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Design of Metamaterials for Optics

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DESIGN OF METAMATERIALS FOR OPTICS

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Abstract

First part of this dissertation studies the problem of designing metamaterial crystals with double negative effective properties for applications in optics by investigating the conditions necessary for generating novel dispersion properties in a metamaterial crystal with subwavelength microstructure. This provides novel optical properties created through local resonances tied to the geometry of the media in subwavelength regime.

In the second part, this dissertation studies the representation formula used to describe band structures in photonic crystals with plasmonic inclusions. By using layer potential techniques, a magnetic dipole operator describing the tangential component of the electrical field generated by magnetic distribution is studied. Its compactness is proved and used to obtain the spectral representation formula for the magnetic field.

Chapter 1
Notations

Throughout this thesis, we will denote by $\mathbb{R}^3$ and $\mathbb{C}^3$ the $3-$ dimensional real and complex Euclidean spaces respectively. The differential operators ∇, $\nabla\cdot$ and $\nabla\times$ are defined in the usual way for a complex vector valued function defined on a bounded domain D in $\mathbb{C}^3$. The integrals discussed in this thesis are always understood in the sense of Lebesgue integration.

We denote by $L^2(D, \mathbb{C}^3)$ the complex vector valued square integrable functions defined on a bounded domain D in $\mathbb{C}^3$ and, for $u \in L^2(D, \mathbb{C}^3)$

$$\|u\|_{L^2} := \left(\int_D |u|^2 \, dx \right)^{\frac{1}{2}}$$

where $|u|$ denotes the Euclidean norm of u. We denote by $L^2_{\#}(D, \mathbb{C}^3)$ the space of functions in $L^2(D, \mathbb{C}^3)$ that are periodic on ∂D. The inner product in $L^2(D, \mathbb{C}^3)$ is given by

$$(u, v) := \int_D u(x) \cdot \overline{v(x)} \, dx, \qquad u, v \in L^2(D, \mathbb{C}^3)$$

We frequently consider vector valued functions satisfying different partial differential equations in different parts of a domain which has certain inclusions as its subdomains. In order to better characterize the problems such functions solve in the whole domain, we often need to consider the jump in the value of these functions across boundaries of inclusions. We denote by $[u]_{\partial K}$ the jump in u across the boundary of K, while $u\big|_{\partial K^-}$ and $u\big|_{\partial K^+}$ represent the restriction of u to ∂K from inside and outside of the domain K respectively. Similarly, we often denote by $n \cdot u\big|_{\partial K^-}$ and $n \times u\big|_{\partial K^-}$ the restriction of the normal component and tangential trace of the vector valued function u to ∂K from inside of the domain K, while $n \cdot u\big|_{\partial K^+}$ and $n \times u\big|_{\partial K^+}$ represent the restriction of the normal component and tangential trace of the vector valued function u to ∂K from outside of the domain

K.

When the restriction of functions to the boundary of a domain is considered, it will be necessary to study surface differential operators acting on tangential vector fields defined the boundary of the domain. We will denote by $\nabla_s u$ and $\Delta_s u$ the surface gradient and surface divergence of the tangential gradient of tangential vector field u on the boundary of the domain under consideration. We follow the definitions for these boundary differential operators given in [12].

Chapter 2
Metamaterial Crystal

2.1 Introduction and Problem Setup

2.1.1 Introduction

Metamaterials are known to be the materials that exhibit novel properties which are not found in nature. They are usually synthesized from patterned composite material components in a periodic structure. When the period length of such microstructures is designed to be much less than the wavelength of incident light, interesting interactions occur between the material and the electromagnetic waves. Due to such interactions, structural geometry of material can be manipulated to obtain desired novel optical properties. One of such novel properties that has been the focus of research in recent years is the double negative metamaterials which has frequency dependent negative effective magnetic permeability and negative effective dielectric permittivity. Optical metamaterials with double negative effective properties have wide range of applications such as biomedical imaging, optical lithography and data storage.

In 1968, the novel properties of materials were studied under the assumption of negative dielectric constant and negative magnetic permeability [13]. The double negative effective properties of periodic array of non-magnetic metallic split-ring resonators at microwave frequency was studied in [14]. Artificial bulk magnetism using infinite periodic array of micro-resonators was studied in [6]. The double negative properties of metamaterials made from arrays of metallic posts and split ring resonators was experimentally demonstrated in [15]. Double negative property of materials obtained for such resonators with different geometric structures in [16, 17, 18, 19, 20, 21, 30, 31]. Double negative properties of metamaterials obtained by employing dielectric material with large permittivity studied in [34, 35, 36].

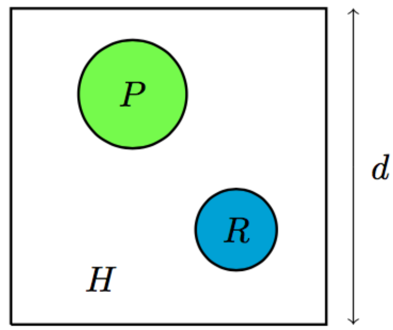
Obtaining double negative properties of metamaterials made from high dielectric core with plasmonic coating at optical frequency studied in [22, 23, 38]. Metamaterial crystals with double negative effective properties are obtained using periodic array of unit cells consisting of two different inclusions in [7] and [8]. Negative bulk dielectric permeability at infrared and optical frequencies using special configurations of plasmonic nanoparticles studied in [39, 40].

The appearance of effective properties for describing scattering problems for metamaterial inclusions made from subwavelength resonators is initiated in [1], [9] in 3D. The dispersion relation and convergent power series representation for Bloch wave in periodic high contrast media with a single inclusion are obtained in [4] and [5]. Recently, the generation of double negative metallic media using two scale expansions can be found in [10] and metamaterials from subwavelength nonmagnetic resonators to control refraction was studied in [41].

Our goal in this chapter of this dissertation is to study the design of metamaterial crystal constructed through the use of high dielectric and frequency-dependent dielectric inclusions in the host material. The novelty in this work is that our analysis is done based on full 3-D model. By using asymptotic expansion approach, we obtain dispersion relation for the propagation of homogenized electromagnetic waves through the mematerial crystal.

2.1.2 Problem Setup

The metamaterial crystal studied in this dissertation is constructed through periodic assemblage of a cube of side length d. In each cube, the region occupied by the host material is denoted by H and it contains two types of non-magnetic inclusions: one with a high dielectric constant and denoted by R, and the other denoted by P has a frequency-dependent dielectric constant. A plane view of a typical cube in the assemblage is shown below. Since we do not assume prescribed



point charges and currents, the Maxwell system of partial differential equations which governs the electromagnetic waves traveling through this media takes the following form,

$$
\begin{aligned}
\nabla \times E &= -\mu_d \frac{\partial H}{\partial t} \\
\nabla \cdot (\varepsilon_d E) &= 0 \\
\nabla \times H &= \varepsilon_d \frac{\partial E}{\partial t} \\
\nabla \cdot H &= 0
\end{aligned}
\tag{2.1}
$$

where H and E denote the magnetic field and the electric field respectively. The magnetic permeability $\mu_d := \mu_0$ is given by its free space value while the dielectric constant $\varepsilon_d(x)$ of metamaterial crystal of period length d with period cell $Y^d = (0,d)^3$ is given by

$$
\varepsilon_d = \begin{cases} \varepsilon_{p,rel}(w) & \text{in} \quad P \\ \dfrac{\varepsilon_{r,rel}}{d^2} & \text{in} \quad R \\ \varepsilon_0 & \text{in} \quad H \end{cases}
\tag{2.2}
$$

where $\varepsilon_{p,rel}$ and $\varepsilon_{r,rel}$ represent the dielectric permittivity of the inclusions P and R respectively, and ε_0 is given by the dielectric permittivity value for the free space. For the electromagnetic waves propagating through the crystal, we consider time-harmonic fields in (2.1) that take the following form

$$E(x,t) = E(x)e^{-i\omega t}, \qquad H(x,t) = H(x)e^{-i\omega t} \tag{2.3}$$

where ω is the wave frequency. Substituting (2.3) into (2.1), we get

$$\begin{aligned} \nabla \times E &= i\omega\mu_d H \\ \nabla \times H &= -i\omega\varepsilon_{_d} E \\ \nabla \cdot \varepsilon_{_d} E &= 0 \\ \nabla \cdot H &= 0 \end{aligned} \tag{2.4}$$

and both the electrical field E and the magnetic field H are continuous across the boundaries of inclusions R and P.

By changing the variable through $y = x/d$, we write (2.4) along with (2.2) as the following problem posed in the unit cell $Y = (0,1)^3$

$$\begin{aligned} \nabla \times E &= id\omega\mu_0 H & [H] &= 0 \\ \nabla \times H &= -id\omega\varepsilon_0\varepsilon^d_{rel} E & [n \times H] &= 0 \\ \nabla \cdot \varepsilon^{d,rel} E &= 0 & [n \cdot H] &= 0 \\ \nabla \cdot H &= 0 \end{aligned} \tag{2.5}$$

$$\varepsilon^d_{_{rel}}(y) = \begin{cases} 1 - \xi_r/\xi & \text{in} \quad P \\ 1/\rho^2 & \text{in} \quad R \\ 1 & \text{in} \quad H \end{cases} \tag{2.6}$$

with $\xi = (\varepsilon_{r,rel}\omega^2)/c^2$, $\xi_r = (\omega_p^2\varepsilon_{r,rel})/(c^2\xi)$, $\rho = d/(\sqrt{\varepsilon_{r,rel}})$ and ω_p is the plasmon frequency of the inclusion P. We set $\eta = (2\pi d)/\lambda$ and write

$$E(y) = \sum_{n=0}^{\infty} \boldsymbol{e_n}(y)\eta^n e^{i\eta\hat{k}\cdot y}, \qquad H(y) = \sum_{n=0}^{\infty} \boldsymbol{h_n}(y)\eta^n e^{i\eta\hat{k}\cdot y}, \qquad \sqrt{\xi} = \sum_{n=0}^{\infty} \sqrt{\xi_n}\eta^n.$$

We substitute them into (2.5) to get power series expansion of the Maxwell's equations. Our main results are obtained by collecting and studying the leading order terms in this expansion.

The rest of this chapter is organized as follows: We present the main results in section 2.2 and the proofs of the main results follow in section 2.3.

2.2 Main Results

2.2.1 Characterization of Leading Order Terms

Before presenting our main theorem, we first make clear the characterization of the leading order terms $\boldsymbol{e}_0$ and $\boldsymbol{h}_0$ in the limit $\eta \to 0$. Such characterizations are realized by identifying the leading order terms as the solutions to certain problems expressed in the following theorems.

Theorem 2.1. *(1) The leading order term $\boldsymbol{e_0} \in \widetilde{H}(curl, Y)$ is characterized as*

$$\begin{aligned} &\boldsymbol{e_0} = 0 && \textit{in } R \\ &\boldsymbol{e_0} = \nabla\varphi + \mathbf{c} && \textit{in } \ Y \\ &\int_Y \boldsymbol{e_0}\, dy = \mathbf{c} && \varphi \in W^{1,2}_{\#}(Y, \mathbb{C}) \end{aligned} \tag{2.7}$$

where the space $W^{1,2}_{\#}(Y, \mathbb{C})$ is given by

$$W^{1,2}_{\#}(Y,\mathbb{C}) := \left\{ u \;\middle|\; u \in L^2_{\#}(Y,\mathbb{C}),\ \partial_i u \in L^2_{\#}(Y,\mathbb{C}),\ \nabla \cdot u = 0 \textit{ in } Y,\ \nabla u + \vec{c} = 0 \textit{ in } R \right\}$$

for $\vec{c} \in \mathbb{C}^3$ with the inner product on $W^{1,2}_{\#}(Y, \mathbb{C})$

$$(u, v) := \int_Y \nabla u \cdot \nabla \overline{v} dy \qquad u, v \in W^{1,2}(Y, \mathbb{C})$$

Also, $\varphi = -\mathbf{c} \cdot \varphi^k$, and φ^k is the solution to

$$\begin{aligned} \nabla \cdot [a_p(y)(\nabla\varphi^k + \tilde{e}^k)] &= 0 \\ n \cdot \varepsilon_{p,rel}(\xi_0)(\nabla\varphi^k + \tilde{e}^k)\Big|_{\partial P^-} &= n \cdot (\nabla\varphi^k + \tilde{e}^k)\Big|_{\partial P^+} \end{aligned} \tag{2.8}$$

where $\tilde{e}^k$, $k = 1, 2, 3$ are the basis vectors, and

$$a_p(y) = \begin{cases} 1 & y \in H \\ \varepsilon_{p,rel}(\xi_0) & y \in P \end{cases} \tag{2.9}$$

(2) The leading order term $\boldsymbol{h_0} \in \widetilde{H}(curl, Y)$ *is characterized as*

$$\begin{aligned} \nabla \times \boldsymbol{h_0} &= 0 && \textit{in } Y \setminus R \\ \nabla \cdot \boldsymbol{h_0} &= 0 && \textit{in } Y \\ \nabla \times \nabla \times \boldsymbol{h_0} &= \xi_0 \boldsymbol{h_0} && \textit{in } Y \end{aligned} \tag{2.10}$$

and

$$[\boldsymbol{h_0}]\Big|_{\partial R} = 0, \qquad [\boldsymbol{h_0}]\Big|_{\partial P} = 0 \tag{2.11}$$

with

$$\widetilde{H}(curl, Y) := \{u \mid u \in \in H_{\#}(curl, Y), \qquad \nabla \cdot u = 0\}$$

$$H_{\#}(curl, Y) = \Big\{u \;\Big|u \in L^2(Y, \mathbb{C}^3), \quad \nabla \times u \in L^2(Y, \mathbb{C}^3), \quad u \textit{ is unit periodic on } \quad \partial Y\Big\}$$

2.2.2 Homogenization Theorem

In our asymptotic analysis, we find that the electric and magnetic activity in the unit cell are determined by the leading order terms $\boldsymbol{e}_0$ and $\boldsymbol{h}_0$. While the homogenized electric activity is given by the volumetric average of $\boldsymbol{e}_0$ over the unit cell, the homogenized magnetic activity is dewscribed by a new quantity, the geometric average $\oint \boldsymbol{h}_0$ which is given by .

$$\left(\oint \boldsymbol{h}_0\right) \cdot e_k := \int_{\Gamma_k} \boldsymbol{h}_0 \cdot e_k \, d\mathscr{H}^1$$

with Γ_k being a curve connecting two opposite points on the faces of the unit cube Y and orthogonal to the unit vector e_k and lying outside of R. The need for the geometric average is due to the fact that the magnetic field is curl free in $Y \setminus R$, meaning that the magnetic field along any curve connecting two points on the opposite faces of Y and lying in $Y \setminus R$ is the same as can be observed from the definition of $\oint \boldsymbol{h}_0$ above.

Now we are ready to present our main theorem in this chapter.

Theorem 2.2. *(1) The plane waves (homogenized H field) $H_{hom}(x,t)$ and the homogenized magnetic field $B_{hom}(x)$ are given by*

$$
\begin{aligned}
H_{hom}(x,t) &= \left(\oint \boldsymbol{h_0}\right) e^{(ik\hat{k}\cdot x - i\omega_0 t)} \\
B_{hom}(x,t) &= \boldsymbol{\mu}_{eff} H_{hom}(x,t)
\end{aligned}
\tag{2.12}
$$

and $H_{hom}(x,t)$ satisfies

$$
\nabla \times \boldsymbol{\varepsilon}_{eff}^{-1} \nabla \times H_{hom}(x,t) = \frac{\omega_0^2}{c^2} \boldsymbol{\mu}_{eff} H_{hom}(x,t) \tag{2.13}
$$

(2) The plane waves (homogenized E field) $E_{hom}(x,t)$ and the homogenized electric displacement field $D_{hom}(x)$ are given by

$$
\begin{aligned}
E_{hom}(x,t) &= \int_Y \boldsymbol{e_0}\ dx\ e^{(ik\hat{k}\cdot x - i\omega_0 t)} \\
D_{hom}(x,t) &= \boldsymbol{\varepsilon}_{eff} E_{hom}(x,t)
\end{aligned}
\tag{2.14}
$$

and $E_{hom}(x,t)$ satisfies

$$
\nabla \times \boldsymbol{\mu}_{eff}^{-1} \nabla \times E_{hom}(x,t) = \frac{\omega_0^2}{c^2} \boldsymbol{\varepsilon}_{eff} E_{hom}(x,t) \tag{2.15}
$$

where $\boldsymbol{\varepsilon}_{eff}$ and $\boldsymbol{\mu}_{eff}$ are effective dielectric permittivity tensor and effective magnetic permeability tensor respectively.

2.2.3 Maxwell's System of Equations For Homogenized Fields

Theorem 2.3. *The homogenized fields satisfy the following Maxwell's equations for a homogeneous effective media*

$$
\begin{aligned}
\nabla \times E_{hom} &= i\varepsilon_0 \omega_0 B_{hom} \\
\nabla \times H_{hom} &= -i\mu_0 \omega_0 D_{hom} \\
\hat{k} \cdot D_{hom} &= 0 \\
\hat{k} \cdot B_{hom} &= 0
\end{aligned}
\tag{2.16}
$$

2.2.4 Homogenized Dispersion Relation

The homogenized dispersion relation for the electromagnetic waves traveling through the metamaterial crystal is given by the following theorem.

Theorem 2.4. *For given $\hat{k}$ and k, the frequencies ξ_0 for which plane waves can propagate with polarization $\oint \boldsymbol{h_0}$ in the direction $\hat{k}$ at wave length $\lambda = \dfrac{k}{2\pi}$ are the roots of the equation*

$$\boldsymbol{det}\left[\varepsilon_r k^2 \mathbf{A} + \xi_0 \boldsymbol{\mu}_{eff}(\xi_0)\right] = 0 \tag{2.17}$$

with $\mathbf{A}_{ij} = \mathcal{E}_{ipm}\hat{k}_p\mathcal{E}_{mnj}\left[\boldsymbol{\varepsilon}_{eff}^{-1}(\xi_0)\right]_{np}\hat{k}_p, \quad i,p,m,n,j = 1,2,3$ *where $\mathcal{E}_{ipm}$ and $\mathcal{E}_{mnj}$ are the symbols for the Levi-Civita tensors.*

The admissible polarization $\mathbf{v} = \oint \boldsymbol{h_0}$ lie in the null space of the matrix in (2.17) *and*

$$\left[\varepsilon_r k^2 \mathbf{A} + \xi_0 \boldsymbol{\mu}_{eff}(\xi_0)\right] \mathbf{v} = 0 \tag{2.18}$$

By equation (2.17), we can find frequency regimes whereh both $\boldsymbol{\mu}_{eff}$ and $\boldsymbol{\varepsilon}_{eff}$ exhibit negative behavior.

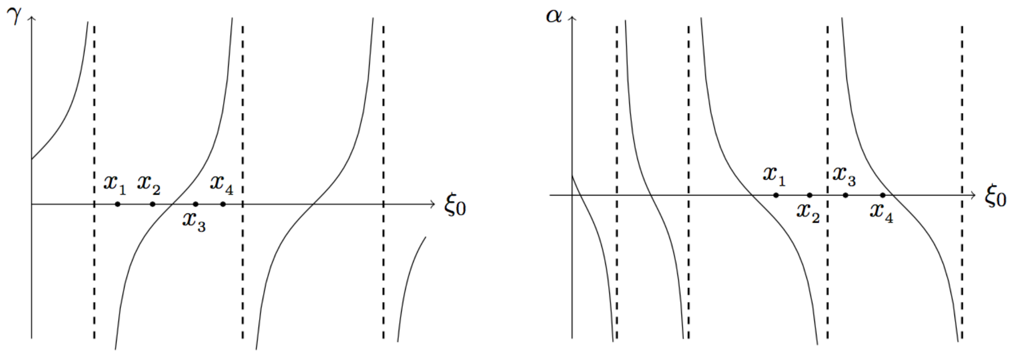
To fix ideas, if we assume that the material is isotropic, then for this media, the dispersion relation is given by

$$\xi_0 = \varepsilon_r k^2 \alpha^{-1} \hat{k} \cdot \hat{k}\ \gamma^{-1} \tag{2.19}$$

where $\alpha,\ \gamma$ are constants appearing in the the formulas $\boldsymbol{\varepsilon}_{eff} = \alpha \mathbf{I}^3, \quad \boldsymbol{\mu}_{eff} = \gamma \mathbf{I}^3$ which is the simplification of formulas of effective properties described in the next theorem.

The equation (2.19) shows the existence of ξ_0 such that both α and γ are negative or positive. The following figures illustrate such $\xi_0 \in [x_1, x_2]$ for which $\alpha < 0, \gamma < 0$

or $\xi_0 \in [x_3, x_4]$ for which $\alpha > 0, \gamma > 0$.



2.2.5 Formulas for Effective Property Tensors

Theorem 2.5. *(1) The effective magnetic permeability tensor $\boldsymbol{\mu}_{eff}$ describing the overall magnetic activity of the electromagnetic wave in the periodic media is given by*

$$\boldsymbol{\mu}_{eff}(\xi_0) = \sum_{1}^{\infty} \frac{\xi_0}{\lambda_n - \xi_0} \left(\int_Y \varphi_n \right) \otimes \left(\int_Y \varphi_n \right) + \mathbf{I}^3 \tag{2.20}$$

where $(\lambda_n,\ \varphi_n),\ n = 1, 2, \cdots$ are the eigenpairs of the following eigenvalue problem

$$\int_Y (\nabla \times \varphi_n) \cdot (\nabla \times \overline{w}) = \lambda_n \int_Y \varphi_n \cdot \overline{w} \qquad\qquad w \in X \tag{2.21}$$

with

$$X := \left\{ u \,\middle|\, u \in W_{\#}^{1,2}(Y, \mathbb{C}^3), \quad \nabla \times u = 0 \;\; in \;\; Y \setminus R, \;\; \nabla \cdot u = 0, \quad \oint u = 0 \right\}$$

and

$$W_{\#}^{1,2}(Y, \mathbb{C}^3) := \left\{ u \;\middle|\; u \in L_{\#}^2(Y, \mathbb{C}^3),\ \partial_i u \in L_{\#}^2(Y, \mathbb{C}^3) \right\}$$

with the inner product

$$(u, v) := \int_Y (\nabla u) : (\nabla \overline{v}) dy \qquad\qquad u, v \in W_{\#}^{1,2}(Y, \mathbb{C}^3)$$

(2) The effective dielectric permittivity tensor $\boldsymbol{\varepsilon}_{eff}$ describing overall electric activity of the electromagnetic wave in the periodic media is given by

$$\boldsymbol{\varepsilon}_{eff}(\xi_0) = (\varepsilon_p(\xi_0)\theta_P + \theta_H)\mathbf{I}^3 - \mathbf{Q}$$

$$\mathbf{Q} = \sum_{0\le\mu_n\le 1} \frac{\varepsilon_p^2(\xi_0)A_{\mu_n}^{P,P} + \varepsilon_p(\xi_0)(A_{\mu_n}^{P,H} + A_{\mu_n}^{H,P}) + A_{\mu_n}^{H,H}}{1-\mu_n+\varepsilon_p(\xi_0)\mu_n}$$

$$A_{\mu_n}^{D_1,D_2} = \left(\int_{D_1} \nabla\overline{\psi_{\mu_n}}\right)\otimes\left(\int_{D_2}\nabla\psi_{\mu_n}\right)$$

where θ_P, θ_H re the volumes of P and H respectively, and ψ_{μ_n} are the eigenfunctions associated to the eigenvalues μ_n of the following eigenvalue problem

$$\mu_n\int_{Y\setminus R}\nabla\psi_{\mu_n}\cdot\nabla v = \int_P \nabla\psi_{\mu_n}\cdot\nabla v, \qquad v\in W_{\#}^{1,2}(Y\setminus R,\mathbb{C})$$

where $W_{\#}^{1,2}(Y\setminus R,\mathbb{C})$ is the restriction of $W_{\#}^{1,2}(Y,\mathbb{C})$ to $Y\setminus R$.

2.3 Proof of Main Results

2.3.1 Proof of Theorem 2.1

We first prove (1) of Theorem 2.1.

Proof. Substitution of

$$E(y) = \sum_{n=0}^{\infty}\boldsymbol{e_n}(y)\eta^n e^{i\eta\hat{k}\cdot y}, \qquad H(y) = \sum_{n=0}^{\infty}\boldsymbol{h_n}(y)\eta^n e^{i\eta\hat{k}\cdot y}, \qquad \sqrt{\xi} = \sum_{n=0}^{\infty}\sqrt{\xi_n}\eta^n$$

into (2.5) gives the following

- In R

$$\begin{aligned}&\tau[(\nabla\times\boldsymbol{e_0}+i\eta\hat{k}\times\boldsymbol{e_0})+\eta(\nabla\times\boldsymbol{e_1}+i\eta\hat{k}\times\boldsymbol{e_1})+\cdots]\\&\quad= i\eta\sqrt{n_0^{-1}}(\sqrt{\xi_0}+\sqrt{\xi_1}\eta+\sqrt{\xi_2}\eta^2+\cdots)(\boldsymbol{h_0}+\eta\boldsymbol{h_1}+\eta^2\boldsymbol{h_2}+\cdots)\end{aligned} \tag{2.22}$$

$$
\begin{aligned}
&[(\nabla \times \boldsymbol{h_0} + i\eta\hat{k} \times \boldsymbol{h_0}) + \eta(\nabla \times \boldsymbol{h_1} + i\eta\hat{k} \times \boldsymbol{h_1}) + \cdots] \\
&= -i\frac{\sqrt{n_0}}{\rho}(\sqrt{\xi_0} + \sqrt{\xi_1}\eta + \sqrt{\xi_2}\eta^2 + \cdots)(\boldsymbol{e_0} + \eta\boldsymbol{e_1} + \eta^2\boldsymbol{e_2} + \cdots)
\end{aligned} \tag{2.23}
$$

- In P

$$
\begin{aligned}
&\tau[(\nabla \times \boldsymbol{e_0} + i\eta\hat{k} \times \boldsymbol{e_0}) + \eta(\nabla \times \boldsymbol{e_1} + i\eta\hat{k} \times \boldsymbol{e_1}) + \cdots] \\
&= i\eta\sqrt{n_0^{-1}}(\sqrt{\xi_0} + \sqrt{\xi_1}\eta + \sqrt{\xi_2}\eta^2 + \cdots)(\boldsymbol{h_0} + \eta\boldsymbol{h_1} + \eta^2\boldsymbol{h_2} + \cdots)
\end{aligned} \tag{2.24}
$$

$$
\begin{aligned}
&\tau[(\nabla \times \boldsymbol{h_0} + i\eta\hat{k} \times \boldsymbol{h_0}) + \eta(\nabla \times \boldsymbol{h_1} + i\eta\hat{k} \times \boldsymbol{h_1}) + \cdots] \\
&= -i\eta\sqrt{n_0}(\sqrt{\xi_0} + \eta\sqrt{\xi_1} + \cdots)(1 - \frac{\xi_r}{\sqrt{\xi_0} + \eta\sqrt{\xi_1} + \cdots})(\boldsymbol{e_0} + \eta\boldsymbol{e_1} + \cdots)
\end{aligned} \tag{2.25}
$$

- In H

$$
\begin{aligned}
&\tau[(\nabla \times \boldsymbol{e_0} + i\eta\hat{k} \times \boldsymbol{e_0}) + \eta(\nabla \times \boldsymbol{e_1} + i\eta\hat{k} \times \boldsymbol{e_1}) + \cdots] \\
&= i\eta\sqrt{n_0^{-1}}(\sqrt{\xi_0} + \sqrt{\xi_1}\eta + \sqrt{\xi_2}\eta^2 + \cdots)(\boldsymbol{h_0} + \eta\boldsymbol{h_1} + \eta^2\boldsymbol{h_2} + \cdots)
\end{aligned} \tag{2.26}
$$

$$
\begin{aligned}
&\tau[(\nabla \times \boldsymbol{h_0} + i\eta\hat{k} \times \boldsymbol{h_0}) + \eta(\nabla \times \boldsymbol{h_1} + i\eta\hat{k} \times \boldsymbol{h_1}) + \cdots] \\
&= -i\eta\sqrt{n_0}(\sqrt{\xi_0} + \sqrt{\xi_1}\eta + \sqrt{\xi_2}\eta^2 + \cdots)(\boldsymbol{e_0} + \eta\boldsymbol{e_1} + \eta^2\boldsymbol{e_2} + \cdots)
\end{aligned} \tag{2.27}
$$

- On R-H interface

$$
n \cdot (\boldsymbol{e_0} + \eta\boldsymbol{e_1} + \cdots)\Big|_{\partial R^-} = \rho^2 n \cdot (\boldsymbol{e_0} + \eta\boldsymbol{e_1} + \cdots)\Big|_{\partial R^+} \tag{2.28}
$$

$$
\begin{aligned}
&n \times [(\nabla \times \boldsymbol{e_0} + i\eta\hat{k} \times \boldsymbol{e_0}) + \eta(\nabla \times \boldsymbol{e_1} + i\eta\hat{k} \times \boldsymbol{e_1}) + \cdots]\Big|_{\partial R^-} \\
&= n \times [(\nabla \times \boldsymbol{e_0} + i\eta\hat{k} \times \boldsymbol{e_0}) + \eta(\nabla \times \boldsymbol{e_1} + i\eta\hat{k} \times \boldsymbol{e_1}) + \cdots]\Big|_{\partial R^+}
\end{aligned} \tag{2.29}
$$

- On P-H interface

$$
\begin{aligned}
&n \cdot \left[(\sqrt{\xi_0} + \eta\sqrt{\xi_1} + \cdots)^2 - \xi_r\right](\boldsymbol{e_0} + \eta\boldsymbol{\xi_1} + \cdots)\Big|_{\partial P^-} \\
&= n \cdot (\sqrt{\xi_0} + \eta\sqrt{\xi_1} + \cdots)(\boldsymbol{e_0} + \eta\boldsymbol{\xi_1} + \cdots)\Big|_{\partial P^+}
\end{aligned} \tag{2.30}
$$

$$
\begin{aligned}
&n \times [(\nabla \times \boldsymbol{e_0} + i\eta\hat{k} \times \boldsymbol{e_0}) + \eta(\nabla \times \boldsymbol{e_1} + i\eta\hat{k} \times \boldsymbol{e_1}) + \cdots]\Big|_{\partial P^-} \\
&= n \times [(\nabla \times \boldsymbol{e_0} + i\eta\hat{k} \times \boldsymbol{e_0}) + \eta(\nabla \times \boldsymbol{e_1} + i\eta\hat{k} \times \boldsymbol{e_1}) + \cdots]\Big|_{\partial P^+}
\end{aligned} \tag{2.31}
$$

We also have

- In R

$$(\nabla \cdot \boldsymbol{e_0} + i\eta\hat{k} \cdot \boldsymbol{e_0}) + \eta(\nabla \cdot \boldsymbol{e_1} + i\eta\hat{k} \cdot \boldsymbol{e_1}) + \cdots = 0 \tag{2.32}$$

- In P

$$(\sqrt{\xi_0} + \eta\sqrt{\xi_1} + \cdots)[(\nabla \cdot \boldsymbol{e_0} + i\eta\hat{k} \cdot \boldsymbol{e_0}) + \eta(\nabla \cdot \boldsymbol{e_1} + i\eta\hat{k} \cdot \boldsymbol{e_1}) + \cdots] = 0 \tag{2.33}$$

- In H

$$(\nabla \cdot \boldsymbol{e_0} + i\eta\hat{k} \cdot \boldsymbol{e_0}) + \eta(\nabla \cdot \boldsymbol{e_1} + i\eta\hat{k} \cdot \boldsymbol{e_1}) + \cdots = 0 \tag{2.34}$$

- On R-H interface

$$n \cdot \frac{1}{\rho^2}\boldsymbol{e_i}\Big|_{\partial R^-} = n \cdot \boldsymbol{e_i}\Big|_{\partial R^+} \tag{2.35}$$

- On P-H interface

$$n \cdot \varepsilon_{p,rel}\boldsymbol{e_i}\Big|_{\partial P^-} = n \cdot \boldsymbol{e_i}\Big|_{\partial P^+} \tag{2.36}$$

From (2.22), (2.24) and (2.26), we have

$$\begin{aligned} \tau(\nabla \times \boldsymbol{e_0}) &= 0, \qquad \text{in R} \\ \tau(\nabla \times \boldsymbol{e_0}) &= 0, \qquad \text{in P} \\ \tau(\nabla \times \boldsymbol{e_0}) &= 0, \qquad \text{in H} \end{aligned} \tag{2.37}$$

From (2.29), (2.31), and (2.36), we have

$$\begin{aligned} &n \times \nabla \times \boldsymbol{e_0}\Big|_{\partial R^-} = n \times \nabla \times \boldsymbol{e_0}\Big|_{\partial R^+} \\ &n \times \nabla \times \boldsymbol{e_0}\Big|_{\partial P^-} = n \times \nabla \times \boldsymbol{e_0}\Big|_{\partial P^+} \\ &\varepsilon_{p,rel}(\xi_0) n \cdot \boldsymbol{e_0}\Big|_{\partial P^-} = n \cdot \boldsymbol{e_0}\Big|_{\partial P^+} \end{aligned} \tag{2.38}$$

From (2.32)-(2.35), we have

$$\begin{aligned}
\nabla \cdot \boldsymbol{e_0} &= 0, \qquad \text{in R} \\
\nabla \cdot \varepsilon_{p,rel}\boldsymbol{e_0} &= 0, \qquad \text{in P} \\
\nabla \cdot \boldsymbol{e_0} &= 0, \qquad \text{in H} \\
n \cdot \boldsymbol{e_0}\Big|_{\partial R^-} &= 0
\end{aligned} \tag{2.39}$$

Using (2.37), we find that

$$\begin{aligned}
\int_Y |\nabla \times \boldsymbol{e_0}|^2 dy &= \int_R |\nabla \times \boldsymbol{e_0}|^2 dy + \int_P |\nabla \times \boldsymbol{e_0}|^2 dy + \int_H |\nabla \times \boldsymbol{e_0}|^2 dy \\
&= 0
\end{aligned} \tag{2.40}$$

So $\boldsymbol{e_0}$ can be written in Y as

$$\boldsymbol{e_0} = \nabla\varphi + \mathbf{c}, \qquad \varphi \in W^{1,2}_{\#}(Y, \mathbb{C}), \qquad \mathbf{c} \in \mathbb{C}^3 \tag{2.41}$$

Using (2.39), we observe that

$$\begin{aligned}
\int_R |\boldsymbol{e_0}|^2 dy &= \int_R \boldsymbol{e_0} \cdot \nabla(\overline{\varphi + \mathbf{c} \cdot y}) dy \\
&= \int_{\partial R^-} n \cdot \boldsymbol{e_0}(\varphi + \mathbf{c} \cdot y) ds \\
&= 0
\end{aligned} \tag{2.42}$$

and this imples that $\boldsymbol{e_0} = 0$ in R. But since $\boldsymbol{e_0} = \nabla\varphi + \mathbf{c} = 0$ in R, we notice that

$$\begin{aligned}
\varphi &= \mathbf{c} \cdot \varphi^k, \qquad k = 1, 2, 3 \\
\varphi^k &= -y_k, \qquad y_k \in \mathbb{C}^3
\end{aligned} \tag{2.43}$$

with $\varphi = \mathbf{c} \cdot \varphi^k$, and φ^k is the solution to

$$\begin{aligned}
\nabla \cdot [a_p(y)(\nabla\varphi^k + \tilde{e}^k)] &= 0 \\
n \cdot \varepsilon_{p,rel}(\xi_0)(\nabla\varphi^k + \tilde{e}^k)\Big|_{\partial P^-} &= n \cdot (\nabla\varphi^k + \tilde{e}^k)\Big|_{\partial P^+}
\end{aligned} \tag{2.44}$$

where $\tilde{e}^k,\ k = 1, 2, 3$ are the unit normal vectors originated from the origin, and

$$a_p(y) = \begin{cases} 1 & y \in H \\ \varepsilon_{p,rel}(\xi_0) & y \in P \end{cases} \tag{2.45}$$

Finally, we have

$$\int_Y \boldsymbol{e_0} dy = \int_Y (\nabla\varphi + \mathbf{c}) dy$$
$$\int_{\partial Y} n \ \varphi \ ds + \mathbf{c} \tag{2.46}$$
$$= \mathbf{c}$$

which completes the proof of (1). □

Now, we prove (2) of Theorem 2.1.

Proof. The equation (2.11) is obvious.

Next, we prove the first two equations in (2.10). To do this, will collect the 0^{th} order terms of η from (2.22)-(2.36) to identify the problem $\boldsymbol{h_0}$ satisfies.

From (2.25) and (2.27) along with the fact that $[\boldsymbol{h}_0] = 0$ across the boundaries of inclusions, we have

$$\begin{aligned} \nabla \times \boldsymbol{h_0} &= 0 \qquad \text{in} \ \ P \\ \nabla \times \boldsymbol{h_0} &= 0 \qquad \text{in} \ \ H \\ \nabla \cdot \boldsymbol{h_0} &= 0 \qquad \text{in} \ \ Y \end{aligned} \tag{2.47}$$

which directly gives the second equation in (2.10). From the fact that $[\boldsymbol{h}_0] = 0$ across the boundaries of inclusions, we have

$$[n \times \nabla \times \boldsymbol{h_0}]\Big|_{\partial R} = 0$$

$$[n \times \nabla \times \boldsymbol{h_0}]\Big|_{\partial P} = 0$$

and

$$[n \cdot \boldsymbol{h_0}]\Big|_{\partial R} = 0$$

$$[n \cdot \boldsymbol{h_0}]\Big|_{\partial P} = 0$$

Now we observe that, for any $\psi \in W^{1,2}_{\#}(Y \setminus R)$

$$\begin{aligned}
&\int_{Y\setminus R} \nabla \times \boldsymbol{h_0} \cdot \nabla\psi \; dy \\
&= \int_P \nabla \times \boldsymbol{h_0} \cdot \nabla\psi \; dy + \int_H \nabla \times \boldsymbol{h_0} \cdot \nabla\psi \; dy \\
&= \int_{\partial R} (n \cdot \nabla \times \boldsymbol{h_0})\Big|_{\partial R^+} \psi ds + \int_{\partial P} [n \cdot \nabla \times \boldsymbol{h_0}]\Big|_{\partial P} \psi ds
\end{aligned} \tag{2.48}$$

If $\boldsymbol{h_0} = (h_1,\ h_2,\ h_3)$, then

$$\nabla \times \boldsymbol{h_0} = (\partial_2 h_3 - \partial_3 h_2, \quad -(\partial_1 h_3 - \partial_3 h_1), \quad \partial_1 h_2 - \partial_2 h_1)$$

$$n \cdot \nabla \times \boldsymbol{h_0} = (n_1 \partial_2 h_3 - n_2 \partial_1 h_3) - (n_3 \partial_1 h_2 - n_1 \partial_3 h_2) e_2 + (n_2 \partial_3 h_1 - n_3 \partial_2 h_1)$$

and since $n \cdot \nabla \times \boldsymbol{h_0}$ is only involving tangential derivatives, we conclude that

$$\int_{\partial P} [n \cdot \nabla \times \boldsymbol{h_0}]\Big|_{\partial P} \psi ds = 0 \tag{2.49}$$

and we get

$$\int_{\partial R} (n \cdot \nabla \times \boldsymbol{h_0})\Big|_{\partial R^+} \psi ds = 0 \tag{2.50}$$

equations (2.48), (2.49), and (2.50) completes the proof of the first equation in (2.10).

Finally, we prove the third equation in (2.10). To do this, we write (2.23) as

$$\begin{aligned}
&(\nabla + i\eta\hat{k}) \times (\boldsymbol{h_0} + \eta \boldsymbol{h_1} + \cdots) \\
&\qquad = -i\frac{\sqrt{n_0}}{\rho}(\sqrt{\xi_0} + \eta\sqrt{\xi_1} + \cdots)(\boldsymbol{e_0} + \eta \boldsymbol{e_1} + \cdots)
\end{aligned} \tag{2.51}$$

and apply the differential operator $(\nabla + i\eta\hat{k})\times$ to the both side of (2.51) to get

$$\begin{aligned}
&(\nabla + i\eta\hat{k}) \times (\nabla + i\eta\hat{k}) \times (\boldsymbol{h_0} + \eta \boldsymbol{h_1} + \cdots) \\
&\qquad = -i\frac{\sqrt{n_0}}{\rho}(\sqrt{\xi_0} + \eta\sqrt{\xi_1} + \cdots)(\nabla + i\eta\hat{k}) \times (\boldsymbol{e_0} + \eta \boldsymbol{e_1} + \cdots)
\end{aligned} \tag{2.52}$$

Collecting the 0^{th} order term of η in (2.52), we find

$$\nabla \times \nabla \times \boldsymbol{h_0} = -i\tau\sqrt{n_0}\sqrt{\xi_0}(\nabla \times \boldsymbol{e_0} + i\hat{k} \times \boldsymbol{e_0}) \qquad \text{in} \quad R \tag{2.53}$$

Collecting the first order term of η in (2.22), we find

$$\tau\sqrt{n_0}\sqrt{\xi_0}(\nabla \times \boldsymbol{e_0} + i\hat{k} \times \boldsymbol{e_0}) = i\sqrt{n_0^{-1}}\sqrt{\xi_0}\boldsymbol{h_0} \qquad \text{in} \quad R \tag{2.54}$$

Substituting (2.54) into (2.53), we get

$$\nabla \times \nabla \times \boldsymbol{h_0} = \xi_0 \boldsymbol{h_0} \qquad \text{in} \quad R \tag{2.55}$$

Taking advantage of the first equation in (2.10) and the relevant result in [1] completes the proof of (2). □

2.3.2 Proof of Theorem 2.2

Proof. The equations (2.12) and (2.14) are obvious.

Now we prove (2.13).

Comparing the terms with the first power of η in (2.23)-(2.27), we have the following equations

$$\begin{aligned} \tau(\nabla \times \boldsymbol{e_1} + i\hat{k} \times \boldsymbol{e_0}) &= i\sqrt{n_0^{-1}}\sqrt{\xi_0}\boldsymbol{h_0} \qquad && \text{in} \quad R \\ \tau(\nabla \times \boldsymbol{e_1} + i\hat{k} \times \boldsymbol{e_0}) &= i\sqrt{n_0^{-1}}\sqrt{\xi_0}\boldsymbol{h_0} \qquad && \text{in} \quad P \\ \tau(\nabla \times \boldsymbol{e_1} + i\hat{k} \times \boldsymbol{e_0}) &= i\sqrt{n_0^{-1}}\sqrt{\xi_0}\boldsymbol{h_0} \qquad && \text{in} \quad H \end{aligned} \tag{2.56}$$

$$\begin{aligned} (\nabla \times \boldsymbol{h_1} + i\hat{k} \times \boldsymbol{h_0}) &= -i\tau\sqrt{n_0}\sqrt{\xi_0}\boldsymbol{e_1} \qquad && \text{in} \quad R \\ \tau(\nabla \times \boldsymbol{h_1} + i\hat{k} \times \boldsymbol{h_0}) &= -i\sqrt{n_0}\sqrt{\xi_0}\varepsilon_{p,rel}(\xi_0)\boldsymbol{e_0} \qquad && \text{in} \quad P \\ \tau(\nabla \times \boldsymbol{h_1} + i\hat{k} \times \boldsymbol{h_0}) &= -i\sqrt{n_0}\sqrt{\xi_0}\boldsymbol{e_0} \qquad && \text{in} \quad H \end{aligned} \tag{2.57}$$

Integrating equations in (2.56) in their respective domains and adding them up, we get

$$\tau \int_Y i\hat{k} \times \boldsymbol{e_0}\ dy = i\sqrt{n_0^{-1}}\sqrt{\xi_0}\int_Y \boldsymbol{h_0}\ dy \tag{2.58}$$

We define the effective magnetic permeability to be the tensor $\boldsymbol{\mu}_{eff}$ such that

$$\boldsymbol{\mu}_{eff}\left(\oint \boldsymbol{h_0}\right) = \int_Y \boldsymbol{h_0}(y)dy \tag{2.59}$$

Now we can write equation (2.58) as

$$\tau \ i\hat{k} \times \left(\int_Y \boldsymbol{e_0} \right) = i\sqrt{n_0^{-1}}\sqrt{\xi_0}\boldsymbol{\mu}_{eff} \left(\oint \ \boldsymbol{h_0} \right) \tag{2.60}$$

By setting $e^{*,i} = \nabla(\varphi^i + y_i), \ \ i = 1,2,3$ where φ^i and y_i are defined as in (2.43), we notice that

$$e^{*,i} = 0 \ \text{ in R}, \qquad \nabla \cdot e^{*,i} = 0, \ \text{ in Y}, \qquad \int_Y e^{*,i} = \widetilde{e}^i, \quad i = 1,2,3 \tag{2.61}$$

and for any function $p \in W_{\#}^{1,2}(Y, \mathbb{C})$

$$\begin{aligned} \int_Y \nabla p \cdot (\nabla \varphi^i + \widetilde{e}^i) \ dy = \int_{Y \backslash R} p \ \nabla \cdot e^{*,i} \ dy + \int_R p \ \nabla \cdot e^{*,i} \ dy \\ + \int_{\partial R} p \ n_i \cdot [\nabla \varphi^i + \widetilde{e}^i] \ ds + \int_{\partial Y} p \ n_i \cdot [\nabla \varphi^i + \widetilde{e}^i] \ ds = 0 \end{aligned} \tag{2.62}$$

where n_i is the ith component of the outward unit normal vector to ∂R. We multiply $e^{*,i}$ to each equation in (2.57) and integrate them over their respective domains. Adding the resulted equations and noting that $e^{*,i} = 0$ in R, we get

$$\tau \left(\int_Y \nabla \times \boldsymbol{h_1} \cdot e^{*,i} \ dy + \int_Y i\hat{k} \times \boldsymbol{h_0} \cdot e^{*,i} \ dy \right) = -i\sqrt{n_0}\sqrt{\xi_0} \int_Y \varepsilon_{rel}^d(\xi_0, y)\boldsymbol{e_0} \cdot e^{*,i} \ dy \tag{2.63}$$

We first observe that

$$\begin{aligned} &\int_Y \nabla \times \boldsymbol{h_1} \cdot \nabla(\varphi^i + y_i) \ dy \\ &= \int_{Y \backslash R} \nabla \times \boldsymbol{h_1} \cdot \nabla(\varphi^i + y_i) \ dy + \int_R \nabla \times \boldsymbol{h_1} \cdot \nabla(\varphi^i + y_i) \ dy \\ &+ \int_{Y \backslash R} \boldsymbol{h_1} \cdot \nabla \times \nabla(\varphi^i + y_i) \ dy + \int_R \boldsymbol{h_1} \cdot \nabla \times \nabla(\varphi^i + y_i) \ dy \\ &+ \int_{\partial R} \boldsymbol{h_1} \times n_i \cdot [\nabla \varphi^i + \widetilde{e}^i] \ ds + \int_{\partial Y} \boldsymbol{h_1} \times n_i \cdot [\nabla \varphi^i + \widetilde{e}^i] \ ds \\ &= 0 \end{aligned} \tag{2.64}$$

By similar calculation, we find that

$$\begin{aligned} &\int_Y \varepsilon_{rel}^d(\xi_0, y)\boldsymbol{e_0} \cdot e^{*,i} \ dy \\ &= \int_Y \varepsilon_{rel}^d(\xi_0, y)\boldsymbol{e_0} \cdot (\nabla \varphi^i + \widetilde{e}^i) \ dy \\ &= \int_Y \varepsilon_{rel}^d(\xi_0, y)(\nabla \chi + \overline{e}) dy \cdot \widetilde{e}^i \end{aligned} \tag{2.65}$$

where

$$\boldsymbol{e_0} = \nabla\chi + \overline{e}, \qquad \int_Y \boldsymbol{e_0}\; dy = \overline{e}, \qquad \chi \in W^{1,2}_{\#}(Y, \mathbb{C}) \tag{2.66}$$

Also, by the property of geometric average [1],

$$\begin{aligned} & i\tau \int_Y \hat{k} \times \boldsymbol{h_0} \cdot e^{*,i}\; dy \\ & = i\tau \cdot \int_Y \boldsymbol{h_0}\; dy e^{*,i} \times \hat{k} \\ & = i\tau \cdot \int_Y \boldsymbol{h_0}\; dy(\nabla\varphi^i + \widetilde{e}^i) \times \hat{k} \\ & = i\tau \left(\hat{k} \times \oint \boldsymbol{h_0}\right) \cdot \widetilde{e}^i \end{aligned} \tag{2.67}$$

Finally, applying (2.64), (2.65), and (2.67) in (2.63) gives

$$i\tau \left(\hat{k} \times \oint \boldsymbol{h_0}\right) \cdot \widetilde{e}^i = -i\sqrt{n_0}\sqrt{\xi_0} \int_Y \varepsilon^d_{rel}(\xi_0, y)(\nabla\chi + \overline{e})dy \cdot \widetilde{e}^i \tag{2.68}$$

We now define the effective dielectric permittivity tensor to be $\boldsymbol{\varepsilon}_{eff}$ such that

$$\boldsymbol{\varepsilon}_{eff}\overline{e} \cdot \widetilde{e}^i := \int_Y \varepsilon^d_{rel}(\xi_0, y)(\nabla\chi + \overline{e})dy \cdot \widetilde{e}^i \tag{2.69}$$

and write (2.68) as

$$\tau \left(\hat{k} \times \oint \boldsymbol{h_0}\right) = -\sqrt{n_0}\sqrt{\xi_0}\boldsymbol{\varepsilon}_{eff} \left(\int_Y \boldsymbol{e_0}\right) \tag{2.70}$$

which gives

$$\int_Y \boldsymbol{e_0}\; dy = -\tau \frac{1}{\sqrt{\xi_0 n_0}} \boldsymbol{\varepsilon}^{-1}_{eff} \left(\hat{k} \times \oint \boldsymbol{h_0}\right) \tag{2.71}$$

Substituting (2.71) into (2.60), we obtain

$$-\tau^2 \hat{k} \times \boldsymbol{\varepsilon}^{-1}_{eff} \hat{k} \times \oint \boldsymbol{h_0} = \xi_0 \boldsymbol{\mu}_{eff} \oint \boldsymbol{h_0} \tag{2.72}$$

Multiplying $e^{i\eta\hat{k}\cdot\frac{x}{d}}$ to both sides of (2.72), we notice that

$$-\tau^2 \left(\hat{k} \times \boldsymbol{\varepsilon}^{-1}_{eff} \hat{k} \times \oint \boldsymbol{h_0}\right) e^{i\eta\hat{k}\cdot\frac{x}{d}} = \xi_0 \boldsymbol{\mu}_{eff} \oint \boldsymbol{h_0}\; e^{i\eta\hat{k}\cdot\frac{x}{d}} \tag{2.73}$$

By the relation

$$\partial^2_x \left(e^{i\eta\hat{k}\cdot\frac{x}{d}}\right) = \partial^2_x(e^{ik\hat{k}\cdot x}) \tag{2.74}$$

and vector product identity, we arrive at

$$\nabla_x \times \boldsymbol{\varepsilon}_{eff}^{-1} \nabla_x \times \oint \boldsymbol{h_0}\ e^{i\kappa\cdot x} = \frac{\omega_0^2}{c^2}\boldsymbol{\mu}_{eff} \oint \boldsymbol{h_0}\ e^{i\kappa\cdot x} \tag{2.75}$$

Multiplying $e^{-i\omega t}$ to the both sides of (2.75) and noting that $H_{\text{hom}}(x,t) = \left(\oint \ \boldsymbol{h_0}\right) e^{(ik\hat{k}\cdot x - i\omega_0 t)}$ completes the proof of (2.13).

(2) By expressing $\oint \boldsymbol{h}_0$ in (2.60) and substituting it into (2.70), we get

$$-\tau^2 \hat{k} \times \boldsymbol{\mu}_{eff}^{-1} \hat{k} \times \int_Y \boldsymbol{e_0} = \xi_0 \boldsymbol{\varepsilon}_{eff} \int_Y \boldsymbol{e_0}$$

Multiplying $e^{i\eta\hat{k}\cdot\frac{x}{d}}$ to both sides and using relation (2.74), we get

$$\nabla_x \times \boldsymbol{\mu}_{eff}^{-1} \times \int_Y \boldsymbol{e_0}\ e^{i\kappa\cdot x} = \frac{\omega_0^2}{c^2}\boldsymbol{\varepsilon}_{eff} \int_Y \boldsymbol{e_0}\ e^{i\eta\hat{k}\cdot\frac{x}{d}}$$

Multiplying $e^{-i\omega t}$ to the both sides and noting that $E_{\text{hom}}(x,t) = \int_Y \boldsymbol{e_0}\, dx\ e^{(ik\hat{k}\cdot x - i\omega_0 t)}$ completes the proof of (2.15).

□

2.3.3 Proof of Theorem 2.3

Proof. By multiplying $e^{i\eta\hat{k}\cdot\frac{x}{d}}$ to equation (2.60) and substituting $c\varepsilon_0 = \sqrt{n_0^{-1}},\ \ \sqrt{\xi_0} = \frac{\omega_0}{c}\varepsilon_r$, we find that

$$\nabla_x \times \left(\int_Y \ \boldsymbol{e}_0\ e^{i\kappa\cdot x}\right) = i\varepsilon_0\omega_0 \boldsymbol{\mu}_{eff} \left(\oint \boldsymbol{h}_0\ e^{i\kappa\cdot x}\right)$$

Multiplying $e^{-i\omega_0 t}$ to both sides and using (2.12) completes the proof of the first equation in (2.3).

By multiplying $e^{i\eta\hat{k}\cdot\frac{x}{d}}$ to equation (2.70) and substituting $c\varepsilon_0 = \sqrt{n_0^{-1}},\ \ \sqrt{\xi_0} = \frac{\omega_0}{c}\varepsilon_r$, we find that

$$\nabla_x \times \left(\oint_Y \ \boldsymbol{h}_0\ e^{i\kappa\cdot x}\right) = -i\mu_0\omega_0 \boldsymbol{\varepsilon}_{eff} \left(\int \boldsymbol{e}_0\ e^{i\kappa\cdot x}\right)$$

Finally, multiplying $e^{-i\omega_0 t}$ to both sides and using (2.14) completes the proof of the second equation in (2.3).

The third and the fourth equations are the direct consequence of applying the divergence operator to H_{hom} and E_{hom} in (2.12) and (2.14).

□

2.3.4 Proof of Theorem 2.4

Proof. We start with equation (2.72). Using Einstein summation notation and Levi-Civita tensor notations, we can write

$$\Big[\boldsymbol{\varepsilon}_{eff}^{-1}(\xi_0)\hat{k} \times \oint \boldsymbol{h}_0\Big]_m = \mathcal{E}_{mnj}\Big[\boldsymbol{\varepsilon}_{eff}^{-1}(\xi_0)\hat{k}\Big]_n\Big[\oint \boldsymbol{h}_0\Big]_j \tag{2.76}$$

and

$$\left(\hat{k} \times \Big[\boldsymbol{\varepsilon}_{eff}^{-1}(\xi_0)\hat{k} \times \oint \boldsymbol{h}_0\Big]\right)_i = \mathcal{E}_{ipm}\hat{k}_p\Big[\boldsymbol{\varepsilon}_{eff}^{-1}(\xi_0)\hat{k} \times \oint \boldsymbol{h}_0\Big]_m \tag{2.77}$$

Using (2.76) in (2.77), we get

$$\left(\hat{k} \times \Big[\boldsymbol{\varepsilon}_{eff}^{-1}(\xi_0)\hat{k} \times \oint \boldsymbol{h}_0\Big]\right)_i = \mathcal{E}_{ipm}\hat{k}_p\mathcal{E}_{mnj}\Big[\boldsymbol{\varepsilon}_{eff}^{-1}(\xi_0)\hat{k}\Big]_n\Big[\oint \boldsymbol{h}_0\Big]_j \tag{2.78}$$

We can also write

$$\Big[\boldsymbol{\varepsilon}_{eff}^{-1}(\xi_0)\hat{k}\Big]_n = \Big[\boldsymbol{\varepsilon}_{eff}^{-1}(\xi_0)\Big]_{np}\hat{k}_p \tag{2.79}$$

So finally, we get

$$\left(\hat{k}\times\Big[\boldsymbol{\varepsilon}_{eff}^{-1}(\xi_0)\hat{k}\times\oint \boldsymbol{h}_0\Big]\right)_i = \mathcal{E}_{ipm}\hat{k}_p\mathcal{E}_{mnj}\Big[\boldsymbol{\varepsilon}_{eff}^{-1}(\xi_0)\Big]_{np}\hat{k}_p\Big[\oint \boldsymbol{h}_0\Big]_j, \qquad i,p,m,n,j = 1,2,3. \tag{2.80}$$

By the matching of ith component, we can also write

$$\xi_0\boldsymbol{\mu}_{eff}(\xi_0)\oint \boldsymbol{h}_0 = \xi_0\Big[\boldsymbol{\mu}_{eff}(\xi_0)\Big]_{ij}\Big[\oint \boldsymbol{h}_0\Big]_j \tag{2.81}$$

So equation (2.72) is written by components as

$$\begin{aligned} 0 &= \tau^2\ \hat{k} \times \boldsymbol{\varepsilon}_{eff}^{-1}(\xi_0)\hat{k} \times \oint \boldsymbol{h}_0 + \xi_0\boldsymbol{\mu}_{eff}(\xi_0)\oint \boldsymbol{h}_0 \\ &= \tau^2\mathcal{E}_{ipm}\hat{k}_p\mathcal{E}_{mnj}\Big[\boldsymbol{\varepsilon}_{eff}^{-1}(\xi_0)\Big]_{np}\hat{k}_p\Big[\oint \boldsymbol{h}_0\Big]_j + \xi_0\Big[\boldsymbol{\mu}_{eff}(\xi_0)\Big]_{ij}\Big[\oint \boldsymbol{h}_0\Big]_j \end{aligned} \tag{2.82}$$

Equation (2.82) implies that the determinant equation for ξ_0 for a given wave number k can be written as

$$\mathbf{det}\left[\tau^2\mathbf{A}+\xi_0\boldsymbol{\mu}_{eff}(\xi_0)\right]=0, \qquad \mathbf{A}_{ij}=\mathcal{E}_{ipm}\hat{k}_p\mathcal{E}_{mnj}\left[\boldsymbol{\varepsilon}_{eff}^{-1}(\xi_0)\right]_{np}\hat{k}_p, \qquad i,p,m,n,j=1,2,3. \tag{2.83}$$

Noting that $\tau^2=k^2\varepsilon_r$, $\xi_0=\dfrac{\omega_0^2}{c^2}\varepsilon_r$ completes the proof.

□

2.3.5 Proof of Theorem 2.5

Proof. (1) We will show the formula (2.20) by applying the series expansion for $\boldsymbol{h_0}$ in the definition of $\boldsymbol{\mu}_{eff}$ given by

$$\boldsymbol{\mu}_{eff}\left(\oint \boldsymbol{h_0}\right)=\int_Y \boldsymbol{h_0}(y)dy \tag{2.84}$$

The leading order theory shows that $\boldsymbol{h_0}$ belongs to the space $H^*(\text{curl},Y)$ with

$$H^*(\text{curl},Y):=\{u\in H_\#(\text{curl},Y)\mid \quad \nabla\cdot u=0, \qquad \nabla\times u=0 \quad \text{in} \quad Y\setminus R\},$$

$$H_\#(\text{curl},Y)=\left\{u\;\middle|u\in L^2(Y,\mathbb{C}^3),\ \ \nabla\times u\in L^2(Y,\mathbb{C}^3),\ \ u \text{ is unit periodic on } \partial Y\right\}$$

and solves the following problem

$$\int_Y(\nabla\times\boldsymbol{h_0})\cdot(\nabla\times\overline{w})=\int_Y\xi_0\boldsymbol{h_0}\cdot\overline{w}, \qquad w\in H^*(\text{curl},Y) \tag{2.85}$$

If we denote the geometric average of $\boldsymbol{h_0}$ by $\oint \boldsymbol{h_0}:=z\in\mathbb{C}^3$ and write $\boldsymbol{h_0}=h^*+z$ in $Y\setminus R$, then we find that h^* solves

$$\int_Y(\nabla\times h^*)\cdot(\nabla\times\overline{w})-\xi_0\int_Y h^*\cdot\overline{w}=\xi_0\int_Y z\cdot\overline{w} \qquad\qquad w\in X \tag{2.86}$$

It's been shown in [1] that the solution h^* to the problem (2.86) is given by $h^*=\sum_{n=1}^{\infty}a_n\varphi_n$ where $\{\varphi_n\}$ is the family of real valued, orthonormal family of eigenfunctions associated to the eigenvalues λ_n of the eigenvalue problem

$$\int_Y(\nabla\times\varphi_n)\cdot(\nabla\times\overline{w})=\lambda_n\int_Y\varphi_n\cdot\overline{w} \qquad\qquad w\in X \tag{2.87}$$

and $X = \text{span}\{\varphi_n\}$.

We write $\boldsymbol{h_0} = \boldsymbol{h_0^z}$ where $\boldsymbol{h_0^z}$ linearly depends on z. Since $\boldsymbol{h_0^z}$ is curl free in $Y \setminus R$, we have:

$$\nabla \times \nabla \times \boldsymbol{h_0^z} = \xi_0 \boldsymbol{h_0^z}$$

$$\boldsymbol{h_0^z}(y) = h^{*z}(y) + z$$

$$z = \oint \boldsymbol{h_0^z}, \quad h^{*z}(y) \in X$$

Using linearity, we write

$$\boldsymbol{h_0^z}(y) = z_1 \boldsymbol{h_0^1}(y) + z_2 \boldsymbol{h_0^2}(y) + z_3 \boldsymbol{h_0^3}(y), \qquad y \in Y$$

where $\boldsymbol{h_0^i}$ is the solution of the following problem:

$$\nabla \times \nabla \times \boldsymbol{h_0^i} = \xi_0 \boldsymbol{h_0^i}$$

$$\boldsymbol{h_0^i}(y) = h^{*i}(y) + e_i$$

$$e_i = \oint \boldsymbol{h_0^i}, \quad h^{*i}(y) \in X$$

For $z = \vec{\xi} = \oint \boldsymbol{h_0^{\vec{\xi}}} \in \mathbb{C}^3$, we write

$$\begin{aligned} \vec{\xi} &= \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 \\ \boldsymbol{h_0^z} &= \boldsymbol{h_0^{\vec{\xi}}} = \xi_1 \boldsymbol{h_0^1} + \xi_2 \boldsymbol{h_0^2} + \xi_3 \boldsymbol{h_0^3} \end{aligned} \tag{2.88}$$

Substituting $h^{*1}(y) = \sum_{n=1}^{\infty} a_n \varphi_n$ into the strong form of (2.86) with $z = e_1$, we see that

$$\begin{aligned} &\nabla \times \nabla \times \left(\sum_{n=1}^{\infty} a_n \varphi_n + e_1 \right) \\ &= \nabla \times \nabla \times \left(\sum_{n=1}^{\infty} a_n \varphi_n \right) \\ &= \lambda_n \sum_{n=1}^{\infty} a_n \varphi_n \\ &= \xi_0 \sum_{n=1}^{\infty} a_n \varphi_n + \xi_0 e_1 \end{aligned} \tag{2.89}$$

We take the dot product of both sides with φ_k and integrate over Y to get

$$\sum_{n=1}^{\infty}\int_Y a_n\lambda_n\varphi_n\cdot\varphi_k = \xi_0\sum_{n=1}^{\infty}\int_Y a_n\varphi_k\cdot\varphi_n + \int_Y \xi_0 e_1\cdot\varphi_k$$

Using the orthogonality of the basis functions $\{\varphi_n\}$, we find

$$a_n\lambda_n = \xi_0 a_n + \xi_0 e_1\cdot\int_Y\varphi_n$$

Then, we get

$$\begin{aligned}
a_n &= \frac{\xi_0 e_1\cdot\int_Y\varphi_n}{\lambda_n-\xi_0}\\
h^{*1}(y) &= \sum_{n=1}^{\infty}\frac{\xi_0 e_1\cdot\int_Y\varphi_n}{\lambda_n-\xi_0}\varphi_n \qquad (2.90)\\
\boldsymbol{h}_{\boldsymbol{0}}^1 &= \sum_{n=1}^{\infty}\frac{\xi_0 e_1\cdot\int_Y\varphi_n}{\lambda_n-\xi_0}\varphi_n + e_1
\end{aligned}$$

Repeating the above process for $h^{*2}(y) = \sum_{n=1}^{\infty} b_n\varphi_n,\quad$ and $\quad h^{*3}(y) = \sum_{n=1}^{\infty} c_n\varphi_n,$ we get

$$\begin{aligned}
b_n &= \frac{\xi_0 e_2\cdot\int_Y\varphi_n}{\lambda_n-\xi_0}\\
h^{*2}(y) &= \sum_{n=1}^{\infty}\frac{\xi_0 e_2\cdot\int_Y\varphi_n}{\lambda_n-\xi_0}\varphi_n \qquad (2.91)\\
\boldsymbol{h}_{\boldsymbol{0}}^2 &= \sum_{n=1}^{\infty}\frac{\xi_0 e_2\cdot\int_Y\varphi_n}{\lambda_n-\xi_0}\varphi_n + e_2
\end{aligned}$$

and

$$\begin{aligned}
c_n &= \frac{\xi_0 e_3\cdot\int_Y\varphi_n}{\lambda_n-\xi_0}\\
h^{*3}(y) &= \sum_{n=1}^{\infty}\frac{\xi_0 e_3\cdot\int_Y\varphi_n}{\lambda_n-\xi_0}\varphi_n \qquad (2.92)\\
\boldsymbol{h}_{\boldsymbol{0}}^3 &= \sum_{n=1}^{\infty}\frac{\xi_0 e_3\cdot\int_Y\varphi_n}{\lambda_n-\xi_0}\varphi_n + e_3
\end{aligned}$$

Now, we have

$$
\begin{aligned}
\boldsymbol{h}_{\mathbf{0}}^1 &= \sum_{n=1}^{\infty} \frac{\xi_0 e_1 \cdot \int_Y \varphi_n}{\lambda_n - \xi_0} \varphi_n + e_1 \\
\boldsymbol{h}_{\mathbf{0}}^2 &= \sum_{n=1}^{\infty} \frac{\xi_0 e_2 \cdot \int_Y \varphi_n}{\lambda_n - \xi_0} \varphi_n + e_2 \\
\boldsymbol{h}_{\mathbf{0}}^3 &= \sum_{n=1}^{\infty} \frac{\xi_0 e_3 \cdot \int_Y \varphi_n}{\lambda_n - \xi_0} \varphi_n + e_3
\end{aligned}
\tag{2.93}
$$

and

$$
\begin{aligned}
\boldsymbol{h}_{\mathbf{0}}^{\vec{\xi}} &= \xi_1 \boldsymbol{h}_{\mathbf{0}}^1 + \xi_2 \boldsymbol{h}_{\mathbf{0}}^2 + \xi_3 \boldsymbol{h}_{\mathbf{0}}^3 \\
&= \sum_{n=1}^{\infty} \frac{\xi_0 \xi_1 e_1 \cdot \int_Y \varphi_n}{\lambda_n - \xi_0} \varphi_n + \xi_1 e_1 \\
&+ \sum_{n=1}^{\infty} \frac{\xi_0 \xi_2 e_2 \cdot \int_Y \varphi_n}{\lambda_n - \xi_0} \varphi_n + \xi_2 e_2 \\
&+ \sum_{n=1}^{\infty} \frac{\xi_0 \xi_3 e_3 \cdot \int_Y \varphi_n}{\lambda_n - \xi_0} \varphi_n + \xi_3 e_3 \\
&= \sum_{n=1}^{\infty} \frac{\xi_0 \vec{\xi} \cdot \int_Y \varphi_n}{\lambda_n - \xi_0} \varphi_n + \vec{\xi}
\end{aligned}
\tag{2.94}
$$

Since $\boldsymbol{h}_{\mathbf{0}} = \boldsymbol{h}_{\mathbf{0}}^{\vec{\xi}}$ with $\vec{\xi} = \oint \boldsymbol{h}_{\mathbf{0}}$, substituting (2.94) into (2.84) delivers the following formula for $\boldsymbol{\mu}_{eff}$

$$
\boldsymbol{\mu}_{eff}(\xi_0) = \sum_{n=1}^{\infty} \frac{\xi_0}{\lambda_n - \xi_0} \left(\int_Y \varphi_n \right) \otimes \left(\int_Y \varphi_n \right) + \mathbf{I}^3 \tag{2.95}
$$

(2) We recall that the effective dielectric permitivity is defined such that

$$
\boldsymbol{\varepsilon}_{eff}(\xi_0)\vec{c} = \int_H (\nabla \chi + \vec{c}) + \varepsilon_p(\xi_0) \int_P (\nabla \chi + \vec{c}) \qquad \chi \in W_{\#}^{1,2}(Y, \mathbb{C}) \tag{2.96}
$$

where $\vec{c} := \int_Y \boldsymbol{e}_{\mathbf{0}}$ is the volumetric average of the homogenized E field $\boldsymbol{e}_{\mathbf{0}}$. The formula for $\boldsymbol{\varepsilon}_{eff}$ is obtained by applying the series expansion of χ in the definition

(2.96).

By (2.44), we see that the function χ solves the following problem

$$\operatorname{div}[a_p(\xi_0)(\nabla\chi + \vec{c})] = 0 \qquad \text{in } \ Y, \qquad \chi \in W^{1,2}_{\#}(Y, \mathbb{C})$$

or equivalently, in weak form

$$\int_H (\nabla\chi + \vec{c}) \cdot \nabla\overline{w} + \varepsilon_p(\xi_0) \int_P (\nabla\chi + \vec{c}) \cdot \nabla\overline{w} = 0, \qquad w \in W^{1,2}_{\#}(Y, \mathbb{C}) \tag{2.97}$$

The continuity of w in Y and $w = 0$ in R implies that

$$\int_{(\partial R)^+} \partial_n w = 0, \qquad \int_{\partial P} [\partial_n w]^+_- = 0$$

and using integration by parts in (2.97) we get

$$\int_H \nabla\chi \cdot \nabla\overline{w} + \varepsilon_p(\xi_0) \int_P \nabla\chi \cdot \nabla\overline{w} = 0, \qquad w \in W^{1,2}_{\#}(Y, \mathbb{C}) \tag{2.98}$$

We also introduce the bilinear form $B_z(u, w)$ which is given by

$$B_z(u, w) = \int_H \nabla u \cdot \nabla\overline{w} \ dy + z \int_P \nabla u \cdot \nabla\overline{w} \ dy, \qquad u, w \in W^{1,2}_{\#}(Y \setminus R, \mathbb{C}) \tag{2.99}$$

and the bilinear form $B_z(u, \cdot)$ is viewed as a linear form T_z on $W^{1,2}_{\#}(Y \setminus R, \mathbb{C})$.

Our next goal is to find the series expansion of the solution χ expressed through the spectral representation of the linear operator T_z. This done by studying the the following eigenvalue problem

$$\mu_n \int_{Y \setminus R} \nabla\psi_{\mu_n} \cdot \nabla v = \int_P \nabla\psi_{\mu_n} \cdot \nabla v, \qquad v \in W^{1,2}_{\#}(Y \setminus R, \mathbb{C}) \tag{2.100}$$

We first observe that the constant functions satisfy the problem (2.100), and we introduce the following decomposition of $W^{1,2}_{\#}(Y \setminus R, \mathbb{C})$:

$$W^{1,2}_{\#}(Y \setminus R, \mathbb{C}) = W_1 \oplus W_2 \oplus W_3 \oplus \mathbb{C}$$

with the constant solutions being uniquely determined by $\int_{Y\setminus R} u = 0$ and

$$W_1 = \left\{ u \;\middle|\; u \in W^{1,2}_{\#}(Y \setminus R, \mathbb{C}), \;\; \mathrm{Supp}(u) \subset\subset P, \int_{Y\setminus R} u = 0 \right\}$$

$$W_2 = \left\{ u \;\middle|\; u \in W^{1,2}_{\#}(Y \setminus R, \mathbb{C}), \;\; \mathrm{Supp}(u) \subset H \text{ with } \;\mathrm{Supp}(u) \cap \partial P = \varnothing, \int_{Y\setminus R} u = 0 \right\}$$

$W_3 \subset W^{1,2}_{\#}(Y \setminus R, \mathbb{C})$ is defined to be the subspace that satisfies $W_3 \perp (W_1 \oplus W_2)$ with the orthogonality is with respect to the inner product defined as

$$(u, v)_{Y\setminus R} = \int_{Y\setminus R} \nabla u \cdot \nabla \overline{v} dy \qquad\qquad u, v \in W^{1,2}_{\#}(Y \setminus R, \mathbb{C})$$

By orthogonality, we find that $u \in W_3$ is characterized as u is periodic on ∂Y, $\partial_n u$ is antiperiodic on ∂Y and

$$W_3 = \left\{ u \in W^{1,2}_{\#}(Y \setminus R, \mathbb{C}) \;\middle|\; \Delta u = 0 \;\text{ in }\; P \cup H, \quad \partial_n u\Big|_{\partial R^+} = 0, \quad \int_{Y\setminus R} u = 0 \right\}$$

Calculation shows that the problem (2.100) is directly linked to the following eigenvalue problem

$$\lambda_n(u_{\lambda_n}, w) = (Tu_{\lambda_n}, w) = \frac{1}{2}\int_H \nabla u_{\lambda_n} \cdot \nabla \overline{w}\; dy - \frac{1}{2}\int_P \nabla u_{\lambda_n} \cdot \nabla \overline{w}\; dy \qquad (2.101)$$

for $w \in W^{1,2}_{\#}(Y \setminus R, \mathbb{C})$ through the relation $\psi_{\mu_n} = u_{\lambda_n}$ when $\mu_n = \frac{1}{2} - \lambda_n$ and by [8], we have

$$\begin{aligned} \chi &= a_1\psi_1 + a_2\psi_2 + \sum_{0<\mu_n<1} a_{\mu_n}\psi_{\mu_n} \\ w &= b_1\psi_1 + b_2\psi_2 + \sum_{0<\mu_n<1} b_{\mu_n}\psi_{\mu_n} \end{aligned} \qquad (2.102)$$

Substituting (2.102) into (2.97), we get

$$\int_{Y\setminus R} \varepsilon(\xi_0)\Big[(a_1\nabla\psi_1 + a_2\nabla\psi_2 + \sum_{0<\mu_n<1} a_{\mu_n}\nabla\psi_{\mu_n} + \vec{c}) \cdot \overline{(b_1\nabla\psi_1 + b_2\nabla\psi_2 + \sum_{0<\mu_n<1} b_{\mu_n}\nabla\psi_{\mu_n})}\Big] = 0$$

By choosing $b_2 = b_{\mu_n} = 0$, we have

$$\int_{Y\setminus R} \varepsilon(\xi_0)\Big[(a_1\nabla\psi_1 + a_2\nabla\psi_2 + \sum_{0<\mu_n<1} a_{\mu_n}\nabla\psi_{\mu_n} + \vec{c}) \cdot b_1\nabla\overline{\psi}_1\Big] = 0$$

Using orthogonality of eigenfunctions, we have

$$a_1 \int_H \nabla\psi_1 \cdot \nabla\overline{\psi}_1 + a_1\varepsilon_p(\xi_0)\int_P \nabla\overline{\psi}_1 \cdot \nabla\psi_1 + \vec{c}\cdot\int_H \nabla\overline{\psi}_1 + \varepsilon_p(\xi_0)\vec{c}\cdot\int_P \nabla\overline{\psi}_1 = 0$$

$$a_1 \int_H \nabla\overline{\psi}_1 \cdot \nabla\psi_1 + \vec{c}\cdot\int_H \nabla\overline{\psi}_1 = 0.$$

So

$$a_1 = -\vec{c}\cdot\int_H \nabla\overline{\psi}_1.$$

Similarly, we have

$$a_2 = -\vec{c}\cdot\int_P \nabla\overline{\psi}_2.$$

Choosing all but only one of the particular b_{μ_m} to be nonzero on the equation, wee see that

$$0 = \int_{Y\setminus R} \varepsilon(\xi_0)\Big[(a_1\nabla\psi_1 + a_2\nabla\psi_2 + \sum_{0<\mu_n<1} a_{\mu_n}\nabla\psi_{\mu_n} + \vec{c})\cdot b_{\mu_m}\nabla\overline{\psi}_{\mu_m}\Big]$$

$$= \int_{Y\setminus R} \varepsilon(\xi_0)a_{\mu_m}b_{\mu_m}\nabla\psi_{\mu_m}\cdot\nabla\overline{\psi}_{\mu_m} + \vec{c}\cdot\int_{Y\setminus R}\varepsilon(\xi_0)b_{\mu_m}\nabla\overline{\psi}_{\mu_m}$$

So we get

$$a_{\mu_m} = \frac{-\vec{c}\cdot\displaystyle\int_{Y\setminus R}\varepsilon(\xi_0)\nabla\overline{\psi}_{\mu_m}}{\displaystyle\int_{Y\setminus R}\varepsilon(\xi_0)\nabla\psi_{\mu_m}\cdot\nabla\overline{\psi}_{\mu_m}} = \frac{-\vec{c}\cdot\Big[\displaystyle\int_H \nabla\overline{\psi}_{\mu_m} + \varepsilon_p(\xi_0)\int_P \nabla\overline{\psi}_{\mu_m}\Big]}{\displaystyle\int_H \nabla\psi_{\mu_m}\cdot\nabla\overline{\psi}_{\mu_m} + \varepsilon_p(\xi_0)\int_P \nabla\psi_{\mu_m}\cdot\nabla\overline{\psi}_{\mu_m}}$$

Subtracting $\dfrac{1}{2}\displaystyle\int_{Y\setminus R}(\nabla u)\cdot(\nabla\overline{w})$ from both sides of equation (2.101), we find that

$$\mu_n = \int_P \nabla\psi_{\mu_n}\cdot\nabla\overline{\psi}_{\mu_n}, \qquad \int_H \nabla\psi_{\mu_n}\cdot\nabla\overline{\psi}_{\mu_n} = 1-\mu_n$$

and

$$\chi = \left(-\vec{c}\cdot\int_H \nabla\overline{\psi}_1\right)\,\psi_1 + \left(-\vec{c}\cdot\int_P \nabla\overline{\psi}_2\right)\,\psi_2 + \sum_{0<\mu_n<1}\left(-\vec{c}\cdot\frac{\displaystyle\int_H \nabla\overline{\psi}_{\mu_n} + \varepsilon_p(\xi_0)\int_P \nabla\overline{\psi}_{\mu_n}}{1-\mu_n+\varepsilon_p(\xi_0)\mu_n}\right)\psi_{\mu_n}$$

Using this in (2.96) and denote the volume of P, H by θ_P, θ_H respectively, we see

$$\int_H (\nabla\chi + \vec{c}) + \varepsilon_p(\xi_0) \int_P (\nabla\chi + \vec{c})$$

$$= \int_H \left[\left(-\vec{c} \cdot \int_H \nabla\overline{\psi}_1 \right) \ \nabla\psi_1 + \left(-\vec{c} \cdot \int_P \nabla\overline{\psi}_2 \right) \ \nabla\psi_2 \right]$$

$$+ \int_H \left[\sum_{0<\mu_n<1} \left(-\vec{c} \cdot \frac{\int_H \nabla\overline{\psi}_{\mu_n} + \varepsilon_p(\xi_0) \int_P \nabla\overline{\psi}_{\mu_n}}{1 - \mu_n + \varepsilon_p(\xi_0)\mu_n} \right) \ \nabla\psi_{\mu_n} + \vec{c} \right]$$

$$+ \varepsilon_p(\xi_0) \int_P \left[\left(-\vec{c} \cdot \int_H \nabla\overline{\psi}_1 \right) \ \nabla\psi_1 + \left(-\vec{c} \cdot \int_P \nabla\overline{\psi}_2 \right) \ \nabla\psi_2 \right]$$

$$+ \varepsilon_p(\xi_0) \int_P \left[\sum_{0<\mu_n<1} \left(-\vec{c} \cdot \frac{\int_H \nabla\overline{\psi}_{\mu_n} + \varepsilon_p(\xi_0) \int_P \nabla\overline{\psi}_{\mu_n}}{1 - \mu_n + \varepsilon_p(\xi_0)\mu_n} \right) \ \nabla\psi_{\mu_n} + \vec{c} \right]$$

$$= - \left[\left(\int_H \nabla\overline{\psi}_1 \right) \otimes \left(\int_H \nabla\psi_1 \right) + \left(\int_P \nabla\overline{\psi}_2 \right) \otimes \left(\int_H \nabla\psi_2 \right) \right] \ \vec{c} + (\varepsilon_p(\xi_0)\theta_P + \theta_H) \ \vec{c}$$

$$- \left[\sum_{0<\mu_n<1} \frac{\left(\int_H \nabla\overline{\psi}_{\mu_n} \right) \otimes \left(\int_H \nabla\psi_{\mu_n} \right) + \varepsilon_p(\xi_0) \left(\int_P \nabla\overline{\psi}_{\mu_n} \right) \otimes \left(\int_H \nabla\psi_{\mu_n} \right)}{1 - \mu_n + \varepsilon_p(\xi_0)\mu_n} \right] \vec{c}$$

$$- \left[\varepsilon_p(\xi_0) \left(\int_H \nabla\overline{\psi}_1 \right) \otimes \left(\int_P \nabla\psi_1 \right) + \varepsilon_p(\xi_0) \left(\int_P \nabla\overline{\psi}_2 \right) \otimes \left(\int_P \nabla\psi_2 \right) \right] \ \vec{c}$$

$$- \left[\sum_{0<\mu_n<1} \frac{\varepsilon_p(\xi_0) \left(\int_H \nabla\overline{\psi}_{\mu_n} \right) \otimes \left(\int_P \nabla\psi_{\mu_n} \right) + \varepsilon_p^2(\xi_0) \left(\int_P \nabla\overline{\psi}_{\mu_n} \right) \otimes \left(\int_P \nabla\psi_{\mu_n} \right)}{1 - \mu_n + \varepsilon_p(\xi_0)\mu_n} \right] \vec{c}$$

$$= - \left[\left(\int_H \nabla\overline{\psi}_1 \right) \otimes \left(\int_H \nabla\psi_1 \right) + \varepsilon_p(\xi_0) \left(\int_P \nabla\overline{\psi}_2 \right) \otimes \left(\int_H \nabla\psi_2 \right) \right] \ \vec{c} + (\varepsilon_p(\xi_0)\theta_P + \theta_H) \ \vec{c}$$

$$-\sum_{0<\mu_n<1} \frac{\left(\int_H \nabla\overline{\psi}_{\mu_n}\right) \otimes \left(\int_H \nabla\psi_{\mu_n}\right) + \varepsilon_p(\xi_0)\left(\int_P \nabla\overline{\psi}_{\mu_n}\right) \otimes \left(\int_H \nabla\psi_{\mu_n}\right)}{1-\mu_n+\varepsilon_p(\xi_0)\mu_n}\ \vec{c}$$

$$-\left[\varepsilon_p(\xi_0)\left(\int_H \nabla\overline{\psi}_1\right) \otimes \left(\int_P \nabla\psi_1\right) + \varepsilon_p(\xi_0)\left(\int_P \nabla\overline{\psi}_2\right) \otimes \left(\int_P \nabla\psi_2\right)\right]\ \vec{c}$$

$$-\sum_{0<\mu_n<1} \frac{\varepsilon_p(\xi_0)\left(\int_H \nabla\overline{\psi}_{\mu_n}\right) \otimes \left(\int_P \nabla\psi_{\mu_n}\right) + \varepsilon_p^2(\xi_0)\left(\int_P \nabla\overline{\psi}_{\mu_n}\right) \otimes \left(\int_P \nabla\psi_{\mu_n}\right)}{1-\mu_n+\varepsilon_p(\xi_0)\mu_n}\ \vec{c}$$

Finally, we see that

$$\int_H (\nabla\chi+\vec{c}) + \varepsilon_p(\xi_0)\int_P (\nabla\chi+\vec{c})$$

$$= -\left[\left(\int_H \nabla\overline{\psi}_1\right) \otimes \left(\int_H \nabla\psi_1\right) + \varepsilon_p(\xi_0)\left(\int_P \nabla\overline{\psi}_2\right) \otimes \left(\int_P \nabla\psi_2\right)\right]\ \vec{c}$$

$$-\sum_{0<\mu_n<1} \frac{\varepsilon_p^2(\xi_0)A_{\mu_n}^{P,P} + \varepsilon_p(\xi_0)(A_{\mu_n}^{P,H} + A_{\mu_n}^{H,P}) + A_{\mu_n}^{H,H}}{1-\mu_n+\varepsilon_p(\xi_0)\mu_n}\ \vec{c} + (\varepsilon_p(\xi_0)\theta_P + \theta_H)\ \vec{c}$$

$$= -\sum_{0\leq\mu_n\leq 1} \frac{\varepsilon_p^2(\xi_0)A_{\mu_n}^{P,P} + \varepsilon_p(\xi_0)(A_{\mu_n}^{P,H} + A_{\mu_n}^{H,P}) + A_{\mu_n}^{H,H}}{1-\mu_n+\varepsilon_p(\xi_0)\mu_n}\ \vec{c} + (\varepsilon_p(\xi_0)\theta_P + \theta_H)\ \vec{c}$$

where

$$A_{\mu_n}^{D_1,D_2} = \left(\int_{D_1} \nabla\overline{\psi_{\mu_n}}\right) \otimes \left(\int_{D_2} \nabla\psi_{\mu_n}\right) n = 1, 2, \cdots$$

Therefore, we have

$$\boldsymbol{\varepsilon}_{eff}(\xi_0) = (\varepsilon_p(\xi_0)\theta_P + \theta_H)\mathbf{I}^3 - \sum_{0\leq\mu_n\leq 1} \frac{\varepsilon_p^2(\xi_0)A_{\mu_n}^{P,P} + \varepsilon_p(\xi_0)(A_{\mu_n}^{P,H} + A_{\mu_n}^{H,P}) + A_{\mu_n}^{H,H}}{1-\mu_n+\varepsilon_p(\xi_0)\mu_n} \tag{2.103}$$

where

$$A_{\mu_n}^{D_1,D_2} = \left(\int_{D_1} \nabla\overline{\psi_{\mu_n}}\right) \otimes \left(\int_{D_2} \nabla\psi_{\mu_n}\right)$$

And the proof of Theorem 2.5 (2) is concluded. □

Chapter 3
Photonic Crystals

3.1 Introduction and Problem Setup

3.1.1 Introduction

Unlike metamaterial crystals, some types of photonic crystals occur in nature in the form of structural color of animals or matter. But photonic crystals can also be fabricated. Through the use of such fabircated photonic crystals, electromagnetic waves from light can be manipulated to achieve a desired optical proprty.
The study of photonic crystals initiated in [24] by considering multi-layer metallic stacks. But the investigation of three-dimensional photonic crystals in [25, 29] has been followed by significant developments in the research on photnonic crystals [26, 27, 28].

The design of two-dimensional lossless photonic crystals with desired electromagnetic resonant properties studied using variational methods in [3]. Creating band gaps in photonic crystals with different configurations was studied in [44, 42].

Our goal in this chapter of this dissertation is to derive spectral representation formula for the Helmholtz operator for vector wave equation. Using this formula, we show the explicit characterization of the inverse of Helmholtz operator to to obtain spectral representation of the solution to the vector wave equation. To obtain the representation formula, we proved the compactness of magnetic dipole operator along with the use of single layer potential operator.

3.1.2 Problem Setup

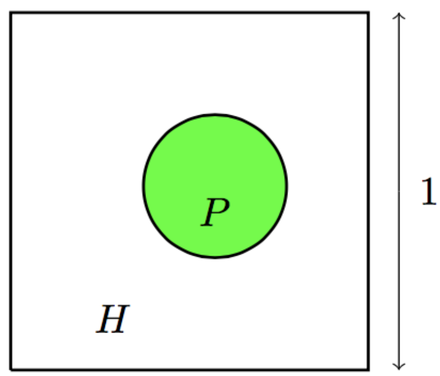
We study the representation formula for the differential operator in the following Helmholtz equation

$$\nabla \times ((\varepsilon(x))^{-1} \nabla \times \boldsymbol{h}) = \xi \boldsymbol{h} \tag{3.1}$$

which is satisfied by the Bloch wave solution $\boldsymbol{h}$ traveling through a photonic crystal made from periodic assemblage of unit cube consisting of a host material denoted by H and a non magnetic plasmonic inclusion denoted by P. The incoming wave frequency is denoted by ξ and the dielectric constant of the material $\varepsilon(x)$ as a function of x is given by

$$\varepsilon = \begin{cases} \varepsilon_p(\omega) & x \in P \\ 1 & x \in H \end{cases}$$

A plain view of a typical unit cube in this photonic crystal is visualized below.



The Bloch wave solution $\boldsymbol{h}$ to (3.1) is sought in the space

$$J^* = \{u \in H_{\#}(\text{curl}, Y), \qquad \nabla \cdot u = 0\} \tag{3.2}$$

It is obvious that the constant function is a solution to (3.1). So, to simplify our analysis, we look for solutions to (3.1) in the space J where

$$J = \left\{ u \in H_{\#}(\text{curl}, Y), \qquad \nabla \cdot u = 0, \qquad \int_Y u = 0 \right\}$$

with the inner product

$$(u, v) = \int_Y \nabla \times u(x) \cdot \nabla \times \overline{w(x)}\, dx \qquad\qquad u, w \in J \tag{3.3}$$

The weak form of equation (3.1) is given by

$$\int_H (\nabla\times\boldsymbol{h})\cdot(\nabla\times\overline{w})\ dx + \varepsilon_p^{-1}\int_P (\nabla\times\boldsymbol{h})\cdot(\nabla\times\overline{w})\ dx = \xi\int_Y \boldsymbol{h}\cdot w\ dx \qquad w\in J \tag{3.4}$$

By observing (3.4), we introduce the bilinear form $B_z : J\times J\to\mathbb{C}$ which is given by

$$B_z(u,w) := \int_H (\nabla\times u)\cdot(\nabla\times\overline{w})\ dx + z\int_P (\nabla\times u)\cdot(\nabla\times\overline{w})\ dx, \quad u,w\in J \tag{3.5}$$

and view it as $B_z(u,w) = (T_z u, w)$ for a linear operator T_z acting on J.

In what follows, our goal is to find the representation formula for the differential operator $\nabla\times((\varepsilon(x))^{-1}\nabla\times$ in (3.1) by expressing the linear operator T_z through eigenpairs of a linear operator T which is directly linked to the following eigenvalue problem

$$\lambda(u,w) = \lambda\int_Y (\nabla\times u)\cdot(\nabla\times\overline{w})dy = \int_P (\nabla\times u)\cdot(\nabla\times\overline{w})dy, \qquad w\in J \tag{3.6}$$

which has nonzero, real eigenvalues λ_n and the corresponding eigenfunctions ψ_n that satisfy

$$\lambda_n\int_Y (\nabla\times\psi_n)\cdot(\nabla\times\overline{w})dy = \int_P (\nabla\times\psi_n)\cdot(\nabla\times\overline{w})dy, \qquad w\in J$$

We have the following property of the eigenfunctions of the eigenvalue problem (3.6).

Lemma 3.1. *The eigenfunctions $\psi_n,\ n = 1,2,3\cdots$ are a complete system of orthogonal functions with respect to the inner product of J.*

Our next goal is to decompose the solution space J into orthogonal invariant subspaces spanned by eigenfunctions associated to the eigenvalues of problem (3.6).

To do this, we proceed by defining J as the direct sum of orthogonal subspaces W_1, W_2, and W_3 as the following:

$$J = W_1 \oplus W_2 \oplus W_3$$

where

$$W_1 = \left\{ u \,\middle|\, u \in J, \ \ \mathrm{Supp}(u) \subset\subset P, \int_Y u = 0 \right\} \tag{3.7}$$

$$W_2 = \left\{ u \,\middle|\, u \in J, \quad \mathrm{Supp}(u) \subset H \text{ with } \quad \mathrm{Supp}(u) \cap \partial P = \varnothing, \int_Y u = 0 \right\} \tag{3.8}$$

and $W_3 \subset J$ is defined to be the subspace that satisfies $W_3 \perp (W_1 \oplus W_2)$ with the orthogonality is with respect to the inner product of J.

In view of (3.7) and (3.8), we notice that the subspaces W_1 and W_2 are spanned by the eigenfunctions associated to the eigenvalues 1 and 0 of the eigenvalue problem (3.6), and $\lambda \in [0, 1]$. The specific characterization of functions in W_3 is given by the next lemma.

Lemma 3.2. *The subspace $W_3 \subset J$ is characterized as the following:*

$$W_3 = \left\{ u \in J \,\middle|\, \begin{array}{ll} \nabla \times \nabla \times u = 0 \text{ in } H \cup P, \quad \nabla \cdot u = 0, & \int_Y u = 0 \\ \\ n \times \nabla \times u \text{ is antiperiodic on } \ \partial Y, & [u]_-^+|_{\partial P} = 0 \end{array} \right\}$$

3.2 Main Results

3.2.1 Mapping Property of Single Layer Potential Operator

In order to characterize the functions in W_3, we parametrize the elements of W_3 by using single layer potential operator defined as

$$\begin{aligned} S(\rho)(x) &= \widetilde{S}(\rho)(x) - \int_Y \widetilde{S}(\rho)(y) ds_y \\ \widetilde{S}(\rho) &= \int_{\partial P} \mathbf{G}(x, y) \rho(y) ds_y, \quad x \notin \partial P \end{aligned} \tag{3.9}$$

where $\rho \in L_t^2(\partial P)^3$ and

$$L_t^2(\partial P)^3 = \Big\{\rho \in L^2(\partial P)^3 \Big| \ \ n \cdot \rho = 0 \ \text{ on } \ \partial P\Big\}$$

In view of Lemma 3.2, we observe that the following must hold for $S(\rho)$:

$$\nabla \cdot S(\rho)(x) = \int_{\partial P} \mathbf{G}(x,y)(\mathrm{Div}\rho(y))ds_y = 0 \tag{3.10}$$

So ρ has to be chosen from $L_{t,0}^2(\partial P)^3$ where

$$L_{t,0}^2(\partial P)^3 := \Big\{\rho \in L_t^2(\partial P)^3 \Big| \ \ \mathrm{Div}\rho = 0, \quad \int_{\partial P} \rho \ ds_y = 0\Big\}$$

and the differential operator Div is understood to be surface divergence.

Also, $\mathbf{G}$ is the dyadic Green's function such that : $\mathbf{G}$ is separately periodic in x and y with unit Y, twice differentiable in each of x and y for $x \neq y$, and

$$\Delta_x \mathbf{G}(x,y) = \sum_{n \in \mathbb{Z}^3} \delta(x - y + n) - 1 \quad \text{ in } \quad \mathbb{D} := \cup_{n \in \mathbb{Z}^3}(Y + n) \tag{3.11}$$

The specific characterization of $\mathbf{G}(x,y)$ is given by the next lemma.

Lemma 3.3. *The dyadic Green's function* $\mathbf{G}(x,y)$ *satisfying* (3.11) *is given by*

$$\mathbf{G}(x,y) = \sum_{n \in \mathbf{Z}^3 \setminus \{0\}} \frac{e^{i2\pi n \cdot (x-y)}}{|2\pi n|^2} \mathbf{I}$$

where $\mathbf{I}$ *is the unit dyadic.*

In order to present the parametrization of elements in W_3, we first establish the following two lemmas that will be useful.

Lemma 3.4. *Let the single layer potential operator* S *be defined as in* (3.9)*. For every* $\rho \in L_{t,0}^2(\partial P)^3$*, we have* $S(\rho) \in W_3$*.*

Lemma 3.5. *The space of tangential vector fields* $L_{t,0}^2(\partial P)^3$ *is a dense subspace of* $V_t^{-\frac{1}{2}}(\partial P)^3$ *where*

$$L_{t,0}^2(\partial P)^3 \subset V_t^{-\frac{1}{2}}(\partial P)^3 = \Big\{(n \times \nabla) f \ ; \ f \in H^{\frac{1}{2}}(\partial P)\Big\} \subseteq H^{-\frac{1}{2}}(\partial P)^3 \tag{3.12}$$

wheret the space $H^{\frac{1}{2}}(\partial P)$ of complex scalar valued functions defined on ∂P is given by

$$H^{\frac{1}{2}}(\partial P) = \left\{ u \in L^2(\partial P), \qquad \frac{|u(x) - u(y)|}{|x-y|^2} \in L^2(\partial P \times \partial P) \right\}$$

with

$$\|u\|_{H^{\frac{1}{2}}(\partial P)} = \left(\int_{\partial P} |u|^2 dx + \int_{\partial P}\int_{\partial P} \frac{|u(x) - u(y)|^2}{|x-y|^4} dxdy \right)^{\frac{1}{2}}.$$

and the norm $\|A\|_{V_t^{-\frac{1}{2}}(\partial P)^3}$ is taken to be

$$\|A\|_{V_t^{-\frac{1}{2}}(\partial P)^3} = inf \left\{ \|\sigma + f\|_{H^{\frac{1}{2}}(\partial P)} \;;\; \sigma \in \mathbb{C}, \quad f \in H^{\frac{1}{2}}(\partial P), \quad (n \times \nabla) f = A \right\} \tag{3.13}$$

Now we are ready to present an important mapping property of the single layer potential operator that will be crucial in characterizing the spectral property of the sesquilinear operator T.

Theorem 3.6. *The single layer potential operator $S : V_t^{-\frac{1}{2}}(\partial P)^3 \to W_3$ is an isomorphism.*

3.2.2 Compactness of Magnetic Dipole Operator

The magnetic dipole operator $M : L^2_{t,0}(\partial P)^3 \to L^2_{t,0}(\partial P)^3$ describing the tangential component of the electric field generated by the change in the magnetic distribution is given by

$$M(\rho) = \int_{\partial P} n \times \nabla_x \times (\mathbf{G}(x,y)\ \rho(y))\, ds_y, \qquad x \in \partial P$$

Theorem 3.7. *$M : V_t^{-\frac{1}{2}}(\partial P)^3 \to V_t^{-\frac{1}{2}}(\partial P)^3$ is a compact operator and*

$$\sigma\left(M;\ V_t^{-\frac{1}{2}}(\partial P)^3\right) = \sigma\left(K^*;\ H_0^{-\frac{1}{2}}(\partial P)\right) \tag{3.14}$$

where K^* is the scalar valued Newmann-Poicaré operator defined on $H_0^{-\frac{1}{2}}(\partial P)$ which is the dual space of $H^{\frac{1}{2}}(\partial P)$, $\sigma\left(M;\ V_t^{-\frac{1}{2}}(\partial P)^3\right)$ and $\sigma\left(K^*;\ H_0^{-\frac{1}{2}}(\partial P)\right)$ are the spectrum of M and K^* respectively.

It's been shown [32] that the spectrum of K^* lies in $\left[-\frac{1}{2},\ \frac{1}{2}\right]$. So by Theorem 3.7 we conclude that

$$\sigma\left(M;\ V_t^{-\frac{1}{2}}(\partial P)^3\right) = \left[-\frac{1}{2},\ \frac{1}{2}\right]$$

3.2.3 Spectral Property of $T = SMS^{-1}$

Theorem 3.8. $T = SMS^{-1} : W_3 \to W_3$ *is a Hermitian, compact operator and*

$$\sigma\left(T;\ W_3\right) = \sigma\left(M;\ V_t^{-\frac{1}{2}}(\partial P)^3\right) \tag{3.15}$$

3.2.4 Spectral Representation Theorem

Now we present the main result on the spectral representation of the Helmholtz operator for the vector wave equation (3.1).

Theorem 3.9. *we have the following identity for the spectral representation of the differential operator*

$$\nabla \times ((\varepsilon(x))^{-1}\nabla \times = -\Delta T_{\varepsilon_p^{-1}}$$

where the Laplace operator is understood to be associated with the bilinear form $B_{\varepsilon_p^{-1}}$ *and we have the following spectral representation of the sesquilinear operator* T_{ε_p} *which separates the effect of the dielectric constant* $\varepsilon(x)$ *from the underlying geometry of the photonic crystal.*

$$T_{\varepsilon_p^{-1}} u = \varepsilon_p^{-1} P_1 u + P_2 u + \sum_{\mu_n} \left[(\frac{1}{2} + \mu_n) + \varepsilon_p^{-1} (\frac{1}{2} - \mu_n) \right] P_{\mu_n} u$$

If $z \neq -(\frac{1}{2} + \mu_n)/(\frac{1}{2} - \mu_n)$, *then we have*

$$T_z^{-1} = z^{-1} P_1 + P_2 + \sum_{\mu_n} z \left[z(\frac{1}{2} + \mu_n) + (\frac{1}{2} - \mu_n) \right]^{-1} P_{\mu_n}$$

3.3 Proof of Lemma 3.1-3.5

Proof of Lemma 3.1

Proof. Letting $n \neq k, \quad w = \psi_k$ and conjugating both sides, we get

$$\lambda_k \int_Y (\nabla \times \psi_k) \cdot (\nabla \times \overline{\psi_n}) dy = \int_P (\nabla \times \psi_k) \cdot (\nabla \times \overline{\psi_n}) dy \tag{3.16}$$

Taking $w = \psi_n$ and conjugating both sides, we get

$$\lambda_n \int_Y (\nabla \times \psi_n) \cdot (\nabla \times \overline{\psi_k}) dy = \int_P (\nabla \times \psi_n) \cdot (\nabla \times \overline{\psi_k}) dy \tag{3.17}$$

Comparing (3.16) and (3.17), we see

$$(\lambda_n - \lambda_k) \int_Y (\nabla \times \psi_n) \cdot (\nabla \times \overline{\psi_k}) dy = 0$$

which shows

$$\int_Y (\nabla \times \psi_n) \cdot (\nabla \times \overline{\psi_k}) dy = 0$$

By (3.6), we have

$$\int_P (\nabla \times \psi_k) \cdot (\nabla \times \overline{\psi_n}) dy = 0$$
$$\int_H (\nabla \times \psi_k) \cdot (\nabla \times \overline{\psi_n}) dy = 0$$

This along with the relevant result in [42] completes the proof. □

Proof of Lemma 3.2

Proof. Since $W_3 \subset J$, we only need to show that for $u \in W_3$, we have

$$\nabla \times \nabla \times u = 0 \quad \text{in} \quad H \cup P$$

$$n \times \nabla \times u \quad \text{is antiperiodic on} \quad \partial Y$$

First, for $u \in W_3$ and $w \in W_2$, we observe

$$\begin{aligned} 0 &= \int_Y (\nabla \times u) \cdot (\nabla \times \overline{w}) dy \\ &= \int_H (\nabla \times u) \cdot (\nabla \times \overline{w}) dy \\ &= -\int_{\partial Y} (n \times \nabla \times u) \cdot \overline{w}\ ds_y \\ &\quad + \int_{\partial R} (n \times \nabla \times u)|^+ \cdot \overline{w}\ ds_y \\ &\quad + \int_H (\nabla \times \nabla \times u) \cdot \overline{w}\ dy \end{aligned} \tag{3.18}$$

Choosing $\mathrm{Supp}(w) \subset\subset H$ in the above equation implies that

$$\nabla \times \nabla \times u = 0 \quad \text{in} \quad H$$

Choosing $\mathrm{Supp}(w) \subset \partial Y$, we have

$$\int_{\partial Y} (n \times \nabla \times u) \cdot \overline{w}\ ds_y = 0$$

By defining the 6 faces of the unit cell as

$L :=$ left face of the unit cell $= \{(0, y_2, y_3)\ |0 < y_2 \leq 1, \quad 0 < y_3 \leq 1\}$

$R :=$ right face of the unit cell $= \{(1, y_2, y_3)\ |0 < y_2 \leq 1, \quad 0 < y_3 \leq 1\}$

$T :=$ top face of the unit cell $= \{(y_1, y_2, 1)\ |0 < y_1 \leq 1, \quad 0 < y_2 \leq 1\}$

$D :=$ downward face of the unit cell $= \{(y_1, y_2, 0)\ |0 < y_1 \leq 1, \quad 0 < y_2 \leq 1\}$

$F :=$ front face of the unit cell $= \{(y_1, 0, y_3)\ |0 < y_1 \leq 1, \quad 0 < y_3 \leq 1\}$

$B :=$ back face of the unit cell $= \{(y_1, 1, y_3)\ |0 < y_1 \leq 1, \quad 0 < y_3 \leq 1\}$

and choosing $\mathrm{Supp}(w)$ such that w vanishes on ∂Y except for the left and right faces of the unit cell and using the periodicity $w(0, y_2, y_3) = w(1, y_2, y_3)$, we get

$$\int_L^R (n \times \nabla \times u(0, y_2, y_3) - n \times \nabla \times u(1, y_2, y_3) \cdot \overline{w}(1, y_2, y_3)\ ds_y = 0$$

Repeating this for top, downward faces and front, back faces of the unit cell respectively, we get

$$\int_D^T (n \times \nabla \times u(y_1, y_2, 1) - n \times \nabla \times u(y_1, y_2, 0) \cdot \overline{w}(y_1, y_2, 1)\ ds_y = 0$$
$$\int_F^B (n \times \nabla \times u(y_1, 1, y_3) - n \times \nabla \times u(y_1, 0, y_3) \cdot \overline{w}(y_1, 1, y_3)\ ds_y = 0$$

The last three equations show that $n \times \nabla \times u$ is antiperiodic on ∂Y.

For $u \in W_3$ and $w \in W_1$, we have

$$\begin{aligned} 0 &= \int_Y (\nabla \times u) \cdot (\nabla \times \overline{w}) dy \\ &= \int_P (\nabla \times u) \cdot (\nabla \times \overline{w}) dy \\ &= -\int_P (\nabla \times \nabla \times u) \cdot \overline{w}\ dy \end{aligned} \tag{3.19}$$

Since $\mathrm{Supp}(w) \subset\subset P$, the above equation implies that

$$\nabla \times \nabla \times u = 0 \quad \text{in} \quad P$$

□

Proof of Lemma 3.3

Proof. we first examine $\nabla \times \nabla \times h(y)$ where

$$h(y) \in W^{1,2}_{\#}(Y)^3, \qquad \int_Y h(y)\ dy = 0$$

where

$$W^{1,2}_{\#}(Y)^3 := \Big\{ u \ \Big|\ u \in L^2_{\#}(Y, \mathbb{C}),\ \partial_i u \in L^2_{\#}(Y, \mathbb{C}) \Big\}$$

By writing $h(y)$ in terms of Fourier series expansion, we notice that

$$h(y) = \sum_{n \in \mathbf{Z}^3} \hat{h}(n)\ e^{i2n\pi \cdot y}$$
$$\hat{h}(n) = \frac{1}{2\pi} \int_Y e^{-i2n\pi \cdot x}\ h(x)\ dx$$

and

$$h(y) = \int_Y \frac{1}{2\pi} \sum_{n \in \mathbf{Z}^3} e^{i2n\pi\cdot(y-x)}\ h(x)\ dx.$$

Since

$$e^{i2n\pi\cdot(y-x)}\ h(x) = (h_1 e^{i2n\pi\cdot(y-x)},\ h_2 e^{i2n\pi\cdot(y-x)},\ h_3 e^{i2n\pi\cdot(y-x)})$$

we see that

$$\nabla_y \times (h(x)\ e^{i2n\pi\cdot(y-x)}) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial y_1 & \partial y_2 & \partial y_3 \\ h_1\ e^{i2n\pi\cdot(y-x)} & h_2\ e^{i2n\pi\cdot(y-x)} & h_3\ e^{i2n\pi\cdot(y-x)} \end{vmatrix}$$

$$= \begin{bmatrix} \partial y_2(h_3\ r(x,y) - \partial y_3(h_2\ r(x,y)) \\ -\partial y_1(h_3\ r(x,y)) - \partial y_3(h_1\ r(x,y)) \\ \partial y_1(h_2\ r(x,y)) - \partial y_2(h_1\ r(x,y)) \end{bmatrix}$$

Then by calculation, we find that

$$\nabla_y \times (h(x)\ e^{i2n\pi\cdot(y-x)}) = \begin{bmatrix} (i2n_2\pi h_3 - i2n_3\pi h_2)r(x,y) \\ -(i2n_1\pi h_3 - i2n_3\pi h_1)r(x,y) \\ (i2n_1\pi h_2 - i2n_2\pi h_1)r(x,y) \end{bmatrix}$$

Setting

$$a_1 = (i2n_2\pi h_3 - i2n_3\pi h_2)r(x,y)$$

$$a_2 = (i2n_3\pi h_1 - i2n_1\pi h_3)r(x,y)$$

$$a_3 = (i2n_1\pi h_2 - i2n_2\pi h_1)r(x,y)$$

$$r(x,y) = e^{i2n\pi\cdot(y-x)}$$

We find

$$\nabla_y \times \nabla_y \times (h(x)\ e^{i2n\pi\cdot(y-x)}) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial y_1 & \partial y_2 & \partial y_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \begin{bmatrix} (\partial y_2 a_3 - \partial y_3 a_2) \\ -(\partial y_1 a_3 - \partial y_3 a_1) \\ (\partial y_1 a_2 - \partial y_2 a_1) \end{bmatrix}$$

and calculation shows that the above is equal to

$$= \begin{bmatrix} [(i2n_2\pi)(i2n_1\pi)h_2 - (i2n_2\pi)^2 h_1 - (i2n_3\pi)^2 h_1 + (i2n_1\pi)(i2n_3\pi)h_3]\, r(x,y) \\ -[(i2n_1\pi)^2 h_2 - (i2n_1\pi)(i2n_2\pi)h_1 - (i2n_2\pi)(i2n_3\pi)h_3 + (i2n_3\pi)^2 h_2 r(x,y)] \\ [(i2n_1\pi)(i2n_3\pi)h_1 - (i2n_1\pi)^2 h_3 - (i2n_2\pi)^2 h_3 + (i2n_2\pi)(i2n_3\pi)h_2 r(x,y)] \end{bmatrix}$$

Using

$$\nabla_y \cdot (r(x,y)h(x)) = (i2n_1\pi h_1 + i2n_2\pi h_2 + i2n_3\pi h_3)r(x,y) = 0$$

repeatedly in the above, we get the following for $\nabla_y \times \nabla_y \times (h(x)\ e^{i2n\pi\cdot(y-x)})$:

$$= \begin{bmatrix} [(i2n_2\pi)(i2n_1\pi)h_2 - (i2n_2\pi)^2 h_1 - (i2n_3\pi)^2 h_1 + (i2n_1\pi)(i2n_3\pi)h_3]\, r(x,y) \\ -[(i2n_1\pi)^2 h_2 - (i2n_1\pi)(i2n_2\pi)h_1 - (i2n_2\pi)(i2n_3\pi)h_3 + (i2n_3\pi)^2 h_2]\, r(x,y) \\ [(i2n_1\pi)(i2n_3\pi)h_1 - (i2n_1\pi)^2 h_3 - (i2n_2\pi)^2 h_3 + (i2n_2\pi)(i2n_3\pi)h_2]\, r(x,y) \end{bmatrix}$$

$$= \begin{bmatrix} [-(i2n_1\pi)^2 h_1 - (i2n_2\pi)^2 h_1 - (i2n_3\pi)^2 h_1]\, r(x,y) \\ -[(i2n_1\pi)^2 h_2 + (i2n_2\pi)^2 h_2 + (i2n_3\pi)^2 h_2]\, r(x,y) \\ [-(i2n_3\pi)^2 h_3 - (i2n_1\pi)^2 h_3 - (i2n_2\pi)^2 h_3]\, r(x,y) \end{bmatrix}$$

$$= \Big[|2n\pi|^2 h_1 e_1 + |2n\pi|^2 h_2 e_2 + |2n\pi|^2 h_3 e_3 \Big]\, r(x,y)$$

This shows that

$$\nabla_y \times \nabla_y \times (h(x)\ e^{i2n\pi\cdot(y-x)}) = |2n\pi|^2 (e^{i2n\pi\cdot(y-x)}\ h(x))$$

provided that $i2n\pi \cdot (e^{i2n\pi\cdot(y-x)}\ h(x)) = 0$ or $\nabla \cdot h(y) = 0$.

So, now switching x and y, we have

$$\mathbf{G}(x,y) = \sum_{n\in\mathbb{Z}^3\setminus 0} \frac{e^{i(2\pi n)\cdot(x-y)}}{|2\pi n|^2}\mathbf{I}.$$

□

Proof of Lemma 3.4

Proof. It's been shown [11] that, for $\rho \in L^2_{t,0}(\partial P)^3$ and $x \in P \cup H$, we have

$$\nabla \cdot S(\rho)(x) = \int_{\partial P} \mathbf{G}(x,y)(\mathrm{Div}\rho(y))ds_y = 0, \qquad x \in H \cup P. \tag{3.20}$$

For $x \notin \partial P$, we have

$$\begin{aligned}\nabla \times \nabla \times S(\rho)(x) &= \int_{\partial P} \nabla_x \times \nabla_x \times \mathbf{G}(x,y)\rho(y)ds_y \\ &= \int_{\partial P} (\delta(x-y) - 1)\rho(y)ds_y \\ &= 0\end{aligned} \tag{3.21}$$

But at the same time

$$\nabla \times \nabla \times S(\rho)(x) = \nabla(\nabla \cdot S(\rho)) - \Delta S(\rho) = -\Delta S(\rho) \tag{3.22}$$

So we get

$$\Delta S(\rho)(x) = 0, \qquad\qquad x \in H \cup P \tag{3.23}$$

By the property of single layer potential operator, we know that

$$[S(\rho)(x)]\Big|_{\partial P} = 0, \tag{3.24}$$

which also implies

$$[n \cdot S(\rho)(x)]\Big|_{\partial P} = 0. \tag{3.25}$$

Since $S(\rho)$ is harmonic on the two sides of ∂P, equation (3.25) along with (3.20) implies that

$$\nabla \cdot S(\rho) = 0, \qquad\qquad x \in Y \tag{3.26}$$

Since $\nabla \times \mathbf{G}(x, y)$ is periodic, noticing that

$$n \times \nabla \times S(\rho)(x) = \int_{\partial P} n \times \nabla \times \mathbf{G}(x, y)\rho(y) ds_y, \tag{3.27}$$

we conclude that $n \times \nabla \times S(\rho)(x)$ is antiperiodic.

Finally,

$$\begin{aligned} n \times \nabla \times S(\rho)|_{\partial R^+} &= \int_{\partial P} n \times \nabla \times \mathbf{G}(x, y)|_{\partial R^+}\ \rho(y) ds_y \\ &= \int_{\partial P} C\ \rho(y) ds_y \\ &= 0 \end{aligned} \tag{3.28}$$

The equations (3.21), (3.24),(3.26), (3.27), (3.28), along with the fact that

$$\int_Y S(\rho)\ dy = 0$$

show that $S(\rho) \in W_3$. □

Proof of Lemma 3.5

Proof. We start with Helmholtz-Hodge decomposition for tangential vector fields, for $\rho \in L^2_t(\partial P)^3$, we have [33]

$$\rho = \nabla_s \varphi + n \times \nabla_s \psi \tag{3.29}$$

for some scalar functions $\varphi,\ \psi \in H^1(\partial P, \mathbb{C})$ and ∇_s is understood to be the surface gradient acting on φ and ψ.

For $\rho \in L^2_{t,0}(\partial P)^3$

$$\text{Div}\rho = 0 \qquad \Rightarrow \qquad \Delta_s \varphi = 0$$

where Δ_s is the surface divergence of tangential gradient of φ. A calculation using integration by parts show that for $\rho \in L^2_{t,0}(\partial P)^3$, the Helmholtz-Hodge decomposition identity (3.29) is reduced to

$$\rho = n \times \nabla_s \psi, \qquad \psi \in H^1(\partial P, \mathbb{C}) \tag{3.30}$$

We see that the density of $W^{1,2}(\partial P)$ in $H^{\frac{1}{2}}(\partial P)$ implies that , for a given $g \in H^{\frac{1}{2}}(\partial P) \setminus \mathbb{C}$, there exists a sequence $\{g_j\}_{j=1}^{\infty} \in H^1(\partial P) \setminus \mathbb{C} \ \subset \ H^{\frac{1}{2}}(\partial P) \setminus \mathbb{C}$ such that

$$\|g - g_j\|_{H^{\frac{1}{2}}(\partial P)\setminus\mathbb{C}} < \varepsilon \tag{3.31}$$

Because $n \times \nabla : H^1(\partial P) \setminus \mathbb{C} \to L^2_{t,0}(\partial P)^3$ is an isomorphism [12], we have that $n \times \nabla g_j \in L^2_{t,0}(\partial P)^3$. By the continuity of the map $n \times \nabla : H^{\frac{1}{2}}(\partial P) \to V_t^{-\frac{1}{2}}(\partial P)^3$ and (3.31), we have that

$$\|n \times \nabla g - n \times \nabla g_j\|_{V_t^{-\frac{1}{2}}(\partial P)^3} < \|g - g_j\|_{H^{\frac{1}{2}}(\partial P)\setminus\mathbb{C}} < \varepsilon \tag{3.32}$$

This shows the density that $L^2_{t,0}(\partial P)^3$ is dense in $V_t^{-\frac{1}{2}}(\partial P)^3$. □

3.4 Proof of Main Results

3.4.1 Proof of Theorem 3.6

Proof. We prove the theorem by the following 2 steps.

First Step. We prove that $S : L^2_{t,0}(\partial P)^3 \to W_3$ is a bounded map.

Given $\rho \in L^2_{t,0}(\partial P)^3$ and $S(\rho) \in W_3$, we have

$$\|S(\rho)\|_{W_3} = \int_Y \nabla \times S(\rho) \cdot \nabla \times \overline{S(\rho)}$$

$$
\begin{aligned}
&= \int_H \nabla \times S(\rho) \cdot \nabla \times \overline{S(\rho)} + \int_P \nabla \times S(\rho) \cdot \nabla \times \overline{S(\rho)} \\
&= \int_{\partial P} [n \times \nabla \times S(\rho)]_+^- \cdot \ \overline{S(\rho)} \\
&= -\int_{\partial P} \rho \cdot \ \overline{S(\rho)}
\end{aligned} \tag{3.33}
$$

For a scalar function f and a vector function g, we observe that

$$
\begin{aligned}
0 &= \int_P \nabla \cdot (\nabla \times (fg)) \\
&= \int_{\partial P} n \cdot \nabla \times (fg) \\
&= \int_{\partial P} [n \cdot (\nabla f \times g) + n \cdot f(\nabla \times g)] \\
&= \int_{\partial P} g \cdot n \times \nabla f + \int_{\partial P} f(n \cdot \nabla \times g)
\end{aligned} \tag{3.34}
$$

So we have

$$
\int_{\partial P} g \cdot n \times \nabla f = -\int_{\partial P} f(n \cdot \nabla \times g) \tag{3.35}
$$

By writing $\rho = n \times \nabla f$ for $f \in H^{\frac{1}{2}}(\partial P) \setminus \mathbb{C}$ and using (3.35) in (3.33), we get

$$
\begin{aligned}
\|S(\rho)\|_{W_3}^2 &= -\int_{\partial P} \rho \ \cdot \overline{S(\rho)} \\
&= -\int_{\partial P} n \times \nabla f \cdot \ \overline{S(\rho)} \\
&= \int_{\partial P} f \ n \cdot \nabla \times \overline{S(\rho)}
\end{aligned} \tag{3.36}
$$

Since $n \cdot \nabla \times S(\rho) \in H_0^{-\frac{1}{2}}(\partial P)$, for $f + \sigma \in H^{\frac{1}{2}}(\partial P)$, it is also true that

$$
\begin{aligned}
\|S(\rho)\|_{W_3}^2 &= \int_{\partial P} (f + \sigma) \ n \cdot \nabla \times \overline{S(\rho)} \\
&\leq \|f + \sigma\|_{H^{\frac{1}{2}}(\partial P)} \|n \cdot \nabla \times S(\rho)\|_{H_0^{-\frac{1}{2}}(\partial P)}
\end{aligned} \tag{3.37}
$$

Because the map $n \cdot \nabla \times S : V_t^{-\frac{1}{2}}(\partial P)^3 \to H^{-\frac{1}{2}}(\partial P)$ is bounded [12], by taking the infimum of $\|f + \sigma\|_{H^{\frac{1}{2}}(\partial P)}$ over all $\sigma \in \mathbb{C}$, we arrive at

$$
\begin{aligned}
\|S(\rho)\|_{W_3}^2 &\leq C\|f + \sigma\|_{H^{\frac{1}{2}}(\partial P)} \|\rho\|_{V_t^{-\frac{1}{2}}(\partial P)^3} \\
&\leq \|\rho\|^2_{V_t^{-\frac{1}{2}}(\partial P)^3}
\end{aligned} \tag{3.38}
$$

and

$$\|S(\rho)\|_{W_3} \leq \|\rho\|_{V_t^{-\frac{1}{2}}(\partial P)^3} \tag{3.39}$$

The inequality (3.39) implies that $S(\rho)$ is bounded for ρ as an element in $\in L^2_{t,0}(\partial P)^3$.

By the theorem on bounded linear extension of a densely defined map, we now see that the extension map $S : V_t^{-\frac{1}{2}}(\partial P)^3 \to W_3$ is bounded.

Second Step. We prove that $S : V_t^{-\frac{1}{2}}(\partial P)^3 \to W_3$ is a bijection.

To do this, we first show that S is one-to-one. For a given $\rho \in V_t^{-\frac{1}{2}}(\partial P)^3$, we have $u = S(\rho) \in W_3$.Furthermore

$$\begin{aligned}
\rho &= n \times \nabla \times u\Big|_{\partial P^+} - n \times \nabla \times u\Big|_{\partial P^-} \\
&= n \times \nabla \times u\Big|_{\partial P^+} - n \times \nabla \times u\Big|_{\partial P^-} \\
&+ n \times \nabla \times u\Big|_{\partial Y} - n \times \nabla \times u\Big|_{\partial Y} \\
&= n \times \nabla \times u\Big|_{\partial H} - n \times \nabla \times u\Big|_{\partial P^-} - n \times \nabla \times u\Big|_{\partial Y}
\end{aligned} \tag{3.40}$$

From the mapping property of tangential trace map, if $f \in L^2(\Omega)$, $\nabla \times f \in L^2(\Omega)$, then we have $n \times f \in H^{-\frac{1}{2}}(\partial\Omega)$ and

$$\|n \times f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|\nabla \times f\|_{L^2(\Omega)}) \tag{3.41}$$

Now we take $f = \nabla \times u$, and using $\nabla \times \nabla \times u = 0$ in $H \cup P$ as well as triangle inequality, we see that

$$\begin{aligned}
\|\rho\|_{H^{-\frac{1}{2}}(\partial P)^3} &= \|n \times \nabla \times u\Big|_{\partial P^+} - n \times \nabla \times u\Big|_{\partial P^-}\|_{H^{-\frac{1}{2}}(\partial P)^3} \\
&\leq \|n \times \nabla \times u\|_{H^{-\frac{1}{2}}(\partial H)^3} + \|n \times \nabla \times u\|_{H^{-\frac{1}{2}}(\partial P)^3} + \|n \times \nabla \times u\|_{H^{-\frac{1}{2}}(\partial(Y))^3} \\
&\leq C_1(\|\nabla \times u\|_{L^2(H)^3} + \|\nabla \times u\|_{L^2(P)^3} + \|\nabla \times u\|_{L^2(Y)^3}) \\
&\leq C_2\|u\|_{H^*(\mathrm{curl},Y)} \\
&= C_2\|S(\rho)\|_{W_3}
\end{aligned} \tag{3.42}$$

Now for $\rho_1, \rho_2 \in V_t^{-\frac{1}{2}}(\partial P)^3$, we have

$$0 \leq \|\rho_1 - \rho_2\|_{H^{-\frac{1}{2}}(\partial P)^3} \leq C_2 \|S(\rho_1) - S(\rho_2)\|_{W_3} \tag{3.43}$$

The inequality (3.43) along with $V_t^{-\frac{1}{2}}(\partial P)^3 \subseteq H^{-\frac{1}{2}}(\partial P)^3$ implies that $S : V_t^{-\frac{1}{2}}(\partial P)^3 \to W_3$ is one-to-one.

Next we show that S is surjective. Assume that $u \in W_3 \subset J$ is given. This means

$$\nabla \times u \in L^2_{\#}(Y, \mathbb{C}^3), \qquad \nabla \times \nabla \times u = 0.$$

By Helmholtz decomposition of vector fields in $L^2(Y, \mathbb{C}^3)$, we can write $\nabla \times u$ as the sum of divergence -free and curl-free parts, that is

$$\nabla \times u = p + \nabla g, \qquad \nabla \cdot p = 0, \quad p \in L^2(Y, \mathbb{C}^3), \quad g \in H^1(Y, \mathbb{C}) \tag{3.44}$$

Since $\nabla \times \nabla \times u = 0$, the decomposition (3.44) gives

$$\nabla \times p = 0 \tag{3.45}$$

Therefore, we can write

$$p = \nabla q, \qquad \text{for some scalar function } \; q \in H^1(Y, \mathbb{C}) \tag{3.46}$$

Now, applying (3.45) and (3.46) in the decomposition (3.44) gives

$$\nabla \times u(x) = \nabla v(x), \qquad v = g + q \in H^1(Y, \mathbb{C}) \tag{3.47}$$

Taking the cross product of (3.47) with the normal vector n and then letting x go to ∂P, we get

$$\begin{aligned} n \times \nabla \times u\Big|_{\partial P^+} &= n \times \nabla_s v\Big|_{\partial P^+} \\ n \times \nabla \times u\Big|_{\partial P^-} &= n \times \nabla_s v\Big|_{\partial P^-} \end{aligned} \tag{3.48}$$

In view of (3.48), we conclude that

$$\begin{aligned} n \times \nabla \times u\Big|_{\partial P^+} &\in V_t^{-\frac{1}{2}}(\partial P)^3 \\ n \times \nabla \times u\Big|_{\partial P^-} &\in V_t^{-\frac{1}{2}}(\partial P)^3 \end{aligned} \tag{3.49}$$

Now we set

$$\rho_u = n \times \nabla \times u\Big|_{\partial P^+} - n \times \nabla \times u\Big|_{\partial P^-} \in V_t^{-\frac{1}{2}}(\partial P)^3$$

and take $w = S(\rho_u)$ to find

$\nabla\times\nabla\times w = 0$ in $P\cup H$, $\qquad \nabla\cdot w = 0$ in Y, $\qquad n\times\nabla\times w$ is antiperiodic on ∂Y.

and

$$\begin{aligned} (w-u, w-u)_Y &= \int_Y \nabla \times (w-u) \cdot \nabla \times (\overline{w} - \overline{u})\, dy \\ &= \int_{H\cup P} \nabla \times (w-u) \cdot \nabla \times (\overline{w} - \overline{u})\, dy \\ &= -\int_{\partial Y} n \times \nabla \times (w-u) \cdot (\overline{w-u})\, ds_y \\ &\quad + \int_{\partial P} [n \times \nabla \times (w-u)]_+^- \cdot (\overline{w-u})\, ds_y \\ &\quad + \int_{H\cup P} \nabla \times \nabla \times (w-u) \cdot (\overline{w-u})\, dy \end{aligned}$$

Using the antiperiodicity and the fact that

$$[(n \times \nabla \times w)]_{\partial P} = [(n \times \nabla \times u)]_{\partial P} = \rho_u, \quad \nabla \times \nabla \times (w-u) = 0 \quad \text{in} \quad H \cup P$$

we see that

$$(w-u, w-u) = 0 \qquad \text{implies} \qquad w - u = C$$

But

$$0 = \int_Y w\, dy - \int_Y u\, dy = C.$$

So we conclude that $w = u$. This shows that S is surjective and thus bijective. The application of open mapping theorem on S yields that S^{-1} is bounded, and then the claim of the theorem follows. $\square$

3.4.2 Proof of Theorem 3.7

Proof. We first establish that M is a bounded map of $V_t^{-\frac{1}{2}}(\partial P)^3$. To do this, we start with the following identity from [12]

$$K^*(n \cdot \nabla \times S) = n \cdot \nabla \times SM, \qquad \text{for} \;\; \rho \in L^2_{t,0}(\partial P)^3 \tag{3.50}$$

where K^* is the scalar valued Newmann-Poicaré operator of $H_0^{-\frac{1}{2}}(\partial P)$ which is a bounded linear operator. The map $n \cdot \nabla \times S : V_t^{-\frac{1}{2}}(\partial P)^3 \to H_0^{-\frac{1}{2}}(\partial P)$ is an isomorphism [12]. On one hand, the boundedness of K^* and the boundedness of the inverse of the operator $n \cdot \nabla \times S$ show that

$$\|K^*(n \cdot \nabla \times S(\rho))\|_{H_0^{-\frac{1}{2}}(\partial P)} \leq \|n \cdot \nabla \times S(\rho)\|_{H_0^{-\frac{1}{2}}(\partial P)} \leq C_1 \|\rho\|_{V_t^{-\frac{1}{2}}(\partial P)^3} \tag{3.51}$$

On the other hand, the boundedness of the inverse of $n \cdot \nabla \times S$ also shows that

$$C_2 \|M\rho\|_{V_t^{-\frac{1}{2}}(\partial P)^3} \leq \|n \cdot \nabla \times SM(\rho)\|_{H_0^{-\frac{1}{2}}(\partial P)} \tag{3.52}$$

In view of (3.51) and (3.52), we have

$$\|M\rho\|_{V_t^{-\frac{1}{2}}(\partial P)^3} \leq C_3 \|\rho\|_{V_t^{-\frac{1}{2}}(\partial P)^3} \tag{3.53}$$

So $M(\rho)$ is bounded for $\rho \in L^2_{t,0}(\partial P)^3 \subset V_t^{-\frac{1}{2}}(\partial P)^3$.

Now we observe that, since $L^2_{t,0}(\partial P)^3$ is dense in $V_t^{-\frac{1}{2}}(\partial P)^3$, by the theorem on bounded linear extension of a densely defined bounded linear map on a Banach space, we can extend M as a map of $V_t^{-\frac{1}{2}}(\partial P)^3$ as

$$M(\rho) = \begin{cases} M(\rho) & \rho \in L^2_{t,0}(\partial P)^3 \\ \\ \lim_{n \to \infty} M(\rho_n) & \rho \in \overline{L^2_{t,0}(\partial P)^3}, \quad \{\rho_n\} \in L^2_{t,0}(\partial P)^3, \;\; \rho_n \to \rho \in V_t^{-\frac{1}{2}}(\partial P)^3 \end{cases}$$

Since $n \cdot \nabla \times S : V_t^{-\frac{1}{2}}(\partial P)^3 \to H_0^{-\frac{1}{2}}(\partial P)$ is an isomorphism, for a bounded sequence $\{\rho_n\} \in V_t^{-\frac{1}{2}}(\partial P)^3$, we have

$$\|n \cdot \nabla \times S(\rho_n)\|_{H_0^{-\frac{1}{2}}(\partial P)} \leq C\|\rho_n\|_{V_t^{-\frac{1}{2}}(\partial P)^3} \tag{3.54}$$

Which shows

$$\{n \cdot \nabla \times S(\rho_n)\}_{n=1}^{\infty} \in H_0^{-\frac{1}{2}}(\partial P),$$

and there exists a bounded sequence

$$\{n \cdot \nabla \times S(\rho_{n_k})\}_{k=1}^{\infty} \in H_0^{-\frac{1}{2}}(\partial P).$$

By the compactness of K^* and (3.50), we have

$$\begin{aligned} &\{K^* n \cdot \nabla \times S(\rho_{n_k})\}_{k=1}^{\infty} \in H_0^{-\frac{1}{2}}(\partial P) \text{ is Cauchy} \\ \Longleftrightarrow &\{n \cdot \nabla \times S(M(\rho_{n_k}))\}_{k=1}^{\infty} \in H_0^{-\frac{1}{2}}(\partial P) \text{ is Cauchy.} \end{aligned} \tag{3.55}$$

Because $n \cdot \nabla \times S : V_t^{-\frac{1}{2}}(\partial P)^3 \to H_0^{-\frac{1}{2}}(\partial P)$ is an isomorphism and K^* is continuous map, we have

$$\begin{aligned} &\|M\rho_{n_k} - M\rho_{n_l}\|_{V_t^{-\frac{1}{2}}(\partial P)^3} \\ &= \|M(\rho_{n_k} - \rho_{n_l})\|_{V_t^{-\frac{1}{2}}(\partial P)^3} \\ &\leq C\|n \cdot \nabla \times S(M(\rho_{n_k} - \rho_{n_l}))\|_{H_0^{-\frac{1}{2}}(\partial P)} \\ &= C\|K^* n \cdot \nabla \times S(M(\rho_{n_k} - \rho_{n_l}))\|_{H_0^{-\frac{1}{2}}(\partial P)} \\ &\leq C_1\|n \cdot \nabla \times S(M(\rho_{n_k} - \rho_{n_l}))\|_{H_0^{-\frac{1}{2}}(\partial P)} \\ &= C_1\|n \cdot \nabla \times S(M(\rho_{n_k})) - n \cdot \nabla \times S(M(\rho_{n_l}))\|_{H_0^{-\frac{1}{2}}(\partial P)} \end{aligned} \tag{3.56}$$

By (3.55), we conclude that

$$\|M\rho_{n_k} - M\rho_{n_l}\|_{V_t^{-\frac{1}{2}}(\partial P)^3} \to 0$$

which shows that $\{M(\rho_{n_k})\}_{n=1}^{\infty} \in V_t^{-\frac{1}{2}}(\partial P)^3$ is Cauchy and thus M is a compact operator on $V_t^{-\frac{1}{2}}(\partial P)^3$.

Finally, the identity (3.14) is the direct consequence of the application of the compactness of $M : V_t^{-\frac{1}{2}}(\partial P)^3 \to V_t^{-\frac{1}{2}}(\partial P)^3$, compactness of Newmann-Poicaré operator $K^* : H_0^{-\frac{1}{2}}(\partial P) \to H_0^{-\frac{1}{2}}(\partial P)$ as well as the isomorphic property of the map $n \cdot \nabla \times S : V_t^{-\frac{1}{2}}(\partial P)^3 \to H_0^{-\frac{1}{2}}(\partial P)$ in the identity (3.50).

□

3.4.3 Proof of Theorem 3.8

Proof. First we show that $T : W_3 \to W_3$ is Hermitian. For $u, w \in W_3$,

$$\begin{aligned}(Tu, w) &= \int_Y (\nabla \times Tu) \cdot (\nabla \times \overline{w})\, dy \\ &= \int_Y (\nabla \times SMS^{-1}u) \cdot (\nabla \times \overline{w})\, dy \\ &= \int_H (\nabla \times SMS^{-1}u) \cdot (\nabla \times \overline{w})\, dy \\ &+ \int_P (\nabla \times SMS^{-1}u) \cdot (\nabla \times \overline{w})\, dy\end{aligned}$$

Using integration by parts and the

$$\nabla \times \nabla \times S(MS^{-1}u) = 0, \qquad \text{in} \;\; H \cup P$$

we see

$$\int_Y (\nabla \times SMS^{-1}u) \cdot (\nabla \times \overline{w})\, dy = \int_{\partial P} \Big[n \times \nabla \times SMS^{-1}u\Big]_+^- \cdot \overline{w}\, ds_y$$

By the jump condition

$$n \times \nabla_x \times S(\rho)\Big|_{\partial P}^{\pm} = \pm\frac{1}{2}\rho + M(\rho)$$

we now have

$$(Tu, w) = -\int_{\partial P} MS^{-1}u \cdot \overline{w}\, ds_y$$

For some $\beta \in V_t^{-\frac{1}{2}}(\partial P)^3$, we write $u = S\beta$, and have

$$\begin{aligned}(Tu, w) &= -\int_{\partial P} MS^{-1}S\beta \cdot \overline{w}\, ds_y \\ &= -\int_{\partial P} M\beta \cdot \overline{w}\, ds_y \\ &= -\frac{1}{2}\int_{\partial P} [n \times \nabla \times S\beta\big|_+ + n \times \nabla \times S\beta\big|_-] \cdot \overline{w}\, ds_y\end{aligned}$$

and integration by parts give

$$-\frac{1}{2}\int_{\partial P}[n\times\nabla\times S\beta\Big|_{+}+n\times\nabla\times S\beta\Big|_{-}]\cdot\overline{w}\;ds_y$$
$$=\frac{1}{2}\int_{H}(\nabla\times S\beta)\cdot(\nabla\times\overline{w})\;dy-\frac{1}{2}\int_{P}(\nabla\times S\beta)\cdot(\nabla\times\overline{w})\;dy$$

So

$$(Tu,w)=\frac{1}{2}\int_{H}(\nabla\times u)\cdot(\nabla\times\overline{w})\;dy-\frac{1}{2}\int_{P}(\nabla\times u)\cdot(\nabla\times\overline{w})\;dy \tag{3.57}$$

Interchanging the position of u and v shows that T is Hermitian, that is

$$(Tu,w)=\overline{(u,Tw)}$$

Now we prove the identity (3.15).

Let

$$\mu\in\sigma\left(M;\;V_t^{-\frac{1}{2}}(\partial P)^3\right),\qquad M\rho=\mu\rho,\quad(\mu,\rho)\in\left[-\frac{1}{2},\frac{1}{2}\right]\times V_t^{-\frac{1}{2}}(\partial P)^3.$$

There exists $u\in W_3$ such that $u=S\rho$, and $\rho=S^{-1}u$. Therefore, we have

$$MS^{-1}u=\mu S^{-1}u$$

This implies that

$$SMS^{-1}u=\mu SS^{-1}u\quad\Rightarrow\quad Tu=\mu u,$$

which shows that $\sigma\left(M;\;V_t^{-\frac{1}{2}}(\partial P)^3\right)\subset\sigma\left(T;\;W_3\right)$.

If we have $Tu=\mu u$, then

$$\mu(u,w)=(Tu,w)=\frac{1}{2}\int_{H}(\nabla\times u)\cdot(\nabla\times\overline{w})\;dy-\frac{1}{2}\int_{P}(\nabla\times u)\cdot(\nabla\times\overline{w})\;dy$$

Subtracting $\int_{Y}(\nabla\times u)\cdot(\nabla\times\overline{w})\;dy$ from both sides, we get

$$\int_{P}(\nabla\times u)\cdot(\nabla\times\overline{w})\;dy=(\frac{1}{2}-\mu)\int_{Y}(\nabla\times u)\cdot(\nabla\times\overline{w})\;dy$$

and we arrive at

$$\lambda\int_{Y}(\nabla\times u)\cdot(\nabla\times\overline{w})\;dy=\int_{P}(\nabla\times u)\cdot(\nabla\times\overline{w})\;dy. \tag{3.58}$$

through the relation $\lambda=\frac{1}{2}-\mu$ and $\lambda\in[0,1]$. □

3.4.4 Proof of Theorem 3.9

Proof. By Theorem 3.8, we can conclude that there exist a countable subset of the real line given by

$$\{\mu_i\} := \left\{-\frac{1}{2}, \frac{1}{2}\right\} \cup \{\mu_n\},\ n = 1, 2, 3, \cdots$$

with a single accumulation point at 0 and an associated family of orthogonal finite dimensional projections$\{P_{\mu_i}\}$ such that

$$(u, v) = (\sum_{i=1}^{\infty} P_{\mu_i}, v), \qquad v \in J$$

or

$$(Tu, v) = (\sum_{i=1}^{\infty} \mu_i P_{\mu_i}, v), \qquad v \in J$$

By (3.7) and (3.8), we have

$$\begin{aligned} (Tu_1, v) &= -\frac{1}{2}(u_1, v), \qquad & u_1 \in W_1 \\ (Tu_2, v) &= \frac{1}{2}(u_2, v), \qquad & u_2 \in W_2 \end{aligned}$$

for all $v \in J$.

If we let P_r be the projection operator acting on the solution $u \in J$ and takes it onto the subspace spanned by the eigenfunctions corresponding to the eigenvalues μ_i, then we have

$$u = P_1 u + P_2 u + \sum_{-\frac{1}{2}<\mu_n<\frac{1}{2}} P_{\mu_n} u = a_1\psi_1 + a_2\psi_2 + \sum_{-\frac{1}{2}<\mu_n<\frac{1}{2}} a_{\mu_n}\psi_{\mu_n} \tag{3.59}$$

with $P_i u,\ i = 1, 2, \mu_n$ are the orthogonal projections of u onto the subspaces W_1, W_2 and W_3.

Now we have the following representation formula for the linear operator T_z associated with the bilinear form in (3.5)

$$(T_z u, w) = (zP_1 u + P_2 u + \sum_{\mu_n} \left[(\frac{1}{2} + \mu_n) + z(\frac{1}{2} - \mu_n)\right] P_{\mu_n} u,\ w) \tag{3.60}$$

for all $u, w \in J$ and.

For $u, w \in J$, by (3.59) we have

$$B_z(P_{\mu_n}u, w) = \int_H (\nabla \times P_{\mu_n}u) \cdot (\nabla \times \overline{w})\ dy + z \int_P (\nabla \times P_{\mu_n}u) \cdot (\nabla \times \overline{w})\ dy \quad (3.61)$$

On the other hand, by (3.57), we know that

$$\begin{aligned}(TP_{\mu_n}u, w) &= \frac{1}{2}\int_H (\nabla \times P_{\mu_n}u) \cdot (\nabla \times \overline{w})\ dy - \frac{1}{2}\int_P (\nabla \times P_{\mu_n}u) \cdot (\nabla \times \overline{w})\ dy \\ &= \mu_n \int_H (\nabla \times P_{\mu_n}u) \cdot (\nabla \times \overline{w})\ dy + \mu_n \int_P (\nabla \times P_{\mu_n}u) \cdot (\nabla \times \overline{w})\ dy\end{aligned}$$

which gives

$$\int_H (\nabla \times P_{\mu_n}u) \cdot (\nabla \times \overline{w})\ dy = \frac{\frac{1}{2} + \mu_n}{\frac{1}{2} - \mu_n} \int_P (\nabla \times P_{\mu_n}u) \cdot (\nabla \times \overline{w})\ dy \quad (3.62)$$

Also, by (3.58), we have

$$\int_P (\nabla \times P_{\mu_n}u) \cdot (\nabla \times \overline{w})\ dy = (\frac{1}{2} - \mu_n) \int_Y (\nabla \times P_{\mu_n}u) \cdot (\nabla \times \overline{w})\ dy \quad (3.63)$$

and (3.62) becomes

$$\int_H (\nabla \times P_{\mu_n}u) \cdot (\nabla \times \overline{w})\ dy = (\frac{1}{2} + \mu_n) \int_Y (\nabla \times P_{\mu_n}u) \cdot (\nabla \times \overline{w})\ dy \quad (3.64)$$

Substituting (3.63) and (3.64) into (3.61), we get

$$B_z(P_{\mu_n}u, w) = \left[(\frac{1}{2} + \mu_n) + z(\frac{1}{2} - \mu_n)\right] \int_Y (\nabla \times P_{\mu_n}u) \cdot (\nabla \times \overline{w})\ dy \quad (3.65)$$

We also note that

$$B_z(P_1u, w) = z \int_P (\nabla \times P_1u) \cdot (\nabla \times \overline{w})\ dy \quad (3.66)$$

$$B_z(P_2u, w) = \int_H (\nabla \times P_2u) \cdot (\nabla \times \overline{w})\ dy \quad (3.67)$$

So we can conclude that

$$B_z(u, w) = (T_zu, w) = (zP_1u + P_2u + \sum_{\mu_n} \left[(\frac{1}{2} + \mu_n) + z(\frac{1}{2} - \mu_n)\right] P_{\mu_n}u, \quad w)$$

By setting $z = \varepsilon(x))^{-1}$, we now see that

$$\nabla \times ((\varepsilon(x))^{-1} \nabla \times = -\Delta T_{\varepsilon_p^{-1}}$$

where the Laplace operator is understood to be associated with the bilinear form $B_{\varepsilon_p^{-1}}$ and

$$T_{\varepsilon_p^{-1}} u = \varepsilon_p P_1 u + P_2 u + \sum_{\mu_n} \left[(\frac{1}{2} + \mu_n) + \varepsilon_p (\frac{1}{2} - \mu_n) \right] P_{\mu_n} u$$

The equations (3.65)-(3.67) shows that we have

$$T_z^{-1} = z^{-1} P_1 + P_2 + \sum_{\mu_n} z \left[z(\frac{1}{2} + \mu_n) + \varepsilon_p (\frac{1}{2} - \mu_n) \right]^{-1} P_{\mu_n}$$

for $z \neq -(\frac{1}{2} + \mu_n)/(\frac{1}{2} - \mu_n)$. This concludes the proof of Theorem 3.9. □

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Vita

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