6-25-2018

Dimers on Cylinders over Dynkin Diagrams and Cluster Algebras

Maitreyee Chandramohan Kulkarni

Louisiana State University and Agricultural and Mechanical College

DIMERS ON CYLINDERS OVER DYNKIN DIAGRAMS AND CLUSTER ALGEBRAS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Maitreyee Chandramohan Kulkarni
B.Sc. in Mathematics, University of Pune, 2009
M.Sc. in Mathematics, University of Pune, 2011
M.S. in Mathematics, Louisiana State University, 2013
August 2018

Acknowledgments

This dissertation would not have been possible without support of many people. First, I am very thankful to my advisor Milen Yakimov for his guidance and support throughout these four years. I am also grateful to him for suggesting this problem and introducing me to the theory of cluster algebras which I really like.

I would like to thank Pramod Achar for being a positive impact in general, and for introducing me to perverse sheaves. Thanks to everyone from the math department at LSU and at Sir Parashurambhau College for being wonderful people to be around. I am grateful to Vinayak Sholapurkar for encouraging me to do math and to pursue a Ph.D. I would like to thank my academic siblings Jesse, Eric, Bach, Trey and Kent for always being understanding and fun. Special thanks to Trey for being the best friend he has been.

I couldn't have been here without the love, patience and support of my family. I am very grateful to my parents, Sarita and Chandramohan Kulkarni, for teaching me the importance of observing every little thing around me; to my grandparents, for teaching me the importance of love and family; and to my husband Jacob Matherne, for keeping me motivated and inspiring me to work hard.

I am thankful to all my friends from India, especially Adwait, Makarand and Mandar, for making every trip back home special and comfortable, and making me feel that no time had passed. These trips were very refreshing and encouraging to work harder when I came back to LSU.

Table of Contents

Abstract

This dissertation describes a general setting for dimer models on cylinders over Dynkin diagrams which in type A reduces to the well-studied case of dimer models on a disc. We prove that all Berenstein–Fomin–Zelevinsky quivers for Schubert cells in a symmetric Kac–Moody algebra give rise to dimer models on the cylinder over the corresponding Dynkin diagram. We also give an independent proof of a result of Buan, Iyama, Reiten and Smith that the corresponding superpotentials are rigid using the dimer model structure of the quivers.

Chapter 1
Introduction

1.1 A motivational problem

Cluster algebras were defined by Fomin and Zelevinsky in 2000 [FZ02] to study Lusztig's dual canonical basis of quantum groups. A cluster algebra is a certain commutative ring that lies somewhere between a polynomial ring and its field of fractions, and it is generated from an initial collection of data (a quiver and a function on each vertex) by a combinatorial procedure called mutation. In particular, a cluster algebra is defined starting from a quiver (or directed graph) with n vertices where each vertex i has a function x_i on it. A process called mutation changes both the quiver and the functions on the vertices, and iteratively produces the generating set of the cluster algebra.

Cluster algebras have also been categorified in certain settings. In 2006, a new category called cluster category was defined by [BMRRT]. This category is Frobenius and it categorifies cluster algebras from acyclic quivers without frozen vertices. In 2009, Amiot categorified cyclic cluster algebras using quivers with potentials that are Jacobi finite [Ami09]. Plamondon extended this result to quivers with potentials that are Jacobi infinite. On the other hand, Geiss, Leclerc and Schröer gave a partial categorification of cluster structure on $\mathbb{C}[G/P]$ with frozen variables. This was the first attempt to include frozen variables in the categorification.

Let G be a Lie group of type ADE and P be a parabolic subgroup. In this setting, Geiss, Leclerc and Schröer proved that the coordinate ring of the partial flag variety G/P has a cluster structure on[GLS08]. This gives a categorification of the coordinate ring of the affine open cell in G/P by a subcategory of modules over the preprojective algebra associated to the Dynkin diagram of G. This categorification is then lifted to the homogeneous coordinate ring on the whole flag variety. In 2016, Jensen, King and Su gave a direct categorification of this homogeneous coordinate ring for Grassmannians, i.e. when G is of type A and P is a maximal parabolic subgroup [JKS16]. This is done using the category of (maximal) Cohen–Macaulay modules T over B, where B is a quotient algebra of a certain preprojective algebra.

Recently, Baur, King and Marsh gave a combinatorial model for this categorification. They used Postnikov diagrams, which were used by Scott to show that the homogeneous coordinate ring of $\mathrm{Gr(k, n)}$ is a cluster algebra [Sco06]. A Postnikov diagram encodes information about seeds of the cluster algebra and its clusters. Each region in a Postnikov diagram is labelled by a k-subset of $\{1, 2, \ldots, n\}$. The quiver obtained from a Postnikov diagram can be shown to be a dimer model on a disk. Let I be a k-subset of $\{1, 2, \ldots, n\}$ corresponding to a minor of the matrix and M_I be a certain Cohen–Macaulay B-module associated to I. To each Postnikov diagram D, associate the module $T_D = \bigoplus_I M_I$. They define a dimer algebra as the Jacobian algebra for the dimer model corresponding to a Postnikov diagram. One of Baur, King, Marsh's main results is that the dimer algebra A_D is isomorphic to the endomorphism ring $\mathrm{End}_B(T_D)$ [BKM16], which gives a combinatorial construction of the endomorphism algebra required for their categorification.

The aim of this project is to generalize this combinatorial setting to any Kac–Moody group and its parabolic subgroup. In order to do this, we would need to look at the quivers that generalize the quivers from Postnikov diagrams. The key idea in this article is to realize cluster algebras associated to symmetric Kac–Moody algebras by Berenstein–Fomin–Zelevinsky quivers defined in [BFZ05]. Let G be a Kac–Moody group and W be its Weyl group. For any pair $(u, v) \in W \times W$, associate a quiver $Q^{u,v}$ following [BFZ05]. In type A, $Q^{u,v}$ is planar for any $u, v \in W$ but in other types, these quivers are not planar in general.

In this thesis, we will introduce conceptual framework for dimer models outside of type A. They will be called dimer models on the cylinder over the Dynkin diagram of G. The dimer models on cylinders over Dynkin diagrams will play the role of dimer models from Postnikov diagrams. Suppose Γ is the Dynkin diagram corresponding to G, then $\Gamma \times \mathbb{R}$ is called the cylinder over the Dynkin diagram Γ. A vertex in the Dynkin diagram is called a branching point if it has more than two edges incident to it. A vertex is called an endpoint if it has exactly one edge incident to it. Let V be the set of endpoints and branching points of a Dynkin diagram. The path $\Gamma_{m,n}$ between any two vertices m and n in V is called a branch in the Dynkin diagram. The space $\Gamma_{m,n} \times \mathbb{R}$ is called the sheet of the cylinder over the branch $\Gamma_{m,n}$. If a Dynkin diagram has k branches then the cylinder over the Dynkin diagram has k sheets glued at a string on every branching point. This realization makes the quiver planar in each sheet of the cylinder.

Theorem 1.1.1. *The quiver $Q^{u,v}$ corresponding to any pair (u, v), where u and v are arbitrary elements in the Weyl group, has the following structure:*

- *Each face of $Q^{u,v}$ is oriented.*
- *Each face of $Q^{u,v}$ on the cylinder $\Gamma \times \mathbb{R}$ projects onto an edge of the Dynkin diagram.*
- *Each edge of $Q^{u,v}$ projects onto a vertex of the Dynkin diagram or an edge of the Dynkin diagram.*

Any quiver on a cylinder over a Dynkin diagram that has the above properties will be called a dimer model on the cylinder [see Definition 3.2] because it will play a similar role to the dimer models of [BKM16]. The dimer algebra of [BKM16] will be replaced by the Jacobian algebra corresponding to a certain potential of the

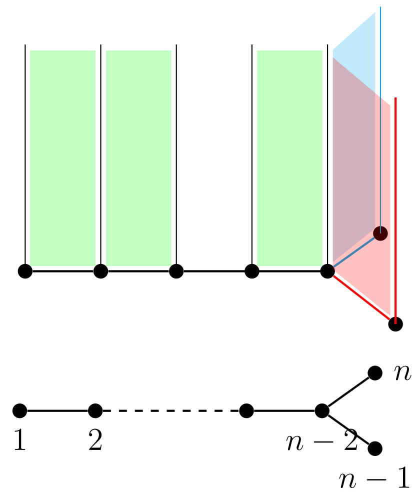


Figure 1.1: The cylinder over Dynkin diagram of type D_n

BFZ quiver. The Jacobian algebra $A(Q, S)$ of the quiver Q depends on the choice of a potential. In this case we use a particularly nice type of potential called a rigid potential. Every rigid potential is non-degenerate which means that any sequence of mutations of the quiver with potential does not create a 2-cycle in the quiver.

We define the superpotential S of a quiver Q as follows:

$$S = \sum \text{clockwise oriented faces} - \sum \text{anti-clockwise oriented faces}.$$

Note that a face of a quiver is a cycle which is not divided by an edge. We will show that this is a rigid potential, i.e. that all cycles in the quiver Q lie in the Jacobian ideal of the potential S. This will be proved in two steps, first for faces, and then for non-self-intersecting oriented cycles. As each cycle in the quiver is oriented, the above two cases cover all cycles in the quiver.

The dimer model structure of these quivers reduces the global problem of verification of rigidity of the super-potential to a local problem on each sheet. Let S_r be the superpotential of the subquiver Q_r of Q drawn on the rth sheet of the cylinder. We show that S_r is rigid in the rth sheet, for each r. As sheets are glued at a string, they only share edges that lie on the gluing string with each other.

Particularly, they do not share any faces, therefore the superpotential S is simply the sum $S = \sum_r S_r$. As each face belongs to a unique sheet, the gluing of sheets does not affect the rigidity of the potential.

In Chapter 2, we give some preliminary definitions. In Chapter 3, we will see the definition of a dimer model. I will also give some history of categorification in this chapter. In Chapter 4, we define a Berenstein–Fomin–Zelevinsky quiver, then give our construction of cylinders on Dynkin diagrams and quivers from double Bruhat cells. In this chapter we prove that the BFZ quivers can be realized as dimer models on the cylinder over a Dynkin diagram. The quiver lies entirely on the cylinder by construction. In the last section, as an application, we give an independent proof of the result in [BIRS11] that, for any Weyl group elements $u \in W$, the superpotential of the BFZ quiver $Q^{u,e}$ is rigid. We will see a more generalised result that the superpotential of the BFZ quiver $Q^{u,v}$ is rigid, for any two Weyl group elements u, v. This will be achieved by a method of obtaining the quiver $Q^{u,v}$ using the quivers $Q^{u,e}$ and $Q^{e,v}$. The construction of this quiver will be explained in the same chapter.

We prove that the quiver is planar in each sheet of the cylinder and each face of the quiver is oriented. We prove the rigidity in each sheet by observing that the faces on the boundary belong to the Jacobian ideal. Then we use induction on the faces of dimers to prove that every face belongs to the ideal. Then we notice that every cycle can be written in terms of faces that the cycle contains, which tells us that each cycle is in the ideal.

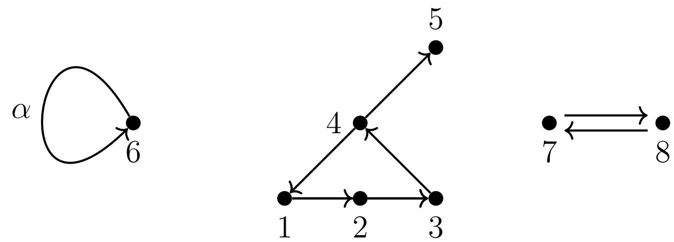
Chapter 2
Notation and Preliminaries

In this chapter, we will define all the terms needed to define cluster algebras. We will introduce cluster algebras and give some examples. We will also define, Jacobian algebras and mutations of quivers with potentials. We will see a few examples of this mutation.

A matrix $M \in \mathbf{GL}_n(\mathbb{R})$ is totally positive (totally nonnegative) if all its minors are positive (nonnegative, respectively). Lusztig defined total positivity and total nonnegativity for an arbitrary split reductive connected algebraic group over $\mathbb{R}$. The theory of canonical bases is the main tools of studying these sets of matrices. Cluster algebras were defined in 2000 by Berenstein, Fomin and Zelevinsky. The initial aim was to approach Lusztig's total positivity for algebraic groups combinatorially and multiplicative properties of the dual canonical basis of the quantised enveloping algebra of a semisimple Lie algebra over $\mathbb{C}$. But more generally, these algebras consist of coordinate rings of various algebraic varieties that play an important role in representation theory, invariant theory. The theory of cluster algebras has also been linked to Poisson geometry, integrable systems, higher Teichmuller theory, commutative and noncommutative algebraic geometry, and representation theory of finite-dimensional algebras.

2.1 Cluster algebras

Cluster algebras can be defined using quivers and their mutations. A quiver is a directed graph. We denote it by Q, its set of vertices by Q_0 and its set of edges by Q_1. The maps $s, t : Q_1 \to Q_0$ assign an arrow its source and target respectively. A loop in a quiver is an arrow from a vertex to itself. A 2-cycle is a pair of arrows between two vertices in opposite directions. A quiver may have multiple arrows between two vertices, all going in one direction. Some examples of quivers are:



Definition 2.1.1. *The process of mutation of a seed at vertex k is defined as follows:*

- *Step 1: Reverse all arrows touching the vertex x_k.*
- *Step 2: Complete triangles, i.e., for every path $i \to k \to j$, and an edge $j \to i$.*
- *Step 3: Cancel any 2-cycles created in Step 2.*
- *Step 4: Replace x_k at the vertex k with $x'_k = \dfrac{\prod_{k\to l} x_l + \prod_{l\to k} x_l}{x_k}$ where the products are over edges with source vertex k and with target vertex k respectively.*

We will denote the mutation at vertex k by μ_k. It can be seen from the definition that mutation μ_k is an involution. We show an example of mutation of the following quiver Q at the vertex labelled x_2 in Figure 2.1.

Definition 2.1.2. *A seed is a quiver together with elements $\{x_i\}_{i\in Q_0}$ of a field on the vertices that together freely generate that field over $\mathbb{Q}$. The elements on the vertices are called cluster variables.*

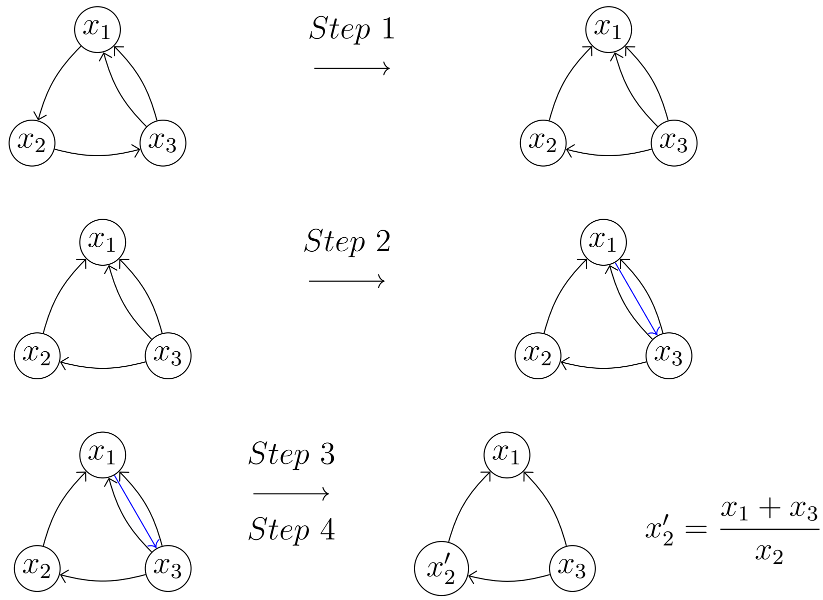


Figure 2.1: Mutation of a quiver

Definition 2.1.3. *Let Q be a finite quiver without loops or 2-cycles with vertices $1, \cdots, n$ and the initial seed $(Q, x_1, \ldots, x_n)$. The cluster algebra $\mathcal{A}_{\mathcal{Q}}$ of quiver Q is the subalgebra of $\mathbb{Q}(x_1, x_2, \ldots, x_n)$ generated by all cluster variables obtained from all possible sequences of mutations applied to the initial seed.*

A cluster algebra can have finite or infinite number of cluster variables. Cluster algebras that are generated by finite number of cluster variables are called cluster algebras of finite type. The exchange graph of a quiver is a graph with seeds as its vertices and edges represent mutations. Let us look at an example of a cluster algebra of finite type. Consider the simplest Dynkin diagram A_2. We will direct the edge of the Dynkin digram as follows:

$$\Gamma : 1 \longrightarrow 2$$

We will now assign a cluster variable to each vertex of Γ (x_1 and x_2 in this case). So the initial seed for A_2 is $(1 \longrightarrow 2, \{x_1, x_2\})$. We mutate at vertex 1. Since the quiver contains only one edge, mutation changes the quiver only by changing the direction of the edge. The variable x_1 changes to $\mu_1(x_1) = \dfrac{1 + x_2}{x_1}$. Hence, the seed becomes

$$(1 \longleftarrow 2, \{x_1' = \frac{1+x_2}{x_1}, x_2\}).$$

If we mutate at the same vertex again, we will get the original seed back, so we mutate at vertex 2, to get a new seed

$$(1 \longrightarrow 2, \{x_1', x_2' = \frac{1+x_2+x_1}{x_1x_2}\}).$$

We continue mutating at vertices 1 and 2 alternatingly. Applying the mutation rule to the cluster variables, we get new cluster variables $x_1' = \frac{1+x_2}{x_1}$, $x_2' = \frac{1+x_1+x_2}{x_1x_2}$, $x_1'' = \frac{1+x_1}{x_2}$ and $x_2'' = x_1$. This brings us back to the same seed that we started with. The exchange graph for A_2 is shown in figure 2.2. So the cluster algebra in this example has five generators that are listed above, i.e there are five cluster variables with two initial cluster variables x_1, x_2 and three non-initial cluster variables x_1', x_2', x_1''. Notice that there are three positive roots for the root system of type A_2. These are in bijection with the non-initial cluster variables. In general, the number of cluster variables for a cluster algebra from any Dynkin digram is the sum of the rank and the number of positive roots. The finite type cluster algebras have been classified using Dynkin diagrams.

Theorem 2.1.1. *The finite type cluster algebras are parametrized by the finite root systems.*

Cluster algebras can also be defined using matrices. To every quiver Q with n vertices, we can associate an $n \times n$ skewsymmetric matrix B_Q such that the entry b_{ij} for $i \leq j$ is the number of arrows from vertex i to vertex j. Then the matrix mutation $\mu_k(B)$ again gives an $n \times n$ skew-symmetric matrix, call B' and is defined

$$
\begin{tikzcd}
& & (x_1 \longrightarrow x_2) \arrow[dll, dash, "\mu_1"'] \arrow[drr, dash, "\mu_1"] & & \\
(x_1'' \longrightarrow x_2'') \arrow[dr, dash, "\mu_2"'] & & & & (x_1' \longleftarrow x_2) \arrow[dl, dash, "\mu_2"] \\
& (x_1'' \longleftarrow x_2') \arrow[rr, dash, "\mu_1"] & & (x_1' \longrightarrow x_2') &
\end{tikzcd}
$$

Figure 2.2: Exchange graph of cluster algebra of type A_2

as follows:

$$
b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \operatorname{sgn}(b_{ik}) \max(b_{ik}b_{kj}, 0) & \text{otherwise} \end{cases}
$$

2.2 Cluster algebras with coefficients

In this section, we generalize the definition of cluster algebras given in section 1. We will generalize two aspects of the definition. First, we will restrict mutations at certain vertices, which will be called frozen vertices. Secondly, we will replace the skew-symmetric matrices by skew-symmetrizable matrices. A skew-symmetrizable matrix represents a quiver in which certain arrows can be seen by their sources, but not by their targets or vice-versa.

We will fix integers m and n such that $1 \leq n \leq m$. Consider an $m \times n$ matrix $\widetilde{B}$

$$
\widetilde{B} = \begin{bmatrix} B \\ C \end{bmatrix}
$$

where B is an $n \times n$ matrix and C is an $(m-n) \times n$ matrix. The matrix B is called the principal part of $\widetilde{B}$. We will call the matrix $\widetilde{B}$ an exchange matrix if B is skew-symmetrizable. The indices $k \leq n$ are called mutable, and the indices that are strictly greater than n are called frozen. We are only allowed to mutate at the mutable indices, i.e. the principal part B of the matrix. The mutation rule for B here is the same as defined earlier. The principal part of the matrix after mutation stays skew-symmetrizable.

If the principal part B of $\widetilde{B}$ is skew-symmetric then the exchange matrix $\widetilde{B}$ corresponds to a quiver $\widetilde{Q}$ with vertex set

$$\{1, 2, \ldots, m\} = \{1, 2, \ldots, n\} \cup \{n + 1, \ldots, m\}.$$

The vertices $k \leq n$ are called mutable and the vertices $k > n$ are called frozen. There are no arrows between frozen vertices. The principal part Q of $\widetilde{Q}$ is the full subquiver with vertices $\{1, 2, \ldots, n\}$ i.e., for any two vertices $i, j \leq n$, the subquiver Q contains all the arrows between them. The cluster algebra $A_{\widetilde{Q}} \subset \mathbb{Q}(x_1, x_2, \ldots, x_m)$ is then defined as before, but with some restrictions:

- Mutations are allowed only at the mutable vertices.
- No arrows are added between frozen vertices during any mutation.
- In a cluster $u = \{u_1, \ldots, u_n, x_{n+1}, \ldots, x_m\}$ only $u_1, \ldots, u_m$ are cluster variables. The x_i's are called coefficients.

The cluster algebras with coefficients include algebras of coordinates on homogeneous varieties. We will see an example below. Consider the quiver in figure 2.3.

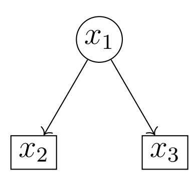


Figure 2.3: Cluster algebra with coefficients

The rectangular vertices denote frozen vertices and circular vertices denote mutable vertices. This quiver has only one mutable vertex, so we can mutate only at vertex 1. The mutation changes the quiver only by reverting the arrows and the new cluster variable $x_1' = \dfrac{1 + x_2x_3}{x_1}$ or $x_1x_1' - x_2x_3 = 1$. Note that having coefficients enables to have extra variables x_2 and x_3 without needing to mutate them. So the cluster algebra represents the algebra of regular functions on the algebraic group $SL_2(\mathbb{C})$, i.e. $\mathbb{C}[a, b, c, d)]/(ad - bc - 1)$

2.3 Path algebras and potentials

In this section we will study Jacobian algebras and quiver with potentials. Given a quiver Q, we can define an algebra using its paths. A collection of arrows $\alpha_1\alpha_2 \dots \alpha_n$ is a path if $t(\alpha_i) = s(\alpha_{i+1})$. Every vertex has a path of length zero that starts and ends at that vertex. This path is called a lazy path and is denoted by e_i. The path algebra $\mathbb{C}(Q)$ of a quiver Q is an algebra generated by paths in the quiver Q with multiplication given by concatenation of paths whenever possible.

A potential $S \in \mathbb{C}(Q)$ is a linear combination of cycles in the quiver. The pair (Q, S) of a quiver and its potential is called a quiver with potential or a QP. Note that all cycles in a potential are simple. This means that no cycle passes through the same vertex twice. We will follow the definition of mutation of quivers with potential in [DWZ08]. To define the mutation, we first need to study some properties of quivers with potentials.

- Two potentials S and S' on Q are cyclically equivalent if $S - S'$ lies in the closure of the vector subspace spanned by all the elements of the form $\alpha_1 \dots \alpha_l - \alpha_2 \dots \alpha_l\alpha_1$ where $\alpha_1 \dots \alpha_l$ is a cycle of positive length.

- Two quivers with potentials (Q, S) and (Q', S') are right equivalent to each other if there exists an isomorphism $\phi : \mathbb{C}(Q) \to \mathbb{C}(Q')$ such that $\phi(S)$ is cyclically equivalent to S'.

- Let us define the cyclic derivative of a potential. For every $a \in Q_1$, the cyclic derivative ∂_a is defined as:

$$\partial_a(a_1 a_2 \cdots a_n) = a_{i+1} \cdots a_n a_1 \cdots a_{i-1},$$

  where $a_1 a_2 \cdots a_n$ is a cycle in the quiver and $a = a_i$. If $a \neq a_i$ for any i, then $\partial_a(a_1 a_2 \cdots a_n) = 0$.

- If S is a potential of Q, we define the Jacobian ideal $J(S)$ to be the ideal generated by $\partial_a(S)$, for all $a \in Q$.

- The Jacobian algebra $P(Q, S)$ is the quotient $\mathbb{C}(Q)/J(S)$.

- A QP is called trivial if it is a sum of cycles of length 2, and the derivatives span Q as a $\mathbb{C}$-vector space.

- A QP is called reduced if the degree-2 component of S is 0, i.e., if the expression of S involves no 2-cycles.

For a quiver with potential (Q, P), we will now define its mutation $\mu_i(Q, S)$ at a vertex i and give an example. Let (Q, S) be a QP as above. Note that Q might contain a loop or a 2-cycle. Let i be a vertex of Q. We assume that no cycle in W starts at vertex i. (We can assume this because if W contains a cycle that starts at i, we can choose a cycle that is cyclically equivalent to the original one.)

We apply the first two steps of the mutation rule from Def 2.1.1 to the quiver Q to get $\widetilde{\mu}_i(Q)$. This quiver is sometimes called the premutation of Q. To mutate the potential, we first define the potential $[S]$ on Q to be the potential obtained from S by replacing any path of length two $\alpha\beta$ passing through i by the arrow $[\alpha\beta]$. Now we define $\Delta_i(S) = \sum \beta^* \alpha^* [\alpha\beta]$, the sum is taken over all paths of length two passing through i. Let $\widetilde{\mu}_i(S) = [S] + \Delta_i(S)$. This is a potential on the quiver $\widetilde{\mu}_i(Q)$.

Now the mutation

$$\mu_i(Q,S) \text{ is the reduced part of the QP } (\widetilde{\mu}_i(Q), \widetilde{\mu}_i(S)).$$

We will give two examples of mutation of a quiver with potential. The second example will be of a degenerate QP. But first, consider the following quiver Q with vertices $1, 2, 3, 4$ and edges a, b, c, d as shown in Figure 2.4. Let $S = abcd$. Let us mutate the QP (Q, S) at vertex 2.

$$\begin{tikzcd}
1 \arrow[r, "a"] & 2 \arrow[d, "b"] \\
4 \arrow[u, "d"] & 3 \arrow[l, "c"]
\end{tikzcd}$$

Figure 2.4: QP mutation: quiver Q

In the premutation $\tilde{\mu}_2(Q)$ as shown in Figure 2.5, a is replaced by $e = a^*$, b is replaced by $f = b^*$ (a^* and b^* are reverse arrows of a and b respectively) and an extra arrow $g = [ab]$ is added. These are the first two steps of the mutation of a quiver.

$$\begin{tikzcd}
1 \arrow[dr, "g"] & 2 \arrow[l, "e = a^*"'] \\
4 \arrow[u, "d"] & 3 \arrow[l, "c"] \arrow[u, "f = b^*"']
\end{tikzcd}$$

Figure 2.5: QP mutation: quiver $\tilde{Q} = \tilde{\mu}_2(Q)$

Now, the potential $\tilde{\mu}_2(S) = \tilde{S}$ is given by

$$\tilde{S} = [S] + \Delta_2(S).$$

Here, $[S] = [ab]cd = gcd$ and $\Delta_2(S) = b^*a^*[ab] = feg$ and hence,

$$\tilde{S} = gcd + feg$$

The potential $\tilde{S}$ is a potential on the quiver $\tilde{Q}$. There are no 2-cycles in the quiver or the potential, so we do not need to reduce the QP. The QP mutation,

$$\mu_2(Q,S) = \tilde{\mu}_2(Q,S) = (\tilde{\mu}_2(Q), \tilde{\mu}_2(S))$$

Let us mutate the QP $(\tilde{Q}, \tilde{S})$ at vertex 3. Note that the cycle feg starts (and ends) at the vertex 3, so in order to mutate the QP $(\tilde{Q}, \tilde{S})$ at vertex 3, we will use the potential $S' = gcd + egf$ which is cyclicly equivalent to $\tilde{S}$. Let us apply the premutation to the quiver $\tilde{Q}$. The edges c, g, f get replaced by their reverse edges c^*, g^*, f^*. New arrows $[gc] : 3 \to 1, [gf] : 1 \to 2$ get added to the quiver. So the QP $(\tilde{\mu}_3(\tilde{Q}), S')$ has vertices 1, 2, 3, 4 and arrows $d, e, c^*, g^*, f^*, [gc], [gf]$.

The potential

$$\tilde{\mu}_3(S') = [S'] + \Delta_3(S')$$

Now, $[S'] = [gc]d + [gf]e$ and $Delta_3(S') = c^*g^*[gc] + f^*g^*[gf]$ Therefore,

$$\tilde{\mu}_3(S') = [gc]d + [gf]e + c^*g^*[gc] + f^*g^*[gf]$$

As we can see $\tilde{\mu}_3(S')$ is not reduced, so we have to remove the trivial part of $\tilde{\mu}_3(\tilde{Q}, S')$ to get $\mu_3(\tilde{Q}, S')$. So the final mutated quiver as shown in Figure 2.7 has no cycles. Consequently the corresponding potential is 0.

$$\begin{tikzcd}
1 \arrow[r, shift left, "{[gc]}"] \arrow[d, shift right, "{[gf]}"'] & 2 \arrow[l, shift left, "d"] \arrow[d, "f^*"] \\
4 \arrow[u, shift right, "e"'] \arrow[r, "c^*"'] & 3 \arrow[ul, "g^*"]
\end{tikzcd}$$

Figure 2.6: QP mutation: quiver $\widetilde{\mu_3}(\tilde{Q})$

Note that as we skip the third step in the mutation process, the quiver might still contain 2-cycles. The potential decides which 2-cycles are deleted at the end.

$$\begin{tikzcd}
1 & 2 \arrow[d, "f^*"] \\
4 \arrow[r, "c^*"'] & 3 \arrow[ul, "g^*"']
\end{tikzcd}$$

Figure 2.7: QP mutation: quiver $\mu_3(\tilde{Q}, S')$

Hence, the mutation of a QP may generate 2-cycles in the quiver. In the above example, we created two 2-cycles which got deleted at the last step. Now we will see an example where a 2-cycle and a loop remains in the final QP.

$$\begin{tikzcd}
 & 1 \arrow[dl, "a"'] & \\
2 \arrow[rr, "b"'] & & 3 \arrow[ul, "c"']
\end{tikzcd}$$

Figure 2.8: QP mutation: quiver Q

Consider the quiver Q as shown in Figure 2.8 with the potential $S = 0$. Let us mutate the QP at vertex 2. To begin with, we want to find $\tilde{\mu}_2(Q)$. So we apply first two steps of mutation to Q, i.e. replace a with its reverse arrow a^*; replace b with its reverse arrow b^* and add an arrow $d = [ab] : 1 \to 3$. This gives us the quiver $\tilde{\mu}_2(Q)$ as shown in Figure 2.9.

The quiver $\tilde{S} = [S] + \Delta_2(S)$. Since S is 0, $[S]$ is also 0. But $\Delta_2(S) = [ab]b^*a^* = db^*a^*$. Therefore,

$$\tilde{S} = db^*a^*$$

Note that $\tilde{S}$ is reduced since it has no 2-cycle. Therefore, $\mu_2(Q, S) = \tilde{\mu}_2(Q, S)$. But the quiver does contain a 2-cycle.

Let us now mutate the QP $(\tilde{Q}, \tilde{S})$ at 3. Let us rename $(\tilde{Q}, \tilde{S})$ as (Q', S') to avoid repetition of notation. We want to find the QP $\mu_3(Q', S')$. Now we apply the first to steps of mutation to the quiver Q' to get the quiver $\tilde{\mu}_3(Q')$ shown in Figure

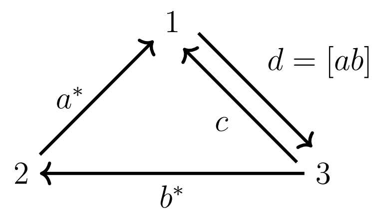


Figure 2.9: QP mutation: quiver $\tilde{\mu}_2(Q)$

2.10. The arrows b^*, c, d get replaced by their reverse arrows b^{**}, c^*, d^*. Note that $b^{**} = b$. Then we add an arrow $[db^*] : 1 \to 2$ and a loop $[dc] : 1 \to 1$

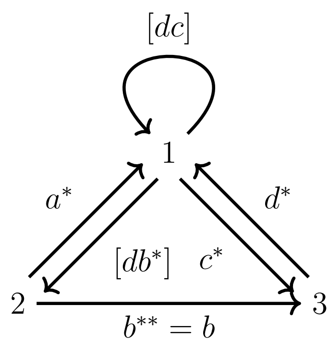


Figure 2.10: QP mutation: quiver $\tilde{\mu}_3(Q')$

The potential $\tilde{S}' = [S'] + \Delta_3(S')$. Recall that $[S']$ is obtained from S' by replacing any path $p \in S'$ of length two passing through 3 by the arrow $[p]$. As db^* passes through 3, $[S'] = [db^*]a^*$.

Also recall that $\Delta_3(S') = \sum \beta^*\alpha^*[\alpha\beta]$, the sum is taken over all paths of length two passing through 3. There are two paths of length 2 passing through 3, bd^* and c^*d^*. So $\Delta_3(S') = [db^*]bd^* + [dc]c^*d^*$. This makes

$$\tilde{S}' = [db^*]a^* + [db^*]bd^* + [dc]c^*d^*.$$

The first term is a 2-cycle in the potential. Let us take the derivative of the potential with respect to $[db^*]$

$$\frac{\partial(\tilde{S}')}{\partial[db^*]} = a^* + bd^*$$

The relation $\frac{\partial(\tilde{S}')}{\partial[db^*]} = 0$ gives $a^* = -bd^*$. Substituting this back in the expression of $\tilde{S}'$ cancels the first two terms in the potential. We can check that the map ϕ defined by $a^* \to a^* - bd^*$ sending all other arrows to themselves is a right equivalence. Hence the reduced quiver with potential $(\tilde{\mu}_3(Q'), [dc]c^*d^*)$ is right equivalent to $(\tilde{\mu}_3(Q', \tilde{S}')$. So, the mutation $\mu_3(Q', S') = (Q'', S'')$ where Q'' is the quiver in Figure 2.11. This quiver with potential contains a 2-cycle. This is an example of a degenerate potential.

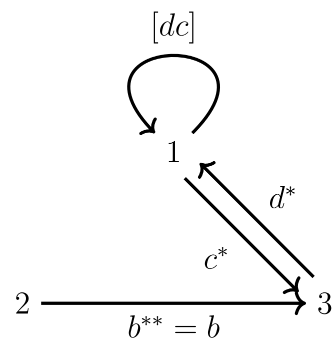


Figure 2.11: QP mutation: quiver Q''

Definition 2.3.1. *A QP (Q, S) is called non-degenerate if every sequence of mutations of (Q, S) is 2-acyclic.*

The process of verifying non-degeneracy is an infinite process in general, as the quiver may not be mutation finite. To verify non-degeneracy of a potential without going through this infinite process, we use a stronger condition on a potential, called rigidity.

Definition 2.3.2. *A QP (Q, S) is rigid if every cycle Q is cyclically equivalent to an element of $J(S)$.*

Rigidity of a potential is much easy to determine. We will give an example of a rigid and a non-rigid potential below. We introduce this notion here because of the following result:

Theorem 2.3.1 ([DWZ08]). *Every rigid potential is non-degenerate.*

$$\begin{tikzcd}
1 \arrow[r, "a"] \arrow[d, "e"'] & 2 \arrow[d, "b"] \\
4 \arrow[r, "d"'] & 3 \arrow[ul, "c"]
\end{tikzcd}$$

Figure 2.12: Rigid and non-rigid potentials

We will end this section with two examples. Consider the potential $S_1 = abc$ in the quiver in Figure 2.12. Then, by differentiating S_1 with respect to the edges a, b and c, we get that the Jacobian ideal is as follows:

$$J(S_1) = \langle bc, ca, ab \rangle.$$

In order for S_1 to be rigid, we need to show that all cycles in the quiver belong to the ideal $J(S_1)$. It is enough to check if the cycles abc and cde belong to the ideal, since all other cycles will be a linear combination or multiplication of these two. As ab is in $J(S_1)$, abc is also in $J(S_1)$ but cde does not belong to $J(S_1)$. Therefore S_1 is not rigid.

On the other hand, let us consider the potential $S_2 = abc + cde$, then $J(S_2) = \langle bc, ca, ab + de, ec, cd \rangle$. As ab and cd both belong to the ideal, abc and cde also belong to $J(S_2)$. Therefore S_2 is rigid.

Chapter 3
Dimer Models

3.1 Dimer models

Dimer models were first defined in statistical physics. They were used as a model to study phase transitions in solid state physics [Fis61] [Kas67] [FT61]. The word dimer corresponds to something that is made of two parts, say black and white. A dimer model is usually a pair of a graph and a directed graph that are dual to each other. Here we will define the quiver from this pair and call it a dimer quiver. The corresponding dual graph will be a bipartite graph. Originally the black and white particles or vertices of the graph were drawn on a square lattice. But we can glue two sides of the square lattice and get a graph on a cylinder. Here we consider graphs on disks.

We will introduce some notation before the definition of a dimer quiver. Let $Q = (Q_0, Q_1)$ be a quiver. We define a face of a quiver as a cycle in which a vertex or an edge doesn't appear more than once, except the start and end vertex of the cycle. The set of faces in the quiver Q is denoted by Q_2 . The number of times an edge of Q_1 appears in the faces of the quiver is called the face multiplicity of that edge.

The incidence graph I of a quiver Q, at a vertex $v \in Q_0$ is an unoriented graph whose vertices are in one-to-one correspondence with the edges of Q incident to v. The edges in I are the length two paths through v. For example, let us consider the following quiver in Figure 3.1, where v is the only internal vertex. All other vertices including u are boundary vertices.

The incidence graph of this quiver at u is just a line as shown in Figure 3.2. On the other hand, the incidence graph of the same quiver at vertex v has four vertices, one for each arrow incident to v. It has four edges as shown by dotted orange edges

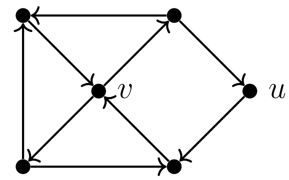


Figure 3.1: A quiver Q with internal and boundary vertices

on the left. The incidence graph is connected in this case and is shown on the right in Figure 3.3. An incidence graph can also be disconnected. For example, Figure 3.4 shows an example of a quiver whose incidence graph is disconnected.

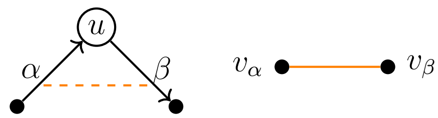


Figure 3.2: A quiver and its incidence graph

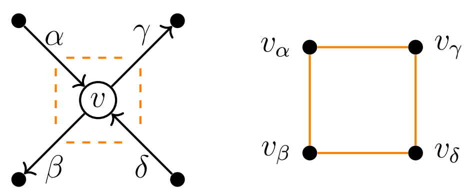


Figure 3.3: A quiver and its incidence graph

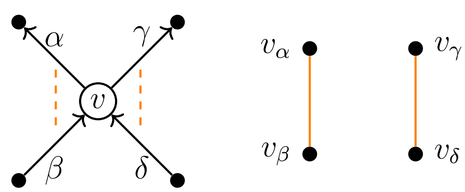


Figure 3.4: A quiver and its incidence graph

Now we are ready to define a dimer quiver.

Definition 3.1.1. *Let $Q = (Q_0, Q_1, Q_2)$ be a finite quiver such that every face in Q_2 is oriented and bounds a disk. Then Q is called a dimer model with boundary if it satisfies the following properties:*

- *the quiver Q has no loops, but 2-cycles are allowed,*
- *each boundary arrow in Q_1 has face multiplicity 1 and each internal arrow has face multiplicity 2,*
- *each internal arrow belongs to two faces oriented in opposite direction,*
- *the incidence graph of Q at each vertex is connected.*

For every dimer model, we define an algebra called a dimer algebra. In a dimer, each internal arrow $a \in Q_1$ belongs to two faces, F^+ and F^- directed anti-clockwise and clockwise, respectively. Let $r_a = p^+ - p^-$, where p^+ and p^- are the paths such that $ap^+ = F^+$ and $ap^- = F^-$. Now set

$$A_Q := \mathbb{C}Q / \langle r_a | a \in Q_1 \rangle$$

where $\mathbb{C}Q$ is the path algebra of Q. A dimer algebra is also known as a Jacobi algebra or a Jacobian algebra. It can be expressed in terms of a superpotential of a quiver as follows: For a quiver Q, any linear combination of its cycles is called its potential. The potential

$$S = \sum \text{clockwise oriented faces} - \sum \text{anti-clockwise oriented faces}$$

is called the superpotential of Q. We will define a notion of differentiation on potentials which matches the differentiation of non-commutative polynomials. The derivative is called cyclic derivative ∂_a with respect to an edge $a \in Q_1$ and is defined on a cycle $a_1 a_2 \cdots a_n$ as follows:

$$\partial_a(a_1 a_2 \cdots a_n) = a_{i+1} \cdots a_n a_1 \cdots a_{i-1},$$

if $a = a_i$ for some i between 1 and n. If $a \neq a_i$ for any i, then $\partial_a(a_1a_2\cdots a_n) = 0$. It can be extended linearly. The cyclic derivatives of a potential S generate an ideal in the path algebra $\mathbb{C}Q$. It is called the Jacobian ideal and is denoted by $J(S)$.

$$J(S) = \langle \partial_a(S) | a \in Q_1 \rangle$$

Then the Jacobian algebra $A(Q, S)$ is the quotient $\mathbb{C}(Q)/J(S)$.

3.2 Postnikov diagrams

For a pair of integers (k, n) with $k < n$, we define a strand diagram called Postnikov diagram. This was defined by Postnikov in 2006 [Pos06]. These were used in [Sco06] by Scott in order to prove that the coordinate ring of a Grassmannian has a cluster algebra structure.

Definition 3.2.1. *A (k, n) Postnikov digram, denoted by D is drawn on a disk with n vertices on its boundary, labeled, $1, 2, \ldots, n$. It has n curves in the disk, called strands which are also labeled $1, 2, \ldots, n$. The strand i starts at vertex i and ends at $i + k$, satisfying two sets of axioms given below:*

- *Local Axioms*

  (L1) Only two strands can cross at a given point. All crossings are transverse.

  (L2) There are only finitely many crossings in a diagram.

  (L3) Given a strand, other strands cross it alternatively from left and from right.

- *Global axioms*

  (G1) No strand can intersect itself.

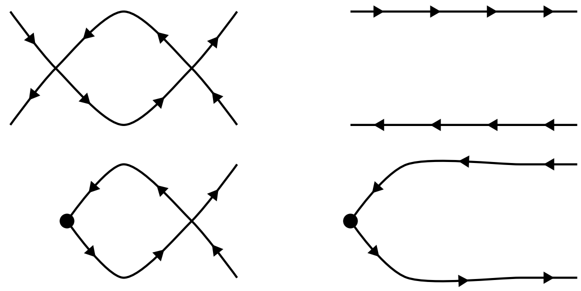


Figure 3.5: Twisting and untwisting moves

(G2) If two strands intersect each other at two distinct points, then they are oriented in opposite directions.

Two Postnikov diagrams are equivalent if one can be obtained from another using twisting and untwisting moves as shown in Figure 3.5. These moves are local and involve only two strands. A Postnikov diagram is called reduced if no untwisting moves can be applied to it. An example of a reduced (3, 6)- Postnikov diagram is shown in Figure 3.6.

The strands of a Postnikov diagram divide the disk into several regions. A region next to the boundary of the disk is called a boundary region and a region which is not adjacent to the boundary is called an internal region. When moving along the boundary of a region, if the strands are oriented alternatively, the region is called alternating. If all strands of the boundary of a region are oriented in one direction (clockwise or anti-clockwise), then the region is called oriented.

Now we will label the alternating regions of a Postnikov diagram. Each strand of a Postnikov diagram divides the disk into two parts. Each alternating region on the left hand side of the strand i gets i in its label. Each such label is a distinct $k-$subset of $[1, 2, \ldots, n]$. The labels in our example are triples that are shown in the figure above. We will denote the set of labels of D by V_D.

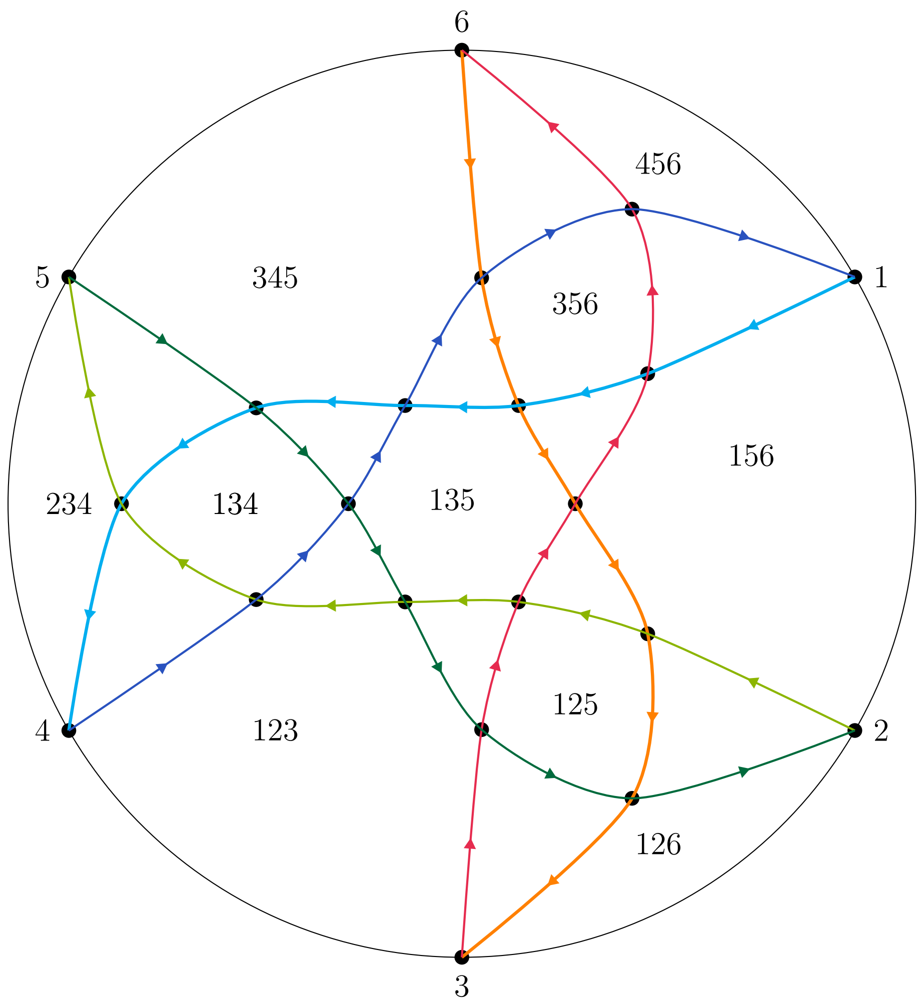


Figure 3.6: A reduced (3,6)-Postnikov diagram

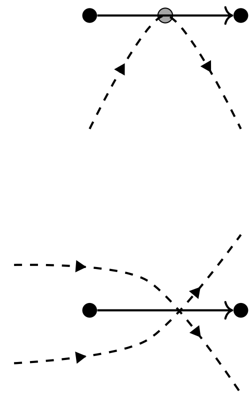


Figure 3.7: Orientation of edges in the quiver

Every Postnikov diagram D can be associated to a quiver $Q(D)$. The set of vertices of this quiver will be given by $Q_0(D) = V_D$. Two vertices in the quiver are connected if the corresponding regions intersect in a point. The orientation of the edges of the quiver will be as shown in the following figure. The dashed curves are the strands in a Postnikov diagram. Following our example, the quiver for $(3,6)-$Postnikov diagram is shown in Figure 3.8.

The vertices corresponding to the boundary regions are called boundary vertices. These will be the frozen vertices of the quiver. All other vertices are called internal. In the example below, the vertices 123, 234, 345, 456, 156, 126 are boundary vertices. Notice that for an internal alternating region, the corresponding vertex of the quiver is placed inside that region. For a boundary alternating region, the vertex is placed on the boundary of the disk. The edges connecting two boundary vertices are drawn along the boundary of the disk. This way, the quiver can be embedded in a disk.

In order to get a dimer model, we need to know how to get a graph corresponding to the quiver defined above. This graph G_D will be a planar, bipartite graph, hence called a plabic graph. The internal vertices of this graph correspond to the oriented regions in the Postnikov diagram D. A vertex is colored white if the boundary of

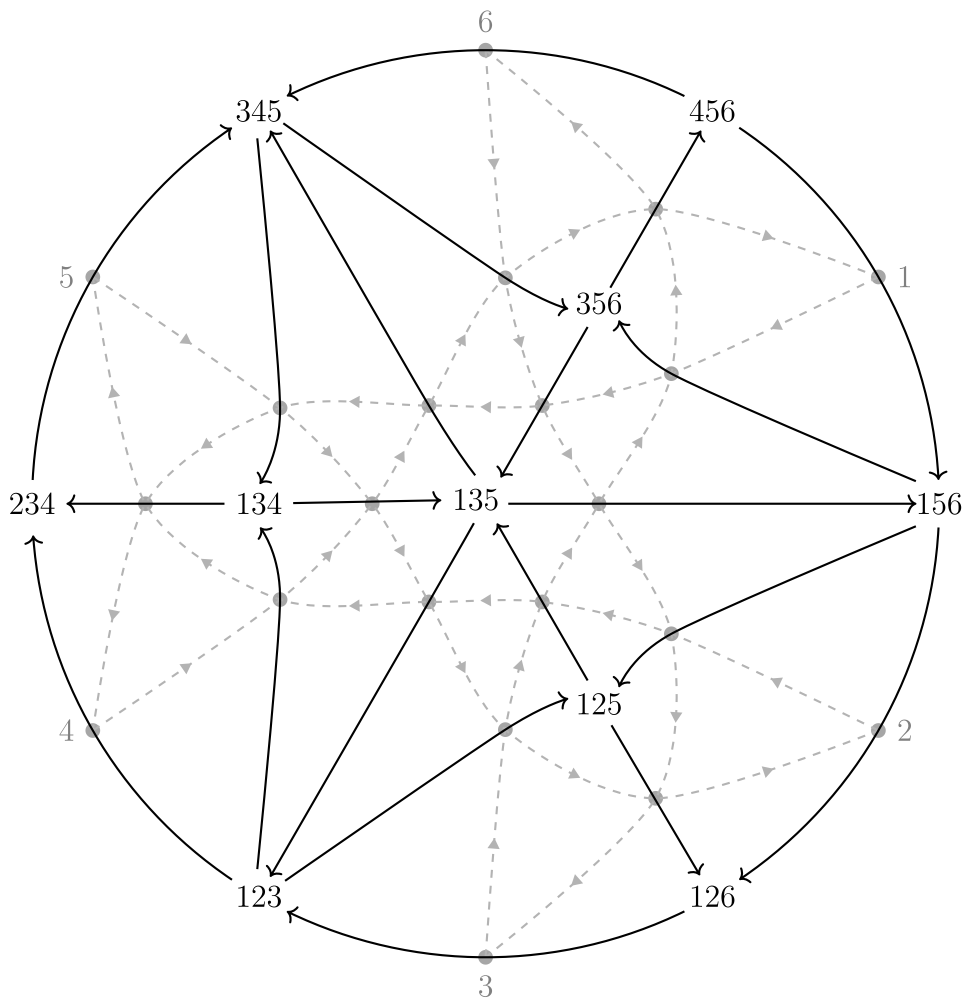


Figure 3.8: Quiver from a Postnikov diagram

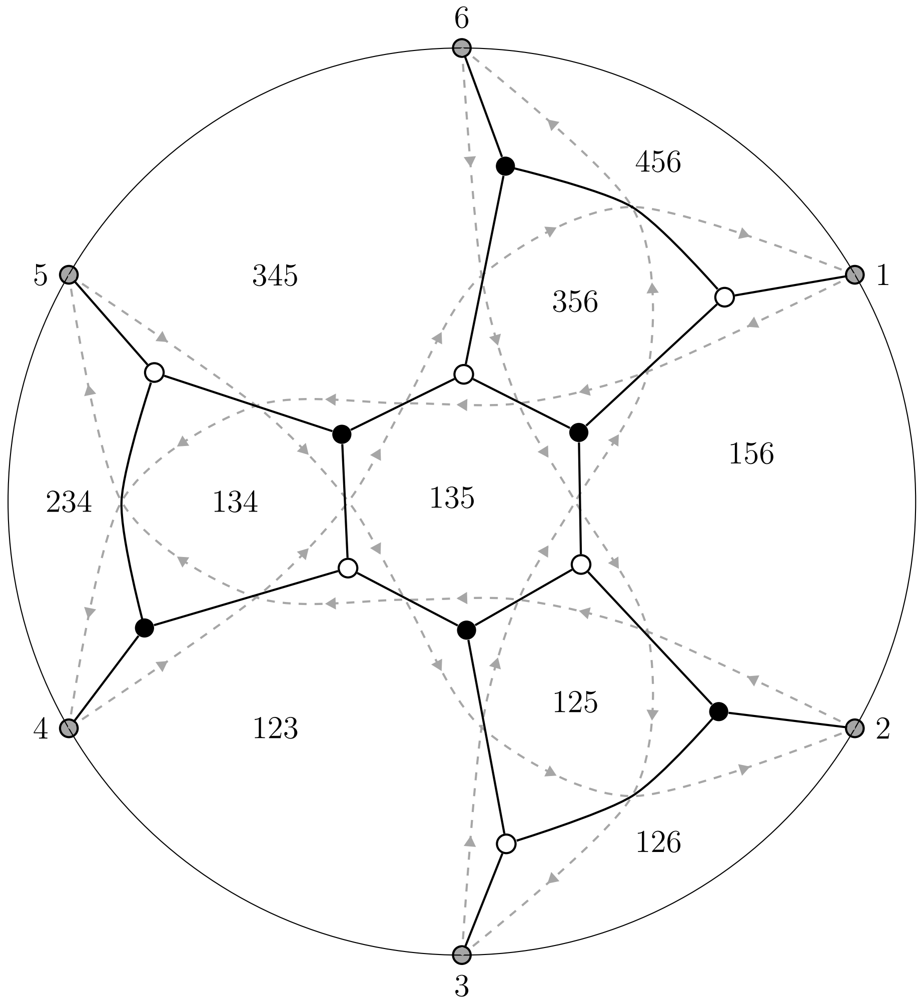


Figure 3.9: Plabic graph from a Postnikov diagram

the region is clockwise, and black if the boundary is oriented anti-clockwise. The boundary vertices of the graph are the boundary vertices of D. There is an edge between two internal vertices, if the two regions intersect at a point. A boundary vertex is connected to the internal vertex that lies in the region of the boundary vertex. An example of the plabic graph for the $(3,6)$-Postnikov diagram is shown in Figure 3.9. The graph can be obtained directly from the quiver Q_D. Each face in the quiver that is clockwise oriented corresponds to a white vertex, and anti-clockwise oriented faces correspond to black vertices.

The dimer quiver and the graph shown in figure 3.8 and 3.9 together can be called a dimer model. But the way we have drawn the quiver on a disk, the quiver in figure 3.8 is also a dimer quiver with boundary in the sense of definition 3.1.1. It is also possible to recover a Postnikov diagram from the quiver Q_D. Given a quiver embedded in a disk, draw strand segments from midpoint of an edge of the quiver to the midpoint of the next edge in the order of orientation of the face. The strand diagram obtained in this way satisfy the local axions, but may not satisfy the global axioms.

These dimer quivers were used to describe the categorification of cluster structure on Grassmannians. Before explaining this particular setting, let us review categorification of cluster algebras in general. In the next two sections, I will introduce certain categories called cluster categories and the work that has been done so far.

3.3 Cluster categorification

Categorification has been used to great success recently to solve a number of problems throughout mathematics. The idea of categorification is to impose more structure on a mathematical object by finding a category that models it. The category is a richer object and studying it can reveal intricate properties of the underlying mathematical object. A particularly fruitful application of categorification has been in the theory of cluster algebras. In a categorical model, the clusters are replaced by objects called cluster-titling objects.

In [IY08], the mutation of cluster titling object was defined and it was shown that the mutated object is also a titling object. If the category is a 2-Calabi–Yau category then the mutation of clusters corresponds to mutation of cluster tilting objects. For cluster algebras without frozen variables, the categorical models are known. In the acyclic case, the categories were defined in [BMRRT]. This setting was generalized to the cyclic case by [Ami09] by considering quivers with potentials.

In 2004, Buan, Iyama, Reineke, Reiten, Todorov defined a new category called cluster category [BMRRT]. A cluster category corresponds to a finite dimensional hereditary algebra H and it is defined as certain quotient of the bounded derived category of finite modules over H. When H is the path algebra associated to a simply-laced Dynkin quiver Γ, the corresponding cluster category $\mathcal{C}$ serves as a model for a cluster algebra $\mathcal{A}$ of type Γ. In particular, the indecomposable objects in the category are in bijection with cluster variables in $\mathcal{A}$. This implies that the clusters of $\mathcal{A}$ are in bijection with the basic tilting objects of the category $\mathcal{C}$.

An object $T \in \mathcal{C}$ is called a cluster tilting object in the category if $\mathrm{Ext}^1_{\mathcal{C}}(T, T) = 0$ and the number of indecomposable summands (up to isomorphism) in T equals the rank of the Grothendieck group of $\mathcal{C}$. The endomorphism ring $B = \mathrm{End}_{\mathcal{C}}(T)$ of a cluster tilting object is known as a cluster tilted algebra. Now consider the cluster tilting algebra B associated to a quiver Q that is mutation equivalent to a Dynkin quiver. Then as shown in [BMRRT], B is isomorphic to a quotient of the path algebra of Q. The relations in this quotient are given by a potential of the quiver. In particular, this algebra is the Jacobian algebra corresponding to a potential called a primitive potential.

To generalize the cluster categorification to allow cycles, Amiot defined the cluster category $\mathcal{C}_{(Q,W)}$ for Jacobi-finite quiver with potential (Q, W). A quiver with potential is called Jacobi-finite if the corresponding Jacobian algebra is finite dimensional. It is Jacobi-infinite if the algebra is not finite dimensional. It is shown that when reduced to appropriate quivers, this category resembles the cluster category defined in [BMRRT], [BIRS09]. The cluster algebras in both the settings above do not have frozen variables and the categories are 2-Calabi–Yau. The setting of this thesis includes cluster algebras with frozen variables as we will see in the next chapter.

In order to obtain a cluster categorification for such algebras, we will use stable 2-Calabi–Yau Frobenius categories instead of 2-Calabi–Yau triangulated categories. In 2008, Geiss, Leclerc and Schröer gave cluster structure on some subalgebra of the homogeneous coordinate ring of the partial flag variety [GLS08]. They also gave a partial categorification of this structure using certain subcategories of the category of modules over a preprojective algebra. These categories are Frobenius whose stable categories are triangulated and 2-Calabi–Yau. They extended this categorification combinatorially to the homogeneous coordinate ring of the whole flag variety. Frobenius categories have also been used in [JKS16] [DI16] [DL16] to give more direct categorifications.

In 2017, Pressland gave a construction of a Frobenius category which starts from the data of an initial seed, instead of depending on the geometry of the partial flag varieties [Pre17]. Given a quiver Q with frozen vertices, we can find a Noetherian algebra A such that its Gabriel quiver matches Q up to some arrows between frozen vertices, the quotient of A by paths through the frozen vertices is finite dimensional and A is internally bimodule 3-Calabi–Yau. This algebra A gives a Frobenius category which categorifies the cluster algebra.

In order to construct such algebra A, we need extra data with the quiver Q. This is done by defining the polarised principal coefficient cluster algebra. This introduces extra frozen vertices to the quiver and hence extra frozen variables in the cluster algebra. Since they differ from the cluster algebras with coefficients only by frozen variables, they still satisfy the universal property stated in [FZ07].

This setting gives a cluster categorification of acyclic cluster algebras with frozen variables, starting from one seed in the cluster algebra or a quiver. The cluster categories for acyclic quivers defined in [BMRRT] are extended according to the extra data in the polarised principal coefficients, to get a Frobenius category. Some of the results in this paper are true for cyclic quivers. In this case the cluster categories come from Amiot's categorification in [Ami09]. An example of a 3-cycle is given in this paper which follows some results of the paper. This is because the Gabriel quiver the corresponding Noetherian algebra is a 3-cycle.

The polarised principal cluster algebra has two frozen vertices for each mutable vertex. Given a quiver Q with mutable vertices, we will assign two frozen vertices i^+ and i^- to each vertex i in the quiver. The edges $\alpha_i : i \to i^+$ and $\beta_i : i^- \to i$ are added to the quiver. There will be some arrows between frozen vertices, such that each cycle created in such a way is oriented. This new quiver $\tilde{Q}$ is then used to define the frozen Jacobian algebra as follows. The full subquiver with frozen vertices is called F with F_0 as the set of frozen vertices. Then $Q_0^m = Q_0 \setminus F_0$ will be the mutable vertices and $Q_1^m = Q_1 \setminus F_1$ will be the arrows in the mutable part of the quiver. We define the cyclic derivative as defined earlier for the arrows in Q_1^m, extended linearly,

$$\partial_a(a_1 a_2 \cdots a_n) = \sum_{a=a_i} a_{i+1} \cdots a_n a_1 \cdots a_{i-1},$$

The ideal generated by the derivatives is called the Jacobian algebra as before. For a potential W of Q, the frozen Jacobian algebra is the quotient:

$$A = J(Q, F, W) = \mathbb{C}(Q)/\langle \partial_a W | a \in Q_1^m \rangle$$

In the last section of the thesis, we will see how this setting can be useful to obtain the goal of this project.

3.4 Connection to semicanonical basis

The cluster algebras were initially defined to understand the (dual) canonical bases of universal enveloping algebras. When an algebra has a cluster structure and a dual canonical basis, a special class of cluster variables called cluster monomials are conjectured to form a subset of the dual canonical basis [FZ02]. Geiss, Leclerc and Schröer have a series of papers studying the connection between the following areas:

- Cluster algebras
- Preprojective algebras
- Representation of quivers
- Semicanonical basis of universal enveloping algebra

Derksen, Weyman and Zelevinsky gave a categorification of cluster algebras using representations of quivers with potentials [DWZ08], [DWZ10]. The categorification of acyclic cluster algebras given in [BMRRT] also establishes a relation between representations of quivers and cluster algebras.

The connection between cluster algebras and preprojective algebras Λ of type ADE was shown in [GLS06]. Let N be a maximal unipotent subgroup of a complex Lie group of the same type as the preprojective algebra. For a preprojective algebra defined as below, let I be the set of vertices in Q and Λ_d be the variety of nilpotent Λ-modules with dimension vector $d = (d_i)_{i \in I}$. Suppose $\mathfrak{n}$ is a Lie algebra of N and $U(\mathfrak{n})$ be the universal enveloping algebra. Lusztig showed that the universal enveloping algebra $U(\mathfrak{n})$ is isomorphic an algebra $\mathcal{M}$ of constructible functions on Λ_d [L00]. This gives a new basis of $U(\mathfrak{n})$ with the irreducible components of these varieties Λ_d. This basis is called a semicanonical basis and is denoted by $\mathcal{S}$. If N is the unipotent group corresponding to $\mathfrak{n}$, then $\mathbb{C}[N]$ can be identified as the graded dual of $U(\mathfrak{n})$. Hence, the basis $\mathcal{S}^*$ can be considered as the dual of the semicanonical basis $\mathcal{S}$ and is called as dual semicanonical basis. It was shown in [GLS06] that all the cluster monomials of $\mathbb{C}[N]$ belong to the dual of Lusztig's semicanonical basis of $U(\mathfrak{n})$. This was generalized to the setting of symmetric Kac–Moody groups and their unipotent cells.

In this generalization, the quiver Q has no oriented cycles and is connected. From Q, its double quiver $\tilde{Q}$ is constructed by adding a reverse arrow a^* to each arrow a of Q. Let c be the element defined as:

$$c = \sum_{a \in Q_1} (a^* a - a a^*)$$

where Q_1 is the set of arrows of Q. Consider the ideal (c) generated by the element c, then the preprojective algebra Λ is

$$\Lambda = \mathbb{C}(\tilde{Q})/(c)$$

Consider the category $\operatorname{nil}(\Lambda)$ of all finite dimensional nilpotent representations of Λ. A representation is nilpotent if its composition series only contains simple representations corresponding to the vertices of Q. This is an abelian category, but does not have projective or injective objects. To every Weyl group element $w \in W$ of the quiver, we can associate a subcategory of $\operatorname{nil}(\Lambda)$, call it $\mathcal{C}_w$. This category is Frobenius and stably 2-Calabi–Yau [BIRS09]. A category is called Frobenius if it has enough projectives and enough injectives, and if the projectives and injectives coincide (i.e. an object is projective if and only if it is injective)

To the subcategory $\mathcal{C}_w$, a cluster algebra $\mathcal{A}_w$ is associated, such that it categorifies the cluster algebra. Suppose $\mathbf{i} = i_1 i_2 \ldots, i_n$ is a reduced expression for the word w. Then the subcategory $\mathcal{C}_w$ contains a maximal rigid module M_i corresponding to this reduced expression. Let $M_i = T_1 \oplus \cdots \oplus T_r$ such that every T_i is an indecomposable object and the last n objects are projective-injective. Let Q_T be the quiver of the endomorphism algebra $\mathrm{End}_\Lambda(T)$. Then the cluster algebra $\mathcal{A}_w = \mathcal{A}(Q_T)$. The objects M_i correspond to the initial seeds of the cluster algebras. The algeba $\mathcal{A}_w$ is shown to be isomorphic to the coordinate ring of the finite dimensional unipotent subgroup of the symmetric Kac–Moody group attached to $\mathfrak{g}$.

3.5 Cluster categorification of coordinate rings on Grassmannians

As mentioned in the introduction, we will follow the combinatorial model for the cluster structure on Grassmannians, to obtain the combinatorial model for coordinate rings on double Bruhat cells. In [JKS16], the authors categorified the cluster structure on the homogeneous coordinate ring of Grassmannian using the category of Cohen–Macaulay modules. They associated a rank one Cohen–Macaulay module M_I over a ring B to every k-subset I of $\{1, 2, \ldots, n\}$. These are the indecomposable objects of the category which correspond to the cluster variables of the cluster structure. Recall that each such k-subset is a vertex in a (k, n)-Postnikov diagram. Hence, to a Postnikov diagram D, one can associate

$$T_D = \oplus M_I$$

where I varies over all k-subsets of of $\{1, 2, \ldots, n\}$. This object is a cluster tilting object in the category of Cohen–Macaulay modules over B, and correspond to the clusters of the cluster structure. Here, B is a quotient of the preprojective algebra of type $\tilde{A}_{n-1}$. The following result by [BKM16] connects this categorification to the combinatorics described above:

Theorem 3.5.1. *The dimer algebra A_D is isomorphic to the endomorphism ring* $\mathrm{End}_B(T_D)$

In order to obtain this isomorphism, the authors define a map $g : A_D \to \mathrm{End}_B(T_D)$. There is a grading defined on A_D using a grading on paths of D. A similar grading can also be defined on $\mathrm{End}_B(T_D)$. The map g is a graded homomorphism corresponding to these gradings. To every edge from I to J in A_D, the map g associates a homomorphism $M_I \to M_J$. For a fixed pair (I, J), this homomorphism generates the space $\mathrm{Hom}_B(M_I, M_J)$ freely as a $\mathbb{C}(t)$-module.

The ring B defined can also be written in terms of this quiver. Consider e in the path algebra to be the some of idempotent elements corresponding to the boundary vertices of A_D. Then eA_De is called the boundary algebra of A_D. Note that the dimer algebra A_D is invariant under twisting and untwisting moves at a boundary vertex or an internal vertex. This leads to the result that

Theorem 3.5.2. *For any two (k, n)-Postnikov diagrams D and D', the boundary algebras eA_De and eA'_De are isomorphic.*

In fact, the ring B is isomorphic to the boundary algebra eA_De. In this way, the categorification of the cluster structure on the homogeneous coordinate ring of Grassmannians is described combinatorially using dimer models. Moreover, a generalized case of Postnikov diagrams on certain surfaces with boundaries is also considered briefly at the end of the paper [BKM16]. In the next chapter, I will give the definition of a dimer model on certain cylinders. These will form a set of combinatorial objects which conjecturally can be used to get a cluster categorification of coordinate rings on double Bruhat cells.

Chapter 4
Dimers over Cylinders

4.1 Quivers from double Bruhat cells

Berenstein, Fomin and Zelevinsky defined certain quivers from double Bruhat cells. In the paper [BFZ05], they showed that the coordinate rings of double Bruhat cells can be identified with certain upper cluster algebras. These algebras are defined using the combinatorial data that comes from the corresponding double Bruhat cells. This combinatorial data can be encoded in a quiver. We recall the definition of this quiver in this section.

Let G be a simply connected, connected, semisimple complex algebraic group of rank r. Let B and B_- be opposite borel subgroups and W be its Weyl group. Then G can be written as

$$G = \cup_{u \in W} BuB = \cup_{v \in W} B_- v B_-.$$

The double Bruhat cell $G_{u,v}$ is defined as the intersection $BuB \cap B_- v B_-$ where $u, v \in W$. To each such pair of Weyl elements $(u, v) \in W \times W$, we can associate a quiver $Q^{u,v}$ as defined in [BFZ05]. There are two ways to define this quiver. We can get the quiver directly from the data given by the words u and v, or we can define a matrix that gives the quiver. We will see both ways in the next section.

4.2 Berenstein–Fomin–Zelevinsky quivers

Let G be a simply connected complex algebraic group. Let W be the Weyl group and $\mathfrak{g}$ be the Lie algebra of G. Every Weyl group can be realized as a Coxeter group with reflections $s_1, s_2, \ldots, s_r$ of simple roots as its generators. Each s_i is an involution and $(s_i s_j)^{m_{ij}} = 1 \in W$, for some integer m_{ij} encoded in the Dynkin diagram. Every element $w \in W$ has a smallest expression in terms of s_i's. A word is a tuple of indices of simple reflections in the smallest expression for w. If for $w = s_{i_1}s_{i_2}\cdots s_{i_l}$ in W is the smallest such expression in terms of the generators of W then the word $i = (i_1, i_2, \cdots, i_l),\ i_j \in [1, \cdots, r]$ is said to be in its reduced form. The length of the word w is denoted by $\ell(w)$ and in this case $\ell(w) = l$.

Fix a pair $(u, v) \in W \times W$. Let us use negative indices for the generators of the first copy of W and positive indices for the second copy of W. Then a reduced word $\mathbf{i} = (i_1, \ldots, i_{\ell(u)+\ell(v)})$ is an arbitrary shuffle of a reduced word for u and a reduced word for v. We add the numbers $(-r, \ldots, -1)$ to the tuple $\mathbf{i}$ to get a new tuple

$$\hat{\mathbf{i}} = (-r, \ldots, -1, i_1, \ldots, i_{\ell(u)+\ell(v)}).$$

For $k \in [-r, -1] \cup [1, \ell(u) + \ell(v)]$, we define k^+ to be the smallest index l such that $k < l$ and $|i_k| = |i_l|$. If $|i_k| \neq |i_l|$ for any $l > k$, then $k^+ = \ell(u) + \ell(v) + 1$. An index is called i-exchangeable if both k and k^+ are in $[-r, -1] \cup [1, \ell(u) + \ell(v)]$. The set if i- exchangeable indices is denoted by $\mathbf{e}(\mathbf{i})$.

Definition 4.2.1. *Let $u, v \in W$. A BFZ quiver $Q^{u,v}$ has set of vertices $Q_0 = \hat{\boldsymbol{i}}$ Vertices k and l such that $k < l$ are connected if and only if either k or l are i-exchangeable. There are two types of edges:*

- *An edge is called horizontal if $l = k^+$ and it is directed from k to l if and only if $\epsilon(i_l) = +1$.*
- *An edge is called inclined if one of the following conditions hold:*

  1. *$l < k^+ < l^+$, $a_{|i_k|,|i_j|} > 0$, $\epsilon(i_l) = \epsilon(i_{k^+})$*
  2. *$l < l^+ < k^+$, $a_{|i_k|,|i_j|} > 0$, $\epsilon(i_l) = -\epsilon(i_{l^+})$*

  An inclined edge is directed from k to l if and only if $\epsilon(i_l) = -1$.

Let $\widetilde{B}(\mathbf{i})$ be the matrix corresponding to this quiver. Its rows are labelled by the set $[-r,-1] \cup [1, \ell(u)+\ell(v)]$ and columns are labelled by $\mathbf{e}(\mathbf{i})$. It is defined as follows:

$$b_{kl} = \begin{cases} -\operatorname{sgn}(k-l)\epsilon(i_p) & \text{if } p = q \\ -\operatorname{sgn}(k-l)\epsilon(i_p)a_{|i_k|,|i_l|} & \text{if } p < q \text{ and } (k-l)(k^+ - l^+)\epsilon(i_p)\epsilon(i_q) > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $p = \max\{k, l\}$, $q = \min\{k^+, l^+\}$ and $a_{|i_k|,|i_l|}$ is the corresponding entry in the Cartan matrix. The vertices of the quiver correspond to the set $[-r,-1] \cup [1, \ell(u)+\ell(v)]$. The edges are given by the matrix entries. Two vertuces k and l are connected if and only if $b_{kl} \neq 0$. If $b_{kl} > 0$ then the edge is directed from k to l. If $b_{kl} < 0$, then the edge goes from l to k.

We will see an example of a BFZ quiver below. Consider the group $SL_4(\mathbb{C})$. Here B is the Borel group of upper-triangular matrices and B_- is the group of lower-triangular matrices. The Weyl group in this case is $W = S_4$, the permutation group on four elements. Let $u = w_0 = s_3s_2s_1s_2s_3$, $v = e \in S_4$. So $\ell(u) = 5$ and $\ell(v) = 0$. The element u is the longest element of W. The quiver $Q^{u,v}$ corresponding to the double Bruhat cell $G_{u,v}$ is as shown below in Figure 4.1. (The Dynkin diagram A_3 is not a part of the quiver.). Following is the detailed computations for this example.

The word $u = w_0 = s_3s_2s_1s_2s_3$ is a reduced word as no braid relation can reduce its length. As the second word v is the identity, it does not contribute any vertex or an edge to the quiver $Q^{u,v}$. So, $\ell(u)+\ell(v) = 5+0$, $r = 3$ and

$$\hat{\mathbf{i}} = (-3, -2, -1,\ 3,\ 2\ , 1,\ 2,\ 3) \quad \text{or}$$

k	-3	-2	-1	1	2	3	4	5
$\hat{\mathbf{i}}$	-3	-2	-1	3	2	1	2	3
i_k	i_{-3}	i_{-2}	i_{-1}	i_1	i_2	i_3	i_4	i_5

Let us compute k^+ for each k. From the definition of k^+, we know that it tells the next entry in $\hat{\mathbf{i}}$ which matches i_k up to sign. For example, for $k = -3$, $k^+ = 1$ because $|i_{-3}| = |i_1| = 3$ and there is no appearance of 3 or -3 between those two (i.e. if $k = -3$, k^+ cannot be 5 even though $|i_{-3}| = |i_5| = 3$ because those are not the consecutive appearances of 3 or -3). The following table shows k^+ for this example.

k	-3	-2	-1	1	2	3	4	5
k^+	1	2	3	5	4	6	6	6

If $|i_k| \neq |i_l|$ for any $l > k$, then $k^+ = \ell(u) + \ell(v) + 1$. For example, for $k = 3$, $i_k = i_3 = 1$ is the last appearance of 1 in $\hat{\mathbf{i}}$ and so $k^+ = 6$. Similarly, $4^+ = 5^+ = 6$. An index is called i-exchangeable if both k and k^+ are in $[-r, -1] \cup [1, \ell(u) + \ell(v)]$. So 3, 4, 5 are not i-exchangeable. Also, $-3, -2, -1$ are not i-exchangeable. The only i-exchangeable indices are $k = 1, 2$. Therefore, $\mathbf{e}(\mathbf{i}) = \{1, 2\}$.

The matrix $\widetilde{B}(\mathbf{i})$ is an 8×2 matrix whose rows are labelled by (-3, -2, -1, 1, 2, 3, 4, 5) and columns are labelled by (1,2). We will compute the entries b_{kl} in the following table. If $p = q$, then the entry in the matrix is zero, and so we do not compute the rest of the values in that particular row. Similarly, if $p > q$, then the corresponding entry in the matrix is zero. For example, the entry $b_{-3,2}$ is zero since $p > q$ in that row, so we do not meed to compute the rest of the entries. When $k = l$, the entry $b_{kl} = b_{kk} = 0$ since the sign of $(k - l)$ determines the entry.

k	l	$p =$ $\max(k,l)$	$q =$ $\min(k^+, l^+)$	$\epsilon(i_p)$	$\epsilon(i_q)$	$\mathrm{sgn}(k-l)$	$\mathrm{sgn}(k^+ - l^+)$	$a_{\lvert i_k\rvert,\lvert i_l\rvert}$	b_{kl}
-3	1	1	1	$+$	$+$	$-$			1
-2	1	1	2	$+$	$+$	$-$	$-$	-1	-1
-1	1	1	3	$+$	$+$	$-$	$-$	0	0
1	1	2	5	$+$	$+$	0			0
2	1	2	4	$+$	$+$	$+$	$-$	-1	0
3	1	3	5	$+$	$+$	$+$	$-$	0	0
4	1	4	5	$+$	$+$	$+$	$-$	-1	1
5	1	5	5	$+$	$+$	$+$			-1
-3	2	2	1						0
-2	2	2	2	$+$	$+$	$-$	$-$		1
-1	2	2	3	$+$	$+$	$-$	$-$	-1	-1
1	2	2	4	$+$	$+$	$-$	$-$	-1	0
2	2	2	4	$+$	$+$	0			0
3	2	3	4	$+$	$+$	$+$	$-$	-1	1
4	2	4	4	$+$	$+$	$+$			-1
5	2	5	4						0

So the matrix $\widetilde{B}(\mathbf{i})$ is:

	1	2
-3	1	0
-2	-1	1
-1	0	-1
1	0	0
2	0	0
3	0	1
4	1	-1
5	-1	0

Since $b_{-3,1} = 1$, the edge in the quiver is directed from -3 to 1. On the other hand, since $b_{-2,1} = -1$, the edge is directed from 1 to -2. There are no edges between k and l if $b_{kl} = 0$. So the quiver $Q^{w_0,e}$ is as shown below:

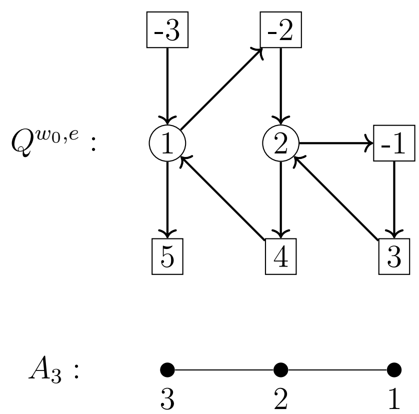


Figure 4.1: A BFZ quiver in type A

The circular vertices are mutable and the square vertices are frozen. The definition above does not include edges between certain frozen vertices. Moreover edges between frozen vertices are usually not shown as they do not contribute any information to the cluster algebra. But in this article, we will add the arrows between frozen vertices that complete simple cycles, as shown in Figure 4.2. Note that every

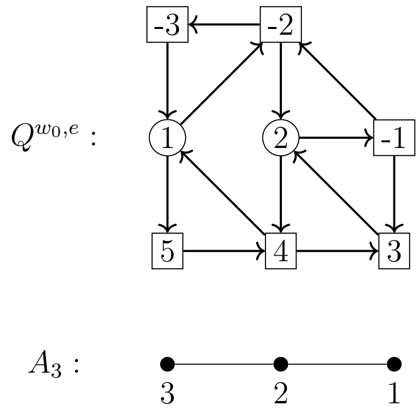


Figure 4.2: A BFZ quiver in type A with extra arrows

cycle in both of these quivers is oriented. The quiver is drawn in such a way that the number of s_i's in the words u and v correspond to the number of arrows of the quiver that lie directly above the vertex i of the Dynkin diagram. For example, there are two s_3's in w_0 and e together, which correspond to the two vertical arrows in the quiver that lie above the vertex 3 of A_3, similarly for s_2 and s_1.

A quiver for double Bruhat cells for A_n can be viewed as a quiver on a plane on A_n as shown in the figure above. Observe that:

- we have drawn the quiver such that all vertices lie on a straight line above a vertex of the Dynkin diagram. Let us call these lines strings;
- all vertical edges in the quiver project onto vertices in the Dynkin diagram, i.e. all vertical edges lie strictly on the strings;
- all inclined edges project onto edges of the Dynkin diagram. In other words, there are no edges that connect two vertices lying on non-adjacent strings.

This structure can be generalized to BFZ quivers outside of type A. In order to do this, we will define quivers on cylinders over Dynkin diagrams, and then show that the BFZ quivers are examples of those.

4.3 Quivers on cylinders over Dynkin diagrams

Let Γ be a Dynkin diagram. A vertex of a Dynkin diagram is called an endpoint if it has only one edge incident to it. A vertex is called a ramification point if it has strictly more than two edges incident to it. A path $\Gamma_{m,n}$ between two vertices m and n in Γ is called a branch if both m and n are branching points or endpoints or if one of them is a branching point and the other is an endpoint.

Definition 4.3.1. *For a Dynkin diagram Γ, we define the cylinder over Γ to be the topological space $\Gamma \times \mathbb{R}$. Let Γ_0 be the set of vertices of Γ. We call the set $\Gamma_0 \times \mathbb{N} \subset \Gamma \times \mathbb{R}$ a grid on the cylinder. The set $\Gamma_{m,n} \times \mathbb{R}$ is called the sheet over the branch $\Gamma_{m,n}$. The length of a sheet is the number of edges on the branch. The subset $\{x_0\} \times \mathbb{R}$ where $x_0 \in \Gamma_0$ is called a string.*

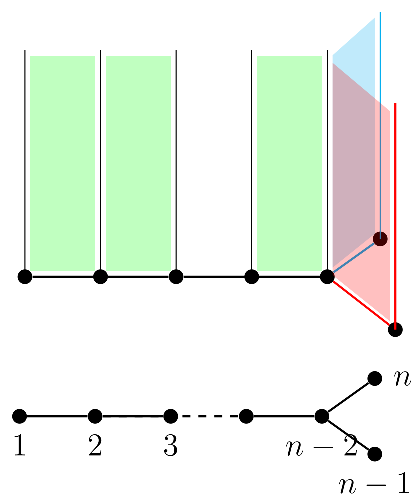


Figure 4.3: The cylinder over Dynkin diagram of type D_n

Example 4.3.1. *A quiver for double Bruhat cells of D_n can be drawn on a book-like structure as shown in the figure below. The cylinder $D_n \times \mathbb{R}$ has n strings and three sheets; one sheet of length $n-3$ and two sheets of length 1 glued together at their boundaries.*

Definition 4.3.2. *A quiver on the cylinder over a Dynkin diagram is called a dimer quiver on the cylinder if*

1. *Each arrow of the quiver projects onto an edge or a vertex of the Dynkin diagram.*

2. *Each face projects onto an edge of the Dynkin diagram.*

3. *Each face is oriented.*

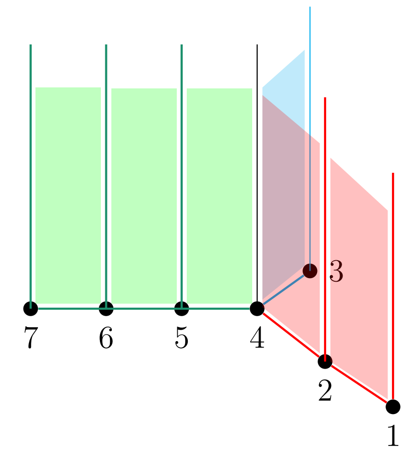
4. *The first and last vetices on each string are frozen. The first and last faces on each stripe have an edge that connects two frozen vertices*

5. *Two faces do not share two edges unless their common string projects onto the ramfication vertex of the Dynkin diagram.*

Theorem 4.3.1. *A BFZ quiver can be realized as a dimer model on the cylinder over the corresponding Dynkin diagram.*

Proof. Notice that the horizontal edges in Definition 4.2.1 lie on the strings of the cylinder over the Dynkin diagram. All inclined edges lie between two adjacent strings such that they project down onto an edge of the Dynkin diagram. According to the definition of the quiver, there is an edge between two vertices of adjacent strings only if the corresponding vertices in the graph are connected by an edge. □

Example 4.3.2. *Quivers for double Bruhat cells of E_7 can be drawn on a book-like structure as shown in Figure 4.4. The cylinder $E_7 \times \mathbb{R}$ has seven strings and three sheets: one sheet of length 3 (green in color), one sheet of length 2(red in color) and one sheet of length 1 (blue in color) glued together at their boundaries (the black string).*



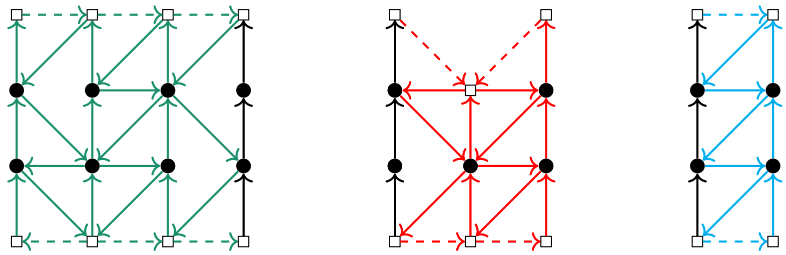
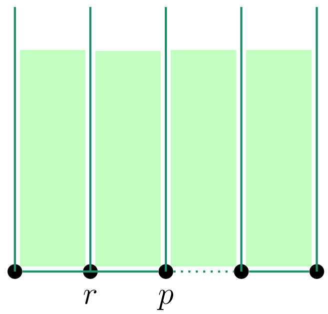


Figure 4.4: A dimer quiver on the cylinder over E_7 corresponding to the element $u = s_1s_3s_2s_4s_5s_7s_3s_6s_1s_5s_7s_6s_4s_3s_2s_1s_4s_5s_6s_7$. The quivers lie on their respective colored sheets, and they share the three black arrows where the sheets intersect. The square vertices are frozen vertices.

Theorem 4.3.2. *For any $u, v \in W$, the quiver $Q^{u,v}$ can be obtained from gluing $Q^{e,v}$ on top of $Q^{u,e}$ in the following way:*

- *On each string of the quiver, identify the bottom frozen vertex of $Q^{e,v}$ to the top frozen vertex of $Q^{u,e}$.*
- *The identified vertices are mutable vertices of $Q^{u,v}$ as they no longer are the boundary vertices.*
- *If the edges between two identified pair of vertices are directed in the same direction, then we keep one edge between them. If the edges are not in the same direction then we delete the edges, so there is no edge between those vertices in $Q^{u,v}$.*

Proof. It is enough to prove this result for any two neighboring strings in the cylinder. Let r, p be two neighboring vertices in the Dynkin diagram, i.e. $M(r,p) < 0$ where M is the Cartan matrix.



Let u and v be two reduced Weyl group elements. Depending on the first and last positions of s_r and s_p in the words u and v, there are four possible cases:

- $u = \_\_\_s_r\_\_\_s_p\_\_\_,\ v = \_\_\_s_r\_\_\_s_p\_\_\_$
- $u = \_\_\_s_r\_\_\_s_p\_\_\_,\ v = \_\_\_s_p\_\_\_s_r\_\_\_$
- $u = \_\_\_s_p\_\_\_s_r\_\_\_,\ v = \_\_\_s_r\_\_\_s_p\_\_\_$
- $u = \_\_\_s_p\_\_\_s_r\_\_\_,\ v = \_\_\_s_p\_\_\_s_r\_\_\_$

Let us consider the first case where

$$u = \_\_\_\ \underset{\substack{\uparrow \\ k_1\text{th}}}{s_r}\ \_\_\_\ \underset{\substack{\uparrow \\ k_2\text{th}}}{s_p}\ \_\_\_,\ v = \_\_\_\ \underset{\substack{\uparrow \\ k_3\text{th}}}{s_r}\ \_\_\_\ \underset{\substack{\uparrow \\ k_4\text{th}}}{s_p}\ \_\_\_.$$

We know that the faces in $Q^{u,e}$ and $Q^{e,v}$ are oriented. In this case, the vertices k_1 and k_2 are frozen vertices of $Q^{u,e}$, l_0 and l_1 are frozen vertices of $Q^{e,v}$.

$$\begin{tikzcd}
 & k_0 \arrow[r] & k_2 \arrow[d, dotted] & l_0 \arrow[dr] & l_3 \arrow[l] \\
 \arrow[rr] & & k_1 \arrow[ul] & l_1 \arrow[u, dotted] & l_2 \arrow[l]
\end{tikzcd}$$

$$
\begin{tikzcd}
{} \arrow[r] & k_2 \arrow[dr] & k_4 \arrow[l] \\
{} \arrow[r] & k_1 \arrow[ul] & k_3 \arrow[l]
\end{tikzcd}
$$

k	i_k	k^+
k_1	r	k_3
k_2	p	k_4
k_3	$-r$	$k_3^+ > k_4$
k_4	$-p$	$k_4^+ \geq k_4 + 1$

Figure 4.5: Gluing faces I

As $k_1 \leq k_2$ and $\text{sgn}(i_{k_2}) \neq \text{sgn}(i_{k_4})$, we need to check for the inequality $k_1 < k_2 < k_2^+ < k_1^+$. But the inequality is not true because $k_2^+ = k_4 > k_3 = k_1^+$. So there is no edge between k_1 and k_2 in $Q^{u,v}$.

For the second case where $u = \_\_\_s_r\_\_\_s_p\_\_\_$, $v = \_\_\_s_p\_\_\_s_r\_\_\_$: Again, the faces in $Q^{u,e}$ and $Q^{e,v}$ are oriented. The vertices k_1 and k_2 are frozen vertices of $Q^{u,e}$, l_0 and l_1 are frozen vertices of $Q^{e,v}$. The following table of k and k^+ in $Q^{u,v}$ shows that $k_1 < k_2 < k_3 < k_1^+$, $sign(i_{k_2}) \neq sign(i_{k_3})$, we also know that $M(|i_{k_1}|, |i_{k_2}|) = M(r, p) < 0$. Hence there exists an edge between k_1 and k_2.

$$
\begin{tikzcd}
k_0 \arrow[r] & k_2 \arrow[d, dotted] & & l_0 \arrow[d, dotted] & l_2 \arrow[l] \\
{} \arrow[r] & k_1 \arrow[ul] & & l_1 \arrow[ur] & l_3 \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
k_0 \arrow[r] & k_2 \arrow[d] & k_3 \arrow[l] \\
{} \arrow[r] & k_1 \arrow[ul] \arrow[ur] & k_4 \arrow[l]
\end{tikzcd}
$$

k	i_k	k^+
k_1	r	k_4
k_2	p	k_3
k_3	$-p$	$k_3^+ > k_4$
k_4	$-r$	k_4^+

Figure 4.6: Gluing faces II

The third and the fourth cases are similar to the first and second respectively. $\square$

Let us see an example of constructing $Q^{u,v}$. The quiver is obtained by attaching the quiver $Q^{e,v}$ on top of the quiver $Q^{u,e}$. We will see this with $u = s_1s_2s_1s_3$, $v = s_2s_3s_3s_1 \in S_4$. Refer to figures 4.7, 4.9.

Lemma 4.3.1. *A BFZ quiver $Q^{u,v}$ is planar in each sheet.*

Proof. Consider the kth and the lth string of the quiver. If the strings are not adjacent on a sheet, then we know that there cannot be edges between the vertices of the strings. If the strings are adjacent, consider the following diagram:

$$\begin{tikzcd}
k \arrow[r] & k^+ \arrow[r] \arrow[dl] & k^{++} \arrow[r] & \cdots \arrow[r] & k^{p+} \\
l \arrow[r] & l^+ \arrow[r] & l^{++} \arrow[r] & \cdots \arrow[r] & l^{r+}
\end{tikzcd}$$

Suppose the vertices l and k^+ are connected. Then depending on whether $k^+ < l$ or $l < k^+$, there will be the following inequalities:

(1) If $k^+ < l$, then $l < k^{++} < l^+$

(2) If $l < k^+$, then $k^+ < l^+ < k^{++}$.

(3) So in both cases above, $k^+ < l^+$.

We want to show that the vertex k is not connected to l^{m+} for any m. Suppose k and l^{m+} are connected. Then again, there are two cases:

(4) If $l^{m+} < k$, then $k < l^{(m+1)+} < k^+$

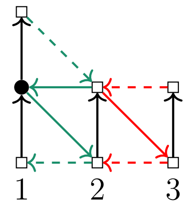


Figure 4.7: $Q^{u,e}$, $u = s_1s_2s_1s_3$

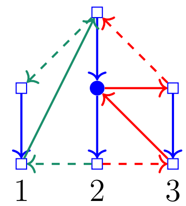


Figure 4.8: $Q^{e,v}$, $v = s_2s_3s_2s_1$

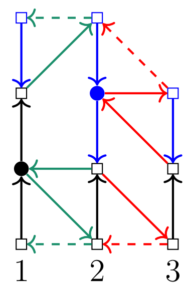


Figure 4.9: $Q^{u,e}$ and $Q^{e,v}$ glued at frozen vertices

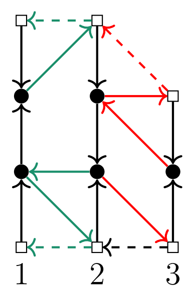


Figure 4.10: $Q^{u,v}$, $u = s_2s_3s_2s_1$, $v = s_2s_3s_2s_1$

(5) If $k < l^{m+}$, then $l^{m+} < k^+ < l^{(m+1)+}$.

(6) Combining inequalities in (4) and (5) with $l^+ < l^{m+}$ we get, $l^+ < k^+$.

As (3) and (6) contradict each other, there cannot be an overlapping edge. Hence the quiver is planar in each sheet. $\square$

Lemma 4.3.2. *For any Kac–Moody algebra $\mathfrak{g}$ and $(u, e) \in W \times W$, all faces of $Q^{u,e}$ are oriented, where e is the identity element in W.*

Proof. Let us assume that there exists a non-oriented n- cycle in the quiver with $p+1$ vertices in jth string, $r+1$ vertices in (a neighboring) kth string and $n = p+r+2$. Note that all edges in all strings are directed in one direction as we are fixing one of the Weyl group elements to be the identity. Two edges between the neighboring strings can be directed in the same or opposite direction (as shown below). Let us consider the first case where the vertical edges have the same direction.

$$\begin{array}{ccccccccc}
j & \longrightarrow & j^{+} & \longrightarrow & j^{++} & \longrightarrow & \cdots & \longrightarrow & j^{p+} \\
\Big\uparrow & & & & & & & & \Big\uparrow \\
k & \longrightarrow & k^{+} & \longrightarrow & k^{++} & \longrightarrow & \cdots & \longrightarrow & k^{r+}
\end{array}$$

Let $r \leq p$. From the direction of the vertical arrows, it is clear that $j < k < j^{+} < k^{+}$ and $j^{p+} < k^{r+} < j^{(p+1)+} < k^{(r+1)+}$. We also know that $j < j^{+} < \cdots < j^{p+} < j^{(p+1)+}$ and $k < k^{+} < \cdots < k^{r+} < k^{(r+1)+}$.

Each inequality $j^{m+} < \cdots < j^{(m+s)+} < k^{n+} < \cdots < k^{(n+t)+} < j^{(m+s+1)}$ creates an edge from $j^{(m+s)+}$ to j. For every such inequality, notice that we get one or more edges in the n-cycle which divides the cycle into smaller oriented cycles. The second case where the two edges between the neighboring strings have opposite directions follows similarly from corresponding inequalities. □

Proposition 4.3.3. *All faces of the quiver $Q^{u,v}$ are oriented.*

Proof. We know from Theorem 4.3.2 that the quiver $Q^{u,v}$ can be obtained from gluing $Q^{e,v}$ on top of $Q^{u,e}$. All faces in the quivers $Q^{e,v}$ and $Q^{u,e}$ are oriented. We need to show that the gluing of two quivers also gives oriented faces.

- Case 1. Suppose the edges between two identified vertices are directed in the same direction. In this case, we identify the two edges. In Figure 2, e is the identified edge, k and l are the identified vertices. The paths p_1 and p_2 are the parts of the oriented faces that contain the edge e in $Q^{u,e}$ and $Q^{e,v}$ respectively. As shown in the figure, after gluing the vertices and the edge, the resulting faces in $Q^{u,v}$ are still oriented.

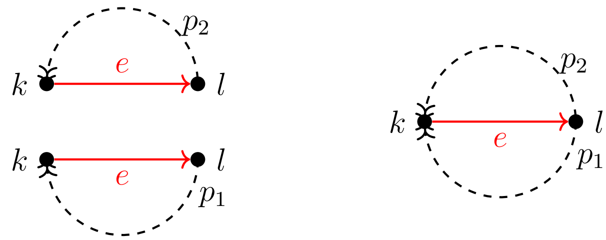


Figure 4.11: Case 1

- Case 2. Suppose the edges between two identified vertices are directed in opposite directions. In this case, we delete the edges. In Figure 3, k and l are the identified vertices. As the two red edges between k and l are oppositely oriented, there is no edge in the glued diagram. This still results into an oriented face F in the quiver $Q^{u,v}$ as shown in the figure.

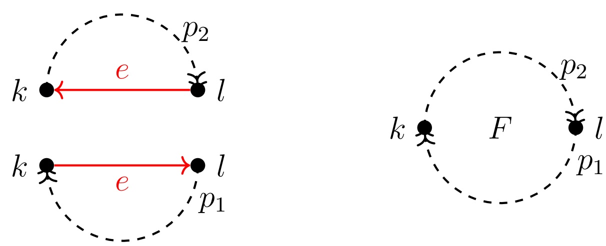


Figure 4.12: Case 2

$\square$

Theorem 4.3.4. *The faces of the quiver on each sheet share at most one edge with each other.*

Proof. Let us consider the following part of the Dynkin diagram. Suppose the two faces share two edges on the rth string. Let $u \in W$ be in its reduced form, then u

has the following form, where each of the spaces do not contain s_{r-1}, s_r ors_{r+1}.

$$u = \underline{\ \ 1st\ \ }s_{r+1}\underline{\ \ 2nd\ \ }s_r\underline{\ \ 3rd\ \ }s_{r-1}\underline{\ \ 4th\ \ }s_r\underline{\ \ 5th\ \ }s_r\underline{\ \ 6th\ \ }s_{r-1}\underline{\ \ 7th\ \ }s_{r+1}\underline{\ \ 8th\ \ }.$$

As the $5th$ space does not contain s_{r-1}, s_r or s_{r+1},

$$s_r\underline{\ \ 5th\ \ }s_r = s_r s_r\underline{\ \ 5th\ \ } = \underline{\ \ 5th\ \ }$$

which implies that u was not in its reduced form. Therefore, it is not possible for two such faces to share two edges.

□

Example 4.3.3. *But two faces can share two edges if the common string projects on to the ramification point of the Dynkin diagram. For example consider the Dynkin diagram D_4. Let $u = s_4s_3s_1s_3s_2s_3s_1s_4$. Then Figure 4.13 shows the quiver $Q^{u,e}$ where two faces share two edges on a ramifying string.*

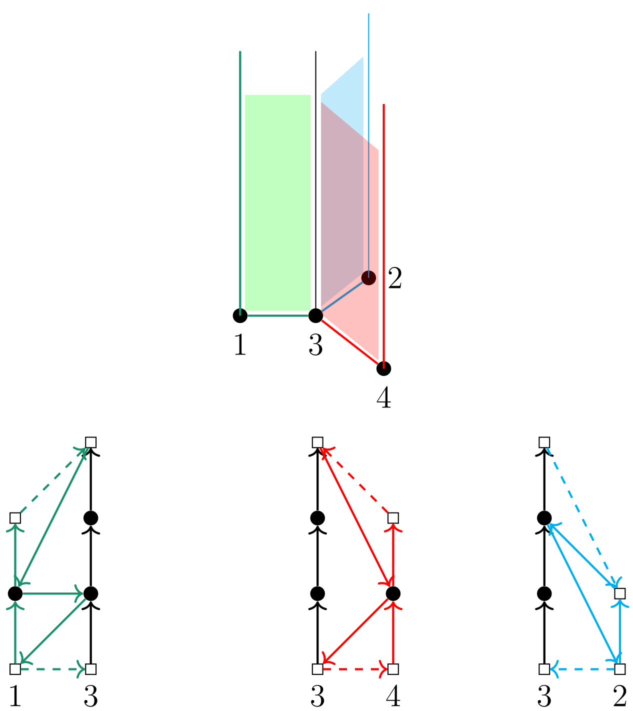


Figure 4.13: The quiver $Q^{u,e}$ for $u = s_4s_3s_1s_3s_2s_3s_1s_4$ in D_4 showing that the top two edges on String 3 are shared by two faces of the quiver.

$$\begin{tikzcd}
j \arrow[r] & j^{+} \arrow[r] & j^{++} \arrow[r] & \cdots \arrow[r] & j^{(n-2)+} \arrow[dll] \\
 & & k \arrow[ull] & &
\end{tikzcd}$$

$$\begin{tikzcd}
j \arrow[r] & j^{+} \arrow[r] & j^{++} \arrow[r] & \cdots \arrow[r] & j^{p+} \arrow[dl] \\
 & & k \arrow[ull] \arrow[r] & k^{+} &
\end{tikzcd}$$

Figure 4.14: Types of faces in a dimer model

Remark 4.3.1. *Each n-face in a quiver has $n-1$ vertices on one string and the remaining one vertex on its adjacent string. The second diagram in Figure 4.14 shows a situation that occurs in $Q^{u,v}$ only on the string at the ramification point of the Dynkin diagram.*

4.4 Rigidity of the superpotential

In this section, we will use the planarity of a dimer models on a cylinder to show that its superpotential is rigid, in certain cases. Recall that a potential of a quiver is a linear combination of cycles in the quiver. The potential

$$S = \sum \text{clockwise oriented faces} - \sum \text{anti-clockwise oriented faces}$$

is called the superpotential of the quiver Q.

Remark 4.4.1. *Each vertex of a quiver $Q^{u,e}$ has at most one edge going to and at most one edge coming from each adjacent string.*

As an application of the theory of dimer models on cylinders we give an independent proof of the following result of [BIRS09]:

Theorem 4.4.1. *The superpotential S of the quiver $Q^{u,e}$ is a rigid potential.*

We first prove that the sub-potential of the superpotential S lying in each sheet is rigid. Recall that two sheets are glued at a string, hence the sub-potentials share edges between them, but they do not share any faces of the quiver. Therefore, rigidity of the sub-potentials indeed implies rigidity of the superpotential. Denote by S_r, the sub-potential of the superpotential S that lies on the r th sheet. In order to prove rigidity of S_r, we need to show that each cycle in the quiver belongs to the Jacobian ideal $J(S_r)$.

Lemma 4.4.1. *Every face belongs to $J(S_r)$.*

Proof. We will prove this by induction on the distance of a face from the boundary of the quiver. The distance of a face F is denoted by $d(F)$ and defined as follows: $d(F) = 0$ if F has a boundary edge as one of its edges. If F is not a boundary face, then $d(F) = d + 1$, where

$$d = \min\{d(F')|\ F' \text{ is an adjacent face to } F\}.$$

Now, if $d(F) = 0$, F has a boundary edge as one of its edges. Let us call the edge f, then $F = f\partial_f(S_r) \in J(S_r)$, which implies all boundary faces are in the Jacobian ideal. Suppose $d(F) = n + 1$, then F has at least one adjacent face whose distance is n. Let that face be E and e be the edge shared by E and F. As $d(E) = n$, by induction, $E \in J(S_r)$, and by the definition of the Jacobian ideal, $e\partial_e(S_r) = E + F \in J(S_r)$, therefore $F \in J(S_r)$. Hence all faces with distance $n+1$ are in the Jacobian ideal, which completes the proof by induction. □

Note that if a cycle is self-intersecting, it can be written a product of two or more non-self-intersecting cycles. If we want to show that the original cycle belongs to the Jacobian ideal, then it suffices to prove that one of its non-self-intersecting cycles belongs to the ideal.

Definition 4.4.1. *We will call a cycle C differentiable with respect to an edge e if e separates the cycle into a face and a smaller cycle.*

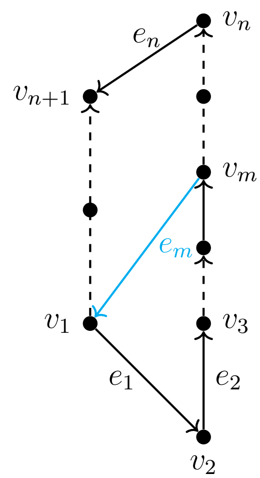
As the quiver is planar in each sheet, we know that the edge e is shared by at most two faces, say F_1 and F_2. If C is differentiable with respect to the edge e, then C contains all edges of either F_1 or F_2 except e.

Lemma 4.4.2. *Every non-self-intersecting cycle in the quiver is differentiable with respect to at least one edge in its interior.*

Proof. Let C be a cycle containing k faces, $F_1, F_2, \ldots, F_k$. Suppose F_i has n_i vertices and n_i edges. We need to show that C contains all but one edge of F_j for some j.

Recall that each face F_i has one of its vertices in a string and the remaining $n_i - 1$ vertices in its adjacent string. Each vertex has at most one edge going to and at most one edge coming from each adjacent string. Lastly, every edge in each string is directed in the same direction.

Let $p : v_2 \to v_n$ be the right-most vertical path in cycle C. Suppose p belongs to the kth string, r_k of the quiver. This path p has exactly one inclined edge $e_1 : v_1 \to v_2$ from the string to its left, r_{k-1}. That means, the face that contains p and e, has its vertex v_1 in string r_k and all remaining vertices in r_{k-1} that belong to path p. As p is the right-most vertical path of C, the edge $e_n : v_n \to v_{n+1}$ lands in the string r_{k-1}. If $v_1 = v_{n+1}$, the cycle is self-intersecting. Hence $v_1 \neq v_{n+1}$. So, for some $2 < m < n$, there is an edge $e_m : v_m \to v_1$, which lies in the interior of C and completes a face in the quiver. This edge e_m separates C into a face (consisting of vertices $v_1, v_2, \ldots, v_m$) and a smaller cycle, and hence C is differentiable with respect to e_m.



$\square$

Lemma 4.4.3. *Any non-self-intersecting cycle C can be written as multiplication of a face and a cycle in the Jacobian algebra.*

Proof. We use induction on k, the number of faces contained inside the cycle. If a cycle contains only one face, then by the lemma above, it belongs to the Jacobian ideal.

Let C be a cycle containing k faces, $F_1, F_2, \dots, F_k$, with $n_1, n_2, \dots, n_k$ number of edges respectively. We know that C contains all vertices of at least one of these k faces. Let that face be F_i, which starts and ends at the vertex e_1. So C contains $n_i - 1$ edges of F_i. Let $p_1 : e_1 \to e_{n_i}$ be the path consisting of $n_i - 1$ edges of F_i that also belong to C. As C is a cycle, there exists a path, say $p_2 : e_{n_i} \to e_1$ such that $C = p_1 p_2$. Let e be the n_ith edge of the face F_i such that $F_i = ep_1$. Now there exists a path $p_1' : e_1 \to e_{n_i}$ such that $\partial_e(S) = p_1 - p_1'$, which implies that $p_1 = p_1'$ in the Jacobian algebra. Hence $C = p_1 p_2 = p_1' p_2$, reducing the number of faces inside C to $k - 2$.

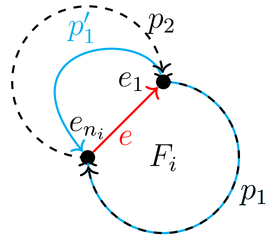


Figure 4.15: Figure for Lemma 4.4.3

□

This shows that every cycle of a quiver $Q^{u,e}$ belongs to the Jacobian ideal corresponding to the superpotential S. Hence S is a rigid potential.

The above result is also true for the superpotential of a quiver $Q^{e,v}$. This is because the identity element e does not contribute anything to the quiver. It only decides whether the indices for the word v are positive or negative. This does not change the quiver itself. Following Theorem 4.3.2, we see that a quiver $Q^{u,v}$ can be obtained by gluing $Q^{u,e}$ and $Q^{e,v}$. Since each cycle in the resulting quiver $Q^{u,v}$ is oriented as shown in Theorem 4.3.3, the superpotential of the quiver contains all its faces. Also the superpotential of each of the quivers $Q^{u,e}$ and $Q^{e,v}$ is rigid, which implies the following:

Corollary 4.4.1. *The superpotential S of the quiver $Q^{u,v}$ is a rigid potential.*

4.5 Future interests

The quivers from double Bruhat cells have many frozen vertices and hence, in order to categorify these cluster algebras with frozen variables, one has to find an appropriate model similar to Pressland's model in [Pre17]. The methods in Pressland's work cannot be applied directly as the number of frozen vertices do not exactly match in these two cases. The quivers defined in [Pre17] have the number of frozen vertices double the number of mutable vertices, whereas the number of frozen and mutable in our case do not have this relation. This is one of the reasons we add the arrows in the quiver between frozen vertices. The quivers in Pressland's setting have arrows connecting frozen vertices following a certain rule. This rule seems to match the current rule we have in our setting. The cycles obtained in this way are also oriented.

The frozen Jacobian algebras can be defined for the BFZ quivers with superpotentials, as defined in [Pre17] or in section 3.4 of this thesis. This algebra does not include derivatives with respect to boundary arrows. In the setting of this thesis, we need to include the cyclic derivatives with respect to the edges between frozen vertices (or the boundary arrows), i.e., for a quiver Q, a superpotential W, and a frozen subquiver F,

$$A = J(Q, F, W) = \mathbb{C}(Q)/\langle \partial_a W | a \in Q_1 \rangle.$$

The boundary algebra B will be eAe where the idempotent element e is the sum of idempotents at every frozen vertex. In order to apply Pressland's result, the first step would be to show that the frozen Jacobian algebra is bimodule internally 3-Calabi–Yau. For a cluster tilting object whose endomorphism algebra is isomorphic to the Jacobian algebra $A(Q, W)$, the mutation of the titling object is not always isomorphic to the Jacobian algebra of a mutation of the potential. In [BIRS11], the necessary conditions for the above statement to hold are given. We would like to study those conditions in this case of BFZ quivers and the superpotentials.

Moreover, the goal of this project is not just to categorify the cluster structure, but also to recognize the cluster monomials as the elements of the Luszting's dual canonical basis. Geiss, Leclerc and Schröer were able to do this using their categorification in type ADE. Even though the result is not exactly the original motivation for the definition of cluster algebras, it brings us closer to it since cluster monomials are shown to be inside the dual of the required basis.

References

[Ami09] C. Amiot, Cluster categories for algebras of global dimension 2 and quivers with potential. *Ann. Inst. Fourier (Grenoble)* 59 (2009), no. 6, 252-52590. MR2640929 (2011c:16026)

[BKM16] K. Baur, Alastair D. King, and R. J. Marsh. Dimer models and cluster categories of Grassmannians. *Proc. Lond. Math. Soc. (3)*, 113(2):213–260, 2016.

[BFZ05] A. Berenstein, S.Fomin, and A. Zelevinsky. Cluster algebras. III. Upper bounds and double Bruhat cells. *Duke Math. J.*, 126(1):1–52, 2005.

[Boc16] R. Bocklandt. A dimer ABC. *Bull. Lond. Math. Soc.*, 48(3):387–451, 2016.

[BIRS09] A. B. Buan, O. Iyama, I. Reiten, and J. Scott. Cluster structures for 2-Calabi–Yau categories and unipotent groups. *Compositio Mathematica*, 145(4):1035–1079, 007 2009.

[BIRS11] A. B. Buan, O. Iyama, I. Reiten, and D. Smith. Mutation of cluster-tilting objects and potentials. *Amer. J. Math.*, 133(4):835–887, 2011.

[BMRRT] A. B. Buan, R. J. Marsh, M. Reineke, I. Reiten, G. Todorov. *Tilting theory and cluster combinatorics*. Adv. Math. 204 (2006), no. 2, 572-618.

[DI16] L. Demonet and O. Iyama, Lifting preprojective algebras to orders and categorifying partial flag varieties *Algebra Number Theory* 10 (2016), no. 7, 15271579. MR3554240.

[DL16] L. Demonet and X. Luo. Ice quivers with potential associated with triangulations and Cohen–Macaulay modules over orders. *Trans. Amer. Math. Soc.* 368 (2016), no. 6, 42574293. MR3453371

[DWZ08] H. Derksen, J. Weyman, and A. Zelevinsky. Quivers with potentials and their representations. I. Mutations. *Selecta Math. (N.S.)*, 14(1):59–119, 2008.

[DWZ10] H. Derksen, J. Weyman, A. Zelevinsky. Quivers with potentials and their representations II: Applications to cluster algebras. *J. Amer. Math. Soc.* 23 (2010), no. 3, 749-790.

[Fis61] M. E. Fisher. Statistical mechanics of dimers on a plane lattice. *Physical Review*, 124(6):1664, 1961.

[FT61] M. E. Fisher. and H. N. V. Temperley. Dimer problem in statistical mechanics-an exact result. *Philosophical Magazine*, 6(68):10611063, 1961.

[FZ02] S. Fomin and A. Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529 (electronic), 2002.

[FZ07] S. Fomin and A. Zelevinsky. Cluster algebras. IV. Coefficients *Compos. Math.* 143 (2007), no. 1, 112 64. MR2295199 (2008d:16049)

[GLS08] C. Geiss, B. Leclerc, and J. Schröer. Partial flag varieties and preprojective algebras. *Ann. Inst. Fourier (Grenoble)*, 58(3):825–876, 2008.

[GLS06] C. Geiss, B. Leclerc, J. Schröer. Rigid modules over preprojective algebras. *Invent. Math.* 165 (2006), no. 3, 589-632.

[HV07] A. Hanany and D. Vegh. Quivers, tilings, branes and rhombi. *J. High Energy Phys.*, (10):029, 35, 2007.

[IY08] O. Iyama and Y. Yoshino. Mutation in triangulated categories and rigid Cohen-Macaulay modules. *Invent. Math.* 172 (2008), no. 1, 117-168.

[JKS16] B. T. Jensen, A. D. King, and X. Su. A categorification of Grassmannian cluster algebras. *Proc. Lond. Math. Soc. (3)*, 113(2):185–212, 2016.

[Kas67] P. W. Kasteleyn. Graph theory and crystal physics. *Graph Theory and Theoretical Physics*, 43110. Academic Press, London, 1967.

[L00] G. Lusztig. Semicanonical bases arising from enveloping algebras. *Adv. Math.* 151 (2000), no. 2, 129–139.

[Pos06] A. Postnikov. Total positivity, Grassmannians, and networks. arXiv:math/0609764v1 [math.CO] Sep 2006.

[Pre17] M. Pressland. Internally Calabi-Yau algebras and cluster-tilting objects. *Math. Z.,* 287(1-2):555-585, 2017.

[Sco06] J. S. Scott. Grassmannians and cluster algebras, *Proc. London Math. Soc.*, (3) 92 (2006) 345–380.

Vita

Maitreyee Kulkarni was born in Pune, India. She completed her undergraduate studies at Sir Parshurambhau College, Pune, earning a Bachelor of Science degree in Mathematics in May 2009. Then she received her Master's degree in Mathematics from University of Pune in 2011. She began graduate studies at Louisiana State University in August 2012. She earned a Master of Science degree in Mathematics from Louisiana State University in December 2013. She is currently a candidate for the Doctor of Philosophy degree in Mathematics to be awarded in August 2018.