W_3-series-minor. By the last lemma, there are subsets X and Y of E(G) such that $G \backslash X/Y \cong W_3$ where every edge of Y is in series with an edge of $G \backslash X$ not in Y. We have the following.

(4.3.2) Lemma: $G \backslash X$ is obtained from W_3 by series extending at most two distinct edges.

Proof: The fact that $G \backslash X$ is homeomorphic from W_3 follows by Lemma (4.3.1). We will show that if $G \backslash X$ is homeomorphic from W_3 with at least three distinct edges of W_3 series extended, then G is not α-outerplanar. To this end, it suffices to show that if G is obtained from W_3 by subdividing three distinct edges of W_3, then G is not α-outerplanar. Suppose, then, G is obtained from W_3 in this way. Then the symmetry of W_3 implies that G is isomorphic to one of the graphs shown in Figure (4.1). It is routine to check that none of the graphs shown in that figure is α-outerplanar. ∎

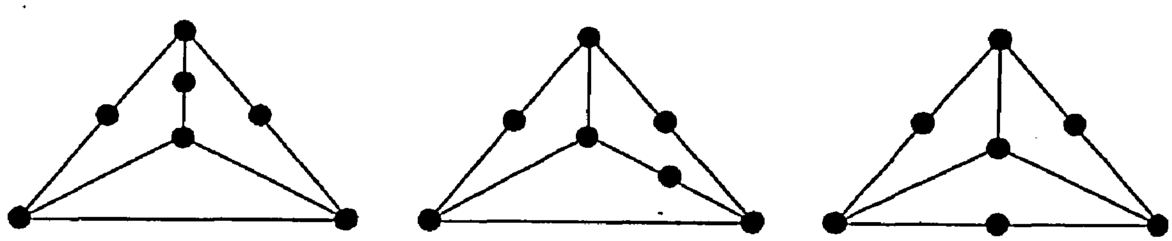


Figure (4.1)

We now know that if G is an α-outerplanar graph with connectivity 2 having W_3 as a series-minor, then, for some subset X of E(G), G\X is isomorphic to one of the graphs shown in Figure (4.2), where each of P_1 and P_2 has length at least two.

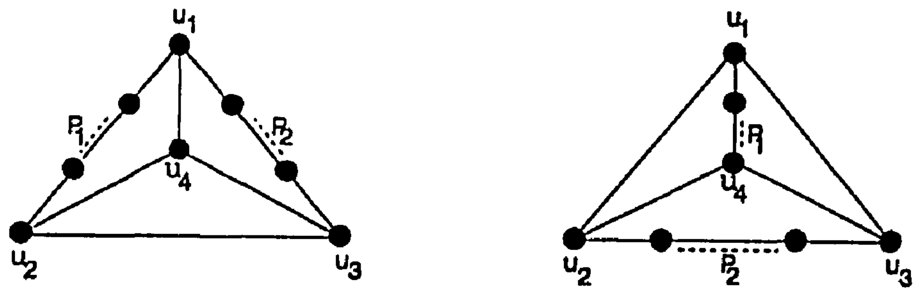


Figure (4.2)

Now every edge e of X must join two vertices of G\X. Hence edges of X either join a vertex of P_1 or P_2 to a vertex not on this path, or are parallel to an edge of G\X that is not in P_1 or P_2, or join vertices of some P_i for i=1 or 2. Edges of the first type described here induce a W_4-minor in G (see Figure (4.3)). Hence, since G cannot have a W_4-minor by Lemma (4.2.1), we conclude the edges of X must be of the second or third type.

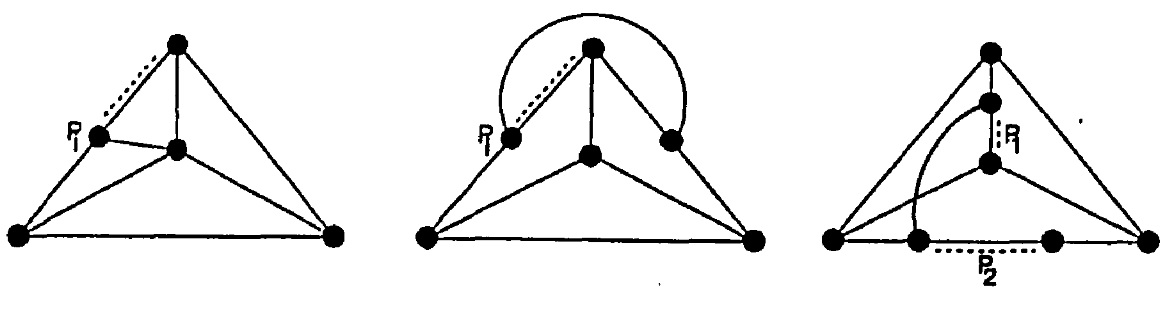


Figure (4.3)

Edges which join vertices of some P_i for i=1 or 2 are either parallel to edges of P_i or join non-adjacent vertices of P_i. Now suppose that X contains an edge e that joins two non-adjacent vertices v and w of P_i, where i=1 or 2. Let [v,w] denote the segment of P_i joining v and w. We will say the edge e <u>spans the interval</u> [v,w] of P_i. Referring to the graph H shown in Figure (4.4) we have the following.

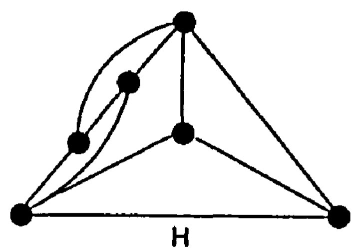


Figure (4.4)

(4.3.3) Lemma: Let H' be a graph having a minor isomorphic to H. Then H' has a $K_{2,4}$-minor and hence H' is not α-outerplanar.

Proof: Clearly $H\backslash e_1,e_2 \cong K_{2,4}$. By Lemma (4.1.1) we conclude H' is not α-outerplanar. ∎

Referring to Figure (4.2), Lemma (4.3.3) says that if edges are added joining vertices of P_i such that two of these edges span intervals sharing a common edge, then one of these intervals is contained in the other, otherwise we obtain a graph that is not α-outerplanar. Hence the edges of X either span intervals of P_i, or are parallel to edges of P_i, or are parallel to an edge of $G\backslash X$ that is not in P_1 or P_2. Moreover, if x and y are in X and both span intervals of P_i then either these intervals do not share a common edge or one is contained in the other. Before describing the class of α-outerplanar graphs which have a W_3-series-minor, we state the following proposition.

(4.3.4) Proposition: Let G be a 2-connected outerplanar graph. Then the simple graph associated with G has a unique embedding in the plane such that its infinite face is a hamiltonian cycle.

Proof: By a result of Tang (1964), a simple 2-connected outerplanar graph has a unique hamiltonian cycle. Hence the simple graph associated with G has a unique embedding in the plane so that the infinite face of G is this hamiltonian cycle. ∎

Accordingly, we will say that G is a canonical embedding of a 2-connected outerplanar graph, or, equivalently, G is a canonical 2-connected outerplanar graph if the infinite face of G is a hamiltonian cycle. We are now ready to state the main result of this section.

(4.3.5) Proposition: Let G be a graph with connectivity 2 having a W_3-series-minor. Then G is α-outerplanar if and only if G is isomorphic to a non-outerplanar minor of the graph H obtained from W_3 as follows:

(i) Take two distinct edges of W_3. At each of these, parallel connect a canonical 2-connected outerplanar graph where the basepoint of the latter graph is any edge on the infinite (hamiltonian) face.

(ii) Parallel extend the edges of $W_3\backslash\alpha$, where α is an edge of W_3 that is not a basepoint of a parallel connection as described in (i).

Proof: Suppose G is α-outerplanar. By Lemma (4.2.3), (4.3.2), (4.3.3), and (4.3.4), and by the remarks preceding Proposition (4.3.4), G is obtained from W_3 by taking the parallel connection of W_3 with canonical 2-connected outerplanar graphs where the basepoints of these parallel connections are distinct edges of the paths P_1 and P_2 shown in Figure (4.2) and any edge on the infinite (hamiltonian) face of these outerplanar graphs. It is easy to see that such a graph is isomorphic to a non-outerplanar minor of the graph H obtained from W_3 as described in (i) and (ii).

Conversely, suppose H is obtained from W_3 as described above. Then H looks like one of the two graphs shown in Figure (4.5), where G_1 and G_2 are canonical outerplanar graphs and where any edge other than the edge labeled α represents a non-trivial parallel class. Using the labeling of that figure, it is routine to check that both $G\backslash\alpha$ and G/α are outerplanar. Hence G is α-outerplanar. ∎

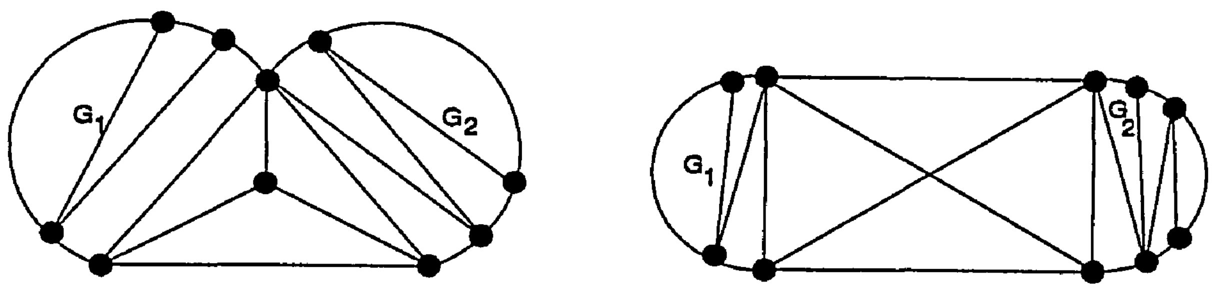


Figure (4.5)

(b) G is a series-parallel network having a $K_{2,3}$-series-minor.

In this section we consider the case when G is an α-outerplanar graph with connectivity 2 having no W_3-minor. By Lemma (4.3.1), there are subsets X and Y of E(G) such that $G\backslash X/Y \cong K_{2,3}$ where every element of Y is in series with an element of $G\backslash X$ not in Y. We begin with the following.

(4.3.6) Lemma: $G\backslash X$ is obtained from $K_{2,3}$ by series extending edges from at most two distinct non-trivial series classes of $K_{2,3}$.

Proof: By Lemma (4.3.1) it suffices to show that if G' is obtained from $K_{2,3}$ by series extending at least one edge of each of the three non-trivial series class of $K_{2,3}$, then G' is not

α-outerplanar. But if G' is obtained from $K_{2,3}$ in this way, it is clear G'/e has a $K_{2,3}$-minor for every edge e of G'. ∎

By Lemma (4.3.5), G\X is as shown in Figure (4.6) where each of the paths P_1 and P_2 has length at least two.

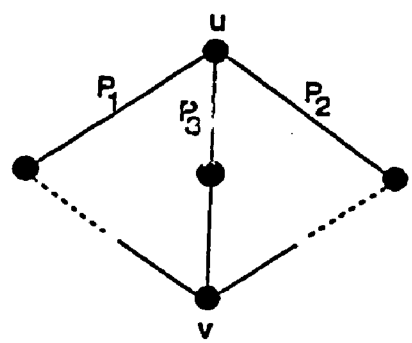


Figure (4.6)

Using the terminology introduced in Section (a), we can now state the main result of this section.

(4.3.7) Proposition: Let G be a graph with connectivity 2 having no W_3-minor. Then G is α-outerplanar if and only if G is isomorphic to a non-outerplanar minor of the graph H obtained from $K_{2,3}$ as follows:

(i) Take distinct edges from exactly two non-trivial series classes of $K_{2,3}$. At each of these, parallel connect a canonical 2-connected outerplanar graph where the basepoint of the latter graph is any edge on the infinite (hamiltonian) face.

(ii) Parallel extend the edges of $K_{2,3}\backslash\alpha$, where α is a member of the non-trivial series class of $K_{2,3}$ in which no member is a basepoint of a parallel connection as described in (i).

(iii) Add any number of new edges joining the two degree three vertices of $K_{2,3}$.

Proof: Suppose G is α-outerplanar. By Lemma (4.3.6), G is obtained from $K_{2,3}$ by series extending edges from at most two distinct non-trivial series classes. Let P_i be one of the paths shown in Figure (4.6). Since G has no W_3-minor, no edge of X joins an internal vertex of one of P_1, P_2, and P_3 to an internal vertex of a different one of these paths. Thus, every edge of X must join vertices of some P_i for $i \in \{1,2,3\}$. Using Lemmas (4.2.3), (4.3.3), (4.3.4), and (4.3.6), it follows that G is obtained from $K_{2,3}$ by taking the parallel connection of $K_{2,3}$ with canonical 2-connected outerplanar graphs where distinct edges of the paths P_1 and P_2 shown in Figure (4.6) and any edge on the infinite (hamiltonian) face of these outerplanar graphs are used as basepoints of these parallel connections. It is easily seen that such a graph is isomorphic to a non-outerplanar minor of the graph H obtained from $K_{2,3}$ as described in (i) - (iii).

Conversely, suppose G is obtained from $K_{2,3}$ as described above. Then G looks like the graph shown in Figure (4.7) where each of G_1, G_2, G_3, and G_4 is a canonical 2-connected outerplanar graph. Using the labeling of that figure, it is routine to check that both $G\backslash\alpha$ and G/α are outerplanar. Hence G is α-outerplanar. ∎

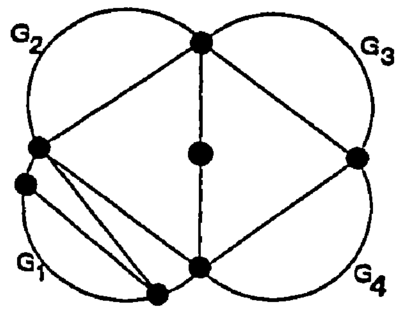


Figure (4.7)

This completes our characterization of α-outerplanar graphs with connectivity 2.

4.4 The excluded minor characterization.

In this section we will obtain an excluded minor characterization for the class $\mathbf{C}_\alpha$ of graphs consisting of α-outerplanar graphs and outerplanar graphs. We showed in Lemma (4.1.1) that $\mathbf{C}_\alpha$ is minor closed. We will now establish the following.

(4.4.1) Proposition: A graph G is a member of $\mathbf{C}_\alpha$ if and only if G has no minor isomorphic to one of the following graphs shown in Figure (4.8).

Proof: Suppose G has a minor isomorphic to one of the graphs shown here. It is routine to check that none of these graphs is α-outerplanar. Hence G $\notin$ $\mathbf{C}_\alpha$.

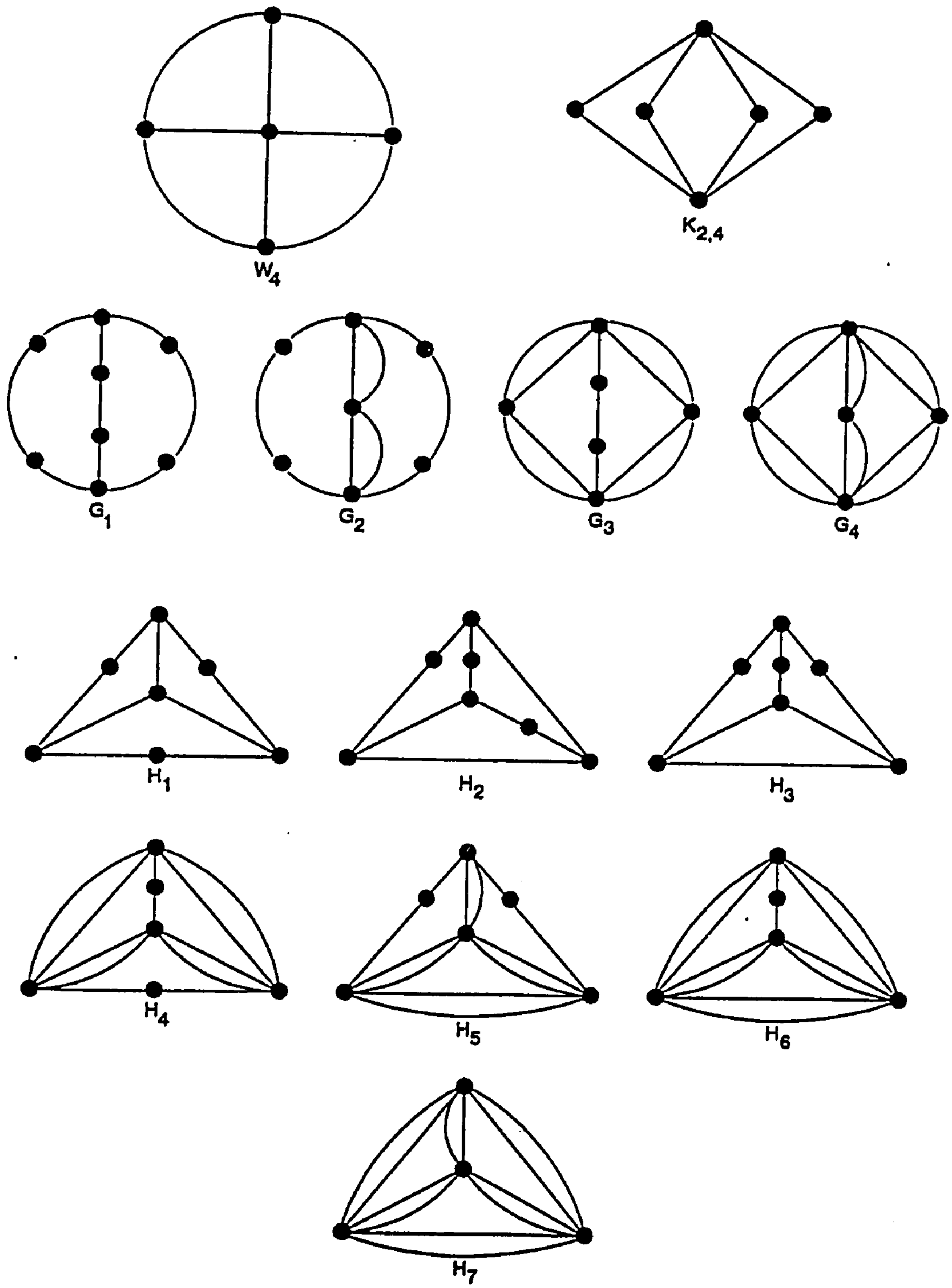


Figure (4.8)

For the converse, suppose that G has no minor isomorphic to one of the graphs shown in Figure (4.8). Suppose also that G is not a member of C_α. Then G is, in particular, non-outerplanar. Hence some block of G has a W_3- or a $K_{2,3}$-minor. By restricting to this block, we may assume in what follows that G is 2-connected . Moreover, since G has no $K_{2,4}$-minor by hypothesis, Lemma (4.3.1) implies that there are subsets X and Y of E(G) such that $G\backslash X/Y$ is isomorphic to W_3 or $K_{2,3}$ where every element of Y is in series with some element of $G\backslash X$ not in Y. Thus $G\backslash X$ is homeomorphic from W_3 or $K_{2,3}$.

First suppose that $G\backslash X$ is homeomorphic from W_3 and that at least three distinct edges of W_3 are series extended. Clearly G has a minor isomorphic to one of the graphs shown in Figure (4.1). But this means G has a minor isomorphic to H_1, H_2, or H_3. This contradicts our choice of G. Similarly, if $G\backslash X$ is homeomorphic from $K_{2,3}$ where at least one edge from each of the three non-trivial series classes of $K_{2,3}$ is series extended, then clearly G has a G_1-minor. Thus we may assume that $G\backslash X$ is homeomorphic from one of the following graphs:

(i) W_3 where at most two distinct edges of W_3 are series extended; or

(ii) $K_{2,3}$ where edges from at most two distinct non-trivial series classes of $K_{2,3}$ are series extended.

We will first consider case (i).

Suppose $G\backslash X$ is homeomorphic from W_3 and that two distinct edges of W_3 are series extended. Referring to Figure (4.2), this means that each edge of X either (a) joins vertices of some P_i for $i = 1$ or 2, or (b) joins a vertex of P_1 to a different vertex of P_2, or (c) joins an internal vertex of P_i to a vertex that is not in P_1 or P_2, or (d) is parallel to an edge of $G\backslash X$ that is not in P_1 or P_2.

In cases (b) and (c), it is clear that G has a W_4-minor. This contradicts our choice of G.

Thus an edge e of X must join vertices of some P_i for $i = 1$ or 2 or is parallel to an edge of G\X that is not in P_1 or P_2. If the former holds, Lemma (4.3.3) says that if edges are added joining vertices of P_i such that two of these edges span intervals that share a common edge and are not contained one in the other, then G has a $K_{2,4}$-minor; a contradiction. Thus the edges of X either span intervals of P_i, or are parallel to edges of P_i, or are parallel to an edge of G\X that is not in P_1 or P_2. Moreover, if x and y are in X and both span intervals of P_i, then either these intervals do not share a common edge or one is contained in the other. But using Proposition (4.3.5) and the fact that G has no H_4- or H_5-minor, this means G must be α-outerplanar; a contradiction.

Similarly, if G\X is homeomorphic from W_3 where at most one edge is series extended, we conclude that either G has a W_4-, $K_{2,4}$-, H_6-, or H_7-minor, or G is α-outerplanar; again, a contradiction.

Now suppose that G\X is homeomorphic from $K_{2,3}$ where edges from at most two distinct non-trivial series classes of $K_{2,3}$ are series extended. Furthermore, we may assume G has no W_3-minor. Referring to Figure (4.6) and Lemma (4.3.3), this means the edges of X are either parallel to edges of P_i for $i = 1$ or 2, or span intervals of P_i which do not share a common edge, or are parallel to an edge of G\X that is not in P_1 or P_2 or span intervals one of which is contained in the other. As in the previous case, using Proposition (4.3.7) we conclude that either G has a G_2-, G_3-, or G_4-minor, or G is α-outerplanar. In either case we contradict our choice of G. We conclude that if G\X is homeomorphic from W_3 or $K_{2,3}$, then either G is α-outerplanar or G has a minor isomorphic to one of the graphs shown in Figure (4.8). This completes the proof of Proposition (4.4.1). ∎

Chapter 5

Regular Matroids with every Circuit Basis Fundamental

5.1 Introduction.

Let M be a binary matroid on the set {1,2,...,n}. The circuit space of M is the subspace of V(n,2) generated by the incidence vectors of the circuits of M. A collection of circuits whose incidence vectors form a basis for the circuit space is called a circuit basis.

Let d be the dimension of the circuit space of M. A collection P of circuits of M, where $|P| = d$, is fundamental if there exists an ordering of the circuits in P such that $C_j \backslash (C_1 \cup C_2 \cup \ldots \cup C_{j-1}) \neq \varnothing$ for $2 \leq j \leq d$.

For graphic matroids, Hartvigsen and Zemel (1989) have obtained the following characterization:

(5.1.1) Theorem: Let M(G) be a graphic matroid. Every circuit basis of M(G) is fundamental if and only if G does not have a minor isomorphic to one of the five graphs shown in Figure (5.1).

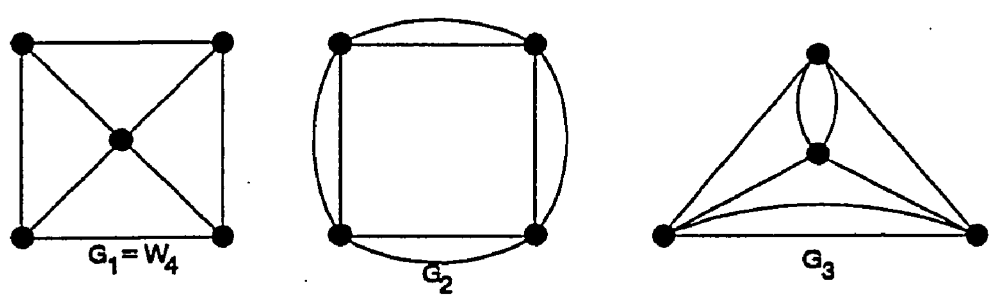


Figure (5.1) (continued over)

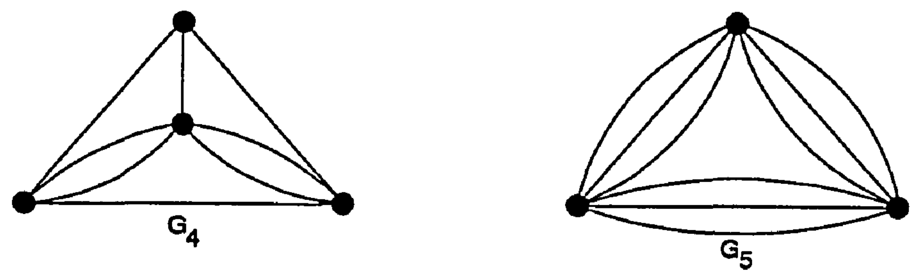


Figure (5.1)

In this chapter we consider the corresponding problem for binary matroids. In general, the class of binary matroids for which every circuit basis is fundamental is not closed under the taking of minors. However, we will show this class is closed under the taking of series-minors. We will also establish some general properties which this class satisfies. We end the chapter by extending Theorem (5.1.1) to the class of regular matroids.

In developing properties which the class of binary matroids with every circuit basis fundamental satisfies we will use the following results about binary matroids. For the proofs of these results, the reader is referred to Oxley (1992), Chapter 9. Throughout this chapter we will assume all matroids are binary.

(5.1.2) Proposition: Let A be a binary representation of M^*. Then the circuit space of M equals the row space of A. Moreover, this space has dimension equal to $r(M^*)$ and is the orthogonal subspace of the cocircuit space of M.

(5.1.3) Corollary: If B is a basis of a rank-r n-element binary matroid M and X is the B-fundamental-circuit incidence matrix of M, then the row spaces of $[I_r \mid X]$ and $[X^T \mid I_{n-r}]$ are the cocircuit and circuit spaces, respectively of M.

In showing that all the circuit bases of a given matroid M are fundamental we will often use the pigeon-hole principle to show that M cannot have a circuit bases which contains each element at least twice. This principle asserts that if $n+1$ objects are placed in n boxes, then at least one of these boxes has two or more of these objects. With this motivation, we proceed with the following.

Let $e \in E(M)$ and let P be a circuit basis of M. We say e is <u>covered m times</u> by P if e is contained in m or more distinct circuits of P. The following is a generalization of a result of Hartvigsen and Zemel (1989).

(5.1.4) Proposition: Let M be a connected binary matroid such that no circuit basis of M covers every element at least twice. Then every circuit basis of M is fundamental.

Proof: We will establish the proposition by a series of lemmas.

(5.1.5) Lemma: Let M be a connected binary matroid such that M has no restriction minor M' with a circuit basis covering every element at least twice. Then every circuit basis of M is fundamental.

Proof: Suppose instead that P is a non-fundamental circuit basis of M. Define $P' \subseteq P$ inductively by the following algorithm:

Step (i). Set $i = |P|$ and $P' = P$.

Step (ii). If there is a circuit C in P' such that C contains an element of M that is not covered twice by P', then choose such a circuit and call it C_i. Replace P' by $P' - \{C_i\}$ and go to Step (iii).

If there is no such circuit C go to Step (iv).

Step (iii). Replace i by i-1 and go to Step (ii).

Step (iv). End.

Clearly, $P' \subseteq P$. If P' is empty at the end of this algorithm, then the algorithm gives an ordering of the circuits of P such that $C_i-(C_1 \cup C_2 \cup \ldots \cup C_{i-1})$ is non-empty for $2 \leq i \leq |P|$, contrary to our choice of P. If P' is non-empty, let M' be the restriction of M to $E(P') = \bigcup_{C \in P'} C$. Because $P' \subseteq P$, the circuits of P' are independent as incidence vectors over GF(2). Moreover, because M has no loops or coloops,the corank of M' is $|P'|$. Thus $|P'|$ is a circuit basis of M' which covers every element at least twice. ∎

(5.1.6) Lemma: Let M' be a 2-connected restriction minor of the 2-connected matroid M. Then if M' has a circuit basis that covers every element at least twice, M has a circuit basis that covers every element at least twice.

Proof: Without loss of generality we can assume $M' = M\backslash x$. Let P be a circuit basis of M' that covers every element at least twice. We will use the following result due to Lehman (1964).

(5.1.7) Proposition: Let M be a connected binary matroid. Let $x \in E(M)$. The circuits of M not containing x are the minimal non-empty sets of the form $C \vartriangle D$, where C and D are circuits of M both containing x.

Since M is connected, x is neither a loop nor a coloop. Let C_i be any member of P, say $i = 1$. Because M is connected, it follows easily from Lehman's Theorem (5.1.7) that M has distinct circuits D_1 and D_2 such that $x \in D_1 \cap D_2$ and $C_1 \subseteq D_1 \cup D_2$. Let $P' = (P - \{C_i\}) \cup \{D_1, D_2\}$. Because the incidence vectors of the members of P are independent as row vectors over GF(2), the incidence vectors of the members of P' are also independent. Moreover, $|P'| = d+1$. Hence P' is a circuit basis of M which covers every element at least twice. This finishes the proof of Lemma (5.1.6). ∎

Returning to the proof of Proposition (5.1.4), let P be a non-fundamental circuit basis of M. By Lemma (5.1.5), M has a restriction minor M' with a circuit basis P' that covers every element at least twice. If M' is not connected, let M_1' be a connected component of M'. Then M_1' is a connected restriction of M with a circuit basis $P'' \subset P$ such that P'' covers every element twice. By Lemma (5.1.6) M has a circuit basis covering every element at least twice. This contradiction completes the proof of Proposition (5.1.4). ∎

Our first result establishes that the class of binary matroids for which every circuit basis is fundamental is closed under the taking of series-minors. We say a matroid N is a <u>series-minor</u> of a matroid M if N can be obtained from M by a sequence of deletions and series contractions. We will use the following proposition.

(5.1.8) Proposition: If N is a series-minor of M, then $N = M\backslash X/Y$ where every element in Y is in series with an element of $M\backslash X$ not in Y.

(5.1.9) Proposition: Let M be a matroid and suppose M has the property that every circuit basis of M is fundamental. Then any series-minor of M also has the property.

Proof: Let $N = M\backslash X/Y$ be a series-minor of M. By Proposition (5.1.8), we can assume that every element of Y is in series with an element of $M\backslash X$. We will use induction on $|Y|$ to establish the proposition.

To begin the induction argument, suppose $|Y| = 1$ and let $Y = \{y\}$. If y is a loop or a coloop of M, then $N = M\backslash X/Y = M\backslash(X \cup y)$ is a deletion-minor of M. We will show the following in this case.

(5.1.10) Lemma: If M is a matroid with the property that every circuit basis of M is fundamental, then $M\backslash S$ has the property for any $S \subseteq E(M)$.

Proof: Suppose $M\backslash S$ has a non-fundamental circuit basis $P = \{C_1, C_2, \ldots C_k\}$. Now for all $1 \leq i \leq k$, $C_i \subseteq E(M\backslash S) \subseteq E(M)$; that is, the C_i's are circuits of M. We will extend P to a non-fundamental circuit basis of M.

Since the C_i's are circuits of M which are independent as incidence vectors over V(n,2), P is a basis for a subspace of the circuit space of M. We can therefore extend P to a basis P' of the circuit space of M, where P' consists of circuits of M. We claim P' is a non-fundamental circuit basis of M. To see this, consider any ordering of the circuits in P'. Since P is a non-fundamental circuit basis, there is an index j such that $C_j \subseteq \bigcup_{i<j} C_i$. Consequently, for any ordering on P' there exists an index j such that $C_j \subseteq (\bigcup_{i<j} C_i) \cup (\bigcup_{i<j} C_i')$ where $C_i' \in P'\backslash P$. Thus P is a non-fundamental circuit basis of M. ▮

We can therefore assume that y is not a loop or coloop of M. Suppose N has a non-fundamental circuit basis $P = \{C_1, C_2, \ldots, C_k\}$. We will extend P to a non-fundamental circuit basis of $M \backslash X$.

By Proposition (5.1.8), y is in a 2-cocircuit, say $C^* = \{y,z\}$ of $M \backslash X$. For every $1 \leq i \leq k$, C_i or $C_i \cup y$ is a circuit of $M \backslash X$. Let $D_i = C_i$ in the former case and let $D_i = C_i \cup y$ in the latter. Let $P' = \{D_1, D_2, \ldots, D_k\}$. We claim P' is a non-fundamental circuit basis of $M \backslash X$. It is immediate that the D_i's are independent because the C_i's are independent:

namely, if the C_i's are viewed as independent m-tuples over GF(2) for some m, then the D_i's are independent (m+1)-tuples. Also, because y is not a loop or coloop of M, the circuit space of $M \setminus X$ and the circuit space of $M \setminus X / y$ have the same dimension. Hence P' is a basis of the circuit space of $M \setminus X$. We want to show P' is non-fundamental.

Assume P' is fundamental and let $D_1, D_2, \ldots, D_k$ be an ordering of the circuits of P' such that, for all j, $D_j - (\bigcup_{i<j} D_i)$ is non-empty. Consider the corresponding ordering $C_1, C_2, \ldots, C_k$ of the circuits of P. Clearly $C_t - (\bigcup_{i<t} C_i)$ is empty for some t. But $(D_t - (\bigcup_{i<t} D_i)) - (C_t - (\bigcup_{i<t} C_i)) \subseteq \{y\}$. Since $D_t - (\bigcup_{i<t} D_i)$ is non-empty, $y \in D_t$. But $\{y,z\}$ is a cocircuit of $M \setminus X$, so $z \in D_t$. Moreover $z \notin D_i$ for $i < t$ otherwise y is in each of these circuits. Thus $z \in C_t - (\bigcup_{i<t} C_i) = \varnothing$. This contradiction completes the proof that P' is a non-fundamental circuit basis of $M \setminus X$. But this contradicts Lemma (5.1.10). Thus Proposition (5.1.9) is established if $|Y| = 1$.

Suppose the proposition holds for all sets Y with $|Y| < n$, and let $|Y| = n$. Let $N = M\setminus X/Y = (M\setminus X/Y')/y_n$ where $Y = \{y_1, y_2, \ldots, y_{n-1}, y_n\} = Y' \cup y_n$. By Lemma (5.1.10), we can assume y_n is not a loop or coloop of $M\setminus X/Y'$. If N has a non-fundamental circuit basis

$P = \{C_1, C_2, \ldots, C_k\}$, then, as in the proof of the previous case, we can extend P to a non-fundamental circuit basis of $M\setminus X/Y'$. Hence $M\setminus X/Y'$ has a non-fundamental circuit basis. This contradiction completes the proof of the proposition. ∎

The following example shows that, in general, the class of matroids for which every circuit basis is fundamental is not closed under the taking of minors.

Let M be the matroid obtained from F_7^* by parallel extending two distinct elements. Thus $M \cong F_7^* \oplus_2 U_{1,3} \oplus_2 U_{1,3}$ where the basepoints of the 2-sum constructions are distinct elements of F_7^*. Up to isomorphism, M has the following representation. (Figure (5.2))

$$M[A] = \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{cccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \end{array}\right] \end{array}$$

Figure (5.2)

Let C be a circuit of M. Then $|C| = 2$ or 4. We also observe that the dimension of the circuit space of M is 5. We will show M does not have a circuit basis which covers every element at least two times.

Suppose $P = \{C_1, C_2, \ldots, C_5\}$ is a circuit basis which covers every element at least twice. Because $|C| = 2$ or 4 for every circuit C of M and because $|E(M)| = 9$, $18 \leq |C_1| + |C_2| + |C_3| + |C_4| + |C_5| \leq 20$. If $|C_i| = 2$ for some $C_i \in P$, then P must cover every element exactly twice. But this implies that as row vectors $C_1 + C_2 + \ldots + C_5 \equiv 0(\text{mod}2)$. Thus, the C_i's are not independent if $|C_i| = 2$ for some $C_i \in P$. Consequently, $|C_i| = 4$ for all $C_i \in P$. Using properties of binary matroids, we must have that $C_1 \triangle C_2 \triangle \ldots \triangle C_5 = \{x,y\}$ contains a circuit of M. Since M has no loops, $\{x,y\}$ is a circuit of M. Thus, $\{x,y\} = \{3,8\}$ or $\{x,y\} = \{4,9\}$. Without loss of generality we may assume $\{x,y\} = \{3,8\}$. Then **3** and **8** appear in at least three of the C_i's in P. But **3** and **8** appear in no common 4-circuit. By the pigeon-hole principle we conclude P cannot be a circuit basis of M covering every element at least twice. By Proposition (5.1.4), every circuit basis of M is therefore fundamental.

Let $M' = M/1 \cong (F_7^*/1) \oplus_2 U_{1,3} \oplus_2 U_{1,3}$. It is routine to check that $M' = M(G)$ where G is the following graph. (Figure (5.3).) Here edge labels correspond to the column labels of

A in Figure (5.2).

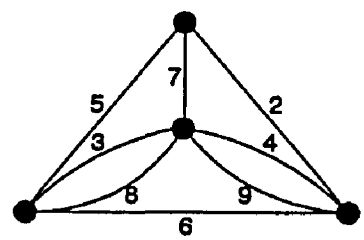


Figure (5.3)

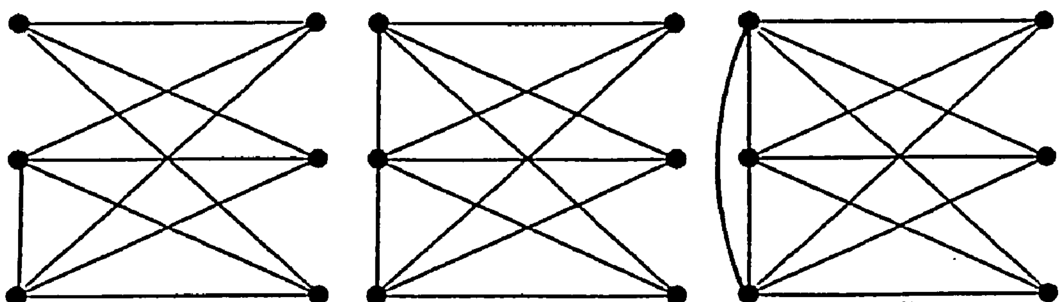
Since $G \cong G_4$ of Theorem (5.1.1), we conclude M' does not have the property that every circuit basis is fundamental. We conclude from this example that the class of matroids for which every circuit basis is fundamental is not closed under the taking of arbitrary minors.

5.2 The regular matroids with every circuit basis fundamental.

In this section we will extend Theorem (5.1.1) to the class of regular matroids. We will need the following result of Bixby (1977) in what follows.

(5.2.1) Theorem: Let M be a regular matroid. Then M is graphic if and only if M has no series-minor isomorphic to $M^*(K_5)$, $M^*(K_{3,3})$, $M^*(K'_{3,3})$, $M^*(K''_{3,3})$, $M^*(K'''_{3,3})$, or R_{10}.

Here $K'_{3,3}$, $K''_{3,3}$, $K'''_{3,3}$ are the graphs shown in Figure (5.4), and R_{10} is the vector matroid induced by the matrix shown in Figure (5.5).



$K'_{3,3}$ $K''_{3,3}$ $K'''_{3,3}$

Figure (5.4)

$$A_{10} = \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

Figure (5.5)

We are now ready to state the main result of this section.

(5.2.2) Theorem: Let M be a regular matroid. Every circuit basis of M is fundamental if and only if M is a graphic matroid having no minor isomorphic to one of $M(G_1)$-$M(G_5)$, where G_1-G_5 are the graphs shown in Figure (5.1).

Proof: Suppose M has the property that every circuit basis is fundamental. We will use

Theorem (5.2.1) to establish that M is graphic. The result will then follow immediately from Theorem (5.1.1).

(5.2.3) Lemma: $M^*(K_5)$ and $M^*(K_{3,3})$ have non-fundamental circuit bases.

Proof: Let $M^*(K_5)$ and $M^*(K_{3,3})$ have the representations over GF(2) shown in Figures (5.6) and (5.7).

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \end{array} \\
M^*(K_5) = & \left[\begin{array}{cccccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{array}\right]
\end{array}
$$

Figure (5.6)

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{array} \\
M^*(K_{3,3}) = & \left[\begin{array}{cccc|ccccc}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1
\end{array}\right]
\end{array}
$$

Figure (5.7)

Using the labeling in Figure (5.6), let $P = \{a_3a_5a_6a_7a_8a_9,\ a_2a_4a_6a_7a_8a_{10},\ a_1a_4a_5a_7a_9a_{10},\ a_1a_2a_3a_8a_9a_{10}\}$. We will show that P is a circuit basis for $M^*(K_5)$ by showing the incidence matrix for P has rank 4. The fact that P covers every element of $M^*(K_5)$ at least twice will then establish that P is non-fundamental.

Let A_P denote the incidence matrix of P. Then

$$A_p = \begin{array}{c} \begin{array}{cccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Let B_P denote the 4 x 4 submatrix of A_P with columns a_1, a_6, a_7 and a_{10}. It is routine to check that det $B_P \neq 0$. We conclude P is a non-fundamental circuit basis for $M^*(K_5)$.

Similarly, using the labeling in Figure (5.7), let P' = $\{a_1a_2a_3a_6,\ a_1a_5a_6a_8a_9,\ a_4a_5a_7a_8a_9,\ a_1a_3a_4a_5a_7,\ a_1a_2a_3a_7a_9\}$. Let $A_{P'}$ denote the incidence matrix of P'. Then

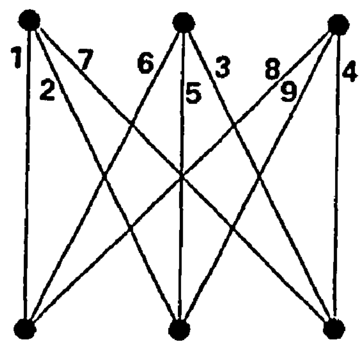
$$A_{p'} = \begin{array}{c} \begin{array}{ccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Let $B_{P'}$ denote the 5 x 5 submatrix of $A_{P'}$ with columns a_1, a_2, a_3, a_4, and a_7. It is routine to check that det $B_{P'} \neq 0$. Hence P is a circuit basis of $M^*(K_{3,3})$. Since P covers every element of $M^*(K_{3,3})$ at least twice, P is non-fundamental. This completes the proof of Lemma (5.2.3). ∎

(5.2.4) Lemma: $M^*(K'_{3,3})$, $M^*(K''_{3,3})$, $M^*(K'''_{3,3})$, and R_{10} do not have the property that every circuit basis is fundamental.

Proof: Let $M^* \in \{ M^*(K'_{3,3}), M^*(K''_{3,3}), M^*(K'''_{3,3}) \}$. Using the labelling in Figure (5.4), the simple matroid associated with $M^*\backslash f$ is a series-minor of M^* isomorphic to $M^*(K_5\backslash e)$. Because $M^*(K_5\backslash e)$ is graphic with $M(G_1)$ as a minor, where G_1 is shown in Figure (5.1), it has a non-fundamental circuit basis by Theorem (5.1.1). Hence each of $M^*(K'_{3,3})$, $M^*(K''_{3,3})$, and $M^*(K'''_{3,3})$ has a non-fundamental circuit basis by Proposition (5.1.9).

Using the labelling in Figure (5.5), $R_{10}/10$ is isomorphic to $K_{3,3}$ as shown in Figure (5.8).



$R_{10}\backslash 10$

Figure (5.8)

As $K_{3,3}$ is graphic and has G_1 as a minor, it has a non-fundamental circuit basis by Theorem (5.1.1). We conclude R_{10} has a series-minor with a non-fundamental circuit basis. Hence R_{10} cannot have the property that every circuit basis is fundamental by Proposition (5.1.9). This completes the proof of Lemma (5.2.4). By our remarks preceding Lemma (5.2.3), the proof of Theorem (5.2.2) is now complete. ∎

Bibliography

Bixby, R. E. (1976). A strengthened form of Tutte's characterization of regular matroids. *J. Combin. Theory Ser.B* **20**, 216-221.

Bixby, R. E. (1977). Kuratowski's and Wagner's theorems for matroids. *J. Combin. Theory Ser.B* **22**, 31-53.

Bondy, J. A. and Murty, U. S. R. (1976). **Graph Theory with Applications.** North-Holland, New York.

Chartrand, G. and Harary, F. (1967). Planar permutation graphs. *Ann. Inst. H. Poincáré Sect. B (N.S.)* **3**, 433-438.

Dirac, G. A. (1952). A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc.* **27**, 85-92.

Gubser, B. (1990). Some problems for graph minors. Ph.D. dissertation, Louisiana State University.

Hartvigsen, D. and Zemel, E. (1989). Is every cycle basis fundamental ? *J. Graph Theory* **13**, 117-137.

Lehman, A. (1964). A solution of the Shannon swithching game. *J. Soc. Indust. Appl. Math* **12**, 687-725.

Oxley, J. G. (1992). **Matroid Theory.** Oxford University Press, Oxford.(In press.)

Robertson, N. and Seymour, P. D. (1984). Generalizing Kuratowski's Theorem. *Congressus Numerantium* **45**, 129-138.

Seymour, P. D. (1977). A note on the production of matroid minors. *J. Combin. Theory Ser.B* **22**, 289-295.

Seymour, P. D. (1980). Decomposition of regular matroids. *J.Combin. Theory Ser.B* **28**, 305-359.

Tang, D. T. (1964). Bi-path networks and multi-commodity flows. *IEEE Trans. Circuit Theory* **11**, 468-474.

Welsh, D. J. A. (1976). **Matroid Theory.** Academic Press, London.

Index of Definitions and Symbols.

VITAE

Lawrence Alan Wargo was born in Philadelphia, Pennsylvania on April 12, 1958. He received a Bachelor of Arts degree from LaSalle College in May, 1981 and a Master of Arts degree from the State University of New York at Stony Brook in May, 1984. He entered Louisiana State University in August, 1985.

DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Lawrence Wargo

Major Field: (Combinatorics) Mathematics

Title of Dissertation: "Some Results on Minors for Graphs and Matroids."

Approved:

James G. Oxley

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

Date of Examination:

October 1, 1991