The spectrum of GRB 930131 ("Superbowl Burst") from 20 keV to 200 MeV

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ABSTRACT

We have constructed a broad-band spectrum for GRB 930131 (the “Superbowl Burst”), ranging from 20 keV to 200 MeV, by combining spectral information from the Gamma Ray Observatory’s BATSE, COMPTEL and EGRET instruments. We present general methods for combining spectra from different time intervals obtained by the same instrument as well as for combining spectra from the same time interval taken by different instruments. The resulting spectrum is remarkably flat (in $\nu F_\nu$-space) up to high energies. We find that the spectral shape can be successfully fitted by the shocked synchrotron emission model of Tavani. We present evidence that the flatness of the spectrum at high energies is not due to spectral time-variability.

Subject headings: gamma rays: bursts

1. INTRODUCTION

Broad-band spectra of gamma-ray bursts (GRBs) pose a difficult challenge to any theoretical model trying to explain them. Looking only at a limited range of energy, as, for example, each of the different instruments on board the Gamma Ray Observatory (GRO) does individually, results in a featureless power law perhaps with some curvature. However, a broad-band spectrum, ranging over many decades in energy, typically contains interesting features like peaks, curvature and breaks. Such features will be diagnostic of the physical processes in the burst fireball and the spectra can be used to directly test models of burst emission. Only a few broad-band spectra have been produced (Schaefer et al. 1998; Greiner et al. 1995; Hurley et al. 1994), as only bright bursts detected by multiple instruments on board the GRO have a wide enough range of available data. The brightest such burst is GRB 930131 which reached a peak flux of $10^5$ ph s$^{-1}$ cm$^{-2}$ (Meegan et al. 1996). This burst has BATSE trigger number 2151 and has been called the “Superbowl Burst” after its time of occurence. EGRET and COMPTEL spectra have already appeared in the literature (Sommer et al. 1994; Ryan et al. 1994), but no BATSE spectrum has been presented due to severe deadtime problems.

This paper is organized in the following way. In §2, we provide general methods for combining spectra obtained by the same instrument during different time intervals (2.1.), as well as for combining spectra taken by different instruments covering the same time interval (2.2.). These methods can be used in many common GRB applications, provided the necessary requirements are met. In §3, we carry out the construction of the broad-band spectrum of GRB 930131 from 20 keV to 200 MeV. First, we describe how
the individual (BATSE, COMPTEL, and EGRET) spectra have been obtained (3.1.). Then, we argue why these independently reduced spectra can be combined with the method of §2, where we point out the non-obliging nature of this procedure in the present case. After presenting the resulting spectrum (3.2.), we compare this to theoretical models of the GRB emission mechanism (3.3.). Subsequently, we discuss evidence for spectral evolution (3.4.). Finally, §4 summarizes the spectral properties of this remarkable burst.

2. COMBINING SPECTRA

Often the problem occurs to combine individual spectra into either a time-averaged or an instrument-averaged spectrum. The second case arises in cross-calibrating spectral information from instruments that are sensitive in different energy ranges. In this section, it is assumed that the observed count spectra have already been reduced into photon spectra. In the following, we describe the method of combining spectra and give the relevant formulae, which are then applied to the case of GRB 930131 in Section 3.

2.1. Combining Across Time

Suppose the time over which one wants to average is divided up into smaller time intervals $k$ with respective livetimes $\tau_k$. For each time interval $k$ and energy bin $i$ the photon flux (in units of photons/area/energy/time) is $(\frac{dN}{dE})_{ik}$ with standard deviation $\sigma_{ik}$. Then, constructing the time-averaged spectrum is straightforward. With the total livetime given by $\tau_{total} = \sum_k \tau_k$, the time-averaged photon flux in energy bin $i$ is

$$\left( \frac{dN}{dE} \right)_i = \tau_{total}^{-1} \sum_k \left( \frac{dn}{de} \right)_{ik} \tau_k,$$

and the resulting standard deviation is

$$\sigma_i^2 = \tau_{total}^{-1} \sqrt{\sum_k (\sigma_{ik}\tau_k)^2}.$$ 

2.2. Combining Across Different Instruments

Spectra from different instruments can be combined just as can spectra from multiple detectors on the same instrument. We here assume that the combination process is robust, i.e., that the resulting spectrum is not greatly obliging (cf., Section 3.2.). This has to be justified on a case by case basis. Another requirement is that either the input spectra are for identical time intervals, or they cover the entire burst. This combination can be described as a four-step process:

Step A:
The spectra from different instruments are divided into energy bins in different ways. Therefore, as a first step, all the bin boundaries ($E_{low}$ and $E_{high}$) from all the instruments are put into increasing order and then used to define subbins. Assume that after the ordering, the following sequence arises: $... < E_{k-1} < E_k < E_{k+1} < ...$ Then define the $k$th subbin to cover an energy interval between $E_k$ and $E_{k+1}$. Figure 1 illustrates this procedure for the case of two instruments.
Step B:
It is preferable to conduct the combining in \( \nu F_\nu \)-space, where \( \nu F_\nu \propto \left( \frac{dN}{dE} \right)^2 \). Then the spectrum is roughly constant over a given energy bin, as opposed to the usual steep decline in ordinary \( \frac{dN}{dE} \)-space. Now, for energy bin \( i \) of instrument \( m \), having lower and higher energies \( E_{\text{low}}^m \) and \( E_{\text{high}}^m \), respectively, define the energy flux per logarithmic energy interval

\[
\left( \frac{d\varphi}{dE} \right)_{m_i} = \left( \frac{dN}{dE} \right)_{m_i} \left( E_{m_i}^{\text{mid}} \right)^2 \pm \sigma_{m_i},
\]

where \( E_{m_i}^{\text{mid}} = \sqrt{E_{\text{low}}^m \cdot E_{\text{high}}^m} \) and \( \sigma_{m_i} \) is the uncertainty of \( \left( \frac{d\varphi}{dE} \right)_{m_i} \). Our procedure presumes that \( \left( \frac{d\varphi}{dE} \right)_{m_i} \) changes little across each energy bin, as is the case for energy bins that are small compared to either the detector resolution or the structure in the spectrum. This covers virtually all GRB applications, although a simple interpolation scheme might be appropriate for a particularly steep spectrum observed with very broad bins.

Step C:
Now, we want to cross-combine the spectra of different instruments. In constructing the spectrum for subbin \( k \), we first determine whether a given instrument \( m \) has an energy bin \( i \) overlapping the subbin. If this is the case, we set

\[
\left( \frac{d\varphi}{dE} \right)_{m_k} = \left( \frac{d\varphi}{dE} \right)_{m_i}, \quad \text{and} \quad \sigma_{m_k} = \sigma_{m_i}.
\]

Figure 1 shows the case of two instruments having overlapping energy bins with subbin \( k \). The energy flux of the cross-combined spectrum is the weighted average of all contributing spectra:

\[
\left( \frac{d\phi}{dE} \right)_k = \sigma_k^2 \cdot \sum_m \frac{1}{\sigma_{m_k}^2} \left( \frac{d\varphi}{dE} \right)_{m_k},
\]

where

\[
\sigma_k = \left( \sum_m \sigma_{m_k}^{-2} \right)^{-1/2}.
\]

Step D:
As a last step, put together the subbins into larger bins of width appropriate for the spectral resolution and features. Rebinning, e.g., two subbins \( k \) and \( k+1 \) into a larger bin \( l \) with boundaries \( E_l^{\text{low}} \) and \( E_l^{\text{high}} \) is accomplished by the following:

\[
\left( \frac{d\Phi}{dE} \right)_l = w_k \left( \frac{d\phi}{dE} \right)_k + w_{k+1} \left( \frac{d\phi}{dE} \right)_{k+1},
\]

where one has for the respective weights

\[
w_k = \frac{E_{k+1} - E_l^{\text{low}}}{E_l^{\text{high}} - E_l^{\text{low}}},
\]

and

\[
w_{k+1} = \frac{E_l^{\text{high}} - E_{k+1}}{E_l^{\text{high}} - E_l^{\text{low}}}.\]

The resulting standard deviation is

\[
\sigma_l^2 = w_k^2 \sigma_k^2 + w_{k+1}^2 \sigma_{k+1}^2.
\]

If the output bin covers more than two subbins, then equations (7)-(10) can be easily generalized or used repeatedly.
3. THE SPECTRUM OF GRB 930131

3.1. The Individual Spectra

For all 3 instruments (BATSE, EGRET, COMPTEL), their photon spectra have been obtained by the traditional forward-folding technique (Loredo & Epstein 1989). This technique assumes a variety of spectral models $M$, and convolves them with the respective detector response matrix (DRM), symbolically $C_{\text{model}} = DRM \ast M$, where $C_{\text{model}}$ is the count spectrum predicted by the model. The parameters of the model are then adjusted to obtain the best fit to the observed count spectrum, $C_{\text{obs}} = DRM \ast P_{\text{true}}$, where $P_{\text{true}}$ is the true (photon) spectrum of the source. Alternatively, a model-independent inverse technique could have been adopted, where $P_{\text{true}} = DRM^{-1} \ast C_{\text{obs}}$. Attempts at doing so have proven unconvincing, and the nearly universal practice in gamma-ray astronomy is to use forward-folding techniques. One exception is the direct inversion method of Pendleton et al. (1996), which has only been applied to low resolution (4-channel) data and introduces considerable additional error (10-15%).

3.1.1. BATSE Spectrum

The “Superbowl Burst” suffers from severe deadtime effects, which is the reason why the original discovery paper (Kouveliotou et al. 1994) does not present a spectrum for the BATSE energy range. For this bright burst, most of the flux arrives in the first 0.06 seconds, a situation which saturates the BATSE Large Area Detectors (LADs), whereas the smaller but thicker Spectroscopy Detectors (SDs) can reliably record the intense photon flux. In constructing our spectrum, we have selected the two burst-facing Spectroscopy Detectors (SD 4 and 5), for which there are available the well suited STTE-data (SD Time-Tagged Events), which cover the first $\sim 1.5$ s of the burst and which have a time resolution of 128 $\mu$s. Therefore, we can correct for the deadtime effects by subdividing the total time into 53 individual spectra with a duration of as short as a few ms around the first, intense peak. For each time interval, the photon spectrum is obtained by following the procedure described in Schaefer et al. (1994). In carrying out the forward-folding, we assume a single power-law spectral model. Then, by applying the methods of Section 2.1., we constructed time-averaged spectra for SD 4 and 5, which were then in turn combined (as described in Section 2.2.) to give the overall spectrum for the BATSE energy-range (21 keV to 1.18 MeV, above which the flux-errors exceed 100%).

3.1.2. COMPTEL And EGRET Spectra

The COMPTEL and EGRET spectra have previously been published (Ryan et al. 1994, and Sommer et al. 1994, respectively) and we refer the reader to these papers for details. We have chosen to work with the spectrum reported by the EGRET Total Absorption Shower Counter (TASC), since the EGRET spark chamber is too severely affected by deadtime effects. The TASC spectrum covers an energy range from 1 MeV to 180 MeV. The overlap region between BATSE and TASC is nicely covered by the COMPTEL instrument, where the COMPTEL Telescope spectrum covers the range from 0.75 Mev to 30 MeV. Both spectra have been obtained by the forward-folding technique with a power-law model, and are corrected for deadtime effects.
3.2. The Combined Spectrum

To construct the combined spectrum with the method described in Section 2.2., we first have to ascertain the robustness of this procedure. It is well known (Fenimore et al. 1983) that the resulting spectral shape can possibly depend sensitively on the details of the fitting technique (i.e., that the spectra might be “obliging”). In principle, it could make a big difference whether the low- and high-energy parts, covered by different instruments, are first unfolded separately and only then combined together, or whether the unfolding is done simultaneously to all instruments. The physical reason for this is that high-energy photons might masquerade as low-energy ones, and that, consequently, the low-energy part of the spectrum cannot be accurately unfolded independently of the high-energy part. For the present case, however, this problem does not occur. It has been convincingly shown that the BATSE Spectroscopy Detectors are non-obliging (Schaefer et al. 1994; cf., their Figures 11 and 52). This is primarily due to their thickness, which largely minimizes photon energies being underreported. The TASC and COMPTEL spectra, on the other hand, are not affected by the lower energy BATSE range. Finally, treating the COMPTEL and TASC spectra independently of each other is rendered possible by the fact that the model fitting leads to almost identical results \(\frac{dN}{dE} \propto E^{-2}\). We are therefore justified in combining the independently obtained spectra from the 3 GRO instruments (BATSE-EGRET-COMPTEL) into the overall, broad-band spectrum of GRB 930131.

This combination is carried out with the method of section 2.2., where we have been careful to construct our BATSE spectrum such that it exactly matches the time coverage of the EGRET TASC instrument, and approximately that of COMPTEL. To evaluate how well the instruments agree in the mutual overlap region around 1 MeV, we compare the fluxes at 1 MeV for the 3 instruments (in units of \(10^{-3}\) photons cm\(^{-2}\) sec\(^{-1}\) keV\(^{-1}\)): BATSE 2\(\pm\)2, COMPTEL 8\(\pm\)3, and TASC 2\(\pm\)0.5. The agreement between BATSE and TASC is good, although the BATSE errors approach 100% at these high energies. The COMPTEL flux is somewhat high, but due to its uncertainties it does not contribute significantly to the weighted average of the final, combined spectrum.

Table 1 and Figure 2 present the \(\nu F_\nu \propto \left(\frac{dN}{dE}\right) E^2\) spectrum in units of \((\text{photons s}^{-1} \text{ cm}^{-2} \text{ keV}^{-1}) \ast (E^{\text{mid}}/100 \text{ keV})^2\). The resulting spectrum is remarkably flat, as compared to other published broad-band spectra, which have a much more peaked appearance (cf., Schaefer et al. 1998). In the following section we ask, whether this rather unusual spectral shape is consistent with the model of shocked synchrotron emission, which successfully fits the characteristics of other broad-band GRB spectra. Subsequently, we investigate whether the flat spectrum of GRB 930131 can be understood as a result of spectral evolution.

3.3. Model-Fits

We fit our combined spectrum to the shocked synchrotron model of Tavani (1996a, b), which gives the following analytical expression for the energy flux:

\[
\psi_{\text{model}} \equiv \left(\frac{d\Phi}{dE}\right) = \nu F_\nu = C\nu \left[I_1 + \frac{1}{e}I_2\right] \tag{11}
\]

\[
I_1 = \int_0^1 y^2 e^{-y} F\left(\frac{\nu}{\nu_c y^2}\right) dy \tag{12}
\]
\[ I_2 = \int_1^\infty y^{-\delta} F \left( \frac{\nu}{\nu_c^* y^2} \right) \, dy, \]

where \( F(x) \equiv x \int_1^\infty K_{5/3}(w) \, dw \) is the usual synchrotron spectral function with \( K_{5/3} \) being the modified Bessel-function of order \( \frac{5}{3} \) and \( e = 2.718 \ldots \). The normalization constant \( C \) has units of specific flux.

Equations (12) and (13) are summing up the synchrotron emission from a Maxwellian distribution of electron energies which breaks to a power law at high energies. Here, \( \delta \) is the index of the supra-thermal power-law distribution of particles, resulting from relativistic shock-acceleration. The critical frequency \( \nu_c^* \) describes where most of the synchrotron power is emitted. We apply the Levenberg-Marquardt method of non-linear \( \chi^2 \) fitting (cf., Numerical Recipes, Press et al. 1992) to minimize

\[ \chi^2 = \sum_{i=1}^{N} \left( \frac{\psi_i - \psi_{\text{model}}(\nu_c^{* \text{mid}}, C, \delta, \nu_c^*)}{\sigma_i} \right)^2. \]

Our observed spectrum with flux \( \psi_i = \frac{d\Phi}{dE} \), and uncertainty \( \sigma_i \) contains \( N = 37 \) data points. Our best-fit parameters are:

\[ C = 104 \pm 8 \text{ erg cm}^{-2} \text{ sec}^{-1} \text{ Hz}^{-1} \]
\[ \delta = 3.3 \pm 0.1 \]
\[ h\nu_c^* = 98 \pm 14 \text{ keV} \]

The fit has a chi-squared of \( \chi^2 = 38 \) with 34 degrees of freedom. Therefore, we can conclude that the spectrum of GRB 930131 is consistent with the Tavani-model. At low energies, the spectrum is asymptotically approaching \( \nu F_\nu \propto \nu^{4/3} \), as is usual for burst spectra (Schaefer et al. 1998). This behavior is predicted by optically thin synchrotron theory (Katz 1994).

### 3.4. Spectral Evolution

All of the published broad-band spectra (Schaefer et al. 1998; Greiner et al. 1995; Hurley et al. 1994) are strongly peaked and fall off steeply above the peak energy. GRB 930131, on the other hand, has a spectrum which remains constant (within a factor of 4) over four orders of magnitude in energy. Can this behavior be understood as the result of a superposition of many spectra, which individually show the usual, strongly peaked shape and whose peak energy evolves with time? For the BATSE energy range, the number of received photons is sufficiently large to allow the construction of time-resolved spectra, whereas for COMPTEL and EGRET, the dearth of photons renders this detailed treatment impossible.

In Figure 3, we present the resulting BATSE spectra for 4 different times. The lightcurve of GRB 930131, as amply documented in the literature (Kouveliotou et al. 1994; Ryan et al. 1994; Sommer et al. 1994), shows a sharp, intense first pulse, lasting for \( \sim 0.06 \) s after the BATSE trigger, followed by a second, less intense and less sharp pulse, lasting from \( \sim 0.75 \) s to \( \sim 1.00 \) s after the trigger. In between, the “interpulse” region of Figure 3, there is significant yet faint flux. Finally, there is again relatively little flux subsequently to the second pulse (lasting for another 50 s). In Figure 3, the first pulse is further subdivided into the spectrum for the time before the maximum flux is reached (0.00 - 0.03 s) and that for the time after the maximum (0.03 - 0.06 s).

Since these time-resolved spectra cover only the low-energy range, a meaningful fit to the Tavani-model (cf., Section 3.3.) cannot be done, since the value of the power-law extension \( \delta \) and the location of the peak energy \( h\nu_c^* \) are mostly constrained by the high-energy regime. Both the spectra for the first and
second pulses are consistent, though, with the spectral fit (besides the normalization \( C \)) obtained for the overall spectrum (cf., Figure 2). Consequently, there is no evidence that the unusual flat morphology of the “Superbowl-Burst” spectrum is caused by the superposition of individually strongly-peaked, time-variable spectra.

The spectrum between pulses is inconsistent with the average burst spectral shape. The observed \( \nu F_\nu \) is close to \( \nu^0 \) from 21 keV to 1 MeV with no significant curvature or maximum. The extreme brightness of GRB 930131 allows for this unique measure of the interpulse spectrum.

4. SUMMARY AND CONCLUSIONS

After having given the relevant formulae for combining individual spectra, we applied these methods to construct the broad-band spectrum of GRB 930131. With appropriate deadtime corrections we first obtained the spectrum for the BATSE energy range, which we then combine with the already published spectra from the COMPTEL and EGRET TASC instruments. Broad-band spectra are fortunate occurrences (multiple instruments on board the GRO have to see a bright burst), available for only a handful of bursts.

Within the general framework of an expanding relativistic fireball, impacting on a surrounding medium (Mészáros & Rees 1993), an attractive model for the production of the \( \gamma \)-ray photons is synchrotron emission from a shocked and highly magnetized plasma (Tavani 1996a, b). This model is successful in fitting the strongly peaked spectral shapes (in \( \nu F_\nu \)-space) of the GRBs for which broad-band spectra have been obtained. Since our resulting spectrum is so unusually flat, it poses an interesting challenge to the Tavani-model. As described in Section 3.3., the model does fit well, although with a value for the power-law component, which lies at the extreme end of the typically encountered range, \( 3 < \delta < 6 \). In the BATSE energy-range, we were able to construct time-resolved spectra, which show no evidence for significant evolution.

We thank D. Palmer for his suggestions concerning the severe deadtime problem in the BATSE data, as well as M. Kippen and E. Schneid for their helpful discussions.
REFERENCES


Fig. 1.— Constructing the combined spectrum in subbin $k$, ranging from $E_k$ to $E_{k+1}$. Instruments $m$ and $m+1$ have overlapping energy bins (heavy lines) with subbin $k$, and are consequently contributing to the averaged spectrum. Note how the subbins of the combined spectrum are defined in accordance with the bin boundaries of the various instruments.

Fig. 2.— Composite spectrum of GRB 930131. The spectrum shows a low-energy portion which approaches a $\nu^{4/3}$ power law and a peak $\nu F_{\nu}$ around 200 keV. The high-energy tail is remarkably flat. Solid line: Best-fit to Tavani shocked synchrotron model.

Fig. 3.— Spectra in the BATSE energy range for various times during the burst. The panels correspond to the following durations: (a) 0.00-0.03 s; (b) 0.03-0.06 s; (c) 0.06-0.75 s; (d) 0.75-1.00 s. The spectra for the first and second pulse are consistent with the overall spectrum of Fig. 1, whereas the interpulse spectrum is not.
Table 1.
Spectrum of GRB 930131

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<tr>
<th>$E_{\nu}$ (keV)</th>
<th>$\nu F_{\nu}$</th>
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<tr>
<td>21................</td>
<td>0.079±0.029</td>
</tr>
<tr>
<td>25................</td>
<td>0.070±0.018</td>
</tr>
<tr>
<td>31................</td>
<td>0.089±0.016</td>
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<tr>
<td>38................</td>
<td>0.112±0.020</td>
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<tr>
<td>45................</td>
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<td>61................</td>
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<tr>
<td>85................</td>
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<tr>
<td>241300..........</td>
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</table>
Instrument $m$: $E_{mi}^{low}$ \hspace{1cm} $E_{mi}^{high}$

Instrument $m+1$: $E_{m+1,j}^{low}$ \hspace{1cm} $E_{m+1,j}^{high}$

Combined: $E_k$ \hspace{1cm} $E_{k+1}$