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Quantum Cluster Algebras at Roots of Unity, Poisson-Lie Groups, and Discriminants

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QUANTUM CLUSTER ALGEBRAS AT ROOTS OF UNITY, POISSON–LIE GROUPS, AND DISCRIMINANTS

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The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way.
– Godfrey Hardy

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Abstract

This dissertation studies quantum algebras at roots of unity in regards to cluster structure and Poisson structure. Moreover, quantum cluster algebras at roots of unity are rigorously defined. The discriminants of these algebras are described, in terms of frozen cluster variables for quantum cluster algebras and Poisson primes for specializations of quantum algebras. The discriminant is a useful invariant for representation theoretic and algebraic study, whose laborious computation deters direct evaluation. The discriminants of quantum Schubert cells at roots of unity will be computed from the two distinct approaches. These methods can be applied to many other quantum algebras.

Chapter 1
Introduction

This work studies quantum algebras at roots of unity from two perspectives, the cluster structure of quantum cluster algebras at roots of unity and the Poisson structure of quantum algebras specialized from an indeterminate to a root of unity. There is overlap and interplay between these points of view, but each is distinct and interesting in itself. We consider the discriminants of these algebras and give methods to determine them. Discriminants have seen much use recently as a tool in studying noncommutative algebras. For example they have been used to determine automorphism groups [7, 8], to resolve the Zariski cancellation problem for certain algebras [1], and to classify Azumaya loci [6].

For many quantum algebras that are specialized to a "good" root of unity, there is a canonical central subalgebra over which the quantum algebra is a free module. The discriminant of these quantum algebras at roots of unity is closely related to the canonical Poisson structure that the discriminant inherits. In particular the discriminant is a Poisson normal element. In the case that the quantum algebra is a UFD or a Poisson UFD, we can give a description of the composition of the discriminant.

Theorem *Let R be a $\mathbb{K}[q^{\pm 1}]$-algebra for a field $\mathbb{K}$ of characteristic 0 and $\epsilon \in \mathbb{K}^{\times}$. Assume that $R_\epsilon := R/(q-\epsilon)R$ is a free module of finite rank over a Poisson subalgebra C_ϵ of its center, and that C_ϵ is a unique factorization domain as a commutative algebra or a noetherian Poisson unique factorization domain. Then, $d(R_\epsilon/C_\epsilon) = 0$ or*

$$d(R_\epsilon/C_\epsilon) =_{C_\epsilon^\times} \prod_{i=1}^{m} p_i$$

for some (not necessarily distinct) Poisson prime elements $p_1, \ldots, p_m \in C_\epsilon$.

Quantum cluster algebras at roots of unity will be given a rigorous definition in chapter 4. Properties fundamental to cluster theory are proven, such as the (quantum) Laurent phenomenon. Moreover, it is shown that classical, commutative cluster algebras embed into these quantum cluster algebras when one avoids certain roots of unity.

Theorem *Suppose the quantum seed* $(M_\epsilon, \widetilde{B}, \Lambda)$ *and the primitive* ℓ^{th} *root of unity* ϵ *satisfy a certain condition* $\boldsymbol{C}$*. Then the* $\mathbb{Z}$*-subalgebra*

$$\mathbb{Z}\langle\, M'_\epsilon(e_i)^\ell, M'_\epsilon(e_j)^{-\ell} \mid (M'_\epsilon, \Lambda', \widetilde{B}') \sim (M_\epsilon, \Lambda, \widetilde{B}),\ i \in [1, N],\ j \in \boldsymbol{inv}\,\rangle$$

of $\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}, \boldsymbol{inv})$ *is isomorphic to* $\mathcal{A}(\widetilde{B}, \boldsymbol{inv})$*.*

Initial steps to study the representation theory of these algebras are taken by describing their discriminant. The discriminant is actually given for a large class of subalgebras of the quantum cluster algebra. In particular, it is expressed in terms of noninverted frozen cluster variables.

Theorem *Suppose the quantum seed* $(M_\epsilon, \widetilde{B}, \Lambda)$ *of rank* N *and the primitive* ℓ^{th} *root of unity* ϵ *satisfy a certain condition* $\boldsymbol{C}$*. Suppose that the collection of seeds* Θ *is a nerve and that* $\mathcal{A}_\epsilon(\Theta)$ *is free and finite rank over* $C_\epsilon(\Theta)$*. Then the discriminant of* $\mathcal{A}_\epsilon(\Theta)$ *over* $C_\epsilon(\Theta)$ *is given as a product of noninverted frozen variables raised to the* ℓ^{th} *power,*

$$d\left(\mathcal{A}_\epsilon(\Theta)/C_\epsilon(\Theta)\right) =_{C_\epsilon(\Theta)^\times} \ell^{(N\ell^N)} \prod_{i\in[1,N]\setminus \boldsymbol{ex}\sqcup\boldsymbol{inv}} \left(M_\epsilon(e_i)^\ell\right)^{a_i} \quad \textit{for some integers } a_i.$$

The Quantum Schubert cell algebra $\mathcal{U}^-[w]$ was introduced by Lusztig and De Concini, Kac, and Procesi for a simple Lie group $\mathfrak{g}$ and Weyl group element $w \in W$. This algebra is a subalgebra of the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ and is, itself, a deformation of $\mathcal{U}(\mathfrak{n}_- \cap w(\mathfrak{n}_+))$ for the nilradicals $\mathfrak{n}_\pm$ of a pair of opposite Borel algebras of $\mathfrak{g}$. After specialization to an ℓ^{th} root of unity ϵ, a canonical central subalgebra generated by the ℓ^{th} powers of Lusztig's root vectors appears that is isomorphic to the coordinate ring of the Schubert cell $B_+ \cdot wB_+$. The quantum Schubert cells are anti-isomorphic to the quantum unipotent cells defined by Geiß, Leclerc, and Schröer [24]. Recently, Goodearl and Yakimov have given an explicit (integral) quantum cluster algebra structure on these quantum unipotent cells [27].

In this dissertation, the two distinct methods that have been introduced will be illustrated by determining the discriminant of $\mathcal{U}_\epsilon^-[w]$ over $C_\epsilon^-[w] \simeq \mathbb{K}[B_+w \cdot B_+]$. For the method using the Poisson structure of $C_\epsilon^-[w]$, we will rely De Concini, Kac, and Procesi's work on specializations of $\mathcal{U}^-[w]$ and quantum groups and will also rely on results for Poisson-Lie groups and Poisson homogeneous spaces. Because of this, we will require that $\mathfrak{g}$ be a simple Lie algebra. For the method using the Cluster structure, the explicit quantum cluster algebra structure on quantum unipotent cells will be translated to a cluster structure on $\mathcal{U}^-[w]$. From there, we will describe $\mathcal{U}_\epsilon^-[w]$ as a quantum cluster algebra at root of unity. The following theorem for quantum Schubert cells will be proven for the finite dimensional case from a Poisson geometric viewpoint and in the general case by the cluster approach.

Theorem *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra, w a Weyl group element and $\ell > 2$ an odd integer which is coprime to d_{i_k} for each k. Assume that $\mathbb{K}$ is a field of characteristic 0 which contains a primitive ℓ^{th} root of unity ϵ. Then*

$$d(\mathcal{U}_\epsilon^-[w]/C_\epsilon^-[w]) =_{\mathbb{K}^\times} \Delta^L_{\rho,w\rho} =_{\mathbb{K}^\times} \prod_{i\in\mathcal{S}(w)} \Delta^L_{\varpi_i,w\varpi_i} =_{\mathbb{K}^\times} \prod_{k\notin \boldsymbol{ex}} D^{\ell L}_{\varpi_{i_k},w\varpi_{i_k}}$$

where $L := \ell^{N-1}(\ell-1)$, $\Delta_{\varpi_i,w\varpi_i}$ are certain generalized minors, and $D_{\varpi_{i_k},w\varpi_{i_k}}$ are certain quantum minors.

Chapter 2
Preliminaries on Poisson Geometry

Poisson algebraic and geometric structures have been studied since the nineteenth century by Poisson, Jacobi, and Lie. However, the modern study of these structures began in 1980s with fundamental work by Weinstein [37] and others. For an excellent survey on Poisson geometry and Poisson-Lie groups, see Weinstein's *Poisson Geometry* [38]. For an informative introduction to quantum groups from the viewpoint of Poisson geometry, see the texts *Lectures on Quantum Groups* by Etingof and Schiffmann [18] and also *A Guide to Quantum Groups* by Chari and Pressley [10].

2.1 Poisson Algebras

Definition 2.1.1. A commutative, associative algebra A over $\mathbb{K}$ is a *Poisson algebra* when it is equipped with a $\mathbb{K}$-bilinear Lie bracket $\{\ ,\ \} : A \otimes A \to A$ that satisfies the Leibniz identity. More explicitly, the bracket satisfies for any $e, f, g \in A$,

$$\{ef, g\} = e\{f, g\} + \{e, g\}f.$$

When it is clear in context which Poisson structure is being, we will denote their Poisson algebra by A rather than $(A, \{\ ,\ \})$. When there are two (or more) Poisson algebras, A and B, it may be necessary to denote the respective Poisson brackets by $\{\ ,\ \}_A$ and $\{\ ,\ \}_B$. A map $\phi : A \to B$ between Poisson algebras is *Poisson* if it preserves the Poisson bracket:

$$\phi(\{x, y\}_A) = \{\phi(x), \phi(y)\}_B.$$

For any element f in a Poisson algebra A, the Leibniz identity gives us a derivation on the algebra defined by

$$X_f(g) = \{f, g\} \text{ for } g \in A.$$

The map X_f is called the *Hamiltonian derivation of f on A*.

We will consider some examples. First, we examine the algebra of polynomial functions on the symplectic plane $\mathbb{R}^2$, $A = \mathbb{K}[x,p]$. Let $e, f, g \in A$. Define the bracket by

$$\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}.$$

It is clear that this bracket is bilinear over $\mathbb{K}$ and that it is alternating, $\{f,f\} = 0$ for all $f \in A$. That this bracket satisfies the Leibniz identity follows from the product rule,

$$\begin{aligned}
\{ef,g\} &= \frac{\partial ef}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial ef}{\partial p}\frac{\partial g}{\partial x} \\
&= \left(e\frac{\partial f}{\partial x} + \frac{\partial e}{\partial x}f\right)\frac{\partial g}{\partial p} - \left(e\frac{\partial f}{\partial p} + \frac{\partial e}{\partial p}f\right)\frac{\partial g}{\partial x} \\
&= e\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}\right) + \left(\frac{\partial e}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial e}{\partial p}\frac{\partial g}{\partial x}\right) f \\
&= e\{f,g\} + \{e,g\}f.
\end{aligned}$$

Using the Leibniz identity and bilinearity, we can verify the Jacobi identity,

$$\{e,\{f,g\}\} + \{f,\{g,e\}\} + \{g,\{e,f\}\} = 0.$$

Thus we have shown that the algebra equipped with the bracket is a Poisson algebra.

This simple example of the algebra of polynomial functions on the symplectic plane can be generalized to the algebra of smooth functions of a symplectic manifold. This will form an important class of Poisson algebras. A symplectic manifold M is a smooth manifold with a nondegenerate closed 2-form ω. A Poisson bracket can be defined on $\mathcal{C}^\infty(M)$ using ω. For $f, g \in \mathcal{C}^\infty(M)$, we defined the bracket by

$$\{f,g\} = \omega(V_f, V_g)$$

where V_f is the vector field determined by

$$df(U) = \omega(U, V_f) \text{ for all vector fields } U.$$

In the case of $M = \mathbb{R}^{2n}$ with coordinates (x_i, p_i) and $\omega = \sum_i dx_i \wedge dp_i$,

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} \right).$$

Another important example of a Poisson algebra, for our purposes, will be the center of a specialization of an algebra A over $\mathbb{K}[q^{\pm 1}]$. Suppose $\epsilon \in \mathbb{K}^\times$ and denote the quotient algebra $A/(q-\epsilon)A$ by A_ϵ. Let $\kappa_\epsilon : A \to A_\epsilon$ be the canonical quotient map. The center $Z(A_\epsilon)$ carries a canonical Poisson structure. Define the bracket for $a, b \in Z(A_\epsilon)$ by

$$\{a, b\} = \kappa_\epsilon \left(\frac{xy - yx}{q - \epsilon} \right) \tag{2.1}$$

where $\kappa_\epsilon(x) = a$ and $\kappa_\epsilon(y) = b$. Note that the right hand side is defined as long as either a or b is central, as $[x, y] \in (q-\epsilon)A$. To see the bracket is well defined, consider $x, x' \in \kappa^{-1}(a)$. We have that $x = x' + (q-\epsilon)w$ for some $w \in A$ and thus

$$\kappa_\epsilon \left(\frac{xy - yx}{q - \epsilon} \right) - \kappa_\epsilon \left(\frac{x'y - yx'}{q - \epsilon} \right) = \kappa_\epsilon(wy - yw) = 0,$$

remembering that $b = \kappa_\epsilon(y) \in Z(A_\epsilon)$. To see that $\{a, b\} \in Z(A_\epsilon)$, let $c \in A_\epsilon$ and $\kappa_\epsilon(z) = c$. We have that

$$\begin{aligned}
[\{a, b\}, c] &= \kappa_\epsilon \left(\left[\frac{[x, y]}{q - \epsilon}, z \right] \right) \\
&= \kappa_\epsilon \left(\frac{1}{q - \epsilon} [\, [x, y], z] \right) \\
&= \kappa_\epsilon \left(-\frac{1}{q - \epsilon} [\, [y, z], x] - \frac{1}{q - \epsilon} [\, [z, x], y] \right) \\
&= - \left[\kappa_\epsilon \left(\frac{[y, z]}{q - \epsilon} \right), a \right] - \left[\kappa_\epsilon \left(\frac{[z, x]}{q - \epsilon} \right), b \right] \\
&= 0
\end{aligned}$$

since $a, b \in Z(A_\epsilon)$. Thus $\{\ ,\ \} : Z(A_\epsilon) \otimes Z(A_\epsilon) \to Z(A_\epsilon)$ is a well defined map. It is then a Poisson bracket, since the commutator bracket is a Lie bracket that satisfies the Leibniz identity.

2.2 Poisson Manifolds and Poisson-Lie Groups

Definition 2.2.1. A *Poisson manifold* is a smooth manifold M that admits a Poisson structure on its algebra of smooth functions $\mathcal{C}^\infty(M)$.

An alternative definition would be given in terms of a Poisson bivector $\Pi \in \Gamma(M, \Lambda^2 TM)$, which is uniquely determined by

$$\{f, g\} = \Pi(df \wedge dg).$$

When Poisson structure is understood, we will just denote the Poisson manifold by M instead of (M, Π).

A smooth map between Poisson manifolds is called *Poisson* when the induced pullback map on algebras of smooth functions is a Poisson algebra map. Explicitly for $\phi : M \to N$, this means for $f, g \in \mathcal{C}^\infty(N)$ that

$$\{f \circ \phi, g \circ \phi\}_M = \{f, g\}_N \circ \phi.$$

In terms of the Poisson bivector, $\phi : (M, \Pi) \to (N, \pi)$ is Poisson if π is ϕ-related to Π, i.e.

$$T_x\phi(\Pi_x) = \pi_{\phi(x)} \text{ for all } x \in M$$

where $T_x\phi$ is the tangent map expanded to multi-tangent vectors.

When we have two Poisson manifolds, (M, Π) and (N, π), the direct product inherits a Poisson structure, $(M \times N, \Pi + \pi)$. Rephrased in terms of Poisson brackets, $\{\ ,\ \}_M$ and $\{\ ,\ \}_N$, the Poisson bracket between $f, g \in C^\infty(M \times N)$ on any point

$(x, y) \in M \times N$ is given by

$$\{f, g\}_{M\times N}(x, y) = \{f(\cdot, y), g(\cdot, y)\}_M(x) + \{f(x, \cdot), g(x, \cdot)\}_N(y).$$

This Poisson structure on $M \times N$ is the unique one such that the projections $\mathrm{pr}_1 : M \times N \to M$ and $\mathrm{pr}_2 : M \times N \to N$ are Poisson maps.

This notion of a Poisson manifold can be generalized to the notion of a *Poisson variety* Y. If Y is an affine algebraic variety such that its coordinate ring, $\mathbb{K}[Y]$, is a Poisson algebra, then Y is a Poisson variety. When $\mathbb{K} = \mathbb{C}$ and Y is smooth, the two notions align by extending the Poisson bracket from $\mathbb{C}[Y]$ to $\mathcal{C}^\infty(Y)$.

Any symplectic manifold is a Poisson manifold. Moreover, any Poisson manifold M can be decomposed into a disjoint union of immersed symplectic manifolds of various dimension. These immersed manifolds will be called the *symplectic leaves* of M.

A symplectic leaf in a Poisson manifold is a maximal submanifold such that the Poisson structure restricts to a symplectic structure on the submanifold. Another description is in terms of Hamiltonian paths. For a Poisson manifold M, a smooth path $\gamma : [0, 1] \to M$ is called Hamiltonian if there is some $f \in \mathcal{C}^\infty(M)$ such that $\gamma'(t) = X_f$ for $0 < t < 1$. The symplectic leaves of M are then equivalence classes of points under the relation of being connected by Hamiltonian curves.

When a manifold has a group structure that is smooth, it is a Lie group. In the case that a manifold is both a Poisson manifold and a Lie group, we would be interested when the structures are compatible.

Definition 2.2.2. A *Poisson-Lie Group* is a Lie group G with a Poisson structure Π such that multiplication $m : G \times G \to G$ is a map of Poisson manifolds. Analogously for a *Poisson algebraic group*, an algebraic group is equipped with a Poisson structure that is compatible with multiplication.

Explicitly, a Lie group G with a Poisson structure is a Poisson-Lie group if

$$\{f,g\}(xy) = \{f \circ \rho_y, g \circ \rho_y\}(x) + \{f \circ \lambda_x, g \circ \lambda_x\}(y)$$

for any $f, g \in \mathcal{C}^\infty(G)$ and $x, y \in G$, where $\lambda_x,\ \rho_y : G \to G$ are the smooth maps given, respectfully, by left multiplication by x and right multiplication by y. Note in particular that

$$\{f,g\}(e) = \{f \circ \rho_e, g \circ \rho_e\}(e) + \{f \circ \lambda_e, g \circ \lambda_e\}(e) = 2\{f,g\}(e),$$

and hence $\{f,g\}(e) = 0$ for any $f, g \in \mathcal{C}^\infty(G)$. Thus Π vanishes at e (i.e. $\Pi_e = 0$) for a Poisson-Lie group (G, Π).

The tangent space $\mathfrak{g} = T_eG$ at e for a Lie group G is a Lie algebra. So the tangent space $\mathfrak{g} = T_eG$ for a Poisson-Lie group G is as well, but the Poisson structure endows more structure on $\mathfrak{g}$ making it into a Lie bialgebra.

For Lie groups, the correspondence between Lie groups and Lie algebras is a crucial result. In particular, the category of simply connected Lie groups is equivalent to the category of finite dimensional Lie algebras via the functor $G \to Lie(G)$. In the Poisson-Lie case, Drinfeld proved that the category of simply connected Poisson-Lie groups is equivalent to the category of finite dimensional Lie bialgebras (via the functor $G \to Lie(G)$ again).

The action of a Poisson-Lie group (G, π) on a Poisson manifold (M, Π) is Poisson if the map

$$(G, \pi) \times (M, \Pi) \to (M, \Pi)$$

is Poisson. The Poisson manifold (M, Π) is then a *Poisson homogeneous space* of (G, π) if M is a homogeneous G-space. If Π vanishes at a point x in the Poisson homogeneous G-space M, then (M, Π) is a Poisson quotient of (G, π) via the map

$$(G, \pi) \to (M, \Pi), \quad g \mapsto g \cdot x. \tag{2.2}$$

In the case that Π vanishes at x, we have the map $(G, \pi) \to (G, \pi) \times (M, \Pi)$ given by $g \mapsto (g, x)$ is Poisson. Hence (2.2) is Poisson as it is the composition of this Poisson map and the group action, and it is surjective since M is a homogeneous G-space. Moreover, for (2.2) to be Poisson, it is necessary that Π vanishes at x, since π vanishes at e for a Poisson-Lie group (G, π).

2.3 Poisson Prime and Normal Elements

The concept of Poisson primes will be central to our work on discriminants. Let $(A, \{\cdot, \cdot\})$ be a Poisson algebra over a base field $\mathbb{K}$ of characteristic 0.

Definition 2.3.1. An element $a \in A$ is called *Poisson normal* if for every $x \in A$,

$$\{a, x\} = ay \quad \text{for some } y \in A.$$

In other words, a is Poisson if the principal ideal (a) is a Poisson ideal,

$$\{(a), x\} \subseteq (a) \text{ for all } x \in A.$$

When A is an integral domain as a commutative algebra, then a is Poisson normal if and only if there is some Poisson derivation ∂ such that

$$\{a, x\} = a\partial(x) \text{ for all } x \in A.$$

Definition 2.3.2. Assume that A is an integral domain as an algebra. An element $p \in A$ is called *Poisson prime* if it is a prime element of the algebra which is Poisson normal.

Equivalently, an element p is Poisson prime if and only if the ideal (p) is nonzero, prime and Poisson. There is also a geometrical interpretation of Poisson prime, which will be very useful.

Lemma 2.3.3. *Assume that the base field is* $\mathbb{C}$ *and* Spec$A$ *is smooth. View the elements of* A *as regular functions on the Poisson variety* Spec$A$*. A prime element* $p \in A$ *is Poisson prime if and only if its zero locus* $\mathcal{V}(p)$ *is a union of symplectic leaves of* Spec$A$*.*

Proof. If $\mathcal{V}(p)$ is a union of symplectic leaves of Spec$A$, then for all $g \in A$, $\{p, g\}$ vanishes on the smooth locus of $\mathcal{V}(p)$. Thus $\{p, g\}$ vanishes on $\mathcal{V}(p)$ and belongs to (p). In the opposite direction, assume that (p) is Poisson. If $\mathcal{L}$ is a symplectic leaf of Spec$A$ such that $\mathcal{L} \cap \mathcal{V}(p) \neq \varnothing$ and $\mathcal{L} \not\subseteq \mathcal{V}(p)$, then for every smooth point $m \in \mathcal{L} \cap \mathcal{V}(p) \subsetneq \mathcal{L}$ there will exist $g \in A$ such that $\{p, g\}(m) \neq 0$. This would contradict the assumption that (p) is a Poisson ideal. □

The Poisson algebra A is called *noetherian* if it is noetherian when considered as a commutative algebra. A noetherian Poisson algebra A is called a *Poisson unique factorization domain* if it is an integral domain as an algebra and every non-zero Poisson prime ideal of A contains a Poisson prime element.

Lemma 2.3.4. *Assume that* A *is a Poisson algebra over a field of characteristic 0 which is a unique factorization domain as a commutative algebra. If* $a \in A$ *is a Poisson normal element and* $p \in A$ *is a prime element such that* $p \mid a$*, then* p *is a Poisson prime element.*

Proof. Let $a = p^k b$ for some $b \in A$ such that $p \nmid b$. For every $x \in A$, there exists $y \in A$ such that $\{a, x\} = ay$. Then

$$k\{p, x\}p^{k-1}b + p^k\{b, x\} = p^k by.$$

Since the base field has characteristic 0, we have $p^k \mid \{p, x\}p^{k-1}b$ for every $x \in A$, and so $p \mid \{p, x\}$. □

This lemma shows that in integral domains, Poisson normal elements are the products of Poisson primes.

Proposition 2.3.5. *Let A be a Poisson algebra over a field of characteristic 0, satisfying one of the following 2 conditions:*

- *A is a unique factorization domain as a commutative algebra or*
- *A is a noetherian Poisson unique factorization domain.*

Then every non-zero, non-unit Poisson normal element $a \in A$ has a unique factorization of the form

$$a = \prod_{i=1}^{m} p_i$$

for some set of (not necessarily distinct) Poisson prime element $p_1, \ldots, p_m \in A$. The uniqueness is up to taking associates and permutations.

The case that A is a UFD follows from the previous lemma. The case of noetherian Poisson UFDs is analogous to the unique factorization property of normal elements in (noncommutative) noetherian UFDs proved by Chatters [11, Proposition 2.1], see also [26, Proposition 2.1].

Chapter 3
Preliminaries on Cluster Structures

Cluster algebras were introduced by Fomin and Zelevinsky in the early 2000s as a tool in the study of total positivity in semisimple groups and canonical bases in their quantum analogs [22]. A quantized version of cluster algebras were created by Berenstein and Zelevinsky in 2005 to prepare the algebraic foundations for the notion of a canonical basis of a cluster algebra [3]. This single parameter version with ground ring $\mathbb{Z}[q^{\pm 1/2}]$ has been generalized to a multiparameter version over any commutative domain [26]. In this chapter, we will briefly review cluster algebras and the single parameter quantum cluster algebras, which will be the model for root of unity quantum cluster algebras.

3.1 Cluster Algebras

Here we recall the definition of the classical, commutative cluster algebras (of geometric type). Let N be a positive integer, $\mathbf{ex}$ be an n-element subset of $[1, N]$, and $\mathcal{F}$ be the field of rational functions in N variables over $\mathbb{Q}$. We will often use $\mathbf{ex}$ instead of n to give a labeling of $[1, n]$. A seed is a pair $(\widetilde{\mathbf{x}}, \widetilde{B})$ if

- $\widetilde{\mathbf{x}} = \{x_1, \dots, x_N\}$ is a transcendence basis of $\mathcal{F}$ over $\mathbb{Q}$ which generates $\mathcal{F}$
- $\widetilde{B} \in M_{N \times \mathbf{ex}}(\mathbb{Z})$ has a skew-symmetrizable $\mathbf{ex} \times \mathbf{ex}$ submatrix B given by rows labeled by $\mathbf{ex}$ (called the principal part of $\widetilde{B}$).

Such a transcendence basis $\widetilde{\mathbf{x}}$ will be called a free generating set of $\mathcal{F}$. In the context of a seed $(\widetilde{\mathbf{x}}, \widetilde{B})$, we call $\widetilde{\mathbf{x}}$ the cluster of the seed and call the elements x_i the cluster variables.

Mutation of a matrix in direction $k \in \mathbf{ex}$ (the exchangeable indices) is given as $\mu_k(\widetilde{B}) = E_s \widetilde{B} F_s$ where $s = \pm$ is a sign and matrices $E_s \in M_N(\mathbb{Z})$, $F_s \in M_{\mathbf{ex}}(\mathbb{Z})$

depend on $\widetilde{B}$:

$$E_s = (e_{ij}) = \begin{cases} \delta_{ij} & \text{if } j \neq k \\ -1 & \text{if } i = j = k \\ \max(0, -sb_{ik}) & \text{if } i \neq j = k \end{cases}$$

$$F_s = (f_{ij}) = \begin{cases} \delta_{ij} & \text{if } i \neq k \\ -1 & \text{if } i = j = k \\ \max(0, sb_{kj}) & \text{if } j \neq i = k \end{cases}$$

Note that for $\widetilde{B} = (b_{ij})$,

$$\mu_k(\widetilde{B}) = (b'_{ij}) = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{else} \end{cases},$$

but we will follow the matrix interpretation of mutation as it will be useful in defining the quantum version.

Lemma 3.1.1 ([2], [22])**.** *Mutation μ_k has the following properties.*

1. *The principal part of $\mu_k(\widetilde{B})$ is the mutation of the principal part of $\widetilde{B}$, $\mu_k(B)$.*
2. *Mutation is an involution, $\mu_k^2(\widetilde{B}) = \widetilde{B}$*
3. *B is integer and skew-symmetrizable if and only if $\mu_k(B)$ is.*
4. *The rank of $\mu_k(\widetilde{B})$ equals the rank of $\widetilde{B}$*

Mutation μ_k of the seed $(\widetilde{\mathbf{x}}, \widetilde{B})$ in the direction of $k \in \mathbf{ex}$ is given by $\mu_k(\widetilde{\mathbf{x}}, \widetilde{B}) = (\widetilde{\mathbf{x}}', \mu_k \widetilde{B})$ where the mutation of $\widetilde{\mathbf{x}}$ depends on $\widetilde{B}$: $\widetilde{\mathbf{x}}' = \{x'_k\} \cup \widetilde{\mathbf{x}} \backslash \{x_k\}$ where

$$x_k x'_k = \prod_{b_{ik}<0} x_i^{-b_{ik}} + \prod_{b_{ik}>0} x_i^{b_{ik}}. \tag{3.1}$$

Since $\mu_k(\widetilde{B})$ has a skew-symmetrizable principal part by the first and third parts of the lemma and $\widetilde{\mathbf{x}}'$ is a free generating set, then $\mu_k(\widetilde{\mathbf{x}}, \widetilde{B})$ is a seed. Considering the mutation equation (3.1) for $(x_k')' \in \mu_k^2(\widetilde{\mathbf{x}}, \widetilde{B})$, one concludes that μ_k is an involution on seeds from the second part of the lemma.

We say that two seeds $(\widetilde{\mathbf{x}}_1, \widetilde{B}_1)$, $(\widetilde{\mathbf{x}}_2, \widetilde{B}_2)$ are mutation-equivalent if $(\widetilde{\mathbf{x}}_2, \widetilde{B}_2)$ can be obtained from $(\widetilde{\mathbf{x}}_1, \widetilde{B}_1)$ via a finite sequence of mutations. Denote this by $(\widetilde{\mathbf{x}}_1, \widetilde{B}_1) \sim (\widetilde{\mathbf{x}}_2, \widetilde{B}_2)$. All seeds that are mutation-equivalent to $(\widetilde{\mathbf{x}}, \widetilde{B})$ contain the same subset $\mathbf{c} \subset \widetilde{\mathbf{x}}$ of cluster variables corresponding to indices $[1, N] \backslash \mathbf{ex}$. These cluster variables are called the frozen variables.

For a mutation-equivalence class $\mathcal{S}$, the cluster algebra $\mathcal{A}(\mathcal{S})$ is defined as the $\mathbb{Z}[\mathbf{c}^{\pm}]$-subalgebra of $\mathcal{F}$ generated by all cluster variables from seeds in $\mathcal{S}$. Since $\mathcal{S}$ is uniquely determined by any seed $(\widetilde{\mathbf{x}}, \widetilde{B}) \in \mathcal{S}$, we often denote $\mathcal{A}(\mathcal{S})$ by $\mathcal{A}(\widetilde{\mathbf{x}}, \widetilde{B})$. We may even denote it by $\mathcal{A}(\widetilde{B})$, as changing the free generating set from $\widetilde{\mathbf{x}}_1$ to $\widetilde{\mathbf{x}}_2$ will induce an automorphism of $\mathcal{F}$ and $\mathcal{A}(\widetilde{\mathbf{x}}_1, \widetilde{B}) \simeq \mathcal{A}(\widetilde{\mathbf{x}}_2, \widetilde{B})$. Instead of inverting all frozen variables, we could pick a subset $\mathbf{inv} \subseteq \mathbf{c}$ to invert. Then $\mathcal{A}(\widetilde{\mathbf{x}}, \widetilde{B}, \mathbf{inv})$, or $\mathcal{A}(\widetilde{B}, \mathbf{inv})$, denotes the $\mathbb{Z}[\mathbf{c}, \mathbf{inv}^{-1}]$-subalgebra generated by all cluster variables from seeds mutation equivalent to $(\widetilde{\mathbf{x}}, \widetilde{B})$.

3.2 Quantum Cluster Algebras

Here we will review the construction and properties of quantum cluster algebras in the generic case, which will guide the construction when dealing with roots of unity. Let $\Lambda : \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{Z}$ be a skew-symmetric bilinear form, which at times will be treated as a matrix $\Lambda = (\lambda_{ij})$. Using a formal variable $q^{1/2}$, we work with Laurent polynomials $\mathbb{Z}[q^{\pm 1/2}]$.

Definition 3.2.1. The *based quantum torus* $\mathcal{T}_q(\Lambda)$ associated with Λ is defined as the $\mathbb{Z}[q^{\pm 1/2}]$-algebra with a $\mathbb{Z}[q^{\pm 1/2}]$-basis $\{X^f \mid f \in \mathbb{Z}^N\}$ and multiplication given by

$$X^f X^g = q^{\Lambda(f,g)/2} X^{f+g} \text{ where } f, g \in \mathbb{Z}^N.$$

Note that $X^f X^g = q^{\Lambda(f,g)} X^g X^f$ and that Λ can be recovered from the commutation relations of $\{X^{e_1}, \ldots, X^{e_N}\}$. In particular, this commutation relation is why we chose to define the based quantum torus over $\mathbb{Z}[q^{\pm 1/2}]$ rather than $\mathbb{Z}[q^{\pm 1}]$. We denote by $\mathcal{F}$ the skew-field of fractions of $\mathcal{T}_q(\Lambda)$, which is a $\mathbb{Q}(q^{1/2})$-algebra.

Given $\sigma \in GL_N(\mathbb{Z})$, we can create another based quantum torus $\mathcal{T}_q(\Lambda')$ where $\Lambda'(f,g) = \Lambda(\sigma f, \sigma g)$ is a skew-symmetric form. Note that if we consider Λ' as a matrix, then $\Lambda' = \sigma^T \Lambda \sigma$. Also, we have a $\mathbb{Z}[q^{\pm 1/2}]$-algebra isomorphism $\Psi_\sigma : \mathcal{T}_q(\Lambda) \to \mathcal{T}_q(\Lambda')$ given by $X^{\sigma f} \mapsto X^f$.

Definition 3.2.2. Let $\mathcal{F}$ be a division algebra over $\mathbb{Q}(q^{1/2})$. A *toric frame* M is defined as a map $M : \mathbb{Z}^N \to \mathcal{F}$ such that there exists a skew-symmetric matrix $\Lambda \in M_N(\mathbb{Z})$ satisfying

1. there is a $\mathbb{Z}[q^{\pm 1/2}]$-algebra embedding $\phi : \mathcal{T}_q(\Lambda) \hookrightarrow \mathcal{F}$ with $\phi(X^f) = M(f)$ for all $f \in \mathbb{Z}^N$

2. $\mathcal{F} = \mathrm{Fract}(\phi(\mathcal{T}_q(\Lambda)))$.

A toric frame M is then an embedding of a based quantum torus into an algebra isomorphic to its skew-field of fractions. The skew-symmetric matrix associated to a toric frame M is often denoted by Λ_M. For any $\sigma \in GL_N(\mathbb{Z})$, $\rho \in \mathrm{Aut}(\mathcal{F})$, and toric frame M, the map $\rho M \sigma$ is a toric frame with $\Lambda_{\rho M \sigma} = \sigma^T \Lambda \sigma$. The embedding ϕ for M gives rise to an embedding $\phi' : \mathcal{T}_q(\Lambda_{\rho M \sigma}) \hookrightarrow \mathcal{F}$ by $\phi' = \rho \circ \phi \circ \Psi_{\sigma^{-1}}$, which satisfies the two properties above for $\rho M \sigma$.

Let $\mathbf{ex}$ be an n-element subset of $[1, N]$, view Λ as a skew-symmetric matrix, and let $\widetilde{B}$ be an $N \times \mathbf{ex}$ matrix. We call a pair $(\Lambda, \widetilde{B})$ *compatible* if

$$\sum_{k=1}^{N} b_{kj}\lambda_{ki} = \delta_{ij}d_j \text{ for all } i \in [1, N],\ j \in \mathbf{ex}$$

for some positive integers d_j. Equivalently $\widetilde{B}^T\Lambda = \widetilde{D}$ where $d_{ii} = d_i$ for $i \in \mathbf{ex}$ are positive integers and otherwise $d_{ij} = 0$. Similar to the principal part B of $\widetilde{B}$, we denote by D the $\mathbf{ex} \times \mathbf{ex}$ submatrix of $\widetilde{D}$. When $\widetilde{B}$ is part of a compatible pair $(\Lambda, \widetilde{B})$, then $\widetilde{B}$ is nice in the following sense.

Lemma 3.2.3 ([3, Proposition 3.3])**.** *If $(\Lambda, \widetilde{B})$ is a compatible pair, then $\widetilde{B}$ has full rank and its principle part B is skew-symmetrized by D.*

A pair $(\Lambda, \widetilde{B})$ may be mutated in the direction of $k \in \mathbf{ex}$, by $\mu_k(\Lambda, \widetilde{B}) = (\Lambda', \widetilde{B}')$ where $\widetilde{B}' = E_s\widetilde{B}F_s$ as in the classical case and $\Lambda' = E_s^T\Lambda E_s$ (note skew-symmetric). The pair $\mu_k(\Lambda, \widetilde{B})$ is independent of choice of sign s, $\mu_k(\Lambda, \widetilde{B})$ is compatible if $(\Lambda, \widetilde{B})$ was, and mutation μ_k of compatible pairs is an involution [3, Propositions 3.4, 3.5].

We call a pair $(M, \widetilde{B})$ a *quantum seed* if the pair $(\Lambda_M, \widetilde{B})$ is compatible. We call $\{M(e_j) \mid j \in [1, N]\}$ the *cluster variables* of the seed $(M, \widetilde{B})$. The subset of cluster variables $\{M(e_j) \mid j \notin \mathbf{ex}\}$ are called frozen variables.

Lemma 3.2.4. *Suppose M is a toric frame, $k \in [1, N]$ and $g = \sum_{i=1}^{N} n_i e_i \in \mathbb{Z}^N$ is such that $\Lambda_M(g, e_j) = 0$ for $j \neq k$ and $n_k = 0$. Then for each $s = \pm$, there is an automorphism $\rho_{g,s} = \rho_{g,s}^M$ of $\mathcal{F}$, such that*

$$\rho_{g,s}(M(e_j)) = \begin{cases} M(e_k) + M(e_k + sg) & \text{if } j = k \\ M(e_j) & \text{if } j \neq k \end{cases}$$

Proof. The proof follows similarly to [3, Proposition 4.3], starting by constructing a $\mathbb{Z}[q^{\pm 1/2}]$-linear map $\mathcal{T}_q(\Lambda_M) \to \mathcal{F}$ by defining on the basis elements

$$M(f) \mapsto \begin{cases} P^{M,m_k}_{g,s,+} M(f) & \text{if } m_k \geq 0 \\ (P^{M,-m_k}_{g,s,-})^{-1} M(f) & \text{if } m_k < 0 \end{cases}$$

for $f = \sum_{j=1}^{N} m_j e_j \in \mathbb{Z}^N$, where

$$P^{M,m_k}_{g,s,\pm} = \prod_{p=1}^{m_k} \big(1 + q^{\mp s(2p-1)\Lambda_M(g,e_k)/2} M(sg)\big).$$

This map is a $\mathbb{Z}[q^{\pm 1/2}]$-algebra homomorphism since $M(g)M(e_j) = M(e_j)M(g)$ if $j \neq k$ and $M(sg)M(e_k) = q^{-s\Lambda_M(g,e_k)} M(e_k)M(sg)$, and it extends to an endomorphism of $\mathcal{F}$. Lastly, the map is an automorphism as $\rho_{g,s} \circ \rho' = \rho' \circ \rho_{g,s} = \mathrm{Id}_{\mathcal{F}}$ where one defines the endomorphism ρ' in a similar fashion such that

$$\rho'(M(e_j)) = \begin{cases} (P^{1}_{g,s,+})^{-1} M(e_k) & \text{if } j = k \\ M(e_j) & \text{if } j \neq k \end{cases}.$$

□

An important application of these maps are for $g = b^k$, where b^k is the k^{th} column of a matrix $\widetilde{B}$ that forms a compatible pair with Λ_M. It is easily checked that the compatibility condition ensures the conditions of the lemma are met.

Remark 3.2.5. The notation used here does not match perfectly with that of [3]. Unlike [3], $\rho^M_{-g,-s} \neq (\rho^M_{g,s})^{-1}$. However, these automorphisms match up with the multiparameter case of [26]. It should be noted that for compatible $(\Lambda_M, \widetilde{B})$, the map $\rho_{b^k,s}$ from [3] matches $\rho^M_{b^k,s}$ on generators $M(e_i)$ and hence is the same automorphism. This does not cause a contradiction, since $\rho^M_{-b^k,-s}$ is not the map $\rho_{-b^k,-s}$ of [3]. The bilinear form Λ_M cannot be compatible with a matrix that has $-b^k$ as the k^{th} column, as we know $\sum_{l=1}^{N} -b_{lk}\lambda_{lk} = -d_k < 0$.

Mutation $\mu_k(M, \widetilde{B})$ of a quantum seed in the direction of k is defined as $(\mu_k(M), \mu_k(\widetilde{B})) = (\rho^M_{b^k,s} M E_s, E_s \widetilde{B} F_s)$. Here the mutation of the toric frame uses the automorphisms above and as in the classical case, depends on $\widetilde{B}$. Since $E_s \in GL_N(\mathbb{Z})$ ($E_s^2 = \mathrm{Id}$) and $\rho^M_{b^k,s} \in Aut(\mathcal{F})$, then $\rho^M_{b^k,s} M E_s$ is a toric frame. Now the skew-symmetric matrix $\Lambda_{\mu_k(M)}$ is equal to $\mu_k(\Lambda_M)$ by [3, Proposition 4.7]. So the pair $(\Lambda_{\mu_k(M)}, \mu_k(\widetilde{B}))$ is compatible, and $\mu_k(M, \widetilde{B})$ is a quantum seed.

Lemma 3.2.6. *Quantum seed mutation is independent of sign and is an involution.*

Proof. These properties are clear for matrix mutation, and so we show them for the toric frame mutation. As noted above, a toric frame is determined by where it sends the standard basis vectors of $\mathbb{Z}^N$. Note that $E_s e_j = e_j$ for $j \neq k$. From their construction, $\rho^M_{b^k,+}(M(e_j)) = M(e_j) = \rho^M_{b^k,-}(M(e_j))$ for $j \neq k$. For e_k,

$$\begin{aligned}
\rho^M_{b^k,-} M E_-(e_k) =& \rho^M_{b^k,-} M(-e_k + [b^k]_+) = (P^{M,1}_{b^k,-,-})^{-1} M(-e_k + [b^k]_+) \\
=& (1 + q^{-\Lambda_M(b^k,e_k)/2} M(-b^k)) M(-e_k + [b^k]_+) \\
=& M(-e_k + [b^k]_+) + M(-e_k - [b^k]_-) \\
\rho^M_{b^k,+} M E_+(e_k) =& \rho^M_{b^k,+} M(-e_k - [b^k]_-) = (P^{M,1}_{b^k,+,-})^{-1} M(-e_k - [b^k]_-) \\
=& (1 + q^{\Lambda_M(b^k,e_k)/2} M(b^k)) M(-e_k - [b^k]_-) \\
=& M(-e_k - [b^k]_-) + M(-e_k + [b^k]_+)
\end{aligned}$$

where $[b^k]_- = \sum_{b_{ik}<0} b_{ik} e_i$ and $[b^k]_+ = \sum_{b_{ik}>0} b_{ik} e_i$.

For the involutive property, note that $E'_\pm$ coming from $\mu_k \widetilde{B}$ is in fact $E_\mp$ since μ_k negates the k^{th} column. Then

$$\begin{aligned}
\mu_k(\mu_k(M)) = \rho^{\mu_k M}_{-b^k,+} \rho^M_{b^k,-} M E_- E'_+ &= \rho^{\mu_k M}_{-b^k,+} \rho^M_{b^k,-} M E_- E_- \\
= \rho^{\mu_k M}_{-b^k,+} \rho^M_{b^k,-} M \qquad &= M
\end{aligned}$$

which can be checked on the standard basis. □

A key property of mutation of quantum seeds was shown in the above proof (see also [3, Proposition 4.9]),

$$\begin{aligned} \mu_k(M)(e_j) &= M(e_j) \text{ for } j \neq k \\ \mu_k(M)(e_k) &= M(-e_k + [b^k]_+) + M(-e_k - [b^k]_-) \end{aligned} \tag{3.2}$$

From this, we see that setting $q = 1$ would "recover" the commutative case of mutation.

Because mutation is involutive, we again can consider the equivalence classes of quantum seeds under finite sequences of mutation. Since $\mu_k(M)(e_j) = M(e_j)$ if $j \neq k$, then for any $(M', \widetilde{B}') \sim (M, \widetilde{B})$ we have that $M'(e_j) = M(e_j)$ for $j \notin \mathbf{ex}$. So the frozen variables of these quantum seeds are the same and only depend on the mutation-equivalence class. We will fix a subset $\mathbf{inv} \subseteq [1, N]\backslash\mathbf{ex}$ corresponding to frozen variables that will be set as invertible.

Definition 3.2.7. The *quantum cluster algebra* $\mathcal{A}_q(M, \widetilde{B}, \mathbf{inv})$ is the $Z[q^{\pm 1/2}]$-subalgebra of $\mathcal{F}$ generated by all cluster variables $M'(e_j), j \in [1, N]$ of quantum seeds $(M', \widetilde{B}')$ mutation equivalent to $(M, \widetilde{B})$ and by the inverses $M(e_j)^{-1}$ for $j \in \mathbf{inv}$.

Some authors use a different domain, such as $\mathbb{K}[q^{\pm 1/2}]$ for a field $\mathbb{K}$, in defining their quantum tori and quantum cluster algebra. We will refer to these as *non-integral* quantum cluster algebras. Quantum cluster structures defined as above using $\mathbb{Z}[q^{\pm 1/2}]$ for the base domain will be called *integral*. If one has an integral quantum cluster algebra, extending scalars will give a non-integral cluster structure.

As before, when the subset $\mathbf{inv}$ is understood, we often leave it out of the notation. If M' is another toric frame such that $(\Lambda_{M'}, \widetilde{B})$ is compatible, then there is an isomorphism $\mathcal{A}_q(M, \widetilde{B}, \mathbf{inv}) \simeq \mathcal{A}_q(M', \widetilde{B}, \mathbf{inv})$.

The *upper quantum cluster algebra* $\mathcal{U}_q(M, \widetilde{B}, \mathbf{inv})$ is defined as the intersection over quantum seeds $(M', \widetilde{B}') \sim (M, \widetilde{B})$ of all $\mathbb{Z}[q^{\pm 1/2}]$-subalgebras of $\mathcal{F}$ of the form

$$\mathbb{Z}[q^{\pm 1/2}]\langle M'(e_i), M'(e_j)^{-1} \mid i \in [1, N],\ j \in \mathbf{ex} \sqcup \mathbf{inv} \rangle.$$

These subalgebras of $\mathcal{F}$ are called mixed quantum tori. We denote by $\mathcal{T}_q(M)$ the based quantum torus with basis $\langle M(f) \mid f \in \mathbb{Z}^N \rangle$. The relation between generators is given by $M(f)M(g) = q^{\Lambda_M(f,g)}M(f+g)$. So, we have an isomorphism $\mathcal{T}_q(M) \simeq \mathcal{T}_q(\Lambda_M)$. For $\mathbf{inv} = [1, N]\backslash\mathbf{ex}$, we can rephrase the above definition for the quantum upper cluster algebra as

$$\mathcal{U}_q(M, \widetilde{B}) = \bigcap_{(M', \widetilde{B}') \sim (M, \widetilde{B})} \mathcal{T}_q(M').$$

Theorem 3.2.8 (The Quantum Laurent Phenomenon [3], [26])**.** *The quantum cluster algebra $\mathcal{A}_q(M, \widetilde{B}, \boldsymbol{inv})$ is contained in the mixed quantum torus*

$$\mathbb{Z}[q^{\pm 1/2}]\langle M'(e_i), M'(e_j)^{-1} \mid i \in [1, N],\ j \in \boldsymbol{ex} \sqcup \boldsymbol{inv} \rangle$$

for any quantum seed $(M', \widetilde{B}') \sim (M, \widetilde{B})$, and we have an inclusion

$$\mathcal{A}_q(M, \widetilde{B}, \boldsymbol{inv}) \hookrightarrow \mathcal{U}_q(M, \widetilde{B}, \boldsymbol{inv}).$$

Berenstein and Zelevinsky proved the theorem when all frozen variables are inverted, $\mathbf{inv} = [1, N]\backslash\mathbf{ex}$. The quantum Laurent phenomenon was shown for the more general setting in [26].

The exchange graph for a cluster algebra $\mathcal{A}(\mathbf{x}, \widetilde{B})$ or quantum cluster algebra $\mathcal{A}_q(M, \widetilde{B})$ is the graph with vertices corresponding to (quantum) seeds mutation-equivalent to $(\mathbf{x}, \widetilde{B})$, or $(M, \widetilde{B})$ respectfully, and with edges given by seed mutation. We will denote the exchange graph for a cluster algebra $\mathcal{A}(\widetilde{B})$ by $E(\widetilde{B})$ and the exchange graph of $\mathcal{A}_q(M, \widetilde{B})$ by $E_q(\Lambda_M, \widetilde{B})$

Theorem 3.2.9 ([3]). *There is a canonical isomorphism between the exchange graphs $E(\Lambda_M, \widetilde{B})$ and $E_q(\widetilde{B})$ obtained by matching an initial seed of the two algebras $\mathcal{A}(\boldsymbol{x}, \widetilde{B})$ and $\mathcal{A}_q(M, \widetilde{B})$ and then matching the rest of the seeds by following the same mutation sequences.*

The identification of exchange graphs was shown in [3] for $\mathbf{inv} = [1, N] \backslash \mathbf{ex}$. The statement then holds for any $\mathbf{inv}$, as the localization of $\mathcal{A}_q(M, \widetilde{B}, \mathbf{inv})$ to $\mathcal{A}_q(M, \widetilde{B}, [1, N] \backslash \mathbf{ex})$ does not change the set of seeds.

Chapter 4
Quantum Cluster Algebras at Roots of Unity

The quantum cluster algebra at a root of unity will be constructed from quantum seeds similarly to the generic case. Specializing a quantum cluster algebra to a root of unity requires care and attention. In the case that a (non-integral) quantum cluster algebra $A = \mathcal{A}_q(M, \widetilde{B})$ is equal to its quantum upper cluster algebra $\mathcal{U}_q(M, \widetilde{B})$ and the algebra satisfies some grading criteria, then the specialization to 1, $A_1 = A/(q^{1/2} - 1)A$, is isomorphic to the cluster algebra $\mathcal{A}(\widetilde{B})$ as shown by Geiß, Leclerc, and Schröer [25]. However, it is not true that specializing any quantum cluster algebra to 1 will recover a cluster algebra. For instance, a quantum Weyl algebra has a cluster structure. However, its specialization to 1 is isomorphic to the Weyl algebra, a noncommutative algebra which cannot be a cluster algebra.

4.1 The Root of Unity Case

To match the presentation of quantum cluster algebras, we let $\epsilon^{1/2}$ be a primitive ℓ^{th} root of unity. Let Λ be a skew-symmetric bilinear form, often thought of as a $N \times N$ matrix. Let **ex** be an n-element subset of $[1, N]$. As before, we will often use **ex** instead of n to give a labeling of $[1, n]$. The *root of unity based quantum torus* $\mathcal{T}_\epsilon(\Lambda)$ is the $\mathbb{Z}[\epsilon^{1/2}]$-algebra with a $\mathbb{Z}[\epsilon^{1/2}]$-basis $\{X^f \mid f \in \mathbb{Z}^N\}$ and multiplication given by

$$X^f X^g = \epsilon^{\Lambda(f,g)/2} X^{f+g} \text{ where } f, g \in \mathbb{Z}^N.$$

Hence $X^f X^g = \epsilon^{\Lambda(f,g)} X^g X^f$. Unlike the previous case, Λ cannot be recovered from commutation relations of basis elements. If $\Lambda \equiv \Lambda'$ as elements of $M_N(\mathbb{Z}/\ell\mathbb{Z})$, then $\mathcal{T}_\epsilon(\Lambda) = \mathcal{T}_\epsilon(\Lambda')$.

Lemma 4.1.1. *There is an isomorphism of algebras $\mathcal{T}_q(\Lambda)/(\Phi_\ell(q^{1/2})) \simeq \mathcal{T}_\epsilon(\Lambda)$ where $\Phi_\ell(q^{1/2})$ is the ℓ^{th} cyclotomic polynomial. We denote by $\kappa_\epsilon : \mathcal{T}_q(\Lambda) \to \mathcal{T}_\epsilon(\Lambda)$ the canonical projection.*

Proof. We have $\mathbb{Z}[q^{\pm 1/2}]/(\Phi_\ell(q^{1/2})) \simeq \mathbb{Z}[\epsilon^{1/2}]$. It follows that $\mathcal{T}_q(\Lambda)/(\Phi_\ell(q^{1/2})) \simeq \mathcal{T}_\epsilon(\Lambda)$ since the free $\mathbb{Z}[q^{\pm 1/2}]$-module $\mathcal{T}_q(\Lambda)$ and the free $\mathbb{Z}[\epsilon^{1/2}]$-module $\mathcal{T}_\epsilon(\Lambda)$ both have the basis $\{X^f \mid f \in \mathbb{Z}^N\}$. □

Let $\mathcal{F}_\epsilon$ be a division algebra over $\mathbb{Q}(\epsilon^{1/2})$. A *root of unity toric frame* M_ϵ is defined as a map $M_\epsilon : \mathbb{Z}^N \to \mathcal{F}_\epsilon$ such that there is a skew-symmetric matrix $\Lambda \in M_N(\mathbb{Z})$ satisfying

1. There is a $\mathbb{Z}[\epsilon^{1/2}]$-algebra embedding $\phi : \mathcal{T}_\epsilon(\Lambda) \hookrightarrow \mathcal{F}_\epsilon$ with $\phi(X^f) = M_\epsilon(f)$ for all $f \in \mathbb{Z}^N$
2. $\mathcal{F}_\epsilon = \mathrm{Fract}\,(\phi(\mathcal{T}_\epsilon(\Lambda)))$.

We will now denote toric frames by M_q to distinguish them from root of unity toric frames M_ϵ. We will typically call root of unity toric frames just toric frames when context is clear.

Again, $\rho M_\epsilon \sigma$ is a toric frame for any $\sigma \in GL_N(\mathbb{Z})$, $\rho \in \mathrm{Aut}(\mathcal{F}_\epsilon)$, and toric frame M_ϵ. If conditions (1) and (2) for M_ϵ are satisfied by Λ and ϕ, then the conditions for $\rho M_\epsilon \sigma$ are satisfied by $\sigma^T \Lambda \sigma$ and $\rho \circ \phi \circ \Psi_{\sigma^{-1}}$ where $\Psi_{\sigma^{-1}}$ is the isomorphism $\mathcal{T}_\epsilon(\sigma^T \Lambda \sigma) \xrightarrow{\sim} \mathcal{T}_\epsilon(\Lambda)$.

We say $\Lambda' \in M_N(\mathbb{Z})$ is *related* to toric frame M_ϵ if $M_\epsilon(f) M_\epsilon(g) = \epsilon^{\Lambda'(f,g)/2} M_\epsilon(f+g)$. Then Λ' is related to M_ϵ if and only if conditions (1) and (2) are satisfied for M_ϵ by Λ'. We denote by $\mathcal{T}_\epsilon(M_\epsilon)$ the based quantum torus with basis $\{M_\epsilon(f) \mid f \in \mathbb{Z}^N\} \subseteq \mathcal{F}_\epsilon$, noting that for Λ related to M_ϵ we have an isomorphism $\mathcal{T}_\epsilon(M_\epsilon) \simeq \mathcal{T}_\epsilon(\Lambda)$.

We will call a triple $(M_\epsilon, \Lambda, \widetilde{B})$ a *root of unity quantum seed* if M_ϵ is a toric frame, $\widetilde{B} \in M_{N \times \mathbf{ex}}(\mathbb{Z})$, and Λ is a skew-symmetric bilinear form such that Λ is related to

M_ϵ and $(\Lambda, \widetilde{B})$ is a compatible pair. Similar to before, we have automorphisms on $\mathcal{F}_\epsilon$ that will be used in mutating seeds.

Lemma 4.1.2. *Suppose M_ϵ is a toric frame, Λ is related to M_ϵ, $k \in [1, N]$ and $g = \sum_{i=1}^{N} n_i e_i \in \mathbb{Z}^N$ is such that $\Lambda(g, e_j) = 0$ for $j \neq k$ and $n_k = 0$. Then for each $s = \pm$, there is an automorphism $\rho_{g,s} = \rho_{g,s}^{M_\epsilon}$ of $\mathcal{F}_\epsilon$, such that*

$$\rho_{g,s}(M_\epsilon(e_j)) = \begin{cases} M_\epsilon(e_k) + M_\epsilon(e_k + sg) & \text{if } j = k \\ M_\epsilon(e_j) & \text{if } j \neq k \end{cases}$$

Proof. The proof follows the same argument as before but with the change

$$P_{g,s,\pm}^{M_\epsilon, m_k} = \prod_{p=1}^{m_k} \left(1 + \epsilon^{\mp s(2p-1)\Lambda(g, e_k)} M_\epsilon(sg)\right).$$

□

We define mutation in the direction of $k \in \mathbf{ex}$ by

$$\mu_k(M_\epsilon, \Lambda, \widetilde{B}) = (\rho_{b^k,s}^{M_\epsilon} M_\epsilon E_s, E_s \widetilde{B} F_s, E_s^T \Lambda E_s).$$

Let $\phi_\Lambda : \mathcal{T}_\epsilon(\Lambda) \to \mathcal{F}_\epsilon$ denote the appropriate embedding.

Lemma 4.1.3. *Given a root of unity quantum seed $(M_\epsilon, \Lambda, \widetilde{B})$, the following relations hold in $\mathcal{F}_\epsilon$ for either sign $s = \pm$:*

$$\rho_{b^k,s}^{M_\epsilon} M_\epsilon E_s(e_j) = M_\epsilon(e_j) \text{ for } j \neq k$$
$$\rho_{b^k,s}^{M_\epsilon} M_\epsilon E_s(e_k) = M_\epsilon(-e_k + [b^k]_+) + M_\epsilon(-e_k - [b^k]_-)$$

Proof. Since $E_s e_j = e_j$ for $j \neq k$, the first equation follows from the definition of $\rho_{b^k,s}^{M_\epsilon}$. We will use the generic case of the second equation to show the equation holds in the root of unity as well, although one could directly copy the proof steps from the generic case.

Since Λ is related to $M : \mathbb{Z}^N \to \mathcal{F}_\epsilon$, there is an embedding $\phi : \mathcal{T}_\epsilon(\Lambda) \hookrightarrow \mathcal{F}_\epsilon$. Consider $T_q(\Lambda)$ and label $\mathrm{Fract}(T_q(\Lambda))$ by $\mathcal{F}_q$. Construct a toric frame $M_q : \mathbb{Z}^N \to \mathcal{F}_q$ by $M_q(f) = X^f$, noting $\Lambda_{M_q} = \Lambda$. Compatibility of $\widetilde{B}$ and Λ gives us a quantum seed $(M_q, \widetilde{B})$. Mutation in the k^{th} direction is given by $(\rho^{M_q}_{b^k,s} M_q E_s, E_s \widetilde{B} F_s)$ where $\rho^{M_q}_{b^k,s} \in \mathrm{Aut}(\mathcal{F}_q)$. Let $\kappa_\epsilon : \mathcal{T}_q(\Lambda) \to \mathcal{T}_\epsilon(\Lambda)$ be the quotient map of Lemma 4.1.1. Viewing $M_\epsilon(f)$ as an element of $\mathcal{T}_\epsilon(\Lambda)$, we may write $\kappa_\epsilon(M_q(f)) = M_\epsilon(f)$.

We know that $\rho^{M_q}_{b^k,s} : \mathcal{F}_q \to \mathcal{F}_q$ is defined by its action on $M_q(f)$. The $\mathbb{Z}[q^{\pm 1/2}]$-subalgebras generated by the images of M_q and $M_q\sigma$ are the same for any $\sigma \in GL_N(\mathbb{Z})$. So we may reduce the problem of understanding $\mu_k(M_\epsilon)$ in terms of $\mu_k(M_q)$ to the problem of understanding $\rho^{M_\epsilon}_{b^k,s} M_\epsilon$ in terms of $\rho^{M_q}_{b^k,s} M_q$

Note that $\kappa_\epsilon(P^{M_q,m_k}_{b^k,s,\pm}) = P^{M_\epsilon,m_k}_{b^k,s,\pm}$. For $f = \sum m_i e_i$ with $m_k \geq 0$,

$$\rho^{M_q}_{b^k,s}(M_q(f)) = P^{M_q,m_k}_{g,s,+} M(f) \in \mathcal{T}_q(\Lambda) \subseteq \mathcal{F}_q$$

which we are viewing as an element of $\mathcal{T}_q(\Lambda)$. We may apply κ_ϵ, and we find that

$$\kappa_\epsilon(\rho^{M_q}_{b^k,s}(M_q(f)) = \rho^{M_\epsilon}_{b^k,s}(M_\epsilon(f)).$$

If $m_k < 0$, consider

$$P^{M_q,-m_k}_{b^k,s,-} \rho^{M_q}_{b^k,s}(M_q(f)) = M_q(f) \in \mathcal{T}_q(\Lambda).$$

Applying κ_ϵ which we derive,

$$\kappa_\epsilon \left(P^{M_q,-m_k}_{b^k,s,-} \rho^{M_q}_{b^k,s}(M_q(f)) \right) = M_\epsilon(f) = P^{M_\epsilon,-m_k}_{b^k,s,-} \rho^{M_\epsilon}_{b^k,s}(M_\epsilon(f)).$$

Hence the desired equation holds for the mutation of a root of unity toric frame by considering $f = E_s e_k$. □

More properties of root of unity quantum seeds follow from the generic case such as the next natural result.

Lemma 4.1.4. *If $(M, \widetilde{B}, \Lambda)$ is a root of unity quantum seed, then so is $\mu_k(M, \widetilde{B}, \Lambda)$. Moreover, mutation is an involution and does not depend on the sign used.*

Proof. The pair $(E_s\widetilde{B}F_s, E_s^T\Lambda E_s)$ is the mutation of the compatible pair of matrices $(\widetilde{B}, \Lambda)$, which is compatible and independent of sign by [3, Proposition 3.4]. The involutive property of mutation of compatible pairs comes from [3, Proposition 3.6].

As discussed previously, $\mu_k(M_\epsilon) = \rho^M_{b^k,s}ME_s$ is a toric frame since $\rho^M_{b^k,s} \in \mathrm{Aut}(\mathcal{F}_\epsilon)$ and $E_s \in GL_N(\mathbb{Z})$, and $\Lambda' = E_s^T\Lambda E_s$ is related to $\mu_k(M)$. From Lemma 4.1.3, we have that $\mu_k(M_\epsilon)$ does not depend on sign s. Moreover, from the proof of Lemma 4.1.3, we see that the equations satisfied by $\mu_k^2(M_q)$ must also be satisfied by $\mu_k^2(M_\epsilon)$. Hence $\mu_k^2(M_\epsilon) = M_\epsilon$ by its image of e_j. □

We consider the equivalence classes under finite sequences of mutations of root of unity quantum seeds. Fix a subset $\mathbf{inv} \subseteq [1, N] \backslash \mathbf{ex}$ corresponding to frozen variables that will set as invertible.

Definition 4.1.5. Given a root of unity quantum seed $(M_\epsilon, \Lambda, \widetilde{B})$, we define the *quantum cluster algebra at a root of unity* $\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}, \mathbf{inv})$ as the $\mathbb{Z}[\epsilon^{1/2}]$-subalgebra of $\mathcal{F}_\epsilon$ generated by all cluster variables $M'_\epsilon(e_j)$, $j \in [1, N]$ of quantum seeds $(M'_\epsilon, \Lambda', \widetilde{B}')$ mutation equivalent to $(M_\epsilon, \Lambda, \widetilde{B})$ and by the inverses of appropriate frozen variables $M_\epsilon(e_j)^{-1}$, $j \in \mathbf{inv}$.

We will sometimes denote $\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}, \mathbf{inv})$ by $\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B})$ if the subset $\mathbf{inv}$ is understood. If $(M'_\epsilon, \Lambda', \widetilde{B})$ is another quantum seed for the same ambient skew-field $\mathcal{F}_\epsilon$, there is an induced isomorphism $\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}, \mathbf{inv}) \simeq \mathcal{A}_\epsilon(M'_\epsilon, \Lambda', \widetilde{B}, \mathbf{inv})$. As such, we may denote the exchange graph of $\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B})$ by $E_\epsilon(\Lambda, \widetilde{B})$.

We will need the following result concerning opposite algebras when working with the cluster structure of quantum Schubert cells. Note for the opposite algebras of quantum tori that $\mathcal{T}_q(\Lambda)^{op} \simeq \mathcal{T}_q(\Lambda^T)$ via $X^f \mapsto X^f$ as

$$X^f \bullet X^g = X^g X^f = q^{-\Lambda(f,g)/2} X^{f+g} = q^{-\Lambda(f,g)} X^g \bullet X^f \text{ for any } f, g \in \mathbb{Z}^N,$$

where $\bullet$ is the opposite multiplication. Similarly $\mathcal{T}_\epsilon(\Lambda)^{op} \simeq \mathcal{T}_\epsilon(\Lambda^T)$. From these facts, we see that $\text{Fract}(\mathcal{T}_q(\Lambda))^{op} \simeq \text{Fract}(\mathcal{T}_q(\Lambda^T))$ and $\text{Fract}(\mathcal{T}_\epsilon(\Lambda))^{op} \simeq \text{Fract}(\mathcal{T}_\epsilon(\Lambda^T))$.

Theorem 4.1.6. *The opposite algebra of a quantum cluster algebra $\mathcal{A}_q(M_q, \widetilde{B}, \textbf{\textit{inv}})^{op}$ is a quantum cluster algebra isomorphic to $\mathcal{A}_q(M_q^{op}, -\widetilde{B}, \textbf{\textit{inv}})$, where M_q^{op} is the toric frame to Fract($\mathcal{T}_q(\Lambda^T)$) whose image is equal to M_q under the canonical vector space isomorphism. Likewise, the opposite algebra of a quantum cluster algebra at a root of unity $\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}, \textbf{\textit{inv}})^{op}$ is isomorphic to $\mathcal{A}_\epsilon(M_\epsilon^{op}, \Lambda^T, -\widetilde{B}, \textbf{inv})$.*

Proof. Given a cluster algebra $\mathcal{A}_q(M_q, \widetilde{B})$, we have a compatible pair $(\widetilde{B}, \Lambda_{M_q})$ and a quantum seed $(M_q, \widetilde{B})$. Denote Λ_{M_q} by Λ for clarity. Since $(\Lambda, \widetilde{B})$ is compatible, we can deduce $(\Lambda^T, -\widetilde{B})$ is compatible from

$$D = \widetilde{B}^T \Lambda = (-\widetilde{B})^T(-\Lambda) = (-\widetilde{B})^T \Lambda^T.$$

Since $\mathcal{T}_q(\Lambda^T) \simeq \mathcal{T}_q(\Lambda)$ as $\mathbb{Q}(q^{1/2})$-vector spaces, let M_q^{op} be the map that sends $f \in Z^N$ to the canonical image of $M_q(f)$ in $(\mathcal{T}_q(\Lambda^T))$. Since $\text{Fract}(\mathcal{T}_\epsilon(\Lambda))^{op}$ is isomorphic to $\text{Fract}(\mathcal{T}_\epsilon(\Lambda^T))$, then

$$M_q^{op}(f) M_q^{op}(g) = q^{-\Lambda(f,g)} M_q^{op}(g) M_q^{op}(f) = q^{\Lambda^T(f,g)} M_q^{op},(g) M_q^{op}(f)$$

Hence the map $X^f \mapsto M_q^{op}(f)$ is indeed a embedding of $\mathcal{T}_q(\Lambda^T)$ into $\text{Fract}(\mathcal{T}_q(\Lambda^T))$, and M_q^{op} is a toric frame. Since $\Lambda_{M_q^{op}} = \Lambda^T$, the pair $(M_q^{op}, -\widetilde{B})$ is a quantum seed.

We now show that the exchange graphs for $\mathcal{A}_q(M_q, \widetilde{B})$ and $\mathcal{A}_q(M_q^{op}, -\widetilde{B}))$ are isomorphic. First we consider toric frame mutation. Let $(\widehat{M}_q, \widehat{B}) \in E_q(\Lambda, \widetilde{B})$. As elements of $\text{Fract}\mathcal{T}_q(\Lambda^T)$, we have the following, $\mu_k(\widehat{M}^{op})(e_j) = \widehat{M}^{op}(e_j) = \widehat{M}(e_j) =$

$u_k\widehat{M}(e_j) = (\mu_k\widehat{M})^{op}(e_j)$ for $j \neq k$. For $j = k$,

$$\begin{aligned}\mu_k(\widehat{M}^{op})(e_k) &= \widehat{M}^{op}(-e_k + [-\widehat{b}^k]_+) + \widehat{M}^{op}(-e_k - [-\widehat{b}^k]_-)\\ &= \widehat{M}(-e_k - [\widehat{b}^k]_-) + \widehat{M}(-e_k + [\widehat{b}^k]_+)\\ &= \mu_k\widehat{M}(e_k) = (\mu_k\widehat{M})^{op}(e_k)\end{aligned}$$

where $\widehat{M}$ mutates with respect to $\widehat{B}$, $\widehat{M}^{op}$ mutates with respect to $-\widehat{B}$, and $\widehat{b}^k$ is the k^{th} column of $\widehat{B}$.

Now for matrix mutation, consider matrices C and $-C$. Then $\mu_k(C) = E_\pm C F_\pm$ and $\mu_k(-C) = E'_\pm(-C)F'_\pm$ for the appropriate matrices $E_\pm$, $F_\pm$, $E'_\pm$, and $F'_\pm$. But from their definition, $E'_\pm = E_\mp$ and $F'_\pm = F_\mp$. As matrix mutation doesn't depend on sign, $\mu_k(-C) = -\mu_k(C)$. Similarly for skew-symmetric matrices, $\mu_k(\widehat{\Lambda}^T) = \mu_k(\widehat{\Lambda})^T$.

Hence the map between exchange graphs $E_q(\Lambda, \widetilde{B})$ and $E_q(\Lambda^T, -\widetilde{B})$ given by ${}^{op} : (\widehat{M}_q, \widehat{B}) \mapsto (\widehat{M}_q^{op}, -\widehat{B})$ is an isomorphism of graphs with edges mapping appropriately since μ_k and ^{op} commute appropriately for seeds. For the roots of unity case, everything follows the same, except now the note on skew-symmetric matrices is necessary. The isomorphism of exchange graphs $E_\epsilon(\Lambda, \widetilde{B})$ and $E_\epsilon(\Lambda^T, -\widetilde{B})$ is given by ${}^{op} : (\widehat{M}_\epsilon, \widehat{\Lambda}, \widehat{B}) \mapsto (\widehat{M}_\epsilon^{op}, \widehat{\Lambda}^T, -\widehat{B})$.

Now for any finite sequence of mutations $\mu_{i_1}...\mu_{i_m}$, we have $\mu_{i_1}...\mu_{i_m}(M^{op}) = (\mu_{i_1}...\mu_{i_m}M)^{op}$. Thus the vector space map from $\Psi : \mathrm{Fract}(\mathcal{T}_q(\Lambda)) \to \mathrm{Fract}(\mathcal{T}_q(\Lambda^T))$, defined by $M(e_i) \mapsto M^{op}(e_i)$ for all i, is an anti-isomorphism that restricts to an anti-isomorphism between $\mathcal{A}_q(M_q, \widetilde{B})$ and $\mathcal{A}_q(M_q^{op}, -\widetilde{B})$. It is also clear that Ψ restricted to $\mathcal{A}_q(M_q, \widetilde{B}, \mathbf{inv})$ induces an anti-isomorphism with $\mathcal{A}_q(M_q^{op}, -\widetilde{B}, \mathbf{inv})$. The root of unity case follows similarly.

□

4.2 Exchange Graphs and the Quantum Laurent Phenomenon

Recall that for a root of unity quantum seed $(M_\epsilon, \Lambda, \widetilde{B})$, we can construct a quantum torus M_q with $\Lambda = \Lambda_{M_q}$ and hence a quantum seed $(M_q, \widetilde{B})$.

Proposition 4.2.1. *For a root of unity quantum seed $(M_\epsilon, \Lambda, \widetilde{B})$, there is a canonical surjection of graphs from $E_q(\Lambda, \widetilde{B})$ onto $E_\epsilon(\Lambda, \widetilde{B})$.*

Proof. Denote the skew-field of $\mathcal{T}_q(\Lambda)$ by $\mathcal{F}_q$. Let $M_q : \mathbb{Z}^N \to \mathcal{F}_q$ be the toric frame given by $M_q(f) = X^f$, so that (M_q, Λ) is a quantum seed. We fix the initial seeds $(M_\epsilon, \Lambda, \widetilde{B})$ in $E_\epsilon(\Lambda, \widetilde{B})$ and $(M_q, \widetilde{B})$ in $E_q(\Lambda, \widetilde{B})$. By the quantum Laurent phenomenon 3.2.8,

$$\mathcal{A}_q(M_q, \widetilde{B}) \hookrightarrow \mathcal{U}_q(M_q, \widetilde{B}) \subseteq \mathcal{T}_q(M_q)$$

so that $\kappa_\epsilon(\mu_{i_m} \cdots \mu_{i_1} M_q(e_j))$ is well defined, where $\kappa_\epsilon : \mathcal{T}_q(M_q) \to \mathcal{T}_\epsilon(M_\epsilon)$ is the canonical projection adjusted by isomorphisms $\mathcal{T}_q(M_q) \simeq \mathcal{T}_q(\Lambda)$ and $\mathcal{T}_\epsilon(M_\epsilon) \simeq \mathcal{T}_\epsilon(\Lambda)$. We will prove that for any sequence of mutations $\mu_{i_m} \cdots \mu_{i_1}$ that

$$\kappa_\epsilon\left(\mu_{i_m} \cdots \mu_{i_1} M_q(e_j)\right) = \mu_{i_m} \cdots \mu_{i_1} M_\epsilon(e_j) \quad \text{for all } j \in [1, N],$$

by induction on m. Hence $\mu_{i_m} \cdots \mu_{i_1}(M_q, \widetilde{B}) = \mu_{j_p} \cdots \mu_{j_1}(M_q, \widetilde{B})$ implies that $\mu_{i_m} \cdots \mu_{i_1}(M_\epsilon, \Lambda, \widetilde{B}) = \mu_{j_p} \cdots \mu_{j_1}(M_\epsilon, \Lambda, \widetilde{B})$, and we have our surjective map from $E_q(\Lambda, \widetilde{B})$ to $E_\epsilon(\Lambda, \widetilde{B})$.

The base case of $m = 1$ is given by Lemma 4.1.3 and (3.2). We will denote $M_q' = \mu_{i_m} \cdots \mu_{i_1} M_q$ and $M_q'' = \mu_{i_{m-1}} \cdots \mu_{i_1} M_q$. Denote M_ϵ' and M_ϵ'' similarly. By (3.2), we have that $M_q'(e_j) = M_q''(e_j)$ for $j \neq i_m$ and that

$$M_q'(e_{i_m}) = M_q''(-e_{i_m} + [c^k]_+) + M_q''(-e_{i_m} - [c^k]_-)$$

where c^k is the k^{th} column of $\widetilde{C} = \mu_{i_{m-1}} \cdots \mu_{i_1} \widetilde{B}$. Hence

$$M_q''(e_{i_m}) M_q'(e_{i_m}) = q^{-\Lambda(-e_{i_m},\ [c^k]_+)/2} M_q''([c^k]_+) + q^{-\Lambda(-e_{i_m},\ -[c^k]_-)/2} M_q''(-[c^k]_-).$$

The left and right hand side are contained in $\mathcal{T}_q(M_q)$ by the quantum Laurent phenomenon, so we may apply κ_ϵ. It follows from induction that

$$M''_\epsilon(e_{i_m})\kappa_\epsilon\left(M'_q(e_{i_m})\right) = \epsilon^{-\Lambda(-e_{i_m},\ [c^k]_+)/2}M''_\epsilon([c^k]_+) + \epsilon^{-\Lambda(-e_{i_m},\ -[c^k]_-)/2}M''_\epsilon(-[c^k]_-).$$

Thus

$$\kappa_\epsilon\left(M'_q(e_{i_m})\right) = M''_\epsilon(-e_{i_m} + [c^k]_+) + M''_\epsilon(-e_{i_m} - [c^k]_-) = M'_\epsilon(e_{i_m}).$$

which finishes the proof. □

The proof of the proposition also gives us that the quantum Laurent phenomenon holds in the root of unity case. We naturally define the upper quantum cluster algebra at a root of unity $\mathcal{U}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}, \mathbf{inv})$ as

$$\mathcal{U}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}, \mathbf{inv}) = \bigcap_{(M_\epsilon,\Lambda,\widetilde{B})\sim(M'_\epsilon,\Lambda',\widetilde{B}')} \mathbb{Z}[\epsilon^{1/2}]\langle M'_\epsilon(e_i), M'_\epsilon(e_j)^{-1} \mid i \in [1,N],\ j \in \mathbf{ex} \sqcup \mathbf{inv}\rangle.$$

for a root of unity quantum seed $(M_\epsilon, \Lambda, \widetilde{B})$ and a choice $\mathbf{inv} \subseteq [1,N] \backslash \mathbf{ex}$.

Theorem 4.2.2. *The quantum cluster algebra $\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}, \boldsymbol{inv})$ is contained in the mixed root of unity quantum torus*

$$\mathbb{Z}[\epsilon^{1/2}]\langle M'_\epsilon(e_i), M'_\epsilon(e_j)^{-1} \mid i \in [1,N],\ j \in \boldsymbol{ex} \sqcup \boldsymbol{inv}\rangle$$

for any quantum seed $(M'_\epsilon, \Lambda', \widetilde{B}') \sim (M_\epsilon, \Lambda, \widetilde{B})$, and we have an inclusion

$$\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}, \boldsymbol{inv}) \hookrightarrow \mathcal{U}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}, \boldsymbol{inv}).$$

Proof. Let $(M'_\epsilon, \Lambda', \widetilde{B}') \sim (M_\epsilon, \Lambda, \widetilde{B})$. As $\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}) = \mathcal{A}_\epsilon(M''_\epsilon, \Lambda'', \widetilde{B}'')$ for any seed $(M''_\epsilon, \Lambda'', \widetilde{B}'') \sim (M_\epsilon, \Lambda, \widetilde{B})$, we need only show that

$$M_\epsilon(e_j) \in \mathbb{Z}[\epsilon^{1/2}]\langle M'_\epsilon(e_i), M'_\epsilon(e_j)^{-1} \mid i \in [1,N],\ j \in \mathbf{ex} \sqcup \mathbf{inv}\rangle.$$

Note for $i \notin \mathbf{ex}$, that $M_\epsilon(e_i) = M'_\epsilon(e_i)$ and $M''_\epsilon(e_j)^{-1} = M'_\epsilon(e_j)^{-1}$ for $j \in \mathbf{inv}$.

Consider a quantum seed $(M'_q, \widetilde{B}', \Lambda')$ such that $\kappa_\epsilon(M'_q) = M'_\epsilon$. Since $(M_\epsilon, \Lambda, \widetilde{B}) \sim (M'_\epsilon, \Lambda', \widetilde{B}')$, there is a finite series of mutations such that $\mu_{i_m} \dots \mu_{i_1} M'_\epsilon = M''_\epsilon$. The quantum Laurent phenomenon, theorem 3.2.8, gives us that

$$\mathcal{A}_q(M, \widetilde{B}, \mathbf{inv}) \hookrightarrow \mathbb{Z}[q^{\pm 1/2}]\langle M'(e_i), M'(e_j)^{-1} \mid i \in [1, N],\ j \in \mathbf{ex} \sqcup \mathbf{inv} \rangle.$$

Hence for some finite sum,

$$\mu_{i_m} \dots \mu_{i_1} M'_q(e_j) = \sum_{f \in \mathbb{Z}^N} p_f(q^{1/2}) M'_q(f),$$

where $p_f(q^{1/2}) \in \mathbb{Z}[q^{\pm 1/2}]$. Applying κ_ϵ it follows from Proposition 4.2.1 that

$$M''_\epsilon(e_j) = \sum_{f \in \mathbb{Z}^N} p_f(\epsilon^{1/2}) M'_\epsilon(f).$$

Thus each generator of $\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}, \mathbf{inv})$ is contained in the mixed quantum torus $\mathbb{Z}[\epsilon^{1/2}]\langle M'(e_i), M'(e_j)^{-1} \mid i \in [1, N],\ j \in \mathbf{ex} \sqcup \mathbf{inv} \rangle \subseteq \mathcal{F}_\epsilon$. □

4.3 Embedding Commutative Cluster Algebras

The main result of this section is the identification of a central $\mathbb{Z}$-subalgebra of $\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B})$ that is isomorphic to the classical cluster algebra $\mathcal{A}(\widetilde{B})$.

Lemma 4.3.1. *If $(M'_\epsilon, \Lambda', \widetilde{B}')$ is mutation-equivalent to $(M_\epsilon, \Lambda, \widetilde{B})$, then the element $M'_\epsilon(e_j)^l \in \mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B})$ is central for any $j \in [1, N]$.*

Proof. We need only show $M_\epsilon(e_j)^l \in Z(\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}))$ for $j \in [1, N]$, since $\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}) = \mathcal{A}_\epsilon(M'_\epsilon, \Lambda', \widetilde{B}')$. Now $M_\epsilon(e_j)^l$ is central in $\mathcal{T}_\epsilon(M_\epsilon)$ as

$$M_\epsilon(e_j)^l M_\epsilon(f) = M_\epsilon(le_j) M_\epsilon(f) = \epsilon^{\Lambda(le_j, f)} M_\epsilon(f) M_\epsilon(le_j) = M_\epsilon(f) M_\epsilon(e_j)^l.$$

Thus, it is central in $\mathrm{Fract}(\mathcal{T}_\epsilon(M_\epsilon))$ and in $\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B})$. □

For a root of unity quantum seed $(M_\epsilon, \Lambda, \widetilde{B})$ and for $j \in \mathbf{ex}$, consider the commutation of elements $M_\epsilon(-e_j + [b^j]_+)$ and $M_\epsilon(-e_k - [b^j]_-)$. The relation in the quantum torus is

$$M_\epsilon(-e_j - [b^j]_-)M_\epsilon(-e_j + [b^j]_+) = \epsilon^{\Lambda(-e_j - [b^j]_-, -e_j + [b^j]_+)} M_\epsilon(-e_j + [b^j]_+)M_\epsilon(-e_j + [b^j]_-).$$

We set $t_j = \Lambda(-e_j - [b^j]_-, -e_j + [b^j]_+)$ as a convenient way to express this exponent.

Lemma 4.3.2. *Let $(M_\epsilon, \Lambda, \widetilde{B})$ be a root of unity quantum seed. Denote by $D = (d_j)$ the $\boldsymbol{ex} \times \boldsymbol{ex}$ diagonal submatrix of $\widetilde{B}^T \Lambda$, which skew-symmetrizes B. Then for $j \in \boldsymbol{ex}$, $t_j = d_j$.*

Proof. We have that

$$\begin{aligned} t_j &= \Lambda(-e_j - [b^j]_-, -e_j + [b^j]_+) \\ &= \Lambda(-e_j, -e_j) + \Lambda(-e_j, [b^j]_+) + \Lambda(-[b^j]_-, -e_j) + \Lambda(-[b^j]_-, [b^j]_+) \\ &= 0 + \Lambda([b^j]_+, e_j) + \Lambda([b^j]_-, e_j) + \Lambda([b^j]_+, [b^j]_-) \\ &= \Lambda(b^j, e_j) + \Lambda([b^j]_+, [b^j]_-) \\ &= d_j + \Lambda([b^j]_+, [b^j]_-) \end{aligned}$$

Now we must consider $\Lambda([b^j]_+, [b^j]_-)$. Note that $b^j - [b^j]_+ = [b^j]_-$ and

$$\Lambda([b^j]_+, [b^j]_-) = \Lambda([b^j]_+, b^j) - \Lambda([b^j]_+, [b^j]_+) = \Lambda([b^j]_+, b^j).$$

Since $b_{jj} = 0$,

$$\begin{aligned} \Lambda([b^j]_+, [b^j]_-) &= \Lambda([b^j]_+, b^j) \\ &= \sum_{b_{ij} > 0} b_{ij} \Lambda(e_i, b^j) \\ &= \sum_{b_{ij} > 0} -b_{ij} \delta_{i,j} d_j \\ &= 0. \end{aligned}$$

Thus $t_j = d_j$. □

We will often require the following condition on our root of unity quantum seed $(M_\epsilon, \Lambda, \widetilde{B})$.

(**C**) Denoting the $\mathbf{ex} \times \mathbf{ex}$ submatrix of $\widetilde{B}^T\Lambda$ which skew-symmetrizes B by $D = (d_j)$, ℓ is coprime to d_k for each $k \in \mathbf{ex}$.

We may view $\mathbf{C}$ as a condition on an odd, primitive ℓ^{th} root of unity ϵ and a compatible pair $(\Lambda, \widetilde{B})$ rather than on a quantum seed. The main use of Lemma 4.3.2 is the following result. The formula appearing should be compared to the mutation relation of (3.1).

Corollary 4.3.3. *Let $(M_\epsilon, \Lambda, \widetilde{B})$ be a root of unity quantum seed satisfying condition $\boldsymbol{C}$.*

Then

$$M_\epsilon(e_k)^\ell \left(\mu_k M_\epsilon(e_k)\right)^\ell = \prod_{b_{ik}>0} (M_\epsilon(e_i)^\ell)^{b_{ik}} + \prod_{b_{ik}<0} (M_\epsilon(e_i)^\ell)^{|b_{ik}|}.$$

Proof. Let $M_q : \mathbb{Z}^N \to \mathrm{Fract}(T_q(\Lambda))$ be the toric frame given by $M_q(f) = X^f$, and consider the quantum seed $(M_q, \widetilde{B})$. As $\mathcal{T}_\epsilon(M_\epsilon) \simeq \mathcal{T}_q(M_q)/(\Phi_\ell(q^{1/2}))$, denote $Y = M_q(-e_k + [b^k]_+)$ and $Z = M_q(-e_k - [b^k]_-)$ in $\mathcal{T}_q(M_q)$. Since $ZY = q^{d_k} YZ$,

$$(Y+Z)^\ell = \sum_{p=0}^{\ell} \begin{bmatrix} \ell \\ p \end{bmatrix}_{q^{d_k}} Y^p Z^{k-p}$$

where

$$\begin{bmatrix} \ell \\ p \end{bmatrix}_x = \frac{(x^\ell - 1)\dots(x-1)}{(x^p-1)\dots(x-1)(x^{\ell-p}-1)\dots(x-1)} \in \mathbb{Z}[x^{\pm 1}]$$

Denote the canonical projection by $\kappa_\epsilon : \mathcal{T}_q(M-q) \to \mathcal{T}_\epsilon(M_\epsilon)$ as before. Since d_k is coprime to ℓ, we may evaluate $\kappa_\epsilon(\left[\begin{smallmatrix} \ell \\ p \end{smallmatrix}\right]_{q^{d_k}})$ using the rational form of $\left[\begin{smallmatrix} \ell \\ p \end{smallmatrix}\right]_{q^{d_k}}$ above. Hence for $1 \le p \le \ell - 1$,

$$\kappa_\epsilon\left(\begin{bmatrix} \ell \\ p \end{bmatrix}_{q^{d_k}}\right) = \frac{(\epsilon^{\ell d_k} - 1)\dots(\epsilon^{d_k}-1)}{(\epsilon^{p d_k}-1)\dots(\epsilon^{d_k}-1)(\epsilon^{(\ell-p)d_k}-1)\dots(\epsilon^{d_k}-1)} = 0.$$

Thus

$$\begin{aligned}
(\mu_k M_\epsilon(e_k))^\ell &= \left(M_\epsilon(-e_k + [b^k]_+) + M_\epsilon(-e_k - [b^k]_-)\right)^\ell \\
&= \kappa_\epsilon\left((Y+Z)^\ell\right) \\
&= M_\epsilon(-e_k + [b^k]_+)^\ell + M_\epsilon(-e_k - [b^k]_-)^\ell \\
&= M_\epsilon(-\ell e_k + \ell[b^k]_+) + M_\epsilon(-\ell e_k - \ell[b^k]_-) \\
&= M_\epsilon(-\ell e_k)\prod_{b_{ik}>0} M_\epsilon(\ell b_{ik} e_i) + M_\epsilon(-\ell e_k)\prod_{b_{ik}<0} M_\epsilon(\ell |b_{ik}| e_i)
\end{aligned}$$

□

Remark 4.3.4. If the constraint that ℓ is coprime to d_k is dropped, then $\kappa_\epsilon\left(\begin{bmatrix}\ell\\p\end{bmatrix}_{q^{d_k}}\right)$ need not be zero as ϵ^{d_k} need not be a primitive ℓ^{th} root of unity. Consider the following example when $\ell = 9$. Let

$$\epsilon^{1/2} = e^{2\pi i/9},\ \ \Lambda = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ \ \widetilde{B} = B = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}.$$

Setting $\mathcal{F} = \mathrm{Fract}\,(\mathcal{T}_\epsilon(\Lambda))$ and $M_\epsilon : \mathbb{Z}^2 \to \mathcal{F}$ by $M_\epsilon(f) = X^f$, we see that (M_ϵ, B, Λ) is a root of unity quantum seed. Here we have

$$B^t\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

and $d_1 = 3$. Then denoting $Y = M_\epsilon(-e_1 + [b^1]_+) = M_\epsilon(-e_1)$ and $Z = M_\epsilon(-e_1 - [b^1]_-) = M_\epsilon(-e_1 + 3e_2)$, a direct computation shows

$$(Y+Z)^9 = Y^9 + 3Y^6Z^3 + 3Y^3Z^6 + Z^9.$$

In a similar way, dropping the odd root of unity condition will result in a failure of the statement. Consider the same case as above except for $\epsilon^{1/2} = i$, a primitive fourth root of unity. Then $\epsilon = -1$ and

$$\begin{aligned}
(Y+Z)^4 &= Y^4 + (1 + \epsilon + 2\epsilon^2 + \epsilon^3 + \epsilon^4)Y^2Z^2 + Z^4 \\
&= Y^4 + 2Y^2Z^2 + Z^4.
\end{aligned}$$

The issue in the even case is that ϵ is a primitive $\frac{\ell}{2}^{th}$ root of unity and not a primitive ℓ^{th} root of unity.

The following theorem holds in light of Corollary 4.3.3.

Theorem 4.3.5. *Suppose $(M_\epsilon, \Lambda, \widetilde{B})$ satisfies condition **C**. Then the $\mathbb{Z}$-subalgebra*

$$\mathbb{Z}\langle\ M'_\epsilon(e_i)^\ell, M'_\epsilon(e_j)^{-\ell} \mid (M'_\epsilon, \widetilde{B}', \Lambda') \sim (M_\epsilon, \Lambda, \widetilde{B}),\ i \in [1, N],\ j \in \boldsymbol{inv}\rangle$$

of $\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}, \boldsymbol{inv})$ is isomorphic to $\mathcal{A}(\widetilde{B}, \boldsymbol{inv})$.

Proof. Since $\mathcal{A}(\{x_1, \ldots, x_N\}, \widetilde{B}, \varnothing)$ is constructed as a subalgebra of $\mathbb{Q}(x_1, \ldots, x_N)$, consider the isomorphism $\phi : \mathbb{Q}(x_1, \ldots, x_N) \to \text{Fract}(\mathbb{Z}[M_\epsilon(e_1)^\ell, \ldots, M_\epsilon(e_N)^\ell])$ given by $x_j \mapsto M_\epsilon(e_j)^\ell$. Corollary 4.3.3 gives us that $\phi(\mu_i(x_j)) = (\mu_i M(e_j))^\ell$ for all $i \in \mathbf{ex}$, $j \in [1, N]$. By induction on the length of the mutation sequence, $\phi(\mu_{i_k} \ldots \mu_{i_1}(x_j)) = (\mu_{i_k} \ldots \mu_{i_1} M_\epsilon(e_j))^\ell$.

Since the generators of $\mathbb{Z}\langle M'_\epsilon(e_i)^\ell \,|\, (M'_\epsilon, \widetilde{B}', \Lambda') \sim (M_\epsilon, \Lambda, \widetilde{B}),\ i \in [1, N]\rangle$ are the images of the generators of $\mathcal{A}(\{x_1, \ldots, x_N\}, \widetilde{B}, \varnothing)$ under the isomorphism ϕ, then we have an isomorphism of $\mathbb{Z}$-algebras. The more general case, when $\mathbf{inv} \neq \varnothing$, is obtained by adjoining the appropriate inverses of frozen variables. □

Corollary 4.3.6. *Let $(\widetilde{B}, \Lambda)$ be a compatible pair and ϵ be a primitive, odd ℓ^{th} root of unity. When condition **C** is satisfied, there are canonical isomorphisms between the exchange graphs $E_q(\Lambda, \widetilde{B})$, $E_\epsilon(\Lambda, \widetilde{B})$, and $E(\widetilde{B})$.*

Proof. In light of the canonical isomorphism between $E_q(\Lambda, \widetilde{B})$ and $E(\widetilde{B})$ of Theorem 3.2.9, only the injectivity of the surjection $E_q(\Lambda, \widetilde{B}) \to E_\epsilon(\Lambda, \widetilde{B})$ from Proposition 4.2.1 is needed. Pick an initial seed $(M_q, \widetilde{B})$ in $E_q(\Lambda, \widetilde{B})$. Label its image in $E_\epsilon(\Lambda, \widetilde{B})$ by $(M_\epsilon, \widetilde{B})$.

Suppose $\mu_{i_m} \ldots \mu_{i_1}(M_q, \widetilde{B})$ and $\mu_{j_p} \ldots \mu_{j_1}(M_q, \widetilde{B})$ are distinct seeds in $E_q(\Lambda, \widetilde{B})$. These correspond to distinct seeds in $E(\widetilde{B})$. Consider their images in $E_\epsilon(\Lambda, \widetilde{B})$.

By the isomorphism of Theorem 4.3.5, $(\mu_{i_m} \dots \mu_{i_1} M_\epsilon(e_k))^\ell$ must be distinct from $(\mu_{j_p} \dots \mu_{j_1} M_\epsilon(e_k))^\ell$ for some k. Thus $\mu_{i_m} \dots \mu_{i_1}(M_\epsilon, \widetilde{B})$ and $\mu_{j_p} \dots \mu_{j_1}(M_\epsilon, \widetilde{B})$ must be distinct seeds.

$\square$

Chapter 5
Discriminants

Discriminants of number fields, introduced by Dedekind in 1871, have proven an invaluable tool in number theory. The notion of a discriminant has also been an important tool in the study of orders and lattices in central simple algebras, see Reiner's book *Maximal Orders* [35]. More recently, this discriminant has found new applications in the noncommutative setting. Bell, Ceken, Palmieri and Zhang used the discriminant as an invariant in determining the automorphism groups of certain polynomial identity algebras [7, 8]. In particular, if the discriminant has certain properties, they showed that the automorphisms must be tame. Bell and Zhang used the discriminant to resolve the Zariski cancellation problem ($A[t] \simeq B[t]$ implies $A \simeq B$) in the case of several classes of Artin–Schelter regular algebras [1]. Brown and Yakimov have shown for a prime affine algebra finitely generated over its center (with some mild conditions), that the zero set of the discriminant ideal equals the complement of the Azumaya locus [6].

5.1 Definition and Motivation

For an algebraic number field K, consider an $\mathbb{Z}$- basis $\{y_1, y_2, \ldots, y_N\}$ of its ring of integers O_K. Let $\mathrm{Tr} = \mathrm{Tr}_{K/\mathbb{Q}}$ be the trace map from K to $\mathbb{Q}$. One definition of the discriminant Δ_K of K is

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$$\Delta_K = \det \begin{pmatrix} \mathrm{Tr}(y_1y_1) & \mathrm{Tr}(y_1y_2) & \dots & \mathrm{Tr}(y_1y_N) \\ \mathrm{Tr}(y_2y_1) & \mathrm{Tr}(y_2y_2) & \dots & \mathrm{Tr}(y_2y_N) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Tr}(y_Ny_1) & \mathrm{Tr}(y_Ny_2) & \dots & \mathrm{Tr}(y_Ny_N) \end{pmatrix}.$$

This definition is easily generalized to algebras with trace.

Definition 5.1.1. An *algebra with trace* is an algebra R with a linear map $\mathrm{tr}\colon R \to R$ such that for all $x, y \in R$:

$$\mathrm{tr}(xy) = \mathrm{tr}(yx), \ \mathrm{tr}(x)y = y\,\mathrm{tr}(x), \ \text{and} \ \mathrm{tr}(\mathrm{tr}(x)y) = \mathrm{tr}(x)\,\mathrm{tr}(y).$$

In particular, the image of *tr* is a subalgebra of the center of R. We can now define the discriminant of a set and the discriminant ideal, for an algebra with such a map.

Definition 5.1.2. The *discriminant of the set* $\mathcal{Y} := \{y_1, \dots, y_N\} \subseteq R$ is defined to be

$$d_N(\mathcal{Y} : \mathrm{tr}) := \det\left([\mathrm{tr}(y_iy_j)]_{i,j=1}^N\right) \in Z(R).$$

The *N-discriminant ideal* $D_N(R/C)$ is the C-submodule of $Z(R)$ generated by

$$d_N(\mathcal{Y} : \mathrm{tr}) \quad \text{for the } N\text{-element subsets } \mathcal{Y} \subseteq R.$$

It is clear that matrix algebras M_N with the traditional trace map Tr are examples of algebras with trace. From these, we can equip more algebras with a trace map. Consider an algebra R that is free and of finite rank N over some central subalgebra $C \subseteq Z(R)$. Then the embedding $R \hookrightarrow M_N(C)$ gives rise to a trace map tr for R:

$$\mathrm{tr}\colon R \hookrightarrow M_n(C) \xrightarrow{\mathrm{Tr}} C \subseteq R$$

Note that this map is independent of the choice of C-basis used to construct the embedding.

Definition 5.1.3. Say R is a free, rank N module over a subalgebra C of the center $Z(R)$. The *discriminant* of R over C is defined by

$$d(R/C) :=_{C^\times} d_N(\mathcal{Y} : \mathrm{tr})$$

where $\mathcal{Y} = \{y_1, \dots, y_N\}$ is a C-basis of R and tr is the canonical trace map.

This definition of $d(R/C)$ is well defined up to associates in C, which we denote by "$=_{C^\times}$". More specifically, if we were to choose a different C-basis $\mathcal{X} := \{x_1, \dots, x_N\}$ of R, then the discriminant would change by multiplication by a unit in C. In particular,

$$d_N(\mathcal{X} : \mathrm{tr}) = \det(b)^2 d_N(\mathcal{Y} : \mathrm{tr}) \tag{5.1}$$

where $b := (b_{ij}) \in M_N(C)$ is the change of bases matrix given by $x_i = \sum_j b_{ij} y_j$.

5.2 Discriminants of Specializations

5.2.1 General Theorems

In this section we prove two general theorems on discriminants and discriminant ideals of algebras obtained as specializations. We also give a recipe about computing discriminants from the first theorem. In the last subsection we obtain extensions of these results to the setting of Poisson orders introduced by Brown and Gordon [5]. In order to keep the exposition more transparent we first prove the results in the more common setting of specializations, and then extend them to the setting of Poisson orders.

Proposition 5.2.1. *Assume that R is an algebra with trace* $\mathrm{tr}\colon R \to C \subseteq Z(R)$ *which is a free module over $C \subseteq Z(R)$ of rank N.*

(i) If ∂ is a derivation of R such that $\operatorname{tr}(\partial x) = \partial \operatorname{tr}(x)$ for all $x \in R$, then

$$\partial d_N(\mathcal{Y} : \operatorname{tr}) = 2 \operatorname{tr}(b) d_N(\mathcal{Y} : \operatorname{tr})$$

for any C-basis $\mathcal{Y} := \{y_1, \dots, y_N\}$ of R, where $b = (b_{ij}) \in M_N(C)$ is the matrix with entries given by $\partial y_i = \sum_j b_{ij} y_j$.

(ii) Let $\operatorname{tr}\colon R \to C$ be the canonical trace map from the embedding $R \subseteq M_N(C)$ associated to any C-basis of R. Then every derivation ∂ of R, satisfying $\partial(C) \subseteq C$, has the property $\operatorname{tr}(\partial x) = \partial \operatorname{tr}(x)$, $\forall x \in R$.

Proof. (i) Since $\mathcal{Y}$ is a C-basis of R, it is also a $C[t]/(t^2)$ of $R[t]/(t^2)$. We can extend our trace map to $\operatorname{tr} : R[t]/(t^2) \to C[t]/(t^2)$, in which case it is clear that the discriminant $d_N(\mathcal{Y} : \operatorname{tr})$ is the same when it is computed for the pair of algebras (R, C) and $(R[t]/(t^2), C[t]/(t^2))$. Consider $(1 + t\partial)\mathcal{Y} := \{(1 + t\partial)y_1, \dots, (1 + t\partial)y_N\}$ which is another $C[t]/(t^2)$-basis of $R[t]/(t^2)$. We can extend our derivation to $R[t]/(t^2)$ by $\partial(t) = 0$, so that $\partial(tx) = t\partial x$ for $x \in R$. Note that $(1 + t\partial)$ is an isomorphism of $R[t]/(t^2)$,

$$(1 + t\partial)(xy) = xy + t\partial(xy) = (1 + t\partial)x \cdot (1 + t\partial)y.$$

Since the derivation ∂ commutes with trace, then $(1+t\partial)$ will as well, $\operatorname{tr}((1+t\partial)x) = \operatorname{tr}(x) + t \operatorname{tr}(\partial x) = (1+t\partial)tr(x)$. It is now clear from the definition of the discriminant that

$$d_N((1 + t\partial)\mathcal{Y} : \operatorname{tr}) = (1 + t\partial)d_N(\mathcal{Y} : \operatorname{tr}).$$

But by (5.1) we get

$$d_N((1 + t\partial)\mathcal{Y} : \operatorname{tr}) = \det(I_N + tb)^2 d_N(\mathcal{Y} : \operatorname{tr}).$$

The first part now follows from the fact that $d_N(\mathcal{Y} : \operatorname{tr}) \in C$ by comparing the coefficients of t in

$$(1 + t\partial)d_N(\mathcal{Y} : \operatorname{tr}) = \det(I_N + tb)^2 d_N(\mathcal{Y} : \operatorname{tr}).$$

(ii) The trace map is independent of basis used for the embedding $R[t]/(t^2) \hookrightarrow M_N(C[t]/(t^2))$. In particular, consider a basis $\mathcal{Y}$ of R over C. Comparing $\mathrm{tr}(x)$ calculated by embedding x with respect to $\mathcal{Y}$ to $\mathrm{tr}((1+t\partial)x)$ calculated by embedding $(1+t\partial)x$ into $M_N(C[t]/(t^2))$ with respect to $(1+t\partial)\mathcal{Y}$, we find that $\mathrm{tr}((1+t\partial)x) = (1+t\partial)\,\mathrm{tr}(x)$ for all $x \in R$. This implies the statement since $\mathrm{tr}(x), \mathrm{tr}(\partial x) \in C$. □

The second part is valid in much greater generality for orders in central simple algebras [35, Ch. 9-10], but we will not need this here.

Let R be an algebra over $\mathbb{K}[q^{\pm 1}]$. For $\epsilon \in \mathbb{K}^\times$, we denote the specialization of R at ϵ by $R_\epsilon := R/(q-\epsilon)R$ and the canonical projection by $\kappa_\epsilon \colon R \to R_\epsilon$. The center $Z(R_\epsilon)$ has a canonical Poisson algebra structure defined by 2.1. Recall that is given as follows. For $z_1, z_2 \in Z(R_\epsilon)$, choose $x_i \in \kappa_\epsilon^{-1}(z_i)$ and set

$$\{z_1, z_2\} := \kappa_\epsilon \left(\frac{x_1 x_2 - x_2 x_1}{q - \epsilon} \right).$$

Proposition 5.2.2. *[12, 28] For every $z \in Z(R_\epsilon)$, the Hamiltonian derivation $y \mapsto \{z, y\}$ of the Poisson algebra $(Z(R_\epsilon), \{\cdot,\cdot\})$ has a lift to an algebra derivation of R_ϵ given by*

$$\partial_x(\kappa_\epsilon(\widetilde{y})) := \kappa_\epsilon \left(\frac{x\widetilde{y} - \widetilde{y}x}{q - \epsilon} \right), \quad x \in \kappa_\epsilon^{-1}(z), \widetilde{y} \in R.$$

Note that $\kappa_\epsilon(x) \in Z(R_\epsilon)$ implies that $\kappa_\epsilon(x\widetilde{y} - \widetilde{y}x) = 0$, so $x\widetilde{y} - \widetilde{y}x \in (q-\epsilon)R$. The lifts coming from different elements $x, x' \in \kappa_\epsilon^{-1}(z)$ differ by the inner derivation of R_ϵ corresponding to $\kappa_\epsilon((x - x')/(q-\epsilon))$.

Theorem 5.2.3. *Let R be a $\mathbb{K}[q^{\pm 1}]$-algebra for a field $\mathbb{K}$ of characteristic 0 and $\epsilon \in \mathbb{K}^\times$. Assume that $R_\epsilon := R/(q-\epsilon)R$ is a free module of finite rank over a Poisson subalgebra C_ϵ of its center.*

(i) Then $d(R_\epsilon/C_\epsilon)$ is a Poisson normal element of $(C_\epsilon, \{\cdot,\cdot\})$.

(ii) Assume, in addition, that C_ϵ is a unique factorization domain as a commutative algebra or a noetherian Poisson unique factorization domain. Then, $d(R_\epsilon/C_\epsilon) = 0$ *or*

$$d(R_\epsilon/C_\epsilon) =_{C_\epsilon^\times} \prod_{i=1}^{m} p_i$$

for some (not necessarily distinct) Poisson prime elements $p_1, \ldots, p_m \in C_\epsilon$.

As usual, a product of 0 primes is considered to be 1. Let $(A, \{\cdot, \cdot\})$ be a Poisson algebra and $u \in A^\times$. Then $a \in A$ is Poisson normal if and only if ua is Poisson normal. The discriminant $d(R_\epsilon/C_\epsilon)$ is defined up to a unit of C_ϵ, but because of this property it does not matter which representative is considered in part (i) of the theorem.

If R_ϵ is an order in a central simple algebra, then the discriminant $d(R_\epsilon/C_\epsilon)$ is nonzero. Specializations of iterated skewpolynomial extensions fall in this class by [14, Theorem 1.5]. In particular, this is true for the families of algebras considered in the next two sections. The nonvanishing of the discriminants of those algebras also follows from the fact that these algebras have filtrations whose associated graded algebras are quasipolynomial algebras, see (6.19); by [8, Proposition 4.10] the leading terms of the discriminants are nonzero. Generally, nonvanishing of discriminants for Cayley–Hamilton algebras follows from the description of the kernel of the trace form in [16, Proposition 3.4 (2)].

By [30, Example 5.12], there are examples of Poisson structures on polynomial algebras that are not Poisson UFDs. In the opposite direction, it is easy to construct Poisson UFDs that are not UFDs as commutative algebras. In other words the two classes of algebras in Theorem 5.2.3 (ii) are not properly contained in each other.

The next result is an explicit version of the statement in Theorem 5.2.3 (i).

Proposition 5.2.4. *In the setting of Theorem 5.2.3 (i), let* $\mathcal{Y} := \{y_1, \dots, y_N\}$ *be a* C_ϵ*-basis of* R_ϵ*. For all* $z \in C_\epsilon$ *and* $x \in \kappa_\epsilon^{-1}(z)$*, we have*

$$\{z, d_N(\mathcal{Y} : \mathrm{tr})\} = 2\,\mathrm{tr}(b(x))d_N(\mathcal{Y} : \mathrm{tr})$$

where $b(x) := (b_{ij}) \in M_N(C_\epsilon)$ *is the matrix with entries given by*

$$\partial_x(y_i) := \sum_j b_{ij} y_j.$$

Proof. Set $\delta := d_N(\mathcal{Y} : \mathrm{tr})$. The proposition follows by combining Propositions 5.2.1 and 5.2.2:

$$\{z, \delta\} = \partial_x \delta = 2\,\mathrm{tr}(b(x))\delta.$$

□

Part (i) of Theorem 5.2.3 follows from Proposition 5.2.4. The second part follows from the first and Proposition 2.3.5.

5.2.2 Scheme for Determining Discriminants

In the situations in which the problem for computing the discriminant $d(R_\epsilon/C_\epsilon)$ was posed, C_ϵ differs only slightly from the full center $Z(R_\epsilon)$. The restriction of the Poisson structure $\{\cdot, \cdot\}$ to C_ϵ is very nontrivial because of the nature of the definition in (2.1). This causes the collection of Poisson primes of C_ϵ to be a small subset of the set of all prime elements of C_ϵ. Theorem 5.2.3 places a strong restriction on the possible form of the discriminant $d(R_\epsilon/C_\epsilon)$. One can fully determine it using the following 4 methods and sets of existing results from Poisson geometry and algebra:

(1) If the algebra R_ϵ is $\mathbb{Z}^n$-graded and C_ϵ is a homogeneous subalgebra, then one can choose a homogeneous C_ϵ-basis $\mathcal{Y}$ of R_ϵ. Since, in this case, the trace map $\mathrm{tr}\colon R_\epsilon \to C_\epsilon$ will be homogeneous, $d_N(\mathcal{Y} : \mathrm{tr})$ will be graded and

$$\deg d_N(\mathcal{Y} : \mathrm{tr}) = 2 \sum_{y \in \mathcal{Y}} \deg y.$$

Furthermore, the grading assumption implies $C_\epsilon^\times = (C_\epsilon)_0^\times$, thus the class of associates for $d(R_\epsilon/C_\epsilon)$ will consist of homogeneous elements of the same degree. The primes in Theorem 5.2.3 (ii) will need to be homogeneous and their degrees will satisfy

$$\sum_{i=1}^{m} \deg p_i = \deg d(R_\epsilon/C_\epsilon) = 2 \sum_{y \in \mathcal{Y}} \deg y.$$

(2) **(A)** The symplectic foliations of the Poisson manifolds coming up in the theory of quantum groups are well understood: the Belavin–Drinfeld Poisson structures [39], the varieties of Lagrangian subalgebras [19, 20], the Poisson homogeneous spaces of non-standard Poisson structures on simple Lie groups [32]. In light of Lemma 2.3.3, these facts can be translated into results for the Poisson primes of the corresponding coordinate rings. The results will be for the case when the base field is $\mathbb{C}$, but the algebras in the theory of quantum groups are defined over $\mathbb{Q}[q^{\pm 1}]$ and by base change one can convert the results to any base field of characteristic 0.

(B) The Poisson primes of all algebras in the very large class of so called Poisson–CGL extensions are described in [26].

Combining (A) and (B), gives a description of the Poisson primes needed for Theorem 5.2.3 (ii) for broad classes of algebras.

(3) If C_ϵ is a domain, Theorem 5.2.3 (i) implies that $d_N(\mathcal{Y} : \mathrm{tr})$ gives rise to a derivation ∂_{discr} of C_ϵ such that

$$\{d_N(\mathcal{Y} : \mathrm{tr}), z\} = d_N(\mathcal{Y} : \mathrm{tr}) \partial_{discr}(z), \quad \forall z \in C_\epsilon.$$

This derivation is explicitly given by Proposition 5.2.4. Every Poisson prime $p \in C_\epsilon$ also gives rise to a derivation ∂_p of C_ϵ such that

$$\{p, z\} = p \partial_p(z), \quad \forall z \in C_\epsilon.$$

The primes in Theorem 5.2.3 need to satisfy

$$\sum_{i=1}^{m} \partial_{p_i} = \partial_{discr}.$$

In fact, when phrased differently, the procedure (3) can be applied to the general situation when the conditions in Theorem 5.2.3 (ii) are not satisfied. Proposition 5.2.4 determines the Poisson brackets of $d_N(\mathcal{Y} : \mathrm{tr})$ with all Hamiltonians on SpecC_ϵ from which one can determine the evolution of $d_N(\mathcal{Y} : \mathrm{tr})$ under all Hamiltonian flows on SpecC_ϵ.

(4) Say A is a filtered algebra that is free of finite rank over a central subalgebra Z with Z-basis $\mathcal{Y} = \{y_1, \ldots, y_N\}$. Then the leading term of the discriminant is closely related to the discriminant of the associated graded algebra gr A, given the nontrivial condition that gr A is a free gr Z-module with a basis gr $\mathcal{Y} = \{\mathrm{lt}\ y_1, \ldots, \mathrm{lt}\ y_N\}$ where lt takes the leading term of an element with respect to the filtration.

Proposition 5.2.5. *[8] Under the notation above, assuming gr A is a free gr Z-module with a basis gr $\mathcal{Y}$. If the discriminant $d(A/Z)$ is nonzero, then*

$$lt\ d(A/Z) = disc(gr\ A/gr\ Z).$$

Filtrations of R or R_ϵ can then be used to obtain leading term results for $d(R_\epsilon/C_\epsilon)$. This puts further restrictions on what Poisson primes can appear in the expansion in Theorem 5.2.3 (ii) by comparing the leading terms of the two sides. In concrete situations these filtrations are different from the gradings in (1).

Remark 5.2.6. In [7, 8, 9] the more general problem of computing discriminants of algebras over integral domains A was considered. One can obtain extensions of Theorems 5.2.3 (ii) and the results below for specializations of algebras R over $A[q^{\pm 1}]$ for an integral domain A as follows. First, apply the theorems to the algebras

$R \otimes_A Q(A)$ over $Q(A)[q^{\pm 1}]$ where $Q(A)$ is the field of fractions of A; this would compute the discriminants $d_N(R_\epsilon \otimes_A Q(A), C_\epsilon \otimes_A Q(A))$. Then compute the leading term of $d(R_\epsilon/C_\epsilon)$ over A using [8, Proposition 4.10], i.e., step (4) in section 5.2, and convert the formula for $d_N(R_\epsilon \otimes_A Q(A), C_\epsilon \otimes_A Q(A))$ to one for $d(R_\epsilon/C_\epsilon)$ by clearing the denominators and introducing the necessary extra factor from A in $d(R_\epsilon/C_\epsilon)$.

5.2.3 Generalizations to Discriminant Ideals

Next we prove two general results for the n-discriminant ideals of specializations of algebras. Recall Definition 5.1.2 and see [35, p. 126] for more background on this notion. These results do not assume any freeness conditions like the one in Theorem 5.2.3.

Theorem 5.2.7. *Let R be a $\mathbb{K}[q^{\pm 1}]$-algebra for an infinite field $\mathbb{K}$ and $\epsilon \in \mathbb{K}^\times$. Assume that C_ϵ is a Poisson subalgebra of the center of $R_\epsilon := R/(q-\epsilon)R$ and that R_ϵ is equipped with a trace function $\mathrm{tr} : R_\epsilon \to C_\epsilon$ which commutes with all derivations ∂ of R_ϵ such that $\partial(C_\epsilon) \subseteq C_\epsilon$.*

Then, for all positive integers n, the discriminant ideal $D_n(R_\epsilon/C_\epsilon)$ is a Poisson ideal of C_ϵ. Furthermore, it has the property that $\partial(D_n(R_\epsilon/C_\epsilon)) \subseteq D_n(R_\epsilon/C_\epsilon)$ for all derivations ∂ of R_ϵ such that $\partial(C_\epsilon) \subseteq C_\epsilon$.

The first statement in the theorem follows from the second in view of Proposition 5.2.2. The second statement of the theorem follows from the next proposition.

Proposition 5.2.8. *Assume that $\mathrm{tr}\colon S \to C \subseteq Z(S)$ is a trace for an algebra S over an infinite field $\mathbb{K}$ which commutes with all derivations ∂ of S such that $\partial(C) \subseteq C$. Then $\partial(D_n(S/C)) \subseteq D_n(S/C)$ for all derivations ∂ of S such that $\partial(C) \subseteq C$.*

Given a positive integer n, define

$$\langle\cdot,\cdot\rangle\colon S^n\times S^n\to C \quad \text{by} \;\; \langle\mathcal{X},\mathcal{Y}\rangle := \det\left([\operatorname{tr}(x_iy_j)]_{i,j=1}^n\right)$$

for $\mathcal{X}:=(x_1,\dots,x_n)$, $\mathcal{Y}:=(y_1,\dots,y_n)\in S^n$. This is obviously a symmetric form on S^n which is C-polylinear in the sense that $\langle(x_1,\dots,cx_k,\dots,x_n),\mathcal{Y}\rangle = c\langle\mathcal{X},\mathcal{Y}\rangle$ for all $c\in C$ and $k\in[1,n]$. For a derivation ∂ of S, define

$$\overline{\partial}(\mathcal{X}) := (\partial(x_1),\dots,\partial(x_n))$$

and

$$\partial(\mathcal{X}) := \sum_{k=1}^{n}(x_1,\dots,\partial(x_k),\dots,x_n).$$

Proof of Proposition 5.2.8. For $p(t)\in S[t]$, denote by $\operatorname{coeff}_{t^i}p(t)\in S$ the coefficient of t^i in $p(t)$. Using several times the differentiation property of ∂ and the assumption that ∂ commutes with tr, gives

$$\partial(d_n(\mathcal{X}:\operatorname{tr})) = 2\langle\mathcal{X},\partial(\mathcal{X})\rangle = \operatorname{coeff}_t(d_n(\mathcal{X}+t\overline{\partial}(\mathcal{X}):\operatorname{tr}))$$

for all $\mathcal{X}\in S^n$. The proposition follows from the fact that $d_n(\mathcal{X}+t\overline{\partial}(\mathcal{X}):\operatorname{tr})\in D_n(S/C)$, $\forall t\in\mathbb{K}$ and the assumption that $\mathbb{K}$ is infinite. □

Theorem 5.2.7 and Proposition 5.2.8 have natural bilinear analogs. Let S be an algebra with trace $\operatorname{tr}\colon S\to C$ where C is a subalgebra of $Z(S)$. Following [8, Definition 1.2 (2)], define the n-th *modified discriminant ideal* $MD_n(S/C)$ to be the ideal of C, generated by

$$\langle\mathcal{X},\mathcal{Y}\rangle \quad \text{for all} \;\; \mathcal{X},\mathcal{Y}\in S^n.$$

Thus, $D_n(S/C)\subseteq MD_n(S/C)$. If S is a free rank N module over C with a basis $\mathcal{X}\in S^N$, then

$$MD_N(S/C) = D_N(S/C) = (d_N(\mathcal{X}:\operatorname{tr}))$$

by an argument similar to the identity (5.1). We refer the reader to [8, Sect. 1] for other properties of modified discriminant ideals.

Theorem 5.2.9. *Assume that k is an infinite field and n is a positive integer.*

(i) Let $\operatorname{tr}\colon S \to C \subseteq Z(S)$ *be a trace for a k-algebra S which commutes with all derivations ∂ of S such that $\partial(C) \subseteq C$. Then $\partial(MD_n(S/C)) \subseteq MD_n(S/C)$ for all derivations ∂ of S such that $\partial(C) \subseteq C$.*

(ii) In the setting of Theorem 5.2.7, for all $\epsilon \in k^\times$, the modified discriminant ideal $MD_n(R_\epsilon/C_\epsilon)$ is a Poisson ideal of C_ϵ with respect to the induced Poisson structure on C_ϵ. Moreover, $\partial(MD_n(R_\epsilon/C_\epsilon)) \subseteq MD_n(R_\epsilon/C_\epsilon)$ for all derivations ∂ of R_ϵ such that $\partial(C_\epsilon) \subseteq C_\epsilon$.

Theorem 5.2.9 (i) is proved analogously to Proposition 5.2.8 using the identity

$$\partial\langle \mathcal{X}, \mathcal{Y}\rangle = \langle \mathcal{X}, \partial(\mathcal{Y})\rangle + \langle \partial(\mathcal{X}), \mathcal{Y}\rangle = \operatorname{coeff}_t\langle \mathcal{X} + t\overline{\partial}(\mathcal{X}), \mathcal{Y} + t\overline{\partial}(\mathcal{Y})\rangle$$

for all $\mathcal{X}, \mathcal{Y} \in S^n$, obtained by applying the differentiation property of ∂ and the assumption that ∂ commutes with tr. The second part of the theorem follows from the first.

5.2.4 In the Setting of Poisson Orders

We finish the section with a generalization of the results in sections 5.2.1 and 5.2.3 to the framework of Poisson orders introduced by Brown and Gordon.

Definition 5.2.10. [5] Assume that S is an affine algebra over a field k of characteristic 0 which is a finite module over a central subalgebra C. The algebra S is called a *Poisson C-order* if there is a k-linear map $\Delta\colon C \to \operatorname{Der}_k(S)$ such that

1. C is stable under Δ_z for all $z \in C$ and

2. the induced bracket $\{\cdot, \cdot\}$ on C given by $\{z_1, z_2\} := \Delta_{z_1}(z_2)$ turns C into a Poisson algebra.

Proposition 5.2.2 implies that, in the setting of the proposition, R_ϵ is a Poisson $Z(R_\epsilon)$-order. The map Δ is given as follows. Choose a $\mathbb{K}$-linear map $\omega\colon Z(R_\epsilon) \to R$ such that $\kappa_\epsilon \omega = \mathrm{id}_{Z(R_\epsilon)}$ and set $\Delta_z = \partial_{\omega(z)}$.

Theorem 5.2.11. *Let S be a $\mathbb{K}$-algebra over a field $\mathbb{K}$ (of characteristic 0) which is a Poisson C-order. Let $\mathrm{tr}\colon S \to C$ be a trace map that commutes with all derivations of S that preserve C.*

(i) If S is a free C-module (of finite rank), then $d(S/C)$ is a Poisson normal element of C. If, in addition, C is a unique factorization domain as a commutative algebra or a noetherian Poisson unique factorization domain, then either $d(S/C) = 0$ or

$$d(S/C) =_{C^\times} \prod_{i=1}^{m} p_i$$

for some (not necessarily distinct) Poisson prime elements $p_1, \ldots, p_m \in C$.

(ii) For all positive integers n, the discriminant and modified discriminant ideals $D_n(S/C)$ and $MD_n(S/C)$ are Poisson ideals of C. Furthermore, $\partial(D_n(S/C)) \subseteq D_n(S/C)$ and $\partial(MD_n(S/C)) \subseteq MD_n(S/C)$ for all derivations ∂ of S such that $\partial(C) \subseteq C$.

The theorem follows from Propositions 5.2.1, 2.3.5 and 5.2.8 and Theorem 5.2.9 (i).

5.3 Discriminants of Quantum Cluster Algebras

We introduce certain subalgebras of quantum cluster algebras at roots of unity that contain canonical central subalgebras. In special cases, one of these subalgebras might be the whole quantum cluster algebra with the canonical central subalgebra corresponding to the classical cluster algebra (with extended scalars).

Let $\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}, \mathbf{inv})$ be a quantum cluster algebra with exchange graph $E_\epsilon(\Lambda, \widetilde{B})$. Let Θ be a collection of seeds in $E_\epsilon(\Lambda, \widetilde{B})$, in other words $\Theta \subseteq E_\epsilon(\Lambda, \widetilde{B})_0$. We define $\mathcal{A}_\epsilon(\Theta, \mathbf{inv})$ to be the $\mathbb{Z}[\epsilon^{1/2}]$-subalgebra of $\mathcal{A}_\epsilon(M_\epsilon, \Lambda, \widetilde{B}, \mathbf{inv})$ generated by $M'_\epsilon(e_j)$ for $j \in [1, N]$ and $M'_\epsilon(e_i)^{-1}$ for $i \in \mathbf{inv}$, for all $(M'_\epsilon, \Lambda', \widetilde{B}') \in \Theta$.

We define $C_\epsilon(\Theta, \mathbf{inv})$ to be the $\mathbb{Z}[\epsilon^{1/2}]$-subalgebra generated by $M'_\epsilon(e_j)^\ell$ for $j \in [1, N]$ and $M'_\epsilon(e_i)^{-\ell}$ for $i \in \mathbf{inv}$, for all $(M'_\epsilon, \Lambda', \widetilde{B}') \in \Theta$. Assuming that $(M_\epsilon, \Lambda, \widetilde{B})$ meets condition $\mathbf{C}$, then $C_\epsilon(\Theta, \mathbf{inv})$ is a central subalgebra of $\mathcal{A}_\epsilon(\Theta, \mathbf{inv})$. When the context is clear, we will drop $\mathbf{inv}$ from the notation and write $\mathcal{A}_\epsilon(\Theta)$ or $C_\epsilon(\Theta)$.

Definition 5.3.1. A finite set of seeds Θ that meets the following condition is called a *nerve*:

$$(\mathbf{N}) \begin{cases} \text{The subgraph in } E_\epsilon(\Lambda, B) \text{ induced by } \Theta \text{ is connected.} \\ \text{For each mutable direction } k \in \mathbf{ex}, \text{ there are at least two seeds in } \Theta \\ \text{mutation equivalent by } \mu_k. \end{cases}$$

The concept of nerves was introduced in [21] for a practical way of specifying a quasi-homomorphism of a cluster algebra. A basic example of a nerve would be a star neighborhood in $E_\epsilon(\Lambda, B)$ of any particular seed.

Theorem 5.3.2. *Let $\epsilon^{1/2}$ be an ℓ^{th} root of unity. Suppose $(M_\epsilon, \Lambda, \widetilde{B})$ satisfies condition $\boldsymbol{C}$ and that Θ is a nerve. Suppose $\mathcal{A}_\epsilon(\Theta)$ is a free $C_\epsilon(\Theta)$-module of rank N. Then the discriminant of $\mathcal{A}_\epsilon(\Theta)$ over $C_\epsilon(\Theta)$ is given as a product of noninverted frozen variables raised to the ℓ^{th} power,*

$$d\left(\mathcal{A}_\epsilon(\Theta)/C_\epsilon(\Theta)\right) =_{C_\epsilon(\Theta)^\times} \ell^{(N\ell^N)} \prod_{i \in [1,N] \setminus \boldsymbol{ex} \sqcup \boldsymbol{inv}} \left(M_\epsilon(e_i)^\ell\right)^{a_i} \quad \text{for some integers } a_i.$$

Proof. Suppose $(M_\epsilon, \Lambda, \widetilde{B}) \in \Theta$, noting that all seeds mutation equivalent will satisfy condition C. From a result on discriminants of skew polynomial algebras [8, Proposition 2.8], we deduce that

$$d(\mathcal{T}_\epsilon(\Lambda)/\mathbb{Z}[\epsilon^{1/2}][x_i^{\pm 1}]_{i=1}^N) =_{(\mathbb{Z}[\epsilon^{1/2}][x_i^{\pm 1}]_{i=1}^N)^\times} \ell^{(N\ell^N)}.$$

where x_i is identified with X_i^ℓ. Note that our algebras have the following isomorphisms,

$$\mathcal{A}_\epsilon(\Theta)[M_\epsilon(e_i)^{-\ell}]_{i=1}^N = \mathcal{T}_\epsilon(M_\epsilon) \simeq \mathcal{T}_\epsilon(\Lambda)$$

and similarly

$$C_\epsilon(\Theta)[M_\epsilon(e_i)^{-\ell}]_{i=1}^N \simeq \mathbb{Z}[\epsilon^{1/2}][x_i^{\pm 1}]_{i=1}^N.$$

By inverting the ℓ^{th} powers of the cluster variables, the discriminant $d(\mathcal{A}_\epsilon(\Theta)/C_\epsilon(\Theta))$ in comparison to $d(\mathcal{T}_\epsilon(\Lambda)/\mathbb{Z}[\epsilon^{1/2}][x_i^{\pm 1}]_{i=1}^N)$ was reduced to $\ell^{(N\ell^N)}$. Hence $d(\mathcal{A}_\epsilon(\Theta)/C_\epsilon(\Theta))$ must be $\ell^{(N\ell^N)}$ multiplied by a unit of $(C_\epsilon(\Theta)[M_\epsilon(e_i)^{-\ell}]_{i=1}^N)^\times$. Hence for some integers a_i,

$$d(\mathcal{A}_\epsilon(\Theta)/C_\epsilon(\Theta)) =_{C_\epsilon(\Theta)^\times} \ell^{(N\ell^N)} \prod_{i\in[1,N]} \left(M_\epsilon(e_i)^\ell\right)^{a_i}$$

We will assume the convention $a_i = 0$ for $i \in \mathbf{inv}$. Thus all a_i are non-negative. For some $k \in \mathbf{ex}$, the seed $\mu_k(M_\epsilon, \Lambda, \widetilde{B})$ lies in Θ. We could have similarly computed the discriminant as

$$d(\mathcal{A}_\epsilon(\Theta)/C_\epsilon(\Theta)) =_{C_\epsilon(\Theta)^\times} \prod_{i\in[1,N]} \left(\mu_k M_\epsilon(e_i)^\ell\right)^{a_i'}.$$

Since $\mu_k M_\epsilon(e_i) = M_\epsilon(e_i)$ for $i \neq k$ and $\mu_k M_\epsilon(e_k) = M_\epsilon(-e_k + [b^k]_+) + M_\epsilon(-e_k - [b^k]_-)$ is not a monomial in terms of $M_\epsilon(e_j)$'s, we must have

$$a_k = 0 = a_k'$$
$$a_i = a_i' \text{ for } i \neq k.$$

If we compute $d(\mathcal{A}_\epsilon(\Theta)/C_\epsilon(\Theta))$ in terms of any $(M_\epsilon', \Lambda', \widetilde{B}') \in \Theta$, then $M_\epsilon'(e_k)$ is absent from the determinant expression since the nerve Θ is connected. Because every possible mutation direction $k' \in \mathbf{ex}$ occurs as $\mu_{k'}(M_\epsilon', \Lambda', \widetilde{B}') = (M_\epsilon'', \Lambda'', \widetilde{B}'')$ for some seeds $(M_\epsilon', \Lambda', \widetilde{B}')$ and $(M_\epsilon'', \Lambda'', \widetilde{B}'')$ in Θ, then $a_k = 0$ for all $k \in \mathbf{ex}$ and only frozen variables occur in the discriminant. □

Remark 5.3.3. If we extend scalars for the quantum cluster algebra from $\mathbb{Z}[\epsilon^{1/2}]$ to another domain, we will only have to adjust the discriminant by the integer outside the product of non-inverted frozen variables. Say we extend scalars of $\mathcal{A}_\epsilon(\Theta)$ and $C_\epsilon(\Theta)$ to $\mathbb{Q}(\epsilon^{1/2})$, i.e. set

$$\mathcal{A}_\epsilon(\Theta)_{\mathbb{Q}(\epsilon^{1/2})} = \mathcal{A}_\epsilon(\Theta) \otimes_{\mathbb{Z}} \mathbb{Q},$$

$$C_\epsilon(\Theta)_{\mathbb{Q}(\epsilon^{1/2})} = C_\epsilon(\Theta) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Then the discriminant of this algebra is given by

$$d(\mathcal{A}_\epsilon(\Theta)_{\mathbb{Q}[\epsilon^{1/2}]}/C_\epsilon(\Theta)_{\mathbb{Q}[\epsilon^{1/2}]}) = \prod_{i\in[1,N]\setminus\mathbf{ex}\sqcup\mathbf{inv}} \left(\mu_k M_\epsilon(e_i)^\ell\right)^{a_i}$$

up to a multiplication by a unit in $C_\epsilon(\Theta)^\times_{\mathbb{Q}[\epsilon^{1/2}]}$ as usual.

Chapter 6
Quantum Schubert Cells

In this section we prove an explicit formula for the discriminants of quantum Schubert cell algebras at roots of unity for all symmetrizable Kac–Moody algebras $\mathfrak{g}$ and Weyl group elements $w \in W$. Quantum Schubert cells $\mathcal{U}^-[w]$ are defined as a subalgebra of $\mathcal{U}_q(\mathfrak{g})$, but can also be seen as a deformation of $\mathcal{U}(\mathfrak{n}_- \cap w(\mathfrak{n}_+))$ for the nilradicals $\mathfrak{n}_\pm$ of a pair of opposite Borel algebras of $\mathfrak{g}$. The specialization of these algebras to roots of unity $\mathcal{U}_\epsilon^-[w]$ were studied by De Concini, Kac, and Procesi [14]. In particular, a central subalgebra $C_\epsilon^-[w]$ was identified, over which $\mathcal{U}_\epsilon^-[w]$ is a free module of finite rank. It is in the context of this central subalgebra that we will find the discriminants of quantum Schubert cells. To illustrate each method given in chapter 5, these discriminants will be found first by using the Poisson structure and second by using the cluster structure of these algebras.

6.1 Background

For two subgroups B_1 and B_2 of a group G, we will denote by $g \cdot B_2$ the elements of G/B_2, by $B_1 g \cdot B_2$ the B_1-orbit of $g \cdot B_2 \in G/B_2$, and by $B_1 g B_2$ the corresponding double coset in G. Let $[c_{ij}] \in M_r(\mathbb{Z})$ be a Cartan matrix (of finite type for now) and $\mathcal{U}_q(\mathfrak{g})$ be the corresponding quantized universal enveloping algebra defined over $\mathbb{K}(q)$ where $\mathbb{K}$ is a field of characteristic 0. We will follow the notation of [29] except for denoting the Chevalley generators of $\mathcal{U}_q(\mathfrak{g})$ by $E_i, F_i, K_i^{\pm 1}$, $i \in [1, r]$ (in [29] they were indexed by the simple roots of $\mathfrak{g}$). Continuing to follow the notation of [29], we denote the subalgebras generated by $\{E_i\}$, $\{K_i\}$, and $\{F_i\}$ respectfully

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by $\mathcal{U}^+$, $\mathcal{U}^0$, and $\mathcal{U}^-$. The Hopf subalgebras generated by $\{E_i, K_i\}$ and $\{F_i, K_i\}$ are denoted as $\mathcal{U}^{\geq}$ and $\mathcal{U}^{\leq}$. Note that some authors denote these subalgebras by $\mathcal{U}_q(\mathfrak{n}_+)$, $\mathcal{U}_q(\mathfrak{h})$, $\mathcal{U}_q(\mathfrak{n}_-)$, $\mathcal{U}_q(\mathfrak{b}_+)$, and $\mathcal{U}_q(\mathfrak{b}_-)$.

Let W be the Weyl group of $[c_{ij}]$, $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ the set of simple roots (often not denoted by Π to limit confusion with Poisson structures), and $\{s_1, \ldots, s_r\} \subseteq W$ the corresponding set of simple reflections. Denote by $\langle \cdot, \cdot \rangle$ the W-invariant bilinear form on $\bigoplus_j \mathbb{Q}\alpha_i$ normalized by $\|\alpha_i\|^2 = 2$ for short roots α_i. Set $q_i := q^{\|\alpha_i\|/2}$.

Given a Weyl group element w and a reduced expression

$$w = s_{i_1} \ldots s_{i_N},$$

consider the roots $\beta_j = s_{i_1} \ldots s_{i_{j-1}}(\alpha_{i_j})$, $j \in [1, N]$. They are precisely the roots of the nilpotent Lie algebra $\mathfrak{n}_+ \cap w(\mathfrak{n}_-)$ where $\mathfrak{n}_\pm$ are the nilradicals of a pair of opposite Borel subalgebras of $\mathfrak{g}$. The quantum Schubert cell algebras $\mathcal{U}^-[w]$ are the $\mathbb{K}[q^{\pm 1}]$-subalgebras of $\mathcal{U}_q(\mathfrak{g})$ generated by the quantum root vectors

$$F_{\beta_j} := T_{i_1} \ldots T_{i_{j-1}}(F_{i_j}), \quad j \in [1, N] \tag{6.1}$$

where T_i refers to the action [29, 33] of the braid group of W on $\mathcal{U}_q(\mathfrak{g})$. In particular,

$$\begin{aligned} T_i(E_i) &= -F_i K_i, & T_i(E_j) &= ad(E_i^{-\langle \alpha_i, \alpha_j \rangle})(E_j) \text{ for } j \neq i, \\ T_i(F_i) &= -K_i^{-1}, E_i & T_i(F_j) &= ad(F_i^{-\langle \alpha_i, \alpha_j \rangle})(F_j) \text{ for } j \neq i, \\ T_i(K_\lambda) &= K_{s_i(\lambda)} \end{aligned}$$

where $K_\lambda = K_1^{a_1} K_2^{a_2} \ldots K_r^{a_r}$ for $\lambda = \sum_{i=1}^r a_i \alpha_i$. Similarly, we have the quantum Schubert cells $\mathcal{U}^+[w]$ generated by root vectors

$$E_{\beta_j} := T_{i_1} \ldots T_{i_{j-1}}(E_{i_j}), \quad j \in [1, N].$$

We will restrict our attention to $\mathcal{U}^-[w]$ for now. Of course, all statements for $\mathcal{U}^-[w]$ have appropriate corresponding statements for $\mathcal{U}^+[w]$. We will denote, especially in section 6.4, $T_{i_1} \dots T_{i_{k-1}}$ by $T_{w_{[1,k-1]}}$ or $T_{w_{\leq k-1}}$. More generally, for $w' = s_{i'_1} \dots s_{i'_m} \in W$, we write $T_{w'_{[j,k]}} = T_{i'_j} \dots T_{i'_k}$.

The algebras $\mathcal{U}^-[w]$ do not depend on the choice of a reduced expression of w [33, 14]. Their generators satisfy the Levendorskii–Soibelman straightening relations: for $1 \leq j < m \leq N$,

$$F_{\beta_m} F_{\beta_j} - q^{-\langle \beta_m, \beta_j \rangle} F_{\beta_j} F_{\beta_m} = \sum_{k_{j+1}, \dots, k_{n-1} \in \mathbb{N}} t_{k_{j+1}, \dots, k_{n-1}} F_{\beta_{j+1}}^{k_{j+1}} \dots F_{\beta_{n-1}}^{k_{n-1}} \tag{6.2}$$

for some $t_{k_{j+1}, \dots, k_{n-1}} \in \mathbb{Q}[q^{\pm 1}]$. As a consequence, $\mathcal{U}^-[w]$ has the PBW basis

$$\left\{ F_{\beta_1}^{k_1} \dots F_{\beta_N}^{k_N} \mid k_1, \dots, k_N \in \mathbb{N} \right\}. \tag{6.3}$$

Remark 6.1.1. The algebras $\mathcal{U}^-[w]$ can be defined as the algebras with generators $F_{\beta_1}, \dots, F_{\beta_N}$ and relations (6.2). In particular, they are defined over $\mathbb{Q}[q^{\pm 1}]$ and their specializations at a root of unity ϵ are defined over $\mathbb{Q}(\epsilon)$. All formulas for discriminants proved for one field of characteristic 0 are valid for any other field of characteristic 0 by a direct base change.

Let $\epsilon \in \mathbb{K}$ be a primitive ℓ^{th} root of unity. Denote the specialization $\mathcal{U}_\epsilon^-[w] := \mathcal{U}^-[w]/(q-\epsilon)\mathcal{U}^-[w]$ and the canonical projection $\kappa_\epsilon \colon \mathcal{U}^-[w] \to \mathcal{U}_\epsilon^-[w]$. Set

$$z_{\beta_j} := (\epsilon^{\|\alpha_{i_j}\|/2} - \epsilon^{-\|\alpha_{i_j}\|/2})^\ell \kappa_\epsilon(F_{\beta_j})^\ell \in \mathcal{U}_\epsilon^-[w], \quad j \in [1, N]. \tag{6.4}$$

Denote by $C_\epsilon^-[w]$ the $\mathbb{K}$-subalgebra of $\mathcal{U}_\epsilon^-[w]$ generated by z_{β_j}, $j \in [1, N]$.

Theorem 6.1.2. *[12] For all integers $\ell > 1$, $C_\epsilon^-[w]$ is a subalgebra of $Z(\mathcal{U}_\epsilon^-[w])$. It is isomorphic to the polynomial algebra in the generators z_{β_j}, $j \in [1, N]$ and is independent of the choice of reduced expression of w.*

The last part was stated in [12, Proposition 3.3] for the longest element of W; the proof works for all $w \in W$. The algebra $\mathcal{U}_\epsilon^-[w]$ is a free $C_\epsilon^-[w]$-module with basis

$$\mathcal{Y} := \{\kappa_\epsilon(F_{\beta_1})^{m_1} \dots \kappa_\epsilon(F_{\beta_N})^{m_N} \mid m_1, \dots, m_N \in [0, \ell - 1]\}. \tag{6.5}$$

This and the second part of Theorem 6.1.2 follow from the PBW basis (6.3).

Denote by G the split, connected, simply connected algebraic $\mathbb{K}$-group with Lie algebra $\mathfrak{g}$. Let $B_\pm$ be a pair of opposite Borel subgroups of G and $U_\pm$ be their unipotent radicals. Let $\{e_i, f_i\}$ be a set of Chevalley generators of $\mathfrak{g}$ that generate $\mathrm{Lie}\,(U_\pm)$. Denote by $\dot{s}_i$ the representatives of s_i in the normalizer of the maximal torus $H := B_+ \cap B_-$ of G given by

$$\dot{s}_i := \exp(f_i) \exp(-e_i) \exp(f_i).$$

They are extended (in a unique way) to Tits' representatives of the elements $u \in W$ in $N_G(H)$ by setting $\dot{v} := \dot{u}\dot{s}_i$ if $v = us_i$ and $l(v) = l(u) + 1$ where $l\colon W \to \mathbb{N}$ is the length function. For a positive root β of $\mathfrak{g}$, denote the root vectors

$$e_\beta = \mathrm{Ad}_{\dot{u}}(e_{\alpha_i}) \quad \text{and} \quad f_\beta := \mathrm{Ad}_{\dot{u}}(f_{\alpha_i}) \tag{6.6}$$

where $u \in W$ is any element satisfying $\beta = u(\alpha_i)$ (it is well known that this does not depend on the choice of $u \in W$ and α_i).

Consider the Schubert cell $B_+ w \cdot B_+$ in the full flag variety G/B_+ and the isomorphisms

$$C_\epsilon^-[w] \simeq \mathbb{K}[U_+ \cap w(U_-)] \simeq \mathbb{K}[B_+ w \cdot B_+]. \tag{6.7}$$

The first one is given by

$$f \in \mathbb{K}[U_+ \cap w(U_-)] \quad \mapsto \quad f\big(\exp(z_{\beta_1} e_{\beta_1}) \dots \exp(z_{\beta_N} e_{\beta_N})\big) \in C_\epsilon^-[w]$$

and the second is the pull-back map for the algebraic isomorphism $B_+ w \cdot B_+ \simeq U_+ \cap w(U_-)$, given by $g \in U_+ \cap w(U_-) \mapsto gw \cdot B_+$. (The first isomorphism is the presentation of $U_+ \cap w(U_-)$ as the product of the one-parameter unipotent subgroups of G corresponding to the roots $\beta_1, \dots, \beta_N$ whose coordinate rings are identified with $\mathbb{K}[z_{\beta_j}]$.)

Denote by $\mathcal{P}^+$ the set of dominant integral weights of $\mathfrak{g}$ and by $\{\varpi_1, \dots, \varpi_r\}$ the set of fundamental weights. Let

$$\rho = \varpi_1 + \cdots + \varpi_r.$$

For $\lambda \in \mathcal{P}^+$ and $u, v \in W$, one defines the generalized minors

$$\Delta_{u\lambda, v\lambda} \in \mathbb{K}[G]$$

as follows. Consider the irreducible highest weight $\mathfrak{g}$-module $L(\lambda)$ with highest weight λ. Let b_λ be a highest weight vector of $L(\lambda)$ and ξ_λ be a vector in the dual weight space, normalized by $\langle \xi_\lambda, b_\lambda \rangle = 1$. Set

$$\Delta_{u\lambda, v\lambda}(g) := \langle \xi_\lambda, \dot{u}^{-1} g \dot{v} b_\lambda \rangle, \quad g \in G.$$

Finally, recall that the support of a Weyl group element w is defined by

$$\mathcal{S}(w) := \{i \in [1, r] \mid s_i \text{ occurs in one and thus in any reduced expression of } w\}.$$

Theorem 6.1.3. *Let $\mathfrak{g}$ be a simple Lie algebra, w a Weyl group element and $\ell > 2$ an odd integer which is $\neq 3$ in the case of G_2. Assume that $\mathbb{K}$ is a field of characteristic 0 which contains a primitive ℓ^{th} root of unity ϵ. Then*

$$d(\mathcal{U}_\epsilon^-[w]/C_\epsilon^-[w]) =_{\mathbb{K}^\times} \Delta^L_{\rho, w\rho} =_{\mathbb{K}^\times} \prod_{i \in \mathcal{S}(w)} \Delta^L_{\varpi_i, w\varpi_i}$$

in the first isomorphism in (6.7) *where $L := \ell^{N-1}(\ell - 1)$.*

More explicitly, under the first isomorphism in (6.7), the minor $\Delta_{\lambda, w\lambda}$ corresponds to

$$\langle \xi_\lambda, \exp(z_{\beta_1} e_{\beta_1}) \dots \exp(z_{\beta_N} e_{\beta_N}) \dot{w} b_\lambda \rangle. \tag{6.8}$$

The equality between the second and third term in Theorem 6.1.3 follows from the product property

$$\Delta_{u\lambda,v\lambda}\Delta_{u\mu,v\lambda} = \Delta_{u(\lambda+\mu),v(\lambda+\mu)}, \quad u,v \in W, \lambda,\mu \in \mathcal{P}^+$$

and the fact that $\Delta_{\varpi_i,w\varpi_i}|_{U_+\cap w(U_-)} = 1$ for $i \notin \mathcal{S}(w)$.

The algebras $\mathcal{U}_\epsilon^-[w]$ and $C_\epsilon^-[w]$ are defined over $\mathbb{Q}(\epsilon)$ and the structure constants for the $C_\epsilon^-[w]$-action on the basis $\mathcal{Y}$ belong to $\mathbb{Q}(\epsilon)$ because of (6.2). This implies that it is sufficient to prove Theorem 6.1.3 for any extension $\mathbb{K}$ of $\mathbb{Q}(\epsilon)$.

In sections 6.4 and 6.5, we will generalize the discriminant result to any symmetrizable Kac–Moody. In particular, we will give the discriminant as a product of the frozen variables which are quantum minors rather than the generalized minors. The ℓ^{th} powers of these quantum minors can be matched up with generalized minors to reconcile the two theorems, see Remark 6.5.4 for more details.

6.2 Poisson Structure

Here and in 6.3, we will assume that $\mathbb{K} = \mathbb{C}$ to avoid technicalities with Poisson manifolds over general fields of characteristic 0. (All arguments work for general fields of characteristic 0.)

For a G-action on a manifold M, denote by $\chi\colon \mathfrak{g} \to \Gamma(M, TM)$ the corresponding infinitesimal action and its extension to multi-tangent vectors, $\wedge^\bullet\mathfrak{g} \to \Gamma(M, \wedge^\bullet TM)$.

Let Δ_+ denote the set of positive roots of $\mathfrak{g}$. Recall the definition of the root vectors (6.6) of $\mathfrak{g}$. The standard r-matrix for $\mathfrak{g}$ is the element

$$r := \sum_{\beta\in\Delta_+} \frac{\|\beta\|^2}{2} e_\beta \wedge f_\beta \in \wedge^2\mathfrak{g}. \tag{6.9}$$

Define the Poisson bivector field

$$\pi := -\chi(r) \in \Gamma(G/B_+, \wedge^2 T(G/B_+)),$$

called the standard Poisson structure of the flag variety G/B_+. Denote the open Richardson varieties

$$R_{v,w} := B_- v \cdot B_+ \cap B_+ w \cdot B_+ \subseteq G/B_+, \quad v \le w \in W,$$

see [4, 17, 36]. We will make repeated use of the following facts:

(A) The H-orbits of symplectic leaves of $(G/B_+, \pi)$ are $R_{v,w}$.

(B) $\overline{R_{v,w}} \cap B_+ w \cdot B_+ = \bigsqcup_{u \in W, u \le v} R_{u,w}$,

see [20, Theorem 4.14] and [36, Theorem 3.2].

Theorem 6.2.1. *The composition of the two isomorphisms in* (6.7) *is an isomorphism of Poisson algebras*

$$(C_\epsilon^-[w], \{\cdot,\cdot\}) \to (\mathbb{C}[B_+ w \cdot B_+], \ell^2 \epsilon^{-1} \{\cdot,\cdot\}_\pi).$$

For the proof of Theorem 6.2.1 we will need several constructions for Poisson algebraic groups and Poisson homogeneous spaces, see [10, Ch. 1] for background. The standard Poisson structure on G is defined by

$$\pi_{\mathrm{st}} := \chi_R(r) - \chi_L(r) \in \Gamma(G, \wedge^2 TG),$$

in terms of (6.9). Here χ_R and χ_L denote the infinitesimal actions for the actions of G on itself on the right and the left. The groups $B_\pm$ are Poisson algebraic subgroups of (G, π_{st}). The r-matrix for the Drinfeld double of the Poisson algebraic group (G, π_{st}) is

$$r_D := \sum_{\beta \in \Delta_+} \frac{\|\beta\|^2}{2} \left((e_\beta, e_\beta) \wedge (f_\beta, 0) - (f_\beta, f_\beta) \wedge (0, e_\beta) \right) + \frac{1}{2} \sum_i (h_i, h_i) \wedge (h_i, -h_i) \in (\mathfrak{g} \oplus \mathfrak{g})^{\otimes 2}$$

where $\{h_i\}$ is an orthonormal basis of $\mathrm{Lie}\,(H)$ with respect to the bilinear form $\langle \cdot, \cdot \rangle$, extending the one in section 6.1. The double of (G, π_{st}) is the group $G \times G$

equipped with the Poisson structure

$$\pi_D := \chi_R(r_D) - \chi_L(r_D).$$

The group

$$G^* := \{(u_- h^{-1}, u_+ h) \mid u_\pm \in U_\pm, h \in H\} \subseteq G \times G$$

is a Poisson submanifold of $(G \times G, \pi_D)$; the pair $(G^*, -\pi_D)$ is the dual Poisson algebraic group of $(G, \pi_{\rm st})$. The projection onto the first component $\eta\colon (G \times G, \pi_D) \to (G, \pi_{\rm st})$, $\eta(g_1, g_2) = g_1$ is Poisson. It restricts to the Poisson quotient map

$$\eta\colon (G^*, -\pi_D) \to (B_-, -\pi_{\rm st}). \tag{6.10}$$

Denote by τ and θ the unique anti-automorphism and automorphism of G which on the Lie algebra level are given by

$$\tau(e_i) = e_i, \tau(f_i) = f_i, \tau(\alpha_i^\vee) = -\alpha_i^\vee \quad \text{and} \quad \theta(e_i) = f_i, \theta(f_i) = e_i, \theta(\alpha_i^\vee) = -\alpha_i^\vee$$

for the Chevalley generators of $\mathfrak{g}$. It follows from the definition of $\pi_{\rm st}$ that $\theta\tau\colon (G, \pi_{\rm st}) \to (G, -\pi_{\rm st})$ is a Poisson map. This gives rise to the Poisson isomorphism

$$\theta\tau\colon (B_-, -\pi_{\rm st}) \xrightarrow{\simeq} (B_+, \pi_{\rm st}). \tag{6.11}$$

The commutation relations

$$\tau \mathrm{Ad}_{\dot{s}_i} = \mathrm{Ad}_{\dot{s}_i^{-1}} \tau \quad \text{and} \quad \theta \mathrm{Ad}_{\dot{s}_i} = \mathrm{Ad}_{\dot{s}_i^{-1}} \theta$$

and the involutivity of τ and θ imply

$$\theta\tau(f_\beta) = e_\beta \quad \text{for} \;\; \beta \in \Delta_+. \tag{6.12}$$

The *nonrestricted rational form* of $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{C}[q^{\pm 1}]$-subalgebra, generated by E_i, F_i, $K_i^{\pm 1}$ and $(K_i - K_i^{-1})/(q_i - q_i^{-1})$. It will be denoted by $\mathcal{U}_q^{\rm nrf}(\mathfrak{g})$. Consider the specialization $\mathcal{U}_\epsilon^{\rm nrf}(\mathfrak{g}) := \mathcal{U}_q^{\rm nrf}(\mathfrak{g})/(q-\epsilon)\mathcal{U}_q^{\rm nrf}(\mathfrak{g})$ and the canonical projection $\nu\colon \mathcal{U}_q^{\rm nrf}(\mathfrak{g}) \to \mathcal{U}_\epsilon^{\rm nrf}(\mathfrak{g})$. De Concini, Kac and Procesi [12, 13] proved that

$\nu(E_i)^\ell, \nu(F_i)^\ell, \nu(K_i)^{\pm\ell} \in Z(\mathcal{U}_\epsilon^{\mathrm{nrf}}(\mathfrak{g}))$ and that, for good integers ℓ, the subalgebra $C_\epsilon^{\mathrm{nrf}}(\mathfrak{g})$, generated by them, is a Poisson subalgebra of $Z(\mathcal{U}_\epsilon^{\mathrm{nrf}}(\mathfrak{g}))$ that contains all elements $\nu(F_{\beta_j})^\ell$.

Extend the reduced expression $w = s_{i_1} \dots s_{i_N}$ to a reduced expression $w_\circ = s_{i_1} \dots s_{i_M}$ of the longest element of W (here $M := \dim \mathfrak{n}_-$). Extend the set of root vectors $F_{\beta_1}, \dots, F_{\beta_N}$ to a set of root vectors $F_{\beta_1}, \dots, F_{\beta_N}, \dots, F_{\beta_M}$ by (6.1) applied for $j \in [1, M]$. The algebra $\mathcal{U}^-[w_\circ]$ is the $\mathbb{C}[q^{\pm 1}]$-subalgebra of $\mathcal{U}_q^{\mathrm{nrf}}(\mathfrak{g})$ generated by all negative Chevalley generators $F_1, \dots, F_r$.

By the definition of the induced Poisson structure for specializations (see section 2.1), the embeddings of $\mathbb{C}[q^{\pm 1}]$-algebras $\mathcal{U}^-[w] \hookrightarrow \mathcal{U}^-[w_\circ] \hookrightarrow \mathcal{U}_q^{\mathrm{nrf}}(\mathfrak{g})$ give rise to the canonical embeddings of Poisson algebras

$$(C_\epsilon^-[w], \{\cdot,\cdot\}) \hookrightarrow (C_\epsilon^-[w_\circ], \{\cdot,\cdot\}) \hookrightarrow (C_\epsilon^{\mathrm{nrf}}(\mathfrak{g}), \{\cdot,\cdot\}) \tag{6.13}$$

where all three Poisson structures are the ones from (2.1). The first embedding is given by sending $z_{\beta_j} \in C_\epsilon^-[w]$ to $z_{\beta_j} \in C_\epsilon^-[w_\circ]$ for $j \in [1, N]$, recall (6.4). The second one is given by $\kappa_\epsilon(F_{\beta_j})^\ell \mapsto \nu(F_{\beta_j})^\ell$.

Proof of Theorem 6.2.1. De Concini, Kac and Procesi [12, Theorem 7.6] constructed an explicit isomorphism of Poisson algebras

$$I_{DKP} \colon (C_\epsilon^{\mathrm{nrf}}(\mathfrak{g}), \{\cdot,\cdot\}) \xrightarrow{\simeq} (\mathbb{C}[G^*], -\ell^2\epsilon^{-1}\{\cdot,\cdot\}_{\pi_D}).$$

It restricts to the Poisson isomorphism

$$I_{DKP} \colon (C_\epsilon^-[w_0], \{\cdot,\cdot\}) \xrightarrow{\simeq} (\mathbb{C}[F\backslash G^*], -\ell^2\epsilon^{-1}\{\cdot,\cdot\}_{\pi_D})$$

where $F := \{(h^{-1}, hu_+) \mid u_+ \in U_+, h \in H\}$ and $\mathbb{C}[F\backslash G^*]$ is viewed as a Poisson subalgebra of $\mathbb{C}[G^*]$. The second isomorphism is explicitly given by $f(z_{\beta_j}) = z_j,\ j \in [1, M]$ where $z_1, \dots z_M$ are the coordinate functions on $F\backslash G^*$ from the parametrization

$$F\backslash G^* = \{F \cdot \exp(z_M f_{\beta_M}) \ldots \exp(z_1 f_{\beta_1}) \mid z_1, \ldots, z_M \in \mathbb{C}\}.$$

The explicit statement of this result is given in [13, Eq. (4.4.1)]. The factor $-\ell^2\epsilon^{-1}$ comes from the normalization made in [12, §7.3] and [13, p. 420] for the induced Poisson bracket on $Z(\mathcal{U}_\epsilon^{\mathrm{nrf}}(\mathfrak{g}))$. The extra factor of 2 in [12, 13] comes from the fact that the Poisson structure π_D differs by a factor of 2 from that in [12, 13]. Composing I_{DKP} with the Poisson maps η^* and $\tau^*\theta^*$ (see (6.10) and (6.11)) gives the Poisson isomorphism

$$\tau^*\theta^*\eta^* I_{DKP} \colon (C_\epsilon^-[w_0], \{\cdot,\cdot\}) \xrightarrow{\simeq} (\mathbb{C}[B_+/H], l^2\epsilon^{-1}\{\cdot,\cdot\}_{\pi_{\mathrm{st}}}) \tag{6.14}$$

where $\mathbb{C}[B_+/H]$ is viewed as a Poisson subalgebra of $(\mathbb{C}[B_+], l^2\epsilon^{-1}\{\cdot,\cdot\}_{\pi_{\mathrm{st}}})$. The definition of I_{DKP} and the property (6.12) of $\tau\theta$ imply that the explicit form of the isomorphism (6.14) is $\tau^*\theta^*\eta^* I_{DKP}(z_{\beta_j}) = \widetilde{z}_j$, $j \in [1, M]$ where $\widetilde{z}_j$ are the coordinate functions on $\mathbb{C}[B_+/H]$ from the parametrization

$$B_+/H = \{\exp(\widetilde{z}_1 e_{\beta_1}) \ldots \exp(\widetilde{z}_M e_{\beta_M}) \cdot H \mid z_1, \ldots, z_M \in \mathbb{C}\}.$$

The flag variety $(G/B_+, \pi)$ is a Poisson homogeneous space for (G, π_{st}). Thus, it is a Poisson (B_+, π_{st})-space. The property (A), from earlier in the section, implies that the Schubert cell $(B_+ w \cdot B_+, \pi)$ is a Poisson homogeneous space for Poisson algebraic group (B_+, π_{st}). By a direct calculation one checks that π vanishes at the base point $w \cdot B_+$. Thus, the fact (2.2) implies that the quotient map

$$(B_+, \pi_{\mathrm{st}}) \to (B_+ w \cdot B_+, \pi), \quad b_+ \mapsto b_+ w \cdot B_+$$

is Poisson. In the $\widetilde{z}_j$ coordinates the map is given by $\exp(\widetilde{z}_1 e_{\beta_1}) \ldots \exp(\widetilde{z}_M e_{\beta_M}) \mapsto \exp(\widetilde{z}_1 e_{\beta_1}) \ldots \exp(\widetilde{z}_N e_{\beta_N}) w \cdot B_+$. The pull-back map is an embedding of Poisson algebras

$$(\mathbb{C}[B_+ w \cdot B_+], \{\cdot,\cdot\}_\pi) \hookrightarrow (\mathbb{C}[B_+/H], \{\cdot,\cdot\}_{\pi_{\mathrm{st}}}).$$

The theorem follows by combining this embedding, the isomorphism (6.14) and the first embedding in (6.13). $\square$

Denote by $\mathcal{Q}$ the root lattice of $\mathfrak{g}$. The algebras $\mathfrak{g}, \mathcal{U}_q(\mathfrak{g}), \mathcal{U}_\epsilon^-[w]$ and $C_\epsilon^-[w]$ are $\mathcal{Q}$-graded and the projection $\kappa_\epsilon \colon \mathcal{U}^-[w] \to \mathcal{U}_\epsilon^-[w]$ is graded. The graded components of these algebras of degree $\gamma \in \mathcal{Q}$ will be denoted by $(.)_\gamma$.

Proposition 6.2.2. *The homogeneous prime elements of* $(C_\epsilon^-[w], \{\cdot,\cdot\})$ *are* $\Delta_{\varpi_i, w\varpi_i}$ *for* $i \in \mathcal{S}(w)$*, in terms of the first identification in* (6.7)*. They satisfy*

$$\{\Delta_{\varpi_i, w\varpi_i}, z\} = -\ell\epsilon^{-1}\langle (w+1)\varpi_i, \gamma\rangle \Delta_{\varpi_i, w\varpi_i} z, \quad \forall z \in (C_\epsilon^-[w])_\gamma.$$

Proof. For $i \in \mathcal{S}(w)$, the vanishing ideal of $\overline{R_{s_i,w}} \cap B_+ w \cdot B_+$ in $\mathbb{C}[B_+ w \cdot B_+]$ is $(\Delta_{\varpi_i, w\varpi_i})$, [40, Theorem 4.7]. Each of these sets is irreducible and is a union of H-orbits of symplectic leaves. This follows from the properties (A)-(B) from the beginning of the section and the well known fact that the open Richardson varieties $R_{v,w}$ are irreducible. Lemma 2.3.3 implies that $\Delta_{\varpi_i, w\varpi_i} \in C_\epsilon^-[w]$ are homogeneous Poisson prime elements.

Assume that $f \in C_\epsilon^-[w]$ is another homogeneous Poisson prime element. By Lemma 2.3.3, the zero locus $\mathcal{V}(f)$ of f should be a union of H-orbits of symplectic leaves of $(B_+ w \cdot B_+, \pi)$. Since

$$B_+ w \cdot B_+ = \bigsqcup_{v \in W, v \leq w} R_{v,w}$$

and $\dim R_{v,w} = \dim(B_+ w \cdot B_+) - l(v)$, either $\mathcal{V}(f) \cap R_{1,w} \neq \varnothing$ or $\mathcal{V}(f) \cap R_{s_i,w} \neq \varnothing$ for some $i \in \mathcal{S}(w)$. The first case is impossible since by (A)-(B), $R_{1,w}$ is a single H-orbit of symplectic leaves and $\overline{R_{1,w}} \supset B_+ w \cdot B_+$. In the second case, $\mathcal{V}(f) \supseteq \overline{R_{s_i,w}} \cap B_+ w \cdot B_+$ because $R_{s_i,w}$ is a single H-orbit of leaves. Since f is prime, $f =_{\mathbb{C}^\times} \Delta_{\varpi_i, w\varpi_i}$.

The formulas for Poisson brackets in the proposition are the specializations at $q = 1$ of eq. (5.1) in [41] for $y_1 = 1$. $\square$

6.3 Proof via Poisson Geometry

We proceed with the proof of Theorem 6.1.3. Recall the $C_\epsilon^-[w]$-basis $\mathcal{Y}$ of $\mathcal{U}_\epsilon^-[w]$ from (6.5). By Theorem 5.2.3 (ii) and Proposition 6.2.2,

$$d_{\ell^N}(\mathcal{Y} : \mathrm{tr}) =_{\mathbb{C}^\times} \Delta_{\lambda, w\lambda} \tag{6.15}$$

for some $\lambda \in \mathcal{P}^+$. ($C_\epsilon^-[w]$ is a polynomial algebra and thus a UFD.) We determine λ by using the methods (1) and (3) in section 5.2: We compare the degrees of the two sides of the equality (in the $\mathcal{Q}$-grading) and their Poisson brackets with the elements of $C_\epsilon^-[w]$. (Since $\Delta_{\varpi_i, w\varpi_i}|_{U_+ \cap w(U_-)} = 1$ for $i \notin \mathcal{S}(w)$, λ is only defined up to adding an element of $\oplus_{i \notin \mathcal{S}(w)} \mathbb{Z}\varpi_i$.) Firstly,

$$\deg \Delta_{\lambda, w\lambda} = \ell(w-1)\lambda.$$

This follows for instance from (6.8) by using that $\deg z_{\beta_j} = -\ell\beta_j$. For the reduced expression $w = s_{i_1} \dots s_{i_N}$, recall the notation

$$w_{\leq j} := s_1 \dots s_{i_j}.$$

Then we have

$$-\beta_j = -w_{\leq j-1}(\alpha_{i_j}) = w_{\leq j}\rho - w_{\leq j-1}\rho. \tag{6.16}$$

Since the map $\mathrm{tr}\colon \mathcal{U}_\epsilon^-[w] \to C_\epsilon^-[w]$ is graded,

$$\begin{aligned} \deg d_{l^N}(\mathcal{Y} : \mathrm{tr}) &= 2\sum_{y \in \mathcal{Y}} \deg y \ = \ 2 \sum_{k_1, \dots, k_N = 0}^{\ell-1} \deg \kappa_\epsilon(F_{\beta_1}^{k_1} \dots F_{\beta_N}^{k_N}) \\ &= 2 \sum_{k_1, \dots, k_N = 0}^{\ell-1} k_1(w_{\leq 1}\rho - \rho) + \dots + k_N(w_{\leq N}\rho - w_{\leq N-1}\rho) \\ &= (\ell-1)\ell^N(w-1)\rho. \end{aligned}$$

Hence, by comparing degrees in (6.15),

$$(w-1)(\lambda - (\ell-1)\ell^{N-1}\rho) = 0. \tag{6.17}$$

Proposition 6.2.2 and the fact that $\deg z_{\beta_j} = -\ell\beta_j$ imply

$$\{\Delta_{\lambda,w\lambda}, z_{\beta_j}\} = \ell^2\epsilon^{-1}\langle(w+1)\lambda, \beta_j\rangle\Delta_{\lambda,w\lambda}z_{\beta_j}, \quad j \in [1,N]. \tag{6.18}$$

To evaluate $\{d_{l^N}(\mathcal{Y} : \mathrm{tr}), z_{\beta_j}\}$, we use Proposition 5.2.4. Since $\mathcal{Y}$ is a $C_\epsilon^-[w]$-basis of $\mathcal{U}_\epsilon^-[w]$ and $C_\epsilon^-[w] \simeq \mathbb{C}[z_{\beta_1}, \dots, z_{\beta_N}]$,

$$\mathcal{U}_\epsilon^-[w] = \oplus_{y\in\mathcal{Y}}\mathbb{C}[z_{\beta_1}, \dots, z_{\beta_N}]y.$$

For a monomial μ in $z_{\beta_1}, \dots, z_{\beta_N}$, a basis element $y \in \mathcal{Y}$ and $r \in \mathcal{U}_\epsilon^-[w]$, denote by $\mathrm{coeff}_{\mu,y}(r)$ the coefficient of μy in r. For $\mathbf{k} = (k_1, \dots, k_N) \in \mathbb{N}^N$, denote the PBW basis element

$$F^{\mathbf{k}} := F_1^{k_1} \dots F_N^{k_N} \in \mathcal{U}^-[w].$$

Lemma 6.3.1. *For all* $\mathbf{k} \in [1, \ell-1]^{\times N}$ *and* $j \in [1,N]$,

$$(\epsilon^{\|\alpha_{i_j}\|/2} - \epsilon^{-\|\alpha_{i_j}\|/2})^\ell \,\mathrm{coeff}_{z_{\beta_j},\kappa_\epsilon(F^{\mathbf{k}})}\big(\partial_{F^l_{\beta_j}}(\kappa_\epsilon(F^{\mathbf{k}}))\big) = \Big(\sum_{m=1}^N \mathrm{sign}(m-j)k_m\langle\beta_m,\beta_j\rangle\Big)\ell\epsilon^{-1}.$$

Proof. Consider the right-to-left lexicographic order $\prec$ on $\mathbb{N}^N$ given by

$$(k_1, \dots, k_N) \prec (m_1, \dots, m_N) \text{ if } k_N = m_N, \dots, k_{j+1} = m_{j+1} \text{ and } k_j < m_j \text{ for some } j.$$

Recursively applying the straightening law (6.2) gives

$$F^{\mathbf{k}}F^{\mathbf{m}} = q^{-\sum_{j>a}k_jm_a\langle\beta_j,\beta_a\rangle}F^{\mathbf{k}+\mathbf{m}} + \sum_{\mathbf{k}'\prec\mathbf{k}+\mathbf{m}} F^{\mathbf{k}'}. \tag{6.19}$$

Thus,

$$F^\ell_{\beta_j}F^{\mathbf{k}} - F^{\mathbf{k}}F^\ell_{\beta_j} = \big(q^{-\ell\sum_{a<j}k_j\langle\beta_j,\beta_a\rangle} - q^{-\ell\sum_{a>j}k_j\langle\beta_j,\beta_a\rangle}\big)F^{\mathbf{k}+\ell e_j} + \sum_{\mathbf{k}'\prec\mathbf{k}+\ell e_j} F^{\mathbf{k}'}$$

where $\{e_1, \dots, e_N\}$ denotes the standard basis of $\mathbb{Z}^N$. The lemma follows from this by dividing by $q - \epsilon$ and applying κ_ϵ. $\square$

It follows from (6.15) and (6.18) that

$$\frac{\{d_{\ell^N}(\mathcal{Y}:\mathrm{tr}), z_{\beta_j}\}}{d_{\ell^N}(\mathcal{Y}:\mathrm{tr}) z_{\beta_j}} \in \mathbb{C}.$$

Now, from Proposition 5.2.4 we have

$$\begin{aligned}\frac{\{d_{\ell^N}(\mathcal{Y}:\mathrm{tr}), z_{\beta_j}\}}{d_{\ell^N}(\mathcal{Y}:\mathrm{tr}) z_{\beta_j}} &= -2 \sum_{k_1,\dots,k_N=0}^{\ell-1} (\epsilon^{\|\alpha_{i_j}\|/2} - \epsilon^{-\|\alpha_{i_j}\|/2})^{\ell} \operatorname{coeff}_{z_{\beta_j}, \kappa_\epsilon(F^{\mathbf{k}})} \big(\partial_{F^{\ell}_{\beta_j}} (\kappa_\epsilon(F^{\mathbf{k}}))\big) \\ &= -2 \sum_{k_1,\dots,k_N=0}^{\ell-1} \Big(\sum_{m=1}^{N} \operatorname{sign}(m-j) k_m \langle \beta_m, \beta_j \rangle \Big) \ell \epsilon^{-1} \\ &= (\ell-1)\ell^{N+1} \langle (w+1)\rho, \beta_j \rangle \epsilon^{-1}.\end{aligned}$$

In the last equality we used the identity $-\langle (w+1)\rho, \beta_j\rangle = \sum_{m=1}^{N} \operatorname{sign}(m-j) \langle \beta_m, \beta_j \rangle$ which follows from (6.16). Comparing this with (6.18), leads to

$$\langle (w+1)(\lambda - (\ell-1)\ell^{N-1}\rho), \beta_j \rangle = 0 \quad \text{for } \ j \in [1, N].$$

The definition of β_j implies $\beta_j - \alpha_{i_j} \in \oplus_{m<j} \mathbb{Z}\alpha_{i_m}$, and thus,

$$\bigoplus_{j=1}^{N} \mathbb{Z}\beta_j = \bigoplus_{i \in \mathcal{S}(w)} \mathbb{Z}\alpha_i.$$

Therefore,

$$\langle (w+1)(\lambda - (\ell-1)\ell^{N-1}\rho), \alpha_i \rangle = 0 \quad \text{for } \ i \notin \mathcal{S}(w).$$

This and the degree formula (6.17) give

$$\langle \lambda - (\ell-1)\ell^{N-1}\rho, \alpha_i \rangle = 0 \quad \text{for } \ i \notin \mathcal{S}(w),$$

that is

$$\lambda - (\ell-1)\ell^{N-1}\rho \in \bigoplus_{i \notin \mathcal{S}(w)} \mathbb{Z}\varpi_i.$$

Theorem 6.1.3 now follows from the fact that $\Delta_{\varpi_i, w\varpi_i}|_{U_+ \cap w(U_-)} = 1$ for $i \notin \mathcal{S}(w)$.

6.4 Cluster Structure

6.4.1 Quantum Unipotent Cells and Integral Forms

Recently, Goodearl and Yakimov have given an integral cluster algebra structure to the quantum unipotent cell algebras $\mathcal{A}_q(\mathfrak{n}_+(w))$ for any symmetrizable Kac–Moody algebra $\mathfrak{g}$ and Weyl element w. As quantum unipotent cells are anti-isomorphic to quantum Schubert cells, we will use Theorem 4.1.6 to translate this cluster structure appropriately.

We now let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra rather than a simple Lie algebra. Fixing notation, we will let $[c_{ij}] \in M_r(\mathbb{Z})$ be its generalized Cartan matrix, $\mathcal{P}$ be its weight lattice, $\mathcal{P}^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathcal{P}, \mathbb{Z})$ be its coweight lattice, $\Pi = \{\alpha_1, \dots, \alpha_r\}$ the set of simple roots, $\Pi^\vee$ the set of simple coroots, and $\{\varpi_1, \dots, \varpi_r\}$ the set of fundamental weights.

We have two anti-isomorphisms $*$ and ϕ of $\mathcal{U}_q(\mathfrak{g})$, which are given by

$$*(E_i) = E_i,\ *(F_i) = F_i, *(K_i) = K_i^{-1},$$

$$\text{and } \varphi(E_i) = F_i,\ \varphi(F_i) = E_i, \varphi(K_i) = K_i.$$

The image of the quantum Schubert cells $*(\mathcal{U}^\pm[w])$ can be denoted by $\mathcal{U}_q(\mathfrak{n}_\pm(w))$. These are also called quantum Schubert cells by some authors and, for $w = s_{i_1} \dots s_{i_N}$, can given as the subalgebras generated by the root vectors

$$T^{-1}_{w^{-1}_{\leq k-1}}(E_{i_k}) \text{ or } T^{-1}_{w^{-1}_{\leq k-1}}(F_{i_k}) \text{ for all } i \in [1, N]$$

appropriately for $\mathcal{U}_q(\mathfrak{n}_+(w))$ and $\mathcal{U}_q(\mathfrak{n}_-(w))$

The Rosso-Tanisaki form will be used in defining quantum unipotent cells and in establishing integral forms for them and for quantum Schubert cells. Recall that a *Hopf pairing* between Hopf $\mathbb{K}$-algebras A and H is a bilinear form $(\cdot, \cdot) : A \times H \to \mathbb{K}$ such that for any $a, b \in A$ and $g, h \in H$,

1. $(ab, h) = (a, h_{(1)})(b, h_{(2)})$

2. $(a, gh) = (a_{(1)}, g)(a_{(2)}, h)$

3. $(a, 1) = \varepsilon_A(a)$ and $(1, h) = \varepsilon_H(h)$

in terms of Sweedler notation.

Let $d \in \mathbb{Z}_{>0}$ be the integer such that $(\mathcal{P}^\vee, \mathcal{P}^\vee) \subseteq \mathbb{Z}/d$. The Rosso-Tanisaki form $(\cdot, \cdot)_{RT} : \mathcal{U}^{\leq} \times \mathcal{U}^{\geq} \to \mathbb{Q}(q^{1/d})$ is the Hopf pairing defined by

$$(F_i, E_j)_{RT} = \delta_{ij}\frac{1}{q_i - q_i^{-1}},\ (K_i, K_j)_{RT} = q^{-(\alpha_i, \alpha_j)},\ (F_i, K_\lambda)_{RT} = 0 = (K_\lambda, E_i)_{RT}$$

for all $i \in [1, r]$. Its restriction to $\mathcal{U}^< \times \mathcal{U}^>$ takes values in $\mathbb{Q}(q)$. The Rosso-Tanisaki form has the following useful properties,

$$(xK_\lambda, yK_\mu)_{RT} = (x, y)_{RT}q^{-(\lambda,\mu)},\ (\mathcal{U}^<_{-\gamma}, \mathcal{U}^>_\delta)_{RT} = 0$$

for $x \in \mathcal{U}^<$, $y \in \mathcal{U}^>$, and γ, $\delta \in \mathcal{Q}_+$ with $\gamma \neq \delta$, see [30].

Let $A_q(\mathfrak{n}_+)$ be the subalgebra of the full dual $(\mathcal{U}^{\geq})^*$ of elements f that satisfy

1. $f(xK_\lambda) = f(x)$ for any $x \in \mathcal{U}^>$ and $\lambda \in \mathcal{P}$ and

2. $f(x) = 0$ for all $x \in \mathcal{U}^>_\gamma$ for $\gamma \in \mathcal{Q}_+ \setminus S$ where S is a finite subset of $\mathcal{Q}_+$.

Then the map $\iota : \mathcal{U}^< \to (\mathcal{U}^{\geq})^*$ given by

$$\langle \iota(x), y\rangle = (x, y)_{RT} \text{ for all } x \in \mathcal{U}^<,\ y \in \mathcal{U}^{\geq},$$

is an algebra homomorphism since the Rosso-Tanisaki form is a Hopf pairing. The image of ι is contained in $A_q(\mathfrak{n}_+)$ by the properties highlighted above for the form. Since the Rosso–Tanisaki form is non-degenerate, ι can be shown to be an isomorphism onto $A_q(\mathfrak{n}_+)$. The quantum unipotent cells $A_q(\mathfrak{n}_+(w)) \subseteq A_q(\mathfrak{n}_+)$

are then defined as the image of $\mathcal{U}(\mathfrak{n}_+(w)) \subseteq \mathcal{U}^<$ under ι. Moreover, we have an anti-isomorphism

$$\iota \circ * : \mathcal{U}^-[w] \to A_q(\mathfrak{n}_+(w)).$$

Kashiwara defined the quantized coordinate ring $A_q(\mathfrak{g})$ for a Kac-Moody algebra $\mathfrak{g}$ as a subalgebra of the full dual of the quantized enveloping algebra of $\mathfrak{g}$, $A_q(\mathfrak{g}) \subseteq \mathcal{U}_q(\mathfrak{g})^*$ [31]. The dual $\mathcal{U}_q(\mathfrak{g})^*$ inherits an algebra structure from the coalgebra structure of $\mathcal{U}_q(\mathfrak{g})$, i.e. for $c, d \in \mathcal{U}_q(\mathfrak{g})^*$ and $x \in \mathcal{U}_q(\mathfrak{g})$

$$cd(x) = c \otimes d(\Delta(x)) = c(x_{(1)})d(x_{(2)})$$
$$\varepsilon d(x) = \varepsilon(x_{(1)})d(x_{(2)}) = d(x) = d\varepsilon(x)$$

where Δ, ε form the coalgebra structure for $\mathcal{U}_q(\mathfrak{g})$.

Moreover, $\mathcal{U}_q(\mathfrak{g})^*$ is a $\mathcal{U}_q(\mathfrak{g})$-bimodule by

$$\langle x \cdot c \cdot y, z \rangle = \langle c, yzx \rangle \text{ for all } c \in \mathcal{U}_q(\mathfrak{g})^*,\ x, y, z \in \mathcal{U}_q(\mathfrak{g}).$$

Recall that a $\mathcal{U}_q(\mathfrak{g})$-module is integrable if E_i and F_i act locally nilpotent. The quantized coordinate ring $A_q(\mathfrak{g})$ is then defined as the unital subalgebra of $\mathcal{U}_q(\mathfrak{g})^*$ of elements $c \in \mathcal{U}_q(\mathfrak{g})^*$ such that

$$\mathcal{U}_q(\mathfrak{g}) \cdot f \in \mathcal{O}_{\text{int}}(\mathfrak{g}) \ \text{ and } \ f \cdot \mathcal{U}_q(\mathfrak{g}) \in \mathcal{O}_{\text{int}}(\mathfrak{g}^{op})$$

where $\mathcal{O}_{\text{int}}(\mathfrak{g})$ is the category of integrable left $\mathcal{U}_q(\mathfrak{g})$-modules with a condition on graded subspaces (nontrivial graded subspaces have weights in $\cup_j(\mu_j + \mathcal{Q})$ for finitely many weights $\mu_1, \ldots, \mu_n \in \mathcal{P}$) and $\mathcal{O}_{\text{int}}(\mathfrak{g}^{op})$ is similarly the category of integrable right $\mathcal{U}_q(\mathfrak{g})$-modules meeting the condition.

Another way to express the quantized coordinate algebra is in terms of matrix coefficients, which was how they were first defined in the finite dimensional case. For a module $M \in \mathcal{O}_{\text{int}}(\mathfrak{g})$, define $D_\varphi M$ to be the restricted dual module with respect to φ,

$$D_\varphi M := \oplus_{\mu\in\mathcal{P}} V_\mu^*$$

where $\mathcal{U}_q(\mathfrak{g})$ is given an action on this dual via φ. The matrix coefficient $c_{\xi v} \in \mathcal{U}_q(\mathfrak{g})^*$ is defined by

$$\langle c_{\xi v}, x\rangle = \langle \xi, x\cdot v\rangle$$

for $v \in M \in \mathcal{O}_{\text{int}}(\mathfrak{g})$ and $\xi \in D_\varphi M$. The quantized coordinate ring is then the subalgebra of $\mathcal{U}(\mathfrak{g})^*$ consisting of matrix coefficients,

$$A_q(\mathfrak{g}) = \{c_{\xi v} \mid M \in \mathcal{O}_{\text{int}}(\mathfrak{g}),\ \xi \in D_\varphi M,\ v \in M\}.$$

It is $\mathcal{P}\times\mathcal{P}$ graded by

$$A_q(\mathfrak{g})_{\mu,\lambda} = \{c_{\xi v} \mid M \in \mathcal{O}_{\text{int}}(\mathfrak{g}),\ \xi \in (M_\mu)^*,\ v \in M_\lambda\}$$

for any $\mu, \lambda \in \mathcal{P}$.

Let v_μ be a highest weight vector of $V(\mu)$. For $w \in W$, denote $v_{w\mu} = T_w v_\mu$. In $V(\mu)^*_{w\mu}$, let $\xi_{w\mu}$ be such that $\langle \xi_{w\mu}, v_{w\mu}\rangle = 1$ The quantum minors of $A_q(\mathfrak{g})$ are the specific matrix coefficients $c_{\xi_{u\mu}, v_{w\mu}}$ for $u, w \in W$ and $\mu \in \mathcal{P}$. The set of minors $E_w = \{c_{\xi_{w\mu}, v_\mu} \mid \mu \in \mathcal{P}_+\}$ form a multiplicative set. For a $\mathcal{P}\times\mathcal{P}$ graded algebra R, let $R_0 = \oplus_{\nu\in P} R_{\nu,0}$. In the case of $A_q(\mathfrak{g})[E_w^{-1}]$, there is an induced $\mathcal{Q}$-grading on $(A_q(\mathfrak{g})[E_w^{-1}])_0$ rather than just a $\mathcal{P}$-grading. There is a $\mathcal{Q}$-graded surjection $\psi_w : (A_q(\mathfrak{g})[E_w^{-1}])_0 \to A_q(\mathfrak{n}_+(w))$.

The quantum minors of the quantum unipotent cell $A_q(\mathfrak{n}_+(w))$ are defined for $u \in W$, $\mu \in \mathcal{P}_+$ by

$$D_{u\mu, w\mu} := \psi_w(c_{\xi_{u\mu}, v_\mu} c^{-1}_{\xi_{w\mu}, v_\mu}).$$

These can be described directly as the elements of $A_q(\mathfrak{n}_+(w))_{(u-w)\mu}$ such that

$$\langle D_{u\mu, w\mu}, xK_\lambda\rangle = \langle \xi_{u\mu}, xv_{w\mu}\rangle$$

for any $x \in \mathcal{U}^{>}$ and $\lambda \in \mathcal{Q}$. The image of these minors under the anti-automorphism $*^{-1} \circ \iota^{-1}$ will be defined as the quantum minors of the quantum Schubert cell $\mathcal{U}^{-}[w]$. These quantum minors will play a crucial role in describing the cluster structure of quantum unipotent cells and quantum Schubert cells.

6.4.2 Quantum Cluster Structure

To discuss the integral quantum cluster algebra structure, we need to establish integral forms of quantum Schubert cells, $\mathcal{U}^{-}[w]^{\vee}_{\mathbb{Z}[q^{\pm 1}]}$, and quantum unipotent cells, $A_q(\mathfrak{n}_+(w))_{\mathbb{Z}[q^{\pm 1}]}$. These should be $\mathbb{Z}[q^{\pm 1}]$-algebras such that extending scalars to $\mathbb{Q}[q^{\pm 1}]$ recovers the appropriate algebra, i.e.

$$\mathcal{U}^{-}[w]^{\vee}_{\mathbb{Z}[q^{\pm 1}]} \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Q}[q^{\pm 1}] \simeq \mathcal{U}^{-}[w]$$
$$A_q(\mathfrak{n}_+(w))_{\mathbb{Z}[q^{\pm 1}]} \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Q}[q^{\pm 1}] \simeq A_q(\mathfrak{n}_+(w)).$$

These integral forms are given by the Rosso-Tanisaki form. For the quantum Schubert cell the *dual integral form* is given by

$$\mathcal{U}^{-}[w]^{\vee}_{\mathbb{Z}[q^{\pm 1}]} \coloneqq \{x \in \mathcal{U}^{-}[w] \mid (x, \mathcal{U}^{>})_{RT} \subseteq \mathbb{Z}[q^{\pm 1}]\}.$$

It is generated as an algebra by rescaled root vectors,

$$F'_{\beta_k} = \frac{1}{(F_{\beta_k}, E_{\beta_k})_{RT}} F_{\beta_k} = (q_{i_k}^{-1} - q_{i_k}) F_{\beta_k},$$

which generate the dual PBW basis of $\mathcal{U}^{-}[w]^{\vee}_{\mathbb{Z}[q^{\pm 1}]}$. The integral form of the quantum unipotent cell is then given by

$$A_q(\mathfrak{n}_+(w))_{\mathbb{Z}[q^{\pm 1}]} \coloneqq \iota \circ *(\mathcal{U}^{-}[w]^{\vee}_{\mathbb{Z}[q^{\pm 1}]}).$$

The quantum cluster algebra structure technically needs an integral form of the quantum algebras over $\mathbb{Z}[q^{\pm 1/2}]$. So we will extend the scalars $\mathbb{Q}[q^{\pm 1}]$ to $\mathbb{Q}[q^{\pm 1/2}]$ for the algebras and also extend the scalars of the integral forms to $\mathbb{Z}[q^{\pm 1/2}]$. For ease of notation we will often denote $\mathcal{U}^-[w]^\vee_{\mathbb{Z}[q^{\pm 1/2}]}$ by $\mathcal{U}^-[w]_\mathbb{Z}$ and $A_q(\mathfrak{n}_+(w))_{\mathbb{Z}[q^{\pm 1/2}]}$ by $A_q(\mathfrak{n}_+(w))_\mathbb{Z}$.

We introduce the quantum seed for quantum unipotent cells in a fashion that will ease our presentation of the cluster structure of quantum Schubert cells and will differ slightly from [27] for that purpose. Recall that we have fixed a reduced expression $w = s_{i_1} \dots s_{i_N}$. Let $p : [1, N] \to [1, N-1] \cup \{-\infty\}$ and $s : [1, N] \to [2, N] \cup \{\infty\}$ be the predecessor and successor maps given by

$$p(k) = \max\{j < k \mid i_j = i_k\} \text{ where } \max \varnothing = -\infty$$
$$\text{and } s(k) = \min\{j > k \mid i_j = i_k\} \text{ where } \min \varnothing = \infty.$$

The mutable directions will be given by

$$\mathbf{ex}(w) = \{k \in [1, N] \mid s(k) \neq \infty\}.$$

This set includes $|\mathbf{ex}(w)| = N - \mathcal{S}(W)$ indices, as each $t \in \mathcal{S}(w)$ will have one and only $j \in [1, N]$ such that $i_j = t$ and $s(j) = \infty$. Let $\widetilde{B}^w$ be the $N \times \mathbf{ex}(w)$ matrix with entries

$$(\widetilde{B}^w)_{j,k} = \begin{cases} -1, & if\ j = p(k) \\ 1, & if\ j = s(k) \\ -c_{i_j i_k} & if\ j < k < s(j) < s(k) \\ c_{i_j i_k} & if\ k < j < s(k) < s(j) \\ 0, & otherwise \end{cases}$$

The principal part B^w is skew-symmetric. Moreover, $\widetilde{B}^w$ is compatible with the $N \times N$ matrix,

$$(\Lambda_w)_{j,k} = \big((w_{\leq j} + 1)\varpi_{i_j}, (w_{\leq k} - 1)\varpi_{i_k} \big)$$

see [27, Proposition 7.2]. The specific quantum minors $D_{\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}$, for $k \in [1, N]$, q-commute in the following way,

$$D_{\varpi_{i_j}, w_{\leq j}\varpi_{i_k}} D_{\varpi_{i_k}, w_{\leq k}\varpi_{i_k}} = q^{(-(\Lambda_w)_{j,k})} D_{\varpi_{i_k}, w_{\leq k}\varpi_{i_k}} D_{\varpi_{i_j}, w_{\leq j}\varpi_{i_j}}, \quad k < j.$$

There is a unique toric frame $\widehat{M}_q^w : \mathbb{Z}^N \to \mathrm{Fract}(\mathcal{T}_q(\Lambda_w^T)) \simeq \mathrm{Fract}(A_q(\mathfrak{n}_+(w))_{\mathbb{Z}})$ with corresponding matrix Λ_w^T given by

$$\widehat{M}_q^w(e_k) = q^{a[1,k]} D_{\varpi_{i_k}, w_{\leq k}\varpi_{i_k}} \quad \text{for any } k \in [1, N]$$

where $a[1, k] = \|(w_{[j,k]-1}\varpi_{i_k})\|/4 \in \mathbb{Z}/2$.

Theorem 6.4.1 ([27])**.** *Let $\mathfrak{g}$ be any symmetrizable Kac–Moody algebra and $w \in W$ a Weyl element with a fixed reduced expression, $w = s_{i_1} \ldots s_{i_N}$. Then the integral form of quantum unipotent cells has a cluster structure, $A_q(\mathfrak{n}_+(w))_{\mathbb{Z}[q^{\pm 1/2}]} = \mathcal{A}_q(\widehat{M}_q^w, -\widetilde{B}^w, \varnothing)$. Moreover, for the nerve Ξ_N, $\mathcal{A}_q(\widehat{M}_q^w, -\widetilde{B}^w, \varnothing) = \mathcal{A}_q(\Xi_N)$.*

The subset Ξ_N of the symmetric group S_N is the collection of permutations σ such that $\sigma([1, k])$ is an interval for any k. We can combinatorially describe this subset in terms of one-line notation: first move 1 as far right as desired, then move 2 as far right as desired up to where 1 now is, then moving 3 right possibly up to 2, et cetera.

$$\begin{array}{ccccccccc}
[1\ 2\ 3\ 4 \ldots N] & \to & [2\ 1\ 3\ 4 \ldots N] & \to & [2\ 3\ 1\ 4 \ldots N] & \to & [2\ 3\ 4\ 1 \ldots N] & \to & \ldots \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & [3\ 2\ 1\ 4 \ldots N] & \to & [3\ 2\ 4\ 1 \ldots N] & \to & \ldots \\
 & & & & & & \downarrow & \searrow & \\
 & & & & & & \vdots & & \ddots
\end{array}$$

The nerve denoted by Ξ_N is a collection of quantum seeds linked by sequences of one-step mutations from the seed $(\widehat{M}_q^w, -\widetilde{B}^w)$, see [27, Theorem 7.3]. These quantum seeds contain toric frames given by cluster variables

$$\widehat{M}_{q,\sigma}^w(e_l) = q^{a[j,k]} D_{w_{\leq j-1}\varpi_{i_j}, w_{\leq k}\varpi_{i_j}} = q^{a[j,k]} T_{w_{\leq j-1}} D_{\varpi_{i_j}, w_{[j,k]}\varpi_{i_j}}$$

where $j = \min\{m \in \sigma([1,l]) \mid i_m = i_{\sigma(l)}\}$ and $k = \max\{m \in \sigma([1,l]) \mid i_m = i_{\sigma(l)}\}$ (noting that $i_j = i_k$). In particular, $\widehat{M}^w_{q,\mathrm{id}} = \widehat{M}^w_q$.

The quantum seeds in Ξ_N are connected by mutations in the following way. Let $\sigma, \sigma' \in \Xi_N$ be permutations such that $\sigma' = \sigma \circ (k, k+1)$ in cycle notation. Note that the permutation group S_N acts on seeds on the right by reordering the basis, denoted by $(M, \widetilde{B}) \cdot \sigma$ or $M \cdot \sigma$. If $i_{\sigma(k)} \neq i_{\sigma(k+1)}$, then the corresponding quantum seeds are equivalent up to ordering the basis by switching k and $k+1$,

$$\widehat{M}^w_{q,\sigma'} = \widehat{M}^w_{q,\sigma} \cdot (k, k+1).$$

If $i_{\sigma(k)} = i_{\sigma(k+1)}$, then the seeds are linked by mutation,

$$\widehat{M}^w_{q,\sigma'} = \mu_k(\widehat{M}^w_{q,\sigma}) \cdot (k, k+1).$$

It is clear by the combinatorial description of $\Xi_N \subset S_N$ that any two permutations of Ξ_N are linked by a finite sequence of simple transpositions. Thus we see that the corresponding collection of seeds Ξ_N is a nerve.

By Theorem 4.1.6 and the anti-isomorphism $\iota \circ *$, we get a cluster structure for dual integral form quantum Schubert cells, given by quantum seed $((\widehat{M}^w_q)^{op}, \widetilde{B}^w)$. Denote $(\widehat{M}^w_q)^{op}$ by M^w_q for convenience. We will abuse notation and denote the images of the quantum minors $\iota \circ *(D_{\mu,u\mu})$ in $\mathcal{U}^-[w]_{\mathbb{Z}}$ by $D_{\mu,u\mu}$. In particular, we are writing

$$M^w_q(e_k) = q^{a[1,k]} D_{\varpi_{i_k}, w_{\leq k}\varpi_{i_k}} \in \mathcal{U}^-[w]_{\mathbb{Z}}.$$

The nerve Ξ_N in the exchange graph of $A_q(M^w, \widetilde{B}^w)$ will be the connected subset of seeds that is isomorphic to Ξ_N in the exchange graph of $A_q(\widehat{M}^w, -\widetilde{B}^w)$, mapping $(M^w, \widetilde{B}^w)$ to $(\widehat{M}^w, -\widetilde{B}^w)$ and matching mutation appropriately. We record this all as the following corollary.

Corollary 6.4.2. *The dual integral form of quantum Schubert cells has a cluster structure and it is equal to the subalgebra given by nerve* Ξ_N*,*

$$\mathcal{U}^-[w]_{\mathbb{Z}} = \mathcal{A}_q(M^w, \widetilde{B}^w, \varnothing) = \mathcal{A}_q(\Xi_N)$$

We now argue that this quantum cluster structure descends to a root of unity quantum cluster structure on quantum Schubert cells at a root of unity.

Proposition 6.4.3. *There exists an integral form of the quantum Schubert cell at a root of unity* $\mathcal{U}_\epsilon^-[w]_{\mathbb{Z}}$ *isomorphic to the quantum cluster algebra at a root of unity* $\mathcal{A}_\epsilon(M_\epsilon^w, \Lambda_w, \widetilde{B}^w)$*. Moreover, it is equal to the subalgebra given by the nerve* Ξ_N*,*

$$\mathcal{U}_\epsilon^-[w]_{\mathbb{Z}} = \mathcal{A}_\epsilon(M_q^w, \Lambda_w, \widetilde{B}^w) = \mathcal{A}_\epsilon(\Xi_N).$$

Proof. Let $\mathcal{U}^-[w]$ be defined over a $\mathbb{K} = \mathbb{Q}(\epsilon^{1/2})$, a field of characteristic zero that contains an ℓ^{th} root of unity. The choice of this field will make the argument cleaner to express, but is not necessary.

We have that the integral quantum cluster algebra $\mathcal{U}^-[w]_{\mathbb{Z}}$ is a subalgebra of $\mathcal{T}_q(\Lambda_w)$ by the quantum Laurent phenomenon, where we are identifying $\mathcal{T}_q(\Lambda_w) \simeq \mathcal{T}_q(M^w)$. Consider $\kappa_\epsilon : \mathcal{T}_q(\Lambda_w) \to \mathcal{T}_\epsilon(\Lambda_w)$, the quotient map with kernel $(\Phi_\ell(q^{1/2}))$ as in Lemma 4.1.1.

$$\begin{array}{ccccc} \mathcal{U}^-[w]_{\mathbb{Z}} & \hookrightarrow & \mathcal{T}_q(\Lambda_w) & \hookrightarrow & \mathrm{Fract}(\mathcal{T}_q(\Lambda_w)) \\ & \searrow & \downarrow \kappa_\epsilon & & \\ & & \mathcal{T}_\epsilon(\Lambda_w) & & \end{array}$$

The generators $M_q^w(e_i)$ of $\mathcal{T}_q(\Lambda_w)$ map to the canonical generators of $\mathcal{T}_\epsilon(\Lambda_w)$. Note this gives us a root of unity toric frame $M_\epsilon^w : \mathbb{Z}^N \to \mathrm{Frac}(\mathcal{T}_\epsilon(\Lambda_w))$, where $M_\epsilon^w(e_i) = \kappa_\epsilon(M_q^w(e_i))$.

Restricting κ_ϵ to $\mathcal{U}^-[w]_{\mathbb{Z}}$, we get a subalgebra of $\mathcal{T}_\epsilon(\Lambda_w)$. The kernel of this restriction is the ideal generated by $\Phi_\ell(q^{1/2})$ inside of $\mathcal{U}^-[w]_{\mathbb{Z}}$. Examining the $\mathbb{K}[q^{\pm 1/2}]$ map $\mathcal{U}^-[w] \to \kappa_\epsilon(\mathcal{U}^-[w]_{\mathbb{Z}}) \otimes \mathbb{Q}$ which maps generators to generators and $q^{1/2} \mapsto \epsilon^{1/2}$, we find that the kernel is $(q^{1/2} - \epsilon^{1/2})$ and $\kappa_\epsilon(\mathcal{U}^-[w]_\mathbb{Z}) \otimes \mathbb{Q} \simeq \mathcal{U}_\epsilon^-[w]$. Hence this subalgebra of $\mathcal{T}_\epsilon(\Lambda_w)$ is an integral form of $\mathcal{U}_\epsilon^-[w]$, which we will label by $\mathcal{U}_\epsilon^-[w]_\mathbb{Z}$.

Since we have a surjection $E_q(\Lambda_w, \widetilde{B}^w) \twoheadrightarrow E_\epsilon(\Lambda_w, \widetilde{B}^w)$, let Xi_N in the context of $E_\epsilon(\Lambda_w, \widetilde{B})$ mean the image of Ξ_n. For instance, we have $M_\epsilon^w \in \Xi_N$. As the image of κ_ϵ restricted to $\mathcal{U}^-[w]_\mathbb{Z}$, it is clear that $\mathcal{U}_\epsilon^-[w]_\mathbb{Z}$ is generated by $M'_\epsilon(e_i)$ for $M'_\epsilon \in \Xi_N$ as $\mathcal{U}^-[w]_\mathbb{Z} = \mathcal{A}_q(\Xi_N)$. Thus as subalgebras of $\mathrm{Fract}(\mathcal{T}_\epsilon(\Lambda_w))$, we have $\mathcal{U}_\epsilon^-[w]_\mathbb{Z} = \mathcal{A}_\epsilon(\Xi_N)$. Since $\mathcal{A}_q(\Xi_N) = \mathcal{A}_q(M_q^w, \widetilde{B}^w)$ and $E_q(\Lambda_w, \widetilde{B}^w) \twoheadrightarrow E_\epsilon(\Lambda_w, \widetilde{B}^w)$, we must have $\mathcal{A}_\epsilon(\Xi_N) = \mathcal{A}_\epsilon(M_\epsilon^w, \Lambda_w, \widetilde{B}^w)$. □

6.5 Proof via Quantum Cluster Algebras

By Proposition 6.4.3, we have a quantum cluster algebra structure on $\mathcal{U}_\epsilon^-[w]_\mathbb{Z}$. Moreover, it is equal to its subalgebra $\mathcal{A}_\epsilon(\Xi)$. In using the cluster structure and Theorem 5.3.2 to solve for the discriminants $d(\mathcal{U}_\epsilon^-[w]/C_\epsilon^-[w])$, we first need to show that the subalgebras $C_\epsilon(\Xi)$ and $C_\epsilon^-[w]$ are equal after extending scalars appropriately. We start with the following exercise.

Lemma 6.5.1. *The quantum minor $D_{\varpi_{i_j}, w_{[j,j]}\varpi_{i_j}}$ is a scalar multiple of F_{i_j}. Moreover, the root vectors F_{β_j} are given by the cluster variables $M_{q,\sigma}^w$ (or $M_{\epsilon,\sigma}^w$ as appropriate), up to rescaling, for $\sigma = [j\ j+1\ \dots N\ j-1 \dots\ 2\ 1] \in \Xi_N$.*

Proof. Consider the $\mathcal{U}_{q_{i_j}}(\mathfrak{sl}_2)$-subalgebra of $\mathcal{U}_q(\mathfrak{g})$ given by

$$E \mapsto E_{i_j},\ F \mapsto F_{i_j},\ K \mapsto K_{i_j}.$$

Note $w_{[j,j]} = s_{i_j}$ and the quantum minor gets mapped $D_{\varpi_1, s_1\varpi_1} \mapsto D_{\varpi_{i_j}, s_{i_j}\varpi_{i_j}}$.

Let us consider $\iota(F) \in A_{q_{i_j}}(\mathfrak{n}_+) \subseteq (\mathcal{U}^{\geq}_{q_{i_j}}(\mathfrak{sl}_2))^*$. Recall that

$$\langle \iota(F), y\rangle = (F, y)_{RT} \text{ for any } y \in \mathcal{U}^{\geq}_{q_{i_j}}(\mathfrak{sl}_2),$$

$$\langle \iota(F), xK^n\rangle = \langle \iota(F), x\rangle \text{ for any } x \in \mathcal{U}^{>}_{q_{i_j}}(\mathfrak{sl}_2),$$

$$\text{and } \langle \iota(F), x\rangle = 0 \text{ for any } x \in \mathcal{U}^{>}_{q_{i_j}}(\mathfrak{sl}_2)_\mu,\ \mu \neq 0.$$

This along with $\langle \iota(F), E\rangle = (F, E)_{RT} = (q_{i_j} - q_{i_j}^{-1})^{-1}$ completely describes $\iota(F)$.

Now $v_{s_1\varpi_1} = T_{s_1} v_{\varpi_1} = -q_{i_j} F v_{\varpi_1}$. Hence for $x \in \mathcal{U}^{>}_{q_{i_j}}(\mathfrak{sl}_2)$, we have

$$\langle D_{\varpi_1, s_1\varpi_1}, xK^n\rangle = -q_{i_j}\langle \xi_{\varpi_1}, xFv_{\varpi_1}.\rangle$$

From this we see that $\langle D_{\varpi_1,s_1\varpi_1}, E\rangle = -q_{i_j}$ and $\langle D_{\varpi_1,s_1\varpi_1}, x\rangle = 0$ for $x \in (\mathcal{U}^{>}_{q_{i_j}}(\mathfrak{sl}_2))_\mu$ where $\mu \neq 1$. Hence $D_{\varpi_1,s_1\varpi_1} = q_{i_j}(q_{i_j}^{-1} - q_{i_j})\iota(F)$ for $\mathcal{U}_{q_{i_j}}(\mathfrak{sl}_2)$. Thus in $\mathcal{U}^-[w]$

$$\iota^{-1}(D_{\varpi_{i_j}, w_{[j,j]}\varpi_{i_j}}) = q_{i_j}(q_{i_j}^{-1} - q_{i_j})F_{i_j}.$$

We have that $T_{w_{\leq j-1}}\iota^{-1}(D_{\varpi_{i_j}, w_{[j,j]}\varpi_{i_j}}) = q_{i_j}(q_{i_j}^{-1} - q_{i_j})F_{\beta_j}$. For

$$\sigma = [j\ j+1 \ldots N\ 1\ 2 \ldots j-1],$$

recall that the cluster variables associated with σ are

$$M^w_{q,\sigma}(e_l) = q^{a[n,k]} T_{w_{\leq n-1}} D_{\varpi_{i_n}, w_{[n,k]}\varpi_{i_n}}$$

where $n = \min\{m \in \sigma[1,l] \mid i_m = i_{\sigma(l)}\}$ and $k = \max\{m \in \sigma[1,l] \mid i_m = i_{\sigma(l)}\}$. So for $l = 1$, we have $n = j$, $k = j$, and

$$M^w_{q,\sigma}(e_l) = q^{a[j,j]} T_{w_{\leq j-1}} D_{\varpi_{i_j}, w_{[j,j]}\varpi_{i_j}} = q_{i_j}(q_{i_j}^{-1} - q_{i_j}) q^{a[j,j]} F_{\beta_j}.$$

This implies the root of unity case as well by considering κ_ϵ. □

Proposition 6.5.2. *Suppose ϵ is a root of unity such that $(M^w_\epsilon, \Lambda_w, \widetilde{B}^w)$ satisfies condition C. Given any Kac–Moody algebra $\mathfrak{g}$ and Weyl element w, the canonical central subalgebra $C_\epsilon(\Xi_N) \otimes \mathbb{Q}$ of $\mathcal{A}_\epsilon(\Xi) \otimes \mathbb{Q} = \mathcal{U}^-_\epsilon[w]$ is equal to the canonical central subalgebra $C^-_\epsilon[w]$.*

Proof. By Lemma 6.5.1, we have that $C_\epsilon^-[w] \subseteq C_\epsilon(\Xi_N)$. To show the reverse inclusion, we will show that the ℓ^{th} powers of the quantum minors of $C_\epsilon(\Xi_N)$ can be written in terms of $F_{\beta_1}^\ell, \ldots, F_{\beta_N}^\ell$.

We need only show that $D^\ell_{\varpi_{i_N}, w\varpi_{i_N}}$ can be written in terms of the ℓ^{th} powers of Lusztig's root vectors. The cases of $D^\ell_{\varpi_{i_j}, w_{\leq j}\varpi_{i_j}}$ will follow by induction on the length of w, noting that $\mathcal{U}_\epsilon^-[w_{\leq k}] \hookrightarrow \mathcal{U}_\epsilon^-[w]$. Then the general case of $D^\ell_{w_{\leq j-1}\varpi_{i_j}, w_{\leq k}\varpi_{i_j}}$ (noting $i_k = i_j$) also follows by induction on the length of w, since $D_{w_{\leq j-1}\varpi_{i_j}, w_{\leq k}\varpi_{i_j}} = T_{w_{\leq j-1}} D_{\varpi_{i_j}, w_{[j,k]}\varpi_{i_j}}$ can be seen as an element of $T_{w_{\leq j-1}}(\mathcal{U}_\epsilon^-[w_{[j,k]}])$.

Suppose that $p(N) = -\infty$, i.e. $i_n \neq i_j$ for all $j < N$. Then $w_{\leq N-1}\varpi_{i_N} = \varpi_{i_N}$ and $D_{\varpi_{i_N}, w\varpi_{i_N}} = T_{w_{\leq N-1}} D_{\varpi_{i_N}, s_{i_N}\varpi_{i_N}}$. Thus $D^\ell_{\varpi_{i_N}, w\varpi_{i_N}}$ is a scalar multiple of $F^\ell_{\beta_N}$ by Lemma 6.5.1.

Now suppose that $p(N) = j$ for some $j < N$. In this case, $s(j) = N$ and $\widetilde{B}^w_{N,j} = -1$. By Lemma 4.3.1, we have that

$$M_\epsilon^w(e_j)^\ell \left(\mu_j M_\epsilon^w(e_j)\right)^\ell = \prod_{\widetilde{B}^w_{ij} > 0} \left(M_\epsilon^w(e_i)^\ell\right)^{b_{ij}} \; + \; M_\epsilon^w(e_N)^\ell \prod_{\widetilde{B}^w_{ij} < 0, \; i \neq N} \left(M_\epsilon^w(e_i)^\ell\right)^{|\widetilde{B}^w_{ik}|}$$

and hence

$$D^\ell_{\varpi_{i_N}, w\varpi_{i_N}} = q^{-\ell a[1,N]} M^w_{\epsilon,\sigma}(e_j)^\ell = \frac{P(F^\ell_{\beta_1}, \ldots, F^\ell_{\beta_N})}{Q(F^\ell_{\beta_1}, \ldots, F^\ell_{\beta_N})}$$

for some polynomials P, Q in N variables over $\mathbb{Z}(q^{\pm 1/2})$. However, $D^\ell_{\varpi_{i_N}, w\varpi_{i_N}}$ is an element of $\mathcal{U}_\epsilon^-[w]$ and can be written in terms of $F_{\beta_1}, \ldots, F_{\beta_N}$. Hence Q must divide P, and it must be possible to write $D^\ell_{\varpi_{i_N}, w\varpi_{i_N}}$ in terms of $F^\ell_{\beta_1}, \ldots, F^\ell_{\beta_N}$. □

We are now ready to prove the main theorem on discriminants for quantum Schubert algebras. We could present the theorem for the integral version, $\mathcal{U}_\epsilon^-[w]_\mathbb{Z}$, but for clarity of the proof we will present it for $\mathcal{U}_\epsilon^-[w]$ over $\mathbb{Q}(\epsilon^{1/2})$. Recall from Remark 6.1.1, that finding the formula of the discriminant $d(\mathcal{U}_\epsilon^-[w]/C_\epsilon^-[w])$ for any field that contains a primitive ℓ^{th} root of unity ϵ will solve the formula for any other. From Remark 5.2.6, we are able to compute the discriminant in the integral version from the discriminant over the field. In this particular case, the only difference between the two discriminants is a scalar that can calculated from the discriminant $d(\mathcal{T}_\epsilon(\Lambda_w)/\mathbb{Z}[\epsilon^{1/2}][X_i^{\pm\ell}]_{i=1}^N)$.

Theorem 6.5.3. *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra, w a Weyl group element and $\ell > 2$ an odd integer which is coprime to d_{i_k} for all k. Assume that $\mathbb{K}$ is a field of characteristic 0 which contains a primitive ℓ^{th} root of unity ϵ. Then*

$$d(\mathcal{U}_\epsilon^-[w]/C_\epsilon^-[w]) =_{\mathbb{K}^\times} \prod_{k \notin \boldsymbol{ex}} D^L_{\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}$$

where $L := \ell^N(\ell - 1)$.

Proof. By Proposition 6.4.3, the quantum Schubert cell has a (non-integral) cluster structure,

$$\mathcal{U}_\epsilon^-[w] = \mathcal{A}_\epsilon(M_q^w, \Lambda_w, \widetilde{B}^w) \otimes \mathbb{Q} = \mathcal{A}_\epsilon(\Xi_N) \otimes \mathbb{Q}$$

noting that $\mathbb{Z}[\epsilon^{1/2}] = \mathbb{Z}[\epsilon]$ and $\mathbb{Q}(\epsilon^{1/2}) = \mathbb{Q}(\epsilon)$ since ℓ is odd. From Proposition 6.5.2, our central subalgebras align $C_\epsilon(\Xi_N) \otimes \mathbb{Q} = C_\epsilon^-[w]$. By the PBW basis of $\mathcal{U}^-[w]$, we have that the algebra $\mathcal{A}_\epsilon(\Xi_N) \otimes \mathbb{Q}$ is a free $C_\epsilon(\Xi_N) \otimes \mathbb{Q}$-module with same basis from (6.5),

$$\mathcal{Y} = \{\kappa_\epsilon(F_{\beta_1})^{m_1} \ldots \kappa_\epsilon(F_{\beta_N})^{m_N} \mid m_1, \ldots, m_N \in [0, \ell - 1]\}.$$

Since ℓ is coprime to all d_{i_k}, we have that $(M_q^w, \Lambda_w, \widetilde{B}^w)$ meets condition **C** from section 4.3. It is now clear that all hypotheses of Theorem 5.3.2 are met and that the discriminant is given by a product of frozen variables,

$$d\left(\mathcal{A}_\epsilon(\Xi_N) \otimes \mathbb{Q}/C_\epsilon(\Xi_N) \otimes \mathbb{Q}\right) = \prod_{k \notin \mathbf{ex}} (M_\epsilon^w(e_k))^{a_k}$$

up to multiplication by a unit in $C_\epsilon(\Xi_N) \otimes \mathbb{Q}$, keeping in mind Remark 5.3.3. As these cluster variables ϵ-commute and ϵ is a unit of central subalgebra, there is no ambiguity about order of multiplication.

In particular, the frozen variables are $M^w_\epsilon(e_k) = D_{\varpi_{i_k}, w_{\leq k-1}\varpi_{i_k}}$ for $k \in [1, N]$ such that $s(k) = \infty$. Note that there is one and only one $t \in \mathcal{S}(w)$ with $i_k = t$. Hence the number of frozen variables is $|\mathcal{S}(w)|$. Finding the multiplicities of these quantum minors will finish the proof.

A $\mathbb{Z}^N$-filtration can be put on $\mathcal{U}^-[w]$ by the reverse lexicographical ordering on the PBW basis (6.3). This descends to a $\mathbb{Z}^N$-filtration on $\mathcal{U}^-_\epsilon[w]$. Note that the associated graded algebra gr $\mathcal{U}^-_\epsilon[w]$ also is graded by $G = (\mathbb{Z}/\ell\mathbb{Z})^N$, and with respect to this grading, $C^-_\epsilon[w]$ is homogeneous of degree 0. Note that gr $\mathcal{U}^-_\epsilon[w]$ is a free gr $C^-_\epsilon[w]$ algebra with basis gr $\mathcal{Y}$ since both the basis and the subalgebra are homogeneous. By Proposition 5.2.5, the discriminant satisfies

$$\begin{aligned}\deg_G\left(\text{lt } d(\mathcal{U}^-_\epsilon[w]/C^-_\epsilon[w])\right) &= \deg_G\left(d(\text{gr } \mathcal{U}^-_\epsilon[w]/\text{gr } C^-_\epsilon[w])\right)\\ &= 2\sum_{0\leq m_i\leq \ell-1} \deg_G\left(\kappa_\epsilon(F_{\beta_1})^{m_1}\dots\kappa_\epsilon(F_{\beta_N})^{m_N}\right)\\ &= \ell^N(\ell-1)(e_1+e_2+\dots+e_N)\end{aligned}$$

where $\{e_i\}$ is the standard basis of G.

The leading terms of the quantum minors $D_{\varpi_{i_k}, w_{\leq k-1}\varpi_{i_k}}$ are given in terms of the predecessors $p^n(k)$ of k,

$$\text{lt } D_{\varpi_{i_k}, w_{\leq k-1}\varpi_{i_k}} = F_{\beta_{p^{m_k}(k)}}\dots F_{\beta_{p(k)}}F_{\beta_k}$$

where m_k is the maximal integer such that $p^{m_k}(k) \neq -\infty$, see [27]. We find for the discriminant that

$$\begin{aligned}\text{lt } d(\mathcal{U}^-_\epsilon[w]/C^-_\epsilon[w]) &= \prod_{k\notin \mathbf{ex}} \text{lt } D^{a_k}_{\varpi_{i_k}, w_{\leq k-1}\varpi_{i_k}}\\ &= \prod_{k\notin \mathbf{ex}} (F_{\beta_{p^{m_k}(k)}}\dots F_{\beta_{p(k)}}F_{\beta_k})^{a_k}\end{aligned}$$

in the associated graded algebra gr $\mathcal{U}^-_\epsilon[w]$. Since each $t \in \mathcal{S}(w)$ corresponds to only one $k \notin \mathbf{ex}$, then F_{β_j}, for each $j \in [1, N]$, is a component of only one of the multiplicands above. In terms of degree,

$$\ell^N(\ell-1)\sum_{i=1}^{N} e_i = \deg_G \left(\text{lt } d(\mathcal{U}_\epsilon^-[w]/C_\epsilon^-[w])\right)$$

$$= \sum_{k\notin \mathbf{ex}} a_k(e_{p^{m_k}(k)} + \cdots + e_{p(k)} + e_k).$$

Hence the multiplicities are given by $a_k = \ell^N(\ell-1) = L$. □

Remark 6.5.4. We can reconcile Theorem 6.1.3 and Theorem 6.5.3 by matching up ℓ^{th} powers of quantum minors $D^{\ell}_{\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}$ with appropriate generalized minors $\Delta_{\varpi_{i_k}, w\varpi_{i_k}}$. In the context of the theorem, $D_{\varpi_{i_k}, w_{\leq k}\varpi_{i_k}} = D_{\varpi_{i_k}, w\varpi_{i_k}}$ since k is such that $s(k)=\infty$. The matching of these quantum and generalized minors can be seen by noting that the ℓ^{th} powers of the frozen quantum cluster variables of $\mathcal{A}_\epsilon(M_q^w, \Lambda_w, \widetilde{B}^w)$ align with the frozen cluster variables of $\mathcal{A}(\widetilde{B}^w)$ via Theorem 4.3.5. These frozen variables of the cluster structure on the coordinate ring of the Schubert cell are the generalized minors. Note this explains that the L's differ by a factor of ℓ in the two theorems.

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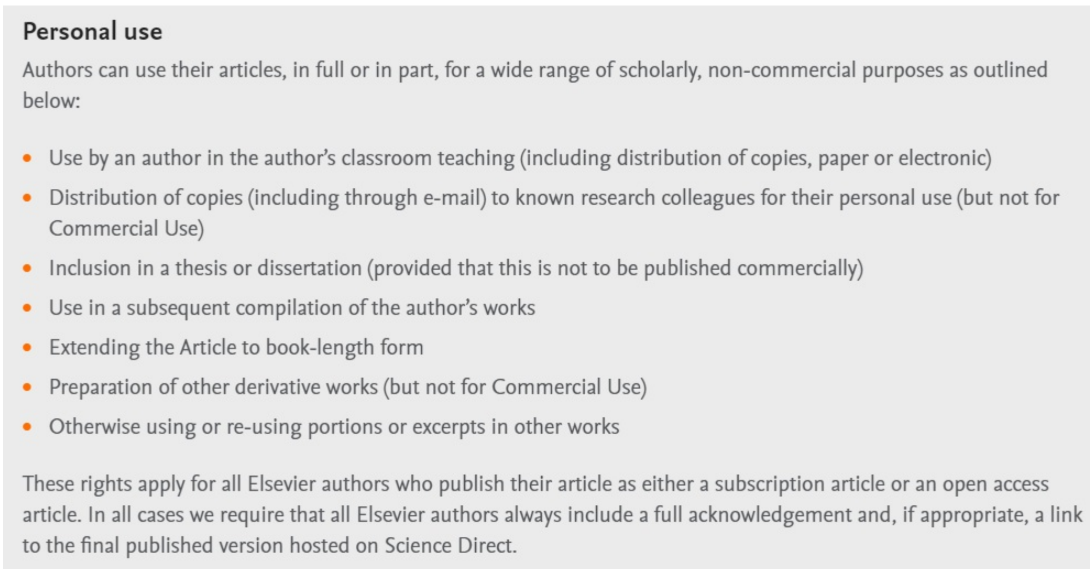

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