Density Theorems for Reciprocity Equivalences.

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Density theorems for reciprocity equivalences

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DENSITY THEOREMS FOR
RECIPROCITY EQUIVALENCES

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by

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ABSTRACT

The author studies reciprocity equivalence and the wild set of a reciprocity equivalence. He proves that if two algebraic number fields $K$ and $L$ are reciprocity equivalent then there exists a reciprocity equivalence between them with an infinite wild set. In particular, there always exists a self-equivalence with an infinite wild set on any algebraic number field. Even though a wild set of an equivalence can be infinite, he proves that its Dirichlet density is always zero.

The structure of a reciprocity equivalence is examined. He proves that the bijection on primes determines the group isomorphism on global square classes in an equivalence. Indeed he proves more. Namely, if $(t_1, T_1)$ and $(t_2, T_2)$ are reciprocity equivalences from $K$ to $L$ and the bijections $T_1$ and $T_2$ on primes agree on a set of Dirichlet density bigger than zero, then the global square class group isomorphisms $t_1$ and $t_2$ agree everywhere, and $T_1$ agrees with $T_2$ at every noncomplex prime.
INTRODUCTION

In 1937 Witt [W] showed that the collection of all quadratic forms over a given field $F$ forms a ring. Today this ring is called the Witt ring of $F$ and is denoted by $W(F)$. The study of quadratic forms over fields has largely become an investigation of Witt rings. Recently Perlis, Szymiczek, Conner, and Litherland investigated the Witt rings associated to algebraic number fields. They proved that two number fields $K$ and $L$ have isomorphic Witt rings if and only if the fields are reciprocity equivalent, which is defined as follows.

Two algebraic number fields, $K$ and $L$, are reciprocity equivalent when there is a 1 to 1 correspondence

$$T : \Omega_K \rightarrow \Omega_L$$

between the set $\Omega_K$ of primes of $K$ and the set $\Omega_L$ of primes of $L$, and a group isomorphism

$$t : K^*/K^{*2} \cong L^*/L^{*2}$$

of global square classes such that Hilbert symbols are preserved; that is

$$(a, b)_P = (ta, tb)_{TP}$$

for every $P$ in $\Omega_K$ and $a, b$ in $K^*/K^{*2}$. We call the pair of maps $(t, T)$ a reciprocity equivalence.

Let $P$ denote a finite prime of $K$. When $(t, T)$ preserves $P$-orders, i.e., when

$$\text{ord}_P(a) \equiv \text{ord}_{TP}(ta) \pmod{2}$$

for each $a$ in $K^*/K^{*2}$, then we say that $(t, T)$ is tame at $P$. Otherwise $(t, T)$ is wild at $P$. The wild set of the reciprocity equivalence $(t, T)$ is the collection of all finite primes $P$ where $(t, T)$ is wild. If the wild set is empty, we say that the reciprocity equivalence $(t, T)$ is tame.
In her dissertation Carpenter [C] has proved that if $K$ and $L$ are reciprocity equivalent, then there exists a reciprocity equivalence between them with a finite wild set. Carpenter's result begs the question: can there exist reciprocity equivalences with infinite wild sets, and if so just how infinite can wild sets be? My first result is that if $K$ and $L$ are reciprocity equivalent, then there exists a reciprocity equivalence between them with an infinite wild set. Since any number field $K$ is always equivalent to itself by letting both $t$ and $T$ be identity maps, it follows that there always exists a (different) self-equivalence $(t_1, T_1)$ on $K$ with an infinite wild set.

In order to quantify how big the wild set can be, we recall the following definition. Let $P$ be a finite prime of $K$ and let $N_P$ denote the norm of $P$, that is, the index of $P$ in the ring $O_K$ of algebraic integers of $K$. If $M$ is a set of primes of $K$, then we define $\delta(M)$ the Dirichlet density of $M$ to be

$$\lim_{s \to 1^+} \frac{\sum_{P \in M} (N_P)^{-s}}{\log(\frac{1}{1-s})},$$

provided that the limit exists. My second result is that the wild set of a reciprocity equivalence always has a Dirichlet density, and that this density is always zero. In other words, a wild set can be finite or infinite, but when it is infinite, it is always "thin."

Then we turn to the structure of a reciprocity equivalence $(t, T)$. The question arises: in what sense are $t$ and $T$ dependent on each other? Perlis, Szymiczek, Conner, and Litherland show that the square class map $t$ determines the map of primes $T$ (up to an arbitrary permutation of the complex primes). As my final result, I show the converse: $T$ determines $t$. In fact, I show somewhat more. If $(t_1, T_1)$ and $(t_2, T_2)$ are reciprocity equivalences from...
$K$ to $L$ and if $T_1 P = T_2 P$ for every prime $P$ in a set $M$ of primes in $\Omega_K$ of Dirichlet density bigger than zero, then $t_1 = t_2$ identically, and $T_1 = T_2$ except possibly at the complex primes of $K$. 
CHAPTER 1
Summary of Previous Results

This section contains a summary of results from the paper [P] that will be used in this dissertation. Let $P$ be a prime, finite or infinite, of the number field $K$, and let $K_P$ denote the completion of $K$ at $P$. Let $(t, T)$ be a reciprocity equivalence from $K$ to $L$. The following is Lemma 1 of [P]:

**Lemma 1:** 1. There are local symbol preserving isomorphisms

$$t_P : K_P^*/K_P^{*2} \rightarrow L_T^*/L_T^{*2}$$

for $P \in \Omega_K$ making the following diagram commute:

\[
\begin{array}{ccc}
K^*/K^{*2} & \longrightarrow & K_P^*/K_P^{*2} \\
\downarrow t & & \downarrow t_P \\
L^*/L^{*2} & \longrightarrow & L_T^*/L_T^{*2}
\end{array}
\]

2. The map $T$ sends real primes to real primes, complex primes to complex primes, dyadic primes to dyadic primes, and finite non-dyadic primes to finite non-dyadic primes.

Let $S$ be a finite set of primes of $K$. Then $S$ is said to be *sufficiently large* when $S$ contains all real and dyadic primes of $K$ and when the class number of the ring

$$O_S = \{X \in K | ord_P(X) \geq 0 \text{ for all primes } P \text{ of } K \text{ outside of } S\}$$

of $S$-integers is odd. (To insure that the latter condition holds, add to $S$ any set of prime generators of the Sylow 2-subgroup of the ideal class group of $K$.) Let $U_S$ be the group of units of $O_S$. That is,
\( U_S = \{ X \in K | \text{ord}_P(X) = 0 \text{ for all primes } P \text{ of } K \text{ outside of } S \} \). We define an 
*S-equivalence* from \( K \) to \( L \) to consist of the following:

1. a bijection \( T \) from a sufficiently large set \( S \) of primes of \( K \) to a sufficiently large set \( TS \) of primes of \( L \);
2. a group isomorphism 
   \[
   t_S : U_S/U_S^2 \to U_{TS}/U_{TS}^2
   \]
3. for each prime \( P \) of \( S \) a symbol preserving isomorphism 
   \[
   t_P : K_P^*/K_P^{*2} \to L_{TP}^*/L_{TP}^{*2}
   \]
4. a commutative diagram

\[
\begin{array}{c}
U_S/U_S^2 \\
\downarrow t_S
\end{array} \xrightarrow{\text{diag}} \prod_{P \in S} K_P^*/K_P^{*2} \\
\downarrow t_P
\begin{array}{c}
U_{TS}/U_{TS}^2 \\
\downarrow
\end{array} \xrightarrow{\text{diag}} \prod_{P \in S} L_{TP}^*/L_{TP}^{*2}
\]

When we want to deemphasize the set \( S \), we call an *S-equivalence* a small equivalence.

The following results will be important in the sequel. They appear as Lemma 6 and Theorem 1, respectively, in [P].

**Lemma 2:** Let \( S \) be a sufficiently large set of primes of \( K \). Then the map 
\[
U_S/U_S^2 \xrightarrow{\text{diag}} \prod_{P \in S} K_P^*/K_P^{*2}
\]
is injective.

**Theorem 1:** An *S-equivalence* from \( K \) to \( L \) can be extended to a reciprocity equivalence that is tame outside of \( S \).
CHAPTER 2

Preliminary Results

Let $F$ be an algebraic number field and let $M$ be a set of primes of $F$. The natural density of $M$ is defined to be

$$\lim_{n \to \infty} \frac{\text{Number of finite } \mathcal{P} \in M \text{ with } N_{\mathcal{P}} \leq n}{\text{Number of finite } \mathcal{P} \text{ of } F \text{ with } N_{\mathcal{P}} < n}$$

provided that the limit exists.

It is well-known that when the natural density exists, then the Dirichlet density also exists and the two densities are equal (see [L], p. 167). Hence, Dirichlet density generalizes natural density. In this dissertation, we will only be concerned with Dirichlet density, and will call it density.

The terminology almost all elements of $M$ signifies: except possibly on a subset of $M$ of density 0. Using this terminology we define

$$G(M) = \{X \in F^*/F_{*2}^2 \text{ such that } X = 1 \text{ in } F_{\mathcal{P}}^*/F_{\mathcal{P}^2}^2 \text{ for almost all } \mathcal{P} \in M\}.$$

Our main goal of this section is to prove the following Lemma.

Main Lemma: If $G(M)$ is an infinite set, then the density of $M$ is zero.

We observe the following Fact.

Fact 1: The number of subgroups of index 2 of a group $G$ is precisely the number of nontrivial homomorphisms from $G$ onto $\mathbb{Z}/2\mathbb{Z}$ which is equal to

$$|\#\text{Hom}(G, \mathbb{Z}/2\mathbb{Z})| - 1.$$

Setting $G = (\mathbb{Z}/2\mathbb{Z})^k$ and observing $\#\text{Hom}((\mathbb{Z}/2\mathbb{Z})^k, \mathbb{Z}/2\mathbb{Z}) = 2^k$, we obtain

Fact 2: The number of nontrivial homomorphisms from $(\mathbb{Z}/2\mathbb{Z})^k$ onto $\mathbb{Z}/2\mathbb{Z}$ is $2^k - 1$.

Now let $F$ be an algebraic number field. Since $F^*/F_{*2}^2$ is an infinite group,
for any natural number $k$ there exist square classes $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_k$ in $F^*/F^{*2}$ such that $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_k$ are linearly independent over $\mathbb{Z}/2\mathbb{Z}$. The following lemma is elementary, but plays a crucial role in what follows, so we give a complete proof.

**Lemma 3:**

1. $F(\sqrt{a}) = F(\sqrt{b})$ if and only if $\bar{a} = \bar{b}$ in $F^*/F^{*2}$.

2. If $\bar{X}_1, \ldots, \bar{X}_k \in F^*/F^{*2}$ are linearly independent over $\mathbb{Z}/2\mathbb{Z}$, then $[F(\sqrt{X_1}, \ldots, \sqrt{X_k}) : F] = 2^k$ where $X_i \in \bar{X}_i$ for $1 \leq i \leq k$.

**Proof:** If $a$ and $b$ represent the same square class, then certainly adjoining their square roots to $F$ produces the same quadratic extension. Conversely, suppose these quadratic extensions are equal. If $b$ is a square in $F$ then so is $a$, in which case $a$ and $b$ represent the trivial square class. Suppose that $b$ is not a square in $F$. Then $1, \sqrt{b}$ forms an $F$-basis of $F(\sqrt{b})$, so $\sqrt{a} = c + d\sqrt{b}$ for some $c, d$ in $F$. Squaring then gives $a = c^2 + d^2b + 2cd\sqrt{b}$. It follows that either $c$ or $d$ is 0. If $d = 0$, then $\sqrt{a} = c$ lies in $F$, contrary to assumption. So $c = 0$, from which it follows that $\sqrt{a} = d\sqrt{b}$, which means that $a$ and $b$ represent the same square class. This proves 1.

Let $G$ be the Galois group of $F(\sqrt{X_1}, \ldots, \sqrt{X_k})/F$. The top field is the compositum of $k$ subfields, each of degree 2 over the base field, so the Galois group $G$ is an elementary 2-group of order $2^j$ for some number $j < k$. It is to be shown that $j = k$. By Galois theory, the number of quadratic extensions of $F$ contained in the top field equals the number of subgroups of index 2 in $G = (\mathbb{Z}/2\mathbb{Z})^j$ which equals $2^j - 1$ by Fact 2. However, for each of the $2^k - 1$ non-trivial choices of vectors $(a_1, \ldots, a_k) \in (\mathbb{Z}/2\mathbb{Z})^k$ the elements

$$X_1^{a_1} \cdot \ldots \cdot X_k^{a_k}$$
represent pairwise distinct square classes. So by part 1 of this lemma, adjoining their square roots to $F$ produces $2^k - 1$ distinct quadratic extensions of $F$ contained in the top field. Hence, $j = k$, proving the lemma.

**Lemma 4:** Let $A$ be a set of primes of a number field such that there exist sets of primes $A_n$ for every natural number $n$ whereby $A$ is almost contained in $A_n$ for each $n$ and $\lim_{n \to \infty} \delta(A_n) = 0$. Then $\delta(A) = 0$.

**Proof:** Fix a natural number $k$. Then there exists a natural number $n$ such that $\delta(A_n) < \frac{1}{2k}$. Also there exists a set $B_n$ of density 0 such that $A \subseteq A_n \cup B_n$. Let $\epsilon > 0$ be such that $\delta(A_n) + \epsilon < \frac{1}{2k}$. Then there exists a $\delta > 0$ such that if $1 < s < \delta$, then

$$\left| \sum_{\text{finite } P \in A_n} \frac{(N_P)^{-s}}{\log\left(\frac{1}{s-1}\right)} \right| < \delta(A_n) + \epsilon$$

and the similar expression with $A_n$ replaced by $B_n$ being less than $\frac{1}{2k}$. Since $A \subseteq A_n \cup B_n$, and by the triangle inequality, it follows that if $1 < s < \delta$, then

$$\left| \sum_{\text{finite } P \in A} \frac{(N_P)^{-s}}{\log\left(\frac{1}{s-1}\right)} \right| < \frac{1}{k}.$$

Since $\lim_{k \to \infty} \frac{1}{k} = 0$, it follows that $\delta(A)$ exists and equals 0. This proves Lemma 4.

Let $E$ and $F$ be algebraic number fields such that $E$ is a finite normal extension of $F$ of degree $n$. Recall that if $P_1, P_2, \ldots, P_g$ are the distinct finite primes of $E$ which lie over $P$, a finite prime of $F$, then $Gal(E|F)$ the Galois group of $E$ over $F$ acts transitively on $P_1, P_2, \ldots, P_g$. Fix $i$ where $1 \leq i \leq g
and define $G_P_i$ the decomposition group of $P_i$ to be the stabilizer of $P_i$ in $Gal(E|F)$. That is, $G_{P_i} = \{ \sigma \in Gal(E|F) | \sigma(P_i) = P_i \}$. Since $Gal(E|F)$ acts transitively on $\{P_1, P_2, ..., P_g\}$, if $P_j$ and $P_i$ are elements of $\{P_1, P_2, ..., P_g\}$, then there exists some $\sigma \in Gal(E|F)$ such that $\sigma^{-1}G_{P_i}\sigma = G_{P_j}$. If $Gal(E|F)$ is abelian, then $G_{P_j} = G_{P_i}$ and we denote $G_{P_j}$ by $G_P$.

The following is a collection of well-known facts from Algebraic Number Theory.

Let $E$, $F$, $n$, and $P, P_1, P_2, ..., P_g$ be as above.

**Fact 3:** (a) If $P$ is unramified in $E$, then $G_{P_i}$ is cyclic for $1 \leq i \leq g$ and is canonically generated by the Frobenius automorphism, denoted $\sigma_{P_i}$.

(b) There are only finitely many primes which ramify in $E$ over $F$, and they are precisely the ones which divide the discriminant of $E$ over $F$.

(c) Let $i$ be fixed such that $1 \leq i \leq g$ and $e$ and $f$ respectively denote the ramification index and inertial degree of $P_i$ over $P$. Then both $e$ and $f$ are independent of $i$, $efg = n$ and $[E_{P_i} : F_P] = ef$.

**Cebotarev's Density Theorem:** If $\tau$ is a fixed element of $Gal(E|F)$, then there exists an unramified finite prime $P$ of $F$ and a finite prime $P_1$ of $E$ lying over $P$ such that $\tau = \sigma_{P_1}$. In addition, the set of all finite primes $P$ of $F$ such that there exists a finite prime $P_1$ of $E$ lying over $P$ with $\sigma_{P_1}$ equal to $\tau$ has density given by

$$\frac{\# \text{ conjugates of } \tau \text{ in } Gal(E|F)}{\#Gal(E|F)}$$

The following is an immediate corollary.

**Special Case of Cebotarev's Density Theorem:** Let $\tau$ be as above. If $Gal(E|F)$ is abelian, then the density of all finite primes $P$ of $F$ such that
there exists a finite prime $P_1$ of $E$ lying over $P$ with $\tau = \sigma_{P_1}$ is equal to \frac{1}{\#Gal(E|F)}.

**Lemma 5:** Let $E$ be a finite normal abelian extension of degree $n$ of the number field $F$. If $P$ is a finite prime of $F$, then $G_P$ is trivial if and only if $P$ splits completely.

**Proof:** If $G_P$ is trivial, then the number of elements in the orbit of a finite prime $P_1$ of $E$ lying over $P$ is $\frac{|Gal(E|F)|}{|G_P|} = \frac{n}{1} = n$, and hence $P$ splits completely. Conversely if $P$ splits completely, then $n =$ the number of elements in the orbit of $P_1 = \frac{|Gal(E|F)|}{|G_P|} = \frac{n}{|G_P|}$. Hence $|G_P| = 1$ and $G_P$ is trivial, proving the lemma.

By the Special Case of Chebotarev's Density Theorem with $\tau =$identity and by Lemma 5, we have the following corollary.

**Corollary:** The density of all the finite primes of $F$ which split completely in $E$ is $\frac{1}{\#Gal(E|F)}$.

**Main Lemma:** Let $M$ be a set of primes of a number field $F$. If $G(M) = \{\bar{X} \in F^*/F^{*2} \text{ such that } \bar{X} = \bar{1} \text{ in } F_P^*/F_P^{*2} \text{ for almost all primes } P \text{ in } M\}$ is infinite, then $M$ has density zero.

**Proof:** Let $k$ be a natural number. By Lemma 3, there exist $\bar{X}_1, \bar{X}_2, ..., \bar{X}_k$ in $G(M)$ such that the extension $E_k = F(\sqrt{X_1}, \sqrt{X_2}, ..., \sqrt{X_k})$ has degree $[E_k : F] = 2^k$, where $X_i$ is any representative of $\bar{X}_i$. Let $D_k$ be the set of finite primes of $F$ that split completely in $E_k$.

We assert that $M$ is almost contained in $D_k$.

For this, fix $k$ and let $i$ and $j$ be such that $1 \leq i, j \leq k$. Also let $S_i$ be the set of all primes $P$ of $M$ for which $X_i$ is not a square in $F_P$ and let $S_j$ be the set of all finite primes of $F$ that ramify in $E_j$. Since any finite collection
of primes has density zero, Fact 3 (b) shows each $\tilde{S}_j$ has density zero. Since each $S_i$ has density zero, and since any finite union of sets of density zero has density zero, then
\[ S = \bigcup_{i=1}^{k} S_i \cup \bigcup_{j=1}^{k} \tilde{S}_j \]
has density zero. Let $P$ be a finite prime in $M - S$ and let $P_1$ be a finite prime of $E_k$ that lies over $P$. Since $X_i$ is a square in $F_p$ for $1 \leq i \leq k$, $(E_k)_{P_1} = F_p(\sqrt{X_1}, \sqrt{X_2}, \ldots, \sqrt{X_k}) = F_p$. By Fact 3 (c) $P$ splits completely in $E_k$, and hence $M$ is almost contained in $D_k$. By the Corollary to Lemma 5, $\delta(D_k) = \frac{1}{2^k}$. By Lemma 4 the density of $M$ is zero, proving the lemma.
Our goal in this chapter is to prove that if two number fields are reciprocity equivalent then there exists a reciprocity equivalence between them with an infinite wild set and that any wild set has density zero. The zero density of a wild set is a consequence of the Main Lemma and the following Fact.

**Fact 4:** Let $K$ be an algebraic number field and let $P$ be a prime of $K$.

(a) If $P$ is finite and nondyadic then $K_P^*/K_P^{*2}$ has exactly 4 representatives: $1, u_P, \pi_P, \pi_P u_P$ where $u_P$ is a local nonsquare unit of the integers of $K_P$ and $\pi_P$ is a local uniformizing parameter of $P$.

(b) If $P$ is dyadic, then $K_P^*/K_P^{*2}$ has order $2^{d+2}$ where $d = [K_P : Q_2]$.

(c) If $P$ is real infinite, then $K_P^*/K_P^{*2}$ has precisely 2 representatives: $1$ and $-1$.

(d) If $P$ is a complex infinite prime, then $K_P^*/K_P^{*2}$ is trivial.

**Lemma 6:** An element $X$ in $K^*$ is a local square at every wild prime of $(t, T)$ with at most finitely many exceptions.

**Proof:** Suppose not. Then, since $X$ is locally a unit at all but finitely many primes, there is an infinite set $C$ of finite nondyadic wild primes of $K$ such that $X$ is locally a non-square unit at every prime in $C$. Applying the square class map $t$ then shows that $tX$ is locally the square class of a local prime element at $TP$ for an infinite set of primes $TP$ of $L$. This is impossible, proving the lemma.

**Theorem A:** If $(t, T)$ is a reciprocity equivalence from $K$ to $L$, then the density of its wild set is zero.

**Proof:** Let $M$ be the wild set of $(t, T)$.
We assert that $G(M)$ is equal to the infinite group $K^*/K^{*2}$.

The inclusion $G(M) \subset K^*/K^{*2}$ is clear. Conversely, take $X$ in $K^*/K^{*2}$ and let $X$ be an element of $\bar{X}$. By Lemma 6 with the possible exception of a finite subset of $M$, $X$ is a local square at every element of $M$. Thus $X$ lies in $G(M)$. Hence $G(M)$ is infinite and so by the Main Lemma, $M$ has density zero, proving Theorem A.

Lemma 7: If $S$ is a finite set of primes containing the infinite primes of a number field $K$, then $U_S/U_S^2$ injects into $K^*/K^{*2}$.

Proof: Define a homomorphism $\alpha$ from $U_S$ into $K^*/K^{*2}$ by $\alpha(u) = u \mod K^{*2}$ for $u$ in $U_S$. Since $U_S^2$ is contained in the kernel of $\alpha$, $\alpha$ can be considered to be a homomorphism from $U_S/U_S^2$ into $K^*/K^{*2}$. If $\bar{u}$ in $U_S/U_S^2$ is an element of the kernel of $\alpha$ and $u \in \bar{u}$, then for some $y \in K^*$, $u = y^2$. Hence for every prime $P$ not in $S$, $0 = \text{ord}_P(u) = \text{ord}_P(y^2) = 2 \text{ord}_P(y)$, and so $\text{ord}_P(y) = 0$. Thus $y \in U_S$, $u \in U_S^2$ and $\alpha$ is an injection. Thus we have proved the lemma.

We now recall the following from [P].

Theorem 1: Let $S$ be a sufficiently large set and $K$ and $L$ be algebraic number fields. Then an $S$-equivalence from $K$ to $L$ can be extended to a reciprocity equivalence that is tame outside of $S$.

After stating Theorem 1 in [P], the following statement is proved (see [P]).

Lemma 8: Let $(t, T)$ be a reciprocity equivalence between two number fields $K$ and $L$ with at most a finite wild set $W$, and let $S$ be a sufficiently large set of primes of $K$ containing $W$. If $TS$ is also sufficiently large, then $(t, T)$ restricted to $U_S/U_S^2$ is an $S$-equivalence.

Let $A$ be a multiplicative group. Then we define the rank of $A$ to be the
smallest number of generators of $A$. If no finite set of elements of $A$ generates $A$, then we define the rank of $A$ to be $\infty$.

**Lemma 9:** If $S$ is a finite set of primes containing all infinite primes of a number field and $T = S \cup \{P\}$ for some prime $P$ not in $S$, then rank of $U_T \leq$ rank of $U_S + 1$.

**Proof:** Let $\alpha$ be an element of $U_T$ such that $\text{ord}_P(\alpha)$ is minimal and bigger than or equal to 1.

We assert that $U_T = U_S \times <\alpha>$. Namely, let $X = y\alpha^k$ where $y \in U_S$ and $k \in \mathbb{Z}$. If $P_0$ is a prime of $K$ not in $T$, then $\text{ord}_{P_0}(X) = \text{ord}_{P_0}(y) + k\text{ord}_{P_0}(\alpha) = 0 + k \cdot 0 = 0$. Thus $X$ is in $U_T$. Let $a \in U_T$ and let $\lambda = \text{ord}_P(a)$. By the Division Algorithm there exist positive integers $m$ and $r$ such that $\lambda = m \cdot \text{ord}_P(\alpha) + r$ where $0 \leq r < \text{ord}_P(\alpha)$. Since $r = \text{ord}_P(\frac{a}{\alpha^m})$ and by definition of $\alpha$, $r = 0$ and so $\text{ord}_P(a) = m \cdot \text{ord}_P(\alpha) = \text{ord}_P(\alpha^m)$. Hence $\text{ord}_P(a \cdot \alpha^{-m}) = 0$, and since $U_T$ is a group, $\text{ord}_{P_0}(a \cdot \alpha^{-m}) = 0$. Thus $a \cdot \alpha^{-m} \in U_S$ and so there exists some $\beta$ in $U_S$ such that $a = \beta \alpha^m$. Hence $U_T = U_S \times <\alpha>$ and so rank of $U_T \leq$ rank of $U_S$, proving the lemma.

**Lemma 10:** Let $S_\infty$ denote the set of infinite primes of the number field $K$. Then $U_{S_\infty}$ has finite rank.

**Proof:** Let $a$ be in $K^\times$. Then $a \in U_{S_\infty}$ if and only if $\text{ord}_P(a) = 0$ for every finite prime $P$ of $K$, which is true if and only if $aO_K = O_K$ where $O_K$ denotes the ring of integers of $K$, which is true if and only if $a$ is a unit in $O_K$. By Dirichlet's Unit Theorem, the units of $O_K$ have finite rank, proving the lemma.

From the last two lemmas we arrive at the following corollary.
**First Corollary:** If \( S \) is a finite set of primes of \( K \) containing all infinite primes, then \( U_S \) is finitely generated.

The following corollary follows immediately.

**Second Corollary:** If \( S \) is as above, then \( U_S/U_S^2 \) is finitely generated.

Let us recall the following fact concerning sufficiently large sets of a number field \( K \). If \( S \) is a finite set of primes containing all real and dyadic primes of \( K \), then to insure that \( S \) is sufficiently large add to \( S \) any set of prime generators of the Sylow 2-subgroup of the ideal class group of \( K \). Any sufficiently large set containing such a set of prime generators will be called specially sufficiently large. The following lemma is now immediate.

**Lemma 11:** If \( S_1 \) is a specially sufficiently large set of primes of a number field \( K \) and \( S_2 \) is a finite set of primes of \( K \) containing \( S_1 \), then \( S_2 \) is also specially sufficiently large.

**Fact 5:** Let \( P \) be a finite nondyadic prime of a number field \( K \) and let \( u \) be a nonsquare unit of the integers of \( K_P \) and \( \pi \) be a local uniformizing parameter of \( P \). Then \( (u,u)_P = 1, (\pi,u)_P = -1, (u\pi,u)_P = -1 \). Moreover \( (\pi,\pi)_P = 1 \) if and only if \(-1\) is a square in \( K_P^*/K_P^{*2} \).

**Lemma 12:** If \( S \) is a specially sufficiently large set of primes of a number field \( K \), then there exists a small equivalence \((t_{S_1}, \tilde{T})\) from \( U_{S_1}/U_{S_1}^2 \) onto \( U_{S_1}/U_{S_1}^2 \) where \( S_1 = S \cup \{P_0\} \) for some finite nondyadic prime \( P_0 \) outside of \( S \) and where \((t_{S_1}, \tilde{T})\) is defined as follows:

1. \( t_{S_1} \) is the identity map on \( U_{S_1}/U_{S_1}^2 \).
2. \( \tilde{T} \) is the identity map on \( S_1 \).
3. For every prime \( P \) in \( S \) the local map \( t_P \) of \((t_{S_1}, \tilde{T})\) is the identity on \( K_P^*/K_P^{*2} \).
4. The local map \( t_{P_0} : K_{P_0}^*/K_{P_0}^{*2} \to K_{P_0}^*/K_{P_0}^{*2} \) is defined by \( t_{P_0}(\bar{1}) = \bar{1}, \)
\( t_{P_0}(\bar{u}_0) = \bar{u}_0\pi_0, \)
\( t_{P_0}(\pi_0) = \pi_0 \) and \( t_{P_0}(\bar{u}_0\pi_0) = \bar{u}_0 \) where \( u_0 \) denotes \( u_{P_0} \) and \( \pi_0 \) denotes \( \pi_{P_0} \).

**Proof:** By the Second Corollary to Lemma 10, there exist \( a_1, a_2, \ldots, a_n \) in \( K \) which generate \( U_S/U_S^2 \). Let \( L_S \) denote the composite of the fields \( K(\sqrt{a_1}), K(\sqrt{a_2}), \ldots, K(\sqrt{a_{n+1}}) \) where \( a_{n+1} = -1 \). By the Corollary to Lemma 5 and since any finite set of primes has density zero, there exist infinitely many primes of \( K \) which split completely in \( L_S \). Now let \( P_0 \) be one such finite nondyadic prime. If \( P_i \) is a prime of \( K(\sqrt{a_i}) \) that lies over \( P_0 \), then we know that \( K(\sqrt{a_i})_{P_i} = K_{P_i}(\sqrt{a_i}) \), and hence \( 1 = [K(\sqrt{a_i})_{P_i} : K_{P_i}] = [K_{P_i}(\sqrt{a_i}) : K_{P_0}] \). Thus for \( 1 \leq i \leq n + 1, a_i \) is a square at \( P_0 \). Let \( S_1 = S \cup \{ P_0 \} \) and let \((t_{S_1}, \bar{T})\) be as in the statement of this claim. Hence the following diagram commutes:

\[
\begin{array}{ccc}
U_{S_1}/U_{S_1}^2 & \xrightarrow{diag} & \prod_{P \in S_1} K_P^*/K_P^{*2} \\
\downarrow & & \downarrow \\
U_{S_1}/U_{S_1}^2 & \xrightarrow{diag} & \prod_{P \in S_1} K_P^*/K_P^{*2}
\end{array}
\]

where the two unidentified maps are identities.

By Fact 5 we have the following Hilbert symbol equalities:

\((u_0, u_0)_{P_0} = (u_0, u_0)_{P_0}(u_0, \pi_0)_{P_0}(\pi_0, \pi_0)_{P_0} = (u_0\pi_0, u_0\pi_0)_{P_0},\)

and \((u_0, \pi_0)_{P_0} = (u_0, \pi_0)_{P_0}(\pi_0, \pi_0)_{P_0} = (u_0\pi_0, \pi_0)_{P_0}.\)

From the two equalities above and the definition of the group isomorphism \( t_{P_0}, \)
\( t_{P_0} \) clearly preserves Hilbert symbols. By Lemma 11 \( S_1 \) is sufficiently large, and hence \((t_{S_1}, \bar{T})\) is a small equivalence, proving the lemma.
The following lemma is clearly true.

**Lemma 13:** (a) Let \((t_1, T_1)\) and \((t_2, T_2)\) respectively be small equivalences from \(U_S/U_S^2\) onto \(U_{T_1S}/U_{T_1S}^2\) and from \(U_{T_1S}/U_{T_1S}^2\) onto \(U_{T_2T_1S}/U_{T_2T_1S}^2\) where \(S, T_1S\) and \(T_2S\) are respectively sufficiently large sets of number fields \(K, L\) and \(M\). Then \((t_2 \circ t_1, T_2 \circ T_1)\) is an \(S\)-equivalence.

(b) If \(P\) is a wild prime of \((t_1, T_1)\) and \(T_1P\) is a tame prime of \((t_2, T_2)\), then \(P\) is a wild prime of \((t_2 \circ t_1, T_2 \circ T_1)\). Also if \(P\) is a tame prime of \((t_1, T_1)\) and \(T_1P\) is a wild prime of \((t_2, T_2)\), then \(P\) is a wild prime of \((t_2 \circ t_1, T_2 \circ T_1)\).

**Lemma 14:** Let \((t, T)\) be reciprocity equivalence from the number field \(K\) to the number field \(L\) with a wild set \(W\) comprised of \(n\) elements (where \(n\) can be zero). Suppose that \(S\) and \(TS\) are respectively specially sufficiently large sets of primes of \(K\) and \(L\) and that \(S\) contains \(W\). Then there exists a prime \(P_0\) of \(K\) outside of \(S\), a set of primes \(S_1 = S \cup \{P_0\}\) and a small equivalence \((t_{S_1}, T_{S_1})\) from \(U_{S_1}/U_{S_1}^2\) onto \(U_{TS_1}/U_{TS_1}^2\) satisfying the following properties:

1. \((t_{S_1}, T_{S_1})\) has exactly \(n + 1\) wild primes;
2. \((t_{S_1}, T_{S_1})\) restricted to \(U_S/U_S^2\) is precisely \((t, T)\) restricted to \(U_S/U_S^2\).

**Proof:** By Lemma 8 \((t, T)\) restricted to \(U_S/U_S^2\) is an \(S\)-equivalence onto \(U_{TS}/U_{TS}^2\). By Lemma 12 there exists a prime \(P_0\) of \(K\) outside of \(S\) such that if \(S_1 = S \cup \{P_0\}\) (and hence \(T_{S_1} = TS \cup \{TP_0\}\)), then there exists a small equivalence \((t_{TS_1}, \overline{T})\) from \(U_{TS_1}/U_{TS_1}^2\) onto \(U_{TS_1}/U_{TS_1}^2\), which satisfies properties 1, 2, 3, and 4 of Lemma 12. By Lemma 13 if we let \((t_{S_1}, T_1)\) denote \((t_{TS_1} \circ t, \overline{T} \circ T)\) from \(U_{S_1}/U_{S_1}^2\) onto \(U_{TS_1}/U_{TS_1}^2\), then \((t_{S_1}, T_1)\) is an \(S_1\)-equivalence with exactly \(n + 1\) wild primes. Since \(U_S \subseteq U_{S_1}\) and by definition of \((t_{S_1}, T_1)\), property 2 of this lemma is satisfied. Hence we have
established the lemma.

**Lemma 15:** Let \((t, T)\) be a reciprocity equivalence from \(K\) to \(L\) with a finite wild set \(W\). Let \(p_1, p_2, p_3, \ldots\) denote an ordering of the rational primes and for every natural number \(n\), let \(A_n\) and \(B_n\) respectively denote the set of all primes of \(K\) and \(L\) which contain \(p_n\). Then there exist two sequences \(\{S_n\}_{n=1}^{\infty}\) and \(\{TS_n\}_{n=1}^{\infty}\) of specially sufficiently large sets of primes respectively of \(K\) and \(L\) such that \(S_n\) contains \(\bigcup_{i=1}^{n} A_i\) and \(TS_n\) contains \(\bigcup_{i=1}^{n} B_i\), and if \(i \leq j\), then \(S_i \subseteq S_j\) and \(TS_i \subseteq TS_j\). Moreover there exists a sequence of small equivalences \(\{(t_n, T_n)\}_{n=1}^{\infty}\), where \((t_n, T_n)\) is a \(S_n\)-equivalence whose wild set is comprised of at least \(n\) primes satisfying the following property:

1. If \(i \leq j\), then \((t_j, T_j)\) restricted to \(U_{S_i}/U_{S_i}^2\) is precisely \((t_i, T_i)\).

**Proof:** We will prove the lemma by induction. Let \(C\) be a finite set of primes which generates the Sylow 2-subgroup of the ideal class group of \(K\). Define \(S_0\), a set of primes of \(K\), to be the union of all infinite primes, dyadic primes and the set \(C\). Similarly define \(\tilde{S}_0\), a set of primes of \(L\), to be the union of all infinite primes, dyadic primes and a set \(D\), where \(D\) generates the Sylow 2-subgroup of the ideal class group of \(L\). By Lemma 11 any finite set of primes containing either \(S_0\) or \(\tilde{S}_0\) is specially sufficiently large. Let \(\tilde{S}_1 = S_0 \cup T^{-1}(\tilde{S}_0 \cup B_1) \cup W \cup A_1\). By Lemma 14 there exists a prime \(P_0\) of \(K\) outside of \(\tilde{S}_1\), a set \(S_1 = \tilde{S}_1 \cup \{P_0\}\) and a small equivalence \((t_{S_1}, T_{S_1})\) from \(U_{S_1}/U_{S_1}^2\) onto \(U_{TS_1}/U_{TS_1}^2\) which has at least one wild prime. Since \(S_1\) contains \(A_1\) and \(TS_1\) contains \(B_1\), the lemma is true for \(n = 1\).

Assume that the lemma is true for \(n\). By Theorem 1 extend \((t_n, T_n)\) to a reciprocity equivalence \((\tilde{t}, \tilde{T})\) from \(K\) to \(L\) that is tame outside of \(S_n\). Let \(\tilde{S}_{n+1} = S_n \cup A_{n+1} \cup \tilde{T}^{-1}(B_{n+1})\). By Lemma 14 there exists a prime
$P_{n+1}$ of $K$ outside of $\tilde{S}_{n+1}$, a set $S_{n+1} = \tilde{S}_{n+1} \cup \{P_{n+1}\}$ and a small equivalence $(t_{n+1}, T_{n+1})$ from $U_{S_{n+1}}/U_{S_{n+1}}^2$ onto $U_{\tilde{S}_{n+1}}/U_{\tilde{S}_{n+1}}^2$ which has at least $n + 1$ wild primes and which satisfies the following property: $(t_{n+1}, T_{n+1})$ restricted to $U_{\tilde{S}_{n+1}}/U_{\tilde{S}_{n+1}}^2$ is precisely $(\tilde{t}, \tilde{T})$ restricted to $U_{\tilde{S}_{n+1}}/U_{\tilde{S}_{n+1}}^2$. Hence $(t_{n+1}, T_{n+1})$ restricted to $U_{S_n}/U_{S_n}^2$ is exactly $(t_n, T_n)$. Since $S_{n+1} \supseteq S_n \cup A_{n+1}$ and $T_{n+1}S_{n+1} \supseteq T_nS_n \cup B_{n+1}$, the lemma is true for $n + 1$. By induction the lemma is valid.

**Theorem B:** If $K$ is reciprocity equivalent to $L$, then there exists a reciprocity equivalence between them with an infinite wild set.

**Proof:** Let $(t, T)$ be a reciprocity equivalence from $K$ to $L$. If the wild set $W$ of $(t, T)$ is infinite, then this theorem is true. So assume that $W$ is at most finite and employ the notation of the previous lemma.

We define the reciprocity equivalence $(t_0, T_0)$ from $K$ to $L$ by the following procedure. Let $\tilde{\alpha}$ be an element of $K^*/K^{*2}$ and let $\alpha$ be a representative of $\tilde{\alpha}$. Since $S_n \supseteq \bigcup_{i=1}^n A_i$ for $m \in N$ and only finitely many primes of $K$ divide $\alpha$, $\alpha \in U_{S_n}$ for some natural number $n$. By Lemma 7 we can consider $\tilde{\alpha}$ in $K^*/K^{*2}$ to be an element of $U_{S_n}/U_{S_n}^2$. Let $n_0$ be the smallest positive integer such that $\tilde{\alpha} \in U_{S_{n_0}}/U_{S_{n_0}}^2$. Define $t_0(\tilde{\alpha}) = t_{n_0}(\tilde{\alpha})$ considered as an element of $L^*/L^{*2}$. Note that $t_0$ is clearly well-defined.

Let $P$ be a prime of $K$. Then $P$ contains some rational prime $p_n$ and thus $P \in S_n$. Let $l$ be the smallest positive integer such that $P \in S_l$. Define $T_0(P) = T_l(P)$ and note that $T_0$ is clearly well-defined.

We assert that the map $t_0$ is an isomorphism.

Namely, let $x, y$ be elements of $K^*/K^{*2}$ and let $m$ be a positive integer such that $x, y$ and $xy$ are elements of $U_{S_m}/U_{S_m}^2$. By 1 in Lemma 15, $t_0(x) = t_m(x)$,
Suppose that \( x \) is in the kernel of \( t_0 \) and let \( r \) be the smallest positive integer such that \( x \in U_{S_r}/U_{S_r}^2 \). Since \( t_r \) is injective, \( t_0 \) is injective. Let \( z \) be in \( L^*/L^{*2} \) and let \( a \) be the smallest positive integer such that \( x \in U_{T_a S_a}/U_{T_a S_a}^2 \). Since \( t_a \) is surjective, \( t_0 \) is also. Hence \( t_0 \) is an isomorphism.

We assert that \( T_0 \) is a bijection.

Namely, let \( P_0 \) be a prime of \( L \). Then \( P_0 \) contains some rational prime \( p_n \) and hence \( P_0 \in T_n S_n \). Let \( m \) be the smallest integer such that \( P_0 \in T_m S_m \) and let \( P_1 = T_m^{-1}(P_0) \). Then \( T_0(P_1) = P_0 \) and thus \( T_0 \) is surjective. Let \( P_2 \) and \( P_3 \) be primes of \( K \) such that \( T_0(P_2) = T_0(P_3) \), and let \( b \) be a positive integer such that \( P_2 \) and \( P_3 \) are both elements of \( S_{t_a} \). By 1 in the Lemma 15, \( T_b(P_2) = T_b(P_3) \) and since \( T_b \) is a bijection, \( P_2 = P_3 \). Hence \( T_0 \) is a bijection.

We assert that \( (t_0, T_0) \) preserves Hilbert symbols.

Namely, let \( a \) and \( b \) be elements of \( K^*/K^{*2} \), \( P \) be a prime of \( K \), and \( n \) be a positive integer such that \( a \) and \( b \) are elements of \( U_{S_n}/U_{S_n}^2 \) and \( P \) is an element of \( S_{t_a} \). By Theorem 1 \((t_n, T_n)\) can be extended to a reciprocity equivalence \((\tilde{t}_0, \tilde{T}_0)\) from \( K \) to \( L \), which in particular preserves Hilbert symbols. Hence \((a, b)_P = (\tilde{t}_0(a), \tilde{t}_0(b))_{\tilde{T}_0 P} = (t_n(a), t_n(b))_{T_n P} = (t_0(a), t_0(b))_{T_0 P}\), and thus \((t_0, T_0)\) preserves Hilbert symbols.

Since \((t_n, T_n)\) contains at least \( n \) wild primes and \((t_0, T_0)\) is an extension of \((t_n, T_n)\), \((t_0, T_0)\) possesses infinitely many wild primes. By our assertions \((t_0, T_0)\) is a reciprocity equivalence from \( K \) to \( L \) with an infinite wild set. Hence the theorem is established.
CHAPTER 4

Main Results: Part 2

Our goal in this chapter is to prove the following theorem.

**Theorem C:** Let \((t_1, T_1)\) and \((t_2, T_2)\) be reciprocity equivalences from \(K\) to \(L\).

1. If \(t_1 = t_2\), then \(T_1 = T_2\) except possibly at the complex primes of \(K\).

2. If \(T_1P = T_2P\) for every \(P\) in a set \(M\) of primes of \(K\) where the density of \(M\) is bigger than zero, then \(T_1 = T_2\) except possibly at the complex primes of \(K\).

If \(P\) is a prime of \(K\), let \(K_P\) denote the completion of \(K\) with respect to \(P\) and \(O_{K_P}\) denote the integers of \(K_P\). In addition, let us recall the Approximation Theorem for pairwise inequivalent valuations.

**Approximation Theorem:** Let \(P_1, P_2, \ldots, P_n\) be distinct primes of \(K\), \(\vert \cdot \vert_{P_i}\) be the unique valuation of \(K_{P_i}\) up to equivalence, \(X_i \in K\) for \(1 \leq i \leq n\), and \(\epsilon > 0\). Then there exists an \(X \in K\) such that \(\vert X - X_i \vert_{P_i} < \epsilon\) for \(1 \leq i \leq n\).

**Corollary:** Let \(P_1, P_2, \ldots, P_n\) be distinct primes of \(K\), \(X_i \in K_{P_i}\) for \(1 \leq i \leq n\) and \(\epsilon > 0\). Then there exists an \(X \in K\) such that \(\vert X - X_i \vert_{P_i} < \epsilon\) for \(1 \leq i \leq n\).

**Proof:** Recall that for every prime \(P\) of \(K\), \(K\) is dense in \(K_P\) with respect to \(\vert \cdot \vert_P\). Hence for \(1 \leq i \leq n\) there exist \(y_i \in K\) such that \(\vert y_i - X_i \vert_{P_i} < \frac{\epsilon}{2}\). By the Approximation Theorem there exists an \(X \in K\) such that \(\vert X - y_i \vert_{P_i} < \frac{\epsilon}{2}\) for \(1 \leq i \leq n\). Hence \(\vert X - X_i \vert_{P_i} \leq \vert X - y_i \vert_{P_i} + \vert y_i - X_i \vert_{P_i} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\). Thus \(\vert X - X_i \vert_{P_i} < \epsilon\) for \(1 \leq i \leq n\).

Let us recall the following Theorem [O].
Local Square Theorem: Let $\alpha$ be an integer in the local field $F$. Then there exists some $\beta$ in the integers of $F$ such that $1 + 4\pi\alpha = (1 + 2\pi\beta)^2$ where $\pi$ is a local uniformizer of $F$.

Corollary: Let $P$ be a finite prime of $K$. Then there exists a natural number $n$ such that if $X$ is an integer of $K_P$ and $X \equiv 1 \mod P^n$, then $X = a^2$ for some $a \in K_P^*$.

Proof: Suppose that $P$ is a finite nondyadic prime. Then 4 is a unit of $O_{K_P}$ and hence $4P = P$. By the above Theorem if $X \equiv 1 \mod P$, then $X = a^2$ for some $a \in K_P^*$.

Suppose that $P$ is a dyadic prime. Let $e$ be the ramification index of $K_P$ over $Q_2$. By definition of $e$, $2O_{K_P} = P^e$, and hence 2 is an element of $P^e$. Let $\pi$ be a local uniformizing parameter of $P$. Then $2 = \pi^e \cdot u$ for some unit $u$ in $O_{K_P}$. Hence $4 = \pi^{2e} \cdot v$ where $v = u^2$ is a unit of $O_{K_P}$, and so $4P = P^{2e+1}$. If $X \equiv 1 \mod P^{2e+1}$, then $X \equiv 1 \mod 4P$ and by the above Theorem, $X = a^2$ for some $a$ in $K_P^*$. Thus the corollary is established.

Let $P$ be a finite prime, $x$ and $y$ be in $K_P$, and $0 < \varepsilon < 1$. We recall that $|x - y|_P < \varepsilon$ means that there exists a natural number $n$ such that $x - y$ is in $P^n$.

Lemma 16: Let $x, y$ be in $K_P^*$ where $P$ is a prime of $K$. There exists an $\varepsilon > 0$ such that if $|x - y|_P < \varepsilon$, then $x = y \cdot a^2$ for some $a \in K_P^*$.

Proof: Suppose that $P$ is complex. Since $K_P^*/K_P^{*2} = \{1\}$, if $x$ and $y$ are in $K_P^*$, then $x = y \cdot a^2$ for some $a$ in $K_P^{*2}$, and hence in this case the Claim is trivially true.

Suppose that $P$ is real infinite. Then $K_P$ is isomorphic to the real numbers $\mathbb{R}$. Let $\phi$ be an isomorphism from $K_P$ onto $\mathbb{R}$ such that if $a \in K_P$, then
$|a|_p = |\phi(a)|$ where $| |$ denotes the standard absolute value. Let $\epsilon = |\phi(x)|$.

If $|x - y|_p < \epsilon$, then both $x$ and $y$ are positive or both $x$ and $y$ are negative. Hence $x = ya^2$ for some $a \in K_p^*$.

Suppose that $P$ is a finite prime. By the previous corollary choose $n$ so that if $z$ is in $O_{K_p}$ and $z \equiv 1 \mod P^n$, then $z = a^2$ for some $a \in K_p^*$.

First assume that $x$ is an integer of $K_p^*$ and that $y$ is a unit of the integers of $K_p^*$. Choose $\epsilon$ so that if $|x - y|_p < \epsilon$, then $x - y \in P^n$. Hence $\frac{x}{y} - 1 \in \frac{1}{y}P^n$.

Since $y$ is a unit of $O_{K_p}$, $\frac{1}{y}$ is likewise and $\frac{1}{y}P^n = P^n$. Thus $\frac{x}{y} = b^2$ for some $b \in K_p^2$ and hence $x = yb^2$.

Secondly assume that $x$ and $y$ are both integers of $K_p^*$ neither of which is a unit of $O_{K_p}$. Let $\pi$ be a local uniformizing parameter of $P$. Then $x = \pi^\alpha \cdot u$, $y = \pi^\beta \cdot v$ where $\alpha, \beta > 0$ and $u, v$ are units of $O_{K_p}$. We may assume that $\alpha \geq \beta$. Choose $\epsilon$ such that if $|x - y|_p < \epsilon$, then $x - y \in P^{(\beta + n)}$. Since $\pi^\alpha u - \pi^\beta v \in P^{(\beta + n)}$, then $\pi^\beta (\pi^{(\alpha - \beta)} u - v) \in P^{(\beta + n)}$. Hence $\pi^{(\alpha - \beta)} \cdot \frac{u}{v} \equiv 1 \mod P^n$. Since $\alpha \geq \beta$, $\pi^{(\alpha - \beta)} \cdot \frac{u}{v}$ is an integer of $K_p^*$, and hence exists some $b \in K_p^*$ such that $\pi^{(\alpha - \beta)} \cdot \frac{u}{v} = b^2$, and so $\pi^{(\alpha - \beta)} u = vb^2$. Hence $\pi^\beta \pi^{(\alpha - \beta)} u = \pi^\beta vb^2$ and thus $x = yb^2$.

By our last two assumptions, if $x$ and $y$ are integers of $K_p^*$, then there exists an $\epsilon > 0$ such that if $|x - y|_p < \epsilon$, then $x = yb^2$ for some $b \in K_p^*$.

Finally assume that $x$ and $y$ are in $K_p^*$ with no further restriction. Then there exists an $\alpha \geq 0$ such that $\pi^\alpha x$ and $\pi^\alpha y$ are both integers of $K_p^*$. Hence there exists an $\epsilon_0 > 0$ such that if $|\pi^\alpha x - \pi^\alpha y|_p < \epsilon_0$, then $\pi^\alpha x = \pi^\alpha y b^2$ for some $b \in K_p^*$. Let $\epsilon = \frac{\epsilon_0}{|\pi|_p}$. Hence if $|x - y|_p < \epsilon$, then $|\pi^\alpha x - y|_P = |\pi^\alpha x - \pi^\alpha y|_P < \epsilon_0$, and so $\pi^\alpha x = \pi^\alpha y b^2$. Thus $x = yb^2$ and we have established the lemma.
Corollary: Let $P_1, P_2, ..., P_n$ be distinct primes of $K$ and $X_i$ be an element of $K^*_p$ for $1 \leq i \leq n$. Then there exists an $X$ in $K^*$ such that $X = X_i \cdot \alpha_i^2$ where $\alpha_i \in K^*_p$ for $1 \leq i \leq n$.

Proof: By Lemma 16 there exist $\epsilon_1, \epsilon_2, ..., \epsilon_n$ such that if $Y_i \in K^*_p$ and $|Y_i - X_i|_{P_i} < \epsilon_i$, then $Y_i = X_i \alpha_i^2$ for some $\alpha_i \in K^*_p$ where $1 \leq i \leq n$. By the Corollary to the Approximation Theorem, there exists an $X \in K^*$ such that $|X - X_i|_{P_i} < \epsilon$ where $\epsilon = \text{minimum of } \{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$. Hence $X = X_i \alpha_i^2$ for $1 \leq i \leq n$, proving the corollary.

Lemma 17: Let $P$ be a tame nondyadic prime of a reciprocity equivalence $(t, T)$ from $K$ to $L$ and let $u_P$ and $u_{TP}$ respectively be nonsquare units of $O_{K_p}$ and $O_{L_{TP}}$. Then $t_P(u_P) = \bar{u}_{TP}$.

Proof: Since $t_P$ is a local homomorphism, $t_P(\bar{1}) = \bar{1}$. By Fact 4(a), $t_P(u_P) = \bar{u}_{TP}$, $\bar{u}_{TP}$ or $\bar{u}_{TP} \cdot \bar{u}_{TP}$. Since $P$ is tame, the latter two possibilities are ruled out, and hence the lemma is established.

Corollary: Let $(t, T)$ be a self-equivalence on $K$, and define a homomorphism $\eta$ from $K^*/K^{*2}$ into $K^*/K^{*2}$ by $\eta(x) = \bar{t}(\bar{x})$ for $x$ in $K^*/K^{*2}$. If $\bar{y}$ is a fixed element of $K^*/K^{*2}$, then $\eta(\bar{y}) = \bar{1}$ in $K^*_p/K^{*2}_p$ for every tame prime $P$ of $K$ such that $TP = P$ which lies outside of a finite exceptional set.

Proof: Let $\bar{y} \in K^*/K^{*2}$, $y \in \bar{y}$ and $O_K$ denote the integers of $K$. For at most finitely many primes $P_1, P_2, ..., P_n$, $ord_P(y) \neq 0$. Suppose that $P$ is a tame nondyadic prime of $K$ such that $TP = P$ which is not one of the $P_i$ where $1 \leq i \leq n$. Hence $\bar{y} = \bar{1}$ in $K^*_p/K^{*2}_p$ or $\bar{y} = \bar{u}_P$ in $K^*_p/K^{*2}_p$. By Lemma 17 $\eta(\bar{y}) = \bar{1} \cdot \bar{1}$ or $\bar{u}_P \cdot \bar{u}_P$ in $K^*_p/K^{*2}_p$, and hence $\eta(\bar{y}) = \bar{1}$ in $K^*_p/K^{*2}_p$. Since there are only finitely many dyadic primes of $K$, we have established the corollary.

Fact 6: Let $P$ be a noncomplex prime of $K$. Then there exist $x$ and $y$ in
Lemma 18: Let $(t, T)$ be a self-equivalence on $K$. If $t$ is the identity map, then $T$ is also except possibly at the complex primes.

Proof: Suppose that the lemma is false. Then there exists a noncomplex prime $P$ of $K$ such that $TP$ is not equal to $P$. By Fact 6, there exist $x$ and $y$ in $K^*$ such that $(x, y)_P = -1$. By the Corollary to Lemma 16 there exists an $a$ in $K^*$ such that $a = x\alpha^2$ for some $\alpha \in K_P^*$ and $a = 1 \cdot \beta^2$ for some $\beta$ in $K_T^*$. Since $t_P(\bar{a}) = \bar{1}$ in $K_T^*/K_{TP}^*$ and $(t, T)$ preserves Hilbert symbols, $(a, y)_P = (t\bar{a}, t\bar{y})_{TP} = 1$. Since $(x, y)_P = (a, y)_P = -1$, we have a contradiction. Thus we have established the lemma.

Lemma 19: Let $(t, T)$ be a self-equivalence on $K$ and let $\eta$ be as before. Suppose that the image of $\eta$ is a finite set. Then $TP = P$ for every prime $P$ of $K$ outside of a finite exceptional set.

Proof: Let $\{\bar{X}_1, \bar{X}_2, ..., \bar{X}_n\}$ denote the image of $\eta$ where each $\bar{X}_i$ is distinct for $1 \leq i \leq n$. Let $TP$ be a finite nondyadic prime of $K$ such that $ord_{TP}(\bar{X}_i) \equiv 0 \pmod{2}$ for $1 \leq i \leq n$. Assume that $TP$ is not equal to $P$. By the Corollary to Lemma 16, there exist $\bar{a}, \bar{b}$ in $K^*/K^*_{P}$ such that $\bar{a} \equiv \bar{a}_P \pmod{K^*_P}$, $\bar{b} \equiv \bar{1} \pmod{K^*_P}$, $\bar{b} \equiv \bar{1} \pmod{K^*_P}$. Let $\bar{X}_j$ be $\eta(\bar{a})$ and $\bar{X}_k$ be $\eta(\bar{b})$. By Fact 5, $-1 = (\bar{a}, \bar{b})_P = (t\bar{a}, t\bar{b})_{TP} = (\bar{X}_j\bar{a}, \bar{X}_k\bar{b})_{TP} = (\bar{a}, \bar{b})_{TP}(\bar{X}_j, \bar{X}_k)_{TP}(\bar{X}_j, \bar{X}_k)_{TP} = 1 \cdot 1 \cdot 1(\bar{X}_j, \bar{X}_k)_{TP} = (\bar{X}_j, \bar{X}_k)_{TP}$. Since $ord_{TP}(\bar{X}_i) \equiv 0 \pmod{2}$ for $1 \leq i \leq n$, by Fact 5 $(\bar{X}_j, \bar{X}_k)_{TP} = 1$, and we have a contradiction. Hence $TP = P$. Since $ord_{P_0}(\bar{X}_i) \not\equiv 0 \pmod{2}$ for at most finitely many finite nondyadic primes $P_0$ of $K$ (where $X_i$ is fixed), and since there are only finitely many infinite and dyadic primes of $K$, we have established the lemma.
Lemma 20: Let $X$ be in $K^*$ and suppose that $X$ is a local square at all but possibly finitely many primes of $K$. Then $X$ is a global square.

Proof: Suppose that the degree of $K(\sqrt{X})$ over $K$ is 2. Let $P$ be a finite prime of $K$, $P_1$ be a prime of $K(\sqrt{X})$ that lies over $P$, $e$ and $f$ respectively be the ramification index and inertial degree of $P_1$ over $P$.

We assert that $P$ splits completely in $K(\sqrt{X})$ if and only if $\sqrt{X} \in K_P$.

Namely, recall that $P$ splits completely in $K(\sqrt{X})$ if and only if $ef = 1$. Suppose that $ef = 1$. Since $K_P(\sqrt{X}) = K(\sqrt{X})_{P_1}$ and by Fact 3 (c), the degree of $K_P(\sqrt{X})$ over $K_P$ is 1. Hence $\sqrt{X} \in K_P$. Conversely if $\sqrt{X} \in K_P$, then $ef = 1$ and we have proved the assertion.

By the Corollary to Lemma 5, the density of all primes of $K$ which split completely in $K(\sqrt{X})$ is $\frac{1}{2}$. Since $X$ is a local square at all but possibly finitely many primes of $K$, by the assertion and since any finite set of primes has density zero, the density of all primes of $K$ which split completely in $K(\sqrt{X})$ is 1. Hence we have a contradiction, and thus $K(\sqrt{X})$ over $K$ is an extension of degree 1. Thus $X$ is a global square and we have proved the lemma.

Lemma 21: Let $(t, T)$ be a self-equivalence on $K$ and $\eta$ be as before. If $t \neq$ identity map or $T \neq$ identity map except possibly at the complex primes of $K$, then the image of $\eta$ is infinite.

Proof: We partition the finite primes of $K$ into three disjoint sets $A, B$ and $C$ defined as follows: $A$ is comprised of all tame primes $P$ of $(t, T)$ such that $TP \neq P$; $B$ is comprised of all tame primes $P$ of $(t, T)$ such that $TP = P$; $C$ is comprised of all wild primes of $(t, T)$. Suppose that the image of $\eta$ is a finite set. By Lemma 19, $A$ is at most a finite set. Let $X \in K^*/K^{*2}$. By the Corollary to Lemma 17, $\eta(X) = 1$ in $K_P^*/K_P^{*2}$ for every prime $P$ in $B$ excluding
a finite exceptional set. By Lemma 6, \(\eta(X) = 1\) in \(K_p^*/K_p^{*2}\) for every prime \(P\) of \(C\) outside of a finite exceptional set. Since \(K\) has only finitely many infinite primes, \(\eta(X) = 1\) in \(K_p^*/K_p^{*2}\) for every prime \(P\) of \(K\) outside of a finite exceptional set. By Lemma 20, \(\eta(X) = Xt(X) = 1\) in \(K^*/K^{*2}\) and hence \(t(X) = X\). Hence \(t =\) identity map and by Lemma 18, \(T =\) identity map except possibly at the complex primes of \(K\). Thus we have a contradiction and so the image of \(\eta\) is an infinite set, proving the lemma.

**Lemma 22:** Let \((t, T)\) be a self-equivalence on \(K\) and let \(M\) be the set of primes \(P\) of \(K\) such that \(TP = P\). Then \(\eta(X) \in G(M)\) for every \(X \in K^*/K^{*2}\).

**Proof:** Let \(X\) be a fixed element of \(K^*/K^{*2}\) and let \(A\) and \(B\) respectively be the set of all tame and wild primes \(P\) of \((t, T)\) contained in \(M\) such that \(\eta(X) \neq 1\) in \(K_p^*/K_p^{*2}\). By the Corollary to Lemma 17, the density of \(A\) is zero. By Theorem A and since any subset of a set with density zero has density zero, the density of \(B\) is zero. Since the density of \(A \cup B\) is zero, we have established the lemma.

**Corollary:** Let \((t, T)\) be a self-equivalence on \(K\) and \(M\) be the set of primes \(P\) of \(K\) such that \(TP = P\). If the density of \(M\) is bigger than zero, then \(t =\) identity map and \(T =\) identity map except possibly at the complex primes of \(K\).

**Proof:** Suppose that \(t \neq\) identity map or \(T \neq\) identity map except possibly at the complex primes of \(K\). By Lemma 21 the image of \(\eta\) is infinite. By the Main Lemma and by Lemma 22, the density of \(M\) is zero. Hence we have a contradiction, and thus we have established the corollary.

**Theorem C:** Let \((t_1, T_1)\) and \((t_2, T_2)\) be reciprocity equivalences from \(K\) to \(L\).
1. If \( t_1 = t_2 \), then \( T_1 = T_2 \) except possibly at the complex primes of \( K \).

2. If \( T_1P = T_2P \) for every prime \( P \) contained in a set \( M \) of primes of \( K \) such that the density of \( M \) is bigger than zero, then \( t_1 = t_2 \) and \( T_1 = T_2 \) except possibly at the complex primes of \( K \).

Proof: Note that \( (t_2^{-1}t_1, T_2^{-1}T_1) \) is a self-equivalence on \( K \). If \( t_1 = t_2 \), then \( t_2^{-1}t_1 = \text{identity map} \), and by Lemma 18 \( T_2^{-1}T_1 = \text{identity map} \) except possibly at the complex primes of \( K \), and 1 follows.

If \( T_1P = T_2P \) for every prime \( P \) contained in a set \( M \) of primes of \( K \) such that the density of \( M \) is bigger than zero, then \( T_2^{-1}T_1P = P \) for \( P \in M \). By the previous corollary, \( t_2^{-1}t_1 = \text{identity map} \) and \( T_2^{-1}T_1 = \text{identity map} \) except possibly at the complex primes of \( K \), and 2 follows. Hence we have established the theorem.

Corollary: Let \( (t, T) \) be a reciprocity equivalence from \( K \) to \( L \). Fix two distinct noncomplex primes \( P, Q \) of \( K \). Define a new map \( T_1 \) on primes by \( T_1(P_0) = T(P_0) \) for \( P_0 \) not in \( \{P, Q\} \), \( T_1(P) = T(Q) \) and \( T_1(Q) = T(P) \). Then for any square class map \( t_1 \), the pair \( (t_1, T_1) \) is not a reciprocity equivalence.

Proof: If \( (t_1, T_1) \) were a reciprocity equivalence, then by Theorem C \( T = T_1 \), contrary to the definition of \( T_1 \), proving the corollary.
References


Vita

Thomas C. Palfrey was born in Baton Rouge, Louisiana on May 6, 1959. In his youth and adolescence, he spent much of his time competing in tennis tournaments. He attended college at Louisiana State University in Baton Rouge and completed his B.S. in August, 1981. After studying Greek and Latin for a year, he first entered graduate school at the University of Utah in Salt Lake City where he received an M.A. in December, 1984. After he left Utah, he briefly resumed his Latin studies and later played a satellite tennis circuit in the summer of 1985. Since the fall of 1985, he has been pursuing a doctorate in mathematics. Upon its completion he intends to join the seminary of the Legionaries of Christ to prepare for the catholic priesthood.
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