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# Microwave conductivity of a d-wave superconductor disordered by extended impurities: a real-space renormalization group approach

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Using a real-space renormalization group (RSRG) technique, the microwave conductivity of a d-wave superconductor disordered by extended impurities is calculated. To do this, a semiclassical approximation is invoked which naturally accesses the Andreev bound states localized near each impurity. Tunneling corrections (which are captured using the RSRG) lead to a delocalization of these quasiparticles and an associated contribution to the microwave conductivity.

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## I. INTRODUCTION

One aspect of the electronic properties of the cuprate superconductors which remains mysterious is their frequency-dependent microwave conductivity<sup>1,2</sup>, the measurement of which sheds light on the effect of disorder on quasiparticle properties. A common feature of many theoretical approaches to disorder (see, e.g., Refs. 3,4) in d-wave superconductors (although there have been a number of exceptions<sup>5,6,7,8</sup>) is the assumption that the scattering may be modelled by *pointlike* impurities. Recent work<sup>7</sup> found that the spectrum of quasiparticle excitations is qualitatively different when the disorder is taken to consist of *extended* impurities, i.e., impurities characterized by a typical size  $a \gg \lambda_F$ , where  $\lambda_F$  is the Fermi wavelength. While the density of states  $\rho(E)$  is believed to vanish linearly at low energies for the case of pointlike impurities<sup>9,10</sup>, in Ref. 7 it was found that, for disorder consisting of extended impurities,  $\rho(E)$  is divergent ( $\sim 1/E \log^3 E$ ) at low-energies. This low-energy buildup of states arises from the hybridization of Andreev bound states occurring near each such extended impurity.

Before proceeding, we provide two specific experimental motivations for studying the effect of extended impurities on the electronic properties of d-wave superconductors: Firstly, recent technical advances have allowed the fabrication of ultra-pure YBCO which has almost no atomic disorder in the  $\text{CuO}_2$  planes<sup>2</sup>. A significant source of disorder may originate away from the  $\text{CuO}_2$  planes and can be expected to project a relatively long-wavelength potential on the  $\text{CuO}_2$  planes. Secondly, there is also the possibility of deliberately creating extended defects via ion irradiation techniques<sup>11</sup>. Apart from these specific possibilities, we note that although the calculation presented here strictly applies to the case of disorder satisfying  $a \gg \lambda_F$ , the physical effects may persist beyond this regime.

The purpose of this Paper is to further explore the consequences of extended impurities on the electronic properties of a d-wave superconductor by computing the associated contribution to the microwave conductivity.

To accomplish this, we invoke a semiclassical approximation<sup>6,7,12</sup> which accesses the low-energy Andreev-bound states occurring near each impurity. Although the individual Andreev bound state wavefunctions are localized near the impurities, they become delocalized at low energies due to tunneling corrections<sup>7</sup>, leading to a concomitant contribution to the conductivity. We find that a natural way to study such tunneling effects between Andreev bound states (and to compute their associated contribution to the conductivity) is via a real-space renormalization group (RSRG) technique which is a variant of the decimation technique familiar from random spin systems<sup>13,14,15,16,17</sup>. Thus, although the Andreev bound states near individual impurities are strongly coupled to each other, the RSRG reveals effective Andreev bound states associated with *many* impurities which are weakly coupled and which contribute strongly to the low-energy microwave response.

This paper is organized as follows. In Sec. II we employ a semiclassical approximation which reduces the Bogoliubov-de Gennes eigenproblem for a disordered d-wave superconductor to a family of one dimensional random pair potential models (represented by a Hamiltonian  $H_A$ ). In Sec. III we introduce the RSRG decimation procedure for computing average properties of  $H_A$ . In Sec. IV we apply this procedure to the computation of the microwave conductivity of a d-wave superconductor and discuss our results. In Sec. V, we make some brief concluding remarks. In the Appendix, we present a calculation of the average single-particle Green function of  $H_A$ .

## II. QUASIPARTICLES OF A DISORDERED D-WAVE SUPERCONDUCTOR

We assume that the quasiparticles of a d-wave superconductor with extended impurities are governed by the following Bogoliubov-de Gennes (BdG) action:

$$S = \int_0^\beta d\tau \int d^2r \Psi^\dagger \left[ \partial_\tau + \begin{pmatrix} \hat{h} & \hat{\Delta} \\ \hat{\Delta} & -\hat{h} \end{pmatrix} \right] \Psi, \quad (1)$$

where  $\hat{h} \equiv -\nabla^2 - \epsilon_F + V(\mathbf{r})$ ,  $\epsilon_F = p_F^2$  is the Fermi energy [i.e.  $p_F (\equiv 2\pi/\lambda_F)$  is the Fermi wavevector], and we have adopted units in which  $\hbar^2/2m = 1$ . The d-wave pair potential operator  $\hat{\Delta} = \Delta_0(p_x^2 - p_y^2)/p_F^2$  with  $\Delta_0$  being the pair-potential maximum. We take the disorder potential  $V(\mathbf{r})$  to arise from randomly located impurities of typical size  $a$  and typical spacing  $\ell$ , and focus on the regime  $\ell \gg a$ .

To study the Andreev bound states occurring near each extended impurity, we next invoke a semiclassical approximation (valid for  $p_F a \gg 1$ ) which has been extensively discussed elsewhere<sup>6,7,12</sup> and which exchanges the BdG Hamiltonian for an effective problem residing on a classical trajectory  $\mathbf{r}_c(s) = (x_c(s), y_c(s))$  solving Newton's equation [i.e.  $2p_F^2 \partial_s^2 \mathbf{r}_c(s) = -\nabla V(\mathbf{r}_c(s))$ ]. This is implemented by writing for a particular incoming momentum direction  $\hat{n} (= \dot{\mathbf{r}}_c(s) \text{ for } s \rightarrow -\infty)$  the field  $\Psi$  as

$$\Psi = \mathcal{A} e^{ip_F S} \psi, \quad (2)$$

and expanding to leading order in derivatives. Here, the eikonal  $S$  satisfies  $\nabla S = \dot{\mathbf{r}}_c(s)$  with the overdot denoting differentiation with respect to the parameter  $s$  along a trajectory. The amplitude  $\mathcal{A} \approx 1$ . A given realization of  $V(\mathbf{r})$  leads to a set of classical trajectories labelled by  $\hat{n}$  and an impact parameter  $b$ .

By inserting Eq. (2) into Eq. (1), we see that the dynamics of the trajectory-dependent quasiparticles (represented by the field  $\psi$ ) is governed by the Andreev<sup>18</sup> Hamiltonian  $H_A$

$$H_A = \begin{pmatrix} -2ip_F \partial_s & \Delta(s) \\ \Delta(s) & 2ip_F \partial_s \end{pmatrix}, \quad (3)$$

which has the form of a one-dimensional superconductor with an effective pair potential  $\Delta(s) = \Delta_0(\dot{x}_c(s)^2 - \dot{y}_c(s)^2)$ . For a given classical trajectory,  $\Delta(s)$  exhibits rapid variations that we wish to treat via an averaging procedure. To properly compute the disorder-averaged correlators of  $\Delta(s)$  one must solve Newton's equation for  $\dot{\mathbf{r}}_c(s)$  in the presence of a given realization of  $V(\mathbf{r})$  and then average over all such realizations. We shall not attempt such a difficult calculation; however, on general grounds we expect that  $\Delta(s)$  has zero mean and short-range correlations beyond  $\ell$ . We shall analyze  $H_A$  by studying the Andreev bound states associated with sign changes in  $\Delta(s)$  using the RSRG. Before discussing how this works, let us briefly review previous results for this one-dimensional random pair-potential Hamiltonian (we note that  $H_A$  has appeared in several other condensed-matter contexts<sup>19</sup>). The disorder-averaged low-energy density of states  $\rho(E) \sim 1/E \ln^3 E$  of  $H_A$  was first obtained by Ovchinnikov and Érikhman<sup>20</sup> using a Fokker-Planck equation. In Ref. 7, this same result was obtained by mapping  $H_A$  to a one-dimensional random hopping model. More recently, the disorder-averaged single-particle Green function  $\bar{G}(x-y; E)$  of  $H_A$  was obtained by Balents and Fisher<sup>21</sup> using a supersymmetry method. They found that  $\bar{G}(x-y; E) \sim \exp[-|x-y|/\log^2 E]$  for

$|x-y| \rightarrow \infty$ . In the Appendix, we reproduce these results using the RSRG technique; here we note that, in the present context, this behavior signifies that the Andreev bound state wavefunctions are exponentially localized but with a characteristic length scale that *diverges* at low energies (i.e. they become delocalized). Finally, as discussed in Refs 6,7,  $H_A$  is closely related to models of supersymmetric quantum mechanics<sup>22,23</sup>.

### III. REAL-SPACE RENORMALIZATION GROUP APPROACH

In this section we discuss how the disorder-averaged low-energy properties of  $H_A$  may be analyzed via a RSRG approach. Of central importance in such an analysis are the sign-changes (i.e., zeroes) of  $\Delta(s)$ <sup>24</sup>. Henceforth, our qualitative picture of  $\Delta(s)$  along a trajectory is depicted in Fig. 1, i.e., it exhibits many sign changes. Before considering the many sign-change case, however, let us consider an infinite system in which  $\Delta$  has exactly one sign change. In this case,  $H_A$  has an exact zero-energy (ZE) eigenstate of the form<sup>6,22,23</sup>

$$\Psi_n(s) \propto \begin{pmatrix} 1 \\ \pm i \end{pmatrix} e^{\pm \frac{1}{2p_F} \int_{s_n}^s \Delta(s) ds}, \quad (4)$$

where the  $+$ ( $-$ ) corresponds to the case  $\Delta(s) < 0$  ( $\Delta(s) > 0$ ) for  $s > s_n$  and we have omitted an overall normalization factor. Indeed, the normalizability of Eq. (4) relies on the existence of the sign change in  $\Delta(s)$ . The subscript  $n$  enters when we consider the case in which the pair potential has multiple sign changes; henceforth it shall refer to the  $n$ th zero  $s_n$  of  $\Delta(s)$  and refer to  $\Psi_n(s)$  as the ZE state associated with  $s_n$ . In the case in which  $\Delta(s)$  has multiple sign changes, however, we must account for tunneling corrections  $t_n = \langle \Psi_n | H_A | \Psi_{n+1} \rangle$  between nearest-neighbor ZE states (i.e. they are no longer at zero energy). A direct calculation gives<sup>6</sup>

$$t_n = \sqrt{\frac{2p_F}{\pi}} |\Delta'(s_n) \Delta'(s_{n+1})|^{1/4} \times \exp\left(-\frac{1}{2p_F} \left| \int_{s_n}^{s_{n+1}} \Delta(s) ds \right|\right), \quad (5)$$

where the prime indicates differentiation with respect to  $s$ .

Having identified the ZE states  $\Psi_n$  associated with the zeroes of  $\Delta(s)$ , we now turn to the RSRG analysis of such states. Although it is formally similar to that studied in Refs. 13,14,15,16,17, there are conceptual differences which are important. The essential result of such an analysis is the following: Although the matrix elements  $t_n$  coupling the  $\Psi_n$  may be large, there are effective ZE states associated with *many* sign changes which are weakly coupled and which may be treated via perturbation theory. Furthermore, the couplings between such effective ZE states have a probability distribution which

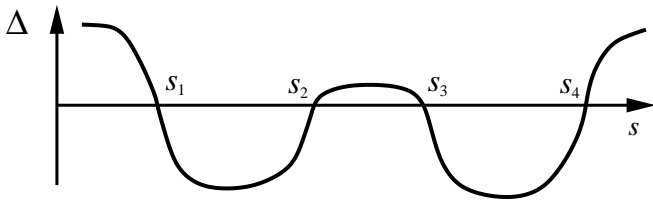


FIG. 1: Sketch of the effective pair potential along a trajectory. The pair-potential has been deliberately drawn smaller between sites  $s_2$  and  $s_3$  to illustrate a point; see the text for details.

is universal in a sense discussed below. To begin, we note that for a given realization of the pair potential, there exists a strongest matrix element  $t_{\max}$ . The ZE states associated with this matrix element are strongly coupled, forming symmetric and antisymmetric wavefunctions at energies  $\pm t_{\max}$ . The difference between these energies defines our initial bandwidth  $\Omega_0 = 2t_{\max}$ . To estimate the magnitude of  $\Omega_0$ , we first assume the impurities are dense enough such that the exponential factor in Eq. (5) is negligible. Then, we assume that  $\Delta'(s_n) \approx \Delta_0/a$  near an extended impurity of size  $a$ . Using  $\Delta_0 \sim 50\text{meV}$ , we have  $\Omega_0 \sim 10^{14}\text{s}^{-1}/\sqrt{\rho_F a}$ .

Generally, we are interested in an observable at some low-energy  $\omega \ll \Omega_0$ ; the RSRG progressively eliminates all pairs of ZE states coupled by matrix elements larger than  $\omega$ . To see what this means, let us examine Fig. 1. According to Eq. (5), the matrix elements  $t_n$  are exponentially dependent on the pair-potential integral  $|\int_{s_n}^{s_{n+1}} \Delta(s) ds|$  between pairs of sites. Thus, in Fig. 1, the strongest matrix element is that connecting sites  $s_2$  and  $s_3$ , (i.e.,  $t_2$ ) since the aforementioned integral is smallest for them. To obtain the decimation procedure, we recall that the sites  $s_n$  are *defined* by the sign changes of  $\Delta(s)$  along a particular trajectory. The fact that  $\Delta$  is only slightly above zero for the segment between  $s_2$  and  $s_3$  suggests that we could have simply *ignored* the sign changes at  $s_2$  and  $s_3$  without making any appreciable error in computing physical quantities. This defines a new description of  $H_A$ : a basis of ZE states that are associated with all sign changes of  $\Delta(s)$  *except* those at sites  $s_2$  and  $s_3$ . The effect of these zeroes of  $\Delta(s)$  is incorporated into a weak matrix element between the *effective* ZE states at sites  $s_1$  and  $s_4$  (that are each associated with three sign changes of  $\Delta(s)$ ). This procedure exchanges four ZE states for two weakly-coupled ZE states (and lowers the bandwidth to  $\Omega < \Omega_0$ ). The associated matrix element  $\tilde{t}$  between the effective ZE states at sites  $s_1$  and  $s_4$  is still given by the general formula Eq. (5); it is easy to see that this formula has the following recursion property:

$$\tilde{t} = t_1 t_3 / t_2. \quad (6)$$

Remarkably, this recursion relation is the same as that obtained by Fisher<sup>14</sup> for the random antiferromagnetic

spin chain; the existence of such a relation is the reason the RSRG works in this seemingly unrelated system. It is convenient to define (as was done in the random spin chain case<sup>14</sup>) logarithmic matrix-element strengths  $\zeta_n \equiv \log(\Omega/2) - \log t_n$  and arc lengths  $l_n \equiv s_{n+1} - s_n$ . It is useful to note that up to a constant (defined so that the minimum  $\zeta_n$  is zero),  $\zeta_n$  is essentially the abovementioned pair-potential integral  $|\int_{s_n}^{s_{n+1}} \Delta(s) ds|$  that motivated the decimation scheme. A particular (trajectory-dependent) form for  $H_A$  defines an initial probability distribution  $P(\zeta, l; \Omega_0)$  for the  $\{\zeta_n\}$  and  $\{l_n\}$ , in which ZE states have been assigned to all sign changes of  $\Delta$ . How do these values evolve under the RSRG decimation? Once sites  $s_2$  and  $s_3$  have been decimated, the new effective arc length is clearly  $\tilde{l} = s_4 - s_1$ . This, along with Eq. (6), implies that

$$\tilde{\zeta} = \zeta_1 + \zeta_3 - \zeta_2 = \zeta_1 + \zeta_3, \quad (7a)$$

$$\tilde{l} = l_1 + l_2 + l_3, \quad (7b)$$

where in Eq. (7a) we have used the fact that (as noted in Ref. 14) since the matrix element connecting  $s_2$  and  $s_3$  is assumed to be the strongest,  $\zeta_2 \equiv 0$  by definition. By iterating this procedure, we produce a basis set of ZE states which are more weakly coupled each iteration (and which are associated with progressively more sign changes of  $\Delta(s)$ ) but which still contain all the low energy physics of  $H_A$ . As mentioned above, although their physical origin is slightly different, the recursion rules for the  $\{\zeta_n\}$  and the  $\{l_n\}$  are the same as those studied by Fisher for random spin chains. The essential results of this work (for our purposes) are the number of sites  $n_\Omega = 1/\ln^2 \Omega_0/\Omega$  remaining after having lowered the bandwidth (by decimating) to energy  $\Omega$  and the asymptotic ( $\Omega \rightarrow 0$ ) probability distribution  $P(\zeta, l; \Omega)$ , which is given by<sup>14</sup>

$$P(\zeta, l; \Omega) = \frac{1}{\Gamma^3} Q(\zeta/\Gamma, l/\Gamma^2), \quad (8a)$$

$$\hat{Q}(\eta, \hat{y}) = \frac{\sqrt{c\hat{y}}}{\sinh \sqrt{c\hat{y}}} e^{-\eta \sqrt{c\hat{y}} \coth \sqrt{c\hat{y}}}. \quad (8b)$$

Here,  $\hat{Q}(\eta, \hat{y})$  is the Laplace transform of  $Q(\eta, y)$  and  $\Gamma \equiv \ln \Omega_0/\Omega$ . The integration constant  $c$  defines the length scale over which  $l$  varies. We assume this is given by the typical value of  $l$  in the initial distribution, i.e.,  $c \approx \ell$ .

#### IV. MICROWAVE CONDUCTIVITY

Having outlined the RSRG technique, we now apply it to the calculation of a specific quantity: The microwave conductivity of a d-wave superconductor disordered by extended impurities, averaged over all realizations of the disorder. Before embarking on the calculation, let us discuss the physical picture. We are interested in the possibility of the Andreev bound state quasiparticles contributing to the microwave conductivity. From the outset

it is not obvious that they will do so, since they are nominally localized [see Eq. (4)] near a particular sign change. The results of Balents and Fisher for the Green function  $\bar{G}(x-y; E)$  of  $H_A$  give hope to this possibility, revealing that the quasiparticles become delocalized at asymptotically low energies. As the conductivity is related to a two-particle Green function (see below), the results of Ref. 21 are not sufficient to compute the conductivity. However, as we shall see the peculiar way in which tunneling corrections conspire to form *effective* Andreev bound states associated with many sign changes conspires to give an interesting contribution to the microwave conductivity.

For a given realization of the disorder, the real part of the microwave conductivity  $\sigma_1$  is given by the Kubo formula<sup>25</sup>:

$$\sigma_1(\omega) = -\frac{1}{\omega} \text{Im} \Pi^{\text{R}}(\omega), \quad (9a)$$

$$\Pi(i\Omega) \equiv -\frac{1}{A} \int_0^\beta d\tau e^{i\Omega\tau} \langle \mathbf{j}(\mathbf{q}, \tau) \cdot \mathbf{j}(\mathbf{q}, 0) \rangle |_{\mathbf{q} \rightarrow 0}, \quad (9b)$$

Here,  $A$  is the area of the system and to obtain the retarded polarization tensor we use the usual analytic continuation  $\Pi^{\text{R}}(\omega) = \Pi(i\Omega \rightarrow \omega + i0^+)$ . The angle brackets represent the thermodynamic average with respect to the action  $\mathcal{S}$ . We shall denote the disorder average by square brackets; however, before attempting to evaluate this average we first simplify Eq. (9b) for  $\Pi(i\Omega)$ .

Within the semiclassical approximation of Sec. II,  $\mathbf{j}(\mathbf{q}, \tau)$  is obtained by inserting Eq. (2) into the usual current operator  $\mathbf{j}(\mathbf{r}, \tau) = ie(\Psi^\dagger \nabla \Psi - h.c.)$ . The Fourier transform  $\mathbf{j}(\mathbf{q}, \tau)$  is (in the  $\mathbf{q} \rightarrow \mathbf{0}$  limit)

$$\mathbf{j}(\mathbf{0}, \tau) = -ep_{\text{F}}^2 \sum_{\hat{n}} \int db ds \dot{\mathbf{r}}_c(s) \psi^\dagger(s, \tau) \psi(s, \tau). \quad (10)$$

Standard manipulations then yield the following expression for  $\sigma_1(\omega)$ :

$$\sigma_1(\omega) = \frac{e^2 k_{\text{F}}^3}{\omega A} \sum_{\hat{n}} \int db ds ds' \dot{\mathbf{r}}_c(s) \cdot \dot{\mathbf{r}}_c(s') \text{Im} K^{\text{R}}(s, s'; \omega), \quad (11a)$$

$$K(s, s'; \tau) \equiv \langle \psi^\dagger(s, \tau) \psi(s, \tau) \psi^\dagger(s', 0) \psi(s', 0) \rangle, \quad (11b)$$

Next, we use Wick's theorem ( $\mathcal{S}$  is a Gaussian action) to express the Matsubara Fourier transform  $K(s, s'; i\Omega)$  of Eq. (11b) as a sum of two terms, one of which gives a vanishing contribution to  $\Pi$ . The other contribution to  $K(s, s'; i\Omega)$  may conveniently be expressed in terms of the normalized eigenstates  $\{\psi_i(s)\}$  (with eigenvalues  $\{E_i\}$ ) of  $H_A$ :

$$\text{Im} K^{\text{R}}(s, s'; \omega) = \pi \sum_{i,j} \psi_i^\dagger(s) \psi_j(s) \psi_j^\dagger(s') \psi_i(s') \times (n_{\text{F}}(E_i) - n_{\text{F}}(E_j)) \delta(\omega + E_i - E_j), \quad (12)$$

where, since we are interested in  $\text{Im} \Pi^{\text{R}}(\omega)$ , we have displayed  $\text{Im} K^{\text{R}}(s, s'; \omega)$  directly.

Turning to Eq. (11a), we see that for the conductivity we require the disorder average of  $\text{Im} K^{\text{R}}(s, s'; \omega)$  multiplied by  $\dot{\mathbf{r}}_c(s) \cdot \dot{\mathbf{r}}_c(s')$ . At low energies, the ZE states hybridize and become delocalized along the classical trajectories. By contrast, the average of  $\dot{\mathbf{r}}_c(s) \cdot \dot{\mathbf{r}}_c(s')$  is expected to vanish on long length scales as the classical trajectories bounce off the extended impurities. Thus, it is sensible to approximate the average of the product of these quantities by the product of the averages; furthermore we shall approximate for the classical problem  $[\dot{\mathbf{r}}_c(s) \cdot \dot{\mathbf{r}}_c(s')]_{\text{dis.}} \simeq \exp(-|s-s'|/\ell)$  i.e., exponentially decaying correlations. We believe that the precise form of this correlator is unimportant at low energies. However, we note that the virtue of this form is that it naturally interpolates between ballistic motion on short length scales (i.e.  $[(\mathbf{r}_c(s) - \mathbf{r}_c(0))^2]_{\text{dis.}} = s^2$  for  $s \rightarrow 0$ ) and diffusive motion on long length scales (i.e.  $[(\mathbf{r}_c(s) - \mathbf{r}_c(0))^2]_{\text{dis.}} = 2s/\ell$  for  $s \rightarrow \infty$ ).

Next, we turn to the computation of the disorder average of  $\text{Im} K^{\text{R}}(s, s'; \omega)$ , for which we require the RSRG analysis. The purpose of the RSRG is to eliminate (decimating until  $\Omega = \omega$ ) all the strong matrix elements so that we are left with an effective theory having only weak (i.e.  $t_n \lesssim \omega$ ) matrix elements between ZE states, allowing the use of perturbation theory. Following Fisher<sup>14</sup>, we assume that the broadness of  $P(\zeta, l; \Omega)$  indicates that the disorder average is dominated by a class of rare pair-potential configurations that provide a large contribution. For the disorder average of Eq. (12), the class we have in mind is when an undecimated pair of states is separated by  $l = |s-s'|$  *exactly*; the relative number of such configurations with logarithmic matrix element strength  $\zeta$  is simply given by  $n_\omega P(\zeta, |s-s'|; \omega)$ . For a given  $\zeta$  and to leading-order in perturbation theory (a similar calculation was done in Ref. 17), the ZE states at  $s$  and  $s'$  form symmetric ( $S$ ) and antisymmetric ( $A$ ) wavefunctions with energies  $\pm t = \pm \frac{\omega}{2} e^{-\zeta}$  (recall we have set  $\Omega = \omega$ ). At low energies, the only terms which contribute appreciably to the sum in Eq. (12) are these states (i.e.  $i, j = S, A$ ). Thus, we have (upon summing over all such configurations which amounts to integrating over  $\zeta$ )

$$[\text{Im} K^{\text{R}}(s, s'; \omega)]_{\text{dis.}} \propto n_\omega \int_0^\infty d\zeta P(\zeta, |s-s'|; \omega) \quad (13a)$$

$$\times \delta(\omega - \omega e^{-\zeta}) \tanh(\omega/4T), \\ \propto P(0, |s-s'|; \omega) \frac{\tanh(\omega/4T)}{\omega \log^2 \Omega_0/\omega} \quad (13b)$$

where we have omitted overall temperature- and frequency-independent prefactors, since the RSRG is not expected to capture these correctly. In Eq. (13a), we see that the delta function constraint restricts attention to  $\zeta = 0$ , i.e., to those pairs of states that are about to be decimated at energy  $\omega$ , leading to Eq. (13b).

Having obtained an expression for  $[\text{Im} K^{\text{R}}(s, s'; \omega)]_{\text{dis.}}$ , we now insert it into Eq. (11a). We can immediately evaluate one of the integrations over the parameters  $s$  and

$s'$  giving the Laplace transform of  $P(0, |s - s'|; \omega)$ . Up to a constant of order unity, the remaining integrations  $\int db ds \propto A$ , so that we have

$$[\sigma_1(\omega)]_{\text{dis.}} \simeq \sigma_0 \frac{\Omega_0 \tanh(\omega/4T)}{\omega \log^2 \Omega_0/\omega}, \quad (14)$$

where  $\sigma_0$  is a dimensionful prefactor that is difficult to estimate (in part due to the uncertainty in the prefactor of  $n_\Omega$ ); a very rough estimate along the lines of our estimate of  $\Omega_0$  gives  $\sigma_0 \sim e^2 \epsilon_F \Delta_0 \ell / \Omega_0^2 a$ .

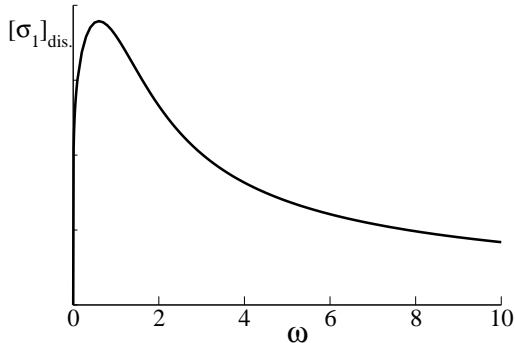


FIG. 2: Plot of  $[\sigma_1(\omega)]_{\text{dis.}}$  (in arbitrary units). Here,  $\omega$  is measured in units of  $10^9 \text{ s}^{-1}$  and we have chosen  $T = 10 \text{ mK}$  and  $\Omega_0 = 10^{13} \text{ s}^{-1}$ .

This formula (plotted in Fig. 2) applies at  $\omega \ll \Omega_0$  and includes *only* the contribution due to extended impurities. It exhibits a peak in  $\omega$  which becomes larger and narrower and moves to zero as  $T \rightarrow 0$ . Also,  $[\sigma_1(0)]_{\text{dis.}} = 0$ . This contribution would occur in parallel to those arising from pointlike impurities<sup>3,4</sup>, and would therefore be most noticeable at low temperatures where the peak is sharpest. For example, according to the calculations of Durst and Lee<sup>4</sup>, the microwave conductivity due to pointlike impurities should attain a universal value at low temperatures and frequencies, in striking contrast to the structure of Eq. (14) which would exhibit strong variations in this regime. At the typical temperature scales of the Turner et al experiments<sup>2</sup>, however, the extended impurity contribution would be particularly broad and small making a quantitative comparison difficult. To isolate the contribution considered here, we make two suggestions: (1) This contribution is largest at  $\omega \rightarrow 0$  with  $T \lesssim \omega$  so that the argument of the tanh in Eq. (14) is large. (2) By deliberately introducing extended impurities via ion irradiation<sup>11</sup> or by perhaps introducing disorder away from the  $\text{CuO}_2$  planes, this contribution will be enhanced.

## V. CONCLUDING REMARKS

In an effort to understand the electronic properties of disordered d-wave superconductors, we have computed

the contribution to the microwave conductivity arising from extended impurities. This contribution has its origin in the hybridization of ZE Andreev bound states occurring near each such extended impurity. To capture this hybridization, we have applied a real space renormalization group technique that generalizes the work of Refs. 13,14,15,16,17 on lattice models to the continuum Hamiltonian  $H_A$ . From a theoretical point of view, this generalization is possible because the underlying supersymmetry of  $H_A$ <sup>6,22,23</sup> implies that there is an approximate zero-energy state near *each* zero of  $\Delta(s)$  for every realization of the random pair potential, providing a natural basis for studying the low-energy properties of  $H_A$  via such tunneling corrections. We found that the singular density of states associated with these Andreev bound states found in Ref. 7 leads to a singular contribution to the microwave conductivity and discussed the possibility of observing this experimentally. Finally, we note that whereas the low-energy quasiparticles studied here rely on the d-wave symmetry of the pair potential, they are not associated with the nodes of the pair potential.

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## APPENDIX A: SINGLE-PARTICLE GREEN FUNCTION

In the present section, we apply the RSRG technique to the calculation of the Green function of  $H_A$  averaged over all realizations of the random pair potential. These results apply more generally to any system in which  $H_A$  arises<sup>19,20,21</sup>. This calculation is similar to the one presented in the main body of the text and obtains a result which has already been obtained by Balents and Fisher<sup>21</sup>. Nevertheless we present it here for the sake of clarity and completeness, as well as to highlight the strengths and weaknesses of the RSRG method. Indeed, the fact that we are able to reproduce (up to numerical prefactors) the exact results of Ref. 21 lends support for this technique. The disorder-averaged Green function  $\bar{G}(x-y; i\omega)$  is given by

$$\bar{G}(x-y; i\omega) = [G(x, y; i\omega)]_{\text{dis.}}, \quad (\text{A1})$$

$$(i\omega - H_A)G(x, y; i\omega) = \delta(x-y), \quad (\text{A2})$$

where the square brackets in Eq. (A1) denote the average over all realizations of the pair potential. It is useful to express  $G(x, y; i\omega)$  in terms of the normalized eigenstates  $\psi_n$  of  $H_A$ :

$$G(x, y; i\omega) = \sum_n \frac{\psi_n(x)\psi_n(y)}{i\omega - E_n}. \quad (\text{A3})$$

Before proceeding with the RSRG calculation, we emphasize that it does not correctly obtain overall prefactors associated with the contributions to  $\bar{G}(x-y; i\omega)$ .

Nonetheless, it obtains in a simple way the correct (non-trivial) asymptotic behavior of  $\bar{G}(x-y; i\omega)$  as well as the correct matrix structure. In the following, these are the features we are primarily interested in obtaining.

The RSRG simplifies the computation of  $\bar{G}(x-y; i\omega)$  in two distinct ways. Firstly, by decimating until  $\Omega = \omega$ , we eliminate all strongly coupled zero-energy (ZE) states. Since the remaining ZE states are weakly coupled, it is plausible to approximate the exact low-energy eigenstates  $\psi_n(x)$  by perturbed ZE states. Secondly, due to the fact that  $P(\zeta, l; \omega)$  is very broad<sup>14</sup>, certain rare pair-potential configurations dominate the average over all random pair-potential configurations. Here, we focus on two leading contributions; analogous contributions were considered in Ref. 16 in the context of the related problem of computing the single particle Green function for a one-dimensional random hopping problem. However, we obtain slightly different results for the second contribution. The first we denote by  $\bar{G}_1(x-y; i\omega)$ . This contribution is due to pair potential configurations such that the coordinates  $x$  and  $y$  are near sign changes that are about to be decimated (owing to the fact that the pair potential connecting them is relatively small) in the sense discussed in Sec. III. The second contribution,  $\bar{G}_2(x-y; i\omega)$ , is due to pair potential configurations having only one of  $x$  or  $y$  be about to be decimated along with a ZE state at a third site. Let us first, however, discuss  $\bar{G}_1(x-y; i\omega)$ .

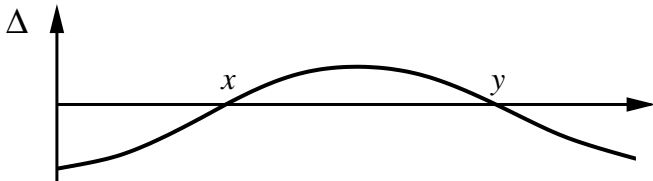


FIG. 3: Class of pair potential configurations providing the contribution to  $\bar{G}_1(x-y; i\omega)$ . The smallness of  $\Delta$  between  $x$  and  $y$  is meant to indicate that the ZE states associated with these sites are about to be decimated.

The rare pair-potential configuration that leads to  $\bar{G}_1(x-y; i\omega)$  is the same as was considered in Sec. IV for  $[\text{Im}K^R(s; \omega)]_{\text{dis.}}$  and is depicted schematically in Fig. 3; the fact that both  $x$  and  $y$  are about to be decimated at energy  $\omega$  is depicted pictorially by having  $\Delta$  be relatively low between  $x$  and  $y$ . Of course, it is not necessary that the sites  $x$  and  $y$  be nearest-neighbor sign changes as depicted in Fig. 3, but merely that they are nearest neighbors after having decimated until  $\omega$ . The relative number of such configurations is given by  $n_\omega P(0, |x-y|; \omega)$ ; to obtain  $\bar{G}_1(x-y; i\omega)$  we simply multiply this by the magnitude of the contribution given by Eq. (A3). For concreteness, let us assume that  $\Delta > 0$  on average (as in Fig. 3) between  $x$  and  $y$  ( $y > x$ ); once we have computed this it is straightforward to obtain the  $\Delta < 0$  configurations. The ZE states associated with these points are given by Eq. (4) with the  $-$  sign for the ZE state  $\Psi_x$  at

$x$  and the  $+$  sign for the ZE state  $\Psi_y$  at  $y$ :

$$\Psi_x \propto \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\phi_x} \quad (\text{A4})$$

$$\Psi_y \propto \begin{pmatrix} 1 \\ i \end{pmatrix} e^{i\phi_y}, \quad (\text{A5})$$

where we have suppressed unimportant normalization factors and displayed phase factors in these expressions which were suppressed in Eq. (4). These factors are chosen to satisfy  $\exp(i(\phi_x - \phi_y)) = -i$  so that  $t_n$  in Eq. (5) is real and positive; one must account for these properly to obtain the correct phase of  $\bar{G}_1(x-y; i\omega)$ . Under the influence of the tunneling matrix element  $t$ , (which is small by virtue of the decimation procedure), these states form symmetric ( $S$ ) and antisymmetric ( $A$ ) combinations  $\psi_{S,A} = \frac{1}{\sqrt{2}}(\Psi_x \pm \Psi_y)$  with energies  $\pm t (= \pm\omega/2$  since they are about to be decimated). As in Sec. IV we take these approximate eigenstates to be the lowest energy states appearing in the sum Eq. (A3) (i.e.,  $n = S, A$ ). Thus, we find (including the contribution due to configurations with  $\Delta < 0$  between  $x$  and  $y$ )

$$\begin{aligned} \bar{G}_1(x-y; i\omega) &\propto \hat{\sigma}_3 \frac{i}{\omega} \frac{1}{\Gamma^5} \\ &\times \sum_{n=1}^{\infty} (-1)^{n+1} n^2 e^{-n^2 \pi^2 |x-y|/\Gamma^2}, \quad (\text{A6}) \end{aligned}$$

where  $\Gamma \equiv \log \Omega_0/\omega$  and by “ $\propto$ ” we specifically mean equality up to a real and positive prefactor not captured within the RSRG approach. We emphasize that the  $\sigma_3$  structure of this contribution comes from correctly identifying the spinor structure of the ZE states in Eq. (A4) and Eq. (A5). The infinite sum arises from the inverse Laplace transform required to compute  $P(0, |x-y|; \omega)$ .

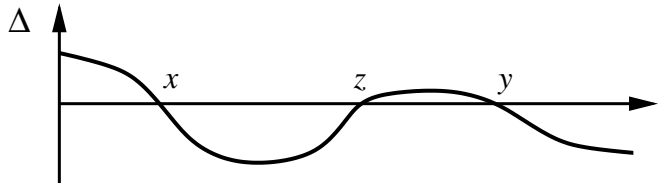


FIG. 4: Second-order class of pair potential configurations contributing to  $\bar{G}_2(x-y; i\omega)$ . The ZE states at sites  $z$  and  $y$  are about to be decimated due to the fact that they are strongly coupled (indicated by the relative smallness of  $\Delta$  between them) but the states at sites  $x$  and  $z$  may still be weakly coupled.

Next, we turn to the evaluation of  $\bar{G}_2(x-y; i\omega)$ . The type of pair-potential configuration we have in mind is depicted in Fig. (4). Again, these states do not have to be nearest neighbor sign changes but merely nearest neighbors after having decimated until  $\omega$ . The ZE states at  $x$  and  $z$  are weakly coupled (by a matrix element  $\delta$ ), owing to the relative largeness of  $\Delta$  between them. The

ZE states at  $z$  and  $y$  are strongly coupled with matrix element  $t = \omega/2$  and are thus about to be decimated. Let us consider the Hamiltonian  $H$  for our system in the subspace of these three ZE states:

$$H = \begin{pmatrix} 0 & \delta & 0 \\ \delta & 0 & t \\ 0 & t & 0 \end{pmatrix}. \quad (\text{A7})$$

The eigenvalues of this matrix are 0 and  $\pm\sqrt{t^2 + \delta^2}$ ; as we are interested in low energies, for the purposes of the sum in Eq. (A3) we proceed by keeping only the state  $\psi_0$  having 0 eigenvalue in Eq. (A3), which has the explicit form

$$\psi_0 = \frac{1}{\sqrt{\delta^2 + t^2}} \begin{pmatrix} t \\ 0 \\ -\delta \end{pmatrix}. \quad (\text{A8})$$

Of course, the elements of the column vector in Eq. (A8)

denote the amplitude of each ZE state contribution associated with each site. Since the ZE state spinors at  $x$  and  $y$  are each of the form of Eq. (A5), the term appearing in the sum has the form (switching to a slightly different notation and neglecting normalization factors as usual)

$$\frac{\psi_0(x)\psi_0(y)}{i\omega} = \frac{t\delta}{\sqrt{\delta^2 + t^2}} \frac{1}{i\omega} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad (\text{A9})$$

where we note that an extra  $-$  sign in Eq. (A9) came from the phase factors in Eq. (A4) and Eq. (A5). By including the possibility that  $\Delta < 0$  between  $z$  and  $y$ , it is straightforward to see that the sum of these contributions is proportional to the unit matrix. Since it is about to be decimated,  $t = \omega/2$ . The other matrix element  $\delta$  may be labeled by  $\zeta$  (as usual) via  $\delta = \frac{\omega}{2}e^{-\zeta}$ . Thus, we have [neglecting  $\delta \ll t$  in the denominator of Eq. (A9)]

$$\bar{G}_2(x-y; i\omega) \propto n_\omega \frac{\hat{\sigma}_0}{i\omega} \int_x^y dz \int_0^\infty d\zeta P(\zeta, x-z; \omega) P(0, z-y; \omega), \quad (\text{A10})$$

where  $\sigma_0$  is the unit matrix. Taking the Laplace transform of both sides of Eq. (A10) and evaluating the integral over  $\zeta$ , we find that the Laplace transform of  $\bar{G}_2(x-y; i\omega)$  is

$$\bar{G}_2(\hat{y}; i\omega) \propto \frac{\sigma_0}{i\omega} \frac{1}{\Gamma^2} \frac{\hat{y}}{\sinh \Gamma \sqrt{\hat{y}}} \frac{1}{\sinh \Gamma \sqrt{\hat{y}} + \sqrt{\hat{y}} \coth \Gamma \sqrt{\hat{y}}}, \quad (\text{A11})$$

with  $\Gamma \equiv \log \Omega_0/\omega$  as before. To obtain  $\bar{G}_2(x-y; i\omega)$ , we must evaluate the inverse Laplace transform of Eq. (A11). To do this, we use the standard technique of identifying the poles  $\hat{y}_n$  of Eq. (A11); these occur when  $\sinh \Gamma \sqrt{\hat{y}_n} = 0$ , i.e.,  $y_n = -n^2\pi^2/\Gamma^2$  (we must exclude the case  $n = 0$ ). Multiplying the associated residues by  $\exp(|x-y|\hat{y}_n)$  and summing over all  $n$  yields the result

$$\bar{G}_2(x-y; i\omega) \propto \frac{i\sigma_0}{\omega} \frac{1}{\Gamma^5} \sum_{n=1}^{\infty} n^2 e^{-n^2\pi^2|x-y|/\Gamma^2}, \quad (\text{A12})$$

Our final result for  $\bar{G}(x-y; i\omega) = \bar{G}_1(x-y; i\omega) + \bar{G}_2(x-y; i\omega)$  agrees with the exact results of Ref. (21), as can be seen by taking the  $M \rightarrow 0$  ( $M$  being proportional to  $[\Delta(s)]_{\text{dis}}$  in our notation) limit of Eq. (5.30) in that paper. In the present context, this form for  $\bar{G}(x-y; i\omega)$  indicates that the Andreev bound state Green function decays exponentially as a function of  $|x-y|$  but with a characteristic length scale  $\log^2 \Omega_0/\omega$  that diverges as  $\omega \rightarrow 0$ , indicating the delocalization of low-energy quasiparticles along a trajectory.

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