
Weizhen Gu

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Exterior vertices in graphs and realization of plurality preference digraphs

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Exterior Vertices in Graphs

and

Realization of Plurality Preference Digraphs

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Abstract

Part I: Peripheral and eccentric vertices of graphs

For a connected graph $G$, let $P(G)$ and $EC(G)$ denote the sets of peripheral vertices and eccentric vertices of $G$, respectively. In 1988 F. Buckley initiated the study of the class $S$ of graphs $G$ for which $P(G) = EC(G)$.

We provide several families of graphs which are in $S$. For certain graphs $G$ with diameter equal to $2r(G)$ or $2r(G) - 1$, we give criteria for $P(G)$ and $EC(G)$ to be equal. Also, for certain products, we characterize those pairs of graphs so that their product is in $S$. We present graphs which are then used to show that all possible set-inclusion relations between $P(G)$ and $EC(G)$ may occur. Additionally, we estimate the smallest order of a graph $H$ having a given graph $G$ as an induced subgraph so that $P(H) = EC(H)$ (or $P(H) = V(G)$).

Part II: Realization of plurality preference digraphs

A digraph $D$ with vertex set $\{x_1, x_2, \ldots, x_k\}$ is $(n, h, k)$-realizable if there exists a connected graph $G$ of order $n$, a subset $V \subseteq V(G)$ with $|V| = h$, and a subset $C = \{c_1, c_2, \ldots, c_k\} \subseteq V(G)$ so that for all distinct $i$ and $j$ in $\{1, 2, \ldots, k\}$, $(x_i, x_j)$ is an arc of $D$ if and only if more vertices in $V$ are closer to $c_i$ than to $c_j$ in $G$. In particular, if $h = n$, then we simply say that $D$ is realizable by $G$ or that $G$ realizes $D$.

In 1988, Johnson and Slater proved that any oriented graph is realizable by a graph. We provide two constructions of graphs which realize a given oriented graph and show that each of these has a smaller order than the example due to Johnson and Slater. The best known construction, due recently to W.
Schnyder, is also provided. Secondly, we characterize digraphs which are realizable by trees. Additionally, we derive some properties of a digraph which is \((n,n,n)\)-realizable by a tree and describe a class of such digraphs. Finally, let \(\mathcal{F}_n\) denote the family of digraphs of order \(n\) which are realizable by trees. For a fixed \(D \in \mathcal{F}_n\), let \(\alpha(D)\) be the smallest order of a tree which realizes \(D\). We determine the value of \(\alpha(\mathcal{F}_n) = \max\{\alpha(D) : D \in \mathcal{F}_n\}\) explicitly.
Introduction

The graphs considered here are simple and connected graphs. As usual, \( V(G) \) denotes the set of vertices of \( G \), and \( E(G) \) denotes the set of edges of \( G \). The number of vertices of a graph \( G \) is called the order of \( G \). The distance \( d_G(u,v) \) (\( d(u,v) \) for short) between vertices \( u \) and \( v \) is the number of edges in a shortest path in \( G \) between \( u \) and \( v \). For a subset \( X \) of \( V(G) \) and a vertex \( v \), \( d_G(v,X) = \min \{ d_G(v,x) : x \in X \} \). For \( u, v \in V(G) \), \( V_{u,v} = \{ x : x \in V(G), d(x,u) < d(x,v) \} \). If \( P \) is a path from a vertex \( x \) to a vertex \( y \), then vertices \( x \) and \( y \) are called endvertices of \( P \), and the edges on \( P \) incident with \( x \) and \( y \) are called end edges of \( P \). A graph \( G \) is a subgraph of a graph \( H \), and \( H \) is a supergraph of \( G \), if \( V(G) \subseteq V(H) \) and \( E(G) \subseteq E(H) \). Let \( S \) be a nonempty subset of \( V(G) \). The subgraph of \( G \) whose vertex set is \( S \) and whose edge set is the set of those edges of \( G \) that have both ends in \( S \) is called the subgraph of \( G \) induced by \( S \) and is denoted by \( (S) \). For any finite set \( S \), the number of elements of \( S \) is denoted by \(|S|\).

A simple graph in which each pair of distinct vertices is joined by an edge is called a complete graph. The complete graph of order \( n \) is denoted by \( K_n \). The complement of a graph \( G = (V,E) \) is the graph \( \bar{G} = (V,\bar{E}) \), where \( \bar{E} = \{ uv : u, v \in V, u \neq v, uv \notin E \} \). For terminology not defined here, one is referred to the text by Bondy and Murty [2].

Graphs are often used to model such things as road networks and communication networks, and many mathematical problems have arisen from different instances of the question of what is an optimal location for a facility in a graph. In most cases the type of facility to be established is one for which a "central" location is optimal. One location problem, considered first by Hakimi [8], is to determine a vertex \( u \in V(G) \) so as to minimize

\[
e(u) = \max \{ d(u,v) : v \in V(G) \}.
\]
The value \( e(u) \) is called the eccentricity of the vertex \( u \). The minimum eccentricity and the maximum eccentricity are called the radius and the diameter of \( G \), denoted \( r(G) \) and \( \text{dia}(G) \) respectively. A vertex \( v \) of \( G \) is called a center vertex if \( e(v) = r(G) \). The set of center vertices of \( G \), denoted \( C(G) \), is simply called the center of \( G \). A standard application using center vertices is to minimize a maximum response time from an emergency facility, such as a hospital or a fire station.

The first notable result concerning center vertices of graphs is due to Jordan [13].

**Theorem 0.1.** If \( T \) is a tree, then \( C(T) \) consists of either a single vertex or two adjacent vertices.

One hundred years after the theorem of Jordan, the problem of characterizing centers of graphs from a specified class has attracted some attention. Most recent work on centers has dealt with structure (what the subgraphs induced by the center look like), embedding (determining when a supergraph \( H \) can be built around a graph \( G \) so that the vertices of \( G \) are precisely the center vertices of \( H \)), and algorithms. Many similar types of "central" vertices, such as centroid vertices and medians, have been studied by P. J. Slater, K. B. Reid, F. Buckley, and other authors (see [3], [6], [7], [10], [20], [23], [24], [25], [26]).

In addition to "central" vertices of a graph, "exterior" vertices of a graph are also currently being investigated. One type of an "exterior" vertex is a peripheral vertex, a concept which was introduced by O. Ore in 1962 [16]. A vertex \( v \) of \( G \) is called a peripheral vertex of \( G \) if \( e(v) = \text{dia}(G) \). K. R. Parthasarthy and R. Nandakumar in 1983 [17] defined a vertex \( x \) to
be an eccentric vertex of $y$ if $d(x, y) = e(y)$. This concept is specified to yeild another type of an "exterior" vertex. In this work, $x$ is an eccentric vertex if, in the sense of K. R. Parthasarthy and R. Nandakumar, $x$ is an eccentric vertex of a center vertex, i.e., $d(x, y) = e(y)$ for some center vertex $y$. Let $P(G)$ and $EC(G)$ denote the sets of peripheral vertices and eccentric vertices respectively. In location of facility theory, these two sets may be thought of as demand sites which are on the "outer edge" or "rim" of a network in which emergency facilities are located at center vertices.

Since a longest path in a nontrivial connected graph has two distinct end vertices, any nontrivial connected graph has at least two peripheral vertices. Also, any nontrivial connected graph has at least two eccentric vertices (see [4]). The concepts of center, peripheral, and eccentric vertices of a graph $G$ are illustrated in Fig. 0.1, where $r(G) = 2$, $dia(G) = 4$, $C(G) = \{e, f\}$, $P(G) = \{a, h\}$, and $EC(G) = \{a, b, d, h\}$.

![Graph diagram](image)

(The number beside a vertex is the eccentricity of the vertex)

Fig. 0.1

Another location problem, considered first by Johnson and Slater [12], concerns the realization of a digraph by user preferences based on distances in graphs. Consider a graph $G$ with two distinguished subsets of the vertex set: one set called the set of candidates and another set called the set of
voters. A voter $v$ prefers candidate $x$ to candidate $y$ if $d_G(v, x) < d_G(v, y)$. This preferences relation defines a digraph whose vertices are candidates, and in which there is an arc from a candidate $x$ to a candidate $y$ if and only if more voters prefer $x$ to $y$ than prefer $y$ to $x$. The set of candidates may be thought as possible sites for the location, at vertices, of one or more desirable facilities, and the set of voters may be thought as the users, also located at vertices, entitled to cast a preference for the placement of those facilities.

One candidate site will be prefered to another candidate site by a voter if the first site is closer than the second, and one site will be prefered to another by the set of voters if more voters prefer the first site over the second. That is, plurality preference, rather than majority preference, is considered here. To be more explicit we use the following definition [11].

Definition: An oriented graph $D$ with vertex set $X = \{x_1, x_2, \ldots, x_k\}$ is $(n, h, k)$-realizable if there exists a connected graph $G$ of order $n$, a subset $V$ of $h$ vertices of $G$ (voters), and a subset $C = \{c_1, c_2, \ldots, c_k\}$ of vertices of $G$ (candidates) so that for all distinct $i$ and $j$ in $\{1, 2, \ldots, k\}$, $x_i x_j \in A(D)$ if and only if more voters in $V$ are closer to $c_i$ than to $c_j$ in $G$. In particular, if $D$ is $(n, n, k)$-realizable by a graph $G$, then we simply say that $D$ is realized by $G$ or that $G$ realizes $D$.

Fig. 0.2.
Example. The digraph $D$ given in Fig. 0.2 (a) is $(7, 6, 3)$-realizable. To see this consider the graph $G$ given in Fig. 0.2 (b), where the 6-set $V$ of voters is as shown and the 3-set $C$ of candidates is \{$c_1 = v_2$, $c_2 = v_3$, $c_3 = v_4$\}.

Although peripheral and eccentric vertices in graphs are not directly related to realizations of plurality preference digraphs, both themes are motivated by issues in facility locations in networks. This dissertation consists of two parts: the results on peripheral and eccentric vertices of graphs are in the first two chapters, and the results on realizations are in the last two chapters.

F. Buckley [4] first introduced and studied graphs for which the set of peripheral vertices is the same as the set of eccentric vertices. He proved the following results:

**Theorem 0.2.** If $(C(G))$ is a bridge, then $P(G) = EC(G)$.

**Theorem 0.3.** If $C(G) = \{x\}$ and $x$ does not lie on a cycle, then $P(G) = EC(G)$.

From the above two theorems and Jordan's theorem (Theorem 0.1), it is easy to see that for any tree $T$, $P(T) = EC(T)$.

As illustrated by the graph in Fig. 0.1, not every connected graph has the property that the set of peripheral vertices is the same as the set of eccentric vertices. In fact, any set-inclusion relations between $P(G)$ and $EC(G)$ may occur. Graphs for which $P(G) = EC(G)$ or $P(G) \subseteq EC(G)$, or $P(G) \supseteq EC(G)$ are given in Fig. 0.3.
Finding a useful characterization of graphs $G$ so that $P(G) = EC(G)$ appears to be a difficult task. A more tractable problem is characterization of these graphs for various classes. In Chapter 1 we provide a description of several families of graphs which have the property that the sets of peripheral vertices and eccentric vertices are the same. Indeed, one of these families is a super-family of the family of graphs described in Theorems 0.2 and 0.3. Also, for certain graphs $G$ with diameter equal to $2r(G)$ or $2r(G) - 1$, we are able to give criteria for $P(G)$ and $EC(G)$ to be equal. In addition, we characterize those pairs of graphs so that, for certain of their products, the set of peripheral vertices is the same as the set of eccentric vertices. As mentioned before, all possible set-inclusion relations between $P(G)$ and $EC(G)$ may occur. In Chapter 2, for each pair of positive integers $a$ and $b$ with $a \leq b \leq 2a$, we construct a graph $G_i$, with $r(G_i) = a$ and $dia(G_i) = b$, which satisfies the condition (i) below:
(1) $P(G_1) = EC(G_1)$ and $a \leq b \leq 2a$;

(2) $P(G_2) \subseteq EC(G_2)$, if $a < b \leq 2a$;

(3) $P(G_3) \supsetneq EC(G_3)$, if $a < b \leq 2a$;

(4) $P(G_4) \cap EC(G_4) \neq \emptyset$, $P(G_4) \subseteq EC(G_4)$, $P(G_4) \nsubseteq EC(G_4)$, if $a < b \leq 2a$; and

(5) $P(G_5) \cap EC(G_5) = \emptyset$, if $a + 2 < b \leq 2a - 2$.

Additionally, we estimate the smallest order of a graph $H$ having a given graph $G$ as an induced subgraph so that $P(H) = EC(H)$ and investigate the possibility of embedding a given graph $G$ into a supergraph $H$ so that $P(H) = S$ for a given proper subset $S$ of $V(G)$.

In Chapter 3, we provide three constructions of graphs which realize a given oriented graph and show that each of these has a smaller order than provided by the construction due to Johnson and Slater (see [12]). One construction is substantially better and based on joint work with W. Schnyder [11]. In Chapter 4 we investigate the structure of digraphs realized by trees. First of all, by describing a criterion for the existence of an arc in a digraph $D$ realized by a tree, we are able to characterize digraphs of order $k$ which are $(n,n,k)$-realizable by trees. In additional, we derive some properties of an oriented graph $D$ which is $(n,n,n)$-realizable by a tree and give a class of graphs which are $(n,n,n)$-realizable by trees. Finally, for a positive integer $n$, let $\mathcal{F}_n$ denote the family of digraphs of order $n$ which are realizable by trees. For a fixed $D \in \mathcal{F}_n$, the realization number of $D$, denoted $\alpha(D)$, is the smallest order of a tree which realizes $D$. Let $\alpha(\mathcal{F}_n) = max\{\alpha(D) : D \in \mathcal{F}_n\}$. We determine the value of $\alpha(\mathcal{F}_n)$.
The problem of describing graphs $G$ with the property that $P(G) = EC(G)$ was first considered by F. Buckley in [4]. He provided two families of such graphs, for example, the family of trees. Hence, the existence problem of a graph $G$ with $P(G) = EC(G)$ was settled. However, extending the families of graphs described in Theorems 0.2 and 0.3 is the object of this chapter. We first discuss graphs in which the block containing the center is complete and then prove that these graphs satisfy $P(G) = EC(G)$. Secondly, we give criteria so that certain graphs $G$ with $\text{dia}(G) = 2r(G)$ or $\text{dia}(G) = 2r(G) - 1$ satisfy $P(G) = EC(G)$. Thirdly, we describe some other graphs $G$ with $P(G) = EC(G)$. Finally, we characterize those pairs of graphs so that, for certain of their products, the set of peripheral vertices is the same as the set of eccentric vertices.

\ §1 Graphs whose center is in a complete block

A connected graph that has no cut vertices is called a block. A block of a graph is a subgraph that is a block and is maximal with respect to this property (see [2]). A block of a graph is called a complete block if the block is a complete graph.

Let $G$ be a graph, and let $B$ be a block of $G$. For any vertex $v$ in $B$, let $G(v)$ (or $G(v, B)$, when the block $B$ is to be emphasized) denote the component of $G - E(B)$ which contains $v$. Let $a_B(v) = \max\{d(u, v) : u \in V(G(v))\}$ and $a(B) = \max\{a_B(v) : v \in V(B)\}$. Let $G$ be the graph shown in Fig. 1.1.1 (a). The notation of $G(v)$ is illustrated in Fig. 1.1.1 (b). It is easy to check that $a_B(u) = 1$, $a_B(v) = 3$, and $a_B(w) = 1$. Thus, $a(B) = 3$. 

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Lemma 1.1.1. Let $G$ be a connected, non-complete graph. Suppose that $(C(G))$ is contained in a complete block $B$. If $|C(G)| \geq 2$, then

1. $r(G) = a(B) + 1$, and
2. There exist distinct vertices $u$ and $v$ in $B$ so that $a_B(u) = a_B(v) = r(G) - 1$.

Proof: (1) Let $V(B) = \{v_1, v_2, \ldots, v_m\} (m \geq 2)$. Without loss of generality, we may assume that $v_1, v_2 \in C(G)$. Then since $B$ is complete,

$$e(v_1) = \max_{2 \leq j \leq m} \{a_B(v_1), a_B(v_j) + 1\}.$$  

Note that, $e(v_1) = e(v_2)$, and, since $B$ is complete, $e(v_2) \geq a_B(v_1) + 1$. Thus, $e(v_1) \geq a_B(v_1) + 1$. Hence,

$$e(v_1) = \max_{2 \leq j \leq m} \{a_B(v_j) + 1\} = \max_{2 \leq j \leq m} \{a_B(v_j)\} + 1.$$ 

Similarly,

$$e(v_2) = \max_{1 \leq j \leq m, j \neq 2} \{a_B(v_j)\} + 1.$$ 

Since $e(v_1) = e(v_2)$, $\max_{2 \leq j \leq m} \{a_B(v_j)\} = \max_{1 \leq j \leq m, j \neq 2} \{a_B(v_j)\} = a(B)$. Hence, $e(v_1) = e(v_2) = a(B) + 1$. That is, $r(G) = a(B) + 1$.

(2) Since $a(B) = \max_{1 \leq j \leq m} \{a_B(v_j)\}$, there is $\ell$, $1 \leq \ell \leq m$, such that $a_B(v_\ell) = a(B)$. Suppose that for any $j \in \{1, 2, \ldots, m\}$, $j \neq \ell$, $a_B(v_j) < a(B)$. Then

$$e(v_\ell) = \max_{1 \leq j \leq m, j \neq \ell} \{a_B(v_\ell), a_B(v_j) + 1\} = a(B).$$
By part (1) above, \( a(B) = r(G) - 1 \). So \( e(v_t) < r(G) \), which is impossible. Hence, there are at least two vertices \( u \) and \( v \) in \( B \) such that \( a_B(u) = a_B(v) = a(B) \). By part (1) again, \( a_B(u) = a_B(v) = r(G) - 1 \). \( \blacksquare \)

Corollary. Let \( G \) be a non-complete, connected graph whose subgraph induced by \( C(G) \) is contained in a complete block \( B \). If \( C(G) \) contains at least two vertices, then \( C(G) = V(B) \).

Proof: Let \( V(B) = \{v_1, v_2, \ldots, v_m\} \) \((m \geq 2)\). For any \( v_i \in V(B) \),

\[
e(v_i) = \max_{1 \leq j \leq m, \; j \neq i} \{a_B(v_i), a_B(v_j) + 1\}
= \max_{1 \leq j \leq m, \; j \neq i} \{a_B(v_j) + 1\} = a_B + 1 = r(G).
\]

The second and the last equalities follow from Lemma 1.1.1 (2) and (1) respectively. Thus, \( V(B) \subseteq C(G) \). Since \( C(G) \subseteq V(B) \) by hypothesis, \( C(G) = V(B) \). \( \blacksquare \)

Lemma 1.1.2. Let \( G \) be a connected graph having a unique center vertex \( v^* \). Suppose that any block containing \( v^* \) is complete, then

(a) \( e(x) \geq 2r(G) \), for any \( x \in EC(G) \), and

(b) \( \text{dia}(G) = 2r(G) \).

Proof: (a) Let \( x \in EC(G) \). Then there exists \( w \in V(G) \setminus \{v^*\} \) which lies on a shortest path between \( x \) and \( v^* \) (\( w \) might be \( x \), but not \( v^* \)) and so that \( wv^* \in E(G) \).

Let \( B \) be a block of \( G \) containing the edge \( wv^* \). Then by assumption, \( B \) must be complete. Let \( V(B) = \{v^*, u_1, u_2, \ldots, u_s\} \). Without loss of generality, we may assume that \( u_1 = w \). We make two observations:

(1) \( a_B(u_i) \leq r(G) - 1 \), for \( 1 \leq i \leq s \);
To see this, suppose that there exists \( j \in \{1, 2, \ldots, s\} \) so that \( a_B(u_j) \geq r(G) \), then

\[
r(G) = e(v^*) \geq 1 + a_B(u_j) \geq 1 + r(G) > r(G),
\]
a contradiction. Thus, (1) follows.

(2) \( a_B(v^*) = r(G) \).

To see this, note that \( a_B(v^*) \leq e(v^*) = r(G) \). If \( a_B(v^*) < r(G) \), then

\[
e(u_1) = e(w) = \max_{2 \leq i \leq s} \{a_B(u_i), a_B(v^*) + 1, a_B(u_i) + 1\} \leq r(G).
\]

But \( r(G) \leq e(x) \) for all \( x \), so \( e(u_1) = r(G) \). This implies that \( u_1 \) is a center vertex of \( G \), i.e., \( w \in C(G) \). Since \( w \neq v^*, |C(G)| \geq 2 \), a contradiction to the assumption that \( |C(G)| = 1 \). So (2) follows.

Note that \( d(x, v^*) = r(G) \) as \( x \in EC(G) \). So by (2) above,

\[
e(x) \geq d(x, v^*) + a_B(v^*) = r(G) + r(G) = 2r(G),
\]
i.e., \( e(x) \geq 2r(G) \), as required.

(b) Since \( \text{dia}(G) = \max\{e(v) : v \in V(G)\} \geq 2r(G) \), by part (a),
and since it is well known that \( \text{dia}(G) \leq 2r(G) \), \( \text{dia}(G) = 2r(G) \).

**Proposition 1.1.1.** Let \( G \) be a connected graph. If \( \text{dia}(G) = 2r(G) \), then \( P(G) \subseteq EC(G) \).

**Proof:** Suppose that \( P(G) \not\subseteq EC(G) \), then there is \( x \in P(G) \setminus EC(G) \).
That means \( e(x) = \text{dia}(G) \) and \( d(x, v) \neq r(G) \) for any \( v \in C(G) \). Also, for any \( v \in C(G) \), since \( d(x, v) \leq e(v) = r(G) \), \( d(x, v) \leq r(G) - 1 \). Let \( y \) be a vertex so that \( d(x, y) = e(x) = \text{dia}(G) \). Then by the triangle inequality,

\[
\text{dia}(G) = d(x, y) \leq d(x, v) + d(v, y) \\
\leq r(G) - 1 + r(G) = 2r(G) - 1,
\]
a contradiction to the hypothesis. Therefore, $P(G) \subseteq EC(G)$. □

**Theorem 1.1.1.** Let $G$ be a connected graph having a unique vertex center $v^*$. Suppose that any block containing $v^*$ is complete, then $P(G) = EC(G)$.

**Proof:** Let $x \in EC(G)$. By Lemma 1.1.2, $e(x) \geq dia(G)$. But $dia(G) \geq e(x)$ is always true, so $e(x) = dia(G)$, and $x \in P(G)$. So, $EC(G) \subseteq P(G)$. The result follows from Proposition 1.1.1. □

**Corollary.** Let $G$ be a connected graph having a unique center vertex $v^*$. If $v^*$ is not on a cycle of $G$ then $P(G) = EC(G)$.

**Proof:** Without loss of generality, we assume that $G$ is not a single vertex. Since $v^*$ is not on any cycle of $G$, $v^*$ must be a cut vertex, and, moreover, each edge containing $v^*$ is not on any cycle of $G$. This implies that each edge containing $v^*$ must be a cut edge of $G$. So each block containing $v^*$ is $K_2$.

By Theorem 1.1.1, $P(G) = EC(G)$. □

**Lemma 1.1.3.** Let $G$ be a connected graph whose subgraph induced by $C(G)$ is contained in a complete block $B$. If $|C(G)| \geq 2$, then

(a) $e(x) \geq 2r(G) - 1$, for any $x \in EC(G)$, and

(b) $dia(G) = 2r(G) - 1$.

**Proof:** (a) Let $x \in EC(G)$. There exists $v \in C(G)$ so that $d(x, v) = e(v) = r(G)$. Pick $w$ from a shortest path between $x$ and $v$ so that $w \neq v$ and $w$ is adjacent to $v$. Since $|C(G)| \geq 2$, by the Corollary to Lemma 1.1.1, $C(G) = V(B)$. According to Lemma 1.1.1 (2), $w \in V(B)$ and there is $u \in V(B)$ with $u \neq w$ such that $a_B(u) = r(G) - 1$. Take $y \in V(G(u))$ so that
\[ d(u, y) = a_B(u) = r(G) - 1. \] Since \( w \) separates \( x \) from any vertex in \( G(u) \),
\[ d(x, y) = d(x, w) + d(w, u) + d(u, y) = r(G) - 1 + 1 + r(G) - 1 = 2r(G) - 1. \]

So, \( e(x) \geq d(x, y) \geq 2r(G) - 1 \), for any \( x \in EC(G) \).

(b) By the definition of \( \text{dia}(G) \) and part (a), \( \text{dia}(G) \geq 2r(G) - 1 \). On the other hand, any pair of vertices \( x \) and \( y \) must lie in some components of \( G - E(B) \), say \( x \in G(u), y \in G(v) \) where \( u, v \in V(B) \). For \( u = v \),
\[ d(x, y) \leq d(x, u) + d(u, y) \leq 2a_B(u) \leq 2a(B) = 2(r(G) - 1). \]

For \( u \neq v \),
\[ d(x, y) \leq d(x, u) + d(u, v) + d(v, y) \leq a_B(u) + 1 + a_B(v) \leq 2a(B) + 1 = 2r(G) - 1. \]

Therefore, \( \text{dia}(G) = \max\{d(x, y) : x, y \in V(G)\} \leq 2r(G) - 1 \). Thus,
\[ \text{dia}(G) = 2r(G) - 1. \]

**Lemma 1.1.4.** Let \( B \) be a block of a graph \( G \). For any \( y \in V(G) \setminus V(B) \), there are two distinct vertices \( u \) and \( v \) in \( B \) such that
\[ d(v, y) \geq d(u, y) + 1. \]

**Proof:** Let \( u \in V(B) \) such that \( d(u, y) = \min\{d(w, y) : w \in V(B)\} \).
Then \( u \) separates \( y \) from the remaining vertices of \( B \). Since \( B \) is a block of \( G \), \( B \) contains at least two vertices. Pick \( v \in V(B) \) with \( v \neq u \). Then
\[ d(v, y) = d(v, u) + d(u, y) \geq 1 + d(u, y). \]
That is, \( d(v, y) \geq d(u, y) + 1 \). \( \blacksquare \)

**Theorem 1.1.2.** Let \( G \) be a non-self-centered graph whose subgraph induced by \( C(G) \) is contained in a complete block \( B \). If \( |C(G)| \geq 2 \), then \( EC(G) = P(G) \).

**Proof:** Let \( x \in EC(G) \), then by Lemma 1.1.3, \( e(x) \geq \text{dia}(G) \). But \( \text{dia}(G) \geq e(x) \), so \( e(x) = \text{dia}(G) \). This implies that \( x \in P(G) \). Hence, \( EC(G) \subseteq P(G) \).

Suppose that \( P(G) \nsubseteq EC(G) \). Pick \( x \in P(G) \setminus EC(G) \), so, \( e(x) = \text{dia}(G) \) and \( d(x,v) \neq r(G) \) for any \( v \in C(G) \). Hence, for any \( v \in C(G) \),

\[
d(x,v) \leq r(G) - 1. \tag{1}
\]

Pick \( y \in V(G) \) so that \( d(x,y) = e(x) = \text{dia}(G) \). Clearly \( y \in P(G) \). Since \( G \) is not self-centered, \( y \notin C(G) \). As \( C(G) = V(B) \) by the Corollary to Lemma 1.1.1, \( y \notin V(B) \). According to Lemma 1.1.4, there exist two distinct vertices \( u \) and \( v \) in \( B \) such that \( d(v,y) \geq d(u,y) + 1 \). So

\[
d(u,y) \leq d(v,y) - 1 \leq e(v) - 1 = r(G) - 1. \tag{2}
\]

Then by the triangle inequality, (1) and (2),

\[
\text{dia}(G) = d(x,y) \leq d(x,u) + d(u,y) \\
\leq r(G) - 1 + r(G) - 1 = 2r(G) - 2.
\]

This contradicts Lemma 1.1.3 (b). Therefore, \( P(G) \subseteq EC(G) \). \( \blacksquare \).

From Theorems 1.1.1 and 1.1.2, we obtain the following:
Corollary. For any block graph $G$, $P(G) = EC(G)$. In particular, for any tree $T$, $P(T) = EC(T)$.

![Diagram](image)

(a number beside a vertex is the eccentricity of the vertex)

Fig. 1.1.2

Theorems 1.1.1 and 1.1.2 contain a slight difference in assumptions. Theorem 1.1.1 requires each block that contains a single center vertex is complete. Theorem 1.1.2, however, requires only one block that contains center to be complete. The word “each” cannot be changed to the word “a”, otherwise the result of Theorem 1.1.1 might be false. The graph $G$ in Fig. 1.1.2 shows that the word “each” cannot be changed to the word “a”. Obviously, for that $G$, $P(G) = V(G) \setminus C(G)$, but $EC(G)$ consists of only vertices on the top path of $G$.

§2. Criteria for a graph $G$ with $dia(G) = 2r(G)$ or $2r(G) - 1$ to satisfy $P(G) = EC(G)$

A shortest path $P$ of a graph $G$ between two vertices is a *diametrical path of $G$* provided the length of $P$ is equal to the diameter of $G$. Let $\mathcal{L}$ denote the collection of (connected) graphs in which each diametrical path contains a center vertex.

**Theorem 1.2.1.** Let $G$ be a connected graph.
(1) If \(|C(G)| \geq 2\) and the block containing \(C(G)\) is complete, then each diametrical path of \(G\) contains an edge of \(\langle C(G) \rangle\) and hence \(G \in \mathcal{L}\);

(2) If \(|C(G)| = 1\) and any block containing the center vertex is complete, then \(G \in \mathcal{L}\).

**proof:** (1) Suppose that there exists a diametrical path \(P\) which does not contain an edge of \(\langle C(G) \rangle\). Let \(B\) be the block that contains \(C(G)\). Then \(P\) must lie in a component \(G(v)\) of \(G - E(B)\) for some \(v \in V(B)\). Let \(x\) and \(y\) be end vertices of \(P\). By Lemma 1.1.1, \(\max\{d(x,v),d(y,v)\} \leq r(G) - 1\).

Hence, by the triangle inequality,

\[
\ell(P) = d(x,y) \leq d(x,v) + d(v,y)
\]

\[
\leq r(G) - 1 + r(G) - 1
\]

\[
= 2r(G) - 2 < \text{dia}(G).
\]

This fact contradicts the choice of \(P\). Thus, each diametrical path contains an edge of \(\langle C(G) \rangle\) and moreover \(G \in \mathcal{L}\).

(2) Let \(C(G) = \{v^*\}\). Suppose that there exists a diametrical path \(P\) from \(x\) to \(y\) so that \(P\) does not contain \(v^*\). Let \(B\) be a block containing \(v^*\). Then \(x\) and \(y\) lie in the same component or different components of \(G - E(B)\). In the first case, let \(z\) be the vertex on \(P\) closest to \(V(B)\), i.e.,

\[
d(z,V(B)) = \min\{d(v,V(B)) : v \in V(P)\}.
\]

Note that

\[
a_B(w) \leq \begin{cases} r(G) - 1 & \text{if } w \neq v^*, \\ r(G) & \text{if } w = v^*. \end{cases}
\]

So,

\[
d(x,z) \leq \begin{cases} a_B(w) & \text{if } w \neq v^*, \\ a_B(w) - 1 & \text{if } w = v^*, \end{cases}
\]

\[
\leq r(G) - 1.
\]
Similarly, \( d(y, z) \leq r(G) - 1 \). Thus,

\[
d(z, y) \leq d(x, z) + d(z, y) \\
\leq r(G) - 1 + r(G) - 1 \\
= 2r(G) - 2 < \text{dia}(G),
\]
a contradiction to the choice of \( P \).

In the second case, let \( x \in V(G(u)) \) and \( y \in V(G(v)) \) where \( u \neq v \). Note that neither \( u \) nor \( v \) is \( v^* \). So, neither \( a_B(u) \) nor \( a_B(v) \) is equal to \( r(G) \). It follows that

\[
dia(G) = d(x, y) \leq a_B(u) + 1 + a_B(v) \\
\leq r(G) - 1 + 1 + r(G) - 1 \\
= 2r(G) - 1 < \text{dia}(G),
\]
a contradiction again. This completes the proof. ■

**Theorem 1.2.2.** Suppose that \( G \in \mathcal{L} \) and that \( |C(G)| = 1 \). Then \( P(G) \subseteq EC(G) \) if and only if \( \text{dia}(G) = 2r(G) \).

**Proof:** If \( \text{dia}(G) = 2r(G) \), then by Proposition 1.1.1, \( P(G) \subseteq EC(G) \).

Conversely, suppose that \( \text{dia}(G) \neq 2r(G) \). Then \( \text{dia}(G) < 2r(G) \), i.e., \( \text{dia}(G) \leq 2r(G) - 1 \). Pick two vertices \( x \) and \( y \) so that

\[
d(x, y) = \text{dia}(G) \leq 2r(G) - 1.
\]

By the definition of \( P(G) \), \( x, y \in P(G) \). Let \( P \) be a shortest path between \( x \) and \( y \). \( P \) is a diametrical path. Let \( C(G) = \{v^*\} \). Then \( P \) contains \( v^* \) since \( G \in \mathcal{L} \). Notice that \( 2r(G) - 1 \geq d(x, y) = d(x, v^*) + d(v^*, y) \). So, one of \( d(x, v^*) \) and \( d(v^*, y) \) is strictly less than \( r(G) \). This implies that one of \( x \) and \( y \) is not in \( EC(G) \). This fact contradicts the given condition that \( P(G) \subseteq EC(G) \). Therefore, \( \text{dia}(G) = 2r(G) \). ■
It would be nice if \( P(G) = EC(G) \) when the conditions in Theorem 1.2.2 are satisfied. Unfortunately, there exists a graph \( G \) which satisfies all conditions in Theorem 1.2.2 but \( P(G) \neq EC(G) \) (see Fig. 1.2.1)

In fact, for this graph \( G \), we see that \( P(G) = \{u,v\} \) and \( EC(G) = \{u,v,x,y,z\} \). So, \( P(G) \neq EC(G) \). But the cause of the fact that \( P(G) \neq EC(G) \) seems to be that vertices \( x, y, \) and \( z \) are not in any diametrical path of \( G \). This observation leads to the following:

**Theorem 1.2.3.** Suppose that \( G \in \mathcal{L} \) and that \( |C(G)| = 1 \). Then \( P(G) = EC(G) \) if and only if both \( \text{dia}(G) = 2r(G) \) and for any \( z \in EC(G) \) there exists a diametrical path containing \( z \).

**Proof:** If \( P(G) = EC(G) \), then by Theorem 1.2.2, \( \text{dia}(G) = 2r(G) \). Now let \( x \in EC(G) \). Since \( P(G) = EC(G) \), \( x \in P(G) \). Hence, \( e(x) = \text{dia}(G) \). Pick \( y \in V(G) \) so that \( d(x,y) = e(x) \). Let \( P \) be a shortest path between \( x \) and \( y \), then \( P \) is a diametrical path containing \( x \).

Conversely, if \( \text{dia}(G) = 2r(G) \) and for any \( z \in EC(G) \) there exists a diametrical path containing \( z \), by Theorem 1.2.2, \( P(G) \subseteq EC(G) \). So
to complete the proof, it suffices to prove that $P(G) \supseteq EC(G)$. Pick $x \in EC(G)$. By hypotheses, there exists a diametrical path $P$, say joining vertices $w$ to $z$, so that $P$ contains $x$. Let $\{v^*\} = C(G)$. Since $G \in \mathcal{L}$, it is easy to verify that $v^*$ is at equal distance to $w$ and $z$. Since $x \in EC(G)$ implies that $d(x, v^*) = r(G)$, $x$ must be one of $w$ and $z$. This implies that $e(x) \geq \ell(P) = \text{dia}(G)$. But $e(x) \leq \text{dia}(G)$. So $e(x) = \text{dia}(G)$. It follows that $x \in P(G)$. Hence, $P(G) \supseteq EC(G)$.

This completes the proof. ■

Note: Theorem 1.1.1 and Theorem 1.2.3 are independent. That is because the graph $G$ shown in Fig. 1.2.2, where $P_m$ is a path of length $m$, can be treated by Theorem 1.1.1 and not by Theorem 1.2.3. However, the graph $H$, which is obtained from a cycle of length $2r + 1$ by attaching a path of length $r$ at a vertex of the cycle, can be treated by Theorem 1.2.3 and not by Theorem 1.1.1.

![Fig. 1.2.2](image)

**Lemma 1.2.1.** Let $G$ be a connected graph. If each diametrical path contains an edge of $C(G)$, then $\text{dia}(G) \leq 2r(G) - 1$.

**Proof:** Suppose that $\text{dia}(G) > 2r(G) - 1$. Since $\text{dia}(G) \leq 2r(G)$, we have $\text{dia}(G) = 2r(G)$. Pick vertices $x$ and $y$ so that $d(x, y) = \text{dia}(G)$. By the
triangle inequality, for any \( v \in C(G) \),

\[
2r(G) = d(x, y) \leq d(x, v) + d(v, y) \leq r(G) + r(G) = 2r(G).
\]

So, for any \( v \in C(G) \),

\[
d(x, v) = d(v, y) = r(G). \tag{3}
\]

Let \( P \) be a shortest path from \( x \) to \( y \). \( P \) is a diametrical path and hence contains an edge \( uv \) of \( (C(G)) \) by the hypotheses. Without loss of generality, we may assume that \( u \) is between \( x \) and \( v \) (otherwise interchange \( u \) and \( v \)).

Now \( d(x, v) = d(x, u) + 1 \). By (3), \( d(x, u) = r(G) \). Hence, \( d(x, v) = r(G) + 1 \), a contradiction to (3). Therefore, \( \text{dia}(G) \leq 2r(G) - 1 \). \( \blacksquare \)

**Theorem 1.2.4.** Let \( G \) be a connected graph satisfying

(1) \( (C(G)) \) is complete, and

(2) each diametrical path contains an edge of \( (C(G)) \).

Then \( P(G) \subseteq EC(G) \) if and only if \( \text{dia}(G) = 2r(G) - 1 \).

**Proof:** Let \( P(G) \subseteq EC(G) \). To show \( \text{dia}(G) = 2r(G) - 1 \), it suffices by Lemma 1.2.1 to show \( \text{dia}(G) \geq 2r(G) - 1 \). Suppose that \( \text{dia}(G) \leq 2r(G) - 2 \).

Pick vertices \( x \) and \( y \) so that \( d(x, y) = \text{dia}(G) \). By the definition of \( P(G) \), \( x, y \in P(G) \). Let \( P \) be a shortest path between \( x \) and \( y \). Then \( P \) is a diametrical path, and by the hypotheses, \( P \) contains an edge of \( (C(G)) \), say \( uv \). Without loss of generality, we may assume that \( d(x, v) > d(x, u) \). Notice that

\[
2r(G) - 2 \geq \text{dia}(G) = d(x, y) = d(x, v) + d(v, y) \geq 2\min\{d(x, v), d(v, y)\}.
\]

So, \( \min\{d(x, v), d(v, y)\} \leq r(G) - 1 \).
Case 1: If \( \min\{d(x,v),d(v,y)\} = d(x,v) \), then \( d(x,v) \leq r(G) - 1 \) implies that \( d(x,u) \leq r(G) - 2 \). Since \( (C(G)) \) is complete and \( u \in C(G) \), for any \( w \in C(G) \),
\[
d(x,w) \leq d(x,u) + d(u,w) \leq r(G) - 2 + 1 = r(G) - 1.
\]
Thus, \( x \notin EC(G) \). But recall that \( x \in P(G) \). So \( P(G) \nsubseteq EC(G) \), a contradiction to the given condition that \( P(G) \subseteq EC(G) \).

Case 2: Suppose that \( \min\{d(x,v),d(v,y)\} = d(v,y) \). We may assume that \( d(v,y) \) is not equal to \( d(x,v) \), because otherwise \( d(x,v) = d(v,y) \leq r(G) - 1 \) and Case 1 applies. Note that if \( d(v,y) = r(G) - 1 \), \( d(x,v) > r(G) - 1 \). It follows that
\[
d(x,y) = d(x,v) + d(v,y) > r(G) - 1 + r(G) - 1 = 2r(G) - 2,
\]
a contradiction. We may thus assume that \( d(v,y) \neq r(G) - 1 \). But since \( d(v,y) \leq r(G) - 1 \), \( d(v,y) \leq r(G) - 2 \). Since \( (C(G)) \) is complete and \( v \in C(G) \), for any \( w \in C(G) \),
\[
d(y,w) \leq d(y,v) + d(v,w) \leq r(G) - 2 + 1 = r(G) - 1.
\]
Thus, \( y \notin EC(G) \). But recall that \( y \in P(G) \). So \( P(G) \nsubseteq EC(G) \), a contradiction to the given condition that \( P(G) \subseteq EC(G) \). In any case, \( dia(G) \geq 2r(G) - 1 \).

Conversely, let \( dia(G) = 2r(G) - 1 \). Suppose that \( P(G) \nsubseteq EC(G) \). Then there exists \( x \in P(G) \) so that \( d(x,v) \leq r(G) - 1 \) for any \( v \in C(G) \). Take a vertex \( y \) so that \( d(x,y) = dia(G) = 2r(G) - 1 \). By the triangle inequality, for any \( v \in C(G) \),
\[
2r(G) - 1 = d(x,y) \leq d(x,v) + d(v,y)
\]
\[
\leq r(G) - 1 + r(G) = 2r(G) - 1.
\]
So, for any \( v \in C(G) \),

\[
\begin{align*}
    d(x, v) &= r(G) - 1 \quad \text{and} \quad d(v, y) = r(G). \quad (4)
\end{align*}
\]

Let \( P \) be a shortest path between \( x \) and \( y \). \( P \) is a diametrical path, so by the hypotheses \( P \) contains an edge of \( \langle C(G) \rangle \), say \( uw \). Without loss of generality, we may assume that \( u \) is between \( x \) and \( w \) (otherwise interchange \( u \) and \( w \)). Then \( d(x, w) = d(x, u) + 1 \). By (4), \( d(x, u) = r(G) - 1 \). So, \( d(x, w) = r(G) - 1 + 1 = r(G) \), a contradiction to (4). Hence, \( P(G) \subseteq EC(G) \).

This completes the proof. \( \blacksquare \)

**Theorem 1.2.5.** Let \( G \) be a connected graph satisfying

1. \( \langle C(G) \rangle \) is complete, and
2. each diametrical path contains an edge of \( \langle C(G) \rangle \).

Then \( P(G) = EC(G) \) if and only if both \( \text{dia}(G) = 2r(G) - 1 \) and for any \( x \in EC(G) \) there exists a diametrical path containing \( x \).

**Proof:** Suppose that \( P(G) = EC(G) \), then by Theorem 1.2.4, \( \text{dia}(G) = 2r(G) - 1 \). It is obvious that for any \( x \in EC(G) = P(G) \), there exists a diametrical path containing \( x \). To complete the proof of this theorem, by Theorem 1.2.4 again, it suffices to show \( P(G) \subseteq EC(G) \) if \( \text{dia}(G) = 2r(G) - 1 \) and for any \( x \in EC(G) \) there exists a diametrical path containing \( x \).

Suppose that \( EC(G) \not\subseteq P(G) \). Then there is \( x \in EC(G) \setminus P(G) \). By the hypotheses, there exists a diametrical path \( P \) so that \( x \in V(P) \). From the condition (2), \( P \) contains an edge \( uv \) of \( \langle C(G) \rangle \). Without loss of generality, we may assume that \( d(x, v) > d(x, u) \). Since \( x \not\in P(G) \), i.e., \( e(x) < \text{dia}(G) \), \( x \) is not an endvertex of \( P \). So \( d(x, u) < d(x, v) \leq r(G) - 1 \). It follows that \( d(x, u) \leq r(G) - 2 \). By the triangle inequality, for any \( w \) in \( C(G) \),

\[
    d(x, w) \leq d(x, u) + d(u, w) \leq r(G) - 2 + d(u, w). \quad (5)
\]
Since \((C(G))\) is complete and \(u \in C(G)\), \(d(u,w) = 1\). Replacing \(d(u,w)\) by
1 in (5), we have, for any \(w\) in \(C(G)\),
\[
d(x,w) \leq r(G) - 2 + 1 = r(G) - 1.
\]

Hence, \(x \notin EC(G)\) which contradicts the given condition that \(P(G) \subseteq EC(G)\).

The proof is complete. \(\blacksquare\)

Note: By the Corollary to Lemma 1.1.1, Lemma 1.1.4, and Theorem 1.2.1, we can see that the “if ” part of Theorem 1.2.4 is a generalization of Theorem 1.1.2.

§3. Some other graphs \(G\) with \(P(G) = EC(G)\)

A graph \(G\) is called a self-centered graph if \(r(G) = \text{dia}(G)\). Obviously, each vertex of a self-centered graph is not only a peripheral vertex but also a center vertex. We therefore have the following:

**Theorem 1.3.1.** For any self-centered graph \(G\), \(P(G) = EC(G)\).

An eccentricity-preserving spanning tree of a graph \(G\) is a spanning tree \(T\) for which \(e_T(v) = e_G(v)\) for each vertex \(v\) of \(G\). Namdkumar [15] characterized graphs \(G\) with an eccentricity-preserving spanning tree as follows:

**Theorem 1.3.2.** A connected graph \(G\) has an eccentricity-preserving spanning tree if and only if

1. either \((C(G)) = K_1\) and \(\text{dia}(G) = 2r(G)\), or \((C(G)) = K_2\) and \(\text{dia}(G) = 2r(G) - 1\); and
each $u \in V(G)$ with $e(u) > r(G)$ has a neighbor $v$ for which $e(v) = e(u) - 1$.

**Theorem 1.3.3.** Let $G$ be a connected graph which has an eccentricity-preserving spanning tree, then $P(G) = EC(G)$.

**Proof:** Let $T$ be an eccentricity-preserving spanning tree of $G$. Then for any vertex $v \in V(G) = V(T)$, $e_T(v) = e_G(v)$. Moreover,

1. $r(T) = r(G)$ and $\text{dia}(T) = \text{dia}(G)$;
2. $C(T) = C(G)$ and $P(T) = P(G)$; and, as remarked in the introduction,
3. $P(T) = EC(T)$.

So, to prove $EC(G) \subseteq P(G)$, it is sufficient to prove $EC(G) \subseteq EC(T)$.

Let $x \in EC(G)$, then there exists a center vertex $c \in C(G)$ so that $d_G(x,c) = r(G)$. By (2), $c \in C(T)$ and

$$r(G) = d_G(x,c) \leq d_T(x,c) \leq r(T).$$  \hspace{1cm} (6)

But $r(T) = r(G)$ and (6) imply that $d_T(x,c) = r(T)$ and $x \in EC(T)$. Thus, $EC(G) \subseteq EC(T)$.

To complete the proof, it is sufficient to show that $P(G) \subseteq EC(G)$. Let $x \in P(G)$. Pick a vertex $y$ so that $d_G(x,y) = \text{dia}(G)$. By (1),

$$\text{dia}(T) = \text{dia}(G) = d_G(x,y) \leq d_T(x,y) \leq \text{dia}(T).$$

Thus, $x \in P(T)$. Since $P(T) = EC(T)$, there is $v \in C(T)$ so that $d_T(x,v) = r(T)$. Since $r(T) = r(G)$, $r(G) \leq d_G(x,v) \leq d_T(x,v) = r(G)$. That is, $d_G(x,v) = r(G)$. Recall that $C(G) = C(T)$. Thus, $x \in EC(G)$. Therefore, $P(G) \subseteq EC(G)$.  

This completes the proof. 

![Graphs](image)

(The number beside a vertex is the eccentricity of the vertex)

Fig. 1.3.1

In 1988 F. Buckley [5] introduced a similar concept called a diameter-preserving spanning tree of a graph. A **diameter-preserving spanning tree** of a graph \( G \) is a spanning tree for which \( \text{dia}(T) = \text{dia}(G) \). A curious problem arises: Is Theorem 1.3.2 still true if the word "eccentricity" is changed to "diameter"? The answer is negative. The graph \( G \) shown in Fig. 1.3.1 (a) has a diameter-preserving tree \( T \) shown in Fig. 1.3.1 (b). It is easy to see that \( \mathcal{P}(G) = \{u,v\} \) and \( \mathcal{E}(G) = \{u,v,x,y,z\} \). So, \( \mathcal{P}(G) \neq \mathcal{E}(G) \). This example shows that a graph \( G \) with a diameter-preserving spanning tree might fail to satisfy \( \mathcal{P}(G) = \mathcal{E}(G) \).

Let \( \overline{G} \) denote the complement of a graph \( G \). As observed in the Introduction, \( \mathcal{P}(T) = \mathcal{E}(T) \) for each tree \( T \). An interesting question is, for which trees \( T \), is \( \mathcal{P}(\overline{T}) = \mathcal{E}(\overline{T}) \). In order to answer this question the following two lemmas are useful:

**Lemma 1.3.1.** Let \( T \) be a tree. If \( \text{dia}(T) \geq 4 \), then \( \overline{T} \) is a self-centered graph.
Proof: Assume that $\text{dia}(T) \geq 4$. Pick two nonadjacent vertices $x$ and $y$ in $\check{T}$. Note that $xy \in E(T)$. Since $\text{dia}(T) \geq 4$, there is a vertex $z$ in $T$ which is adjacent to neither $x$ nor $y$. That is, $xy \not\in E(\check{T})$, but $xz, zy \in E(\check{T})$. This means that $d_{\check{T}}(x, y) = 2$ and consequently, $\text{dia}(\check{T}) = 2$.

Now let $v \in V(T)$. Pick a vertex $u$ adjacent to $v$ in $T$. Then $d_{\check{T}}(v, u) \geq 2$. So, $e_{\check{T}}(v) \geq 2 = \text{dia}(\check{T})$. It follows that $r(\check{T}) = \text{dia}(\check{T})$. Therefore, $\check{T}$ is self-centered. \]

Lemma 1.3.2. Let $T$ be a tree. Then $P(\check{T}) = V(T) \setminus P(T)$ if and only if $\text{dia}(T) = 3$.

Proof: Suppose that $P(\check{T}) = V(T) \setminus P(T)$ but $\text{dia}(T) \neq 3$. Then either $\text{dia}(T) \leq 2$ or $\text{dia}(T) \geq 4$. In the first case, $T$ is either $K_2$ or a star. It follows that $\check{T}$ is a disconnected graph which implies that $P(\check{T}) \neq V(T) \setminus P(T)$, a contradiction. So, we may assume that $\text{dia}(T) \geq 4$. By Lemma 1.3.1, $\check{T}$ is self-centered. So, $P(\check{T}) = V(\check{T}) = V(T)$. But $P(\check{T}) = V(T) \setminus P(T)$ implies that $P(T) = \emptyset$ which is impossible. Thus, $\text{dia}(T) = 3$.

Conversely, suppose that $\text{dia}(T) = 3$. Let $xyzw$ be a diametrical path of $T$. Since $T$ is a tree, $N(y) \cap N(z) = \emptyset$ where $N(v) = \{u \in V(T) : uv \in E(T)\}$. So, $\text{dia}(T) = 3$ implies that $T = (\{y, z\} \cup N(y) \cup N(z))$ (see Fig. 1.3.2). Therefore, $P(T) = V(T) \setminus \{y, z\}$. On the other hand, it is easy to verify that

$$d_{\check{T}}(u, v) = \begin{cases} 3 & \text{if } u = y \text{ and } v = z, \\ \leq 2 & \text{otherwise}. \end{cases}$$

So, $P(\check{T}) = \{y, z\} = V(T) \setminus P(T)$ as desired. \]
Theorem 1.3.4. Let $T$ be a tree. Then $P(T) = EC(T)$ if and only if $\text{dia}(T) \geq 3$.

Proof: Note that $\text{dia}(T) \leq 2$ implies that $T$ is disconnected. So, $P(T) = EC(T)$ implies that $\text{dia}(T) \geq 3$.

Conversely, suppose that $\text{dia}(T) \geq 3$. If $\text{dia}(T) \geq 4$, then by Lemma 1.3.1, $T$ is self-centered and hence $P(T) = EC(T)$. If $\text{dia}(T) = 3$, let $xyzw$ be a diametrical path of $T$. Then by the proof of Lemma 1.3.2, $P(T) = \{y, z\}$. But $EC(T) = \{y, z\}$ too. So, $P(T) = EC(T)$. ■

Corollary. Let $T$ be a tree. Then $EC(T) = V(T) \setminus EC(T)$ if and only if $\text{dia}(T) = 3$.

§4. Products of graphs

The following definitions of products of graphs are in [18]. Let $G$ and $H$ be connected graphs. Let $V = V(G) \times V(H)$, $u = (g, h) \in V$, and $v = (g', h') \in V$.

The cartesian product of $G$ and $H$ is $G \times H = (V, E)$ where $uv \in E$ if and only if either both $g = g'$ and $hh' \in E(H)$ or both $h = h'$ and $gg' \in E(G)$.

The symmetric difference of $G$ and $H$ is $G \oplus H = (V, E)$ where $uv \in E$ if and only if either $gg' \in E(G)$ or $hh' \in E(H)$, but not both.
The disjunction of $G$ and $H$ is $G \vee H = (V, E)$ where $uv \in E$ if and only if either $gg' \in E(G)$ or $hh' \in E(H)$, or both.

The composition (Lexicographic product) of $G$ and $H$ is $G[H] = (V, E)$ where $uv \in E$ if and only if $gg' \in E(G)$ or both $g = g'$ and $hh' \in E(H)$.

We will assume that both $G$ and $H$ are connected and contain at least two vertices unless stated otherwise. It is thus easy to verify that any product defined above is connected.

Let $G$ and $H$ be the connected graphs shown in Fig. 1.4.1. Each product of $G$ and $H$ is illustrated in Fig. 1.4.1. Note that for the graph $H$, $P(H) \neq EC(H)$. But $P(G \oplus H) = EC(G \oplus H)$.

So, it is interesting to search for conditions under which a product of two graphs has the property that the set of peripheral vertices is the same as the set of eccentric vertices.

In [4] F. Buckley gave a neccessary and sufficient condition for the Cartesian product of $G$ and $H$ to satisfy $P(G \times H) = EC(G \times H)$. He proved the following:

**Theorem 1.4.1.** $P(G \times H) = EC(G \times H)$ if and only if $P(G) = EC(G)$ and $P(H) = EC(H)$. 
In this section we characterize the other products, among those defined above, for which the set of peripheral vertices is the same as the set of eccentric vertices.

By the definition of the symmetric difference of two graphs and the definition of the disjunction of two graphs, it is easy to see the following:

Remark. For any graphs $G$ and $H$, $G \oplus H$ is a subgraph of $G \vee H$. Therefore, $r(G \vee H) \leq r(G \oplus H)$ and $\text{dia}(G \vee H) \leq \text{dia}(G \oplus H)$.

Theorem 1.4.2. $P(G \oplus H) = EC(G \oplus H)$ for any connected graphs $G$ and $H$.

Proof: Let $u = (g, h)$ and $v = (g', h')$ be vertices of $G \oplus H$. It is easy to verify that

$$d_{G \oplus H}(u, v) = \begin{cases} 1, & \text{if } d_G(g, g') = 1 \text{ or } d_H(h, h') = 1, \text{ but not both}, \\ 2, & \text{otherwise}. \end{cases}$$

So, for each vertex $v$ in $G \oplus H$, $e_{G \oplus H}(v) = 2$. It follows that $r(G \oplus H) = \text{dia}(G \oplus H) = 2$, and that $G \oplus H$ is self-centered. Hence, $P(G \oplus H) = EC(G \oplus H)$. 

Lemma 1.4.1. For any graphs $G$ and $H$, $r(G) = r(H) = 1$ if and only if $r(G \vee H) = 1$.

We obtain immediately:

Corollary. Let $G$ and $H$ be connected graphs with $r(G) = r(H) = 1$. Then $C(G \vee H) = C(G) \times C(H)$.

Lemma 1.4.2. $G \vee H$ is a self-centered graph if and only if $r(G) \neq 1$, or $r(H) \neq 1$, or $\text{dia}(G) = \text{dia}(H) = 1$. 
Proof: Suppose that \( r(G) \neq 1 \), or \( r(H) \neq 1 \), or \( \text{dia}(G) = \text{dia}(H) = 1 \).

By Lemma 1.4.1, \( r(G \lor H) \neq 1 \). Thus, \( r(G \lor H) \geq 2 \). But by the Remark and the proof of Theorem 1.4.2, \( r(G \lor H) \leq r(G \oplus H) = 2 \) and \( \text{dia}(G \lor H) \leq \text{dia}(G \oplus H) = 2 \). Thus, \( r(G \lor H) = 2 \) and \( \text{dia}(G \lor H) = 2 \), so \( G \lor H \) is self-centered.

Conversely, suppose that \( G \lor H \) is self-centered. By the Remark and the proof of Theorem 1.4.2 again, either \( r(G \lor H) = \text{dia}(G \lor H) = 1 \) or \( r(G \lor H) = \text{dia}(G \lor H) = 2 \). If \( r(G \lor H) = 2 \), then by Lemma 1.4.1, \( r(G) \neq 1 \) or \( r(H) \neq 1 \). So, we may assume that \( r(G \lor H) = \text{dia}(G \lor H) = 1 \). Then by the definition of \( G \lor H \), both \( G \) and \( H \) are complete. Thus, \( \text{dia}(G) = \text{dia}(H) = 1 \).

This completes the proof. \( \blacksquare \)

Lemma 1.4.3. Let \( G \) be a connected graph with \( r(G) = 1 \). Then \( P(G) = EC(G) \) if and only if either \( G \) is a complete graph or \( |C(G)| = 1 \).

Proof: Suppose that \( P(G) = EC(G) \), but \( G \) is not a complete graph with \( |C(G)| \geq 2 \). Pick two distinct center vertices \( u \) and \( v \). As \( e(u) = r(G) = 1 \), \( d(u, v) = 1 \) and \( u \in EC(G) \). Since \( G \) is not complete, \( u \notin P(G) \). So, \( u \in EC(G) \setminus P(G) \), a contradiction.

Conversely, if \( G \) is complete, then \( P(G) = EC(G) \). So we may assume that \( G \) is a non-complete with exactly one center vertex. Then \( \text{dia}(G) = 2 \), since \( r(G) = 1 \). Clearly, \( P(G) = V(G) \setminus C(G) \) and \( EC(G) = V(G) \setminus C(G) \).

So, \( P(G) = EC(G) \). \( \blacksquare \)

Theorem 1.4.3. Let \( G \) and \( H \) be connected graphs. Then \( P(G \lor H) = EC(G \lor H) \) if and only if \( G \) and \( H \) satisfy one of the following conditions:
(1) \( r(G) \neq 1 \) or \( r(H) \neq 1 \);
(2) \( r(G) = r(H) = 1 \) and \( \text{dia}(G) = \text{dia}(H) = 1 \), i.e., both \( G \) and \( H \) are complete;
(3) \( r(G) = r(H) = 1 \), \( \text{dia}(G) = \text{dia}(H) = 2 \), and \( |C(G)| = |C(H)| = 1 \).

**Proof:** Suppose that \( P(G \lor H) = EC(G \lor H) \). If \( r(G) \neq 1 \) or \( r(H) \neq 1 \), then \( G \) and \( H \) satisfy condition (1). So we may assume that \( r(G) = r(H) = 1 \).

By Lemma 1.4.1, \( r(G \lor H) = 1 \). Since \( P(G \lor H) = EC(G \lor H) \), Lemma 1.4.3 implies that either \( G \lor H \) is complete or \( |C(G \lor H)| = 1 \). In the first case, both \( G \) and \( H \) are complete and hence satisfy condition (2). In the second case, by the Corollary to Lemma 1.4.1, \( |C(G)| = |C(H)| = 1 \). Since \( r(G) = r(H) = 1 \), both \( G \) and \( H \) contain at least two vertices. So \( \text{dia}(G) \geq 2 \) and \( \text{dia}(H) \geq 2 \). But \( \text{dia}(G) \leq 2r(G) = 2 \) and \( \text{dia}(H) \leq 2r(H) = 2 \). Thus, \( \text{dia}(G) = \text{dia}(H) = 2 \). It follows that \( G \) and \( H \) satisfy condition (3).

Conversely, if \( G \) and \( H \) satisfy either condition (1) or condition (2), Lemma 1.4.2 implies that \( G \lor H \) is self-centered, and hence \( P(G \lor H) = EC(G \lor H) \). So, we may assume that \( G \) and \( H \) satisfy condition (3). Since \( r(G) = r(H) = 1 \) and \( |C(G)| = |C(H)| = 1 \), Lemma 1.4.1 and its Corollary imply that \( r(G \lor H) = 1 \) and \( |C(G \lor H)| = 1 \). It follows from Lemma 1.4.3 that \( P(G \lor H) = EC(G \lor H) \).

This completes the proof.

Consider the composition \( G[H] \) of \( G \) and \( H \). Let \( u = (g,h) \) and \( v = (g'h') \) be two vertices of \( G[H] \). The construction of \( G[H] \) yields the following simple facts:

**Fact 1.**

\[
d_{G[H]}(u,v) = \begin{cases} 
1, & \text{if } g = g' \text{ and } d_H(h,h') = 1, \\
2, & \text{if } g = g' \text{ and } d_H(h,h') \geq 2, \\
d_G(g,g'), & \text{otherwise.}
\end{cases}
\]
Fact 2. If \( r(G) \geq 2 \), then \( e_{G[H]}(u) = e_G(g) \).

Fact 3. 

\[
r(G[H]) = \begin{cases} 
1, & \text{if } r(G) = r(H) = 1, \\
2, & \text{if } r(G) = 1 \text{ and } r(H) \geq 2, \\
r(G), & \text{otherwise}.
\end{cases}
\]

Fact 4. 

\[
dia(G[H]) = \begin{cases} 
\max\{\text{dia}(G), \text{dia}(H)\}, & \text{if } r(G) = r(H) = 1, \\
2, & \text{if } r(G) = 1 \text{ and } r(H) \geq 2, \\
\text{dia}(G), & \text{otherwise}.
\end{cases}
\]

Lemma 1.4.4.

\[
C(G[H]) = \begin{cases} 
C(G) \times C(H), & \text{if } r(G) = r(H) = 1, \\
V(G) \times V(H), & \text{if } r(G) = 1 \text{ and } r(H) \geq 2, \\
C(G) \times V(H), & \text{otherwise}.
\end{cases}
\]

Proof: Case 1: Suppose that \( r(G) = r(H) = 1 \). Let \((g^*, h^*) \in C(G) \times C(H)\), so that \( d_G(g^*, g) = 1 \) for any \( g \in V(G) \) and \( d_H(h^*, h) = 1 \) for any \( h \in V(H) \). Thus, by Fact 1 and Fact 3,

\[
e_{G[H]}((g^*, h^*)) = 1 = r(G[H]).
\]

That is, \((g^*, h^*) \in C(G[H])\). Hence \( C(G) \times C(H) \subseteq C(G[H])\).

Conversely, suppose that \( C(G[H]) \not\subseteq C(G) \times C(H) \), then there exists \((g^*, h^*) \in C(G[H])\) so that either \( g^* \not\in C(G) \) or \( h^* \not\in C(H) \). First assume that \( g^* \not\in C(G) \). Then \( e_G(g^*) \geq 2 \). So there is \( g \in V(G) \) so that \( d_G(g^*, g) \geq 2 \).

By Fact 3 and Fact 1,

\[
1 = r(G[H]) = e_{G[H]}((g^*, h^*)) \\
\geq d_{G[H]}((g^*, h^*), (g, h^*)) = d_G(g^*, g) \geq 2,
\]

a contradiction. If \( h^* \not\in C(H) \), a contradiction is similarly obtained. Therefore, \( C(G[H]) \subseteq C(G) \times C(H) \).
Case 2: Suppose that $r(G) = 1$ and $r(H) \geq 2$. Clearly, $C(G[H]) \subseteq V(G) \times V(H)$. Let $(g', h') \in V(G) \times V(H)$. For any $g, h \in V(G)$, $d_G(g', g) \leq \text{dia}(G) \leq 2r(G) = 2$. By Facts 1 and 3,

$$e_{G[H]}((g', h')) = \max\{d_{G[H]}((g', h'), (g, h)) : (g, h) \in V(G[H])\} = 2 = r(G[H]).$$

So $(g', h') \in C(G[H])$, and therefore $V(G) \times V(H) \subseteq C(G[H])$.

Case 3: Suppose that $r(G) \geq 2$. Then the result follows immediately from Facts 2 and 3.

The proof is complete. ■

**Lemma 1.4.5.**

1. If $r(G) \geq 2$, then $P(G[H]) = P(G) \times V(H)$.

2. If $r(G) > 2$ or both $r(G) = 2$ and $\text{dia}(H) = 1$, then $EC(G[H]) = EC(G) \times V(H)$.

**Proof:**

1. Let $r(G) \geq 2$, let $(g, h) \in P(G[H])$. By Facts 2 and 4,

$$\text{dia}(G) = \text{dia}(G[H]) = e_{G[H]}((g, h)) = e_G(g).$$

This implies that $g \in P(G)$ and hence $(g, h) \in P(G) \times V(H)$. Thus, $P(G[H]) \subseteq P(G) \times V(H)$.

On the other hand, let $(g, h) \in P(G) \times V(H)$. So, $e_G(g) = \text{dia}(G)$. By Facts 2 and 4 again,

$$\text{dia}(G[H]) = \text{dia}(G) = e_G(g) = e_{G[H]}((g, h)).$$

That is, $(g, h) \in P(G[H])$. So, $P(G) \times V(H) \subseteq P(G[H])$.

Consequently, $P(G[H]) = P(G) \times V(H)$. 


(2) Let \((g, h) \in EC(G[H])\). Then there exists \((g', h') \in C(G[H])\) so that 
\[
d_{G[H]}((g, h), (g', h')) = r(G[H]).
\]
By Fact 3,
\[
d_{G[H]}((g, h), (g', h')) = r(G[H]) = r(G) \geq 2. \tag{a}
\]
Since \(r(G) > 2\) or both \(r(G) = 2\) and \(\text{dia}(H) = 1\), \(g \neq g'\). By Fact 1,
\[
d_{G[H]}((g, h), (g', h')) = d_G(g, g'). \tag{b}
\]
Note that \(g' \in C(G)\) by Lemma 1.4.4. So, by (a) and (b), \(g \in EC(G)\). Hence 
\[
EC(G[H]) \subseteq EC(G) \times V(H).
\]
The similar proof that \(EC(G) \times V(H) \subseteq EC(G[H])\) is omitted. 

**Theorem 1.4.4.** \(P(G[H]) = EC(G[H])\) if and only if \(G\) and \(H\) satisfy one of the following conditions:

1. \(\text{dia}(G) = \text{dia}(H) = 1\);
2. \(r(G) = r(H) = 1\) and \(|C(G)| = |C(H)| = 1\);
3. \(r(G) = 1\) and \(r(H) \geq 2\);
4. \(r(G) = 2\), \(\text{dia}(H) = 1\), and \(P(G) = EC(G)\);
5. \(r(G) = \text{dia}(G) = 2\);
6. \(r(G) > 2\) and \(P(G) = EC(G)\).

**Proof:** Suppose that \(G\) and \(H\) satisfy one of the stated conditions. If \(G\) and \(H\) satisfy condition (1), then both \(G\) and \(H\) are complete, \(G[H]\) is complete, and so \(P(G[H]) = EC(G[H])\). If \(G\) and \(H\) satisfy condition (2), then by Fact 3 and Lemma 1.4.4, \(r(G[H]) = 1\) and \(|C(G[H])| = 1\). So, by Lemma 1.4.3, \(P(G[H]) = EC(G[H])\). If \(G\) and \(H\) satisfy condition (3), then by Lemma 1.4.4, \(C(G[H]) = V(G) \times V(H) = V(G[H])\). So, \(G[H]\) is self-centered and hence \(P(G[H]) = EC(G[H])\). If \(G\) and \(H\) satisfy condition (5),
then by Facts 3 and 4, $r(G[H]) = r(G) = \text{dia}(G) = \text{dia}(G[H])$. So, $G[H]$ is self-centered and hence $P(G[H]) = EC(G[H])$. If $G$ and $H$ satisfy condition (4) or condition (6), then by Lemma 1.4.5, $P(G[H]) = EC(G[H])$.

Conversely, suppose that $P(G[H]) = EC(G[H])$. Consider $r(G[H])$.

If $r(G[H]) = 1$, then $r(G) = r(H) = 1$ by Fact 3. Moreover, by Lemma 1.4.3, either $G[H]$ is complete or $|C(G[H])| = 1$. In the first case, both $G$ and $H$ are complete and hence $\text{dia}(G) = \text{dia}(H) = 1$. In the second case, $|C(G)| = |C(H)| = 1$ by Lemma 1.4.4, and therefore $G$ and $H$ satisfy condition (2).

If $r(G[H]) = 2$, then by Fact 3, either $r(G) = 1$ and $r(H) \geq 2$ or $r(G) = 2$. In the first case, $G$ and $H$ satisfy condition (3). In the second case, if, in additional, $\text{dia}(H) = 1$, then by Lemma 1.4.5, $P(G[H]) = P(G) \times V(H)$ and $EC(G[H]) = EC(G) \times V(H)$. So $P(G[H]) = EC(G[H])$ implies that $P(G) = EC(G)$, which is condition (4). Now assume that $r(G) = 2$ and $\text{dia}(H) > 1$. Then there are two distinct vertices $h, h' \in V(H)$ so that $d_H(h, h') \geq 2$. Pick $g \in C(G)$. Then by Lemma 1.4.4, $(g, h), (g, h') \in C(G[H])$ and moreover, by Fact 1, the present assumption, and Fact 3,

$$d_{G[H]}((g, h), (g, h')) = 2 = r(G) = r(G[H]).$$

So, $(g, h) \in EC(G[H])$. By the original hypothesis, $EC(G[H]) = P(G[H])$, and since $(g, h) \in C(G[H])$, $e_{G[H]}((g, h)) = r(G[H])$. So, $\text{dia}(G[H]) = e_{G[H]}((g, h)) = r(G[H]) = 2$. By Fact 4, $\text{dia}(G) = \text{dia}(G[H]) = 2$. Thus, $G$ and $H$ satisfy condition (5).

If $r(G[H]) > 2$, then by Lemma 1.4.5, $P(G[H]) = P(G) \times V(H)$ and $EC(G[H]) = EC(G) \times V(H)$. So, $P(G[H]) = EC(G[H])$ implies that $P(G) = EC(G)$. Therefore, $G$ and $H$ satisfy condition (6).
The proof is complete. ■

The conjunction (Kronecker product) of $G$ and $H$ is $G \wedge H = (V,E)$ where $V = V(G) \times V(H)$ and $(g,h)(g',h') \in E$ if and only if $gg' \in E(G)$ and $hh' \in E(H)$.

Note that the Kronecker product fails to preserve connectivity. For example, the Kronecker of an even cycle and any tree contains 2 connected components (see [27]). In fact, Weichsel's Theorem in [27] states that the Kronecker product of graphs $G$ and $H$ is connected if and only if one of $G$ and $H$ contains an odd cycle. It remains an open question to describe a criterion for the set of peripheral vertices to be the same as the set of eccentric vertices in the Kronecker product of two graphs, one of which contains an odd cycle.
Chapter 2. Constructions

§1. Relations between $P(G)$ and $EC(G)$

In this section we will present several graphs which then are used to show that all possible set-inclusion relations between the set of peripheral vertices and the set of eccentric vertices may occur. More precisely, for each $i, 1 \leq i \leq 5,$ and for each pair of positive integers $a$ and $b,$ $a \leq b \leq 2a,$ we will construct a graph $G_i,$ with $r(G_i) = a$ and $\text{dia}(G_i) = b,$ which satisfies condition (i) below:

1. $P(G_1) = EC(G_1);$
2. $P(G_2) \nsubseteq EC(G_2),$ if $a < b \leq 2a;$
3. $P(G_3) \nsubseteq EC(G_3),$ if $a < b \leq 2a;$
4. $P(G_4) \cap EC(G_4) \neq \emptyset,$ $P(G_4) \nsubseteq EC(G_4),$ $P(G_4) \nsubseteq EC(G_4),$ if $a < b \leq 2a;$ and
5. $P(G_5) \cap EC(G_5) = \emptyset,$ if $a + 2 \leq b \leq 2a - 2.$

F. Buckley [4] proved the following:

Theorem 2.1.1. For any positive integers $a$ and $b$ with $a \leq b \leq 2a,$ there exists a graph $G$ for which $r(G) = a, \text{dia}(G) = b,$ and $P(G) = EC(G).$

Let $C$ be a cycle of length $2n$ and let $v \in V(C).$ A vertex $u \in V(C)$ so that $d_C(u, v) = n$ is called the antipole of $v.$

For positive integers $a, \ell,$ and $s,$ let $G(a, \ell, s)$ denote the graph obtained from a cycle of length $2a$ by attaching one path of length $\ell$ at a vertex of the cycle and attaching another path of length $s$ at the antipole of that vertex. Let $x$ and $y$ be the vertices of degree 1 in $G(a, \ell, s).$
The construction of $G(a, \ell, s)$ yields the following facts.

**Fact 1**: $\text{dia}(G(a, \ell, s)) = a + \ell + s$, and any path between $x$ and $y$ is a diametrical path.

**Fact 2**: $P(G(a, \ell, s)) = \{x, y\}$.

**Fact 3**: $r(G(a, \ell, s)) = \max\{a, \left\lceil \frac{a+\ell+s}{2} \right\rceil\}$ where $\left\lceil \frac{a+\ell+s}{2} \right\rceil$ is the least integer greater than or equal to $\frac{a+\ell+s}{2}$. Moreover, if $v \in V(G(a, \ell, s))$ satisfies $d(v, x) = r(G(a, \ell, s))$ or $d(v, y) = r(G(a, \ell, s))$, then $v \in C(G(a, \ell, s))$.

**Theorem 2.1.2.** For any positive integers $a$ and $b$ with $a < b \leq 2a$, there exists a graph $G$ for which $r(G) = a$, $\text{dia}(G) = b$, and $P(G) \subsetneq E(G)$.

**Proof:** Let $b = a + k$ where $1 \leq k \leq a$.

**Case 1:** If $a = 1$, then $b = 2$. Consider the graph $G$ obtained from $K_4$ by deleting one edge. It is easy to verify that $r(G) = 1$, $\text{dia}(G) = 2$, and $P(G) \subsetneq E(G)$.

**Case 2:** If $k$ is an even integer, then we consider the graph $G(a, \frac{k}{2}, \frac{k}{2})$. By Fact 1, $\text{dia}(G(a, \frac{k}{2}, \frac{k}{2})) = a + \frac{k}{2} + \frac{k}{2} = a + k = b$. By Fact 3 and the assumption that $k \leq a$, $r(G(a, \frac{k}{2}, \frac{k}{2})) = \max\{a, \left\lceil \frac{a+k}{2} \right\rceil\} = \max\{a, \frac{a+k}{2}\} = a$.

To show that $P(G(a, \frac{k}{2}, \frac{k}{2})) \subsetneq E(G(a, \frac{k}{2}, \frac{k}{2}))$, let $v^*$ be a vertex of a diametrical path $P$ so that the distance from $v^*$ to an endvertex of $P$ is $a$. By Fact 3, $v^*$ is a center vertex. So $P(G(a, \frac{k}{2}, \frac{k}{2})) = \{x, y\} \subseteq E(G(a, \frac{k}{2}, \frac{k}{2}))$. Since $k \leq a$, $v^*$ must lie on the cycle. Denote the antipole vertex of $v^*$ by $u^*$, then $d(v^*, u^*) = a$. It follows that $u^* \in E(G(a, \frac{k}{2}, \frac{k}{2}))$. But $u^* \neq x, y$. So $u^* \not\in P(G(a, \frac{k}{2}, \frac{k}{2}))$. Hence, $P(G(a, \frac{k}{2}, \frac{k}{2})) \subsetneq E(G(a, \frac{k}{2}, \frac{k}{2}))$.

**Case 3:** If $k$ is odd, consider the graph $G(a, \frac{k+1}{2}, \frac{k-1}{2})$. Arguments similar to those in Case 2 can be applied here to prove that the $G(a, \frac{k+1}{2}, \frac{k-1}{2})$ satisfies the following conditions: $r(G(a, \frac{k+1}{2}, \frac{k-1}{2})) = a$, $\text{dia}(G(a, \frac{k+1}{2}, \frac{k-1}{2})) = b$, and $P(G(a, \frac{k+1}{2}, \frac{k-1}{2})) \subsetneq E(G(a, \frac{k+1}{2}, \frac{k-1}{2})).$
This completes the proof. ■

Before proving the following results, we will introduce some results concerning Cartesian products of pairs of graphs and describe some properties of hypercubes. These results will be used to construct graphs $G$ with prescribed radius, diameter, and relation between $P(G)$ and $EC(G)$.

The following lemma was noted by F. Buckley [4].

**Lemma 2.1.1.** Let $G$ and $H$ be connected graphs. Then

1. $r(G \times H) = r(G) + r(H)$ and $\text{dia}(G \times H) = \text{dia}(G) + \text{dia}(H)$;
2. $P(G \times H) = P(G) \times P(H)$ and $EC(G \times H) = EC(G) \times EC(H)$.

The second result in Lemma 2.1.1 is equivalent to Theorem 1.4.1.

Some of our constructions make use of $n$-cubes or hypercubes, $Q_n$, defined recursively by $Q_1 = K_2$ and $Q_n = Q_{n-1} \times K_2$. Hypercubes are self-centered and $r(Q_n) = \text{dia}(Q_n) = n$.

**Theorem 2.1.3.** For any positive integers $a$ and $b$ with $a < b \leq 2a$, there exists a graph $G$ for which $r(G) = a$, $\text{dia}(G) = b$, and $P(G) \nsubseteq EC(G)$.

**Proof:** For $b = a + 1$, let $G_{a,b}$ be the graph shown in Fig. 2.1.1, where the length of the horizontal path from $c$ to $x_i$ ($i = 1, 2, 3, 4$) is $a$.

Fig. 2.1.1
It is easy to check that the vertex $c$ is the unique center vertex of $G_{a,b}$, $P(G_{a,b}) = V(G_{a,b}) \setminus \{c\}$ and $EC(G_{a,b}) = \{ x_i : i = 1, 2, 3, 4 \}$. Therefore, $P(G_{a,b}) \supseteq EC(G_{a,b})$.

Before constructing $G_{a,b}$ for any positive integers $a$ and $b$ with $a + 2 \leq b \leq 2a$, consider the graph $G_m$ shown in Fig. 2.1.2, where $P_m$ is a path of length $m$ ($m \geq 2$). The graph $G_m$ has radius $m + 2$ and diameter $2m + 2$. It is easy to check that $P(G_m) = \{u_i, v_j : 1 \leq i \leq 4, 1 \leq j \leq 3\}$ and $EC(G_m) = \{u_i : 1 \leq i \leq 4\}$. Hence, $P(G_m) \supsetneq EC(G_m)$.

Fig. 2.1.2

Now for any pair of positive integers $a$ and $b$ with $a + 2 \leq b \leq 2a$, let $G_{a,b} = G_{b-a} \times Q_{2a-b-2}$, where $Q_{2a-b-2}$ is the hypercube with $2^{2a-b-2}$ vertices. Recall that hypercubes are self-centered graphs and $r(Q_{2a-b-2}) = dia(Q_{2a-b-2}) = 2a - b - 2$. Then by Lemma 2.1.1,

\[ r(G_{a,b}) = r(G_{b-a}) + r(Q_{2a-b-2}) = b - a + 2 + 2a - b - 2 = a \hspace{1cm} \text{and} \]
\[ dia(G_{a,b}) = dia(G_{b-a}) + dia(Q_{2a-b-2}) = 2(b - a) + 2 + 2a - b - 2 = b. \]

By Lemma 2.1.1 (2), $P(G_{a,b}) \supsetneq EC(G_{a,b})$. 
This completes the proof. □

**Theorem 2.1.4.** For any positive integers \( a \) and \( b \), \( a < b \leq 2a \), there exists a graph \( G \), with \( r(G) = a \) and \( \text{dia}(G) = b \), which satisfies \( P(G) \cap EC(G) \neq \emptyset \), \( P(G) \nsubseteq EC(G) \), and \( P(G) \nsubseteq EC(G) \).

**Proof:** If \( b = a + 1 \), let \( H_{a,b} \) be the graph shown in Fig. 2.1.3. Both small cycles of \( H_{a,b} \) are of the length \( 2a + 1 \) and the vertical paths from \( x \) to \( w \), from \( w \) to \( y \), and from \( x \) to \( y \) are of the length \( 2 \), \( a - 2 \) and \( a \) respectively. Let \( C \) be the largest cycle of \( H_{a,b} \). Note that \( d(x, z) = 3 \) and \( d(z, y) = a - 1 \). By computing the eccentricities of vertices of \( H_{a,b} \), we can see that \( P(H_{a,b}) = (V(C) \setminus \{x, y\}) \cup \{z\} \) and \( EC(H_{a,b}) = V(C) \). It follows that \( P(H_{a,b}) \cap EC(H_{a,b}) \neq \emptyset \), \( P(H_{a,b}) \nsubseteq EC(H_{a,b}) \), and \( P(H_{a,b}) \nsubseteq EC(H_{a,b}) \).

![Fig. 2.1.3](image)

If \( b \geq a + 2 \), then take any positive integers \( a_1, a_2, b_1, \) and \( b_2 \) such that \( a_1 < b_1 \leq 2a_1, a_2 < b_2 \leq 2a_2, a = a_1 + a_2, \) and \( b = b_1 + b_2 \). By Theorems 2.1.2 and 2.1.3, there exist graphs \( A_{a_1,b_1} \) and \( B_{a_2,b_2} \) so that

(i) \( r(A_{a_1,b_1}) = a_1, \text{dia}(A_{a_1,b_1}) = b_1, \) and \( P(A_{a_1,b_1}) \nsubseteq EC(A_{a_1,b_1}) \);

(ii) \( r(B_{a_2,b_2}) = a_2, \text{dia}(B_{a_2,b_2}) = b_2, \) and \( P(B_{a_2,b_2}) \nsubseteq EC(B_{a_2,b_2}) \).

Let \( H_{a,b} = A_{a_1,b_1} \times B_{a_2,b_2} \). By Lemma 2.1.1, and results (i) and (ii),

\[
r(H_{a,b}) = r(A_{a_1,b_1}) + r(B_{a_2,b_2}) = a_1 + a_2 = a,
\]
\[
\text{dia}(H_{a,b}) = \text{dia}(A_{a_1,b_1}) + \text{dia}(B_{a_2,b_2}) = b_1 + b_2 = b,
\]

\[
P(H_{a,b}) \cap EC(H_{a,b}) \neq \emptyset, P(H_{a,b}) \not\subseteq EC(H_{a,b}), \text{ and } P(H_{a,b}) \not\supseteq EC(H_{a,b}).
\]

F. Buckley [4] also proved the following two results:

\textbf{Lemma 2.1.2.} For any graph \( G \) with \( P(G) \cap EC(G) = \emptyset \),

\[r(G) + 2 \leq \text{dia}(G) \leq 2r(G) - 2.
\]

\textbf{Theorem 2.1.5.} For any positive integers \( a \) and \( b \), \( a + 2 \leq b \leq 2a - 2 \), there is a graph \( G \) so that \( r(G) = a \), \( \text{dia}(G) = b \), and \( P(G) \cap EC(G) = \emptyset \).

Since a graph \( G \) with \( P(G) \cap EC(G) = \emptyset \) exists, it is natural to ask how far apart the two sets \( P(G) \) and \( EC(G) \) can be. To answer this question, consider the graph \( G_r \) shown in Fig. 2.1.4, where \( r \) is any positive integer larger than 4 and \( P_{r-5} \) (\( P_{r-3} \)) is a path of length \( r - 5 \) (\( r - 3 \), respectively).

(Each number beside a vertex is the eccentricity of the vertex)

\textbf{Fig. 2.1.4}

It is easy to check that \( C(G_r) = \{c\} \), \( P(G_r) = \{w, z\} \), \( EC'(G_r) = \{x, y\} \), and \( d(P(G_r), EC(G_r)) = 2r - 5 \). Since \( r \) is arbitrary, \( d(P(G_r), EC(G_r)) \rightarrow \infty \) (as \( r \rightarrow \infty \)). Thus, if we consider the ratio of \( d(P(G), EC(G)) \) to \( \text{dia}(G) \), we have
Theorem 2.1.6. Let $\mathcal{D} = \{G : P(G) \cap EC(G) = \emptyset\}$. Then

$$
\sup \{ \frac{d(P(G), EC(G))}{\text{dia}(G)} : G \in \mathcal{D} \} = 1.
$$

Proof: It is clear that $\sup \{ \frac{d(P(G), EC(G))}{\text{dia}(G)} : G \in \mathcal{D} \} \leq 1$. To show that the equality holds, it is enough to show that for any $\epsilon > 0$, there exists a $G \in \mathcal{D}$ so that

$$
\frac{d(P(G), EC(G))}{\text{dia}(G)} > 1 - \epsilon.
$$

In fact, for any $\epsilon > 0$, let $r$ be a positive integer so that $r > \frac{3+2\epsilon}{2\epsilon}$. Then $G_r \in \mathcal{D}$ and

$$
\frac{d(P(G_r), EC(G_r))}{\text{dia}(G_r)} = \frac{2r - 5}{2r - 2} = 1 - \frac{3}{2r - 2} > 1 - \epsilon.
$$

Thus,

$$
\sup \{ \frac{d(P(G), EC(G))}{\text{dia}(G)} : G \in \mathcal{D} \} = 1.
$$

Since $P(G) \cap EC(G) = \emptyset$ implies $d(P(G), EC(G)) \leq \text{dia}(G) - 1$, a question that arises is whether there exists a graph $G \in \mathcal{D}$ so that $\frac{d(P(G), EC(G))}{\text{dia}(G)} = \frac{a}{b}$ for any positive integers $a$ and $b$ with $a < b$. The answer is positive. Before proving this, note the following self-evident observations:

Remark. Let $G_1$ and $G_2$ be connected graphs. If $d(P(G_1), EC(G_1)) = \ell_1$ and $d(P(G_2), EC(G_2)) = \ell_2$, then $d(P(G_1 \times G_2), EC(G_1 \times G_2)) = \ell_1 + \ell_2$.

Combining this Remark with Lemma 2.1.1, we obtain the following:

Corollary. Let $G_1, G_2, \ldots, G_m$ $(m \geq 1)$ be connected graphs and let $G = G_1 \times G_2 \times \cdots \times G_m$ $(m \geq 1)$. If $d(P(G_i), EC(G_i)) = \ell_i$ and $\text{dia}(G_i) = d_i$ $(i = 1, 2, \ldots, m)$, then

$$
\frac{d(P(G), EC(G))}{\text{dia}(G)} = \frac{\sum_{i=1}^{m} \ell_i}{\sum_{i=1}^{m} d_i}.
$$
Theorem 2.1.7. For any positive integers $a$ and $b$ with $a < b$, there exists a graph $G \in \mathcal{D}$ so that

$$\frac{d(P(G), \overline{E}(G))}{\text{dia}(G)} = \frac{a}{b}.$$ 

Proof: Note that

$$\frac{a}{b} = \frac{10a}{10b} = \frac{(10a - 5) + 5 + 0}{(10a - 2) + 8 + [10(b - a) - 6]}.$$ 

Take $G = G_{5a} \times G_5 \times P_{10(b-a)-6}$ where $P_{10(b-a)-6}$ is a path of length $10(b-a)-6$. By the Corollary above, the graph $G$ is the graph desired. \(\blacksquare\)

§2. Embeddings

In [6] problems of the following type were studied: given a graph $G$, is it possible to embed $G$ as the central subgraph of some supergraph $H$ so that $H$ has some set of prescribed properties? In [4] F. Buckley described a way to embed a graph $G$ into a graph $H$ so that $P(H) = \overline{E}(H)$. Bielak and Syslo [1] proved that for every graph $G$ there exists a graph $H$ so that $P(H)$ is equal to $V(G)$. In this section, for a given graph $G$, we estimate the smallest order of a graph $H$ so that $G$ is an induced subgraph of $H$ and $P(H) = \overline{E}(H)$. Also, for a given graph $G$ and a proper subset $S$ of $V(G)$, we investigate the existence of a graph $H$ so that $G$ is an induced subgraph of $H$ and $P(H) = S$. Finally, for a given connected graph $G$ and a positive integer $d$ with $d \leq \tau(G)$, we estimate the smallest order of a graph $H$ with $\text{dia}(H) = d$ so that $G$ is an induced subgraph of $H$ and $P(H) = V(G)$.

For a given graph $G$, let $\gamma(G)$ be the minimum number of vertices of a graph $H$ having $G$ as an induced subgraph so that $P(H) = \overline{E}(H)$. 
**Theorem 2.2.1.** For a connected graph $G$ of order $n$,

$$n \leq \gamma(G) \leq n + 1.$$ 

**Proof:** It is obvious that $\gamma(G) \geq n$. So to complete the proof, it suffices to prove that $\gamma(G) \leq n + 1$. Let us consider $r(G)$.

Case 1: If $r(G) \geq 2$, then let $H$ be the graph obtained from $G$ by adding a new vertex $v$ and joining $v$ to each vertex of $G$ (see Fig. 2.2.1 (a)). Clearly, $C(H) = \{v\}$ and $P(H) = V(G) = EC(H)$. Thus, $\gamma(G) \leq n + 1$.

Case 2: Assume that $r(G) = 1$. If $|C(G)| = 1$, let $H = G$ since $P(G) = V(G) \setminus C(G) = EC(G)$. If $|C(G)| \geq 2$ and $G \neq K_n$, let $H$ be the graph obtained from $G$ by adding a new vertex $v$ to $G$ and joining $v$ to each vertex of $V(G) \setminus C(G)$ (see Fig. 2.2.1 (b)). It is easy to verify that $H$ is self-centered.

If $G = K_n$ then clearly, $\gamma(G) \leq n$. So, in each case, $\gamma(G) \leq n + 1$. ■

![Fig. 2.2.1](image)

The following result is due to H. Bielak and M. M. Syslo [1]:

**Theorem 2.2.2.** If a graph $G$ has no vertex adjacent to all other vertices of $G$, then there is a graph $H$ with induced subgraph $G$ so that $P(H) = V(G)$.

Here we consider a more general problem as follows: given a graph $G$ and a proper subset $S$ of $V(G)$, does there exist a graph $H$ having $G$ as an induced subgraph so that $P(H) = S$?
The following result describes some necessary conditions for a graph $G$ to be embedded into a supergraph $H$ so that $P(H) = S$.

**Theorem 2.2.3.** Let $G$ be a non-self-centered graph and let $S$ be a proper subset of $V(G)$ with at least two vertices. Then there is no graph $H$ having $G$ as an induced subgraph so that $P(H) = S$ provided $S$ satisfies one of the following conditions:

1. There exists a vertex $v \in S$ so that $v$ is adjacent to every other vertex of $S$;
2. $\min\{\max\{d_G(u,v) : u \in S\} : v \in S\} = 2$ and $P(G) \neq S$.

**Proof:** Suppose that there exists a graph $H$ so that $G$ is an induced subgraph of $H$ and $P(H) = S$. We show that neither (1) nor (2) hold.

First suppose that (1) holds. Since $S \neq V(G)$, $e_H(x) \geq 2$ for each vertex $x$ of $S$. So, there exists $w \in V(H)$ so that $d_H(v,w) = e_H(v) \geq 2$. Note that $w \notin S$ by choice of $v$. Now since $v \in S = P(H)$, $d_H(v,w) = e_H(v) = \text{dia}(H)$. Thus, $w \in P(H) \setminus S$. So, $S \neq P(H)$, a contradiction.

Next suppose that (2) holds. There exists $v^* \in S$ so that

$$\max\{d_G(u,v^*) : u \in S\} = 2.$$ 

Then $e_H(v^*) \geq 2$. In fact, $e_H(v^*) = 2$. For, if $e_H(v^*) > 2$, then there exists $w \in V(H)$ so that $d_H(v^*,w) = e_H(v^*) > 2$ which implies that $w \notin S$. But $e_H(w) \geq d_H(v^*,w) = e_H(v^*) = \text{dia}(H)$ since $v^* \in S = P(H)$. Thus, $w \in P(H)$, a contradiction to $P(H) = S$. Therefore, $e_H(v^*) = 2$. Note that $v^* \in S$ and $S = P(H)$. So, $\text{dia}(H) = e_H(v^*) = 2$.

Let $x$ be any vertex in $V(G) \setminus S = V(G) \setminus P(H)$. Then $1 \leq e_H(x) < \text{dia}(H) = 2$ and hence $e_H(x) = 1$. That is, $x$ is adjacent to every vertex.
of $H$. Since $G$ is subgraph of $H$, $z$ is adjacent to every vertex of $G$. It follows that $r(G) = 1$. Since $G$ is not self-centered, $\text{dia}(G) = 2$ and hence $(V(G) \setminus S) \cap P(G) = \emptyset$. That is,

$$P(G) \subseteq S. \quad (a)$$

On the other hand, for each vertex $v \in S$,

$$\text{dia}(G) \geq e_G(v) \geq \max\{d_G(u, v) : u \in S\}$$

$$\geq \min\{\max\{d_G(u, v) : u \in S\} : v \in S\}$$

$$= 2 = \text{dia}(G),$$

i.e., $e_G(v) = \text{dia}(G)$ and hence $v \in P(G)$. So,

$$S \subseteq P(G). \quad (b)$$

By (a) and (b), we obtain $P(G) = S$, a contradiction to the given condition. This completes the proof. ■

**Lemma 2.2.1.** Let $H$ be a connected graph. If $G$ is a proper connected subgraph of $H$ and $P(H) = V(G)$, then $2 \leq \text{dia}(H) \leq r(G)$.

**Proof:** Note that if $\text{dia}(H) = 1$, then $H$ is complete so $V(H) = P(H) = V(G)$, a contradiction to the assumption that $G$ is a proper subgraph of $H$. Hence, $\text{dia}(H) \geq 2$.

Let $u$ be a center vertex of $G$. Note that

$$\max\{d_H(u, v) : v \in V(G)\} \leq \max\{d_G(u, v) : v \in V(G)\} = r(G).$$

Since $P(H) = V(G)$,

$$\max\{d_H(u, v) : v \in V(G)\} = \max\{d_H(u, v) : v \in P(H)\}$$

$$= \max\{d_H(u, v) : v \in V(H)\} = e_H(u) = \text{dia}(H).$$

Thus, $\text{dia}(H) \leq r(G)$. ■
Lemma 2.2.2. Let \( G \) be connected graph. Then \( V(G) \) can be partitioned into two disjoint subsets \( A \) and \( B \) so that \( A \) and \( B \) satisfy the following conditions:

(1) for any vertex \( x \) of \( A \), there exists a vertex \( y \) of \( B \) so that \( d(x, y) > r(G) \);

(2) for any vertex \( y \) of \( B \), there exists a vertex \( x \) of \( A \) so that \( d(x, y) > r(G) \).

Proof: Let \( A(u) = \{ v \in V(G) : d(u, v) = e(u) \text{ or } d(u, v) = e(v) \} \) for any \( u \in V(G) \). The following algorithm will produce two required disjoint subsets \( A \) and \( B \) of \( V(G) \).

\[
\begin{align*}
i &:= 1 \\
&\text{Pick } x_1 \in V(G), \text{ find } A(x_1) \\
\text{while } V(G) \setminus \bigcup_{j=1}^{i} (A(x_j) \cup \{x_j\}) \neq \emptyset \\
&\text{ Pick } x_{i+1} \in V(G) \setminus \bigcup_{j=1}^{i} (A(x_j) \cup \{x_j\}) \\
&\text{ Find } A(x_{i+1}) \\
&\text{ Let } A = \bigcup_{j=1}^{i} \{x_j\} \text{ and } B = V(G) \setminus A \\
&\text{ i := i + 1 }
\end{align*}
\]

Clearly, for any \( x_i, x_j \in A \), \( d(x_i, x_j) < \min\{e(x_i), e(x_j)\} \). That is, for any \( x \in A \), there exists \( y \in B \) so that \( d(x, y) = e(x) \geq r(G) \).

On the other hand, for any \( y \in B \) there exists \( x \in A \) so that \( y \in A(x) \). Hence, \( d(x, y) = e(x) \) or \( d(x, y) = e(y) \). That is,

\[d(x, y) \geq \min\{e(x), e(y)\} \geq r(G)\].
The proof is complete. □

**Theorem 2.2.4.** Let $G$ be a connected graph and let $d$ be a positive integer with $2 \leq d \leq r(G)$. Then there exists a connected graph $H$ with $\text{dia}(H) = d$ so that $G$ is an induced subgraph of $H$ and so that $P(H) = V(G)$.

**Proof:** Let $A$ and $B$ be disjoint subsets of $V(G)$ satisfying conditions (1) and (2) in Lemma 2.2.2. Let $P_{d-2}$ be a path of length $d - 2$.

Let $H$ be the graph obtained from $G$ and $P_{d-2}$ by adding edges between one endvertex of $P_{d-2}$ and each vertex in $A$ and adding edges between another endvertex of $P_{d-2}$ and each vertex in $B$ (see Fig. 2.2.2). Obviously, $G$ is an induced subgraph of $H$. Moreover, for any vertex $v$ in $V(G)$, $e_H(v) = d$, and for any vertex $u \in V(H) \setminus V(G)$, $e_H(v) \leq d - 1$. Hence $\text{dia}(H) = d$ and $P(H) = V(G)$. Note that $|V(H)| = |V(G)| + d - 1$. □

![Fig. 2.2.2](image)

For a connected graph $G$ and a positive integer $d$ with $d \leq r(G)$, let $\beta(G,d)$ be the smallest order of a graph $H$ with $\text{dia}(H) = d$ so that $G$ is an induced subgraph of $H$ and $P(H) = V(G)$. From the construction of $H$ in Theorem 2.2.4, we obtain the following:

**Corollary.** For a connected graph $G$ and a positive integer $d$ with $d \leq r(G)$,

$$|V(G)| \leq \beta(G,d) \leq |V(G)| + d - 1.$$
Remark 2.2.1. If $P(G) \neq V(G)$, then $\beta(G,2) = |V(G)| + 1$.

The upper bound in the Corollary is also achieved for certain path. That is, we provided evidence for the following conjecture:

Conjecture. For a positive integer $d \geq 3$, if $P$ denotes a path of length at least $2d - 1$, then $\beta(P,d) = |V(P)| + d - 1$.

Lemma 2.2.3. Let $P$ be a path of length at least 5. If $P$ is an induced subgraph of a connected graph $H$ with $\text{dia}(H) = 3$ so that $P(H) = V(P)$, then $|V(H)| \geq |V(P)| + 2$.

Proof: Suppose that $|V(H)| \leq |V(P)| + 1$. Since not every vertex of $P$ is a peripheral vertex of $P$, $|V(H)| = |V(P)| + 1$. Note that since $P(H) = V(P)$ and $\text{dia}(H) = 3$, $r(H) = 2$. Let $c$ be the vertex of $H$ not on $P$. Since $P(H) = V(P)$, $e_H(x) = \text{dia}(H) = 3$ for all $x \in V(P)$. Hence, since $r(H) = 2$, $c$ must be the only center vertex of $H$, and hence $e_H(c) = 2$. Let

$$N(c) = \{u : d_H(u,c) = 1\} \text{ and } N^2(c) = \{u : d_H(u,c) = 2\}.$$ 

Clearly, $|N^2(c)| \geq 1$.

Case i: If $|N^2(c)| = 1$, let $N^2(c) = \{u^*\}$. Let $v^* \in V(P)$ so that $u^*v^* \in E(P)$. Then $d_H(v^*,v) \leq 2$ for any $v \in V(H)$. So, $e_H(v^*) = 2 = r(H)$ and $v^*$ is another center vertex of $H$. But this contradicts the uniqueness of $c$.

Case ii: If $|N^2(c)| = 2$, say $N^2(c) = \{x,y\}$, then either $x$ is adjacent to $y$ or not. In the first case, adjust notation, if necessary, so that $z \neq x$ is a vertex on $P$ adjacent to $y$. Then $e_H(z) = 2$, again a contradiction to the uniqueness of $c$. So, $x$ is not adjacent to $y$ in $P$. If there exists $w \in V(P)$ such that $xw$, $wy \in E(P)$, then $e_H(w) = 2$, a contradiction again. Thus, $d_P(x,y) \geq 3$. 
But if \( d_P(x,y) > 3 \), then \( d_H(x,y) \geq 4 \), a contradiction to \( \text{dia}(H) = 3 \). So \( d_P(x,y) = 3 \). Let \( w^* \) be the vertex of \( P \) between \( x \) and \( y \) so that \( w^* \) is adjacent to \( x \). Then \( e_H(w^*) = 2 \), a contradiction again.

Case iii: If \( |N^2(c)| = 3 \), say \( N^2(c) = \{x,y,z\} \), we may adjust notation, if necessary, so that \( y \) is between \( x \) and \( z \). As above, since \( \text{dia}(H) = 3 \), \( d_P(x,z) \leq 3 \). If \( d_P(x,z) = 2 \), then \( d_H(c,y) = 3 \leq e_H(c) = r(H) = 2 \), a contradiction. So \( d_P(x,z) = 3 \). Adjust notation, if necessary, so that \( w \) is the vertex of \( P \) adjacent to \( x \) and \( y \). Then \( e_H(w) = 2 \), which again contradicts the uniqueness of \( c \).

Case iv: Suppose that \( |N^2(c)| \geq 4 \). Pick \( x,y \in N^2(c) \) so that

\[
\alpha(x,y) = \max\{d_H(u,v) : u,v \in N^2(c)\}.
\]

Since \( |N^2(c)| \geq 4 \) and \( N^2(c) \subseteq V(P) \), \( d_H(x,y) = d_P(x,y) \geq 3 \). But \( \text{dia}(H) = 3 \). So, \( d_H(x,y) = 3 \). By the choice of \( x \) and \( y \), there exists a vertex \( w \in N^2(c) \) between \( x \) and \( y \). Thus, \( 3 \leq d_H(c,w) \leq e_H(c) = r(H) = 2 \), a contradiction.

All cases show that the assumption that \( |V(H)| \leq |V(P)| + 1 \) was wrong. Therefore, \( |V(H)| \geq |V(P)| + 2 \). \( \blacksquare \)

From Lemma 2.2.3 and Corollary to Theorem 2.2.4, we obtain the following:

**Remark 2.2.2.** Let \( P \) be a path of length at least 5. Then

\[
\beta(P,3) = |V(P)| + 2.
\]

That is, the conjecture above is true when \( d = 3 \).
Chapter 3. Realisation of Digraphs by Preferences Based on Distances in Graphs

§1. Introduction

Let $D$ be an oriented graph with vertex set $V(D)$ and arc set $A(D)$. If whenever $xy, yz \in A(D)$ it is also true that $zx \in A(D)$, then $D$ is said to be transitive. If $D$ contains no directed cycle, then $D$ is said to be acyclic.

An oriented graph can be considered as a model of outcomes of voters preferences. Consider a set $C = \{c_1, c_2, \ldots, c_k\}$ of candidates and a set $V = \{v_1, v_2, \ldots, v_n\}$ of voters. Each voter has a preferential ordering or ranking of the candidates. That is, each voter assigns a numeric value to each candidate and prefers a candidate with a smaller number to any other candidate with a higher number, but ties can result as candidates are allowed to receive the same value. We can represent the outcome of those voters preferences with an oriented graph $D$ with vertex set $C$ in which $c_i c_j$ is an arc of $D$ if and only if the sum of the numeric values assigned to $c_i$ by all the voters is strictly less than the sum of the numeric values assigned to $c_j$ by all the voters. Of course, there might be no arc between two candidates $c_i$ and $c_j$ since each may receive the same sum of numeric values. The following example demonstrates this. Let $V = \{v_1, v_2, v_3\}$, $C = \{a, b, c, d\}$, and let the voter assignments be as in Fig. 3.1.1 (a). That is, for example, voter $v_2$ assigns 1 to both $a$ and $d$, assigns 2 to $b$, and assigns 3 to $c$. The resulting oriented graph $D$ shown in Fig. 3.1.1 (b). Note that if the ordering given by the voters in Fig. 3.1.1 (a) is thought of as preference, then $xy$ is an arc of $D$ if and only if more voters prefer $x$ to $y$ than prefer $y$ to $x$. Thus, we say that the resulting oriented graph is model of voters preferences. It should be noted
that the plurality preference is involved this model. Issues involved with
ing voting with an agenda, in which majority preference is employed resulting in
complete oriented (tournaments) have been studied by Miller [14] and Reid [19].

\begin{align*}
v_1 & \quad v_2 & \quad v_3 & \text{numeric value} \\
c & a, d & b & 1 \\
a & b & c & 2 \\
b & c & a, d & 3 \\
d & 4
\end{align*}

\begin{tikzpicture}
    \node (a) at (0,0) {a};
    \node (b) at (1,0) {b};
    \node (c) at (1,-1) {c};
    \node (d) at (0,-1) {d};
    \draw[->] (a) -- (b);
    \draw[->] (a) -- (c);
    \draw[->] (a) -- (d);
    \draw[->] (b) -- (d);
\end{tikzpicture}

(a) \quad (b)

Fig. 3.1.1

In the remainder of this chapter, voter preferences are based on distances of graphs.

Let \( G \) be a connected graph with vertex set \( V(G) \). For distinct vertices 
\( u \) and \( v \) in \( V(G) \), let \( V_{u,v} = \{ z : z \in V(G), \ d(x,u) < d(x,v) \} \), where \( d(x,y) \)
denotes the distance in \( G \) between vertices \( x \) and \( y \). Let \( V \) and \( C \) denote two
distinguished subsets of \( V(G) \). Consider the oriented graph \( D \) with vertex set \( C \)
where \( cc' \) is an arc of \( D \) if and only if more vertices of \( V \) are closer (in \( G \)) to 
\( c \) than to \( c' \), for each pair of distinct vertices \( c \) and \( c' \) in \( C \). If \( V \) is considered
as a set of sites each of which is occupied by a single voter and \( C \) is considered
as a set of candidate sites, say for the placement of desirable facilities on the
network represented by \( G \), then \( D \) represents plurality preferences of the voters
over all pairs of distinct candidates. Thus, we will say that such a
digraph \( D \) is realized from \( G \) according to the preferences of voters in \( V \) for
candidates in \( C \).
If an oriented graph $D$ is given in advance, one may ask if there exists a connected graph $G$ and vertex subsets $V$ and $C$ so that $D$ is realized from $G$ according to the preferences of voters in $V$ for candidates in $C$. And, if there is, what is the smallest possible order of such a $G$? To be more explicit we recall the following definition stated on page 4.

**Definition.** Let $n, h, k$ be positive integers, $n \geq \max(h, k)$. An oriented graph $D$ with vertex set $\{x_1, x_2, \ldots, x_k\}$ is $(n, h, k)$-realizable if there exists a connected graph $G$ of order $n$, a subset $V \subseteq V(G)$ with $|V| = h$ (the set of voters), and a subset $C = \{c_1, c_2, \ldots, c_k\} \subseteq V(G)$ (the set of candidates) so that $x_i x_j \in A(D)$ if and only if $|V \cap V_{c_i, c_j}| > |V \cap V_{c_j, c_i}|$, for all distinct $i$ and $j$ in $\{1, 2, \ldots, k\}$. In particular, if $D$ is $(n, n, k)$-realizable by a graph $G$, then we simply say that $D$ is realized by $G$ or that $G$ realizes $D$.

**Example.** The digraph $D$ given in Fig. 3.1.2 (a) is $(12, 3, 4)$-realizable. To see this consider the graph $G$ given in Fig. 3.1.2 (b), where the 3-set $V$ of voters is as shown and the set of candidates is $C = \{c_1, c_2, c_3, c_4\}$. Notice that the digraph $D$ is neither transitive nor acyclic.

![Diagram](image)

Fig. 3.1.2.

The concept of $(n, h, k)$-realization was introduced by Johnson and Slater [12] who discussed the case $n = h$. We will also treat the case $n = h$ in the next section. Since ties may well occur between candidates, $D$ should
indeed be restricted to be an oriented graph in the above definition. It should also be pointed out that the decision procedure for the choice between candidates is by plurality decision rather than by majority choice decision, i.e., for $x_i x_j \in A(D)$ it is only required that at least one more voter, rather than a majority of the voters, be closer to $c_i$ than to $c_j$.

§2. On the smallest possible order of a graph realizing a digraph

By construction of a suitable graph, Johnson and Slater [12] proved the following theorem.

**Theorem 3.2.1.** Any oriented graph $D$ with $k$ vertices, $q$ arcs, and maximum degree (as an undirected graph) $\Delta$ is $(n, n, k)$-realizable, where $n = k^2 + k\Delta - q$.

So, the existence question raised in the first section is settled and an upper bound on the smallest order is established. The goal of this section is to provide two constructions of graphs which realize (with $n = h$) a given oriented graph $D$ and to show that each of these has smaller order than the example due to Johnson and Slater. Finally, the best construction due recently to W. Schnyder [11] will be presented.

**Theorem 3.2.2.** Any oriented graph $D$ with $k$ vertices and $q$ arcs is $(n, n, k)$-realizable, where $n = \min\{k^2 + 2, 2k + 2q\}$.

**Proof:** Let $D$ be an oriented graph with vertex set $\{1, 2, \ldots, k\}$ and $q$ arcs.

First we show that $D$ is $(k^2 + 2, k^2 + 2, k)$-realizable by constructing a suitable graph $G$. Start with the complete graph with vertex set $V_1 = \{c_1, c_2, \ldots, c_k\}$ and subdivide each edge $c_i c_j$, $1 \leq i < j \leq k$ with two new
vertices $v_{ij}$ and $v_{ji}$ so that $v_{ij}$ is closer to $c_i$ than is $v_{ji}$. Let $V_2 = \{v_{ij} : 1 \leq i, j \leq k, i \neq j\}$. Adjoin edge $c_i v_{ji}$ if and only if $ij \in A(D)$. Let $G$ be the graph obtained by adjoining two new vertices $w$ and $z$ so that $w$ is adjacent to each vertex of $V_1$ and $z$ is adjacent to each vertex of $V_2$. An example of this construction is shown in Fig. 3.2.1. (Note that the construction due to Johnson and Slater would require $4^2 + (4)(3) - 4 = 24$ vertices to realize the oriented graph $D$ shown in Fig. 3.2.1)

![Diagram](image)

**Fig. 3.2.1**

It is easy to see that

1. $V_1 \cup V_2 \cup \{w, z\}$ is a partition of $V(G)$ and $|V(G)| = k^2 + 2$, and
2. $V_{c_i,c_j} \cap V_1 = \{c_i\}$ and $V_{c_i,c_j} \cap \{w, z\} = \emptyset$, for all $1 \leq i, j \leq k, i \neq j$.

Let $V(G) = V$ and $V_1$ be the voter set and candidate set, respectively. We now check that the voter preferences in $G$ realize $D$. For any pair of distinct voters $u$ and $v$, since $V_{u,v} = V_{u,v} \cap (V_1 \cup V_2 \cup \{w, z\})$, we see that

$$|V_{c_i,c_j}| = |V_{c_i,c_j} \cap V_1| + |V_{c_i,c_j} \cap V_2| + |V_{c_i,c_j} \cap \{w, z\}|.$$
So, by (2) above we see that for any distinct \( i \) and \( j \), \( 1 \leq i, j \leq k \),

\[
|V_{c_i,c_j}| - |V_{c_j,c_i}| = |V_{c_i,c_j} \cap V_2| - |V_{c_j,c_i} \cap V_2|.
\]

Now, for distinct \( i \) and \( j \) in \( \{1, 2, \ldots, k\} \), let

\[
S_1(i, j) = \{ v_{pm} : v_{pm} \in V_2, \{p, m\} \cap \{i, j\} = \emptyset \}
\]

and \( S_2(i, j) = V_2 \setminus S_1(i, j) \).

Since for \( v_{pm} \in S_1(i, j) \), \( d(v_{pm}, c_i) = d(v_{pm}, c_j) = 3 \), \( |V_{c_i,c_j} \cap S_1(i, j)| = |V_{c_j,c_i} \cap S_1(i, j)| \). Thus, by (3)

\[
|V_{c_i,c_j}| - |V_{c_j,c_i}| = |V_{c_i,c_j} \cap S_2(i, j)| - |V_{c_j,c_i} \cap S_2(i, j)|.
\]

For \( 1 \leq l \leq k \), let \( S_l \) denote the set \( \{ v_{pm} : v_{pm} \in V_2, p \) or \( m \) equals \( l \} \). Then for all distinct \( i \) and \( j \), \( 1 \leq i, j \leq k \), \( |S_l| = |S_j| = 2(k-1) \), \( S_i \cap S_j = \{ v_{ij}, v_{ji} \} \),

and \( S_2(i, j) = S_i \cup S_j \). Moreover,

\[
V_{c_i,c_j} \cap S_2(i, j) = \begin{cases} S_i \setminus \{ v_{ij}, v_{ji} \}, & \text{if } ji \in A(D), \\ S_i \setminus \{ v_{ji} \}, & \text{otherwise.} \end{cases}
\]

Combining this with (4), we see that

\[
|V_{c_i,c_j}| - |V_{c_j,c_i}|
= \begin{cases} |S_i \setminus \{ v_{ij} \}| - |S_i \setminus \{ v_{ij}, v_{ji} \}|, & \text{if } ij \in A(D), \\ |S_i \setminus \{ v_{ji} \}| - |S_i \setminus \{ v_{ij} \}|, & \text{if neither } ij \text{ nor } ji \text{ is in } A(D) . \end{cases}
\]

Consequently, for all distinct \( i \) and \( j \), \( 1 \leq i, j \leq k \), \( ij \in A(D) \) if and only if

\[
|V_{c_i,c_j} \cap V| = |V_{c_i,c_j}| > |V_{c_j,c_i}| = |V_{c_j,c_i} \cap V| .
\]

That is, \( D \) is realized by \( G \).

To complete the proof we show that \( D \) is also \( (2k + 2q, 2k + 2q, k) \) - realizable by constructing an appropriate graph. Start with the complete graph with vertex set \( V_1 = \{ v_1, v_2, \ldots, v_k \} \) and for each \( i \), \( 1 \leq i \leq k \), adjoin a new vertex \( c_i \) adjacent to \( v_i \). Set \( V_2 = \{ c_1, c_2, \ldots, c_k \} \). For each arc \( ij \in A(D) \), adjoin two additional new vertices \( v_{ij} \) and \( w_{ij} \) so that \( v_{ij} \) is adjacent to both
$c_i$ and $v_j$ and so that $w_{ij}$ is adjacent to each vertex in $V_i \setminus \{v_i, v_j\}$. Denote the resulting graph by $H$ and let $V_3$ and $V_4$ denote the sets \( \{x : x = v_{ij} \text{ for some arc } ij \in A(D)\} \) and \( \{x : x = w_{ij} \text{ for some arc } ij \in A(D)\} \), respectively. An example of this construction is shown in Fig. 3.2.2. (Note that $H$ contains only 16 vertices, in contrast to the 18 required in the first part of this proof.)

![Diagram of graph $H$ showing vertices and edges]

From the construction of $H$ we conclude that

\[(5)\quad V_{c_i, c_j} \cap V_1 = V_{c_i, c_j} \cap V_2 = \{c_i\}, \text{ for all } 1 \leq i, j \leq k, \ i \neq j.\]

Let $V(H) = V$ and $V_2$ be the voter set and candidate set respectively.

\[(6)\quad V_{c_i, c_j} \cap V_1 = V_{c_i, c_j} \cap V_2 = \{c_i, c_j\}, \text{ for all } 1 \leq i, j \leq k, \ i \neq j.\]

Let $V(H) = V$ and $V_2$ be the voter set and candidate set respectively.

We now check that the voter preferences in $H$ realize $D$. It is convenient to note the preferences of voters in $V_3$ and $V_4$. If $v_{ij} \in V_3$, then $d(v_{ij}, c_l) = 1$, $d(v_{ij}, c_j) = 2$, and $d(v_{ij}, c_l) = 3$ for all $1 \leq l \leq k$, $i \neq l \neq j$. If $w_{ij} \in V_4$, then $d(w_{ij}, c_l) = 2$ for all $1 \leq l \leq k$, $i \neq l \neq j$, and $d(w_{ij}, c_i) = d(w_{ij}, c_j) = 3$.

Suppose that $pm \in A(D)$, then

\[(7)\quad V_{c_p, c_m} \cap V_3 = \{v_{pl} : pl \in A(D)\} \cup \{v_{lp} : lp \in A(D)\},\]

\[(8)\quad V_{c_p, c_m} \cap V_4 = \{w_{ml} : ml \in A(D)\} \cup \{w_{lm} : lm \in A(D), \text{ but } l \neq p\}\]
(9) \( V_{cm,c_p} \cap V_3 = \{v_{ml} : ml \in A(D)\} \cup \{v_{lm} : lm \in A(D), \text{ but } l \neq p\} \), and
(10) \( V_{cm,c_p} \cap V_4 = \{w_{pl} : pl \in A(D), \text{ but } l \neq m\} \cup \{w_{lp} : lp \in A(D)\} \).

For a vertex \( x \) in \( D \), let us use \( d^+_D(x) \) (\( d^-_D(x) \)) to denote the cardinality of the set \( \{y : y \in V(D), \ xy \in A(D)\} \) (\( \{y : y \in V(D), \ yx \in A(D)\} \), respectively.)

So, if \( pm \) is an arc of \( D \), then combining (5) - (10), we obtain
\[
|V_{c_p,c_m} \cap V| = \sum_{i=1}^{4} |V_{c_p,c_m} \cap V_i|
= 2 + d^+_D(p) + d^-_D(p) + d^+_D(m) + (d^-_D(m) - 1),
\]
and
\[
|V_{cm,c_p} \cap V| = \sum_{i=1}^{4} |V_{cm,c_p} \cap V_i|
= 2 + d^+_D(m) + (d^-_D(m) - 1) + (d^+_D(p) - 1) + d^-_D(p).
\]
Thus, \(|V_{c_p,c_m} \cap V| - |V_{cm,c_p} \cap V| = 1 > 0\).

Similarly, if neither \( pm \) nor \( mp \) are arcs in \( A(D) \), then
\[
|V_{c_p,c_m} \cap V| = |V_{cm,c_p} \cap V|
= 2 + d^+_D(p) + d^-_D(p) + d^+_D(m) + d^-_D(m), \text{ so that}
\]
\[
|V_{c_p,c_m} \cap V| - |V_{cm,c_p} \cap V| = 0.
\]
Consequently, for all distinct \( i \) and \( j \), \( 1 \leq i,j \leq k \), \( ij \in A(D) \) if and only if
\(|V_{c_p,c_m} \cap V| > |V_{cm,c_p} \cap V| \). That is, \( D \) is realized by \( H \).

This completes the proof. \( \blacksquare \)

Remark. If \( q > 2 \) in Theorems 3.2.1 and 3.2.2, then
\[
2q = \sum \{d^+_D(x) + d^-_D(x) : x \in V(D)\}
\leq \sum \{\Delta : x \in V(D)\} = k\Delta
\]
which implies that \( q + 2 < 2q \leq k\Delta \) or \( k^2 + 2 < k^2 + k\Delta - q \). That is, Theorem 3.2.2 is indeed an improvement over Theorem 3.2.1. Moreover, if
$q < \binom{k}{2} - \frac{k-2}{2}$, then $2k + 2q < k^2 + 2$, so a further improvement is provided for these values of $q$ by the second part of Theorem 3.2.2.

Recently, Walter Schnyder [11] obtained a graph of linear order which realizes a given oriented graph. He proved the following theorem.

**Theorem 3.2.3.** Any oriented graph $D$ with $k$ vertices is $(3k + 1, 3k + 1, k)$-realizable.

**Proof:** Let $D$ be an oriented graph with vertex set $V(D) = \{1, 2, \cdots, k\}$. For $j \in V(D)$, let $I_n(j) = \{i : ij \in V(D)\}$ and let $I_n(j)^c = V(D) \setminus I_n(j)$. Also, let $V = \{c_1, c_2, \cdots, c_k\}$, $V^+ = \{v^+ : v \in V\}$, and $V^- = \{v^- : v \in V\}$. Construct the graph $G^*$ with vertex set $V^+ \cup V \cup V^-$ as follows: start by adding an edge between any two vertices in $V^+$, then join $c_j^+$ to each vertex in $\{c_i : i \in I_n(j)\} \cup \{c_j\}$ by an edge, and join $c_j^-$ to each vertex in $\{c_i : i \in I_n(j)^c\} \cup \{c_i^+ : i \in I_n(j)\}$ by an edge. Let $G$ be the graph obtained from $G^*$ by adding a new vertex $w$ adjacent to each vertex in $V$. An example of this construction is shown in Fig. 3.2.3.

![Fig. 3.2.3](image-url)

Note that $|V(G)| = 3k + 1$. Consider $V(G)$ and $V$ as voter set and candidate set, respectively. For $i, j \in V(D)$, let
\[ V_1(i,j) = \{ c_\ast : v_\ast \in V \setminus \{ c_i, c_j \}, i, j \in In(s) \}, \]
\[ V_2(i,j) = \{ c_\ast : c_\ast \in V \setminus \{ c_i, c_j \}, i, j \in In(s)^c \}, \]
\[ V_3(i,j) = \{ c_\ast : c_\ast \in V \setminus \{ c_i, c_j \}, i \in In(s), j \in In(s)^c \}, \]
and let
\[ V_4(i,j) = \{ c_\ast : c_\ast \in V \setminus \{ c_i, c_j \}, i \in In(s)^c, j \in In(s) \}. \]

Also, for \( \ell, \ell \in \{ 1, 2, 3, 4 \} \), let \( V^{\ell^+}(i,j) = \{ v^+ : v \in V_\ell(i,j) \} \) and \( V^{\ell^-}(i,j) = \{ v^- : v \in V_\ell(i,j) \} \). Then from the construction of \( G \) we obtain that

\[
\begin{align*}
(11) \quad d_G(c_i, u) &= \begin{cases} 
1, & \text{if } u \in V^{\ell^+}_1(i,j) \cup V^{\ell^-}_2(i,j) \cup V^{\ell^+}_3(i,j) \cup V^{\ell^-}_4(i,j), \\
2, & \text{if } u \in V^{\ell^-}_1(i,j) \cup V^{\ell^+}_2(i,j) \cup V^{\ell^-}_3(i,j) \cup V^{\ell^+}_4(i,j), 
\end{cases} \\
(12) \quad d_G(c_j, u) &= \begin{cases} 
1, & \text{if } u \in V^{\ell^+}_1(i,j) \cup V^{\ell^-}_2(i,j) \cup V^{\ell^-}_3(i,j) \cup V^{\ell^+}_4(i,j), \\
2, & \text{if } u \in V^{\ell^-}_1(i,j) \cup V^{\ell^+}_2(i,j) \cup V^{\ell^+}_3(i,j) \cup V^{\ell^-}_4(i,j), 
\end{cases} \\
(13) \quad d_G(u, v) &= 2 \quad \text{and} \quad d_G(u, w) = 1 \quad \text{for any } u, v \in V.
\end{align*}
\]

If \( ij \in A(D) \), then
\[
d_G(c_i, c_i^+) = 1, \quad d_G(c_j, c_i^+) = 2, \quad d_G(c_i, c_i^-) \geq 3, \quad d_G(c_j, c_i^-) = 1 \\
d_G(c_i^+, c_j^+) = 1, \quad d_G(c_j^+, c_j^+) = 1, \quad d_G(c_i^-, c_j^-) = 2, \quad d_G(c_j^-, c_j^-) \geq 3.
\]

Thus, by (11), (12), and (13), we obtain \( V_{c_i,c_j} = V^{\ell^+}_3(i,j) \cup V^{\ell^-}_4(i,j) \cup \{ c_i^+, c_j^- \} \) and \( V_{c_j,c_i} = V^{\ell^-}_3(i,j) \cup V^{\ell^+}_4(i,j) \cup \{ c_i^-, c_j^+ \} \). So, \(|V_{c_i,c_j}|-|V_{c_j,c_i}| = 1 > 0. \)

If neither \( ij \) nor \( ji \) are arcs of \( D \), then
\[
d_G(c_i, c_i^+) = 1, \quad d_G(c_j, c_i^+) = 2, \quad d_G(c_i, c_i^-) \geq 3, \quad d_G(c_j, c_i^-) = 1 \\
d_G(c_i^+, c_j^+) = 2, \quad d_G(c_j^+, c_j^+) = 1, \quad d_G(c_i^-, c_j^-) = 1, \quad d_G(c_j^-, c_j^-) \geq 3.
\]

Thus, by (11), (12), and (13) again, we obtain \( V_{c_i,c_j} = V^{\ell^+}_3(i,j) \cup V^{\ell^-}_4(i,j) \cup \{ c_i^+, c_j^- \} \) and \( V_{c_j,c_i} = V^{\ell^-}_3(i,j) \cup V^{\ell^+}_4(i,j) \cup \{ c_i^-, c_j^+ \} \). So, \(|V_{c_i,c_j}|-|V_{c_j,c_i}| = 0. \)
Consequently, for all distinct $i$ and $j$, $1 \leq i, j \leq k$, $ij \in A(D)$ if and only if $|V_{c_i,c_j}| > |V_{c_j,c_i}|$. That is, $D$ is realized by $G$. ■

Note that not every oriented graph of order $k$ is $(k,k,k)$-realizable since the tournament of order 3 is not $(3,3,3)$-realizable. Thus, it remains an open question to find the smallest order of a graph which realizes a given oriented graph.
Chapter 4. Plurality Preference Digraphs Realised by Trees

The goal of this chapter is to discuss oriented graphs which are realized only by trees. The first section provides a description of a criterion for the presence of an arc in an oriented graph \( D \) realized by a tree, and then addresses the possible structure of a digraph of order \( k \) which is \((n,n,k)\)-realizable by a tree of order \( n \). It is showed that an oriented graph \( D \) of order \( k \) is \((n,n,k)\)-realizable by a tree of order \( n \) greater than \( k \) if and only if \( D \) is transitive and contains no induced anti-directed path of length 3.

For a positive integer \( n \), let \( T_n \) denote the family of digraphs of order \( n \) which are realizable by trees. For a fixed \( D \in T_n \), the realization number of \( D \), denoted \( \alpha(D) \), is the smallest order of a tree which realizes \( D \). Let \( \alpha(T_n) = \max\{\alpha(D) : D \in T_n\} \). The value of \( \alpha(T_n) \) is determined explicitly in the second section.

Some properties of digraphs \( D \) of order \( n \) which are \((n,n,n)\)-realizable by trees and examples of such digraphs are given in the third section.

In the last section estimates are given for the largest possible order of a tournament contained in a digraph which is realized by a tree of order \( n \) with diameter \( d \) and \( i \) centroid vertices, for any positive integers \( n, d, \) and \( i \) \((n - 1 \geq d \text{ and } i = 1, 2)\).

§1. A characterization of a digraph realizable by a tree

Let \( T \) be a tree. For \( x \in V(T) \), the branch weight of \( x \) is defined by \( b(x) = \max\{|V(T')| : T' \text{ is a subtree of } T - x\} \). The branch weight centroid of \( T \) (centroid of \( T \) for short), denoted \( C_d(T) \), consists of all vertices \( x \) for which \( b(x) \) is a minimum. Each vertex in \( C_d(T) \) is called a centroid vertex of \( T \) (See [23]). In 1869 Jordan [13] proved that the centroid of \( T \) consists of a single vertex or two adjacent vertices.
Lemma 4.1.1. Let \( c \) be a centroid vertex of a tree \( T \). If \( C_1, C_2, \cdots, C_s \) are the components of \( T - c \) and \( b(c) = |V(C_s)| \), then

\[
\sum_{i=1}^{s-1} |V(C_i)| \geq b(c) - 1.
\]

Moreover, the equality holds if and only if \( T \) contains two centroid vertices.

Proof: Suppose \( \sum_{i=1}^{s-1} |V(C_i)| < b(c) - 1 \). Let \( x \in V(C_s) \), so that \( x \) is adjacent to \( c \). Then the subgraph induced by \( (\bigcup_{i=1}^{s-1} V(C_i)) \cup \{c\} \) is a subtree of \( T - x \). The remaining subtrees of \( T - x \) are contained in \( C_1 \).

Since \( b(x) > b(c) \), \( b(x) = \sum_{i=1}^{s-1} |V(C_i)| + 1 \). Thus, by our assumption, \( b(x) = \sum_{i=1}^{s-1} |V(C_i)| + 1 < b(c) - 1 + 1 = b(c) \), a contradiction to the choice of \( c \). Hence, \( \sum_{i=1}^{s-1} |V(C_i)| \geq b(c) - 1 \).

If the equality holds, then, for the vertex \( x \) chosen above,

\[
b(x) = \sum_{i=1}^{s-1} |V(C_i)| + 1 = b(c) - 1 + 1 = b(c).
\]

So, \( x \) is also a centroid vertex of \( T \), and hence \( T \) contains two centroid vertices.

Conversely, suppose \( T \) contains two centroid vertices but

\[
\sum_{i=1}^{s-1} |V(C_i)| > b(c) - 1.
\]

Then for any \( x \in V(C_i) \) (\( 1 \leq i \leq s - 1 \)), \( b(x) \geq |V(C_s)| + 1 > b(c) \), and for any vertex \( x \in V(C_s) \), \( b(x) \geq \sum_{i=1}^{s-1} |V(C_i)| + 1 > b(c) - 1 + 1 = b(c) \).

It follows that \( c \) is the unique centroid vertex of \( T \), a contradiction to the assumption. \( \blacksquare \)

Let \( T \) be a tree. If \( x \) is a vertex of \( T \) and \( w \) is either a vertex or edge of \( T \), then \( T(x, w) \) denotes the subtree of \( T - w \) which contains \( x \).
Lemma 4.1.2. Let T be a tree with two (adjacent) centroid vertices \( c_1 \) and \( c_2 \). For any \( x, y \in V(T) \), \( |V(T(x, c_1 c_2))| = |V(T(y, c_1 c_2))| \).

Proof: The result is obviously true if \( x \) and \( y \) lie in the same component of \( T - \{c_1 c_2\} \). Suppose that \( |V(T(x, c_1 c_2))| \neq |V(T(y, c_1 c_2))| \). Without loss of generality, we may assume that \( |V(T(x, c_1 c_2))| < |V(T(y, c_1 c_2))| \). Since both \( c_1 \) and \( c_2 \) are centroid vertices of \( T \), we may assume that \( c_1 \in T(x, c_1 c_2) \). Then \( T(y, c_1 c_2) \) is a subtree of \( T - c_1 \), and the remaining subtrees of \( T - c_1 \) are contained in \( T(x, c_1 c_2) \). Thus, \( |V(T(x, c_1 c_2))| < |V(T(y, c_1 c_2))| \) implies that \( b(c_1) = |V(T(y, c_1 c_2))| \).

On the other hand, \( T(x, c_1 c_2) \) is a subtree of \( T - c_2 \). So, \( b(c_2) \geq |V(T(x, c_1 c_2))| \). Note that \( C_d(T) = \{c_1, c_2\} \) and \( b(c_1) = b(c_2) \). So, \( b(c_2) > |V(T(x, c_1 c_2))| \). Suppose that \( b(c_2) = |V(T')| \), where \( T' \) is a subtree of a component of \( T - c_2 \). Then \( T' \) is a proper subtree of \( T(y, c_1 c_2) \) which implies \( b(c_2) = |V(T')| < |V(T(y, c_1 c_2))| = b(c_1) \). Thus, \( b(c_1) \neq b(c_2) \), a contradiction. ■

Lemma 4.1.3. Let \( x_1, x_2, x_3 \) be distinct vertices of a tree \( T \), let \( x_0 \) be the unique vertex on the three paths connecting \( x_1, x_2, x_3 \) in pairs (see Fig. 4.1.1), and let \( d_i = d(x_0, x_i) \) (\( i = 1, 2, 3 \)). If \( d_3 < \min\{d_1, d_2\} \) and \( |V_{x_1, x_3}| > |V_{x_2, x_3}| \), then \( |V_{x_3, x_3}| > |V_{x_2, x_3}| \).

![Fig. 4.1.1](image)
Proof: If \( V_{x_1,x_2} \supseteq V_{x_1,x_2} \) and \( V_{x_2,x_3} \subseteq V_{x_2,x_3} \), then

\[
|V_{x_3,x_2}| \geq |V_{x_1,x_2}| > |V_{x_2,x_3}| \geq |V_{x_2,x_1}|.
\]

So, to prove \( |V_{x_3,x_2}| > |V_{x_2,x_3}| \), it is sufficient to prove that \( V_{x_3,x_2} \supseteq V_{x_1,x_2} \) and \( V_{x_2,x_3} \subseteq V_{x_2,x_1} \).

Recall that \( T(x_i,x_0) \) denotes the component of \( T - x_0 \) containing \( x_i \) for \( i = 1, 2, 3 \). If \( d_i = 0 \) for some \( i \in \{1, 2, 3\} \), then we regard \( V(T(x_i,x_0)) \) as the empty set. Note that \( 0 \leq d_3 < \min\{d_1, d_2\} \) implies that \( d_1 > 0 \) and \( d_2 > 0 \).

Now consider \( d_3 \).

Case 1: Suppose that \( d_3 > 0 \). Notice that \( d_3 < d_2 \). So, for any \( w \in V(T) \setminus (V(T(x_2,x_0)) \cup V(T(x_3,x_0))) \),

\[
d(x_3,w) = d(x_3,x_0) + d(x_0,w) = d_3 + d(x_0,w) < d_2 + d(x_0,w) = d(x_2,x_0) + d(x_0,w) = d(x_2,w).
\]

So, \( w \in V_{x_3,x_2} \). Therefore,

\[
V(T) \setminus (V(T(x_2,x_0)) \cup V(T(x_3,x_0))) \subseteq V_{x_3,x_2}.
\]

Pick \( w \in V(T(x_3,x_0)) \), and let \( z \) be the first vertex on both \( P(x_3,w) \) and \( P(x_0,w) \) from \( x_3 \) (\( z \) might be \( w \)), where \( P(x,y) \) is a shortest path from \( x \) to \( y \). Then

\[
d(x_3,w) = d(x_3,z) + d(z,w) \leq d(x_3,x_0) + d(x_0,z) + d(z,w) < d(x_2,x_0) + d(x_0,z) + d(z,w) = d(x_2,w).
\]

So, \( w \in V_{x_3,x_2} \). Therefore,

\[
V(T(x_3,x_0)) \subseteq V_{x_3,x_2}.
\]
If \( w \in V(T(x_2, x_o)) \cap V_{x_1,x_2}, \)
\[
d(x_3, w) = d(x_3, x_o) + d(x_o, w) = d_3 + d(x_o, w)
\]
\[
< d_1 + d(x_o, w) = d(x_1, x_o) + d(x_o, w)
\]
\[
= d(x_1, w) < d(x_2, w).
\]
So, \( w \in V_{x_3,x_2}. \) Therefore,
\[
V(T(x_2, x_o)) \cap V_{x_1,x_2} \subseteq V_{x_3,x_2}. \tag{3}
\]

By (1), (2), and (3), \( V(T) \setminus (V(T(x_2, x_o)) \setminus V_{x_1,x_2}) \subseteq V_{x_3,x_2}. \) It is obvious that \( V_{x_1,x_2} \subseteq V(T) \setminus (V(T(x_2, x_o)) \setminus V_{x_1,x_2}). \) Hence, \( V_{x_1,x_2} \subseteq V_{x_3,x_2}. \)

Case 2: Suppose that \( d_3 = 0. \) The proof is the same as the proof in Case 1 as long as we regard \( V(T(x_3, x_o)) \) as the empty set.

In either case, we have \( V_{x_1,x_2} \subseteq V_{x_3,x_2}. \)

Now we show that \( V_{x_3,x_3} \subseteq V_{x_3,x_1}. \) Note that for any \( w \in V(T(x_1, x_o)), \)
\[
d(x_2, w) = d(x_2, x_o) + d(x_o, w)
\]
\[
> d(x_3, x_o) + d(x_o, w) = d(x_3, w).
\]
So, \( w \in V_{x_3,x_3}. \) Hence, \( V(T(x_1, x_o)) \cap V_{x_3,x_3} = \emptyset. \) Similarly, \( V(T(x_3, x_o)) \cap V_{x_2,x_3} = \emptyset. \) Let \( w \in V_{x_2,x_3}. \) Then
\[
d(x_2, w) < d(x_3, w) = d(x_3, x_o) + d(x_o, w)
\]
\[
< d(x_1, x_o) + d(x_o, w) = d(x_1, w).
\]
So, \( w \in V_{x_2,x_3}. \) It follows that \( V_{x_3,x_3} \subseteq V_{x_3,x_1}. \)

This completes the proof. \( \square \)

**Lemma 4.1.4.** Let \( T, x_i (i = 0, 1, 2, 3), \) and \( d_i = d(x_o, x_i) (i = 1, 2, 3) \) be as in Lemma 4.1.3. If \( d_2 \leq \min\{d_1,d_3\}, \) \( |V_{x_1,x_2}| > |V_{x_2,x_3}|, \) and \( |V_{x_2,x_5}| > |V_{x_3,x_3}|, \) then \( |V_{x_1,x_5}| > |V_{x_3,x_1}|. \)
Proof: If \( d_1 = d_2 \), then \( V_{x_1,x_3} = V_{x_2,x_3} \) and \( V_{x_3,x_1} = V_{x_2,x_2} \). So, \( |V_{x_1,x_3}| = |V_{x_2,x_3}| > |V_{x_2,x_1}| = |V_{x_3,x_1}| \) and we are done. Hence, we may assume that \( d_1 > d_2 \). In this case, if we can prove that
\[
V_{x_1,x_3} \supseteq V_{x_1,x_2}
\]
and
\[
V_{x_3,x_1} \subseteq V_{x_2,x_1}
\]
then \( |V_{x_1,x_3}| \geq |V_{x_2,x_2}| > |V_{x_2,x_1}| \geq |V_{x_3,x_1}| \) and we are done. So, it is sufficient to prove (5) and (6).

We treat (5) first. Suppose that \( d_2 = 0 \). Let \( w \in V_{x_1,x_2} \). Then
\[
d(x_1,w) < d(x_2,w) \leq d(x_2,w) + d(x_2,x_3) = d(x_3,w),
\]
which implies that \( w \in V_{x_1,x_3} \). Hence, \( V_{x_1,x_3} \subseteq V_{x_1,x_3} \). So, we may assume that \( d_2 > 0 \). Since \( d_2 < d_1 \) and \( d_2 \leq d_3 \), \( V_{x_1,x_2} \cap V(T(x_3,x_o)) = \emptyset \) and
\[
V_{x_1,x_2} \cap V(T(x_3,x_o)) = \emptyset.
\]
So, if \( w \in V_{x_1,x_2} \),
\[
d(x_1,w) < d(x_2,w) = d(x_2,x_o) + d(x_o,w)
\]
\[
= d_2 + d(x_o,w) = d(x_3,x_o) + d(x_o,w) = d(x_3,w),
\]
which implies that \( w \in V_{x_1,x_3} \). Hence \( V_{x_1,x_2} \subseteq V_{x_1,x_3} \), and (5) follows.

Now we show that \( V_{x_3,x_1} \subseteq V_{x_2,x_1} \). Recall that \( d_1 > d_2 \). If \( w \in V_{x_3,x_1} \cap V(T(x_3,x_o)) \), then
\[
d(x_2,w) = d(x_2,x_o) + d(x_o,w)
\]
\[
< d(x_1,x_o) + d(x_o,w) = d(x_1,w).
\]
Hence, \( w \in V_{x_2,x_1} \). It follows that
\[
V_{x_3,x_1} \cap V(T(x_3,x_o)) \subseteq V_{x_2,x_1}.
\]
If \( w \in V_{z_3,z_1} \cap (V(T) \setminus V(T(x_3,x_0))) \), then
\[
d(x_2,w) \leq d(x_2,x_0) + d(x_0,w)
< d(x_3,x_0) + d(x_0,w)
= d(x_3,w) < d(x_1,w).
\]

Hence, \( w \in V_{z_3,z_1} \). It follows that
\[
V_{z_3,z_1} \cap (V(T) \setminus V(T(x_3,x_0))) \subseteq V_{z_3,z_1}.
\]  
(8)

Combining (7) and (8), we conclude that \( V_{z_3,z_1} \subseteq V_{z_3,z_1} \).

This completes the proof. \( \square \)

**Lemma 4.1.5.** Let \( T, x_i \ (i = 0, 1, 2, 3) \), and \( d_i = d(x_0, x_i) \ (i = 1, 2, 3) \) be as in Lemma 4.1.3. If \( d_1 = d_2 < d_3 \), then \( |V_{z_3,z_1}| > |V_{z_3,z_2}| \) if and only if \( |V_{z_1,z_3}| > |V_{z_3,z_1}| \).

**Proof:** Since \( d_1 = d_2 \), \( V_{z_1,z_3} = V_{z_3,z_1} \) and \( V_{z_3,z_1} = V_{z_3,z_2} \). The assertion follows. \( \square \)

**Theorem 4.1.1.** Let \( D \) be an oriented graph of order \( k \). If \( D \) is \( (n,n,k) \)-realizable by a tree of order \( n \), then \( D \) is transitive.

**Proof:** Suppose that \( D \) is \( (n,n,k) \)-realizable by the tree \( T \) of order \( n \). By abuse of notation, we may regard \( V(D) \) as the set of candidates in \( T \).

Suppose that \( xy,yz \in A(D) \). Then \( |V_{x,y}| - |V_{y,z}| > 0 \) and \( |V_{y,z}| - |V_{z,y}| > 0 \). To show that \( D \) is transitive, it suffices to show \( xz \in A(D) \), i.e., \( |V_{x,z}| - |V_{z,x}| > 0 \).

Consider the subgraph of \( T \) induced by paths from \( x \) to \( y \) and from \( x \) to \( z \), where \( w \) is the unique vertex on the two paths (see Fig. 4.1.2).
Let $d_z = d_T(x, w)$, $d_y = d_T(y, w)$, and $d_z = d_T(z, w)$. Since $|V_{y,z}| > |V_{z,y}|$, by Lemmas 4.1.3 and 4.1.5, neither $d_z < \min\{d_x, d_y\}$ nor $d_z = d_x < d_y$ nor $d_z = 0$. Thus, one of the following relations among $d_x$, $d_y$, and $d_z$ occurs:

(i) $d_y \leq \min\{d_x, d_z\}$;
(ii) $d_z < \min\{d_y, d_z\}$;
(iii) $d_z = d_y < d_z$;
(iv) $d_z = d_y < d_x$.

But by Lemmas 4.1.3, 4.1.4, and 4.1.5, each of (i) - (iv) implies that $|V_{x,z}| > |V_{z,x}|$, that is, $xz \in A(D)$. □

**Theorem 4.1.2.** Let $D$ be an oriented graph of order $k$ which is $(n,n,k)$-realizable by a tree $T$ of order $n$. Then $xy \in A(D)$ if and only if one of the following statements holds:

(a) $d_T(x, C_d(T)) < d_T(y, C_d(T))$;
(b) If $d_T(x, C_d(T)) = d_T(x, c) = d_T(y, c) = d_T(y, C_d(T))$ for some $c$ in $C_d(T)$, let $w$ be the vertex on the shortest path from $x$ to $y$ in $T$ so that $d_T(x, w) = d_T(y, w)$. Then

$$|V(T(x, w))| > |V(T(y, w))|.$$  \hspace{1cm} (9)

**Proof:** By abuse of notation, we may regard $V(D)$ as the set of candidates in $T$. We prove the "if" part first. Let $x, y \in V(D)$. 

![Diagram of a graph with vertices x, w, y, and z]
Suppose that statement (a) holds. Let \( c \) and \( c' \) be the centroid vertices of \( T \) so that \( d_T(x, c) = d_T(x, C_d(T)) \) and \( d_T(y, c') = d_T(y, C_d(T)) \). Let \( C_1, C_2, \ldots, C_s \) be the components of \( T - c \). Without loss of generality, we may assume that \( y \in V(C_s) \). Since for any \( w \in V(T) \setminus V(C_s) \),

\[
d_T(x, w) \leq d_T(x, c) + d_T(c, w) < d_T(y, c) + d_T(c, w) = d_T(y, w),
\]

\( V_{x,y} \supseteq V(T) \setminus V(C_s) \). Thus, \( V_{y,z} \subseteq V(C_s) \).

If \( c = c' \), then let \( z \in V(C_s) \) be adjacent to \( c \). Clearly \( z \) is not a centroid vertex of \( T \), and hence \( b(z) > b(c) \). Moreover, it is easy to check (analogous to the proof of Lemma 4.1.1) that \( b(z) = \sum_{i=1}^{s-1} |V(C_i)| + 1 \). So,

\[
|V_{x,y}| \geq \sum_{i=1}^{s-1} |V(C_i)| + 1 = b(z) > b(c) \geq |V(C_s)| \geq |V_{y,z}|.
\]

This implies that \( xy \in A(D) \).

If \( c \neq c' \), then \( c' \in V(C_s) \) since \( y \in V(C_s) \). So,

\[
d_T(x, c') = d_T(x, c) + d_T(c, c') < d_T(y, c') + 1,
\]

i.e., \( d_T(x, c') \leq d_T(y, c') \). This implies that \( c' \not\in V_{y,z} \) and hence \( V_{y,z} \subseteq V(C_s) \).

By Lemma 4.1.1, \( |V_{x,y}| \geq \sum_{i=1}^{s-1} |V(C_i)| + 1 \geq b(c) \) and \( |V_{y,z}| < |V(C_s)| \leq b(c) \). So, \( |V_{x,y}| - |V_{y,z}| > 0 \), and consequently \( xy \in A(D) \).

So, statement (a) implies that \( xy \in A(D) \).

Suppose that \( x \) and \( y \) satisfy (b). Then \( V_{x,y} = V(T(x, w)) \) and \( V_{y,z} = V(T(y, w)) \). So, \( |V(T(x, w))| > |V(T(y, w))| \) implies that \( |V_{x,y}| - |V_{y,z}| > 0 \).

That is, \( xy \in A(D) \).

We now prove the "only if" part.

Let \( xy \in A(D) \). Suppose that neither (a) nor (b) holds. Note that by the "if" part above, \( d_T(x, C_d(T)) > d_T(y, C_d(T)) \) implies that \( yx \in A(D) \),
a contradiction to $xy \in A(D)$. So, we may assume that $d_T(x, C_d(T)) = d_T(y, C_d(T))$. Again by the "if" part above, we need only treat the following two cases:

(i) $d_T(x, c) = d_T(y, c)$ for some $c \in C_d(T)$ and the inequality in (9) is equality;

(ii) $d_T(x, c) = d_T(y, c')$, where $\{c, c'\} = C_d(T)$ and $c \neq c'$.

In case (i), as in the proof of (b) above, we have $V_{x,y} = V(T(x, w))$, and $V_{y,x} = V(T(y, w))$. So, $|V(T(x, w))| = |V(T(y, w))|$ implies that $|V_{x,y}| - |V_{y,x}| = 0$. That is, $xy \notin A(D)$, a contradiction to $xy \in A(D)$. Therefore, case (ii) must occur. Then $x$ and $y$ lie in distinct components of $T - cc'$. By Lemma 4.1.2, $|V(T(x, cc'))| = |V(T(y, cc'))|$. Note that $V_{x,v} = V(T(x, cc'))$ and $V_{y,z} = V(T(y, cc'))$. So $|V_{x,v}| - |V_{y,z}| = |V(T(x, cc'))| - |V(T(y, cc'))| = 0$ which implies that $xy \notin A(D)$, a contradiction to $xy \in A(D)$. So, either (a) or (b) must hold. ■

**Theorem 4.1.3.** If an oriented graph $D$ of order $k$ is $(n, n, k)$-realizable by a tree $T$ of order $n$, then $D$ has no induced anti-directed path of length 3.

**Proof:** Suppose that there is an induced anti-directed path $P$ of length 3 between vertices $x$ and $w$. Without loss of generality, we may assume that the arcs of $P$ are $xy$, $zy$, $zw$ (see Fig. 4.1.3), for some vertices $y$ and $z$.

![Fig. 4.1.3](image)

First, we claim that $d_T(x, c) = d_T(y, c) = d_T(z, c) = d_T(w, c)$ for some $c \in C_d(T)$. To see this, note that by Theorem 4.1.2 since there are no arcs
in $D$ between $z$ and $y$ and $w$, and $x$ and $w$, respectively,

$$d_T(x,C_d(T)) = d_T(z,C_d(T)),$$

$$d_T(y,C_d(T)) = d_T(w,C_d(T)),$$

and

$$d_T(x,C_d(T)) = d_T(w,C_d(T)).$$

If $C_d(T) = \{c\}$, our claim follows. Suppose $C_d(T) = \{c, c'\}$ where $c \neq c'$.

By Lemma 4.1.2, the two components of $T - cc'$ have the same size. Since $xy \in A(D)$, Theorem 4.1.2 implies that $x$ and $y$ are in the same component of $T - cc'$. Similarly, $yz \in A(D)$ (respectively, $zw \in A(D)$) implies that $y$ and $z$ (respectively, $z$ and $w$) are in the same component of $T - cc'$. It follows that $x,y,z,$ and $w$ are in the same component of $T - cc'$, say the component of $T - cc'$ containing $c$. Then

$$d_T(x,c) = d_T(y,c) = d_T(z,c) = d_T(w,c).$$

It follows from this that, for each $v \in \{y,z,w\}$, there is a vertex $w_{zv}$ on the shortest path between $x$ and $v$ so that $w_{zv}$ is at equal distance to $x$ and $v$ in $T$. Let $T(x,w_{zv})$ and $T(v,w_{zv})$ be the components of $T - w_{zv}$ containing $x$ and $v$, respectively. By Theorem 4.1.2, $xy \in A(D)$ implies that $|V(T(x,w_{zv}))| > |V(T(y,w_{zv}))|$. Hence, by Theorem 4.1.2 again, since there is no arc between $x$ and $w$ and no arc between $y$ and $w$, $w \notin V(T(x,w_{zv})) \cup V(T(y,w_{zv}))$, and hence $w_{zv} \neq w_{zy}$. Note that $V(T(x,w_{zy}))$ and $V(T(y,w_{zy}))$ are contained in $V(T(x,w_{zv}))$. Since $xw,wx \notin A(D), \text{ by Theorem } 4.1.2, |V(T(x,w_{zv}))| = |V(T(w,w_{zv}))|$. But $zw \notin A(D)$, so $z \notin V(T(x,w_{zv})) \cup V(T(w,w_{zv}))$. Now consider $w_{zz}$. Either $w_{zz} = w_{zw}$ or $w_{zz} \neq w_{zw}$. In the first case, $zw \in A(D)$, which implies, by Theorem 4.1.2, that $|V(T(z,w_{zv}))| > |V(T(w,w_{zv}))| = |V(T(x,w_{zv}))|$. So, by Theorem
4.1.2, \(zz \in A(D)\), a contradiction. In the second case, \(zw \in A(D)\) implies that \(|V(T(z,w_{zz}))| > |V(T(w,w_{zz}))|\). But \(T(w,w_{zz}) = T(x,w_{zz})\), so, by Theorem 4.1.2 again, \(zz \in A(D)\), a contradiction.

The proof is complete. ■

**Definition 4.1.1.** Let \(D\) be a connected digraph. Then \(D\) is said to be **bipartitionable** if there exist two subdigraphs \(D_1\) and \(D_2\) satisfying the following conditions:

1. \(V(D) = V(D_1) \cup V(D_2)\) and \(V(D_1) \cap V(D_2) = \emptyset\); and
2. for any \(v_i \in V(D_i) (i = 1,2)\), \(v_1v_2 \in A(D)\).

In such a case we write \(D = D_1 \Rightarrow D_2\).

For any vertex \(x\) of a digraph \(D\), let \(O(x) = \{v \in V(D) : xv \in A(D)\}\) and \(I(x) = \{v \in V(D) : vx \in A(D)\}\). Set \(d_D^+(x) = |O(x)|\) and \(d_D^-(x) = |I(x)|\).

**Lemma 4.1.6.** Let \(D\) be a connected transitive digraph of order at least 2 without an induced anti-directed path of length 3. If \(x_1,x_2,\ldots,x_k\) are vertices of \(D\) with \(d_D^-(x_i) = 0\), then \(\bigcap_{i=1}^k O(x_i) \neq \emptyset\).

**Proof:** Suppose that \(\bigcap_{i=1}^k O(x_i) = \emptyset\). Clearly, \(O(x_1) \neq \emptyset\) since \(|V(D)| \geq 2\). So, \(k \geq 2\). Pick \(p \in \{1,2,\ldots,k-1\}\) so that \(\bigcap_{i=1}^p O(x_i) \neq \emptyset\) but \(\bigcap_{i=1}^{p+1} O(x_i) = \emptyset\). Then by the transitivity of \(D\), there are no arcs between \(\bigcap_{i=1}^p O(x_i)\) and \(O(x_{p+1})\). Let \(v \in O(x_{p+1})\). Since \(v \notin \bigcap_{i=1}^p O(x_i)\), there exists \(j \in \{1,2,\ldots,p\}\) such that there is no arc between \(x_j\) and \(v\).

If there exists \(x_q\) \((1 \leq q \leq p, q \neq j)\) so that \(x_qv \in A(D)\), then for some \(u \in O(x_j) \cap O(x_q), \{{x_j,u,x_q,v}\}\) is an anti-directed path of length 3, a contradiction to the hypothesis. So, we may assume that \(x_qv \notin A(D)\),
for all $i \in \{1,2,\cdots,p\}$. Since $D$ is connected, there exists $w \in V(D) \setminus \left( \bigcap_{i=1}^{p} O(x_i) \cup O(x_{p+1}) \right) \cup \{x_1,x_2,\cdots,x_{p+1}\}$ so that $wu, wv \in A(D)$. It follows that $\langle \{u,w,v,x_{p+1}\} \rangle$ is an anti-directed path of length 3, a contradiction. This completes the proof. $lacksquare$

**Lemma 4.1.7.** Let $D$ be a connected transitive digraph without an induced anti-directed path of length 3, and let $x_1,x_2,\cdots,x_k$ be all the vertices of $D$ with $d^+_D(x_i) = 0$ $(1 \leq i \leq k)$. Then $D = \langle V(D) \setminus \bigcap_{i=1}^{k} O(x_i) \rangle \Rightarrow \langle \bigcap_{i=1}^{k} O(x_i) \rangle$.

**Proof:** Set $C = \bigcap_{i=1}^{k} O(x_i)$. By Lemma 4.1.6, $C \neq \emptyset$, so pick $v \in C$ so that $x_iv \in A(D)$ for all $i \in \{1,2,\cdots,k\}$. By the transitivity of $D$, there is no arc from $C$ to $\bigcup_{i=1}^{k} O(x_i) \setminus C$. Suppose that there is $v \in C$ and $u \in \bigcup_{i=1}^{k} O(x_i) \setminus C$ joined by no arc. Since $u \notin C$, there are $i,j$ so that $x_iu \in A(D)$ but $x_ju \notin A(D)$. Then $\langle \{x_j,v,x_i,u\} \rangle$ is an anti-directed path of length 3, a contradiction. So, for any $v \in C$ and $u \in \bigcup_{i=1}^{k} O(x_i) \setminus C$, $uv \in A(D)$.

Let $Y = V(D) \setminus (\bigcup_{i=1}^{k} O(x_i) \cup \{x_1,x_2,\cdots,x_k\})$. Then the subdigraph $(Y)$ is transitive. This fact implies that there exists $w \in Y$ so that $d^+_Y(w) = 0$. Note that, by the transitivity of $D$, there is no vertex $z$ in $V(D) \setminus Y$ so that $zw \in A(D)$. So $d^-_D(w) = 0$, a contradiction to the maximality of $k$. Hence, $Y = \emptyset$. This completes the proof. $lacksquare$

**Theorem 4.1.4.** Let $D$ be a transitive digraph of order $k$ without an induced anti-directed path of length 3. Then for some $n$ greater than $k$, $D$ is $(n,n,k)$-realizable by a tree of order $n$.

**Proof:** We use induction on $|V(D)|$. 
If $|V(D)| = 1$, the result is obviously true. Suppose that the result holds for such a digraph of order $k - 1 \geq 1$. Let $|V(D)| = k$. As $D$ is transitive it contains at least one vertex $x$ with $d_D(x) = 0$. Let $C = \bigcap_{i=1}^{m} O(x_i)$ where $x_1, x_2, \ldots, x_m$ are all vertices of $D$ with $d_D(x_i) = 0$ ($i = 1, 2, \ldots, m$). If $|V(D)\setminus C| = k$, then $D = \langle V(D)\setminus C \rangle \Rightarrow \langle C \rangle$ by Lemma 4.1.7. Since $|V(D)\setminus C|$ and $|C|$ are each less than $k$ and since both $\langle V(D)\setminus C \rangle$ and $\langle C \rangle$ are transitive and neither contains an induced anti-directed path of length 3, the induction hypothesis implies that there exists a tree $T_1$ realizing $\langle V(D)\setminus C \rangle$ and a tree $T_2$ realizing $\langle C \rangle$. Without loss of generality we may assume that both $T_1$ and $T_2$ have a single centroid vertex, say $c_1$ and $c_2$, respectively. Pick a vertex $z$ from $V(T_1)$ so that $d_{T_1}(z, c_1) = \max\{d_{T_1}(v, c_1) : v \in V(T_1)\}$. Let $T$ be the tree obtained from $T_1$ and $T_2$ by joining $z$ to $c_2$ by an edge and adjoining $|V(T_2)|$ additional new vertices each adjacent to $c_1$. It is easy to check that $c_1$ is the only centroid vertex of $T$. Therefore, by Theorem 4.1.2, $T$ realizes $D$. □

The next result follows immediately from Theorems 4.1.1, 4.1.3, and 4.1.4.

**Theorem 4.1.5.** Let $D$ be an oriented graph of order $k$. Then $D$ is $(n, n, k)$-realizable by a tree of order $n$ greater than $k$ if and only if $D$ is transitive and contains no induced anti-directed path of length 3.

**Note:** The condition that $n$ is greater than $k$ in Theorem 4.1.5 is necessary. The digraph $D$ shown in Fig. 4.1.4 illustrates this point. Clearly, $D$ is transitive and contains no induced anti-directed paths of length 3. By checking each tree of order 5, one can verify that none realizes $D$. So, $D$ is not $(5, 5, 5)$-realizable by a tree of order 5.
§2. Realization numbers

For a positive integer $n$, let $\mathcal{F}_n$ be the family of oriented graphs of order $n$ which are realizable by trees. For any $D \in \mathcal{F}_n$, $D$ is said to be $m$-realizable if $D$ is realizable by a tree of order $m$. The realization number of $D$, denoted $\alpha(D)$, is the smallest integer $m$ for which $D$ is $m$-realizable. Let $\alpha(\mathcal{F}_n) = \max\{\alpha(D) : D \in \mathcal{F}_n\}$. The following questions arise naturally:

(1) For special $D \in \mathcal{F}_n$, determine $\alpha(D)$ explicitly or find bounds for $\alpha(D)$.

(2) Determine $\alpha(\mathcal{F}_n)$ explicitly or find bounds for $\alpha(\mathcal{F}_n)$.

To solve these two problems, an interesting family of oriented graphs will be introduced, an explicit formula for $\alpha(\mathcal{F}_n)$ will be derived in this section.

An exhaustive examination of all digraphs of small orders yielded the digraphs listed in Table 1 (on page 108) as those with the maximum realization number for each order $n \leq 7$. The labels on the vertices in the trees are to indicate the candidate vertices corresponding to the vertices in the digraphs.

Table 1 suggests the following family of digraphs whose realization numbers attain the maximum values.

For a positive integer $n$, let $H_n$ be the oriented graph defined recursively by $H_1 = K_1$, $H_2 = K_1 \Rightarrow K_1$, and $H_n = (K_1 \Rightarrow H_{n-2}) \cup K_1$. 
### Table 1

<table>
<thead>
<tr>
<th>n</th>
<th>D</th>
<th>T realizing D</th>
<th>( \alpha(D) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \bullet 1 )</td>
<td>( \bullet 1 )</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( \begin{array}{c} 1 \ 2 \ 1 \end{array} )</td>
<td>( \begin{array}{c} 2 \ 1 \end{array} )</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>( \begin{array}{c} 1 \ 2 \ 3 \end{array} )</td>
<td>( \begin{array}{c} 1 \ 2 \ 3 \end{array} )</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>( \begin{array}{c} 1 \ 2 \ 3 \ 4 \end{array} )</td>
<td>( \begin{array}{c} 1 \ 2 \ 3 \ 4 \end{array} )</td>
<td>14</td>
</tr>
<tr>
<td>5</td>
<td>( \begin{array}{c} 1 \ 2 \ 3 \ 4 \ 5 \end{array} )</td>
<td>( \begin{array}{c} 1 \ 2 \ 3 \ 4 \ 5 \end{array} )</td>
<td>28</td>
</tr>
<tr>
<td>6</td>
<td>( \begin{array}{c} 1 \ 2 \ 3 \ 4 \ 5 \ 6 \end{array} )</td>
<td>( \begin{array}{c} 1 \ 2 \ 3 \ 4 \ 5 \ 6 \end{array} )</td>
<td>46</td>
</tr>
<tr>
<td>7</td>
<td>( \begin{array}{c} 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \end{array} )</td>
<td>( \begin{array}{c} 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \end{array} )</td>
<td>88</td>
</tr>
</tbody>
</table>
To simplify the proof of Theorem 4.2.1, we label all vertices of $H_n$ as $1, 2, \cdots, n$ so that $A(H_n) = \{ij : i > j$ and $n - i \equiv 1 (\text{mod} \ 2)\} \cup \{21\}$. Examples of labeled $H_n$ ($n \leq 7$) are given in the second column of Table 1.

In order to determine $\alpha(H_n)$, a tree of the smallest possible order will be constructed to realize $H_n$.

For a given tree $T_0$ and $c_0 \in V(T_0)$, define a sequence of trees $\{W(T_0; i) : i \geq 0\}$ and a sequence of vertices $\{c_i : i \geq 0\}$, where $W(T_0; 0) = T_0$, according to the following rules: For $k \geq 1$,

(i) $W(T_0; 2k - 1)$ is obtained from two vertex-disjoint copies of $W(T_0; 2k - 2)$ by adding a new vertex $c_k$ adjacent to exactly the two copies of $c_{k-1}$ (see Fig. 4.2.1 (a));

(ii) $W(T_0; 2k)$ is obtained from copies of $W(T_0; 2k - 1)$ and $W(T_0; 2k - 2)$ by adding a new vertex adjacent to the copy of the vertex $c_{k-1}$ in $W(T_0; 2k - 2)$ and by adding an edge connecting the copy of $c_k$ in $W(T_0; 2k - 1)$ and the copy of $c_{k-1}$ in $W(T_0; 2k - 2)$ (see Fig. 4.2.1 (b)).

From the construction of the sequence of trees $\{W(T_0; i) : i \geq 0\}$, it is easy to verify the following observations:

\begin{figure}[h]
\centering
\includegraphics{fig4.2.1}
\caption{Fig. 4.2.1}
\end{figure}
(1) For any \( k \geq 1 \), \( W(T_0; k) \) has only one centroid vertex; \( W(T_0; 2k - 1) - c_k \) contains exactly two identical components, each a copy of \( W(T_0; 2k - 2) \), while \( W(T_0; 2k) - c_k \) contains exactly three components, two of which are identical and a third one which has one more vertex than the other two identical components.

\( (2) \quad |V(W(T_0; 2k))| = 3|V(W(T_0; 2k - 2))| + 2 \quad \text{and} \quad |V(W(T_0; 2k - 1))| = 2|V(W(T_0; 2k - 2))| + 1. \)

\( (3) \quad |V(W(T_0; 2k))| = 3^k(|V(T_0)| + 1) - 1 \quad \text{and} \quad |V(W(T_0; 2k - 1))| = 2 \cdot 3^{k-1}(|V(T_0)| + 1) - 1. \)

Of course, as explicitly seen in (3), for any positive integer \( n \), the order of tree \( W(T_0; n) \) is a function of the order of the initial tree \( T_0 \).

Now consider the oriented graph \( H_n \), which was defined above, when \( n = 2k + 1 \) (\( k \geq 1 \)). Let \( A \) denote the path of length three shown in Fig. 4.2.2 (a). Let \( a_0, b_0 \in V(W(A; 1)) \) lie in the same component of \( W(A; 1) - c_1 \) so that \( d_{W(A;1)}(a_0,c_1) = d_{W(A;1)}(b_0,c_1) = 2 \). Let \( a_i \in V(W(A;i)) \) be in the component of \( W(A;i) - c_{[\frac{i}{2}]} \) not containing \( a_j \) (\( 0 \leq j \leq i - 1 \)) so that

\[ d_{W(A;i)}(a_i,c_{\lfloor \frac{i}{2} \rfloor}) = \left\lfloor \frac{i}{2} \right\rfloor + 1. \]

In \( W(A; 2k - 1) \) consider the set \( \{a_0, b_0, a_1, \ldots, a_{2k-1} \} \) as the set of candidate. Then by Theorem 4.1.2, it is straightforward, but tedious, to check that \( W(A; 2k - 1) \) realizes a digraph isomorphic to \( H_n = H_{2k+1} \). Also, by observation (3),

\[ |V(W(A; 2k - 1))| = 2 \cdot 3^{\frac{n-3}{2}} (|V(A)| + 1) - 1 \]
\[ = 2 \cdot 3^{\frac{n-3}{2}} \cdot 5 - 1 = 10 \cdot 3^{\frac{n-3}{2}} - 1. \]
So, for odd \( n \), \( H_n \) is realizable by a tree of order \( 10 \cdot 3^{n/2} - 1 \). By observation (1) above, the tree \( W^*(A; 2k - 1) \) obtained from \( W(A; 2k - 1) \) by replacing the path of length two which contains centroid vertex \( c_k \) as interior vertex with a single edge connecting the two ends of that path is also a tree that realizes \( H_n \); moreover

\[
|V(W^*(A; 2k - 1))| = |V(W(A; 2k - 1))| - 1 = 10 \cdot 3^{n/2} - 2.
\]

Therefore, the following result follows:

**Remark 4.2.1.** For any odd integer \( n \) \((n \geq 3)\), \( H_n \) is \((10 \cdot 3^{n/2} - 2)\)-realizable.

Next consider \( H_n \) when \( n = 2k \) \((k \geq 2)\). Let \( A \) denote the tree of order 7 shown in Fig. 4.2.2 (b). Let \( a_0, b_0, d_0 \in V(W(A; 1)) \) be in the same component of \( W(A; 1) - c_1 \) so that \( d_{W(A; 1)}(a_0, c_1) = d_{W(A; 1)}(b_0, c_1) = d_{W(A; 1)}(d_0, c_1) = 2 \). For \( i \geq 1 \), let \( a_i \) be the vertex of \( W(A; i) \), defined as in the case when \( n = 2k + 1 \). In \( W(A; 2k - 3) \) consider the set \( \{ a_0, b_0, d_0, a_1, \ldots, a_{2k-3} \} \) as the set of candidates. Then by Theorem 4.1.2, it is straightforward, but tedious, to check that the tree \( W(A; 2k - 3) \) realizes a digraph isomorphic to \( H_n = H_{2k} \). Also, by observation (3),

\[
|V(W(A; 2k - 3))| = 2 \cdot 3^{n/4} (|V(B)| + 1) - 1
\]

\[
= 2 \cdot 3^{n/4} \cdot 8 - 1 = 16 \cdot 3^{n/4} - 1.
\]
So, $H_n$ is $(16 \cdot 3^{n/4} - 1)$-realizable. By observation (1) above, the tree $W^*(A; 2k - 3)$ obtained from $W(A; 2k - 3)$ by replacing the path of length two which contains centroid vertex $c_{k-1}$ as interior vertex with a single edge connecting the two ends of that path is also a tree that realizes $H_n$. Moreover,

$$|V(W^*(A; 2k - 3))| - 1 = 16 \cdot 3^{n/4} - 2.$$ 

**Remark 4.2.2.** For any even integer $n$ ($n \geq 4$), $H_n$ is $(16 \cdot 3^{n/4} - 2)$-realizable.

An obvious observation from Theorem 4.1.2 is the following:

**Remark 4.2.3.** Let $D$ be a disconnected digraph. If $D$ is realizable by a tree $T$, then all vertices of $D$ in $T$ have the same distance to $Cd(T)$. Moreover, if $Cd(T) = \{c\}$ and each component of $T - c$ contains a vertex of $D$, then all components of $T - c$ have the same order.

**Lemma 4.2.1.** If $D$ is $m$-realizable by a tree $T$, then $D$ is $(m+1)$-realizable by a tree which contains exactly one centroid vertex.

**Proof:** If $T$ contains a single centroid vertex $c$, then let $T^*$ be the tree obtained from $T$ by adding a new vertex to $c$. Then $T^*$ still realizes $D$ and $|V(T^*)| = |V(T)| + 1 = m + 1$. So, we may assume that $T$ contains two centroid vertices $c_1$ and $c_2$. It is well known that $c_1 c_2$ is an edge of $T$ (see [13]), and that $T - c_1 c_2$ contains two components of the same order. Let $T'$ denote the tree obtained from $T$ by deleting the edge $c_1 c_2$ and adjoining two new edges $c' c_1$ and $c' c_2$, where $c'$ is a new vertex. Then $c'$ is the only centroid vertex of $T'$, $T'$ still realizes $D$ (by Theorem 4.1.2), and $|V(T')| = |V(T)| + 1 = m + 1$. $\blacksquare$
**Lemma 4.2.2.** Let $D$ be a disconnected digraph with components $D_1, D_2, \ldots, D_k$. If $T$ is a tree realizing $D$, then for some centroid vertex $c$ and for each $i$ ($1 \leq i \leq k$), there exists a component $C_i$ of $T - c$ containing all vertices of $D_i$ and so that $|V(C_i)| \leq \frac{1}{2}|V(T)|$.

**Proof:** Since $D$ is disconnected, by Remark 4.2.3, all vertices of $D$ in $T$ are equal distance to $C_d(T)$.

Case 1: If $|C_d(T)| = 2$, let $C_d(T) = \{c_1, c_2\}$. Note that $T - c_1c_2$ contains exactly two components $T(c_1, c_1c_2)$ and $T(c_2, c_1c_2)$ with $|V(T(c_1, c_1c_2))| = |V(T(c_2, c_1c_2))|$. So, by Theorem 4.1.2, the connectivity of $D_i$ implies that $V(D_i)$ must be contained in one of $T(c_1, c_1c_2)$ and $T(c_2, c_1c_2)$, say $T(c_1, c_1c_2)$. Then $|V(T(c_1, c_1c_2))| = \frac{1}{2}|V(T)|$. Thus, $T(c_1, c_1c_2)$ is as required.

Case 2: If $|C_d(T)| = 1$, let $C_d(T) = \{c\}$. Pick two adjacent vertices $x$ and $y$ in $D_i$ (i.e., $xy \in V(D_i)$ or $yx \in V(D_i)$). Suppose that $T(x, c) \neq T(y, c)$. Then by Theorem 4.1.2,

$$|V(T(x, c))| \neq |V(T(y, c))|. \quad (1)$$

Pick a vertex $z$ in $D_j$ where $j \neq i$. If $T(z, c) = T(x, c)$, it follows from (1) that $|V(T(y, c))| \neq |V(T(z, c))|$. By Theorem 4.1.2 again, $y$ is adjacent to $z$, a contradiction to the fact that $y$ and $z$ are in different components of $D$. If $T(z, c) \neq T(x, c)$, since $z$ is not adjacent to $x$ in $D$, by Theorem 4.1.2, $|V(T(x, c))| = |V(T(z, c))|$. Hence, by (1), $|V(T(y, c))| \neq |V(T(z, c))|$. That is, by Theorem 4.1.2 again, $z$ is adjacent to $y$, again a contradiction. Therefore, $T(x, c) = T(y, c)$. Consequently, any vertex $y$ adjacent to $x$ in $D$ is in $T(x, c)$. Applying the same analysis to each vertex adjacent to $y$, we conclude that any vertex of $D_i$ is in $T(x, c)$ since $D_i$ is connected.
Let $C$ be the component of $T - c$ so that $bw(c) = |V(C)|$ ($C$ might be $T(x,c)$). Since $c$ is the only centroid vertex of $T$, By Lemma 4.1.1,

$$|V(T)| - |V(C)| \geq bw(c) = |V(C)|,$$

i.e., $2|V(C)| \leq |V(T)|$. But $2|V(T(x,c))| \leq 2|V(C)|$. So, $|V(T(x,c))| \leq \frac{1}{2}|V(T)|$. Therefore, $T(x,c)$ is as required. 

Lemma 4.2.3. Let $D$ be a disconnected digraph realized by a tree $T$ of the smallest possible order with a single centroid vertex $c$. Then each component of $T - c$ contains a vertex of $D$.

Proof: Since $D$ is disconnected, by Remark 4.2.3, for any $x, y \in V(D)$, $d_T(x,c) = d_T(y,c)$.

Let $S = \{u : d_T(u,x) = d_T(u,y), \text{for all } x, y \in V(D)\}$. Pick a vertex $w \in S$ so that, for $x \in V(D)$,

$$d_T(x,w) = \min\{d_T(x,u) : u \in S\}.$$ 

By the choice of $w$, at least two components of $T - w$ contain a vertex of $D$. Let $C_1, C_2, \cdots, C_k$ ($k \geq 2$) be all components of $T - w$, each of which contains a vertex of $D$. Note that all vertices of $D$ are in $\bigcup_{i=1}^k V(C_i)$. Since $D$ is disconnected, by Theorem 4.1.2, for any $i, j \in \{1, 2, \cdots, k\}$, $C_i$ and $C_j$ have the same order. The subtree induced by $(\bigcup_{i=1}^k V(C_i)) \cup \{w\}$, denoted by $T^*$, has a single centroid vertex $w$. By Theorem 4.1.2 again, $T^*$ realizes $D$. Since $|V(T^*)| \leq |V(T)|$ and $T$ is of the smallest order, $T^* = T$. Therefore, $w = c$ and hence each component of $T - c$ contains a vertex of $D$. 

Theorem 4.2.1. For any integer \( n \) with \( n \geq 3 \),

\[
\alpha(H_n) = \begin{cases} 
10 \cdot 3^{\frac{n-3}{2}} - 2, & \text{if } n \text{ is odd,} \\
16 \cdot 3^{\frac{n-4}{2}} - 2, & \text{if } n \text{ is even.}
\end{cases}
\]

Proof: Assume that \( H_n \) is labeled as before.

(1) Suppose that \( n \) is odd. Let \( T_1(H_n) \) denote the set of trees which realize \( H_n \) and contain exactly one centroid vertex. Pick a tree \( T^{(1)}_n \in T_1(H_n) \) so that

\[
|V(T^{(1)}_n)| = \min\{|V(T)| : T \in T_1(H_n)\}.
\]

Let \( c_n \) be the centroid vertex of \( T^{(1)}_n \). By Lemma 4.2.1, to show \( \alpha(H_n) \geq 10 \cdot 3^{\frac{n-3}{2}} - 2 \), it suffices to show

\[
|V(T^{(1)}_n)| \geq 10 \cdot 3^{\frac{n-3}{2}} - 1.
\]

This is done by induction on odd \( n \).

If \( n = 3 \), it is easy to check that there does not exist a tree of order less than 9, with a single centroid vertex, which realizes \( H_n \). Thus, \( |V(T^{(1)}_3)| \geq 9 \).

Suppose that the result is true for \( H_{n-2} \). Note that \( H_n \) is disconnected with exactly two components \( \{n\} \) and \( \{n-1\} \Rightarrow H_{n-2} \). So, by Lemmas 4.2.2 and 4.2.3, \( T^{(1)}_n - c_n \) contains exactly two components \( T(n, c_n) \) and \( T(n-1, c_n) \). Since there are no arcs between \( n \) and \( n-1 \), by Theorem 4.1.2, \( |V(T(n, c_n))| = |V(T(n-1, c_n))| \). Note that \( T(n-1, c_n) \) is a tree realizing the digraph \( H_n - n \). Let \( c_{n-1} \) be the vertex in \( T(n-1, c_n) \) adjacent to \( c_n \).

Then \( T(n-1, c_n) - c_{n-1} \) contains at least two components each of which contains a vertex of \( D \). For otherwise, let \( w \) and \( T^* \) be the vertex and the subtree of \( T(n-1, c_n) \) defined as in the proof of Lemma 4.2.3. Let \( T \) be the tree obtained from \( T^* \) and a copy of \( T^* \) by adding a new vertex \( v \) adjacent to
Take a vertex in the copy of $T^*$, which is a copy of a vertex of $D$, as the vertex $n$. Then the resulting tree $T$ realizes $D$ and contains a single centroid vertex $v$. But $|V(T)| < |V(T_n^{(1)})|$. This contradicts the minimality of $|V(T_n^{(1)})|$. 

Let $C(i)$ be the component of $T(n - 1, c_n) - c_{n-1}$ containing the vertex $i$ of $H_n - n$. Since $T(n - 1, c_n) - c_{n-1}$ contains at least two components each of which contains a vertex of $D$, there is a vertex $j \in \{1, 2, \ldots, n - 2\}$ so that $C(j) \neq C(n - 1)$. Note that there is no arc between $n - 2$ and $j$. So, by Theorem 4.1.2 and the fact that $C(j)$ is not equal to $C(n - 1)$, $C(n - 2)$ is not equal to $C(n - 1)$.

Case 1: If $C(n - 3) = C(n - 2)$, then since for any $i \in \{1, 2, \ldots, n - 4\}$, $(n - 3)i \in A(H_n)$, but neither $(n - 2)i$ nor $i(n - 2)$ is in $A(H_n)$, $C(i) = C(n - 2)$. Thus, $T(n - 1, c_n) - c_{n-1}$ contains exactly two components $C(n - 1)$ and $C(n - 2)$. (see Fig. 4.2.3 (a)). By the minimality of $|V(T_n^{(1)})|$, 

$$|V(C(n - 1))| = |V(C(n - 2))| + 1.$$
smallest order in $T_1(H_{n-2})$. So,

$$|V(T_n^{(1)})| = 2|V(T(n-1,c_n))| + 1$$

$$\geq 2(2|V(C(n-2))| + 2) + 1$$

$$\geq 4\alpha(H_{n-2}) + 5$$

$$\geq 4(|V(T_{n-2}^{(1)})| - 1) + 5$$

$$\geq 3|V(T_{n-2}^{(1)})| + 2.$$ 

Case 2: If $n - 3 \notin V(C(n - 2))$, then neither $(n - 2)(n - 3)$ nor $(n - 3)(n - 2)$ in $A(H_n)$ implies that $|V(C(n - 2))| = |V(C(n - 3))|$. Since $|V(C(n - 1))| > |V(C(n - 2))|$, $n - 3$ is not in $C(n - 1)$. Note that the vertex $n - 2$ is not adjacent to $i$, but $(n - 3)i \in A(H_n)$, for any $i \in \{1, 2, \cdots, n - 4\}$. So, by Theorem 4.1.2, $i \in C(n - 3)$, for any $i \in \{1, 2, \cdots, n - 4\}$. It follows that $T(n - 1, c_n) - c_{n-1}$ contains exactly three components $C(n - 1)$, $C(n - 2)$, and $C(n - 3)$ (see Fig. 4.2.3 (b)). By the minimality of $|V(T_n^{(1)})|$, 

$$|V(C(n - 1))| = |V(C(n - 2))| + 1 = |V(C(n - 3))| + 1.$$ 

Note that the tree $T'_{n-2}$ obtained from $T(n - 1, c_n)$ by deleting $C(n - 1)$ is in $T_1(H_{n-2})$. It is easy to see that

$$|V(T_n^{(1)})| \geq 2||V(T_{n-2}')| + \frac{1}{2}(|V(T_{n-2}')| + 1)| + 1$$

$$= 3|V(T_{n-2}')| + 2$$

$$\geq 3|V(T_{n-2}^{(1)})| + 2,$$

where $T_{n-2}'$ is a tree of smallest order in $T_1(H_{n-2})$.

Hence, each case yields $|V(T_n^{(1)})| \geq 3|V(T_{n-2}^{(1)})| + 2$. By induction hypothesis,

$$|V(T_n^{(1)})| \geq (3(10 \cdot 3^{\frac{n - 3}{2} - 3} - 1) + 2$$

$$= 10 \cdot 3^{\frac{n - 3}{2}} - 3 + 2$$

$$= 10 \cdot 3^{\frac{n - 3}{2}} - 1.$$
Therefore, \( \alpha(H_n) \geq 10 \cdot 3^{\frac{n-3}{1}} - 2 \). By Remark 4.2.1,

\[
\alpha(H_n) = 10 \cdot 3^{\frac{n-3}{1}} - 2.
\]

(2) Similar analysis can be applied for the case when \( n \) is even. ■

**Lemma 4.2.4.** Let \( D = D_1 \cup D_2 \) be realizable by a tree. Then

\[
\alpha(D) \leq \begin{cases} 
\max_{1 \leq i \leq 2} \{3\alpha(D_i) + 3\}, & \text{if } D_i \text{ is disconnected} \\
\max_{1 \leq i \leq 2} \left\{ \frac{3}{2}\alpha(D_i \cup \{x_i\}) \right\}, & \text{if } D_i \text{ is connected} \\
\max \{3\alpha(D_i) + 3, \frac{3}{2}\alpha(D_j \cup \{x_j\})\}, & \text{if } D_i \text{ is disconnected} \\
& \text{and } D_j \text{ is connected,} \\
& 1 \leq i, j \leq 2, i \neq j,
\end{cases}
\]

where \( x_i \) is a vertex not in \( V(D) \), for \( i = 1, 2 \).

**Proof:** Case 1: Suppose that \( D_i \) is disconnected for \( i = 1, 2 \). Let \( T_i \) be a tree of order \( \alpha(D_i) + 1 \) with a single centroid vertex \( c_i \) so that \( T_i \) realizes \( D_i \). By Remark 4.2.3, all vertices of \( D_i \) are at equal distance to \( c_i \). Denote this distance by \( d_i \).

For \( i = 1, 2 \), let \( k = 2 \), if \( i = 1 \), and \( k = 1 \), if \( i = 2 \). Let \( T \) be the tree obtained from \( T_1 \) and \( T_2 \) by joining \( c_1 \) to \( c_2 \) by a path of length \( |d_2 - d_1| + 1 \) and adding \( \alpha \) vertices and \( \beta \) vertices at \( c_i \) and \( c_k \), respectively, where

\[
\alpha = \begin{cases} 
\left\lfloor \frac{1}{3} |V(T_k)| - |V(T_i)| - (d_k - d_i) \right\rfloor, & \text{if } d_i \leq d_k \& |V(T_i)| \leq |V(T_k)|, \\
\left\lfloor \frac{1}{2} |V(T_k)| - (d_k - d_i) \right\rfloor, & \text{if } d_i \leq d_k \& |V(T_i)| \geq |V(T_k)|
\end{cases}
\]

and

\[
\beta = \begin{cases} 
\left\lfloor \frac{1}{3} |V(T_k)| \right\rfloor, & \text{if } d_i \leq d_k \text{ and } |V(T_i)| \leq |V(T_k)|, \\
|V(T_i)| - \left\lfloor \frac{1}{2} |V(T_k)| \right\rfloor, & \text{if } d_i \leq d_k \text{ and } |V(T_i)| \geq |V(T_k)|.
\end{cases}
\]
The number \[ \lfloor x \rfloor \] (or \( \lceil x \rceil \)) is the least (respectively, greatest) integer greater (respectively, smaller) than or equal to \( x \). The tree shown in Fig. 4.2.4 illustrates the case when \( d_1 \leq d_2 \) and \( |V(T_1)| \leq |V(T_2)| \).

Let \( c \) be the vertex on the path joining \( c_1 \) to \( c_2 \) which is adjacent to \( c_k \). Then by the construction of \( T \), \( c \) and \( c_k \) are centroid vertices of \( T \). Also, all vertices of \( D = D_1 \cup D_2 \) are at equal distance to \( C_d(T) \). By Theorem 4.1.2, it can be verified that \( T \) realizes \( D \). Moreover, by calculating \( |V(T)| \) in each case,

\[
\alpha(D) \leq |V(T)| \leq \max_{1 \leq i \leq 2} \{3|V(T_i)|\}
\]

\[
= \max_{1 \leq i \leq 2} \{3(\alpha(D_i) + 1)\}
\]

\[
= \max_{1 \leq i \leq 2} \{3\alpha(D_i) + 3\}.
\]

Case 2: Suppose that \( D_i \) is connected for \( i = 1, 2 \). Let \( T^* \) be a tree of order \( \alpha(D_i \cup \{z_i\}) \) so that \( T^*_i \) realizes \( D_i \cup \{z_i\} \), where \( z_i \not\in V(D) \). By Lemma 4.2.2 applied to \( D_i \cup \{z_i\} \), there exists a component \( T_i \) of \( T^*_i - c_i^* \) containing all vertices of \( D_i \), where \( c_i^* \in C_d(T^*_i) \). Moreover,

\[
|V(T_i)| \leq \frac{1}{2}|V(T^*_i)|.
\]  

(2)

Let \( c_i \in V(T_i) \) be adjacent to \( c_i^* \) in \( T^*_i \). Let \( T \) be the tree constructed as in

---

Fig. 4.2.4
Case 1. Then \( T \) realizes \( D \). By (2),
\[
\alpha(D) \leq |V(T)| \leq \max_{1 \leq i \leq 2} \{3|V(T_i)|\}
\]
\[
\leq \max_{1 \leq i \leq 2} \{ \frac{3}{2} |V(T_i^*)| \}
\]
\[
= \max_{1 \leq i \leq 2} \{ \frac{3}{2} \alpha(D_i \cup \{x_i\}) \}.
\]

Case 3: Suppose that only one of \( D_1 \) and \( D_2 \) is connected. Combining the two cases above, we can obtain the required result.

This completes the proof. \( \blacksquare \)

Note that the tree \( T' \) obtained from the tree \( T \) constructed above by adding a new vertex to \( c_1 \) realizes the digraph \( D = D_1 \Rightarrow D_2 \). Thus, the next Remark 4.2.4 follows immediately.

**Remark 4.2.4.** Let \( D = D_1 \Rightarrow D_2 \) be realizable by a tree. Then
\[
\alpha(D) \leq \begin{cases} 
\max_{1 \leq i \leq 2} \{3\alpha(D_i) + 4\}, & \text{if } D_i \text{ is disconnected} \\
\max_{1 \leq i \leq 2} \{ \frac{3}{2} \alpha(D_i \cup \{x_i\}) + 1\}, & \text{if } D_i \text{ is connected} \\
\max\{3\alpha(D_i) + 4, \frac{3}{2} \alpha(D_j \cup \{x_j\}) + 1\}, & \text{if } D_i \text{ is disconnected and } D_j \text{ is connected}, i \neq j,
\end{cases}
\]
where \( x_i \) is a vertex not in \( V(D) \), for \( i = 1, 2 \).

**Lemma 4.2.5.** Let \( D = \{u\} \cup D_2 \) (respectively \( D = \{u\} \Rightarrow D_2, D = D_2 \Rightarrow \{u\} \)), where \( D_2 \) is a disconnected digraph. If \( T_2 \) is a tree with a single centroid vertex \( c \), which realizes \( D_2 \) and is of the smallest order, then there exists a tree \( T \) which realizes \( D \) and is of order
\[
|V(T)| \leq \frac{k+1}{k} |V(T_2)| - \frac{1}{k}
\]
(respectively, \(|V(T)| \leq \frac{k+1}{k} |V(T_2)| - \frac{1}{k} + 1\),
\[
|V(T)| \leq \frac{k+1}{k} |V(T_2)| - \frac{1}{k} - 1),
\]
where $k$ is the number of components of $T_2 - c$.

**Proof:** Let $C_1, C_2, \ldots, C_k$ be the components of $T_2 - c$ and let $c_i$ be the vertex of $C_i$ adjacent to $c$ ($1 \leq i \leq k$). By Remark 4.2.3, each component of $T_2 - c$ contains at least one vertex of $D_2$. Moreover, the disconnectedness of $D_2$ implies, by Theorem 4.1.2, that any two components of $T_2 - c$ have the same order. Hence, $|V(C_i)| = \frac{1}{k}(|V(T_2)| - 1) (1 \leq i \leq k)$. Let $T$ be the tree obtained from $T_2$ and a copy of $C_1$ by adding an edge connecting $c$ to the copy of $c_1$. This second copy of $C_1$ in $T$ is denoted by $C_{k+1}$ (see Fig. 4.2.5).

Denote a vertex of $C_{k+1}$ which is a copy of a vertex of $D_2$ by $u$. Consider the vertices of $T_2$ which represent vertices of $D_2$, together with $u$, as the set of candidates, and consider $V(T)$ as the set of voters. Then by Theorem 4.1.2 again, $T$ realizes $D = \{u\} \cup D_2$. Clearly,

$$|V(T)| \leq |V(T_2)| + \frac{1}{k}(|V(T_2)| - 1) \leq \frac{k+1}{k} |V(T_2)| - \frac{1}{k}.$$

![Fig. 4.2.5](image)

If $D = \{u\} \Rightarrow D_2$ (respectively, $D = D_2 \Rightarrow \{u\}$), let $T$ be the tree obtained from the tree constructed above by adding a new vertex adjacent to the vertex of $C_{k+1}$ which is a copy of $c_1$ (respectively, deleting an end vertex of $C_{k+1}$). Hence, $T$ is as required.

Every tree constructed in the previous lemmas contains a single centroid vertex. So, one can always assume that such a tree contains a single centroid vertex whenever those lemmas are used.
**Theorem 4.2.2.** For any $D \in \mathcal{F}_n$,

$$
\alpha(D) \leq \begin{cases} 
16 \cdot 3^{\frac{n-4}{2}} - 2, & \text{if } n \text{ is even} \\
10 \cdot 3^{\frac{n-3}{2}} - 2, & \text{if } n \text{ is odd.}
\end{cases}
$$

**Proof:** The proof is by induction on $n$.

It is easy to check Table 1 to see that the inequality holds for $n = 1, 2, 3, 4$. Suppose that the result is true for any $D \in \mathcal{F}_k$ ($4 \leq k \leq n - 1$). Let $D \in \mathcal{F}_n$. By Theorems 4.1.1 and 4.1.3, $D$ is transitive and contains no anti-directed path of length 3.

First of all, consider the case when $n$ is even. Note that $n \geq 6$.

**Case A:** Suppose that $D$ is connected. By Lemma 4.1.7, $D = D_2 \Rightarrow D_1$ for some subdigraphs $D_1$ and $D_2$ of $D$. Let $n_i = |V(D_i)|$, $i = 1, 2$.

**Subcase A.1:** Suppose that $n_i \geq 2$ ($i = 1, 2$). Let $T$ be a tree realizing $D$ so that $|V(T)| = \alpha(D)$. By Remark 4.2.4,

$$
\alpha(D) \leq \begin{cases} 
\max \{3\alpha(D_1) + 4\}, & \text{if } D_i \text{ is disconnected} \\
\max \{\frac{3}{2}\alpha(D_i \cup \{x_i\}) + 1\}, & \text{if } D_i \text{ is connected}
\end{cases} 
$$

where $x_i$ is a vertex not in $V(D)$, for $i = 1, 2$. Note that the function $f(x) = 3^x$ (or $g(x) = 3^{-x}$) is increasing (respectively, decreasing). By induction hypothesis, for $i = 1, 2$,

$$
3\alpha(D_i) + 3 \leq \begin{cases} 
3(16 \cdot 3^{\frac{n_i-4}{2}} - 2) + 4, & \text{if } n_i \text{ is even,} \\
3(10 \cdot 3^{\frac{n_i-3}{2}} - 2) + 4, & \text{if } n_i \text{ is odd,}
\end{cases}
$$

$$
\leq \begin{cases} 
3 \cdot 16 \cdot 3^{\frac{n_i-3}{2}} - 6 + 4, & \text{if } n_i \text{ is even,} \\
3 \cdot 10 \cdot 3^{\frac{n_i-3}{2}} - 6 + 4, & \text{if } n_i \text{ is odd,}
\end{cases}
$$

$$
\leq 16 \cdot 3^{\frac{n_i-4}{2}} - 2.
$$
Subcase A.2: Suppose that \( r_t = 1 \). Then \( n - n_j = n - 1 \) is odd. By Lemma 4.2.1, \( D_2 \) is \((\alpha(D_2) + 1)\)-realizable by a tree, say \( T_2 \), which contains a single centroid vertex \( c \). Let \( T \) be the tree obtained from \( T_2 \) by adding two vertices \( u \) and \( v \) adjacent to \( c \) and to a vertex \( x \) furthest away from \( c \), respectively. Consider the set \( V(D_2) \cup \{v\} \) as the set of candidates, and consider the set \( V'(T') \) as the set of voters, then \( T \) realizes \( D \). So, by the induction hypothesis,

\[
\frac{3}{2} \alpha(D \cup \{x_i\}) + 1 \leq \begin{cases} 
\frac{3}{2}(16 \cdot 3^{\frac{n_i+1}{2}} - 2) + 1, & \text{if } n_i + 1 \text{ is even,} \\
\frac{3}{2}(10 \cdot 3^{\frac{n_i+1}{2}} - 2) + 1, & \text{if } n_i + 1 \text{ is odd,} \\
\frac{3}{2} \cdot 16 \cdot 3^{\frac{n^2+1}{2}} - 3 + 1, & \text{if } n_i + 1 \text{ is even,} \\
3 \cdot 5 \cdot 3^{\frac{n^2+1}{2}} - 3 + 1, & \text{if } n_i + 1 \text{ is odd,} \\
< 16 \cdot 3^{\frac{n^2}{2}} - 2. 
\end{cases}
\]

Thus, \( |V(T)| \leq 16 \cdot 3^{\frac{n^2}{2}} - 2 \).

Subcase A.3: Suppose that \( n_1 = 1 \). Then \( n - n_1 = n - 1 \) is odd. By Lemma 4.2.1, \( D_1 \) is \((\alpha(D_1) + 1)\)-realizable by a tree, say \( T_2 \), which contains a single centroid vertex \( c \). Let \( T \) be a tree realizing \( T_2 \) and consider the centroid vertex of \( T_2 \) as the only vertex of \( D_2 \) (a candidate). Then \( T \) realizes \( D \) and

\[
|V(T)| = |V(T)| \leq 10 \cdot 3^{\frac{n-1-3}{2}} - 2 < 16 \cdot 3^{\frac{n-4}{2}} - 2.
\]

So, if \( D \) is connected, then \( \alpha(D) < 16 \cdot 3^{\frac{n-4}{2}} - 2 \) when \( n \) is even.

Now we consider the case when \( D \) is disconnected.
Case B: Suppose that $D$ is disconnected. Assume that $D = D_1 \cup D_2$, where $D_1$ is connected. Let $n_i = |V(D_i)|$ ($i = 1, 2$).

Subcase B.1: Suppose that $n_i \geq 2$ ($i = 1, 2$). Let $T$ be a tree realizing $D$ so that $|V(T)| = \alpha(D)$. By Lemma 4.2.4,

$$\alpha(D) \leq \begin{cases} 
\max \{3\alpha(D_i) + 3\}, & \text{if } D_i \text{ is disconnected} \\
\max \left\{ \frac{3}{2} \alpha(D_i \cup \{x_i\}) \right\}, & \text{if } D_i \text{ is connected} \\
\max \{3\alpha(D_i) + 3, \frac{3}{2} \alpha(D_j \cup \{x_j\})\}, & \text{if } D_i \text{ is disconnected and } D_j \text{ is connected}, \\
1 \leq i, j \leq 2, \ i \neq j,
\end{cases}$$

where $x_i$ is a vertex not in $V(D)$, for $i = 1, 2$. A computation very similar to that done in Subcase A.1 yields $|V(T)| \leq 16 \cdot 3^{\frac{n-4}{2}} - 2$.

Subcase B.2: Suppose that $n_1 = 1$.

If $D_2$ is disconnected, let $T_2$ be a tree realizing $D_2$ so that $T_2$ contains a single centroid vertex and is of the smallest order. Then by Lemma 4.2.5, there exists a tree $T$ realizing $D$ and for some integer $k \geq 2$,

$$|V(T)| \leq \frac{k + 1}{k} |V(T_2)| - \frac{1}{k}.$$

By the induction hypothesis,

$$|V(T)| \leq \frac{k + 1}{k} \left(10 \cdot 3^{\frac{n-4}{2}} - 1\right) - \frac{1}{k}$$

$$\leq \frac{k + 1}{k} \cdot 10 \cdot 3^{\frac{n-4}{2}} - 1 - \frac{2}{k}$$

$$< 16 \cdot 3^{\frac{n-4}{2}} - 2.$$

If $D_2$ is connected, then by Lemma 4.1.7, $D_2 = D_{22} \Rightarrow D_{21}$ for some subdigraphs $D_{21}$ and $D_{22}$ of $D_2$. Let $\alpha_i = |V(D_{2i})|$. Note that $\alpha_2 = n - 1 - \alpha_1$. 
Subsubcase B.2.1: Suppose that $\alpha_i \geq 3$ ($i = 1, 2$). Let $T_2$ be a tree of order $\alpha(D_2)$ which realizes $D_2$. Then by Remark 4.2.4, $\alpha(D_2)$ is less than or equal to
\[
\max_{1 \leq i \leq 2} \left\{ \begin{array}{ll}
3\alpha(D_{2i}) + 4, & \text{if } D_{2i} \text{ is disconnected} \\
\frac{3}{2} \alpha(D_{2i} \cup \{x_i\}) + 1, & \text{if } D_{2i} \text{ is connected}
\end{array} \right.
\]
\[
= \max_{1 \leq i \leq 2} \left\{ \begin{array}{ll}
3\alpha(D_{2i}) + 4, & \text{if } D_{2i} \text{ is disconnected} \\
\frac{3}{2} \alpha(D_{2i} \cup \{x_i\}) + 1, & \text{if } D_{2i} \text{ is connected}
\end{array} \right.
\]
\[
\text{if } D_{2i} \text{ is disconnected and } D_{2j} \text{ is connected,}
\]
\[
1 \leq i, j \leq 2, \ i \neq j,
\]
where $x_i \not\in V(D)$, for $i = 1, 2$.

Note that $3 \leq \alpha_i = |V(D_{2i})| \leq n - 4$. So, a computation very similar to that done in Subcase A.1 gives the following inequalities:

\[
3\alpha(D_{2i}) + 4 \leq \left\{ \begin{array}{ll}
3 \cdot 16 \cdot 3^{\frac{n-i}{2}} - 2, & \text{if } \alpha_i \text{ is even,} \\
3 \cdot 10 \cdot 3^{\frac{n-i}{2}} - 2, & \text{if } \alpha_i \text{ is odd,}
\end{array} \right.
\]
\[
= \left\{ \begin{array}{ll}
\frac{1}{3} \cdot 16 \cdot 3^{\frac{n-i}{2}} - 2, & \text{if } \alpha_i \text{ is even,} \\
\frac{1}{3} \cdot 10 \cdot 3^{\frac{n-i}{2}} - 2, & \text{if } \alpha_i \text{ is odd,}
\end{array} \right.
\]

and

\[
\frac{3}{2} \alpha(D_{2i} \cup \{x_i\}) + 1 \leq \left\{ \begin{array}{ll}
\frac{3}{2} \cdot 16 \cdot 3^{\frac{n-i}{2}} - 2, & \text{if } \alpha_i \text{ is odd,} \\
\frac{3}{2} \cdot 10 \cdot 3^{\frac{n-i}{2}} - 2, & \text{if } \alpha_i \text{ is even,}
\end{array} \right.
\]
\[
= \left\{ \begin{array}{ll}
\frac{1}{2} \cdot 16 \cdot 3^{\frac{n-i}{2}} - 2, & \text{if } \alpha_i \text{ is odd,} \\
\frac{1}{2} \cdot 10 \cdot 3^{\frac{n-i}{2}} - 2, & \text{if } \alpha_i \text{ is even}
\end{array} \right.
\]

Hence,

\[
|V(T_2)| < \frac{1}{2} \left(16 \cdot 3^{\frac{n-i}{2}} - 2\right). \quad (3)
\]

As mentioned before this theorem, $T_2$ contains only one centroid vertex, denoted $c_2$. Let $T$ be the tree obtained from $T_2$ and a copy of $T_2$ by joining
c_2 and a copy of c_3 by an edge. Then take all vertices of D_2 in T_2 and a copy of a vertex of D_2 in the copy of T_2 as the set of candidates in T, and take the vertex set of T as the set of voters, then by Theorem 4.1.2, T realizes D. Also |V(T)| = 2|V(T_2)|. By (3),

|V(T)| < 16 \cdot 3^{n/4} - 2.

Subsubcase B.2.2: Suppose that \( \alpha_1 = 2 \) (or \( \alpha_2 = 2 \)). Then \( \alpha_2 = n - 1 - \alpha_1 = n - 3 \) is odd and greater than or equal to 3. Let \( D_{22}^* = D_{22} \cup \{z\} \), where \( z \notin V(D) \). Let \( T_{22}^* \) be a tree, of order \( \alpha(D_{22}^*) \), which realizes \( D_{22}^* \). By Lemma 4.2.2, there exists a component \( T_{22} \) of \( T_{22}^* - c \), for some \( c \in \mathcal{C}(T_{22}^*) \), so that all vertices of \( D_{22} \) are in \( T_{22} \) and \( |V(T_{22})| \leq \frac{1}{2}|V(T_{22}^*)| \). Let \( v \) be the vertex in \( V(T_{22}) \) adjacent to \( c \). Remark 4.2.3 implies that all vertices of \( D_{22} \) in \( T_{22} \) are at equal distance to \( v \). Let \( T_2 \) be the tree obtained from two vertex-disjoint copies of \( T_{22} \) by adding a new vertex \( u \) adjacent to both copies of the vertex \( v \) and adding another new vertex \( w \) adjacent to one of copies of \( v \). As the set of candidates in \( T_2 \), choose two vertices of \( D_{22} \) in the copy of \( T_{22} \) to which \( w \) is not added, together with a copy of \( V(D_{22}) \) in the other copy of \( T_{22} \). Take \( V(T_2) \) as the set of voters in \( T_2 \). Then by Theorem 4.1.2, \( T_2 \) realizes \( D_2 \). Let \( T \) be the tree obtained from two vertex-disjoint copies of \( T_2 \) by adding a new edge connecting two copies of \( u \). Then \( T \) realizes \( D \). Moreover, by the induction hypothesis,

\[
|V(T)| = 2|V(T_2)| = 2(2|V(T_{22})| + 2) \leq 2|V(T_{22}^*)| + 4 \\
\leq 2(16 \cdot 3^{\frac{n-3}{4}+1-4} - 2) + 4 \\
< 16 \cdot 3^{\frac{n-4}{4}} - 2.
\]

Subsubcase B.2.3: Suppose that \( \alpha_2 = 1 \) (or \( \alpha_1 = 1 \)). Then \( \alpha_1 = n - 1 - \alpha_2 = n - 2 \) is even. Let \( V(D_{22}) = \{z\} \). If \( D_{21} \) is connected, then by Lemma 4.1.7,
\(D_{21} = Y \Rightarrow X\), for some subdigraphs \(X\) and \(Y\). So, \(D_2 = (Y \cup \{z\}) \Rightarrow X\), and then this case is contained in the previous cases. Therefore, without loss of generality, assume that \(D_{21}\) is disconnected. Let \(T_{21}\) be a tree, of the smallest order with a single centroid vertex, which realizes \(D_{21}\). By Lemma 4.2.5, there exists a tree \(T_2\) realizing \(D_2\) so that for some integer \(k \geq 2\),

\[
|V(T_2)| \leq \frac{k + 1}{k} |V(T_{21})| - \frac{1}{k} + 1. \tag{4}
\]

Note that \(T_2\) contains a single centroid vertex, say \(c\). Let \(T\) be the tree obtained from \(T_2\) and a copy of \(T_2\) by adding an edge joining \(c\) and its copy. As the set of candidates in \(T\), choose one vertex of \(D_2\) in one copy of \(T_2\), together with a copy of \(V(D_2)\) in the other copy of \(T_2\). Take \(V(T)\) as the set of voters. Then by Theorem 4.1.2, \(T\) realizes \(D\). By the induction hypothesis and (4),

\[
|V(T)| = 2|V(T_2)|
\leq \frac{2(k + 1)}{k} |V(T_{21})| - \frac{2(1 - k)}{k}
\leq \frac{2(k + 1)}{k} (16 \cdot 3^{\frac{n-4}{2}} - 1) - \frac{2(1 - k)}{k}
\leq \frac{2(k + 1)}{3k} \cdot 16 \cdot 3^{\frac{n-4}{2}} - \frac{4}{k}
\leq 16 \cdot 3^{\frac{n-4}{2}} - 2.
\]

This completes the proof that \(\alpha(D) \leq 16 \cdot 3^{\frac{n-4}{2}} - 2\) if \(n\) is even.

Arguments similar to those in the case when \(n\) is even can be used in the case when \(n\) is odd. That is, if \(n\) is odd, then

\[
\alpha(D) \leq 10 \cdot 3^{\frac{n-3}{2}} - 2.
\]

The proof is complete. \(\blacksquare\)
Theorem 4.2.3. For any positive integer $n$,

$$\alpha(F_n) = \begin{cases} 
1, & \text{if } n = 1 \\
3, & \text{if } n = 2 \\
10 \cdot 3^{\frac{n-3}{2}} - 2, & \text{if } n \text{ is odd and } n \geq 3 \\
16 \cdot 3^{\frac{n-4}{2}} - 2, & \text{if } n \text{ is even and } n \geq 4.
\end{cases}$$

Proof: It is easy to check that $\alpha(F_1) = 1$ and $\alpha(F_2) = 3$. For $n \geq 3$, the result follows from Remarks 4.2.1, 4.2.2, and Theorem 4.2.2.

§3. Results on digraphs which are $(n,n,n)$-realizable by trees

In this section, we assume that $D$ is an oriented graph which is $(n,n,n)$-realizable by a tree $T$. Unless explicitly stated, $V(D)$ is used as the vertex set of $D$ as well as the set of candidates in $T$.

Let $D$ be a digraph. If $xy \in A(D)$, then we say that $x$ dominates $y$.

Theorem 4.3.1. If an oriented graph $D$ is $(n,n,n)$-realizable by a tree $T$, then each vertex of $D$ corresponding to a centroid vertex of $T$ dominates every vertex in $V(D) \setminus C_d(T)$.

Proof: Pick a centroid vertex $x$ in $T$. Denote the components of $T - x$ by $C_1, C_2, \ldots, C_s$. Without loss of generality, let $b(x) = |V(C_s)|$. Note that $C_d(T) \cap V(C_i) = \emptyset$ ($1 \leq i \leq s - 1$).

For any $y \in V(C_i)$ ($1 \leq i \leq s - 1$), $V_{x,y} \supseteq V(C_s) \cup \{x\}$ and $V_{y,x} \subseteq V(C_i)$. So $|V_{x,y}| \geq |V(C_s)| + 1 = b(x) + 1$ and $|V_{y,x}| \leq |V(C_i)| \leq b(x)$. Thus, $|V_{x,y}| - |V_{y,x}| \geq b(x) + 1 - b(x) > 0$. It follows from Theorem 4.1.2 that $xy \in A(D)$.

For any $y \in V(C_s) \setminus C_d(T)$, $V_{x,y} \supseteq V(T) \setminus V(C_s)$ and $V_{y,x} \subseteq V(C_s)$. Therefore, by Lemma 4.1.1,

$$|V_{x,y}| \geq \sum_{i=1}^{s-1} |V(C_i)| + 1 \geq b(x) \geq |V(C_s)| \geq |V_{y,x}|.$$
If $|V_{x,y}| = |V_y,z|$, then $|V(C_x)| = |V_y,z|$. This implies that $y$ is adjacent to $z$ in $T$ and hence, by Lemma 4.1.1, $y \in C_d(T)$, a contradiction to the choice of $y$. Thus, $|V_{x,y}| > |V_y,z|$, and so by Theorem 4.1.2, $xy \in A(D)$. This completes the proof. \[ \]

**Corollary.** Let $D$ be an oriented graph whose underlying graph is not a star. If $D$ is $(n,n,n)$-realizable by a tree, then for each vertex $v$ in $D$, $D - v$ is connected.

**Proof:** Suppose that $D$ is $(n,n,n)$-realizable by a tree $T$. Note that if $T$ contains two centroid vertices, then by Theorem 4.3.1, the result is clearly true. Now suppose that $T$ contains a single centroid vertex $c$. Let $v$ be a vertex of $D$. If $v \neq c$, then by Theorem 4.3.1, $D - v$ is connected. On the other hand, suppose that $v = c$ and $D - v$ is disconnected with components $C_1, C_2, \ldots, C_k$. Then none existence of arcs between $C_i$ and $C_j$ $(1 \leq i, j \leq k; i \neq j)$ implies, by Theorem 4.1.2, that $d_T(v_i, v) = d_T(v_j, v)$, for any $v_i \in V(C_i)$ and for any $v_j \in V(C_j)$. Since $d_T(v_i, v) = 1$ for some $v_i$, $d_T(u, v) = 1$, for all $u \in V(D) \setminus \{v\}$. This means that $T$ is a star. It is easy to verify that a star realizes an oriented graph whose underlying graph is a star, a contradiction to the given condition. Therefore, $D - v$ is connected for any vertex of $D$. \[ \]

For $v \in V(T)$ and non-negative integer $k$ no large than the diameter of $T$, let $B(v; k)$ denote the set $\{u \in V(T) : d_T(v, u) \leq k\}$.

**Theorem 4.3.2.** Let $T$ be a tree of order $n$ with a single centroid vertex $c$. If for every non-negative integer $k$ with $k \leq \text{dia}(T)$, $T - B(c; k)$ consists of subtrees with different orders, then $T$ realizes a transitive tournament of order $n$. \[ \]
Proof: Let \( D \) be \((n,n,n)\)-realizable by the given tree \( T \). Then by Theorem 4.1.1, \( D \) is transitive. We now prove that \( D \) is a tournament, i.e., for any \( x, y \in V(D) \), either \( xy \) or \( yx \) in \( A(D) \).

Let \( x, y \in V(D) \), \( x \neq y \). By Theorem 4.1.2., we may assume that \( d_T(x,c) = d_T(y,c) \). Let \( w \) be the vertex on the shortest path from \( x \) to \( y \) in \( T \) so that \( d_T(x,w) = d_T(y,w) \). Let \( d_T(c,w) = k \) so that \( x \) and \( y \) are in different components of \( T - B(c;k) \); say \( C(x) \) and \( C(y) \), where \( x \in C(x) \) and \( y \in C(y) \). By the assumption, since \( k \leq \text{dia}(T) \), \( |C(x)| \neq |C(y)| \). So, by Theorem 4.1.2, either \( xy \) or \( yx \in A(D) \). Therefore, \( D \) is a transitive tournament of order \( n \).

Corollary. If \( D \) denotes the transitive tournament with \( n \) vertices, then \( D \) is \((n,n,n)\)-realizable by a tree unless \( n = 8 \) or \( 2 \leq n \leq 6 \).

Proof: Of course, the result is trivially true for \( n = 1 \).

We will construct such a tree for \( n \geq 7, n \neq 8 \).

If \( n \geq 10 \) is an even integer, then a path \( P \) of length \( n - 3 \) has two centroid vertices, denoted \( c \) and \( c' \). Append a path of length 2 to \( P \) at a centroid vertex, say \( c \). Denote the resulting tree by \( T \). (See Fig. 4.3.1) Then \( c \) is the only centroid vertex of \( T \). Certainly, this tree \( T \) satisfies the conditions in Theorem 4.3.2, so \( T \) realizes a transitive tournament of order \( n \). That is, \( D \) is \((n,n,n)\)-realizable by a tree.

![Fig. 4.3.1](image-url)
If \( n \geq 7 \) is odd, then append to a path \( Q \) of length \( n - 2 \) a new vertex \( w \) at a centroid vertex of \( Q \). Denote the resulting tree by \( T' \). Again by Theorem 4.3.2, it follows that \( T' \) realizes \( D \).

We now need to prove that \( D \) is not \((n,n,n)\)-realizable by a tree when \( n = 8 \) or \( 2 \leq n \leq 6 \).

For \( n = 8 \), let \( D_8 \) denote the transitive tournament of order 8. Suppose that \( D_8 \) is \((8,8,8)\)-realizable by a tree \( T^* \). If \( T^* \) contains two centroid vertices, then by Lemma 4.1.2 and Theorem 4.1.2, there is no arc between those two centroid vertices, a contradiction to the fact that \( D_8 \) is a tournament. Thus \( T^* \) contains only one centroid vertex, say \( c \). Let \( C_1,C_2, \ldots ,C_s \) be components of \( T^* - c \). By Theorem 4.1.2, no two components have the same order. Thus as \( n = 8, s \leq 3 \). On the other hand, without loss of generality, let \( b(c) = |V(C_s)| \). Then by Lemma 4.1.1, \( \sum_{i=1}^{s-1} |V(C_i)| + 1 \geq b(c) \) and hence \( \sum_{i=1}^{s} |V(C_i)| + 1 \geq 2b(c) \). That is, \( 8 \geq 2b(c) \). So, \( b(c) \leq 4 \). But \( s \leq 3 \) implies \( b(c) = 4 \). Pick a vertex \( v \) in \( C_s \) so that \( v \) is adjacent to \( c \), then \( b(v) = 4 = b(c) \). So \( v \) is also a centroid vertex, a contradiction to the uniqueness of \( c \). So \( D_8 \) is not \((8,8,8)\)-realizable by tree.

One can similarly show that a transitive tournament of order \( n \) (\( 2 \leq n \leq 6 \)) is not \((n,n,n)\)-realizable by a tree.

This completes the proof. \( \square \)

§4. Maximum order of a tournament in a digraph realized by a tree

For fixed positive integers \( n, d \), and \( i \) \((n - 1 \geq d, i = 1,2)\), let \( T^{(i)}(n,d) \) denote the collection of trees of order \( n \) with diameter equal to \( d \) and exactly \( i \) centroid vertices. If for any \( T^{(i)}_{n,d} \in T^{(i)}(n,d) \) we consider \( V(T^{(i)}_{n,d}) \) to be the
set of candidates as well as the set of voters, then $T_{n,d}^{(i)}$ realizes a transitive digraph of order $n$, denoted $D_{n,d}^{(i)}$. Let $D^{(1)}(n,d)$ denote the collection of all digraphs which are $(n,n,n)$-realizable by trees in $T^{(i)}(n,d)$. It is obvious that each $D_{n,d}^{(i)}$ contains some transitive tournament as a subdigraph. For any $D_{n,d}^{(i)} \in D^{(i)}(n,d)$, let $ot(D_{n,d}^{(i)})$ be the maximum order of a tournament contained in $D_{n,d}^{(i)}$. Also, let $f_1(n,d) = \min \{ot(D_{n,d}^{(i)}) : D_{n,d}^{(i)} \in D^{(i)}(n,d)\}$.

In this section we estimate the value of $f_1(n,d)$.

**Lemma 4.4.1.** Let $D$ be a digraph realized by a tree $T$. Suppose that $c$ is a centroid vertex of $T$. Suppose that there exist two vertices $u$ and $v$ in $T$ satisfying the following two conditions:

1. $u$ and $v$ are in two components of $T - c$ with different sizes;
2. $d_T(u, C_d(T)) = d_T(u, c)$ and $d_T(v, C_d(T)) = d_T(v, c)$.

If $P$ denotes the shortest path from $u$ to $v$ in $T$, then the subdigraph induced by $V(P)$ is a tournament in $D$ of order $\ell(P) + 1$.

**Proof:** This follows immediately from Theorem 4.1.2. $\blacksquare$

**Theorem 4.4.1.** For any positive integers $n$ and $d$, $n - 1 \geq d$,

$$\left\lfloor \frac{d}{2} \right\rfloor + 1 \leq f_1(n,d) \leq \left\lceil \frac{d}{2} \right\rceil + 2,$$

where $\left\lceil \frac{d}{2} \right\rceil$ is a least integer larger than or equal to $\frac{d}{2}$.

**Proof:** To show that $f_1(n,d) \geq \left\lceil \frac{d}{2} \right\rceil + 1$, pick a digraph $D \in D^{(1)}(n,d)$ which is realized by a tree $T \in T^{(1)}(n,d)$. Let $P$ be a diametrical path in $T$ between two vertices, say $x$ and $y$. Let $c$ be the centroid vertex of $T$. Without loss of generality, let $d_T(x,c) \geq d_T(y,c)$. Clearly,

$$d_T(x,c) \geq \left\lfloor \frac{\ell(P)}{2} \right\rfloor = \left\lfloor \frac{d}{2} \right\rfloor.$$
If $Q$ denotes the shortest path in $T$ from $x$ to $c$, then by Lemma 4.4.1, the subdigraph of $D$ induced by $V(Q)$ is a tournament of order $|V(Q)|$. But $|V(Q)| \geq \lceil \frac{d}{2} \rceil + 1$. Therefore, $f_1(n, d) \geq \lceil \frac{d}{2} \rceil + 1$.

To prove that $f_1(n, d) \leq \lceil \frac{d}{2} \rceil + 2$, first notice that there is no digraph in $D^{(1)}(n, d)$ which is realized by a path of an odd length, since such a path has two centroid vertices. So, if $d$ is odd, or $d$ is even but $d \neq n - 1$, then $d < n - 1$. In either case, let $T$ be the tree obtained from a path $P^*$ of length $d$ by joining $n - d - 1$ new vertices to a centroid vertex of $P^*$, say $c$ (see Fig. 4.4.1). Clearly, the distance between $c$ and an endvertex of $P^*$ is either $\lceil \frac{d}{2} \rceil$ or $\lfloor \frac{d}{2} \rfloor$, where $\lfloor \frac{d}{2} \rfloor$ is the largest integer less than or equal to $\frac{d}{2}$. Without loss of generality, let $d_T(c, b) = \lceil \frac{d}{2} \rceil$, where $b$ is an endvertex of $P^*$. Pick a vertex $w$ adjacent to $c$ but not on $P^*$. $d_T(w, b) = d_T(c, b) + 1 = \lceil \frac{d}{2} \rceil + 1$. Thus, by Lemma 4.4.1, the shortest path $Q^*$ from $w$ to $b$ produces a tournament of order $\lceil \frac{d}{2} \rceil + 2$ in the digraph $D$ realized by $T$. In other words, $(V(Q^*))$ is a tournament of order $\lceil \frac{d}{2} \rceil + 2$ in $D$. Moreover, this tournament is of maximum order and hence $ot(D) = \lceil \frac{d}{2} \rceil + 2$.

If $d$ is even and $d = n - 1$, then the only tree of order $n$ with diameter equal to $d$ is a path of length $n - 1$. It is easy to see that the maximum possible order of a subtournament in the digraph realized by this path is $\frac{d}{2} + 1$ which is less than $\lceil \frac{d}{2} \rceil + 2$.

Consequently, $f_1(n, d) \leq \lceil \frac{d}{2} \rceil + 2$. $\blacksquare$

---

**Fig. 4.4.1**

![Diagram showing a tree with vertices and edges labeled appropriately.](attachment:image.png)
Remark 1. Suppose that \( f_1(n, d) = \left\lceil \frac{d}{2} \right\rceil + 1 \). Let \( D \) be a digraph in \( D^{(1)}(n, d) \) so that \( ot(D) = f_1(n, d) = \left\lceil \frac{d}{2} \right\rceil + 1 \). If \( T \) is a tree realizing \( D \), then \( T - c \) consists of components of the same order, where \( c \) is the centroid vertex of \( T \).

Proof: Suppose that \( f_1(n, d) = \left\lceil \frac{d}{2} \right\rceil + 1 \), but \( T - c \) contains at least two components with different orders. Since \( \text{dia}(T) = d \), there exists a component of \( T - c \), denoted \( C \), containing a path, say \( Q \), from a vertex adjacent to \( c \), of length \( \left\lceil \frac{d}{2} \right\rceil - 1 \). Pick a vertex \( v \) from a component with order different from \( |C| \) so that \( v \) is adjacent to \( c \). By Lemma 4.4.1, the subdigraph of \( D \) induced by \( V(Q) \cup \{c, v\} \) is a tournament of order \( \left\lceil \frac{d}{2} \right\rceil + 2 \), a contradiction to \( ot(D) = \left\lceil \frac{d}{2} \right\rceil + 1 \).

Remark 2. If \( d = 2 \), then \( f_1(n, d) = \left\lceil \frac{d}{2} \right\rceil + 1 = 2 \).

Theorem 4.4.2. Let \( n \) and \( d \) be positive integers with \( d \geq 3 \). Then \( f_1(n, d) = \left\lceil \frac{d}{2} \right\rceil + 1 \) if and only if \( \frac{n-1}{p} \geq \left\lceil \frac{d}{2} \right\rceil \), where \( p \) is the smallest prime factor of \( n - 1 \).

Proof: Suppose that \( f_1(n, d) = \left\lceil \frac{d}{2} \right\rceil + 1 \). Let \( D \) be a digraph realized by a tree \( T \) in \( T^{(1)}(n, d) \) so that \( ot(D) = f_1(n, d) = \left\lceil \frac{d}{2} \right\rceil + 1 \). By Remark 1, \( T - c \) consists of components of the same order, where \( c \) is the centroid vertex of \( T \). Let \( t \) and \( s \) be the number of components of \( T - c \) and the order of each component of \( T - c \), respectively. Note that \( ts = n - 1 \). Since \( \text{dia}(T) = d \), there is a component of \( T - c \) containing a path of length \( \left\lceil \frac{d}{2} \right\rceil - 1 \). Thus, this component contains at least \( \left\lceil \frac{d}{2} \right\rceil \) vertices. It follows that \( t \left\lceil \frac{d}{2} \right\rceil \leq ts = n - 1 \), i.e., \( \frac{n-1}{t} \geq \left\lceil \frac{d}{2} \right\rceil \). But \( p \leq t \), so \( \frac{n-1}{p} \geq \left\lceil \frac{d}{2} \right\rceil \).

Conversely, suppose that \( \frac{n-1}{p} \geq \left\lceil \frac{d}{2} \right\rceil \) and \( n - 1 = pm \) for some integer \( m \). Note that \( m > 1 \), for otherwise, \( n - 1 = p \) and hence \( \frac{n-1}{p} = 1 \geq \left\lceil \frac{d}{2} \right\rceil \) implies that \( d \leq 2 \), a contradiction to the assumption that \( d \geq 3 \).
Case 1: Suppose that $d$ is even. Let $T(m, d)$ be the tree obtained from a star of order $m + 1 - \frac{d}{2}$ and a path $P$ of length $\frac{d}{2} - 3$ by adding a new edge connecting the root of the star with one end vertex of $P$ (see Fig. 4.4.2 (a)). Denote by $u$ the endvertex of $P$ which is not adjacent to the root of the star. Note that $\text{dta}(T(m, d)) = \frac{d}{2} - 1$. Let $T_p^*(n, d)$ be the tree obtained from $p$ vertex-disjoint copies of $T(m, d)$ by adding a new vertex $c_d$ which is adjacent to each copy of $u$ (see Fig. 4.4.2 (b)). It is easy, by Lemma 4.4.1, to verify that $\text{ot}(D^*) = \frac{d}{2} + 1$, where $D^*$ is the digraph realized by the tree $T_p^*(n, d)$.

![Fig. 4.4.2](image)

Case 2: Suppose that $d$ is odd. Let $T'$ be the tree obtained from a copy of $T_{p-1}^*(n - m, d - 1)$ and a copy of $T(m, d + 1)$ by adding an edge between the vertex $c_{d-1}$ of $T_{p-1}^*(n - m, d - 1)$ and the vertex $u$ of $T(m, d + 1)$ (see Fig. 4.4.3). Again by Lemma 4.4.1, $\text{ot}(D') = \frac{d+1}{2} + 1$, where $D'$ is the digraph realized by the tree $T'$.

Therefore, $f_1(n, d) \leq \left\lceil \frac{d}{2} \right\rceil + 1$. This, together with Theorem 4.4.1, implies that $f_1(n, d) = \left\lceil \frac{d}{2} \right\rceil + 1$. ■
Theorem 4.4.3. For any positive integers $n$ and $d$,

$$\left\lfloor \frac{d}{2} \right\rfloor + 1 \leq f_2(n, d) \leq \left\lceil \frac{d}{2} \right\rceil + 2.$$ 

Proof: In order to prove $f_2(n, d) \geq \left\lfloor \frac{d}{2} \right\rfloor + 1$, pick a digraph $D$ from $\mathcal{D}^{(2)}(n, d)$ which is realized by a tree $T \in \mathcal{T}^{(2)}(n, d)$. Pick two vertices $x$ and $y$ from $V(T)$ so that $d_T(x, y) = \text{dia}(T) = d$. Without loss of generality, assume that

$$d_T(x, c_1) = d_T(x, C_d(T)) \geq d_T(y, C_d(T)),$$

where $c_1 \in C_d(T)$. Then $d_T(x, c_1) \geq \left\lceil \frac{d-1}{2} \right\rceil = \left\lceil \frac{d}{2} \right\rceil$.

Let $Q$ be a shortest path from $x$ to $c_1$, then by Lemma 4.4.1, the subdigraph in $D$ induced by $V(Q)$ is a tournament of order $|V(Q)| \geq \left\lfloor \frac{d}{2} \right\rfloor + 1$. Thus, $f_2(n, d) \geq \left\lceil \frac{d}{2} \right\rceil + 1$.

To show that $f_2(n, d) \leq \left\lceil \frac{d}{2} \right\rceil + 2$, notice that if $d$ is even, then $d < n - 1$ since each tree in $\mathcal{T}^{(2)}(n, d)$ contains two centroid vertices. So, if $d$ is even, or $d$ is an odd integer with $d < n - 1$, construct a tree $T'$ as follows (see Fig. 4.4.4): start with a path $P$ of length $d$ whose endvertices are $u$ and $v$. Pick two adjacent vertices on $P$, say $c$ and $c'$ so that $d_P(u, c') = \left\lfloor \frac{d+1}{2} \right\rfloor - 1$ and
\[ d_P(v, c) = \left\lceil \frac{d+1}{2} \right\rceil - 1. \] Add \[ \left\lceil \frac{n-1-d}{2} \right\rceil \] new vertices each adjacent to the vertex \( c \), and then add \[ \left\lceil \frac{n-1-d}{2} \right\rceil \] new vertices each adjacent to the vertex \( c' \). Let \( D' \) be the digraph realized by \( T' \), then by Lemma 4.4.1, it is easy to verify that \( \omega(D') = \left\lceil \frac{d+1}{2} \right\rceil - 1 + 2 = \left\lceil \frac{d+1}{2} \right\rceil + 1 = \left\lfloor \frac{d}{2} \right\rfloor + 2. \]

![Diagram](image)

**Fig. 4.4.4**

If \( d = n - 1 \) is an odd integer, then the only tree of order \( n \) with diameter equal to \( d \) is a path \( P' \) of length \( n - 1 \). Let \( D'' \) be the digraph realized by \( P' \), then \( \omega(D'') = \frac{d-1}{2} + 1 \leq \left\lfloor \frac{d}{2} \right\rfloor + 2. \)

Consequently, \( f_2(n, d) \leq \left\lfloor \frac{d}{2} \right\rfloor + 2. \)

Note that every tree \( T \) of order \( n \) in \( T^{(1)}(n, d) \) contains two adjacent centroid vertices. So, by Lemma 4.1.2, \( n \) is even.

**Theorem 4.4.4.** Let \( n \) and \( d \) be positive integers with \( d \geq 3 \). Then \( f_2(n, d) = \left\lfloor \frac{d}{2} \right\rfloor + 1 \) if and only if \( \frac{d-1}{p} \geq \left\lfloor \frac{d}{2} \right\rfloor \), where \( p \) is the smallest prime factor of \( \frac{n}{2} - 1 \).

**Proof:** Suppose that \( \frac{d-1}{p} \geq \left\lfloor \frac{d}{2} \right\rfloor \), where \( p \) is the smallest prime factor of \( \frac{n}{2} - 1 \).

If \( d \) is even, let \( T_e \) be the tree obtained from a copy of \( T_p^*(\frac{d}{2}, d - 2) \) and a copy of \( T_p^*(\frac{d}{2}, d) \) by joining the vertex \( c_{d-2} \) of \( T_p^*(\frac{n}{2}, d - 2) \) to the vertex \( c_d \) of \( T_p^*(\frac{n}{2}, d) \) by an edge. Let \( D_e \) be the digraph realized by the tree \( T_e \), then by the construction of \( T_e \) and by Lemma 4.4.1, \( \omega(D_e) = \frac{d}{2} + 1 = \left\lfloor \frac{d}{2} \right\rfloor + 1. \)
Similarly, if $d$ is odd, let $T_0$ be the tree obtained from two vertex-disjoint copies of $T_p^*(\frac{n}{2}, d-1)$ by adding an edge between the two copies of the vertex $c_{d-1}$ of $T_p^*(\frac{n}{2}, d-1)$. Let $D_0$ be the digraph realized by the tree $T_0$, then $\text{ot}(D_0) = \frac{d-1}{2} + 1 = \lceil \frac{d}{2} \rceil + 1$.

In either case, $f_2(n, d) \leq \lceil \frac{d}{2} \rceil + 1$. So, by Theorem 4.4.3, $f_2(n, d) = \lceil \frac{d}{2} \rceil + 1$.

Conversely, suppose that $f_2(n, d) = \lceil \frac{d}{2} \rceil + 1$. Pick $D \in \mathcal{D}^{(2)}(n, d)$ so that $D$ is realized by a tree $T \in T^{(2)}(n, d)$ and $\text{ot}(D) = f_2(n, d) = \lceil \frac{d}{2} \rceil + 1$. Let $C_d(T) = \{c_1, c_2\}$, $c_1 \neq c_2$, and let $z$ and $y$ be two vertices of $T$ so that $d_T(z, y) = \text{dia}(T) = d$. Without loss of generality, we may assume that $d_T(z, c_1) = d_T(z, C_d(T)) \geq d_T(y, C_d(T))$. Then $d_T(z, c_1) \geq \lceil \frac{d}{2} \rceil$.

Note that $z$ is in the component $T(c_1, c_1c_2)$. Also, all components of $T(c_1, c_1c_2) - c_1$ have the same order. For otherwise, let $u$ be a vertex adjacent to $c_1$ so that $u$ is in a component of $T(c_1, c_1c_2) - c_1$ different from the component in which $z$ lies. Then the shortest path from $u$ to $z$ corresponds to a tournament in $D$ of order $d_T(z, c_1) + 2 \geq \lceil \frac{d}{2} \rceil + 2$, a contradiction to the assumption that $f_2(n, d) = \lceil \frac{d}{2} \rceil + 1$. Let $t$ and $s$ be the number of components of $T(c_1, c_1c_2) - c_1$ and the order of each component of $T(c_1, c_1c_2) - c_1$, respectively. Then $ts = \frac{n}{2} - 1$. Since the component of $T(c_1, c_1c_2) - c_1$ containing $z$ has at least $\lceil \frac{d}{2} \rceil$ vertices, $s \geq \lceil \frac{d}{2} \rceil$. It follows that $\frac{n}{2} - 1 = ts \geq t\lceil \frac{d}{2} \rceil$, i.e., $\frac{d-1}{t} \geq \lceil \frac{d}{2} \rceil$. But $p \leq t$, so $\frac{d-1}{p} \geq \lceil \frac{d}{2} \rceil$. \qed
Bibliography


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Major Field: Mathematics


Approved:

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EXAMINING COMMITTEE:

[Signatures of committee members]

Date of Examination: May 4, 1990