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Kantowski-Sachs spacetime in loop quantum cosmology: bounds on expansion and shear scalars and the viability of quantization prescriptions

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Using effective dynamics, we investigate the behavior of expansion and shear scalars in different proposed quantizations of the Kantowski-Sachs spacetime with matter in loop quantum cosmology. We find that out of the various proposed choices, there is only one known prescription which leads to the generic bounded behavior of these scalars. The bounds turn out to be universal and are determined by the underlying quantum geometry. This quantization is analogous to the so called ‘improved dynamics’ in the isotropic loop quantum cosmology, which is also the only one to respect the freedom of the rescaling of the fiducial cell at the level of effective spacetime description. Other proposed quantization prescriptions yield expansion and shear scalars which may not be bounded for certain initial conditions within the validity of effective spacetime description. These prescriptions also have a limitation that the “quantum geometric effects” can occur at an arbitrary scale. We show that the ‘improved dynamics’ of Kantowski-Sachs spacetime turns out to be a unique choice in a general class of possible quantization prescriptions, in the sense of leading to generic bounds on expansion and shear scalars and the associated physics being free from fiducial cell dependence. The behavior of the energy density in the ‘improved dynamics’ reveals some interesting features. Even without considering any details of the dynamical evolution, it is possible to rule out pancake singularities in this spacetime. The energy density is found to be dynamically bounded. These results show that the Planck scale physics of the loop quantized Kantowski-Sachs spacetime has key features common with the loop quantization of isotropic and Bianchi-I spacetimes.

I. INTRODUCTION

Kantowski-Sachs spacetime is a homogeneous and anisotropic cosmological model which is of dual importance as it serves as both a setting to study effects of anisotropies in the evolution of the universe and also as a description of the interior of Schwarzschild black hole in the vacuum case. This spacetime classically has a past and a future singularity, which can be an anisotropic structure such as a barrel, cigar or a pancake, or an isotropic point like structure depending on the initial conditions on anisotropic shear and matter[1]. At these classical singularities geodesic evolution ends, which is captured by the divergences in the expansion and shear scalars, and also of the energy density when the matter is present. The occurrence of singularities indicates that general relativity (GR) is being pushed to the limits of its validity, and a quantum gravitational treatment of spacetime is necessary.

Though a full theory of quantum gravity is not yet available, insights on the problem of classical singularities have been gained for various spacetimes in loop quantum cosmology (LQC) in recent years [2]. LQC is a quantization of symmetry reduced spacetimes using techniques of loop quantum gravity (LQG) which is a nonperturbative canonical quantization of gravity based on the Ashtekar variables: the $SU(2)$ connections and the conjugate triads. The elementary variables for

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the quantization are the holonomies of the connection components, and the fluxes of the triads. The classical Hamiltonian constraint, the only non-trivial constraint left after symmetry reduction in the minisuperspace setting, is expressed in terms of holonomies and fluxes and is quantized. Quantization of various isotropic models in LQC demonstrates the resolution of classical singularities when the spacetime curvature reaches Planck scale. The big bang and big crunch are replaced by a quantum bounce, which first found in the case of the spatially flat isotropic model [3–5] is tied to the underlying quantum geometry and has been shown to be a robust phenomena through different analytical [6] and numerical investigations [7–9]. A generalization of these results has been performed for Bianchi models [10–17], where the quantum Hamiltonian constraint also turns out to be non-singular. An interesting feature of LQC is that for sharply peaked states which lead to a macroscopic universe at late times, it is possible to derive an effective spacetime description [18–20]. The resulting effective dynamics has been extremely useful in not only extracting physical predictions, but also to gain insights on the viability of various possible quantizations. In particular it has been shown that for isotropic models there is a unique way of quantization, the so called ‘improved dynamics’ or the $\bar{\mu}$ quantization [5], which results in a consistent ultra-violet and infra-red behavior and is free from the rescalings of the fiducial cell introduced to obtain finite integrations on the non-compact spatial manifold at the level of the effective spacetime description [21, 22].¹ Note that the fiducial cell which acts like an infra-red regulator is an arbitrary choice in the quantization procedure. Hence a consistent quantization prescription must yield physical predictions about observables such as expansion and shear scalars independent of the choice of this cell for suitable semiclassical states if the spatial topology is non-compact.

The improved dynamics quantization of the isotropic LQC results in a generic bound on the expansion scalar of the geodesics in the effective spacetime and leads to a resolution of all possible strong singularities in the spatially flat model [25, 26]. These results have also been extended to Bianchi models, where $\bar{\mu}$ quantization results in generic bounds on expansion and shear scalars [17, 22, 28, 29], and the resolution of strong singularities in Bianchi-I spacetime [28]. There are other possible ways to quantize isotropic and anisotropic models, such as the earlier quantization of isotropic models in LQC – the μ_o quantization [4, 30] and the lattice refined models [31]. In these quantization prescriptions,² quantum gravitational effects can occur at arbitrarily small curvature scales and the expansion and shear scalars are not bounded in general [21, 22]. Further, the physics in these prescriptions is also not free from fiducial cell dependence at the level of effective dynamics.

Loop quantization of Kantowski-Sachs spacetimes has been mostly studied for the vacuum case [33–39], where the quantum Hamiltonian constraint has been found to be non-singular. Ashtekar and Bojowald proposed a quantization of the interior of the Schwarzschild interior and concluded

¹ It should be noted that in the full quantum description, the independence from the fiducial cell is only approximate in the $\bar{\mu}$ prescription in the following sense. At the level of the physical Hilbert space, a mapping taking in to account fiducial rescaling mixes the superselected sectors in the quantum difference equation [21]. However, for volumes much larger than the Planck volume, the rescaling invariance is recovered. This issue has been rigorously discussed in Refs. [23, 24] where the effect of fiducial rescaling on the expectation values has been carefully studied, and rescaling invariance has been shown to be not satisfied at the full quantum level, but to be preserved at a semi-classical level. One can also introduce an approximate rescaling invariance, such that for a suitably chosen parameters of the semi-classical states, the mapping does not yield a distinguishable effect [24]. The quantization of the Kantowski-Sachs spacetime in the $\bar{\mu}$ prescription studied in this manuscript will share this caveat. However, since this issue does not arise at the level of the effective spacetime description which is derived using semi-classical states, the results obtained in this manuscript are unaffected.

² Our usage of term “quantization prescriptions” in loop quantization in this paper is different from an earlier work in isotropic LQC [32]. Here different quantum prescriptions refer to the way the area of the loops over which holonomies in the quantum theory are constructed are constrained with respect to the minimum area gap. Whereas in Ref. [32], different quantum prescriptions were used to distinguish the quantum Hamiltonian constraints in the $\bar{\mu}$ quantization of isotropic LQC.

that the wavefunction of universe can be evolved across the classical central singularity pointing towards singularity resolution [33]. Spherically symmetric spacetimes have been studied in the midisuperspace setting by Campiglia, Gambini, Pullin [36–38], to quantize Schwarzschild black hole [39] and calculate the Hawking radiation [40]. Though these works provide important insights on the quantization of black holes in LQG, it is to be noted that the quantization prescription used in these works is analogous to the earlier works in isotropic LQC (the μ_o quantization) which was found to yield inconsistent physics. In particular, the loop quantization in these models is carried out such that the loops over which holonomies are considered have edge lengths (labelled by δ_b and δ_c) as constant. As in the case of the μ_o quantization in LQC, the constant δ quantization of Schwarzschild interior has been shown to be dependent on the rescalings of the fiducial length L_o in the x direction of the $\mathbb{R} \times \mathbb{S}^2$ spatial manifold [41–43]. To overcome these problems, Boehmer and Vandersloot proposed a quantization prescription motivated by the improved dynamics in LQC [41], which we label as $\bar{\mu}$ quantization in Kantowski-Sachs model. In this prescription, δ_b and δ_c depend on triad components in such a way that the effective Hamiltonian constraint respects the freedom in rescaling of length L_o .³ This prescription has been used to understand the phenomenology of the Schwarzschild interior [44] and has been recently used to loop quantize spherically symmetric spacetimes [43]. It is to be noted that this prescription leads to “quantum gravitational effects” not only in the neighborhood of the physical singularity at the origin, but also at the coordinate singularity at the horizon, which points to the limitation of dealing with Schwarzschild interior in this setting. This problem has been noted earlier, see for eg. Ref. [44] where the problem with the fiducial cell at the horizon in this prescription is noted. However, note that such an issue does not arise in the presence of matter which is the focus of the present manuscript.

In literature, another quantization prescription inspired by the improved dynamics, which we label as the $\bar{\mu}'$ prescription⁴ has been proposed. In this prescription though edge lengths δ_b and δ_c are functions of the triads, problems with fiducial length rescalings persist [42]. These prescriptions have also been analyzed for the von-Neumann stability of the quantum Hamiltonian constraints which turn out to be difference equations [31]. It was found that $\bar{\mu}'$ quantization, in contrast to the $\bar{\mu}$ quantization, does not yield a stable evolution.

These studies indicate that if we consider fiducial length rescaling issues, $\bar{\mu}$ quantization in the Kantowski-Sachs spacetime is preferred over the constant δ quantization [33] and the $\bar{\mu}'$ quantization prescription [42]. However one may argue that these issues which arise for the non-compact spatial manifold, can be avoided if the topology of the spatial manifold is compact ($\mathbb{S}^1 \times \mathbb{S}^2$). Note that for all the models studied so far, it has been found that all three prescriptions lead to singularity resolution. Still, little is known about the conditions under which singularity resolution occurs for the arbitrary matter. Hence, various pertinent questions remain unanswered. In particular, which of these quantization prescriptions promises to generically resolve all the strong singularities⁵ within the validity of the effective spacetime description in LQC? Is it possible that in any of these quantization prescriptions, expansion and shear scalars may not be generically bounded in effective dynamics which disfavor them over others? Are there any other consistent quantization prescriptions for the Kantowski-Sachs model, or is the $\bar{\mu}$ quantization prescription unique as in the isotropic LQC? Finally, what is the fate of energy density if expansion and shear scalar are

³ As remarked in footnote 1, strictly speaking this fiducial length independence is not present when one is considering volumes comparable to the Planck volume in the full quantum description [21, 23, 24]. In the following, the fiducial rescaling independence will be referred to only at the effective level.

⁴ Our labeling of the $\bar{\mu}$ and $\bar{\mu}'$ prescriptions in Kantowski-Sachs spacetime is opposite to that of Ref. [42]. This difference is important to realize to avoid any confusions about the physical implications or the limitations of these prescriptions while relating this work with Ref. [42].

⁵ For a discussion of the strength of the singularities in LQC, see Ref. [25].

generically bounded? Note that in the isotropic LQC, and the Bianchi-I model similar questions were raised in Refs. [21, 25, 28], and the answers led to $\bar{\mu}$ quantization as the preferred choice. It turned out to be a unique quantization prescription leading to generic bounds on expansion and shear scalars, which were instrumental in proving the resolution of all strong singularities in the effective spacetime [25, 28].

The goal of this work is to answer these questions in the effective spacetime description in LQC for Kantowski-Sachs spacetime with minimally coupled matter. The expansion and shear scalars are tied to the geodesic completeness of the spacetime and are independent of the fiducial length at the classical level. We will be interested in finding the quantization prescription which promises to resolve all possible classical singularities generically. Such a quantization prescription is expected to yield bounded behavior of these scalars. It is also reasonable to expect, due to the underlying Planck scale quantum geometry, that in the bounce regime, depending on the approach to the classical singularity, at least one of the scalars takes Planckian value. We find that in the effective dynamics for constant δ and $\bar{\mu}'$ prescriptions, these scalars are not necessarily bounded above. In the cases where the classical singularities are resolved, it is possible that the expansion and shear scalars in these prescriptions can take arbitrary values in the bounce regime. In contrast, for the $\bar{\mu}$ quantization prescription, we show that the expansion and shear scalars turn out to be generically bounded by universal values in the Planck regime. It is to be noted that in the $\bar{\mu}$ prescription, the bounded behavior of the expansion scalar has been mentioned earlier for the Schwarzschild interior [45].⁶

We find that the behavior of expansion and shear scalars in the $\bar{\mu}$ prescription is similar to the improved dynamics of isotropic and Bianchi-I spacetime in LQC where the universal bounds on expansion and shear scalars were found. Next, we address the important question of the uniqueness of the $\bar{\mu}$ prescription. For this we consider a general ansatz to consider edge lengths δ_b and δ_c as functions of triads, allowing a large class of loop quantization prescriptions in the Kantowski-Sachs spacetime. We find that demanding that the expansion and shear scalars be bounded leads to a unique choice – the $\bar{\mu}$ quantization prescription. In this quantization prescription we also investigate the behavior of the energy density and find that its potential divergence is determined only by the vanishing $g_{\Omega\Omega}$ component of the spacetime metric. This is unlike the behavior in the classical GR, and other quantization prescriptions where divergence in energy density can occur when either of g_{xx} or $g_{\Omega\Omega}$ components vanish. An immediate consequence of this behavior is that the pancake singularities which occur when g_{xx} component of the line element approaches zero, and $g_{\Omega\Omega}$ is finite, are forbidden. It turns out that energy density is bounded dynamically, since $g_{\Omega\Omega}$ never becomes zero and approaches an asymptotic value. This property of $g_{\Omega\Omega}$ was first seen in the case of vacuum Kantowski-Sachs spacetime, and turns out to be true for all perfect fluids [46]. These results show that the $\bar{\mu}$ quantization in the Kantowski-Sachs spacetime is strikingly similar to the $\bar{\mu}$ quantization in the isotropic and Bianchi-I spacetimes. It leads to generic bounds on the expansion and shear scalars and is independent of the rescalings of the fiducial cell at the effective level.

This article is organized as follows. In the next section we summarize the Kantowski-Sachs spacetime in terms of Ashtekar variables and obtain the classical equations. In Sec. III, we introduce the effective Hamiltonian constraint, and derive expressions for expansion and shear scalars for three quantization prescriptions. We discuss the boundedness of these scalars and for completeness also discuss their dependence on fiducial cell. Then, in Sec. IV, we consider a general ansatz and investigate the conditions under which a quantization prescription yields bounded behavior of expansion and shear scalars. This leads us to the uniqueness of the $\bar{\mu}$ quantization

⁶ We thank Alejandro Corichi for pointing out Ref.[45] to us.

prescription. The behavior of energy density is discussed in Sec. V, which is followed by a summary of the main results.

II. CLASSICAL HAMILTONIAN OF KANTOWSKI-SACHS SPACE-TIME

We consider the Kantowski-Sachs spacetime with a spatial topology of $\mathbb{R} \times \mathbb{S}^2$. Utilizing the symmetries associated with each spatial slice, the symmetry group $\mathbb{R} \times SO(3)$, and after imposing the Gauss constraint, the Ashtekar-Barbero connection and the conjugate (densitized) triad can be expressed in the following form [33]:

$$A_a^i \tau_i dx^a = \tilde{c} \tau_3 dx + \tilde{b} \tau_2 d\theta - \tilde{b} \tau_1 \sin \theta d\phi + \tau_3 \cos \theta d\phi, \quad (2.1)$$

$$\tilde{E}_i^a \tau_i \partial_a = \tilde{p}_c \tau_3 \sin \theta \partial_x + \tilde{p}_b \tau_2 \sin \theta \partial_\theta - \tilde{p}_b \tau_1 \partial_\phi, \quad (2.2)$$

where $\tau_i = -i\sigma_i/2$, and σ_i are the Pauli spin matrices. The symmetry reduced triad variables are related to the metric components of the line element,⁷

$$ds^2 = -N(t)^2 dt^2 + g_{xx} dx^2 + g_{\Omega\Omega} (d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.3)$$

as

$$g_{xx} = \frac{\tilde{p}_b^2}{\tilde{p}_c}, \quad \text{and} \quad g_{\Omega\Omega} = |\tilde{p}_c|. \quad (2.4)$$

The modulus sign arises because of two possible triad orientations. Without any loss of generality, we will assume the orientation to be positive throughout this analysis. Since the spatial manifold in Kantowski-Sachs spacetime is non-compact, we have to introduce a fiducial length along the non-compact x direction. Denoting this length be L_o , the symplectic structure is given by

$$\Omega = \frac{L_o}{2G\gamma} \left(2d\tilde{b} \wedge d\tilde{p}_b + d\tilde{c} \wedge d\tilde{p}_c \right). \quad (2.5)$$

Here γ is the Barbero-Immirzi parameter whose value is fixed from the black hole entropy calculations in loop quantum gravity to be 0.2375. Since the fiducial length can be arbitrarily rescaled, the symplectic structure depends on L_o . This dependence can be removed by a rescaling of the symmetry reduced triad and connection components by introducing the triads p_b and p_c , and the connections b and c :

$$p_b = L_o \tilde{p}_b, \quad p_c = \tilde{p}_c, \quad b = \tilde{b}, \quad c = L_o \tilde{c}. \quad (2.6)$$

The non-vanishing Poisson brackets between these new variables are given by,

$$\{b, p_b\} = G\gamma, \quad \{c, p_c\} = 2G\gamma. \quad (2.7)$$

Note that p_b and p_c both have dimensions of length squared, whereas b and c are dimensionless. Also note that c and p_b scale as L_o where as other two variables are independent of the fiducial cell.

In Ashtekar variables, the Hamiltonian constraint for the Kantowski-Sachs spacetime with minimally coupled matter corresponding to an energy density ρ_m can be written as

$$\mathcal{H}_{\text{cl}} = \frac{-N}{2G\gamma^2} \left[2bc\sqrt{p_c} + (b^2 + \gamma^2) \frac{p_b}{\sqrt{p_c}} \right] + N 4\pi p_b \sqrt{p_c} \rho_m, \quad (2.8)$$

⁷ This metric can be expressed as the one for the Schwarzschild interior by choosing $N(t)^2 = \left(\frac{2m}{t} - 1\right)^{-1}$ where m denotes the mass of the black hole, and identifying $g_{xx} = \left(\frac{2m}{t} - 1\right)$ and $g_{\Omega\Omega} = t^2$.

and the physical volume of the fiducial cell is $V = 4\pi p_b \sqrt{p_c}$. In the following, the lapse will be chosen as unity.⁸ Using the Hamilton's equations, for $N = 1$, the dynamical equations become,

$$\dot{p}_b = -G\gamma \frac{\partial \mathcal{H}_{\text{cl}}}{\partial b} = \frac{1}{\gamma} \left(c\sqrt{p_c} + \frac{bp_b}{\sqrt{p_c}} \right) \quad (2.9)$$

$$\dot{p}_c = -2G\gamma \frac{\partial \mathcal{H}_{\text{cl}}}{\partial c} = \frac{1}{\gamma} 2b\sqrt{p_c} \quad (2.10)$$

$$\dot{b} = G\gamma \frac{\partial \mathcal{H}_{\text{cl}}}{\partial p_b} = \frac{-1}{2\gamma\sqrt{p_c}} (b^2 + \gamma^2) + 4\pi G\gamma\sqrt{p_c} \left(\rho_m + p_b \frac{\partial \rho_m}{\partial p_b} \right) \quad (2.11)$$

$$\dot{c} = 2G\gamma \frac{\partial \mathcal{H}_{\text{cl}}}{\partial p_c} = \frac{-1}{\gamma\sqrt{p_c}} \left(bc - (b^2 + \gamma^2) \frac{p_b}{2p_c} \right) + 8\pi\gamma G p_b \left(\frac{\rho_m}{2\sqrt{p_c}} + \sqrt{p_c} \frac{\partial \rho_m}{\partial p_c} \right). \quad (2.12)$$

The vanishing of the classical Hamiltonian constraint, $\mathcal{H}_{\text{cl}} \approx 0$, yields

$$\frac{2bc}{\gamma^2 p_b} + \frac{b^2}{\gamma^2 p_c} + \frac{1}{p_c} = 8\pi G \rho_m \quad (2.13)$$

which using the expressions for the directional Hubble rates $H_i = \dot{\sqrt{g_{ii}}}/\sqrt{g_{ii}}$ can be written as the Einstein's field equation for the 0-0 component:

$$2 \frac{\sqrt{\dot{g}_{xx}}}{\sqrt{g_{xx}}} \frac{\sqrt{\dot{g}_{\Omega\Omega}}}{\sqrt{g_{\Omega\Omega}}} + \left(\frac{\sqrt{\dot{g}_{\Omega\Omega}}}{\sqrt{g_{\Omega\Omega}}} \right)^2 + \frac{1}{g_{\Omega\Omega}} = 8\pi G \rho_m. \quad (2.14)$$

Introducing the expansion θ and the shear σ^2 of the congruence of the cosmological observers

$$\theta = \frac{\dot{V}}{V} = \frac{\dot{p}_b}{p_b} + \frac{\dot{p}_c}{2p_c}. \quad (2.15)$$

and

$$\sigma^2 = \frac{1}{2} \sum_{i=1}^3 \left(H_i - \frac{1}{3}\theta \right)^2 = \frac{1}{3} \left(\frac{\dot{p}_c}{p_c} - \frac{\dot{p}_b}{p_b} \right)^2 \quad (2.16)$$

we can rewrite eq.(2.14) as

$$\frac{\theta^2}{3} - \sigma^2 + \frac{1}{g_{\Omega\Omega}} = 8\pi G \rho_m. \quad (2.17)$$

To investigate if the Kantowski-Sachs spacetime is singular, we consider the expansion and the shear scalars of the geodesics. At a singular region one or more of these diverge. This divergence causes the curvature invariants to blow up. To see this, we can compute the Ricci scalar R , which for the Kantowski-Sachs metric turns out to be

$$R = 2 \frac{\ddot{p}_b}{p_b} + \frac{\ddot{p}_c}{p_c} + \frac{2}{p_c}. \quad (2.18)$$

Using the equations for the expansion and the shear scalar, the Ricci scalar can be expressed as

$$R = 2\dot{\theta} + \frac{4}{3}\theta^2 + 2\sigma^2 + \frac{2}{p_c}. \quad (2.19)$$

⁸ To make a connection with the Schwarzschild interior, a convenient choice of lapse is $N = \frac{\gamma\sqrt{p_c}}{b}$ [33]. For studies of the expansion and shear scalars and the phenomenological implications of Kantowski-Sachs spacetime with matter, the choice $N = 1$ is more useful, and is thus considered here.

Thus, a divergence in θ and σ^2 signals a divergence in the Ricci scalar. For this reason, understanding the behavior of expansion and shear scalars is important to gain insights on not only the properties of the geodesic evolution, but it is also useful to understand the behavior of curvature invariants.

The scalars, θ and σ^2 , diverge if either one or both of $\frac{\dot{p}_b}{p_b}$ and $\frac{\dot{p}_c}{p_c}$ diverge. From the Hamilton's equations of motion (2.9) and (2.10), these ratios are,

$$\frac{\dot{p}_b}{p_b} = \frac{1}{\gamma} \left(\frac{c\sqrt{p_c}}{p_b} + \frac{b}{\sqrt{p_c}} \right) \quad (2.20)$$

$$\frac{\dot{p}_c}{p_c} = \frac{2b}{\sqrt{p_c}\gamma}. \quad (2.21)$$

It is clear from equations (2.20) and (2.21) that the expansion and shear scalars diverge as the triad components vanish, and/or the connection components diverge. In the Kantowski-Sachs spacetime with perfect fluid as matter, classical singularities occur at a vanishing volume. The structure of the singularity can be a barrel, cigar, pancake or a point [1]. For all these structures, either p_b or p_c vanish, causing a divergence in θ and σ^2 .⁹

At the above classical singular points, the energy density also diverges. From the vanishing of the Hamiltonian constraint $\mathcal{H}_{cl} \approx 0$, the expression for energy density becomes

$$\rho_m = \frac{1}{8\pi G\gamma^2} \left[\frac{2bc}{p_b} + \frac{b^2 + \gamma^2}{p_c} \right]. \quad (2.22)$$

Thus, if either of p_b or p_c vanishes, ρ_m grows unbounded as the physical volume approaches zero.

III. EFFECTIVE LOOP QUANTUM COSMOLOGICAL DYNAMICS: COMPARISON OF DIFFERENT PRESCRIPTIONS

Due to the underlying quantum geometry, the loop quantization of the classical Hamiltonian of the Kantowski-Sachs spacetime yields a difference equation [33]. The difference equation arises due to non-local nature of the field strength of the connection in the quantum Hamiltonian constraint which is expressed in terms of holonomies of connection components over closed loops. The action of the holonomy operators on the triad states is discrete, leading to a discrete quantum Hamiltonian constraint which is non-singular.¹⁰ The resulting quantum dynamics can be captured using an effective Hamiltonian constraint derived using the geometrical formulation of quantum mechanics [47]. Here one treats the Hilbert space as a quantum phase space and seeks an embedding of the

⁹ Note that for the vacuum Kantowski-Sachs spacetime, the expansion and shear scalars are ill defined at the horizon because of the coordinate singularity. However, $\theta^2/3 - \sigma^2$ is regular at the horizon, and can be used to understand the behavior of the curvature invariants. As an example, in this case, the Kretschmann scalar at the horizon can be written as $K_{t=2m} = 12(\theta^2/3 - \sigma^2)^2$, which being finite shows that the singularity at $t = 2m$ is not physical.

¹⁰ In principle, there can also be inverse triad modifications in the quantum Hamiltonian constraint. However, such modifications can not be consistently defined for spatially non-compact manifolds not only at the full quantum level but even in the effective Hamiltonian description, due to the dependence on the fiducial length. As discussed earlier in footnotes 1 and 2, note that under the rescaling of the fiducial cell, the invariance of the quantum theory does not hold when volume is comparable to the Planck volume. However, at larger volumes in the quantum theory, and also in the approximation of the validity of the effective dynamics $\bar{\mu}$ prescription preserves rescaling under fiducial cell. In this analysis, we do not consider inverse triad modifications. However, these can be consistently included if the spatial topology is compact, and conclusions reached in this manuscript remain unaffected in this case. Further, it is also possible to get rid of terms depending on inverse triad using a suitable choice of lapse.

finite dimensional classical phase space into it. For the isotropic and homogeneous models in LQC, such a suitable embedding has been found using sharply peaked states which probe volumes larger than the Planck volume [19, 20]. For these models, the dynamics from the quantum difference equation and the effective Hamiltonian turn out to be in an excellent agreement for states which correspond to a classical macroscopic universe at late times. Recent numerical investigations show that the departures between the effective spacetime description and the quantum dynamics are negligible unless one consider states which correspond to highly quantum spacetimes, such as states which are widely spread or are highly squeezed and non-Gaussian, or those which do not lead to a classical universe at late times [8, 9]. Though the effective Hamiltonian constraint has not been derived for the anisotropic spacetimes in LQC using the above embedding approach, an expression for it has been obtained by replacing b with $\frac{\sin b\delta_b}{\delta_b}$ and c with $\frac{\sin c\delta_c}{\delta_c}$ in (2.8), where δ_b and δ_c are the edge lengths of the holonomies [41, 42]. Following this procedure for the case of the loop quantization of the vacuum Bianchi-I spacetime, the resulting effective Hamiltonian dynamics turns out to be in excellent agreement with the underlying quantum evolution [48]. In the following we will assume that the effective Hamiltonian constraint for the Kantowski-Sachs spacetime as obtained from the above polymerization of the connection components, and assume it to be valid for all values of triads. For a general choice of δ_b and δ_c , the effective Hamiltonian constraint for the Kantowski-Sachs model with matter is given as [41, 42]:

$$\mathcal{H} = \frac{-N}{2G\gamma^2} \left[2 \frac{\sin(b\delta_b)}{\delta_b} \frac{\sin(c\delta_c)}{\delta_c} \sqrt{p_c} + \left(\frac{\sin^2(b\delta_b)}{\delta_b^2} + \gamma^2 \right) \frac{p_b}{\sqrt{p_c}} \right] + N4\pi p_b \sqrt{p_c} \rho_m. \quad (3.1)$$

Note that (3.1) goes to the classical Hamiltonian (2.8) in the limit $\delta_b \rightarrow 0$ and $\delta_c \rightarrow 0$. However, due to the existence of minimum area gap in LQG, in the quantum theory, one shrinks the loops to the minimum finite area. Different choices of the way holonomy loops are constructed and shrunk lead to different δ_b and δ_c , and different properties of the quantum Hamiltonian constraint. We will identify these choices as different prescriptions to quantize the theory, which lead to different functional forms of δ_b and δ_c in the polymerization of the connection, and hence result in different effective Hamiltonian constraints. This is analogous to the situation in the quantization of isotropic spacetimes in LQC, where the older quantization was based on constant δ (the so called μ_o quantization [4, 30]), and improved quantization is based on a δ which is function of isotropic triad $\delta \propto 1/\sqrt{p}$ (the so called $\bar{\mu}$ quantization [5]). As in the isotropic case, the physics obtained from the theory is dependent on these holonomy edge lengths and hence they have to be chosen carefully. This can be further seen by noting that $\sin(b\delta_b)$ and $\sin(c\delta_c)$ in (3.1) can be expanded in infinite series as $b\delta_b - \frac{b^3\delta_b^3}{3!} + \dots$ and $c\delta_c - \frac{c^3\delta_c^3}{3!} + \dots$. Hence it is required that $b\delta_b$ and $c\delta_c$ should be independent of fiducial length. Else different terms of the expansion will have different powers of L_o and any calculation based on this Hamiltonian will yield results which are sensitive to the choice of L_o . Of the possible choices of holonomy edge lengths that can be motivated, we have to choose the one that gives a mathematically consistent theory which renders the physical scalars such as expansion and shear scalars independent of the choice of fiducial length, as in classical GR. There are three proposed prescriptions in LQC literature for the choice of holonomy edge-lengths in the Kantowski-Sachs model: the constant δ [33], the $\bar{\mu}$ (or the ‘improved dynamics’) prescription [41], and the $\bar{\mu}'$ (inspired from the improved dynamics) quantization prescriptions [42]. Due to their similarities with the notation of the isotropic model, we will label the effective Hamiltonian constraint for constant δ with μ_o . The effective Hamiltonians for ‘improved dynamics’ inspired prescription will be labelled by $\bar{\mu}'$, and that of ‘improved dynamics’ prescription with $\bar{\mu}$.

A. Constant δ prescription

The simplest choice of δ 's is to choose them as constant. The resulting effective Hamiltonian constraint then corresponds to the loop quantization of Kantowski-Sachs spacetime where the holonomy considered over the loop in $x - \theta$ plane, and the loop in the $\theta - \phi$ plane has minimum area with respect to the fiducial metric fixed by the minimum area eigenvalue Δ in LQG: $\Delta = 4\sqrt{3}\pi\gamma l_{\text{Pl}}^2$. In the quantization of the Schwarzschild interior proposed in Ref. [33], the δ 's were chosen equal¹¹ $\delta_b = \delta_c = 4\sqrt{3}$. Loop quantization with constant δ_b and δ_c is also considered in various other works on the loop quantization of black hole spacetimes [36, 37, 40], and is analogous to the μ_o quantization in the isotropic LQC [4, 30]. Here we will assume the same prescription in the presence of matter.

The resulting effective Hamiltonian constraint for $N = 1$ with minimally coupled matter is:

$$\mathcal{H}_{\mu_0} = \frac{-1}{2G\gamma^2} \left[2 \frac{\sin(b\delta_b)}{\delta_b} \frac{\sin(c\delta_c)}{\delta_c} \sqrt{p_c} + \left(\frac{\sin^2(b\delta_b)}{\delta_b^2} + \gamma^2 \right) \frac{p_b}{\sqrt{p_c}} \right] + 4\pi p_b \sqrt{p_c} \rho_m. \quad (3.2)$$

Using the Hamilton's equations, the equations of motion for the triads are

$$\dot{p}_b = -G\gamma \frac{\partial \mathcal{H}_{\mu_0}}{\partial b} = \frac{1}{\gamma} \left(\cos(b\delta_b) \frac{\sin(c\delta_c)}{\delta_c} \sqrt{p_c} + \frac{\sin(b\delta_b) \cos(b\delta_b)}{\delta_b} \frac{p_b}{\sqrt{p_c}} \right), \quad (3.3)$$

$$\dot{p}_c = -2G\gamma \frac{\partial \mathcal{H}_{\mu_0}}{\partial c} = \frac{2}{\gamma} \cos(c\delta_c) \frac{\sin(b\delta_b)}{\delta_b} \sqrt{p_c}. \quad (3.4)$$

From these one can find the expressions for expansion¹² and shear scalars for Kantowski-Sachs spacetime with matter as follows,

$$\theta = \frac{1}{\gamma} \left(\frac{\sqrt{p_c} \cos(b\delta_b) \sin(c\delta_c)}{p_b \delta_c} + \frac{\sin(b\delta_b)}{\sqrt{p_c} \delta_b} (\cos(b\delta_b) + \cos(c\delta_c)) \right) \quad (3.5)$$

$$\sigma^2 = \frac{1}{3\gamma^2} \left((2 \cos(c\delta_c) - \cos(b\delta_b)) \frac{\sin(b\delta_b)}{\delta_b \sqrt{p_c}} - \frac{\cos(b\delta_b) \sin(c\delta_c)}{\delta_c} \frac{\sqrt{p_c}}{p_b} \right)^2. \quad (3.6)$$

It is clear from the above expressions that the expansion and shear scalars are unbounded and blow up as p_b or p_c approach zero, precisely as in the classical Kantowski-Sachs spacetime if the effective spacetime description is assumed to be valid for all values of triads. Note that the effective spacetime description is expected to breakdown in the regime when the volume of the spacetime is less than Planck volume [8]. Hence, in this quantization prescription there are no generic bounds on the expansion and shear scalars within the expected validity of effective dynamics. Even if one considers a specific matter model which results in a singularity resolution and a bounce of the mean volume, the dependence of θ and σ^2 on the triads shows that these scalars may not necessarily take Planckian values in the bounce regime. The spacetime curvature in the bounce regime can in principle be extremely small in this effective dynamics. Note that the maximum value of expansion (3.5) and shear scalars (3.6) depends on the values of p_b and p_c . Since the values of triads at the bounce can be made arbitrarily large or small by the choice of initial conditions and the matter content, the maximum values of expansion and shear scalars, reached near the bounce, can hence

¹¹ Since [33] was using an area gap of $\Delta = 2\sqrt{3}\pi\gamma l_{\text{Pl}}^2$, the corresponding holonomy edge lengths were $2\sqrt{3}$. For $\Delta = 4\sqrt{3}\pi\gamma$, edge lengths should be $4\sqrt{3}$.

¹² The expressions for θ in three prescriptions studied in this section were also obtained for the Schwarzschild interior in Ref.[45], however no physical implications were studied except for noticing the bounded behavior in the case of $\bar{\mu}$ prescription.

take arbitrary values. This problem is analogous to the dependence of energy density at the bounce on the momentum of the scalar field or the triad in the μ_o quantization of isotropic LQC. There too by choosing different initial conditions it is possible to obtain “quantum bounce” at arbitrarily small spacetime curvature.

Let us now consider the issue of fiducial cell dependence for this prescription. Since $\delta_b = \delta_c = 4\sqrt{3}$, they are independent of the rescaling under the fiducial length L_o . However, since c is proportional to L_o , therefore $c\delta_c$ depends on the fiducial length L_o . Due to this reason, the resulting physics from the effective Hamiltonian constraint (3.2), in particular the expressions for expansion and shear scalars, unlike in the classical theory, are not independent of the fiducial length rescaling. Again this problem of constant δ prescription in the Kantowski-Sachs spacetime is analogous to the one for the μ_o quantization of the isotropic LQC, where the resulting physical predictions such as the scale at which the quantum bounce occurs and the infra-red behavior depend on the fiducial volume of the fiducial cell [4, 21]. This problem is tied to the dependence of the expansion and triad scalars in this quantization prescription on triads as discussed above. Since p_b can be rescaled arbitrarily by rescaling L_o , the curvature scale in the bounce regime inevitably depends on the fiducial length L_o and hence can take arbitrary values.

In conclusion, we find that constant δ quantization prescription does not provide a generic bounded behavior of expansion and shear scalars. Further, it is possible to obtain “quantum gravitational effects,” originating from the trigonometric functions in eq.(3.2), at any arbitrary scale.

B. An ‘improved dynamics inspired’ prescription

For the isotropic models in LQC, the problems with constant δ (i.e. μ_o) quantization were overcome in the improved dynamics (the $\bar{\mu}$ quantization) [5], where $\bar{\mu}$ is related to the isotropic triad as $\bar{\mu} = \Delta/\sqrt{p}$ [5]. This quantization turns out to be independent of the various problems of the μ_o quantization, and is also the unique prescription for the quantization of isotropic models in which physical predictions are free of the dependence on the fiducial cell in the effective spacetime description [21]. Motivated by the success of $\bar{\mu}$ quantization, a different prescription for the choice of δ_b and δ_c for Kantowski-Sachs model has been considered [42], where

$$\delta_b = \sqrt{\frac{\Delta}{p_b}}, \quad \text{and} \quad \delta_c = \sqrt{\frac{\Delta}{p_c}}. \quad (3.7)$$

We note that this choice for δ 's is also motivated from the lattice refinement scheme [31].

The effective Hamiltonian constraint for this quantization becomes:

$$\mathcal{H}_{\bar{\mu}'} = \frac{-1}{2G\gamma^2\Delta} \left[2\sin(b\delta_b)\sin(c\delta_c)p_c\sqrt{p_b} + (\sin^2(b\delta_b)p_b + \gamma^2\Delta) \frac{p_b}{\sqrt{p_c}} \right] + 4\pi p_b\sqrt{p_c}\rho_m. \quad (3.8)$$

As we noted above, for the effective Hamiltonian constraint to yield a consistent physics, the argument of trigonometric functions should be independent of the fiducial length. However since b is independent of L_o and p_b is proportional to L_o , $b\delta_b = b\sqrt{\frac{\Delta}{p_b}}$ depends on fiducial length. Similarly $c\delta_c$ also depends on the fiducial length. This clearly shows that this quantization is unsuitable for Kantowski-Sachs spacetime because the resulting physical implications will be sensitive to the fiducial length L_o .

The equations of motion for the triads in this quantization are

$$\dot{p}_b = -G\gamma \frac{\partial \mathcal{H}_{\bar{\mu}'}}{\partial b} = \frac{\cos(b\delta_b)}{\gamma\sqrt{\Delta}} \left(p_c \sin(c\delta_c) + p_b \sqrt{\frac{p_b}{p_c}} \sin(b\delta_b) \right) \quad (3.9)$$

$$\dot{p}_c = -2G\gamma \frac{\partial \mathcal{H}_{\bar{\mu}'}}{\partial c} = \frac{2}{\gamma\sqrt{\Delta}} \sqrt{p_b p_c} \sin(b\delta_b) \cos(c\delta_c), \quad (3.10)$$

using which the expansion and shear scalars turn out to be as follows:

$$\theta = \frac{1}{\gamma\sqrt{\Delta}} \left[\frac{p_c}{p_b} \cos(b\delta_b) \sin(c\delta_c) + \sqrt{\frac{p_b}{p_c}} \sin(b\delta_b) (\cos(b\delta_b) + \cos(c\delta_c)) \right], \quad (3.11)$$

$$\sigma^2 = \frac{1}{3\gamma^2\Delta} \left[\frac{p_c}{p_b} \cos(b\delta_b) \sin(c\delta_c) + \sqrt{\frac{p_b}{p_c}} \sin(b\delta_b) (\cos(b\delta_b) - 2\cos(c\delta_c)) \right]^2. \quad (3.12)$$

We see that the $\bar{\mu}'$ quantization has the same problem as the constant δ quantization as far as the divergence of θ and σ^2 is concerned. These scalars can potentially diverge for $p_b \rightarrow 0$, $p_b \rightarrow \infty$, $p_c \rightarrow 0$ or $p_c \rightarrow \infty$.

As in the constant δ quantization prescription, even if the singularities are resolved, the curvature scale associated with singularity resolution can be arbitrarily small and depends on the initial conditions. Also remembering that it has spurious dependency on the fiducial length we are led to the conclusion that $\bar{\mu}'$ quantization is not apt for Kantowski-Sachs spacetime. The results that constant δ and $\bar{\mu}'$ quantizations do not yield necessarily consistent physics is in accordance with a similar study in FRW model in LQC [21]. As remarked earlier, problems of this prescription have also been noted in the context of the von-Neumann stability analysis of the resulting quantum Hamiltonian constraint [31].¹³

C. ‘Improved Dynamics’ prescription

The improved dynamics prescription is based on noting that the field strength of the Ashtekar-Barbero connection should be computed by considering holonomies around the loop whose minimum area with respect to the physical metric is fixed by the minimum area eigenvalue (Δ) in LQG. This is in contrast to the constant δ prescription where the minimum area with respect to the fiducial metric was fixed with respect to the underlying quantum geometry. In this scheme we obtain the holonomy edge lengths as [41]:

$$\delta_b = \sqrt{\frac{\Delta}{p_c}}, \quad \delta_c = \frac{\sqrt{\Delta p_c}}{p_b}. \quad (3.13)$$

Now the effective Hamiltonian (3.1) becomes,

$$\mathcal{H}_{\bar{\mu}} = \frac{-p_b \sqrt{p_c}}{2G\gamma^2\Delta} \left[2 \sin(b\delta_b) \sin(c\delta_c) + \sin^2(b\delta_b) + \frac{\gamma^2\Delta}{p_c} \right] + 4\pi p_b \sqrt{p_c} \rho_m. \quad (3.14)$$

Before we proceed further, we note an important property of this effective Hamiltonian not shared by \mathcal{H}_{μ_o} and $\mathcal{H}_{\bar{\mu}'}$. Due to the scaling properties of b, c, p_b and p_c , $b\delta_b$ and $c\delta_c$ are invariant under

¹³ For different prescriptions, the problems in the effective dynamics and the numerical instability of the quantum difference equation in the corresponding quantization run in parallel. See Ref. [7] for a discussion of these issues in different quantizations in LQC.

the change of the fiducial length L_o . Thus $\sin(b\delta_b)$ and $\sin(c\delta_c)$ are independent of fiducial length. Due to this reason, we expect that the physical predictions concerning scalars such as expansion and shear scalars will be independent of L_o in this prescription, as in the classical theory.

The evolution equations for triads and cotriads turn out to be as follows:

$$\dot{p}_b = -G\gamma \frac{\partial \mathcal{H}_{\bar{\mu}}}{\partial b} = \frac{p_b \cos(b\delta_b)}{\gamma\sqrt{\Delta}} (\sin(c\delta_c) + \sin(b\delta_b)), \quad (3.15)$$

$$\dot{p}_c = -2G\gamma \frac{\partial \mathcal{H}_{\bar{\mu}}}{\partial c} = \frac{2p_c}{\gamma\sqrt{\Delta}} \sin(b\delta_b) \cos(c\delta_c) \quad (3.16)$$

$$(3.17)$$

Using (2.15), (3.15) and (3.16), we obtain the following expression for the expansion scalar,

$$\theta = \frac{1}{\gamma\sqrt{\Delta}} (\sin(b\delta_b) \cos(c\delta_c) + \cos(b\delta_b) \sin(c\delta_c) + \sin(b\delta_b) \cos(b\delta_b)). \quad (3.18)$$

Unlike the case of \mathcal{H}_{μ_o} and $\mathcal{H}_{\bar{\mu}'}$, the expansion scalar turns out to be independent of the fiducial length L_o , and is generically bounded above by a universal value:

$$|\theta| \leq \frac{3}{2\gamma\sqrt{\Delta}} \approx \frac{2.78}{l_{\text{Pl}}}. \quad (3.19)$$

Similarly for the shear scalar, using (2.16), (3.15) and (3.16), we get

$$\sigma^2 = \frac{1}{3\gamma^2\Delta} (2\sin(b\delta_b) \cos(c\delta_c) - \cos(b\delta_b) (\sin(c\delta_c) + \sin(b\delta_b)))^2. \quad (3.20)$$

As for the expansion scalar, σ^2 turns out to be independent of L_o and has a universal maximum:

$$\sigma^2 \leq \frac{5.76}{l_{\text{Pl}}^2}. \quad (3.21)$$

Hence both shear and expansion scalars are bounded above in this quantization prescription of the Kantowski-Sachs spacetime. Unlike constant δ and $\bar{\mu}'$ quantization prescriptions, the expansion and shear scalars take Planckian values in the bounce regime and curvature scale associated with singularity resolution does not depend on the initial conditions. Note that for the improved dynamics prescription, similar properties of expansion and shear scalar were earlier found for the isotropic model [25] and the Bianchi models [17, 22, 28, 29]. In the isotropic and Bianchi-I model, using the boundedness properties of expansion and shear scalars it was found that strong singularities are generically resolved in the effective spacetime description [25, 28].¹⁴ Above results provide a strong evidence that strong singularities may be generically absent in this quantization of Kantowski-Sachs spacetime.

IV. UNIQUENESS OF $\bar{\mu}$ PRESCRIPTION

In the previous section, we found that of the three proposed quantization prescriptions for the Kantowski-Sachs spacetime in LQC, only the the $\bar{\mu}$ effective Hamiltonian leads to consistent physics and results in generic bounds on expansion and shear scalars. In this section we pose the question whether $\bar{\mu}$ quantization is the only possible choice for which the expansion and shear

¹⁴ These results have also been extended to the effective description of the hybrid quantization of Gowdy models [49].

scalars are generically bounded singularity resolution in the Kantowski-Sachs spacetime? A similar question was posed in the isotropic models in LQC, where the answer turned out to be positive [21, 22]. We will see that in the Kantowski-Sachs spacetime, under the assumption that δ_b and δ_c have a general form given in eq.(4.4), the answer also turns to be in an affirmative in the effective spacetime description.

We start with the effective LQC Hamiltonian (3.1), where the holonomy edge lengths δ_b and δ_c are any general functions of the triads. Then the Hamilton's equations lead to the following expressions for shear and expansion scalars.

$$\theta = \frac{1}{\gamma} \left(\frac{\sqrt{p_c} \cos(b\delta_b(p_b, p_c)) \sin(c\delta_c(p_b, p_c))}{p_b \delta_c(p_b, p_c)} + \frac{\sin(b\delta_b(p_b, p_c))}{\sqrt{p_c} \delta_b(p_b, p_c)} (\cos(b\delta_b(p_b, p_c)) + \cos(c\delta_c(p_b, p_c))) \right) \quad (4.1)$$

$$\sigma^2 = \frac{1}{3\gamma^2} \left[(2 \cos(c\delta_c(p_b, p_c)) - \cos(b\delta_b(p_b, p_c))) \frac{\sin(b\delta_b(p_b, p_c))}{\delta_b(p_b, p_c) \sqrt{p_c}} - \frac{\cos(b\delta_b(p_b, p_c)) \sin(c\delta_c(p_b, p_c)) \sqrt{p_c}}{\delta_c(p_b, p_c) p_b} \right]^2. \quad (4.2)$$

We now find what general choices of $\delta_b(p_b, p_c), \delta_c(p_b, p_c)$ yield a bound on expansion and shear scalars. These scalars become unbounded when either an inverse power of a triad blows up as that triad tends to zero or when a positive power of triad blows up as that triad tend to infinity. In eqs. (4.1) and (4.2), the trigonometric factors are always bounded and hence the terms that will decide the boundedness of the expansion and shear scalars are

$$T_b = \frac{1}{\sqrt{p_c} \delta_b(p_b, p_c)} \quad \text{and} \quad T_c = \frac{\sqrt{p_c}}{p_b \delta_c(p_b, p_c)}. \quad (4.3)$$

Then the task at hand reduces to finding general functions of triads which when chosen as the holonomy edge lengths, give an upper bound on T_c and T_b . To this end we make an assumption that δ_b and δ_c are functions of p_b and p_c such that one can express their inverses as

$$\delta_b^{-1} = \sum B_{ij} p_b^{m_i} p_c^{n_j}, \quad \delta_c^{-1} = \sum C_{ij} p_b^{m_i} p_c^{n_j}, \quad (4.4)$$

where $m_i, n_j \in \mathbb{R}$. This ansatz includes all the three choices of δ_b and δ_c discussed in Sec. III, but is more general. Using (4.4), one can write (4.3) as

$$T_c = \sum C_{ij} p_b^{m_i-1} p_c^{n_j+1/2}, \quad (4.5)$$

$$T_b = \sum B_{ij} p_b^{m_i} p_c^{n_j-1/2}. \quad (4.6)$$

We now require that if θ and σ^2 have to be bounded then T_c and T_b should not diverge as triads tend to zero or infinity. This is possible only if m_i and n_j in (4.5) and (4.6) satisfy certain constraints. We find that these constraints only allow $\delta_b \propto (p_c)^{-1/2}$ and $\delta_c \propto p_c^{1/2}/p_b$, the same as in the $\bar{\mu}$ quantization (3.13).

First let us take a closer look at (4.5) from which we wish to obtain constraints on δ_c . Keeping p_c as nondiverging and nonvanishing, one can obtain bounds on values of m_i , the powers of p_b with nonzero coefficients. As $p_b \rightarrow 0$, for each term in T_c to be nondiverging, they should all have a non-negative power of p_b . Thus, for any nonzero C_{ij} , $m_i \geq 1$. Also, as $p_b \rightarrow \infty$, any positive power of p_b diverges. Hence for T_c to be bounded, for any nonzero C_{ij} , $m_i \leq 1$. Therefore, the only possible value for m_i that leaves T_c bounded for $p_b \rightarrow 0$ and $p_b \rightarrow \infty$ is $m_i = 1$. Similarly, to find the allowed values for n_j , we study the behavior of T_c as p_c goes to zero and infinity for

a finite nonzero value of p_b . It is clear that positive powers of p_c will result in a divergence of T_c as $p_c \rightarrow \infty$ where as negative powers will result in a divergence when $p_c \rightarrow 0$. This implies that the only choice of n_j that leaves T_c bounded for the whole range of p_c is $n_j = -1/2$. Finally, we consider the case of both the triads simultaneously approaching one of the extreme values - zero or infinity. For $m_i = 1$ and $n_j = -1/2$, from (4.5) it can be seen that T_c is independent of triads i.e, it is just a constant. Hence for both the triads simultaneously approaching an extreme value, T_c remains bounded. For any other choice of m_i or n_j , T_c can diverge, causing a divergence in the expansion and shear scalars.

Repeating the same analysis, for T_b in (4.6), it can be seen that the only values of m_i and n_j which keep T_b bounded for the whole domain of p_b and p_c are $m_i = 0$ and $p_c = 1/2$. Thus from (4.4) it can be seen that the only choice of δ 's which keeps θ and σ^2 bounded throughout the entire domain of triads correspond to

$$\delta_c \propto \frac{\sqrt{p_c}}{p_b}, \quad \delta_b \propto \frac{1}{\sqrt{p_c}}. \quad (4.7)$$

These are precisely the functional dependencies of the holonomy edge lengths on these triads in the 'improved dynamics' prescription. (3.14). Thus, for the general ansatz (4.4) we find that the only possible choices of δ_b and δ_c which result in bounded expansion and shear scalars for the geodesics in the effective dynamics correspond to $\bar{\mu}$ prescription. It is important to stress that we found the uniqueness of $\bar{\mu}$ quantization prescription by only demanding that the expansion and shear scalars be bounded, and our argument is not tied to requirements based on fiducial cell rescaling freedom or to the topology of the spatial manifold. But, it is rather interesting that the prescription which results in generic bounds on scalars is the one which is also free from the freedom under rescalings of the fiducial cell. It is straightforward to see that requiring $b\delta_b$ and $c\delta_c$ to be independent of fiducial length L_o , and assuming that δ_b and δ_c are constructed from the triads p_b and p_c , one is led to the $\bar{\mu}$ prescription.

In the above analysis we have seen that by requiring that the expansion and shear scalars be always bounded, one can find the exact dependence of δ_b and δ_c on the triads. The same functional forms of δ_b and δ_c can be obtained from an independent physical motivation. Note that holonomy corrections in the effective Hamiltonian arise from the field strength of the connection components b and c , where one has to take the holonomies around closed loops with edge lengths determined by δ_b and δ_c . To compute the field strength, the loops over which the holonomies are considered are shrunk to the minimum area eigenvalue in LQG. One could in principle form loops from holonomies with constant edgelengths δ_b, δ_c or as in the $\bar{\mu}'$ scheme, where $\delta_b = \sqrt{\frac{\Delta}{p_b}}$ and $\delta_c = \sqrt{\frac{\Delta}{p_c}}$. But loops with such edge lengths do not have physical area matching the minimum area gap from LQG. The constant δ quantization takes the holonomy loops to have constant fiducial area, but not the physical area. However, fiducial area is not independent of rescaling of fiducial length and thus is not a physical quantity. In this quantization, a loop with edges of length δ_b along θ and ϕ directions will have a physical area $\delta_b^2 p_c$.¹⁵ This area is clearly dependent of the triad and can even vanish as $p_c \rightarrow 0$, thus becoming smaller than the minimum area eigenvalue of LQG. Similarly, in $\bar{\mu}'$ quantization, the area of a loop with edge δ_b each along θ and ϕ directions will be $\frac{\Delta p_c}{p_b}$. Once again this area is not constant and can go below the minimum area gap of LQG if p_c/p_b becomes less than unity. In contrast the loops constructed in the improved dynamics with $\delta_b = \sqrt{\Delta/p_c}$ and $\delta_c = \sqrt{\Delta p_c}/p_b$ in $x - \theta$ and $\theta - \phi$ planes have a physical area Δ , which is same as the minimum area gap. Thus, this argument further supports the improved dynamics or the $\bar{\mu}$ prescription for the Kantowski-Sachs spacetime.

¹⁵ It is straightforward to see that the same conclusion is reached or the loop in $x - \theta$ plane.

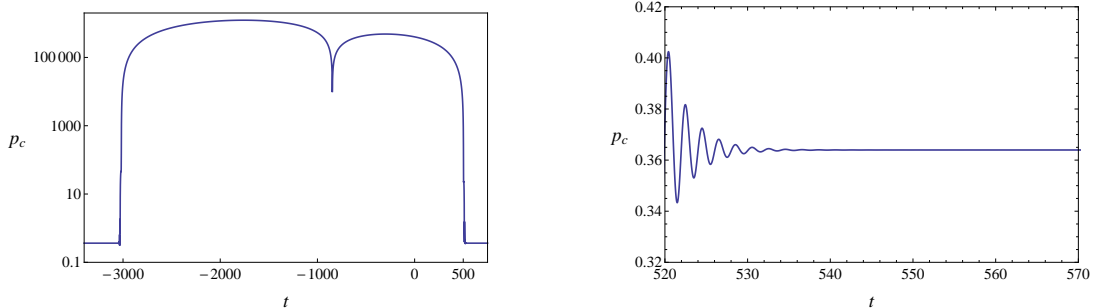


FIG. 1: Evolution of the triad component p_c is shown as a function of proper time for the massless scalar field evolution in the $\bar{\mu}$ effective dynamics. The initial conditions are $p_b(0) = 5 \times 10^5$, $b(0) = -0.1$, $p_c(0) = 4 \times 10^5$, $c(0) = 0.16$ (all in Planck units). Initial value of energy density is obtained by solving the Hamiltonian constraint. We see that the classical singularity is avoided, and p_c is non-zero in the entire evolution. Asymptotic approach of p_c to a finite value is also shown. A similar plot is obtained for the vacuum case, where it was shown that some cycles of classical phases appear before p_c reaches Planck regime [42]. For the above left plot, the two macroscopic turn arounds occur in the classical regime. In the right plot, the wiggles on the left occur in the non-classical regime where the magnitude of $\sin(\delta_b b)$ and $\sin(\delta_c c)$ is not close to zero. In the forward evolution, the wiggles progressively occur in a more quantum regime.

V. ENERGY DENSITY IN THE ‘IMPROVED DYNAMICS’

We have so far seen that out of various possible quantization prescriptions, the $\bar{\mu}$ prescription for the Kantowski-Sachs spacetime is the only one which results in bounded expansion and shear scalars for all the values of triads. Also, the resulting physics turns out to be independent of the rescalings under fiducial length at the level of effective spacetime description. In this sense, this is the preferred choice for the loop quantization in the Kantowski-Sachs model. We now investigate the issue of the boundedness of the energy density in this prescription. It will be useful to recall some features of classical singularities in this context. In classical GR, approach to singularities in the Kantowski-Sachs spacetime is accompanied by a divergence in the energy density for perfect fluids when the volume vanishes [1]. The nature of the singularity – whether it is isotropic or anisotropic depends on the equation of state of matter. Apart from the isotropic or the point like singularity, cigar, pancake and barrel singularities can also form in the classical Kantowski-Sachs spacetime. For the point singularity both g_{xx} and $g_{\Omega\Omega}$ vanish, for the cigar singularity $g_{xx} \rightarrow \infty$ and $g_{\Omega\Omega} \rightarrow 0$, for the barrel singularity g_{xx} approaches a finite value and $g_{\Omega\Omega} \rightarrow 0$, and for the pancake singularity g_{xx} vanishes and $g_{\Omega\Omega}$ approaches a finite value. In terms of the triad components, for point, cigar and barrel singularities both p_b and p_c vanish. However, the pancake singularity occurs at a finite value of p_c , with p_b vanishing.

We now investigate whether the energy density is bounded in the effective spacetime description of the $\bar{\mu}$ quantization. The energy density can be obtained from the Hamiltonian constraint $H_{\bar{\mu}} \approx 0$ as

$$\rho_{\bar{\mu}} = \frac{1}{8\pi G\gamma^2\Delta} \left[2 \sin(b\delta_b) \sin(c\delta_c) + \sin^2(b\delta_b) + \frac{\gamma^2\Delta}{p_c} \right]. \quad (5.1)$$

It is clear that this energy density is bounded for all values of triads and cotriads except when $p_c \rightarrow 0$. Especially, we note that even if the triad p_b is vanishing, the energy density is bounded as far as p_c is nonzero. Since a pancake singularity is attained when p_c remains finite, we can already conclude that such a singularity is absent in the effective description of the Kantowski-

Sachs spacetime for the $\bar{\mu}$ quantization.¹⁶

Let us now return to the properties of the energy density in general, and understand its behavior for the generic singularities. The energy density in $\bar{\mu}$ approach will be bounded if p_c does not vanish. In the non-singular evolution, one expects that the dynamics results in a non-zero value of p_c . The pertinent question is whether in effective dynamics this happens to be true. Numerical analysis of the Hamilton's equations shows that the answer turns out to be positive. The first evidence of this behavior of p_c was reported in the vacuum Kantowski-Sachs case, where it was found that due to holonomy corrections, p_c (as well as p_b) undergo non-singular evolution, and p_c never approaches zero throughout the evolution [41]. It was found that p_c approaches an asymptotic non-zero value after classical singularity is avoided. Detailed numerical analysis of effective Hamiltonian constraint (3.14) for different types of matter fields shows that a similar behavior occurs for p_c in general [46]. An example of this phenomena is shown in Fig. 1, where we plot the behavior of p_c versus proper time for the case of massless scalar field in a typical numerical simulation. Giving the initial date at $t = 0$ we numerically solve the Hamilton's equations for the effective Hamiltonian constraint (3.14). During the past and future evolution, the physical volume does not go to zero when the classical singularity is approached, but instead bounces. The triad p_c never goes to zero in the entire evolution, but asymptotes towards a constant value. These results, and also of Ref. [41], confirm that dynamically p_c is always bounded away from zero. Hence, we conclude that the energy density (5.1) is always bounded in the loop quantization of the Kantowski-Sachs spacetime.

VI. CONCLUSIONS

Classical Kantowski-Sachs spacetime is singular for generic matter choices, which calls upon a quantum gravitational treatment to see if the singularity persists. A good understanding about the geodesic completeness of a spacetime can be obtained via expansion and shear scalars. Any divergence in these scalars indicates presence of a singularity. Since singularity denotes break down of the theory which is used to describe spacetime, it is hoped that the right theory of quantum gravity will resolve these singularities in general. A quantum theory of spacetime should pass various consistency tests. If the spatial manifold is non-compact, then the expansion and shear scalars must be independent of the choice of the fiducial cell. If the singularities are indeed resolved, then the curvature scale associated with singularity resolution should not be arbitrary. Due to quantization ambiguities, various prescriptions can exist for quantization of a spacetime. Is it possible that a particular prescription is favored over others? This question was earlier posed in the isotropic [21] and Bianchi-I spacetime in LQC [22], where it was found that $\bar{\mu}$ quantization prescription in contrast to other quantization prescriptions leads to generic bounded behavior of expansion and shear scalars, and physical predictions free from the rescaling under fiducial cell. The goal of this analysis was to answer this question in the loop quantization of Kantowski-Sachs spacetime assuming the validity of effective spacetime description for minimally coupled matter.

Previous works on loop quantization of Kantowski-Sachs spacetime have been mostly devoted to study the vacuum case, for which the expansion scalar has been partially studied earlier [45]. Little details about the physics of singularity resolution for generic matter were so far available. Three quantization prescriptions were proposed in the literature. Of these, only one was shown to be preferred in the sense that the effective Hamiltonian does not depend on the rescalings of the fiducial length. This quantization prescription (denoted by $\bar{\mu}$) is the analog of the improved

¹⁶ In contrast, this is not true in the constant δ and the ‘improved dynamics inspired’ quantizations discussed earlier. For these prescriptions, the expression of energy density contains inverse power of p_b as well as p_c in the expression for energy density. Thus, allowing all kinds of singularities.

dynamics in isotropic LQC [5]. The other two quantization prescriptions, denoted by μ_o and $\bar{\mu}'$ lead to resolution of singularities in the vacuum case, but were known to be problematic under rescalings of the fiducial cell. Unlike $\bar{\mu}$ prescription, these also yield quantum difference equations which are von-Neumann unstable [31]. We obtained the expansion and shear scalars using the effective dynamics in each of these prescriptions and found that except the case of $\bar{\mu}$ quantization, in both the other choices these scalars are not necessarily bounded in the effective spacetime. Thus it is possible that a strong curvature singularity may not get resolved for μ_o and $\bar{\mu}'$ prescriptions for some choices of matter depending on the initial conditions in effective dynamics. Even if the singularities are resolved, we found that the associated curvature scale is arbitrary. In contrast, the $\bar{\mu}$ quantization leads to universal bounds on the expansion and shear scalars which are dictated by the underlying Planckian geometry for Kantowski-Sachs spacetime with matter. These bounds point towards a generic resolution of singularities in this prescription. Analysis of the behavior of energy density in $\bar{\mu}$ prescription reveals that it is dynamically bounded because p_c is bounded from below. It turns out that this is a generic feature of all types of perfect fluids, whose details will be reported in a future work [46]. It is interesting to note that without solving dynamical equations, it is possible to rule out pancake singularities in the $\bar{\mu}$ prescription. The bounded behavior of expansion and shear scalars and energy density is a strong indication that curvature singularities may be generically resolved in the $\bar{\mu}$ quantization prescription of the Kantowski-Sachs spacetime with matter, as in the case of isotropic and Bianchi-I model [25, 27, 28].

To investigate whether there is another quantization prescription which gives a bounded behavior of expansion and shear scalars, we considered a general ansatz of the edge lengths of the holonomies. It turns out that $\bar{\mu}$ quantization is a unique choice for which the expansion and shear scalars are bounded. For any other prescription, expansion and shear scalars can be unbounded in the effective dynamics. It is remarkable that the demand that these scalars are bounded also chooses the prescription which is free from the rescalings of the fiducial cell. This property is shared by the $\bar{\mu}$ quantization in the isotropic and Bianchi-I spacetime in LQC [21, 22]. All these similarities between the $\bar{\mu}$ quantization of the isotropic, Bianchi-I and Kantowski-Sachs spacetimes bring out a harmonious and robust picture of the loop quantization.

Finally, it is important to stress that though this analysis provides further insights on the loop quantization of Kantowski-Sachs spacetime, singling out the $\bar{\mu}$ prescription on various grounds, more work is needed to rigorously formulate the $\bar{\mu}$ prescription in the quantum theory. It is known that for the Schwarzschild interior, the $\bar{\mu}$ quantization results in quantum gravitational effects at the event horizon where the spacetime curvature in the classical theory can be very small [41]. The existence of these effects is tied to the choice of the coordinates which lead to the classical coordinate singularity at the horizon. Not distinguishing it from the curvature singularity, quantum geometric effects resulting from the holonomies of the connection components thus become significant at the horizon resolving even the coordinate singularity. Note that this coordinate artifact does not arise in the Kantowski-Sachs spacetime in presence of matter. These issues will be closely examined in the $\bar{\mu}$ quantization of the Schwarzschild interior [50]. Further, it has been reported that the Kantowski-Sachs vacuum spacetime in the $\bar{\mu}$ prescription leads to the Nariai-like spacetime after the bounce in the asymptotic approach [41].^{17,18} It turns out that this feature is more general, which reveals some subtle properties of the effective spacetime in LQC [51]. A deeper understanding of these issues is required to gain further insights on the details of the physics of singularity resolution in the Kantowski-Sachs spacetime in LQC.

¹⁷ It is important to make a distinction here with the classical Nariai spacetime, since in the asymptotic approach to Nariai-like spacetime, the spacetime is quantum.

¹⁸ Before the Nariai-like phase is asymptotically approached, the spacetime gives birth to baby blackhole spacetimes [42].

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