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Reflection Positivity: A Quantum Field Theory Connection

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REFLECTION POSITIVITY: A QUANTUM FIELD THEORY CONNECTION

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Abstract

At the heart of constructive quantum field theory lies reflection positivity. Through its use one may extend results for a Euclidean field theory to a relativistic theory. In this dissertation we connect functorial and constructive quantum field theories through reflection positivity. In 2014 Santosh Kandel constructed examples of d-dimensional functorial QFTs when d is even. We define functorial reflection positivity and show that this functorial theory is a reflection positive theory. We go on to show that every reflection positive theory produces a reflection positive Hilbert space. Iterated doubles are then introduced and used as a starting point to produce a four dimensional quantum field theory. The $(0 + 1)$ dimensional theory is then analyzed and shown to correspond to quantum mechanics.

Chapter 1

Introduction

The purpose of this work is to bridge the divide between the constructive and topological quantum field theories. To do so, we use a condition which it is believed every viable theory must obey: reflection positivity. This idea stems from a tool used by mathematicians and physicists called Wick rotation after Gian Carlo Wick. The basic idea is as follows:

Suppose we were given the following integral

$$I_p(x) = \int_0^\infty e^{ixt^p}\, dt.$$

If we were to replace $ixt^p \to -xt^p$ we would be able to integrate easily (assuming we know the Gamma function) and get

$$I'_p(x) = \int_0^\infty e^{-xt^p}\, dt = \frac{1}{x^{1/p}}\Gamma\left(\frac{p+1}{p}\right).$$

The trick, then, is to use a real integral and rotate back into the complex plane. Setting up an appropriate contour integral gives a visual representation of this particular rotation.

In general, if we were given a problem in Minkowski space, $\mathbb{R}^{1,3}$ and a similar problem in Euclidean space$\mathbb{R}^4$, it would be easier to work in Euclidean space. Following this previous line of thought, if we were to be able to make a suitable transformation, one might be able to move between the two spaces. Formally, we have the following:

- Minkowski metric: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$
- Euclidean metric: $ds^2 = d\tau^2 + dx^2 + dy^2 + dz^2$
- Wick Rotation: $t \mapsto i\tau$

Mathematically, we can say Wick rotation is the analytic continuation of a Euclidean solution to a Minkowski solution. In the early 1970s over a span of two papers, Konrad Osterwalder and Robert Schrader formulated precise conditions for Wick rotation between relativistic and Euclidean field theories. This condition is what we now refer to as reflection positivity.

1.1 Historical Background

Quantum mechanics and special relativity have dominated the physics landscape for decades. Since their inception, there have been incredible results and predictions from each. Alongside each discovery, however, have come a myriad of problems. The most famous of these issues is the ongoing incompatibility of the two subjects.

A step towards reconciliation was made in the 1920s with the introduction of quantum field theory. A group of theoreticians (Born, Heisenberg, and Jordan, jointly) applied the methods of the newly formulated quantum mechanics to electromagnetic fields. The quantum field theory of today is a direct descendent of what emerged almost immediately after in a paper by Dirac on quantum electrodynamics.

In quantum electrodynamics (QED) the photon was understood as a field before it was understood as a particle. Additionally, the electron was taken to be relativistic, governed by the Dirac equation. QED, then, was a crucial step toward rectifying the quantum world with the relativistic view.

Field theories are plagued with difficulties, and in nearly every expansion we encounter infinite terms. Typically, when we wish to investigate a system, we start with a non-interacting version called the free theory. To determine how the new system behaves, the Hamiltonian (which describes the total energy of a system) would be "perturbed," and this perturbation would often be expanded as a series. A fundamental issue in QFT is the divergence of these perturbation series expansions.

To rectify the problem of divergence, physicists employ a number of methods. Readers may be familiar with the ideas of regularization, renormalization, integral truncation, cutoffs, etc. One of these methods, renormalization, was popularized by Feynmann and Schwinger in the forties. Renormalization is the method of redefining physical parameters in order to absorb the infinities in these new definitions.

A divide was growing between mathematics and physics. Before the 1800s, the two were essentially inseparable. Around the turn of the century, however, there was already a gap growing between the sciences. This can be seen to some extent in the statement of Hilbert's sixth problem:

"*6. Mathematical Treatment of the Axioms of Physics. The investigations on the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part; in the first rank are the theory of probabilities and mechanics.*"

Around the time of Feynman and Schwinger, technology had already grown to such an extent that results were being churned out rapidly. In response to the sixth problem, among other things, Arthur Wightman developed a set of axioms that formalized the requirements any QFT should satisfy [37].

The axioms of Wightman form the basis of constructive quantum field theory (CQFT). We will return to CQFT, but it should be noted that these axioms were not, and still are not, the solution to Hilbert's sixth problem. Less than a decade after the Wightman axioms came the Haag-Kastler Axioms.

In 1964 Rudolf Haag and Daniel Kastler [16] published their own axiom system dealing specifically with the algebra of observables for a system giving rise to algebraic quantum field theory (AQFT). In essence, the Haag-Kastler system axiomatizes the Heisenberg picture for quantum field theory. The axiomatization of Wightman leans more towards the structure of the fields themselves.

In the late 1980s Michael Atiyah axiomatized topological quantum field theory (TQFT) while Graeme Segal did the same for conformal field theory (CFT). As of the writing of this work, the list of field theories includes QFT, CQFT, TQFT, AQFT, RQFT, FQFT, CFT, EFT, and more.

1.1.1 Recent Developments in Reflection Positivity

The axioms of Wightman tell us how to construct a QFT. The system developed by Osterwalder-Schrader allows us to make an often simpler construction with Euclidean fields and then Wick-rotate back to Minkowski space-time. At the heart of this system is the axiom of reflection positivity.

Reflection positivity can be thought of as a way of understating how to move from a probabilistic to a quantum interpretation of fields. As such, this idea has gained the most ground through its interaction with statistical mechanics and random fields. Through the use of reflection positivity and Osterwalder-Schrader reconstruction, interacting QFTs were shown to exist for dimensions $d = 2$ and $d = 3$ by James Glimm and Arthur Jaffe.

It's important to note that the idea of reflection positivity originated from conversations between Jaffe, Osterwalder, and Schrader. Soon afterwards, a great deal of research on reflection positivity was conducted by Jürg Frölich.

Over the past few decades, reflection positivity has become increasingly important to our understanding of mathematics. For instance, the reflection shows up in the modular Tomita-Takesaki theory. In 2010 a spherical version of reflection positivity was used by Rupert Frank and Elliot Lieb to give new estimates for the Hardy-Littlewood-Sobolev inequality. In 2013 Christian Anderson, a student of Jaffe, showed reflection positivity for certain operators on manifolds.

Since the late 1980s, mathematicians have worked to understand reflection positivity through representations. When one Wick rotates, changing time corresponds to an involution in the symmetry group of a system. In QFT the physical system corresponds to a unitary representation of this symmetry group. Wick rotation, therefore, is essentially a connection between a representation of the Euclidean group of symmetries with the Poincare group of symmetries in Minkowski space-time.

In 1986 Robert Schrader published his work applying reflection positivity to the complementary series of $SL(2n, \mathbb{C})$. This was the beginning of subsequent work led by Palle Jorgenson, Karl-Hermann Neeb, and Gestur Ólafsson. Over the last few decades, these three researchers have published various results on the analytic continuation of representations. Most recently, Neeb and Ólafsson characterized reflection positivity on the interval, circle, and sphere; they also discussed connections to modular theory and the KMS conditions. Part of this work is given in Chapter 3 and can be found in detail in [24].

1.1.2 Recent Developments in Functorial Theories

Reflection positivity was implemented in the original axiom system for TQFTs through unitarity. Though this notion had already been understood, in 2016 Dan Freed and Michael Hopkins focused the discussion squarely on reflection positivity. In their paper the notion of reflection positivity was defined explicitly and then introduced for extended TQFTs.

For topological and other functorial theories, reflection positivity is typically subsumed by the unitarity axiom. In fact one usually begins with a reflection positive measure when working backwards from fields on a manifold to describing a functorial theory.

This is more of less the case in Riemannian functorial quantum field theories (RFQFTs). In Riemannian FQFTs, manifolds are not stripped of their metrics. The additional data ultimately turns the target spaces into infinite dimensional Hilbert spaces.

This program has been carried out in different directions by Stephan Stolz, Peter Teichner, David Ayala and others. In his papers, Ayala identifies the homotopy type of the classifying space of geometric cobordism categories using sheaf theory. This work followed that of Galatius, Madsen, Tillman and Weiss.

A goal of the Stolz-Teichner program is to achieve results in cohomology using a Riemannian bordism category. The program is influenced heavily by Segal's development of CFT. In fact, most papers dealing with functorial field theories (including this one) uses the work of Segal as a reference.

In the past decade, we have poured a great deal of effort into understanding the Riemannian bordism category. The dissertation of Santosh Kandel, published in 2014, is the most recent addition to this body of work. In his dissertation Kandel used the work of Doug Pickrell from 2007 to develop a FQFT for even dimensions. Additionally, Kandel showed the field theory is projective for odd dimensions.

The Riemannian theory may be distinguished from the topological in many ways. First, as described in numerous accounts on topological theories, the Hilbert space obtained is, by necessity, finite dimensional.

Proposition. *Let $\mathcal{F}$ be a TQFT, then $\mathcal{F}(M)$ is finite dimensional for every (object) $M \in \boldsymbol{Bord}_n$.*

This proposition is the fundamental roadblock in describing a "realistic" QFT from a TQFT. That is not to imply that "unrealistic" necessarily means "useless." On the contrary, topological theories have been used to compute a myriad of topological invariants. In terms of physical use, an important application of TQFT is quantum computing, for which finite dimensional target spaces are perfectly acceptable. To find a measure of equivalence between a TQFT and CQFT, however, one must move the target category beyond finite dimensional spaces.

An issue with the category *RBord* is that anti-involutions are ill defined. Because of this, we no longer have a well defined coevaluation map. In fact, since coevaluation is not well defined, an object does not even have a guaranteed identity morphism in *RBord*. For this reason, *RBord* is not truly a category but is instead a semi-category. This is to our benefit, for this removes the finiteness condition.

The target category for our Riemannian theory, then, will be the category of Hilbert spaces, which are infinitely dimensional in our case. This choice of target category is made after first passing through the category T_{pol} of polarized topological vector spaces, ensuring that our linear maps extend. Kandel and others have ensured us that FQFT works, but there is still a lot to be done.

Before we begin we also note that, in the physics terminology, our Riemannian theory would be called Euclidean because our manifolds have a flat connection.

1.2 Organization

The work is laid out as follows:

In Chapter 2 we introduce CQFT and its axioms. After discussing the axioms, we dig deeper into reflection positivity and its many forms in mathematics. We then return to the reflection positivity axiom and provide theorems for Riemannian manifolds, which will be the objects of interest throughout this document. For those familiar with the work of Jaffe, Osterwalder and Schrader and Neeb, Ólafsson et al., Chapter 2 may be skimmed over.

In Chapter 3 we define a TQFT. We begin by briefly introducing category theory, providing only the pertinent definitions. Short examples are given for the reader who, like the author of this work, finds category theory befuddling at first glance. We then define the categorical notion of reflection positivity. The definitions and theorems in the latter half of Chapter 3 follow the 2016 work by Dan Freed and Michael Hopkins. For the categorically inclined reader, Chapter 3 may easily be glossed over.

Chapter 4 presents a segue into the hybridization of TQFT and CQFT. We follow the work of Stephan Stolz, Peter Teichner, Doug Pickrell and Santosh Kandel. We move away from TQFTs by giving our manifolds a metric and using the semi category $\mathbf{RBord}_n$. We provide the necessary material to define a Riemannian FQFT, which are functorial assignments and gluing procedures.

Our main results are found in Chapters 5 and 6. We begin Chapter 5 by expanding some preliminary results of Freed/Hopkins and Kandel from Chapters 3 and 4. We then redefine functorial reflection positivity and connect CQFTs with FQFTs. In Chapter 6 we give a variety of applications and discuss open problems.

1.2.1 Main Results

Reflection positivity is defined in [11] for a TQFT. In this paper the definition of reflection positivity is extended to certain FQFTs. It is then shown that the FQFT given in [19] has a natural Hermitian form. We then prove that this Riemannian FQFT is a reflection positive theory. This implies that the resulting Hilbert spaces coming from this functorial theory are reflection positive. We prove this idea in the main result of this work, theorem 5.2.1:

Theorem. *Suppose M is a complete, connected manifold. Given a reflection positive Hilbert space $(\mathcal{E}, \mathcal{E}_+, \theta)$ corresponding to $L^2(M)$, there is a reflection positive functorial quantum field theory $\mathcal{Z}_{RP}$ such that*

1. $\hat{\mathcal{E}}$ is the target object of $\mathcal{Z}_{RP}$

2. $\mathcal{E}_+$ and θ are recoverable from $\hat{\mathcal{E}}$

Similarly, every reflection positive functorial quantum field theory generates reflection positive Hilbert spaces

Therefore, we have shown that CQFTs may be studied using the tools of homology and category theory.

Having given this correspondence, we then provide applications. In Chapter 6 we begin by introducing a new structure: the iterative double.

Definition. *An iterative double of X is a sequence of null-cobordant manifolds described by the process*

$$X =)\!(X_1 := X_1 \to Dbl(X_1) \xrightarrow{\varnothing^2} X_2 \to Dbl(X_2) \to \cdots \to Dbl(X_{n-1}) \xrightarrow{\varnothing^n} X_n \to \cdots$$

A truncation of an iterative double is termed an iterated double. It is pointed out that any ball or sphere of any dimension can be taken as an iterated double. We give a very brief analysis of these objects, which are used to describe a new recipe for extending or reducing a QFT. Through this work and the contributions of Dimock [9], we give an interesting new decomposition of functions and Gaussian measures in terms of their restrictions to submanifolds.

Next, the $(0+1)$ dimensional free scalar theory is worked out for the RFQFT developed in chapter four. It is generally know in the QFT community that a $(0+1)$ QFT coincides with quantum mechanics. In section 6.3 it is explicitly shown that this $(0+1)$ dimensional RFQFT does indeed coincide with quantum mechanics.

Finally, we give a brief review of the two dimensional case before discussing ongoing research and open questions.

Thank you for your attention and taking the time to peruse this work. Enjoy!

Chapter 2

Constructive Quantum Field Theory

This chapter is foundationally the answer to the question:

Question: For a given classical Euclidean invariant field ϕ with distribution $d\mu(\phi)$, does there exist a corresponding Minkowski QFT? In other words, is there an appropriate representation of the Poincaré group with invariant vacuum?

In the early 1970's Osterwalder and Schrader gave the affirmative in [25] and [26]. This framework is known to work well in dimensions $d = 2$ and $d = 3$. Indeed, rigorous constructions have been provided [see [15]] for the ϕ^4 and Yukawa interactions.

In the first section we recall the foundations of CQFT. The axioms listed are the usual axioms for constructive quantum field theory by Konrad Osterwalder and Robert Schrader. Though the notation has changed over the years, the axioms as given in [15] are still the ones referenced extensively by constructive theorists.

It is important to keep in mind that this approach has note been enough to develop a rigorous QFT in $d = 4$. Additionally, the axioms that follow will rely on the linearity of the space. Therefore, this approach must be modified to develop gauge theories (i.e. Yang Mills theory).

We will use the following notation:

- $f \in \mathcal{D}$ are test functions: smooth functions with compact support
- Fields $\phi \in \mathcal{D}'$ are distributions.
- We denote the canonical pairing between distributions and test functions as

$$\phi(f) = \langle \phi, f \rangle = \int \phi(x) f(x)\, dx$$

- A Euclidean Field is a probability measure μ on $\mathcal{D}'$

- The inverse fourier transform of μ is given by

$$S : \mathcal{D} \to \mathbb{C}, \quad S(f) := \int e^{iD(f)}\, d\mu(D)$$

In the second section we restrict our attention now to **OS3**, reflection positivity, the material for which comes from [24]. There has been extensive investigation into the uses of reflection positivity in mathematics. Applications include sharp estimates of the Hardy-Littlewood-Sobolev inequality [10] and the Cartan duality of symmetric Lie groups [24]. In this section we follow the path laid out in [24]. Numerous advances have been made by Jorgenson, Neeb, Ólafsson, and others in reflection positivity, the definitions and propositions below are only a drop in the bucket.

First we will define a reflection positive Hilbert space and then move through quantization. An important question that must be addressed is: what operators (if any) survive the quantization process? After answering this question we mention reflection positive representations and give two important examples of their use.

For the last section we look at reflection positivity on Riemannian manifolds and provide important results from [1]. Riemannian manifolds will be the setting for the main results.

2.1 Osterwalder-Schrader Axioms

Definition 2.1.1 (OS0: Analyticity)**.** *All functions* $e_f(D) := e^{iD(f)},\ f \in \mathcal{D}_{\mathbb{C}}$*, are* μ*-integrable and the functional*

$$S : \mathcal{D}_{\mathbb{C}} \to \mathbb{C}, \quad S(f) := \int_{\mathcal{D}'} e_f\, d\mu$$

obtained this way is holomorphic on all finite-dimensional complex subspaces.

Definition 2.1.2 (OS1: Regularity)**.** *There exists some* $p \in [1, 2]$ *and a* $c \geq 0$ *such that*

$$|S(f)| \leq e^{c(\|f\|_1 + \|f\|_p^p)} \quad \text{for all } f \in \mathcal{D}_{\mathbb{C}}$$

Definition 2.1.3 (OS2: Euclidean Invariance)**.** *S is invariant under the action of the euclidean group* $Mot(\mathbb{R}^d) \cong \mathbb{R}^d \rtimes O_d(\mathbb{R})$

Definition 2.1.4 (OS3: Reflection Positivity)**.** *Let* $\theta : \mathbb{R}^d \to \mathbb{R}^d, \quad \theta(x_0, x) = (-x_0, x)$ *be the time reflection and* $\mathscr{A}_+ := span\{e_f : f \in \mathscr{D}(\mathbb{R}^d_+)_{\mathbb{C}}\}$*, where* $\mathbb{R}^d_+ = \{(x_0, x) \in \mathbb{R}^d : x_0 > 0\}$ *Then we require that* μ *is reflection positive in the sense that*

$$\langle \theta A, A \rangle \geq 0 \;\; \textit{for all } A \in \mathscr{A}_+$$

Definition 2.1.5 (OS4: Ergodicity)**.** *The unitary one-parameter group* $(U_t)_{t\in\mathbb{R}}$ *defined by the action of* $T(t)$ *on* $L^2(\mathcal{D}', \mu)$ *is ergodic, where*

$$T(t) : \mathbb{R}^d \to \mathbb{R}^d, \qquad (x_0, x) \mapsto (x_0 + t, x)$$

In other words, for all $A \in L^1(\mathcal{D}', \mu)$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t T(s) A T(s)^{-1}\, ds = \int_{\mathcal{D}'} A\, d\mu$$

At this point it is useful to have a correspondence between the mathematical notation and physical language. The descriptions and identifications are, again, from [15].

Ergodicity implies uniqueness of the vacuum. Euclidean invariance analytically continues to Lorentzian invariance. The regularity axiom is often tweaked in investigations of singularities.

- $\mathcal{D}'(\mathbb{R}^d)$ = path space
- $d\mu$ = Feynman-Kac measure on path space
- $\mathcal{D}'(\mathbb{R}^{d-1})$ = configuration space
- $t \mapsto \phi(x, \cdot)$ = a path with values in $\mathcal{D}(\mathbb{R}^{d-1})$

- $H = L^2(\mathcal{D}(\mathbb{R}^{d-1}), d\nu)$ (Schrödinger representation)

Lastly, let μ be a probability measure on $\mathcal{D}'$ which satisfies reflection positivity and is invariant under reflection and time translation. Quantum mechanics is reconstructed through the following:

- Each operator $T(t), t \geq 0$ preserves $\mathcal{E}_+$

- $T(t), t \geq 0$ induces a hermitian contraction on the physical Hilbert space $\mathscr{H}$, $\widehat{T}$

- $\widehat{T(t)} = e^{-tH}$

- H is a positive self-adjoint operator

- $H\Omega = 0$ for $\Omega := \widehat{1}$

- H is the Hamiltonian

2.2 Reflection Positive Hilbert Spaces

Definition 2.2.1. *Let $\mathcal{E}$ be a Hilbert space, $\theta \in U(\mathcal{E})$ be a unitary involution, and $\mathcal{E}_+$ be the closed subspace*

$$\mathcal{E}_+ = \{\eta \in \mathcal{E} : \langle \eta, \theta\eta \rangle \geq 0\}$$

*The triple $(\mathcal{E}, \mathcal{E}_+, \theta)$ is a **reflection positive Hilbert space** with new inner product*

$$\langle \eta, \xi \rangle_\theta = \langle \eta, \theta\xi \rangle$$

From this point forward, a reflection positive Hilbert space will be abbreviated RPHS.

2.2.1 OS Quantization

Let $(\mathcal{E}, \mathcal{E}_+, \theta)$ be an RPHS with inner product $\langle \eta, \theta\xi \rangle$. Then we have the subspace

$$\mathcal{N} := \{\eta \in \mathcal{E}_+ : \langle \eta, \theta\eta \rangle = 0\}.$$

We then mod out be equivalence classes via $q : \mathcal{E}_+ \to \mathcal{E}_+/\mathcal{N}$ and obtain a pre-Hilbert space. Then $\widehat{\mathcal{E}}$ is the Hilbert space completion of $\mathcal{E}_+/\mathcal{N}$ with respect to the norm

$$\|\widehat{\eta}\| = \sqrt{\langle \widehat{\eta}, \theta\widehat{\eta} \rangle}$$

The resulting space, $\widehat{\mathcal{E}}$, is the quantum mechanical Hilbert space of states. This quantization process is given by the exact sequence

$$0 \longrightarrow \mathcal{N} \xrightarrow{\ i\ } \mathcal{E}_+ \xrightarrow{\ \widehat{}\ } \widehat{\mathcal{E}} \longrightarrow 0$$

Where i is an injection and the quantization map "$\widehat{}$" is the composition *completion* $\circ\, q$.

2.2.2 OS Quantization - Operators

Let $S : \mathcal{E}_+ \to \mathcal{E}_+$ be a linear operator with domain $\mathcal{D}(S)$ such that

- $S : \mathcal{D}(S) \cap \mathcal{E}_+ \to \mathcal{E}_+$
- $S : \mathcal{D}(S) \cap \mathcal{N} \to \mathcal{N}$

Then S induces a linear operator $\widehat{S} : \mathcal{D}(\widehat{S}) \to \widehat{\mathcal{E}}$ with $\widehat{S}\widehat{\eta} = \widehat{S\eta}$ where $\mathcal{D}(\widehat{S}) := \widehat{\mathcal{D}(S)} = \{\widehat{v} : v \in \mathcal{D}(S)\}$

Lemma 2.2.1. *Let $(\mathcal{E}, \mathcal{E}_+, \theta)$ be an RPHS and suppose that*

- *$\mathcal{D} \subseteq \mathcal{E}_+$ is a linear subspace such that $\widehat{\mathcal{D}}$ is dense in $\widehat{\mathcal{E}}$*
- *$S, T : \mathcal{D} \to \mathcal{E}_+$ are linear operators*

Then

1. *If* $\langle S\eta, \zeta\rangle_\theta = \langle \eta, T\zeta\rangle_\theta$ *for* $\eta, \zeta \in \mathcal{D}$*, then* $S(\mathcal{N}) \subseteq \mathcal{N}$*,* $\widehat{S}, \widehat{T}$ *are well defined and* $\langle \widehat{S}\widehat{\eta}, \widehat{\zeta}\rangle = \langle \widehat{\eta}, \widehat{T}\widehat{\zeta}\rangle$ *for* $\widehat{\eta}, \widehat{\zeta} \in \widehat{\mathcal{D}}$

2. *If* $S \in U(\mathcal{E}_+)$ *and* $\theta S \theta = S$*, then* $\widehat{S}$ *extends to a unitary operator on* $\widehat{\mathcal{E}}$

3. *If* $T = S$ *in (1), then* $\widehat{S}$ *is a symmetric operator. If* S *is bounded on* $\mathcal{D} = \mathcal{E}_+$*, then so is* $\widehat{S}$ *and* $\|\widehat{S}\| \leq \|S\|$*.*

If $T = S$ in (1), then $\widehat{S}$ is a symmetric operator. If S is bounded on $\mathcal{D} = \mathcal{E}_+$, then so is $\widehat{S}$ and $\|\widehat{S}\| \leq \|S\|$.

Quantization of operators can be represented again by the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{N} & \xrightarrow{i} & \mathcal{E}_+ & \xrightarrow{\widehat{\ }} & \widehat{\mathcal{E}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle \theta S \theta} & & \downarrow{\scriptstyle \widehat{S}} & & \\
0 & \longrightarrow & \mathcal{N} & \xrightarrow{i} & \mathcal{E}_+ & \xrightarrow{\widehat{\ }} & \widehat{\mathcal{E}} & \longrightarrow & 0
\end{array}$$

2.2.3 Representations

Definition 2.2.2. *Let* G *be a Lie group with Lie algebra* $\mathfrak{g}$*. Let* τ *be an involutive automorphism of* G*, then* (G, τ) *is a symmetric Lie group.*

Define $G_\tau := G \rtimes \{1, \tau\}$ and $H := G_0^\tau$, then we immediately have

- τ induces an involution $d\tau : \mathfrak{g} \to \mathfrak{g}$
- eigenspaces $\mathfrak{h}$ and $\mathfrak{q}$ for ± 1
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$
- $\mathfrak{h}$ is the lie algebra of H

With this, we get the Cartan Dual.

Definition 2.2.3. *Let* $\mathfrak{g}$ *be the lie algebra as in the previous definition. Then the Cartan Dual of* $\mathfrak{g}$ *is given by* $\mathfrak{g}^c := \mathfrak{h} \oplus i\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$

Cartan Duals provide fascinating insight into some of the most fundamental groups in physics, the Euclidean, Poincare, and Heisenberg.

Example 2.2.1. The Euclidean group is given by $E(n) = \mathbb{R}^n \rtimes O_n(\mathbb{R})$ with lie algebra $\mathfrak{e}(n)$. Its elements $(b, A) \in E(n)$ act on $\mathbb{R}^n$ by $(x, A) \cdot v = Av + x$. Let $T := \text{diag}(-1, 1, \ldots, 1)$ and define $\tau(x, A) = (Tx, TAT)$.

The Poincare group is given by $P(n) = \mathbb{R}^{1,n-1} \rtimes O_{1,n-1}(\mathbb{R})$ with lie algebra $\mathfrak{p}(n)$.

In terms of Cartan Duals we have

$$\mathfrak{e}(n)^c \simeq \mathfrak{p}(n)$$

The example $n = 4$ is fundamental to our understanding of space time. This relationship at the lie algebra level is a motivating example of the relationship between representations and reflection positivity.

Example 2.2.2. The Heisenberg group has Lie algebra $\mathfrak{heis} = \langle P, X, z \rangle$ with commutation relations $[P, X] = z$. Define the involution

$$\tau(P) = -P, \quad \tau(X) = X, \quad \tau(z) = -z$$

then

$$\mathfrak{heis}^c \simeq \mathfrak{heis}$$

Definition 2.2.4. *A symmetric subsemigroup is a triple* (G, S, τ) *where* (G, τ) *is a symmetric lie group and* S *is a subsemigroup of* G *such that*

- S *is invariant under* $s^\sharp := \tau(s)^{-1}$
- $HS = S$

- $1 \in \bar{S}$

Definition 2.2.5. *Let* $(\mathcal{E}, \mathcal{E}_+, \theta)$ *be an RPHS and* (G, τ) *be a symmetric Lie group. A reflection positive representation is a unitary representation* $\pi : G \to U(\mathcal{E})$ *such that*

- $\pi(\tau(g)) = \theta\pi(g)\theta$
- $\pi(S)\mathcal{E}_+ \subseteq \mathcal{E}_+$

Definition 2.2.6. *A reflection positive unitary one-parameter group is a strongly continuous unitary one-parameter group* $(U_t)_{t\in\mathbb{R}}$ *on* $\mathcal{E}$ *for which* $\mathcal{E}_+$ *is invariant under* U_t *for* $t > 0$ *and* $\theta U_t \theta = U_{-t}$ *for* $t \in \mathbb{R}$

Proposition 2.2.2. $(\widehat{U}_t)_{t\geq 0}$ *is a strongly continuous one-parameter semigroup of symmetric contractions on* $\widehat{\mathcal{E}}$

2.3 Reflection Positivity for Riemannian Manifolds

The following definitions and results for manifolds from [1] will be needed in chapters 5 and 6.

Let M be a Riemannian manifold. We call M static if it possesses a globally defined, hypersurface orthogonal Killing field. Simply put, time translation is well defined on the manifold.

Definition 2.3.1. *A complete, connected, Riemannian manifold* M *is quantizable if it is static and equipped with a reflection. Such a manifold is decomposed as*

$$M = M_- \sqcup \Sigma \sqcup M_+$$

Here Σ *is the time-zero reflection hypersurface and* $\sqcup$ *is disjoint union. The manifold is equipped with a reflection* θ *that fixes* Σ *and exchanges* Ω_+ *and* Ω_-

Definition 2.3.2. *Let* $\mathcal{E}_\pm = \mathcal{E}_{M_\pm} \subset \mathcal{E}$*, then* $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ *is reflection positive when* $\langle \theta f, f \rangle_{\mathcal{E}} \geq 0$

Let $C_M = (\Delta_M + m^2)^{-1}$ and $C_B = (\Delta_B + m^2)^{-1}$ be the resolvent of the Laplacian on a manifold M and the resolvent with boundary conditions, respectively. From [1] we have the following theorems:

Theorem 2.3.1. *Let M be a complete, connected Riemannian manifold with a dissecting reflection θ. Let $M_+ \sqcup \Sigma \sqcup M_-$ denote the partition of the manifold by the reflection hyperplane. For all $f \in C_0^2(M_+)$,*

$$0 \leq \langle \theta f, C_M f \rangle_{L^2}$$

Theorem 2.3.2. *Suppose that Δ_B is the Laplacian on $\mathbb{R}^d$ with boundary data B on a finite union Γ of piece-wise smooth hypersurfaces. Suppose that the boundary data consists of a mixture of Dirichlet and/or Neumann conditions, and suppose that the boundary conditions are symmetric under the reflection θ. Then C_B is reflection positive with respect to θ.*

Chapter 3

Topological Quantum Field Theory

In the early 1980's Edward Witten introduced an informal definition of a Topological Quantum Field Theory, denoted henceforth by TQFT, as a QFT on a smooth manifold M independent of the metric on M. Almost concurrently, Graeme Segal provided an axiomatization of Conformal Field Theory (CFT). Motivated by this, Sir Michael Atiyah axiomatized TQFT in the latter half of the 1980's. The axioms of TQFT published by Atiyah in 1988 have been modified over the years and one such modification is given here (see [6]). The expositions of John Baez, Dan Freed, Jacob Lurie and Constantin Teleman to name a few are great references for information on TQFTs.

It is impossible to express in the space provided just how profoundly this theory has impacted mathematics and physics. Over the past few decades hundreds and hundreds of mathematicians have worked on advancing our understanding of knots and manifold invariants through TQFTs.

In the realm of physics, TQFT has had incredible success in quantum information and quantum computing with the so called fractional quantum Hall effect (see [33]). For a more general physical understanding of the importance of TQFT, John Baez gives a wonderfully concise analogue:

$(n-1)$-dimensional manifold	Hilbert space
cobordism between $(n-1)$-dimensional manifolds	operator
composition of cobordisms	composition of operators
identity cobordism	identity operator

The first two analogues are between "Space $\to$ States" and "Space-time $\to$ Process."

The structure of this chapter is as follows. We begin with a review of basic category theory. The first section is crucial background material for constructive theorists with little to no background in category theory.

In section two we define TQFTs list the axioms such a theory should obey. Only a few comments are made for general TQFTs. These comments will be in contrast to the functorial theory presented in the next chapter.

After working through the axioms and some important properties of TQFTs we introduce topological reflection positivity. This part follows the work done by Dan Freed and Michael Hopkins in the paper [11].

To understand TQFT fully one should have a fairly thorough understanding of homology. We will not pursue this vein here, as our goal is to extract particular definitions and results and apply them to other functorial field theories. In particular, the results in the latter half of this chapter will be the starting point for chapter 6, in which reflection positivity will be investigated for functorial theories.

3.1 Category Theory

Definition 3.1.1. *A Category C consists of a collection of objects, a collection of morphisms or maps between these objects and composition of morphisms. Composition of morphisms, when defined, is associative and there is an identity morphism for each object.*

Definition 3.1.2. *A Semicategory is a category without an identity morphism.*

Example 3.1.1 (Grp). The objects in the category **Grp** are groups and morphisms are group homomorphisms.

Example 3.1.2 ($\mathbf{Vect}_{\mathbb{C}}$). Objects in $\mathbf{Vect}_{\mathbb{C}}$ are, as its namesake implies, vector spaces over the complex field. Morphisms are linear maps. Much of linear algebra is the study of the category $\mathbf{Vect}_{\mathbb{C}}$

Example 3.1.3 (**Hilb**)**.** The category **Hilb** is an extension of $\mathbf{Vect}_{\mathbb{C}}$. The spaces are now not-necessarily-finite-dimensional Hilbert spaces and the morphisms are linear transformations. For our purposes later on, the mappings will be trace-class operators.

The target categories for TQFTs and functorial QFTs are $\mathbf{Vect}_{\mathbb{C}}$ and **Hilb** respectively. These categories have important additional structures, tensor products. The tensor product turns these categories into monoidal categories.

Definition 3.1.3. *A Monoidal Category is a category $\mathcal{C}$ equipped with an object $1 \in \mathcal{C}$, and a tensor product*

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

which is associative up to natural isomorphism

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$$

and which satisfies the pentagon and triangle identities

$$\begin{array}{ccccc}
((w \otimes x) \otimes y) \otimes z & \xrightarrow{a_{w \otimes x, y, z}} & (w \otimes x) \otimes (y \otimes z) & \xrightarrow{a_{w, x, y \otimes z}} & w \otimes (x \otimes (y \otimes z)) \\
\downarrow{\scriptstyle a_{w,x,y} \otimes \mathrm{id}_z} & & & & \uparrow{\scriptstyle \mathrm{id}_w \otimes a_{x,y,z}} \\
(w \otimes (x \otimes y)) \otimes z & & \xrightarrow{\qquad a_{w, x \otimes y, z} \qquad} & & w \otimes ((x \otimes y) \otimes z)
\end{array}$$

$$\begin{array}{ccc}
(x \otimes 1) \otimes y & \xrightarrow{\quad a_{x,1,y} \quad} & x \otimes (1 \otimes y) \\
& {\scriptstyle \rho_x \otimes 1_y} \searrow \quad \swarrow {\scriptstyle 1_x \otimes \lambda_y} & \\
& x \otimes y &
\end{array}$$

where a, ρ, λ are the associator, right unitor, and left unitor respectively.

Furthermore, there are natural isomorphisms in $\mathbf{Vect}_{\mathbb{C}}$ and **Hilb** giving these categories a symmetric structure.

Definition 3.1.4. *A Symmetric Monoidal Category is a monoidal category along with natural isomorphisms*

$$\gamma_{U,V} : U \otimes V \cong V \otimes U$$

satisfying the hexagonal diagram

$$\begin{array}{ccccc}
(x \otimes y) \otimes z & \xrightarrow[a_{x,y,z}]{} & x \otimes (y \otimes z) & \xrightarrow{\gamma_{x,y\otimes z}} & (y \otimes z) \otimes x \\
\downarrow{\scriptstyle \gamma_{x,y} \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle a_{y,z,x}} \\
(y \otimes x) \otimes z & \xrightarrow{a_{y,x,z}} & y \otimes (x \otimes z) & \xrightarrow{\mathrm{id} \otimes \gamma_{x,z}} & y \otimes (z \otimes x)
\end{array}$$

Example 3.1.4 (**Bord**$_n$)**.** The category **Bord**$_n$ is named for its morphisms, bordisms, from the french "bord" for boundary. In **Bord**$_n$, the objects are closed $(n-1)$-dimensional real manifolds M for some fixed n. Typically, M is the spatial slice of n-dimensional space-time. The unit object is the empty manifold and the tensor product is given by disjoint union. A morphism $M \to N$ is an equivalence class of bordism, for which we need the following definition.

Definition 3.1.5. *Let M, N be oriented, closed, $(n-1)$-dimensional smooth manifolds. A bordism Σ from $M \to N$ is an oriented, compact, n-dimensional manifold with boundary $\partial\Sigma$ such that its boundary is the disjoint union*

$$\partial\Sigma \cong M \sqcup N.$$

In this definition M, N, or both could be the empty manifold. It is common to make the following informal definitions. A closed manifold will be a manifold without boundary. A compact manifold will be mean a manifold with boundary. We will keep this convention.

Example 3.1.5 (**Bord**$_n$ continued)**.** Compositions of morphisms in **Bord**$_n$ occur along boundaries. For instance, if $\Sigma_1 : X \to Y$ and $\Sigma_2 : Y \to Z$ then

$$\Sigma_2 \circ \Sigma_1 : X \to Z$$

is given by gluing along Y.

Definition 3.1.6. *A functor is a function between categories which maps objects to objects and morphisms to morphisms, respecting composition and the identity morphism.*

Our purpose being to understand how reflection positivity crosses QFT formulations, we must define involutions at the categorical level. First, we call an isomorphism between functors a natural transformation.

Definition 3.1.7. *Let $\mathcal{C}$ be a category. An involution of $\mathcal{C}$ is a pair (τ, η) of a functor $\tau : \mathcal{C} \to \mathcal{C}$ and a natural isomorphism $\eta : id_{\mathcal{C}} \to \tau^2$ such that for any $x \in \mathcal{C}$ we have $\tau\eta_x = \eta_{\tau x}$ as morphisms $\tau x \to \tau^3 x$. For symmetric monoidal categories, τ is required to be a symmetric monoidal functor.*

Just as an involution on a manifold M would lift to an involution on $C^\infty(M)$, involutions on categories interact as follows.

Definition 3.1.8. *Let $\mathcal{B}$ and $\mathcal{C}$ be categories with involutions $(\tau_{\mathcal{B}}, \eta_{\mathcal{B}})$ and $(\tau_{\mathcal{C}}, \eta_{\mathcal{C}})$ respectively. Let $F : \mathcal{B} \to \mathcal{C}$ be a functor. Equivariance Data for F is an isomorphism $\varphi : F\tau_{\mathcal{B}} \xrightarrow{\cong} \tau_{\mathcal{C}} F$ of functors $\mathcal{B} \to \mathcal{C}$ such that for every object $x \in \mathcal{B}$ the diagram*

$$\begin{array}{ccc} Fx & \xrightarrow{F\eta_{\mathcal{B}}} & F\tau_{\mathcal{B}}^2 x \\ & \searrow^{\eta_{\mathcal{C}}} & \downarrow{\scriptstyle \phi^2} \\ & & \tau_{\mathcal{C}}^2 F x \end{array}$$

commutes.

Let x be an object in a symmetric monoidal category $\mathcal{C}$, then we call $x^\vee$ the 'dual' of x.

Finally, to make sense of an inner product being positive definite we must find a categorical equivalent. We may do so for (semi) categories admitting certain involutions.

Definition 3.1.9. *Let* (τ, η) *be an involution on a symmetric monoidal category* $\mathcal{C}$*. A hermitian structure on an object* $x \in \mathcal{C}$ *is an isomorphism* $h : \tau x \to x^\vee$ *such that the composition*

$$\tau x \cong \tau((x^\vee)^\vee) \xrightarrow{\tau(h^\vee)} \tau(\tau x)^\vee) \cong \tau^2(x^\vee) \xrightarrow{\eta^{-1}} x^\vee$$

is equal to h*.*

3.2 Topological Quantum Field Theory

With the basic tools of category theory we can now define TQFTs.

Definition 3.2.1. *An* n*-dimensional topological quantum field theory (TQFT) is a symmetric monoidal functor*

$$\mathcal{F} : \boldsymbol{Bord}_n \to \boldsymbol{Vect}_{\mathbb{C}}$$

As a functor $\mathcal{F}$ assigns objects to objects, so

$$(n-1)\text{-dimensional manifolds } M \quad \xrightarrow{\mathcal{F}} \quad \text{vector spaces over } \mathbb{C}$$

The vector space $\mathcal{F}(M)$ is typically thought of as the analog of the state space $\mathcal{H}_M$ of the quantum system. We discuss other possibilities in chapter 6.

Secondly, morphisms are mapped to morphisms. For a TQFT that means

$$\text{Bordisms } \Sigma : M \to N \quad \xrightarrow{\mathcal{F}} \quad \text{linear maps } \mathcal{F}(\Sigma) : \mathcal{F}(M) \to \mathcal{F}(N)$$

Linear maps will be thought of as time evolution from M to N and objects will be regarded as spatial slices of our space-time.

3.2.1 Axioms

Again, the axioms of TQFT have been given and modified numerous times in many different directions. The axioms listed here are those that are believed to be most commonly used, for example in [6], or on nLab.

1. Naturality: An orientation preserving diffeomorphism $d : M \to N$ induces an isomorphism $\mathcal{F}(M) \to \mathcal{F}(N)$

2. Multiplicativity: $\mathcal{F}$ comes with isomorphisms

$$\mathcal{F}(\emptyset) = \mathbb{C}; \quad \mathcal{F}(M \sqcup N) \cong \mathcal{F}(M) \otimes \mathcal{F}(N)$$

   compatible which are compatible with associativity of the tensor products and with the symmetries γ.

3. Gluing of manifolds corresponds to composition of linear maps

$$\mathcal{F}(\Sigma_1 \sqcup_M \Sigma_2) = \mathcal{F}(\Sigma_1) \circ \mathcal{F}(\Sigma_2)$$

   where $\sqcup_M$ denotes gluing the disjoint union along the common boundary M. In this case, M is the outgoing boundary of Σ_1 and the in-going boundary of Σ_2.

4. $\mathcal{F}$ is involutory: $\mathcal{F}(M^\vee) \cong \overline{\mathcal{F}(M)}$ where $M^\vee$ is M with the opposite orientation (its dual object) and $\overline{\mathcal{F}(M)}$ is the dual space of $\mathcal{F}(M)$.

In the category $\mathbf{Bord}_n$ every object M has a "dual object" here denoted by $M^\vee$. Additionally, bordisms are n-dimensional and objects are $(n-1)$-dimensional. Each object comes with a normal orientation into n dimensions called the arrow of time. The manifold $M^\vee$, then, is the same as M but with the arrows of time reversed.

With duals at our disposal we can define the evaluation map. We begin graphically by imagining a cylinder. We label both ends, the incoming and outgoing boundaries, as M. The cylinder itself is the bordism Σ. We now "bend" the cylinder in half so that it looks like half of a doughnut pointing to the left (or down, depending on ones preference).

In other words, the "straight cylinder" with incoming and outgoing boundaries M is the same as the bent cylinder with incoming boundary the disjoint union of M and M with its opposite orientation. There are no outgoing boundaries in this version, so the bordism is to the empty manifold. since we can associate the oppositely oriented M with its dual according to axiom (4), we get a natural pairing.

More rigorously, for the evaluation map can write

$$ev_M = [0,1] \times M : M^\vee \sqcup M \to \emptyset^{n-1}.$$

We use a similar argument to understand the coevaluation map. The difference is that now our cylinder is bent the other direction. Taking the incoming boundary as the empty set and the outgoing boundary as the disjoint union of M and $M^\vee$ we write

$$co_M : \emptyset^{n-1} \to M^\vee \sqcup M.$$

Since $\mathbf{Bord}_n$ is a category, we have an identity morphism for every object. This, combined with our description of the evaluation and coevaluation maps gives the following result:

Proposition 3.2.1. *Let $\mathcal{F}$ be a TQFT, then $\mathcal{F}(M)$ is finite dimensional for every (object) $M \in \boldsymbol{Bord}_n$.*

This result is specific to TQFTs. We will see in chapter 4 that by adding a geometric structure back onto our bordisms we lose this finiteness (hence the transition from $\mathbf{Vect}_{\mathbb{C}}$ to **Hilb**).

Corollary 3.2.2. *For an object M in $\textbf{\textit{Bord}}_n$, $\mathcal{F}(M \times S^1) = \dim(\mathcal{F}(M))$*

3.3 Topological Reflection Positivity

As discussed in the introduction and chapter 2, reflection positivity is a key component in any viable quantum field theory. Unitarity had been implemented in the axiom systems of Atiyah and Segal but reflection positivity does not appear to have been given the same importance in TQFT as it has in CQFTs. In 2016, however, Dan Freed and Michael Hopkins published a paper focused on reflection positivity via a Wick-rotated symmetry group H_n. Throughout this section we follow the work done by Freed and Hopkins [11] and make use of their notation:

- M^n: Minkowski space-time with isometry group of $I^n \equiv Isom(M^n)$
- $\mathscr{G}_n$: Global symmetry group
- ρ: a homomorphism $\rho : \mathscr{G}_n \to I_n$
- K: Internal symmetry group $K = \ker \rho$
- G_n: Global symmetry group modulo translations $\mathscr{G}_n/(\text{translation subgroup})$
- H_n: compact real form of complexification $G_n(\mathbb{C})$ of G_n

The symmetry group H_n fits into the short exact sequence of compact Lie groups

$$1 \longrightarrow K \longrightarrow H_n \xrightarrow{\rho_n} O_n$$

We use this symmetry group to augment the definition of a TQFT as follows

Definition 3.3.1. *A topological field theory with Wick-rotated vector symmetry group H_n is a symmetric monoidal functor*

$$F : \boldsymbol{Bord}_n(H_n) \longrightarrow \boldsymbol{Vect}_{\mathbb{C}}$$

to the symmetric monoidal category of complex vector spaces under tensor product.

The previous properties of a TQFT still hold but now our objects are compact $(n-1)$-manifolds M without boundary equipped with an H_n structure $P \to M$ and an arrow of time.

Definition 3.3.2. *An H_n-structure is a pair (P, θ) consisting of a principal H_n-bundle $P \to \Sigma$ equipped with an isomorphism of principal O_n-bundles $\mathcal{B}_O(\Sigma) \xrightarrow{\theta} \rho_n(P)$.*

This structure is enough for the discussion throughout this chapter, but the following definition will be important subsequently.

Definition 3.3.3. *A differential H_n-structure is a connection Θ on $P \to \Sigma$ such that θ maps the Levi-Civita connection to $\rho_n(\Theta)$*

Definition 3.3.4. *An H_n-manifold is a Riemannian n-manifold equipped with an H_n-structure. For an $(n-1)$-dimensional manifold, an H_n-structure is given by stabilizing the tangent bundle of M by summing with a trivial line bundle: $R \oplus TY \to Y$.*

A morphism $\Sigma : M \to N$ in $\mathbf{Bord}_n(H_n)$ is a compact n-manifold Σ with H_n structure and an isomorphism of the boundary $\partial\Sigma \to M \sqcup N$

In order to discuss reflection positivity for this theory we first pass from H_n to the extended symmetry group $H_n \times \{\pm 1\}$. We then extend the principal H_n-bundle $P \to \Sigma$ to a principal $H_n \times \{\pm 1\}$-bundle $i(P) \to \Sigma$ where i is the inclusion map

$i : H_n \to H_n \times \{\pm 1\}$. The isomorphism θ extends to an isomorphism $\hat{\theta}$ over the principal bundles.

Definition 3.3.5. *The opposite H_n-structure, H_n^{op}-structure (P', θ'), is the principal H_n-bundle $P' := i(P)\backslash P \to \Sigma$ and the restriction θ' of $\hat{\theta}$ to $\{-1\} \times \mathcal{B}_O(\Sigma)$*

Definition 3.3.6. *Define the involution of categories*

$$\beta_{\mathcal{B}} : \boldsymbol{Bord}_n(H_n) \to \boldsymbol{Bord}_n(H_n)$$

as fixing the underlying manifold and involuting

$$H_n \textit{structure} \to H_n^{op} \textit{structure}$$

Recall that each object M in $\mathbf{Bord}_n$ has a dual $M^\vee$. We maintain this result in $\mathbf{Bord}_n(H_n)$. Along with the preceding definition we get

Proposition 3.3.1. *For any object Y in $\boldsymbol{Bord}_n(H_n)$, there is a canonical isomorphism $\beta_{\mathcal{B}} Y \xrightarrow{\cong} Y^\vee$*

The proof is given in [11]. The importance of this proposition is that every object in $\mathbf{Bord}_n(H_n)$ has a canonical hermitian structure.

Definition 3.3.7. *Complex conjugation $\beta_{\mathcal{C}}$ is an involution of categories*

$$\beta_{\mathcal{C}} : \boldsymbol{Vect}_{\mathbb{C}} \to \boldsymbol{Vect}_{\mathbb{C}}$$

Definition 3.3.8. *A reflection structure on a functor $\mathcal{F}$ is equivariance data for the involutions $\beta_{\mathcal{B}},\ \beta_{\mathcal{C}}$*

For every closed $(n-1)$-manifold M with H_n-structure we have an isomorphism of vector spaces $\mathcal{F}(\beta_{\mathcal{C}} M) \cong \overline{\mathcal{F}(M)}$. The isomorphisms between reflection structures gives a hermitian form

$$\mathcal{F}(ev_M) = h_M : \mathcal{F}(M^\vee) \otimes \mathcal{F}(M) \cong \mathcal{F}(\beta_{\mathcal{C}} M) \otimes \mathcal{F}(M) \cong \overline{F(Y)} \otimes F(Y) \to \mathbb{C}$$

Hermitian structure in hand, we are now able to define reflection positivity for a topological quantum field theory.

Definition 3.3.9. *A reflection structure is positive if the induced hermitian form* h_Y *is positive definite for all* $Y \in \boldsymbol{Bord}_n(H_n)$

For an object in the bordism category we may form its double. Visually, this is done by first making a duplicate copy of the original manifold and reversing its orientation. The two copies share a common boundary and so are glued along this boundary. Rigorously, we define the double as follows.

Definition 3.3.10. *Let* Σ *be a compact* H_n*-manifold with boundary, viewed as a bordism* $\emptyset^{n-1} \to \partial\Sigma$*. The double of* Σ *is the closed* H_n*-manifold* $\Sigma_{Dbl} = e_{\partial\Sigma}(\Sigma^\vee, \Sigma)$

Reflection positivity in the topological setting gives immediate interesting results about topological invariants (though these will not be discussed here). Of use for these results is the following fact:

Proposition 3.3.2. *If a theory* $\mathcal{F} : \boldsymbol{Bord}_n(H_n) \to \boldsymbol{Vect}_{\mathbb{C}}$ *admits a reflection positive structure, then* $\mathcal{F}(\Sigma_{Dbl}) \geq 0$ *for all compact* H_n*-manifolds with boundary*

Chapter 4

Riemannian Functorial Quantum Field Theory

In this chapter we discuss the semi-category $\mathbf{RBord}_n$ and results by Stephan Stolz, Peter Teichner, Doug Pickrell, and most recently Santosh Kandel. The semicategory $\mathbf{RBord}_n$ differs from the category $\mathbf{Bord}_n$ in that we do not necessarily have identity morphisms. It is because of this that a number of results from TQFTs do not extend.

Recall from chapter 3 that for a TQFT $\mathcal{F}$, every object in the bordism category came with an identity morphism. This led directly to the result $\dim(\mathcal{F}(M)) < \infty$ for every object M. As it is our intention to move closer to CQFTs, we relinquish this control over dimension. In a CQFT the space of states is infinite dimensional and so a logical extension of the target category is **Hilb**.

Though we no longer obtain topological invariants from our functors, we will see that the targets do indeed resemble the spaces seen in CQFTs. As such, it is believed that the sequence

$$\text{TQFT} \longrightarrow \text{RFQFT} \longrightarrow \text{CQFT}$$

will provide valuable insights into complex problems in both physics and mathematics.

4.1 Riemannian Functorial Quantum Field Theory

To move from TQFTs, we now give the metric back to our Riemannian manifolds. The semi-category $\mathbf{RBord}_n$ differs from $\mathbf{Bord}_n$ primarily in that the objects are closed $(n-1)$-dimensional Riemannian manifolds M with metric. A morphisms $\Sigma : M \to N$ is an n-dimensional compact oriented Riemannian manifold with orientation preserving isometry $\overline{M} \sqcup N \to \partial\Sigma$.

In order for composition of morphisms to be well defined, morphisms are required to have product metric near the boundary. This category is discussed in length to varying degrees of rigor in [34], [27], and others.

The target category will be changed from $\mathbf{Vect}_{\mathbb{C}}$ to **Hilb**. The objects of **Hilb** are Hilbert spaces, as the name implies, and morphisms are continuous linear operators. **Hilb** is still a symmetric monoidal category as defined in chapter 3, so disjoint unions will again be mapped to tensor products.

We give the formal definition of an RFQFT as

Definition 4.1.1. *A **Riemannian Functorial Quantum Field Theory (RFQFT)** is a functor $\mathcal{Z}$: $\boldsymbol{RBord}_n \to \boldsymbol{Hilb}$ that maps disjoint unions into tensor products.*

For the RFQFT developed in [19], the action functional addressed is given by

$$S(\phi) = \frac{1}{2}\int_{\Sigma} d\phi \wedge *d\phi + m^2 * \phi^2$$

where $*$ is the Hodge star operator associated to the Riemannian metric, m is a positive real number, and ϕ is a field.

As we will see, target objects are the infinite dimensional Hilbert spaces $L^2(D'(M), d\mu)$ for suitable Gaussian measure on the manifold M with reproducing kernel Hilbert space the Sobolev space $W^{1/2}(M)$. The target morphisms are trace class operators. Composition of bordisms leads directly to composition of operators and disjoint union of objects is the tensor product of Hilbert spaces.

4.2 Assignments in an RFQFT

The ideas in this section come from various sources but most recently the works of Kandel and Pickrell. For proofs of the theorems, lemmas, and corollaries in this section see [19] and [27]. This material will be used extensively in the main results in chapters 6 and 7. The goal of this section is to provide a recipe for the RFQFT $\mathcal{Z}$. To do so we first provide the analytical prerequisites.

Let $C^\infty(\Sigma)$ be the space of real valued smooth functions on a manifold Σ, Δ_Σ be the non-negative Laplacion on Σ, and $m > 0$. Assume $M = \partial\Sigma \neq \emptyset$. Let ν be the outward unit normal vector to M and $i : M \to \Sigma$ be the inclusion map.

It is a fact that the Dirichlet problem

$$(\Delta_\Sigma + m^2)\phi = 0; \quad \phi|_M = \eta$$

has a unique solution $\phi_\eta \in C^\infty(\Sigma)$ for all $\eta \in C^\infty(M)$. The unique solution is the Helmholtz extension of η.

Definition 4.2.1. *The operator on M defined by*

$$D_\Sigma \eta = \frac{\partial \phi_\eta}{\partial \eta}$$

is called the Dirichlet-to-Neumann operator associated to the Helmholtz operator $\Delta_\Sigma + m^2$

Let M be a closed oriented Riemannian manifold, $m > 0$, and $s \in \mathbb{R}$. The bilinear form on $C^\infty(M)$ given by

$$\langle f, g \rangle_s = \int_M f(\Delta_M + m^2)^s g \, d\mathrm{vol}(M)$$

defines an inner product on $C^\infty(M)$.

Definition 4.2.2. *The Sobolev space $W^s(M)$ is the completion of $C^\infty(M)$ with respect to the inner product $\langle f, g \rangle_s$*

Lemma 4.2.1. *Let Σ be a compact oriented Riemannian manifold and $\partial\Sigma = M$. Then*

$$\alpha_\Sigma = D_\Sigma(\Delta_M + m^2)^{1/2}$$

defines a continuous positive operator on $W^{1/2}(M)$

Corollary 4.2.2.

$$\alpha_{\Sigma_2} \circ \alpha_{\Sigma_1} = \alpha_{\Sigma_2 \circ \Sigma_1}$$

Definition 4.2.3. *Let X be a Banach space with norm $\|x\|$ and X^* be the topological dual for X with with norm $\|\xi\|$. A probability measure μ on $(X, \mathfrak{A}_X)$ is Gaussian with variance σ^2 if for every $\xi \in X^*$ and real number α,*

$$\mu[x \in X \mid \xi(x) \leq \alpha] = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}\sigma} e^{-u^2/2\sigma^2} du$$

Lemma 4.2.3. *Let M be an object in $\boldsymbol{RBord}_n$ and consider the inner product on $C^\infty(M)$ given by*

$$\langle f, g \rangle_{W^{1/2}(M)} = \int_M f(\Delta_M + m^2)^{1/2} g \, dvol(M).$$

Let μ_M be the corresponding Gaussian measure on the space of distributions on M, $D'(M)$. The Sobolev space $W^{1/2}(M)$ is the Cameron-Martin space of μ_M

Given a measure μ on a separable Banach space B, The Cameron-Martin space is the associated reproducing kernel Hilbert space [4].

Definition 4.2.4. *The Bosonic Fock space of a Hilbert space H is the Hilbert space direct sum*

$$\bigoplus_{n=0}^{\infty} Sym^n(H) = \{(\alpha_n)_{n=0}^{\infty} \mid \alpha_n \in Sym^n H \text{ with } \sum_{n=0}^{\infty} \|\alpha_n\|^2 < \infty$$

where $Sym^n(H)$ is the closed subspace of $H^{\otimes n}$ that is invariant under the action of the permutation group S_n. We use $Sym^n(H)$ to denote the Bosonic Fock space of H.

Theorem 4.2.4. *Let X be a nuclear space, μ a Gaussian measure on $X^\vee$ and $H(\mu)$ the reproducing kernel Hilbert space of μ. Then there is an isomorphism of Hilbert spaces, the Segal-Ito isomorphism*

$$S : L^2(X^\vee, \mu) \to Sym^* H(\mu)^\vee$$

where $Sym^* H(\mu)^\vee$ is the Bosonic Fock space of $H(\mu)^\vee$

Now, to an object M in $\mathbf{RBord}_n$ we assign the Hilbert space of states

$$\mathcal{Z}(M) = Sym^* W^{1/2}(M)^\vee$$

Thus, $\mathcal{Z}(M)$ is the L^2 space of the Gaussian measure μ on $D'(M)$ whose reproducing kernel Hilbert space is $W^{1/2}(M)$. The complexification of $\mathcal{Z}(M)$ is an irreducible unitary representation of the standard Heisenberg group of $C^\infty(M) \oplus C^\infty(M)$.

By construction we have:

- $\mathcal{Z}(\emptyset) = \mathbb{R}$
- $\mathcal{Z}(\overline{M}) = \mathcal{Z}(M)^\vee$
- $\mathcal{Z}(M \sqcup N) = \mathcal{Z}(M) \otimes \mathcal{Z}(N)$

Let $\Sigma : \emptyset \to M$ be a morphism in $\mathbf{RBord}_n$. To make the proper assignment and obtain a vector $\mathcal{Z}(\Sigma) \in \mathcal{Z}(M)$ one must first address composition of operators.

For $\alpha_\Sigma = D_\Sigma(\Delta_M + m^2)^{1/2}$, we have the result

Lemma 4.2.5. *$\alpha_\Sigma - I$ is a trace class operator on $W^{1/2}(M)$.*

Definition 4.2.5. *Let H be a real Hilbert space and let $A : H \to H$ be a continuous positive operator. The Cayley transform of A is defined by*

$$C(A) = (I - A)(I + A)^{-1}$$

The Cayley transform of α_Σ, $C(\alpha_\Sigma)$, is a Hilbert-Schmidt symmetric operator on $W^{1/2}(M)$ with $\|C(\alpha_\Sigma)\| < 1$. Furthermore,

$$Cay := \mathrm{Exp}(1/2C(\alpha_\Sigma)) \in Sym^* W^{1/2}(M)^\vee$$

If $\Sigma : M \to N$ is a morphism in $\mathbf{RBord}_n$, then - just as we did for a TQFT - Σ can be seen as the morphism

$$\Sigma : \emptyset \to N \sqcup \overline{M}$$

in which case

$$Cay \in \mathcal{Z}(M)^\vee \otimes \mathcal{Z}(N)$$

and we can additionally identify Cay with a Hilbert-Schmidt operator

$$Cay : \mathcal{Z}(M) \to \mathcal{Z}(N)$$

Definition 4.2.6. *A projective representation T of a category $\mathcal{C}$ in **Hilb** assigns to an object $C \in \mathcal{C}$ a Hilbert space $T(C)$ and to a morphism $P : C \to D$ a continuous linear operator $T(P)$ such that for any pair of morphisms $P : C \to D$ and $Q : D \to E$ we have*

$$T(Q \circ P) = \lambda(Q, P)T(Q) \circ T(P)$$

where $\lambda(Q, P)$ is a nonzero complex number.

Theorem 4.2.6. *The assignment $M \to \mathcal{Z}(M)$ where M is an object in $\boldsymbol{RBord_n}$ and $\Sigma \to Cay$ where $\Sigma : \emptyset \to M$ is a morphism in $\boldsymbol{RBord_n}$ defines a projective representation of $\boldsymbol{RBord_n}$ in **Hilb***

The proof, in [19], gives a number of computations which make calculating λ possible.

The next step is to deprojectivize. Starting with morphisms, for $\Sigma : \emptyset \to M$ define

$$\mathcal{Z}(\Sigma) = \frac{1}{\det_\zeta(\Delta_{\Sigma,D} + m^2)^{1/2} \cdot \det_\zeta(2D_\Sigma)^{1/4}} \cdot \frac{Cay}{\|Cay\|}$$

where $\Delta_{\Sigma,D}$ is the operator Δ_Σ with Dirichlet boundary condition. The zeta regularized determinant $\det_\zeta$ is an infinite dimensional extension of the determinant. Definitions and computations for the zeta regularized determinant of the Laplacian can be found in [20].

Lemma 4.2.7. *Let $\Sigma : \emptyset \to M$, then*

$$\det(\alpha_\Sigma) = \frac{\det_\zeta(2D_\Sigma)}{\det_\zeta(2(\Delta_M + m^2)^{1/2})}$$

Theorem 4.2.8. *Let $\Sigma_1 : M \to N$ and $\Sigma_2 : N \to L$ be two morphisms in* $\boldsymbol{RBord}_n$*, then*

1. *There exists a nonzero constant C_{Σ_2,Σ_1} such that*

$$\mathcal{Z}(\Sigma_2) \circ \mathcal{Z}(\Sigma_1) = C_{\Sigma_2,\Sigma_1} \mathcal{Z}(\Sigma_2 \circ \Sigma_1)$$

2. *When n is even, $C_{\Sigma_2,\Sigma_1} = 1$*

Corollary 4.2.9. *Suppose that Σ is a closed $n-$dimensional oriented manifold and n is even. Assume that $\Sigma = \Sigma_2 \circ \Sigma_1$ in* $\boldsymbol{RBord}_n$ *where $\Sigma_1 : \emptyset \to M$ and $\Sigma_2 : M \to \emptyset$. Then*

$$\mathcal{Z}(\Sigma) = \frac{1}{\det_\zeta(\Delta_\Sigma + m^2)^{1/2}}$$

4.3 $\mathbf{RBord}_n(H_n)$

The semi-category $\mathbf{RBord}_n(H_n)$ will be the semi-category $\mathbf{RBord}_n$ enriched with an H_n-structure as defined in chapter 3. The objects in $\mathbf{RBord}_n(H_n)$ are now four-tuples $M \equiv (M, H_n, g, \nabla)$ where M is the manifold, H_n the tangent structure, g the metric on H_n, and ∇ the connection on the tangent structure. The connection will be assumed to be compatible with the Levi-Civita connection. Morphisms $\Sigma : M \to N$ in $\mathbf{RBord}_n(H_n)$ will be assumed to have product metric near the boundary.

All results from the previous section carry through with only minor adjustments. To an object in $\mathbf{RBord}_n(H_n)$ we assign the Hilbert space

$$\mathcal{Z}(M) = Sym^* W^{1/2}(M, H_n)^\vee.$$

To a morphism $\Sigma \equiv (\Sigma, H_n, g, \nabla)$ such that

$$\Sigma : \emptyset \to (M, H_n, g, \nabla)$$

we assign the vector

$$\mathcal{Z}(\Sigma) = \frac{1}{\det_\zeta(\Delta_{\Sigma,D} + m^2)^{1/2} \cdot \det_\zeta(2D_\Sigma)^{1/4}} \cdot \frac{Cay}{\|Cay\|}.$$

As in the previous section, this defines an RFQFT for all dimensions but is projective when the dimension is odd.

Chapter 5

The RP Correspondence

We come now to the main results. We briefly state some preliminary results on TQFTs and RFQFTs, building upon the works of Freed/Hopkins, Kandel, and others. We then extend our definition of an RFQFT to include an H_n-structure.

In the second section we prove the main theorem: There is a correspondence between Reflection Positive Hilbert Spaces and RFQFTs.

5.1 Functorial Reflection Positivity

We begin by extending the definition of reflection positivity to include non-topological theories. Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal (semi)categories with involutions. Let $\mathcal{Z}$ be a functor $\mathcal{Z} : \mathcal{C} \to \mathcal{D}$ with equivariance data.

Definition 5.1.1. *A functorial quantum field theory $\mathcal{Z}$ is said to be* ***Reflection Positive*** *if its induced hermitian form h_M is positive definite for all M in the source category.*

Lemma 5.1.1. *There is a natural hermitian form on the category* $\boldsymbol{RBord}_n$*.*

Proof. Let τ be the functor on $\mathbf{RBord}_n$ that gives a dissecting reflection for each manifold. Since reflections are isometries, $\tau^2 M$ is isometrically isomorphic to M. Hence τ gives an involution on $\mathbf{RBord}_n$. Swapping the arrow of time is a reversal of the normal direction, another isometry. Hence there is a natural isometric isomorphism between dissecting reflections and orientation reversal satisfying definition 3.1.9. In other words, there is a natural hermitian structure on $\mathbf{RBord}_n$.

$$h_M : \tau M \to M^\vee$$

□

For the Riemannian FQFT of the previous chapter we obtain a deeper result.

Lemma 5.1.2. *Let $\mathcal{Z} : \boldsymbol{RBord}_n \to \boldsymbol{Hilb}$ be the Riemannian functorial quantum field theory such that $\mathcal{Z}(M) = Sym^* W^{1/2}(M)^\vee$. Then $\mathcal{Z}$ is a reflection positive functorial quantum field theory.*

Proof. Let M be an object in $\mathbf{RBord}_n$. From 5.1.1, there is an involution τ on $\mathbf{RBord}_n$ which gives a dissecting reflection for each manifold and gives a natural hermitian structure. Let θ be the involution on $L^2(M)$ defined by $\theta f = f \circ \tau$. Then the induced hermitian structure

$$h_M : \mathcal{Z}(\tau M) \otimes \mathcal{Z}(M) \to \mathbb{R}$$

is the pairing in the Sobolev space

$$\langle \theta f, f \rangle_{-1} = \langle \theta f, Cf \rangle$$

From [15] and [1] we know that $C = (\Delta_M + m^2)^{-1}$ is a reflection positive operator for suitable Riemannian manifolds. So, for each manifold M we have $\langle \theta f, f \rangle_s \geq 0$ and $\mathcal{Z}$ is a reflection positive theory. □

When combined with proposition 3.3.2, this gives the following result.

Corollary 5.1.3. *For the RFQFT $\mathcal{Z}$ of chapter 4, $\mathcal{Z}(\Sigma_{Dbl}) \geq 0$ for all compact manifolds with boundary.*

Suppose $\mathcal{Z}$ is an FQFT that is not reflection positive. Collect all objects M in the source category $\mathcal{C}$ such that the induced hermitian form is positive semi-definite. Additionally, take all morphisms between these objects. We call this collection RPn.

Definition 5.1.2. *A subcategory S of a category $\mathcal{C}$ is a collection of objects and morphisms from $\mathcal{C}$ such that for each morphism A in S the domain and codomain are in S. Each object comes with its identity morphism. For each composable pair of morphisms $A : X \to Y$ and $B : Y \to Z$, its composite $BA : X \to Z$ is in S.*

Proposition 5.1.4. *RPn is a subcategory for $\textbf{Bord}_n$ and subsemicategory for $\textbf{RBord}_n$.*

Proof. The hermitian form takes in one object at a time, along with its dual, and disregards other morphisms. If a morphism is in RPn, its source and target are in RPn by construction. For objects in RPn, morphisms between these objects are unaffected by the hermitian structure and so are included, i.e. composable morphisms are still composable. Thus for $\mathbf{Bord}_n$, RPn is a subcategory. For $\mathbf{RBord}_n$, RPn would not include identity morphisms unless they were included in the parent category. Therefore RPn is a subsemicategory of $\mathbf{RBord}_n$. □

Let $\mathcal{Z} : \mathbf{Bord}_n \to \mathbf{Vect}_{\mathbb{C}}$ be an FQFT. The reflection positive theory obtained by restricting $\mathcal{Z}$ to RPn will be denoted $\mathcal{Z}_{RP}$. Note that if the theory is topological and already reflection positive then $RPn = \mathbf{Bord}_n$. We may therefore restrict our attention to RP theories in a natural way.

5.1.1 RFQFTs with H_n-structure

Throughout this section we apply results from chapter three to the RFQFT in chapter four. Recall from chapter four the semi-category $\mathbf{RBord}_n(H_n)$. The objects in $\mathbf{RBord}_n(H_n)$ are four-tuples $M \equiv (M, H_n, g, \nabla)$ where M is the manifold, H_n the tangent structure, g the metric on H_n, and ∇ the connection on the tangent structure. The connection will be assumed to be compatible with the Levi-Civita connection. Morphisms $\Sigma : M \to N$ in $\mathbf{RBord}_n(H_n)$ will be assumed to have product metric near the boundary.To an object in $\mathbf{RBord}_n(H_n)$ we assign the Hilbert space

$$\mathcal{Z}(M) = Sym^* W^{1/2}(M, H_n)^{\vee}.$$

To a morphism $\Sigma \equiv (\Sigma, H_n, g, \nabla)$ such that

$$\Sigma : \emptyset \to (M, H_n, g, \nabla)$$

we assign the vector

$$\mathcal{Z}(\Sigma) = \frac{1}{\det_\zeta(\Delta_{\Sigma,D} + m^2)^{1/2} \cdot \det_\zeta(2D_\Sigma)^{1/4}} \cdot \frac{Cay}{\|Cay\|}.$$

Proposition 5.1.5. *Let $M \in \boldsymbol{RBord}_n(H_n)$, $\epsilon > 0$. Let Θ be a dissecting reflection of $(-\epsilon, \epsilon) \times M$. Then Θ lifts to an involution on the tangent structure and there is an isometric isomorphism between an Hn involution and time reversal*

$$\beta_t M \to M^\vee$$

Hence, there is an induced hermitian form coming from the H_n structure.

Proof. The proof of this proposition in terms of manifolds without metric is done in [11]. The manifold M can be seen as embedded in $(-\epsilon, \epsilon) \times M$ and an involution at this level lifts to the frame bundle. The difference here is that "not every germ admits a reflection which is an isometry." This issue is resolved by demanding that our involution is a dissecting reflection of the manifold $(-\epsilon, \epsilon) \times M$. □

Combining this result with the previous section, we get the following corollary:

Corollary 5.1.6. $\mathcal{Z}_{RP}(X_{Dbl}) \geq 0$ *for all compact H_n-manifolds with boundary*

5.2 RP Correspondence

The main result of this work is the classification of all reflection positive Hilbert spaces in terms of functorial quantum field theories.

$$\text{RPHS} \iff \text{RPFQFT}$$

Theorem 5.2.1. *Suppose M is a complete, connected manifold. Given a reflection positive Hilbert space $(\mathcal{E}, \mathcal{E}_+, \theta)$ corresponding to $L^2(M)$, there is a reflection positive functorial quantum field theory $\mathcal{Z}_{RP}$ such that*

1. *$\hat{\mathcal{E}}$ is the target object of $\mathcal{Z}_{RP}$*

2. $\mathcal{E}_+$ and θ are recoverable from $\hat{\mathcal{E}}$

Similarly, every reflection positive functorial quantum field theory generates reflection positive Hilbert spaces

Proof. Let M be a connected complete Riemannian manifold, $C = (\Delta_M + m^2)^{-1}$, and τ be a dissecting reflection, then we have the reflection positive Hilbert space $(\mathscr{E}, \mathscr{E}_+, \theta)$ where:

- $\mathcal{E}$ is the completion of of $L^2(M)$ with respect to the inner product $\langle Cf, g\rangle_{L^2(M)}$
- $\mathcal{E}_+$ is generated by $C_C^\infty(M_+)$
- θ is an involution on $L^2(M)$ defined by $\theta f = f \circ \tau$

From lemma 5.1.1 τ is an involution on $\mathbf{RBord}_n$ giving a natural hermitian structure. By lemma 5.1.2, the Riemannian FQFT of chapter 4 is a reflection positive theory. Hence $\mathcal{Z}_{RP} \equiv \mathcal{Z}$.

Similarly, let $\mathcal{Z} : \mathbf{RBord}_n \to \mathbf{Hilb}$ be an equivariant functor for involutions τ and θ, respectively. For an object $M \in \mathbf{RBord}_n$

$$ev : \tau M \sqcup M \to \emptyset^{n-1}$$

Applying the RFQFT gives

$$\mathcal{Z}(ev) : \mathcal{Z}(\tau M \sqcup M) \cong \mathcal{Z}(\tau M) \otimes \mathcal{Z}(M) \cong \theta\mathcal{Z}(M) \otimes \mathcal{Z}(M) \to \mathbb{R}$$

Suppose $\mathcal{Z} = \mathcal{Z}_{RP}$ is a reflection positive theory, then the induced hermitian form $\langle \theta \cdot, \cdot \rangle \geq 0$.

Then $\mathscr{E}_+ \cong \mathcal{Z}(M)$ and we have an RPHS $(\mathscr{E}, \mathscr{E}_+, \theta)$

Hence, for every Riemannian manifold M, the functorial QFT corresponding to its $(\mathscr{E}, \mathscr{E}_+, \theta)$ is given by that of chapter 4. □

Chapter 6

Examples and Applications

6.1 Iterated Doubles

Recall from chapter 3 the double of a compact manifold with boundary:

Definition 6.1.1. *Let X be a compact n-manifold with boundary viewed as a bordism $\varnothing^{n-1} \to \partial X$ and let σ be an involution. The **double** of X is*

$$Dbl(X) = X \cup_{\partial X} \sigma(X)$$

To move from doubles to iterative doubles, we employ the following construction. Begin with the compact $1-$manifold X_1 and construct the double $Dbl(X_1)$. We then view the closed manifold $Dbl(X_1)$ as a boundary and construct the bordism $\varnothing^2 \to Dbl(X_1)$. Label the new compact manifold X_2. Again, we construct the double $Dbl(X_2)$ and then continue this process indefinitely. Formally, we define iterative doubles as follows.

Definition 6.1.2. *An iterative double of X is a sequence of null-cobordant manifolds described by the process*

$$X = \rtimes\!\!\ltimes X_1 := X_1 \to Dbl(X_1) \xrightarrow{\varnothing^2} X_2 \to Dbl(X_2) \to \cdots \to Dbl(X_{n-1}) \xrightarrow{\varnothing^n} X_n \to \cdots$$

We will refer to the truncated sequence of an iterative double as iterated and will be denoted

$$X = \rtimes\!\!\ltimes_n X_1 := X_1 \to Dbl(X_1) \xrightarrow{\varnothing^2} X_2 \to Dbl(X_2) \to \cdots \to Dbl(X_{n-1}) \xrightarrow{\varnothing^n} X_n$$

Remark 6.1.1. Every n-sphere is the boundary of an iterated double. Every n-ball is an iterated double.

To see this, create a one-dimensional disk D^1 (a line segment) in $\mathbb{R}$ of radius r. Copy this disk to create a second of the same length. Connect the disks by identifying the endpoints and form the circle S^1. Fill in the circle to create the disk D^2. Copy the new D^2, connect the boundaries, and create the sphere S^2. Fill in the sphere to create the ball D^3, copy, connect, and continue.

6.1.1 Measurements

Throughout this section, k will denote the dimension of the ambient space $\mathbb{R}^k$. Let $V_k(r)$ denote the volume of a sphere in dimension k with beginning radius r (i.e., $V_3(r)$ is the volume of S^2, the sphere in $\mathbb{R}^3$). Similarly, let $S_k(r)$ denote its surface area. $\Gamma(x)$ is the usual gamma function and $\mathrm{B}(x, y)$ is the beta function.

Proposition 6.1.1. *Given a starting radius r, the radius R_k of a sphere in $\mathbb{R}^k$ taken as an iterated double is given by*

$$R_k = \prod_{i=1}^{k-1} \left(\frac{2^{2-i}}{i\mathrm{B}(\frac{i}{2}, \frac{i}{2})} \right)^{1/i} r$$

Proof. Begin with the interval D^1 of radius r and volume (length) $2r$. Then the iterated k-ball D^k is given by the sequence

$$D^k \equiv \rtimes\!\!\ltimes_k D^1 := D^1 \to Dbl(D^1) \equiv S^1 \to D^2 \to \cdots \to D^{k-1} \to Dbl(D^{k-1}) \equiv S^{k-1} \to D^k$$

Let R_k denote the radius of a sphere in $\mathbb{R}^k$, then $S^{k+1} \equiv Dbl(D^k)$ gives

$$S_{k+1}(R_{k+1}) = 2V_k(R_k)$$

Using the well known formulas for volume and surface area, we have

$$\frac{2\pi^{(k+1)/2}}{\Gamma(k/2 + 1/2)} R_{k+1}^k = 2 \frac{\pi^{k/2}}{\Gamma(k/2 + 1)} R_k^k$$

The duplication formula for the gamma function is

$$\Gamma(z)\Gamma\left(z + \tfrac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$$

Using the substitution $z = k/2$ gives

$$\begin{aligned} R_{k+1}^k &= \frac{\Gamma(k/2+1/2)}{2\pi^{(k+1)/2}} \frac{2\pi^{k/2}}{\Gamma(k/2+1)} R_k^k \\ &= \frac{2^{1-k}\sqrt{\pi}\Gamma(k)}{2\pi^{(k+1)/2}} \frac{2\pi^{k/2}}{\Gamma(k/2)\Gamma(k/2+1)} R_k^k \\ &= 2^{1-k}\Gamma(k) \frac{1}{\frac{k}{2}\Gamma(k/2)\Gamma(k/2)} R_k^k \end{aligned}$$

Cleaning up both sides and substituting in the beta function

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

gives

$$\begin{aligned} R_{k+1} &= \left(\frac{2^{2-k}}{kB(k/2,k/2)}\right)^{1/k} R_k \\ &= \left(\frac{2^{2-k}}{kB(k/2,k/2)}\right)^{1/k} \left(\frac{2^{2-(k-1)}}{(k-1)B((k-1)/2,(k-1)/2)}\right)^{1/(k-1)} R_{k-1} \\ &\vdots \\ &= \prod_{i=1}^{k} \left(\frac{2^{2-i}}{i\mathrm{B}(\frac{i}{2},\frac{i}{2})}\right)^{1/i} r \end{aligned}$$

Where the second line onward is achieved by repeating the process for $R_k,\ R_{k-1}$, etc. □

Using this iterated radius in the well known volume and surface area formulas, we find the following simple results

Corollary 6.1.2. *The volume of a sphere in $\mathbb{R}^k$ taken as an iterated double is given by*

$$V_k(r) = \frac{\pi^{k/2}}{\Gamma\left(\frac{k}{2}+1\right)} \prod_{i=1}^{k-1} \left(\frac{2^{2-i}}{i\mathrm{B}(\frac{i}{2},\frac{i}{2})}\right)^{k/i} r^k$$

Corollary 6.1.3. *The surface area of a sphere in $\mathbb{R}^k$ taken as an iterated double is given by*

$$S_k(r) = \frac{2\pi^{k/2}}{\Gamma\left(\frac{k}{2}\right)} \prod_{i=1}^{k-1} \left(\frac{2^{2-i}}{i\mathrm{B}(\frac{i}{2},\frac{i}{2})}\right)^{(k-1)/i} r^{k-1}$$

Now, suppose we begin with a unit sphere in an arbitrary dimension. If we consider the sphere as an iterated double, then setting R_k equal to 1, gives us

Corollary 6.1.4. *To obtain an iterated unit sphere in* $\mathbb{R}^k$*, the starting radius must be*

$$r = \prod_{i=1}^{k-1} \left(2^{i-2} i \mathrm{B} \left(\frac{i}{2}, \frac{i}{2} \right) \right)^{1/i}$$

This process can be repeated for any manifold which admits a double. Care must be taken if one allows for vector fields. In what follows, this will not be needed as we will work backwards. Rather than constructing iterated doubles, it will be useful to decompose a given structure into its constituent parts.

6.2 $n = 4$

Let us now put this iteration to use. Consider the functor

$$\mathcal{Z} : \mathbf{RBord}_n \to \mathbf{Hilb}$$

for $n = 4$. Our objects are now finitely many disjoint unions of closed, simply-connected 3-manifolds, hence 3-spheres.

Remark 6.2.1. There are multiple methods to obtain an extended QFT. One such method is to let S_l be a closed oriented Riemannian manifold of dimension one, i.e. a circle of length l. Each circle induces a functor

$$Comp : \mathbf{RBord}_n \to \mathbf{RBord}_{n+1}$$

via product with S_l. This approach certainly has its uses. In the RFQFT described in chapter 4, each theory of odd dimension is projective. If one were to induce the functor $Comp$, we would get the chain

$$\mathcal{F}_{comp} := \mathcal{Z} \circ Comp : \mathbf{RBord}_n(odd) \to \mathbf{RBord}_{n+1}(even) \to \mathbf{Hilb}.$$

Doing so skirts the projecitivization issue but raises new questions of physical meaning as well as sewing procedures. For $n = 4$ the RFQFT functor developed is not projective, so there is no need.

Additionally, we may take the approach similar to "extended topological quantum field theory" and use n-categories. Unfortunately there is no one agreed upon definition for n-categories. For simplicity, we will say that an n-category is a category with

- Objects
- 1-morphisms between objects
- 2-morphisms between 1-morphisms
- ...
- n-morphisms between (n-1)-morphisms

For a 2-category, this means that morphisms can be composed along objects while 2-morphisms can be composed either along objects or 1-morphisms in a way satisfying an interchange law.

6.2.1 RPn_{ext}

To create an extended theory, we will employ iterated doubles and skirt the issue of $n-$morphisms. We begin working directly with 3-spheres by assigning each to its associated Sobolev space as in chapter 4.

We then consider each 3-sphere to be an iterated double

$$\bullet_+ \xrightarrow{Dbl} S^0 \xrightarrow{\varnothing^1} I \xrightarrow{Dbl} S^1 \xrightarrow{\varnothing^2} D^2 \xrightarrow{Dbl} S^2 \xrightarrow{\varnothing^3} D^3 \xrightarrow{Dbl} S^3.$$

We note that, theoretically, each sphere (S^n such that $n = 1, 2, 3$) comes with an associated theory

$$\mathcal{Z} : \mathbf{RBord}_n \to \mathbf{Hilb}$$

and so for each we get a corresponding Sobolev space $W^{-1/2}(S^n)$. Thus, we have the following diagram:

$$\begin{array}{ccccccc} S^0 & \longrightarrow & S^1 & \longrightarrow & S^2 & \longrightarrow & S^3 \\ \downarrow{\scriptstyle \mathcal{Z}_1} & & \downarrow{\scriptstyle \mathcal{Z}_2} & & \downarrow{\scriptstyle \mathcal{Z}_3} & & \downarrow{\scriptstyle \mathcal{Z}_4} \\ L^2(\mathbb{R}) & \longrightarrow & W^{-1/2}(S^1) & \longrightarrow & W^{-1/2}(S^2) & \longrightarrow & W^{-1/2}(S^3) \end{array}$$

Remark 6.2.2. The assignment $S^0 \xrightarrow{\mathcal{Z}_1} L^2(\mathbb{R})$ is important enough to warrant its own section and so will be justified next. For now, we take this as fact.

Now take the unit 3-sphere in $\mathbf{RBord}_n$ for $n = 4$ and consider a hyperplane reflection. Reflection positivity on the sphere is addressed in [10] and so this raises no issues. We then take S^2 as the boundary of this reflection and decompose the 3-sphere into $S^3 = D^3_+ \sqcup S^2 \sqcup D^3_-$.

We also need the following lemma along with the fact that

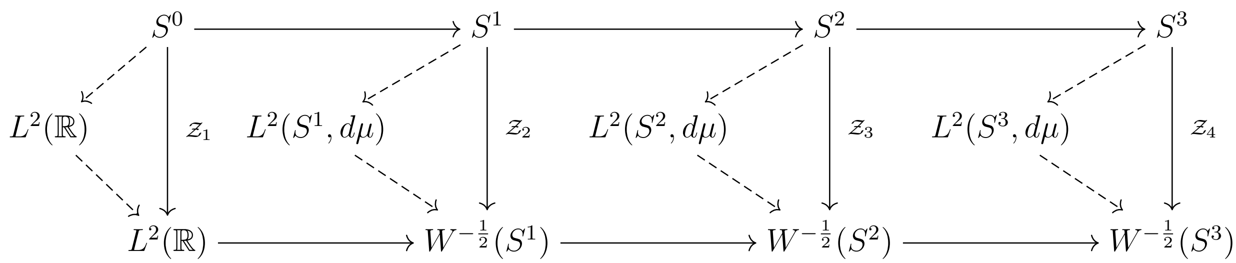
Lemma 6.2.1. *Let M be a manifold, $\Omega \subset M$ open, and let $H^1_0(\Omega)$ be the closure of $C^\infty_0(\Omega)$ in $H^1(M)$, then*

$$H^{-1}(M) = (-\Delta + m^2)H^1_0(ext\ \Omega) \oplus H^{-1}_{\partial\Omega}(M) \oplus (-\Delta + m^2)H^1_0(\Omega)$$

Using this decomposition to restrict measures as in [27] and the preceding lemma from [9], we decompose our functions on $W^{-1/2}(S^3)$ as

$$f_{S^3} = (-\Delta + m^2)f_{D^3_+} + f_{S^2} + (-\Delta + m^2)f_{D^3_-}$$

Since our theory is reflection positive, $f(S^2)$ is in the null space of the RPHS. We now consider $f(S^2)$ as part of an $n = 3$ theory. We note that the projective nature of an $n = 3$ theory is an obstruction to the next decomposition. As such, we sidestep as in the following diagram



Where $d\mu$ is the Gaussian measure $d\mu_C$, the down left arrow is our intermediary assignment, and the down-right arrow is the Segal-Ito isomorphism for $n = 1, 2, 3$.

Note that we could skirt the issue by noting that each hyperplane represents an object in the previous theory. As we are trying to decompose by cycling through H^1, we believe that passing through L^2 is more natural.

Now, we take $f(S^2)$ and decompose as before.

$$f_{S^2} = (-\Delta + m^2) f_{D^2_+} + f_{S^1} + (-\Delta + m^2) f_{D^2_-}$$

Iterating, we get

$$f_{S^3} = f_{\bullet_+} + \sum_{i=1}^{3} (-\Delta_{D^i} + m^2) f_{D^i_\pm}$$

We refer to this as an iterated function. If our sequence of theories is reflection positive then we refer to this as the **reflection positive decomposition** of f.

This is the starting point of future work, the current direction for this material is to analyze this for measures.

Definition 6.2.1. *Let $d\mu_C$ be a Gaussian measure on S^n with mean zero and covariance $(-\Delta + m^2)^{-1}$. Suppose $d\mu_C$ is reflection positive and can be written as*

$$d\mu_C(S^n) = d\mu_C(\mathbb{R}) + \sum_{i=1}^{n} (-\Delta_{D^i} + m^2) f_{D^i_\pm}.$$

We call this the ***completely reflection positive decomposition*** *of $d\mu_C$.*

Suppose now that we are given a Gaussian measure with a completely reflection positive decomposition. We mod out equivalence classes by moving to the corresponding Cameron-Martin space for each measure in the decomposition. We now define an iterated reflection positive FQFT.

Definition 6.2.2. *An iterated reflection positive functorial quantum field theory is a functor*

$$RPn_{ext} : \boldsymbol{RBord}_n \to \boldsymbol{Hilb}$$

mapping disjoint unions into tensor products such that

1. $RPn_{ext}(M) = W^{-\frac{1}{2}}(M)$
2. $RPn_{ext}(S^n) = W^{-\frac{1}{2}}(S^n)$
3. *For each n-sphere, the corresponding Gaussian measure has a completely reflection positive decomposition*

Corollary 6.2.2. *An iterated reflection positive functorial quantum field theory induces a reflection positive theory for each dimension $k < n$.*

This corollary once again puts the impetus back on measures and gives a functorial way of classifying measures.

6.3 $n = 1$:

The one dimensional case for topological theories is a good exercise for beginning category theorists. It has been shown that there is an equivalence of groupoids between one dimensional TQFTs and and the category of Dual Pairs (over a field $\mathbb{K}$) [6].

Question: Is there an equivalence of categories for these one-dimensional RFQFTs?

To start with, in odd dimension our Riemannian FQFT would be projective according to [19]. We show that this issue is avoided entirely in dimension one. Throughout this section we consider the RFQFT $\mathcal{Z} : \mathrm{RBord}_{\langle 0,1 \rangle} \to \mathbf{Hilb}$.

The objects of $\mathrm{RBord}_{\langle 0,1\rangle}$ are points with orientation ($\{\bullet_+\}$ and $\{\bullet_-\}$) and their finite (possibly empty) disjoint unions. Morphisms between points become oriented intervals with length. Suppose the positively oriented point has been assigned the space $X = \mathcal{Z}(\bullet_+)$. An involution on the point flips the orientation and conjugates the assigned space so that $X^\vee = \mathcal{Z}(\bullet_-)$.

Consider the morphism $I_{l/2} : \bullet_+ \to \bullet_+$ with length $l/2$. $I_{l/2}$ is a one dimensional compact Riemannian manifold. We can view $I_{l/2}$ as the bordism $Il/2 : \emptyset \to \partial I_{l/2}$. Preforming the doubling procedure, we get the closed manifold

$$S^1 = ev_{\partial I_{l/2}}(I_l^\vee, I_{l/2}).$$

Remark 6.3.1. A future direction of work is to reproduce reflection positivity on the circle using the $n = 1$ theory. A logical next step would be to enforce $\mathcal{Z}(S^1) \geq 0$. The simplest way to do so would be to state that $\mathcal{Z}$ be a reflection positive theory as a condition.

6.3.1 Assignments

Recall that one of the goals of a QFT is to make sense of the path integral

$$Z = \int_{\mathfrak{F}} e^{-S(\phi)/\hbar} D\phi.$$

In the $n = 1$ theory we have $M = pt$ and so a field on M is a real variable, the space of field configurations is $\mathbb{R}$. So, the path integral becomes

$$Z = \int_{\mathbb{R}} e^{-S(\phi)/\hbar} d\phi$$

We therefore assign $\mathcal{Z}(\bullet_+) := L^2(\mathbb{R})$. We view $L^2(\mathbb{R})$ as the rigged Hilbert space between test functions and distributions (i.e. we have a Gelfand triple (D, L^2, D')).

Since the space of fields is finite dimensional, we do not have to worry about the zeta regularized determinant. We assign

$$\mathcal{Z}(\Sigma) = \det\left(\frac{d^2}{dt^2}|_\Sigma + m^2\right)^{-1/2}.$$

For intervals of length l we impose mixed boundary conditions (B) and make the assignment

$$\mathcal{Z}(I_l) = \det\left(\frac{d^2}{dt^2}|_{I_l,B} + m^2\right)^{-1/2}.$$

6.3.2 Quantum Mechanics

Clearly, $\mathcal{Z} : \mathrm{RBord}_{\langle 0,1\rangle} \to \mathbf{Hilb}$ is a $0+1$ quantum field theory. As an RFQFT, manifolds are mapped to (an isomorphism of) the space of states for a constructive QFT. In practice, an $0+1$ QFT is the playground of burgeoning theorists as a way to test theories and practice techniques. The reason for this is simple, a $0+1$ QFT is quantum mechanics.

Having made the proper assignments, we can see that the $\mathbf{RBord}_1$ theory reduces to Quantum Mechanics. In particular, the bordism

$$S^1 = ev_{\partial I_{l/2}}(I_l^\vee, I_{l/2})$$

gets mapped via $\mathcal{Z}$ to

$$\mathcal{Z}(S^1) = \det\left(\frac{d^2}{dt^2}|_{S^1} + m^2\right)^{-1/2}.$$

In other words, we have periodic boundary conditions and recover the free particle on a ring.

Note that this is only slightly different from the usual assignment for a Euclidean field theory which makes the assignment $\mathcal{Z}(I_t) = e^{-t\Delta}$ and $\mathcal{Z}(S^1) = \mathrm{tr}(e^{-t\Delta})$. The two versions have a natural equivalence and we believe these assignments to be more appropriate for $\mathbf{RBord}_1$.

6.4 $n = 2$:

Once we pass into $\mathbf{RBord}_2$ our objects become closed one dimensional manifolds and so we are dealing with collections of circles. According to the RFQFT theory, we are not concerned with a projective representation since we are in even dimension. Thus, we can focus on obtaining the results one would expect of a CQFT.

In the exposition [24], the connection between reflection positivity on the circle group and KMS states was described. Additionally, the case $n = 2$ was done in [27] for the $P(\phi)_2$ interaction using the RFQFT described in chapter 4. Indeed, the dissertation and subsequent paper by Kandel is an offshoot of that work. Furthermore, Dimock [9] used the Markov property to give sewing procedures for circles, due in great part to reflection positivity.

Therefore, it appears that $\mathrm{RBord}_{\langle 1,2\rangle}$ is a logical backdrop for understanding the connection between constructive and functorial reflection positivity. This road will be traveled in future work and no more will be said about this case.

6.5 Continued Research

The goal of this research program is to achieve a complete classification of constructive quantum field theories via functorial quantum field theory. The hope is that 4d QFT could be investigated from yet another blend of directions so that one day soon we have a working theory. To continue in this direction, there are two primary questions that must be addressed.

6.5.1 Constructing $\mathcal{Z}_{RP}$

Suppose $(\mathscr{E}, \mathscr{E}_+, \theta)$ is an RPHS. Both $\mathcal{E}_+$ and $\theta\mathcal{E}_+$ are objects in the category **Hilb**. Is it possible to construct an RFQFT $\mathcal{Z} : \mathbf{RBord}_n(H_n) \to \mathbf{Hilb}$? Consider the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{\mathcal{Z}} & \widehat{\mathcal{E}} \\ & \overset{?}{\searrow} & \uparrow Q \\ & & \mathcal{E}_+ \end{array}$$

Is there a way to factor through $\mathcal{E}_+$ in a way similar to passing through the category TVS_{pol} on the way to **Hilb**?

6.5.2 Constructing $\mathcal{Z}_C$

Throughout this work, preference has been given to the Laplace operator. As a reflection positive operator, it ensures the corresponding measure is reflection positive and so the theory can be analytically continued to a Lorentzian measure. Along with the zeta regularized determinants, it was shown in [19] and [27] that $\mathcal{Z} : \mathbf{RBord}_n \to \mathbf{Hilb}$ defines a functorial quantum field theory.

Along this vein, we consider the following: let C be an arbitrary reflection positive operator on a manifold M. Does every C induce a functorial QFT? Given C, is there a general prescription of sewing procedures? Under what conditions? If there are reflection positive operators that do not induce FQFTs, in what way can the obstructions be classified?

6.5.3 The Forgetful Functor

Finally, we note that there is a forgetful functor from $\mathbf{RBord}_n$ to $\mathbf{Bord}_n$ as well as a forgetful functor from $\mathbf{Hilb}$ to $\mathbf{Vect}_{\mathbb{C}}$. In the first case, we drop the metric from all manifolds and the category becomes completely topological. In the latter, we drop the Hilbert structure and consider vector spaces in general. We get the following diagram:

$$\begin{array}{ccc} \mathbf{RBord}_n & \xrightarrow{FQFT} & \mathbf{Hilb} \\ {\scriptstyle For}\downarrow & & \downarrow{\scriptstyle For} \\ \mathbf{Bord}_n & \xrightarrow[TQFT]{} & \mathbf{Vect}_{\mathbb{C}} \end{array}$$

This raises a number of questions. First, can this diagram be expanded? Even in the Stolz/Teichner program there is an intermediary category of polarized topological vector spaces. In the works of Pickrell and Kandel we move from these to the category of Hilbert spaces. What special structures cause this shift?

Second, if we begin in $\mathbf{RBord}_n$ and pass to $\mathbf{Bord}_n$ via the forgetful functor, we then end up in *finite dimensional* vector spaces. Going the other direction, the forgetful passage from **Hilb** to $\mathbf{Vect}_{\mathbb{C}}$ does not cause this. Essentially, this diagram is not commutative. The question, then, is does there exist some intermediary step that would make such a diagram commute?

Lastly, and most philosophically, to what extent can physical theories be described by an expansion of this diagram? If low energy theories are described well by topological QFTs and higher energy theories described by constructive QFTs, can careful analysis of this diagram be a way of understanding these physical theories in general?

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Vita

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