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The Congruence Extension Property for Algebraic Semigroups.

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THE CONGRUENCE EXTENSION PROPERTY FOR ALGEBRAIC SEMIGROUPS

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ABSTRACT

A semigroup has the congruence extension property (CEP) provided that each congruence on each subsemigroup can be extended to the semigroup. This property, along with the ideal extension property (IEP) and the group congruence extension property (GCEP) are studied in this work. Whether each of these properties is productive, hereditary or preserved by homomorphisms is determined (except for the homomorphic property for CEP). Conditions under which the homomorphic image of a semigroup with CEP has CEP are established.

Disruptive element and disruptive pair theory is developed and shown to be an important concept in the study of CEP and IEP.

Properties of semigroups with CEP are sought. It is proved that each semigroup with CEP has index less than four, and that this is both necessary and sufficient for a cyclic semigroup to have CEP. It is established that a group has CEP if and only if it is a torsion group with GCEP. In particular, an abelian group has CEP if and only if it is a torsion group.

Introduction

Motivation for this work was provided by the 1972 paper of Albert Stralka regarding the extension of congruences in semigroups. While extensive literature on group congruence extensions has existed for many years, results on semigroup congruence extensions remain limited. Recent results on algebra extension which appear the papers of Biró, Kiss, and Pálfy and of Day have had a significant influence on this work.

The objectives of this research are:

(1) Provide useful methods of detecting whether a semigroup has the congruence extension property (CEP); and

(2) Find conditions under which the homomorphic image of a semigroup with CEP also has CEP.

Groups and cyclic semigroups are considered first, since these are important atoms in all semigroups. These are studied in Chapters 2 and 3.

A property which is similar to the congruence extension property in an algebra and called the ideal extension property (IEP) is studied in Chapter 4.

In Chapter 5, a theory pertaining to the relations between the congruences on a semigroup and those on its homomorphic images is presented. Special tools whose names are borrowed from terms in category theory are developed and studied. Based on this analysis, we have derived results which give conditions for a homomorphic image of semigroup with the congruence extension property (CEP) to also have CEP.

Chapter 6 deals with methods to determine whether a semigroup has CEP.

These are mostly useful to establish that a semigroup does not have CEP.

Examples are the primary concern of the results of Chapter 7. This development is rendered in the spirit of "new examples from old". Many of the examples presented prior to this chapter are computer generated examples of finite semigroups. This chapter opens a path to infinite examples. It is limited in the variety of types of examples however.

Chapter 8 presents some useful diagrams to summarize the results of the other chapters. It also lists some questions which remain open to future research on the congruence extension property for algebraic semigroups.

CHAPTER 1

BASIC CONCEPTS

The primary purpose of this first chapter is to establish the notions that will be used in the rest of this work, to recall previous work done in related areas, and to present a few general results that will be employed in later chapters.

A **congruence** σ on a semigroup S is defined to be an equivalence relation on S which is compatible with the semigroup operation, i.e.,

(1) $\Delta_S \subseteq \sigma$, where $\Delta_S = \{(x,x) : x \in S\}$ is the diagonal of S (σ is reflexive);

(2) $\sigma^{-1} = \sigma$, where $\sigma^{-1} = \{(b,a) : (a,b) \in \sigma\}$ (σ is symmetric);

(3) $\sigma \circ \sigma \subseteq \sigma$, where $\sigma \circ \sigma$ is the composition of σ with itself (σ is transitive); and

(4) If $(a,b) \in \sigma$ and $(c,d) \in \sigma$, then $(ac, bd) \in \sigma$ (σ is compatible).

It is well-known that if condition (1) and (3) are present, then condition (4) is equivalent to the statement: If $(a,b) \in \sigma$ and $s \in S$, then $(as, bs) \in \sigma$ and $(sa, sb) \in \sigma$ (see [Clifford and Preston, 1963]).

If S is a semigroup, T is a subsemigroup of S and σ is a congruence on T, then a congruence $\overline{\sigma}$ on a subsemigroup Q of S is called an **extension** of σ provided $T \subseteq Q$ and $\overline{\sigma} \cap (T \times T) = \sigma$.

A semigroup S is said to have the **congruence extension property** (CEP) provided that for each subsemigroup T of S and each congruence σ on

T, σ has an extension to S.

The following example illustrates that an extension of a congruence need not be unique.

1.1 Example. *Congruence extensions are not always unique.* Let $S = \{1,2,3,4,5\}$ with multiplication defined by the Cayley table:

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{array}$$

Let T be the subsemigroup $\{3,4,5\}$ and let $\sigma = \{(4,5),(5,4)\} \cup \Delta_T$. Then σ is a congruence on T and each of the following is an extension of σ to S:

$$\overline{\sigma} = \{(4,5),(5,4),(1,3),(3,1)\} \cup \Delta_S$$

and

$$\sigma^* = \{(4,5),(5,4),(1,2),(2,1)\} \cup \Delta_S.$$

If S is any semigroup, observe that the diagonal $\Delta_S = \{(s,s) : s \in S\}$ is a minimal congruence on S and $S \times S$ is a maximal congruence on S.

For a semigroup S, let $\mathcal{C}(S) = \{\sigma : \sigma \text{ is a congruence on } S\}$. Then $\bigcap\{\sigma : \sigma \in \mathcal{C}(S)\} = \Delta_S$ and $\bigcup\{\sigma : \sigma \in \mathcal{C}(S)\} = S \times S$.

Observe that a finite semigroup S whose order is less than or equal to three must have the congruence extension property, since the only congruences on a proper subsemigroup T of S are $T \times T$ and Δ_T, and thus $(S \times S) \cap (T \times T) = T \times T$ and $\Delta_S \cap (T \times T) = \Delta_T$.

If S is a semigroup and ρ is a subset of $S \times S$, then the congruence generated by ρ on S (denoted $\langle \rho \rangle_S$) is the minimal congruence on S containing ρ. It is immediate that $\langle \rho \rangle_S = \bigcap\{\sigma : \sigma \in C(S) \text{ and } \rho \subseteq \sigma\}$.

When no confusion seems likely, we write $\langle \rho \rangle$ for $\langle \rho \rangle_S$.

1.2 Proposition. *Let S be a semigroup. Then S has the congruence extension property (CEP) if and only if each subsemigroup of S has the congruence extension property.*

Proof. If each subsemigroup of S has CEP, then since S is a subsemigroup of itself, then S has CEP.

On the other hand, suppose that S has CEP. Let T a subsemigroup of S, let K be a subsemigroup of T, and let σ be a congruence on K. Since K is a subsemigroup of S and S has CEP, there is an extension σ^* of σ to S, i.e., $\sigma^* \cap (K \times K) = \sigma$. Let $\overline{\sigma} = \sigma^* \cap (T \times T)$. Then $\overline{\sigma}$ is a congruence on T, $\overline{\sigma} \cap (K \times K) = \sigma^* \cap (K \times K) = \sigma$, and hence $\overline{\sigma}$ is an extension of σ to T. We conclude that T has CEP. ∎

We will use $\mathbb{N}$ throughout to denote the set of all positive integers and for $n \in \mathbb{N}$, $\rho^{(n)}$ denotes the n-fold composition of a relation ρ.

The following result concerning congruences generated by a given relation can be established using results found in [Clifford and Preston, 1963]. A proof is presented here from basic concepts to illustrate some of the techniques to be employed in later arguments.

1.3 Proposition. *Let S be a semigroup, T a subsemigroup of S, σ a* *congruence on T, $\delta = \sigma \cup \Delta_S$, and let $\rho = \{(xay, xby) : (a,b) \in \delta, x, y \in S^1\}$. Then the congruence generated by σ is given by*

$$\langle\sigma\rangle_S = \bigcup_{n\in\mathbb{N}} \rho^{(n)}.$$

Proof. Let $\sigma^* = \bigcup_{n\in\mathbb{N}} \rho^{(n)}$. We first want to prove that σ^* is a congruence on S.

Since $\Delta_S \subseteq \delta \subseteq \rho \subseteq \sigma^*$, we have that σ^* is reflexive.

In view of the fact that σ is symmetric and Δ_S is symmetric, we see that δ is symmetric. It follows that ρ is symmetric. Let $n \in \mathbb{N}$. To see that $\rho^{(n)}$ is symmetric, let $(c,d) \in \rho^{(n)}$. Then there exist elements $e_1, e_2, \ldots, e_n = d$ in S such that $(c, e_1), (e_1, e_2), \ldots, (e_{n-1}, d)$ are in ρ. Since ρ is symmetric, we have that $(e_1, c), (e_2, e_1), \ldots, (d, e_{n-1})$ are in ρ. Reversing the order of this sequence we have that $(d, e_{n-1}), \ldots, (e_2, e_1), (e_1, c)$ are in ρ, and we see that $(d,c) \in \rho^{(n)}$ and $\rho^{(n)}$ is symmetric. From this it follows that σ^* is symmetric.

To see that σ^* is transitive, let $(a,b), (b,c) \in \sigma^*$. Then $(a,b) \in \rho^{(n)}$ and $(b,c) \in \rho^{(m)}$ for some $n, m \in \mathbb{N}$. Thus $(a,c) \in \rho^{(n)} \circ \rho^{(m)} = \rho^{(m+n)} \subseteq \sigma^*$, and we conclude that σ^* is transitive.

To complete the argument that σ^* is a congruence, we need to show that it is compatible with the multiplication on S. For this purpose, let $(c,d) \in \sigma^*$ and let $s \in S$. Then $(c,d) \in \rho^{(m)}$ for some $m \in \mathbb{N}$. Thus there exists a sequence $e_1, e_2, \ldots, e_m = d$ in S such that $(c, e_1), (e_1, e_2), \ldots, (e_{m-1}, d)$ are in ρ. From the definition of ρ, we have that $(sc, se_1), (se_1, se_2), \ldots, (se_{m-1}, sd)$ are in ρ, and hence $(sc, sd) \in \rho \subseteq \sigma^*$. Similarly, $(cs, ds) \in \sigma^*$. We conclude that σ^* is a congruence on S.

Observe that $\sigma \subseteq \delta \subseteq \rho \subseteq \sigma^*$.

Let β be a congruence on S such that $\sigma \subseteq \beta$. We next show that $\sigma^* \subseteq \beta$. Now $\Delta_S \subseteq \beta$ and hence $\delta = \sigma \cup \Delta_S \subseteq \beta$. Since β is compatible with multiplication on S, $\sigma \subseteq \beta$, and from the definition of ρ, we have $\rho \subseteq \beta$. For each $n \in \mathbb{N}$, we have (in view of the transitivity of β) that $\rho^{(n)} \subseteq \beta$. It follows that $\sigma^* = \bigcup_{n \in \mathbb{N}} \rho^{(n)} \subseteq \beta$.

Now $\langle \sigma \rangle_S$ is defined to be the minimal congruence on S containing σ. Since $\langle \sigma \rangle_S$ contains σ, we have that $\sigma^* \subseteq \langle \sigma \rangle_S$ from the previous paragraph. From the minimality of $\langle \sigma \rangle_S$, we conclude that $\langle \sigma \rangle_S = \sigma^*$. ∎

1.4 Proposition. *Let S be a semigroup, T a subsemigroup of S, and let σ be a congruence on T. Then σ has an extension to S if and only if $\langle \sigma \rangle_S$ is an extension of σ to S.*

Proof. Suppose that σ is a congruence on T and σ has an extension to S Let $\overline{\sigma}$ be an extension of σ to S. Then $\overline{\sigma}$ is a congruence on S containing σ, and hence $\langle \sigma \rangle_S \subseteq \overline{\sigma}$. Now $\langle \sigma \rangle_S \cap (T \times T) \subseteq \overline{\sigma} \cap (T \times T) = \sigma$. Since $\sigma \subseteq (T \times T)$ and $\sigma \subseteq \langle \sigma \rangle_S$, we have that $\sigma \subseteq \langle \sigma \rangle_S \cap (T \times T)$. We conclude that $\langle \sigma \rangle_S \cap (T \times T) = \sigma$.

The converse is immediate. ∎

1.5 Corollary. *Let S be a semigroup. Then S has the congruence extension property (CEP) if and only if for each subsemigroup T of S and each congruence σ on T, $\langle \sigma \rangle_S \cap (T \times T) = \sigma$.*

1.6 Corollary. *A semigroup S has the congruence extension property if and only if for each subsemigroup T of S and each congruence σ on T,*

$\langle\sigma\rangle_S \cap (T \times T) \subseteq \sigma$.

Proof. This follows from the fact that the reverse containment is always valid and from 1.5. ∎

1.7 Proposition. *Let S be a semigroup which is a union of a tower of subsemigroups $\{S_n : n \in \mathbb{N}\}$. Then S has the congruence extension property (CEP) if and only if each S_n has the congruence extension property.*

Proof. If S has CEP, then from 1.2, we see that each S_n has CEP.

On the other hand, suppose that S_n has CEP for each $n \in \mathbb{N}$. Let T be a subsemigroup of S and let σ be a congruence on T. Let $T_n = T \cap S_n$ for each $n \in \mathbb{N}$. Then $T = T \cap S = T \cap \bigcup_{n\in\mathbb{N}} S_n = \bigcup_{n\in\mathbb{N}} T_n$. Now, for each $n \in \mathbb{N}$, T_n is a subsemigroup of S_n. Moreover, since $S_n \subseteq S_{n+1}$, $T \cap S_n \subseteq T \cap S_{n+1}$ and hence $T_n \subseteq T_{n+1}$ for each $n \in \mathbb{N}$. Let $\sigma_n = \sigma \cap (T_n \times T_n)$ for each $n \in \mathbb{N}$. Then σ_n is a congruence on T_n. Since $T_n \subseteq T_{n+1}$, $\sigma \cap (T_n \times T_n) \subseteq \sigma \cap (T_{n+1} \times T_{n+1})$ and thus $\sigma_n \subseteq \sigma_{n+1}$.

Since T_n is a subsemigroup of S_n and S_n has CEP, σ_n has an extension $\overline{\sigma}_n$ to S_n. Without loss of generalization, we can assume that this extension is the minimal extension,i.e., $\overline{\sigma}_n = \langle\sigma_n\rangle_{S_n}$. Note that $\sigma_n \subseteq \sigma_{n+1}$ and $S_n \subseteq S_{n+1}$ for each $n \in \mathbb{N}$ and hence $\overline{\sigma}_n = \langle\sigma_n\rangle_{S_n} \subseteq \langle\sigma_n\rangle_{S_{n+1}} \subseteq \langle\sigma_{n+1}\rangle_{S_{n+1}} = \overline{\sigma}_{n+1}$. Let $\overline{\sigma} = \bigcup_{n\in\mathbb{N}} \overline{\sigma}_n$.

We claim that $\overline{\sigma}$ is a congruence on S. It is clear that $\overline{\sigma}$ is reflexive and symmetric. To show transitivity, suppose that (a,b) and (b,c) are in $\overline{\sigma}$. Then $(a,b) \in \overline{\sigma}_m$ and $(b,c) \in \overline{\sigma}_n$ for some $m,n \in \mathbb{N}$. We can assume that $m \leq n$. Thus $(a,b) \in \overline{\sigma}_m \subseteq \overline{\sigma}_n$, and hence (a,b) and (b,c) are both in $\overline{\sigma}_n$. It follows that $(a,c) \in \overline{\sigma}_n \subseteq \overline{\sigma}$ and $\overline{\sigma}$ is transitive.

To complete the argument that $\overline{\sigma}$ is a congruence, it remains to show that $\overline{\sigma}$ is compatible with the multiplication on S. For this purpose, let $(a,b) \in \overline{\sigma}$ and let $s \in S$. Then $(a,b) \in \overline{\sigma}_m$ and $s \in S_n$ for some $m, n \in \mathbb{N}$. If $m = n$, then $(sa, sb) \in \sigma_m \subseteq \overline{\sigma}$. If $m < n$, then $(a,b) \in \overline{\sigma}_m \subseteq \overline{\sigma}_n$ and hence $(sa, sb) \in \overline{\sigma}_n \subseteq \overline{\sigma}$. If $n < m$, then $s \in S_n \subseteq S_m$ and $(sa, sb) \in \overline{\sigma}_n \subseteq \overline{\sigma}$. In any case, $(sa, sb) \in \overline{\sigma}$. Similarly, $(as, bs) \in \overline{\sigma}$. We conclude that $\overline{\sigma}$ is a congruence on S.

Observe that $\sigma = \bigcup\limits_{n \in \mathbb{N}} \sigma_n \subseteq \bigcup\limits_{n \in \mathbb{N}} \overline{\sigma}_n = \overline{\sigma}$.

To complete the proof that $\overline{\sigma}$ is an extension of σ to S, it remains to demonstrate that $\overline{\sigma} \cap (T \times T) \subseteq \sigma$. Let $(x,y) \in \overline{\sigma} \cap (T \times T)$. Then $(x,y) \in \overline{\sigma}_m$ and $(x,y) \in (T_n \times T_n)$ for some $m, n \in \mathbb{N}$. If $m = n$, then $(x,y) \in \overline{\sigma}_m \cap (T_m \times T_m) = \sigma_m$. If $m < n$, then $(x,y) \in \overline{\sigma}_m \subseteq \sigma_n$ and $(x,y) \in (T_n \times T_n)$. Thus $(x,y) \in \overline{\sigma}_n \cap (T_n \times T_n) = \sigma_n \subseteq \sigma$. If $n < m$, then $(x,y) \in \overline{\sigma}_m$ and $(x,y) \in (T_n \times T_n) \subseteq (T_m \times T_m)$. Thus $(x,y) \in \overline{\sigma}_m \cap (T_m \times T_m) = \sigma_m \subseteq \sigma$.

Thus $\overline{\sigma}$ is an extension of σ to S, and we conclude that S has CEP. ∎

1.8 Lemma. *Let S be a semigroup which is a union of a tower of subsemigroups $\{S_\alpha : \alpha \in A\}$, let σ be a congruence on a subsemigroup T of $\bigcap\limits_{\alpha \in A} S_\alpha$ and let $\sigma_\alpha = \langle \sigma \rangle_{S_\alpha}$ for each $\alpha \in A$. Then $\sigma^* = \bigcup\limits_{\alpha \in A} \sigma_\alpha$ is a congruence on S.*

Proof. Note that $\sigma_\alpha \subseteq \sigma_\beta$ for $\alpha \leq \beta$, $\Delta_S = \bigcup\limits_{\alpha \in A} \Delta_{S_\alpha}$, and for each $\alpha \in A$, $\Delta_{S_\alpha} \subseteq \sigma_\alpha$. Thus $\Delta_S \subseteq \bigcup\limits_{\alpha \in A} \sigma_\alpha = \sigma^*$ and σ^* is reflexive.

From the fact that σ^* is a union of symmetric relations, it follows that σ^* is symmetric.

To see that σ^* is transitive, let (a,b) and (b,c) be in σ^*. Then $(a,b) \in \sigma_\beta$

for some $\beta \in A$, and $(b, c) \in \sigma_\gamma$ for some $\gamma \in A$. Without loss of generalization we can assume that $\beta \leq \gamma$. Then $\sigma_\beta \subseteq \sigma_\gamma$ and hence (a, b) and (b, c) are in σ_γ. Since σ_γ is transitive, we have that $(a, c) \in \sigma_\gamma \subseteq \sigma^*$. It follows that σ^* is transitive.

Let $(a, b) \in \sigma^*$ and let $s \in S$. Then $(a, b) \in \sigma_\alpha$ for some $\alpha \in A$ and $s \in S_\beta$ for $\beta \in A$. If $\alpha = \beta$, then $(sa, sb) \in \sigma_\alpha = \sigma_\beta \subseteq \sigma^*$. If $\alpha < \beta$, then $(a, b) \in \sigma_\alpha \subseteq \sigma_\beta$ and $(sa, sb) \in \sigma_\beta \subseteq \sigma^*$. If $\beta < \alpha$, then $s \in S_\beta \subseteq S_\alpha$ and $(sa, sb) \in \sigma_\alpha \subseteq \sigma^*$. In any case, σ^* is compatible with the multiplcation on S, and thus is a congruence on S. ▮

A congruence α on a semigroup S is called a **principal congruence** if α is generated by a single pair $(a, b) \in (S \times S)$.

Notation. If S is a semigroup and $(a, b) \in (S \times S)$, then the minimal congruence on S containing the pair (a, b) is denoted by $\alpha^S(a, b)$.

If S is a semigroup and $(a, b) \in (S \times S)$, then $\alpha^S(a, b)$ can be constructed as follows:

$$\alpha_0 = \{(a, b), (b, a)\}$$

$$\alpha_1 = \alpha_0 \cup \Delta_S$$

$$\alpha_2 = \{(xcy, xdy) : (c, d) \in \alpha_1,\ x, y \in S^1\}$$

Then

$$\alpha^S(a, b) = \bigcup_{n \in \mathbb{N}} \alpha_2^{(n)}.$$

A semigroup S is said to have the **principal congruence extension**

property (PCEP) provided that for each $(a, b) \in (S \times S)$ and each subsemigroup T of S, $\alpha^S(a, b) \cap (T \times T) = \alpha^T(a, b)$.

The existence of a maximal extension of a congruence for a subsemigroup T of a semigroup S is essential to our proof of the equivalence of the congruence extension property and the principal congruence extension property for semigroups.

Let T be a subsemigroup of a semigroup S and let σ be a congruence on T. A pair (T^*, σ^*) is said to be a **maximal extension** of σ provided that T^* is a subsemigroup of S containing T and σ^* is an extension of σ to T^* which does not extend to a subsemigroup of S which contains T^* as a proper subsemigroup.

1.9 Lemma. *Let S be a semigroup, T a subsemigroup of S, and let σ be a congruence on T. Then σ has a maximal extension.*

Proof. Let $\mathcal{P} = \{(P, \pi): P$ is a subsemigroup of S containing T and π is an extension of σ to $P\}$. Then $(T, \sigma) \in \mathcal{P}$ and hence $\mathcal{P} \neq \emptyset$. Define $\leq$ on $\mathcal{P}$ by $(P, \pi) \leq (R, \rho)$ provided $P \subseteq R$ and ρ is an extension of π to R. Then $\leq$ is a partial order on $\mathcal{P}$. From the Hausdorff Maximality Principal, $\mathcal{P}$ contains a maximal chain $\mathcal{C}$. Let $T^* = \bigcup\{P: (P, \pi) \in \mathcal{C}\}$ and let $\sigma^* = \bigcup\{\langle\sigma\rangle_P: (P, \pi) \in \mathcal{C}\}$. Then σ^* is a congruence on T^* by Lemma 1.8, and since $\langle\sigma\rangle_P$ is an extension of σ by Lemma 1.4, we have that σ^* is an extension of σ.

To see that (T^*, σ^*) is a maximal extension of σ, suppose that σ^* has an extension μ to a subsemigroup M of S containing T^*. Then μ is an extension of σ to M and hence $(M, \mu) \in \mathcal{C}$. In view of the maximality of $\mathcal{C}$, we have that

$M = T^*$, and conclude that (T^*, σ^*) is a maximal extension of σ. ∎

The following example illustrates that a maximal extension need not be unique.

1.10 Example. *A maximal extension congruence is not always unique.* Let S be the semigroup of example 1.1, let $T = \{1,2,4,5\}$, and let $\sigma = \{(1,2),(2,1),(4,5),(5,4)\} \cup \Delta_T$. Then the congruence $\overline{\sigma}$ whose classes are $\{1,2,3\}$ and $\{4,5\}$ are both maximal extensions of σ to S. In view of the fact that S has CEP, each maximal extension of σ must have S as its semigroup.

If $\mathcal{P}$ is a class of relations on a semigroup S, then $\bigvee \mathcal{P}$ denotes the congruence on S generated by $\bigcup \mathcal{P}$.

The next result was presented for algebraic varieties in [Day, 1970]. Its adaptation to semigroups and the maximality argument in the proof that follows is new.

1.11 Theorem. *A semigroup S has the congruence extension property (CEP) if and only if S has the principal congruence extension property (PCEP).*

Proof. Suppose that S has PCEP, let T be a subsemigroup of S and let σ be a congruence on T. In view of 1.9, we see that σ has a maximal extension (M, σ^*). We will produce an extension of σ by producing an extension of σ^* to S.

Let $\mathcal{P} = \{\rho : \rho$ is a congruence on S and $\rho \cap (M \times M) \subseteq \sigma^*\}$. Now $\mathcal{P}$ is not empty, since $\Delta_S \in \mathcal{P}$.

Let $\rho \in \mathcal{P}$, and let $D = \{b \in S: (b,m) \in \rho$ for some $m \in M\}$. Then D is a subsemigroup of S.

We claim that $D = M$. Now $\Delta_M \subseteq \Delta_S \subseteq \rho$ and hence $M \subseteq D$. Define $\gamma = \{(p,q) \in (D \times D): (p,m) \in \rho$ and $(q,m) \in \rho$ for some $m \in M\}$.

We will show that γ is a congruence on D. It is immediate that γ is reflexive and symmetric. To see that γ is transitive let $(p,q),(q,r) \in \gamma$. Then $(p,m),(q,m) \in \rho$ for some $m \in M$. Also $(q,n),(r,n) \in \rho$ for some $n \in M$. We obtain that $(q,m),(n,q) \in \rho$ and hence $(m,q),(q,n) \in \rho$. Since ρ is transitive, we have that $(m,n) \in \rho$. Thus, $(p,m),(m,n) \in \rho$ and $(r,n) \in \rho$. From $(p,n),(r,n) \in \rho$ and the definition of γ we conclude that $(p,r) \in \gamma$ and γ is transitive. Finally, to see that γ is compatible with multiplication on D, let $(p,q) \in \gamma$ and let $d \in D$. Then $(p,m),(q,m) \in \rho$ for some $m \in M$ and $(dp,dm),(dq,dm) \in \rho$. It follows that $(dp,dq) \in \gamma$, and γ is a congruence on D.

Now if $(a,b) \in \rho \cap (M \times M)$, then $(a,b) \in \rho$. Since $(b,b) \in \rho$, we have that $(a,b) \in \gamma$. It follows that γ is an extension of σ^* to D. In view of the maximality of (M,σ^*), we see that $M = D$.

Let $\alpha \in \mathcal{P}$ and let $\beta \in \mathcal{P}$. We claim that $\alpha \vee \beta \in \mathcal{P}$. Let $(a,b) \in (\alpha \vee \beta) \cap (M \times M)$. We want to show that $(a,b) \in \sigma^*$. Since $(a,b) \in \alpha \vee \beta$, there exists a sequence $a = x_0, x_1, x_2, \ldots, x_n = b$ in S such that $(a,x_1) \in \alpha$, $(x_1,x_2) \in \beta, \ldots$. We can assume that $(x_{n-1},b) \in \beta$. Now $a \in M$ and $(a,x_1) \in \alpha$. Thus $x_1 \in M$, since $\alpha \in \mathcal{P}$ and $M = D$. We also have $(x_2,x_1) \in \beta$, since $(x_1,x_2) \in \beta$, and from $\beta \in \mathcal{P}$, we have $x_2 \in M$. Continuing recursively, we have that $x_i \in M$ for $i = 0,1,\ldots,n$. Hence, $(a,b) \in (\alpha \cap (M \times M)) \vee (\beta \cap (M \times M)) \subseteq \sigma^*$. We conclude that $(\alpha \vee \beta) \in \mathcal{P}$.

We next claim that $\bigvee \mathcal{P} \in \mathcal{P}$. Observe that $\bigvee \mathcal{P} \cap (M \times M) = \bigcup\{\rho \cap (M \times M): \rho \in \mathcal{P}\}$. Let $(a,b) \in (\bigvee \mathcal{P}) \cap (M \times M)$. We want to show that $(a,b) \in \sigma^*$. Since $(a,b) \in \bigvee \mathcal{P}$, there exists a sequence $a = x_0, x_1, x_2, \ldots, x_m = b$ in S such that $(a,x_1) \in \alpha_1$, $(x_1,x_2) \in \alpha_2$, $\ldots$, $(x_{m-1},b) \in \alpha_m$ for some $\alpha_i \in \mathcal{P}$ for $i = 0,1,2,\ldots,m$. From $(a,x_1) \in \alpha_1$ and $(x_1,x_2) \in \alpha_2$, we have $(a,x_2) \in (\alpha_1 \vee \alpha_2) \in \mathcal{P}$, and hence $(a,x_2) \in \sigma^*$. From $(a,x_2) \in (\alpha_1 \vee \alpha_2)$ and $(x_2,x_3) \in \alpha_3$, we have $(a,x_3) \in (\alpha_1 \vee \alpha_2 \vee \alpha_3) \in \mathcal{P}$, and hence $(a,x_3) \in \sigma^*$. By recursion, we obtain that $(a,b) \in \sigma^*$, and conclude that $\bigvee \mathcal{P} \in \mathcal{P}$.

Let $\delta = \bigvee \mathcal{P}$. We claim that $\delta \cap (M \times M) = \sigma^*$. To see this, first observe that $\delta \in \mathcal{P}$ from the preceeding paragraph. Thus $\delta \cap (M \times M) \subseteq \sigma^*$. Let $(a,b) \in \sigma^*$. Then $\alpha^S(a,b) \cap (M \times M) = \alpha^M(a,b) \subseteq \sigma^*$. Thus $\alpha^S(a,b) \in \mathcal{P}$ and $(a,b) \in \bigvee \mathcal{P} = \delta$. It follows that $\sigma^* \subseteq \delta$ and $\delta \cap (M \times M) = \sigma^*$.

We conclude that S has CEP.

The converse that CEP implies PCEP is immediate. ∎

1.12 Proposition. *Let S be a semigroup and T a subsemigroup of S such that $S \backslash T$ (the complement of T in S) is an ideal of T. Then every congruence on T can be extended to S.*

Proof. In view of 1.11, it suffices to show that each principal congruence on T extends to S. Let $(a,b) \in (T \times T)$ and let $(c,d) \in \alpha^S(a,b) \cap (T \times T)$. Let $(c,d) = (s_1 a s_2, s_3 b s_4)$ [or $(s_1 b s_2, s_1 a s_4)$ or $(s_1 a s_2, s_3 a s_3)$ or $(s_1 b s_2, s_3 b s_4)$] for some $s_1, s_2, s_3, s_4 \in S$. Since $S \backslash T$ is an ideal of S, $s_i \in T$ for $i = 1,2,3,4$. It follows that $(c,d) \in \alpha^T(a,b)$ and $\alpha^S(a,b)$ is an extension of $\alpha^T(a,b)$. ∎

Observe that if S, T, and σ are as in 1.12, then $\overline{\sigma} = \sigma \cup [(S \backslash T) \times (S \backslash T)]$

is an extension of σ to S.

1.13 Corollary. *Let S be a monoid and σ a congruence on the group of units $H(1)$ of S. Then σ has an extension to S.*

Proof. This follows immediately from 1.12 and the known fact that $S \backslash H(1)$ is an ideal of S. ∎

We adopt the following standard convention: If S is a semigroup, then $S^1 = S$ if S is a monoid (has an identity), and $S^1 = S \cup \{1\}$ (S with an identity adjoined) otherwise.

1.14 Proposition. *A semigroup S has the congruence extension property (CEP) if and only if S^1 has the congruence extension property.*

Proof. Let S be a semigroup. If $S = S^1$, there is nothing to prove. Thus we assume that $S \neq S^1$.

Let us first see that if σ is a congruence on S, then σ has an extension to S^1. Define $\overline{\sigma} = \sigma \cup \{(1,1)\}$. Then $\overline{\sigma}$ is the desired extension.

If S^1 has CEP, then it follows immediately from 1.2 that S has CEP.

On the other hand, suppose that S has CEP. Let T be a subsemigroup of S^1. Now if $1 \notin T$, then T is a subsemigroup of S and in view of the fact that S has CEP, each congruence on T can be extended to S and hence to S^1 as above. We thus can assume that $1 \in T$.

Let $(a, b) \in (T \times T)$. To complete the proof (using 1.2), we need to show that $\alpha^{S^1} \cap (T \times T) \subseteq \alpha^T(a, b)$. If $a = b$, then $\alpha^{S^1} = \Delta_{S^1}$ and $\Delta_{S^1} \cap (T \times T) = \Delta_T \subseteq \alpha^T(a, b)$. We can thus assume that $a \neq b$. If $a \neq 1$ and $b \neq 1$, then

$\alpha^{S^1}(a,b) = \alpha^S(a,b) \cup \{(1,1)\}$ and $\alpha^{S^1} \cap (T \times T) = (\alpha^S(a,b) \cup \{(1,1)\} \cap (T \times T) = \alpha^T(a,b) \cup \{(1,1)\} = \alpha^T(a,b)$. In the case that $b = 1$ (or similarly that $a = 1$), let $(c,d) \in \alpha^{S^1}(a,1) \cap (T \times T)$. We consider three cases:

(1). If $d = 1$, then $c = a^n$ for some $n \in \mathbb{N}$ and hence $(c,d) = (c,1) = (a^n,1) \in \alpha^T(a,1)$.

(2). If $c = 1$, then the argument is dual to that of case (1).

(3) If $c \neq 1$ and $d \neq 1$, then $(c,d) \in \alpha^{S^1}(a,1) \cap (S \times S)$ (which is a congruence on S) and hence $(c,d) \in \alpha^T(a,1)$. ∎

A **semigroup variety** is a class of semigroups which is closed under subsemigroups, homomorphic images, and cartesian products. Much of the study of the congruence extension property to date has been restricted to varieties (see [Biró, Kiss, and Pálfy, 1977]). If we consider the class of semigroups with the congruence extension property, we have seen that it is closed under subsemigroups (1.2) and we will address the problem of homomorphic images later. However, this class is not closed under cartesian products as illustrated in the following example.

1.15 Example. *The congruence extension property (CEP) is not productive.* This is an example of a semigroup S which has CEP but $S \times S$ does not have CEP.

Let $S = \{1,2,3\}$ with multiplication defined by the Cayley table:

$$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{array}$$

Then S has CEP, since each order 3 semigroup must have CEP. Rename the elements of $S \times S$ according to the following scheme:

$$\begin{array}{lll} 1=(1,1) & 4=(1,2) & 7=(1,3) \\ 2=(2,1) & 5=(2,2) & 8=(2,3) \\ 3=(3,1) & 6=(3,2) & 9=(3,3) \end{array}$$

The Cayley table for $S \times S$ is

$$\begin{array}{ccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 4 & 4 & 4 \\ 1 & 1 & 2 & 1 & 1 & 2 & 4 & 4 & 5 \\ 1 & 2 & 3 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 4 & 4 & 4 & 7 & 7 & 7 \\ 1 & 1 & 2 & 4 & 4 & 5 & 7 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

Let $T = \{1,4,5,7,8\}$ and let $\sigma = \{(7,8),(8,7)\} \cup \Delta_T$. Then T is a subsemigroup of $S \times S$ and σ is a congruence on T. The congruence on $S \times S$ generated by σ is

$$\langle\sigma\rangle_{(S\times S)} = \{(1,2),(2,1),(4,5),(5,4),(7,8),(8,7)\} \cup \Delta_{(S\times S)}.$$

Now $\langle\sigma\rangle_{(S\times S)} \cap (T \times T) = \{(4,5),(5,4),(7,8),(8,7)\} \cup \Delta_T \neq \sigma$. It follows that $S \times S$ does not have CEP.

A group G is said to have the **group congruence extension property** (GCEP) provided that for each subgroup H of G and each congruence σ on H, there exists a extension of σ to G.

This property will be discussed in detail in a later chaper.

1.16 Notes. The following is a list of facts about congruences in groups (see [Clifford and Preston, 1961] and [Biró,Kiss, and Pálfy, 1977]).

(1) [Clifford and Preston, 1961]. *Let G be a group and σ a congruence on G. Then there exists a normal subgroup N of G such that $(a,b) \in \sigma$ if and only if $ab^{-1} \in N$.*

(2) [Biró,Kiss, and Pálfy, 1977]. *Let G be a group. Then G has the group congruence extension property (GCEP) if and only if whenever H is a subgroup of G and K is a normal subgroup of H, there exists a normal subgroup N of G such that $N \cap H = K$.*

(3) Corollary. *A group G such that every subgroup H of G is normal in G has the group congruence extension property (GCEP), or equivalently, abelian and hamiltonian groups have GCEP.*

The following study of periodic semigroups is based on some of the work found in [Stralka, 1972], and is included to complete the material in this area.

1.17 Lemma. *Let S be a semigroup, let $x \in S$, and let $n \in \mathbb{N}$ such that $x^n = x$. Then $x^{n+(n-1)k} = x$ for each $k \in \mathbb{N}$.*

Proof. The proof is by induction on k.

For the case that $k = 1$, we have $x^{n+(n-1)} = x^{2n-1} = x^n x^{n-1} = xx^{n-1} = x^n = x$.

Assume that the result is true for $k = m-1$, i.e., $x^{n+(n-1)(m-1)} = x$. Then for $k = m$, we have that $x^{n+(n-1)m} = x^{n+mn-m} = x^{mn-m+1}x^{n-1} = xx^{n-1} = x^n = x$. ∎

An element x of a semigroup S is said to be a **periodic element** provided there exists $n \in \mathbb{N}$ with $1 < n$ such that $x^n = x$. The least such n is called the **period** of x. We say that S is a **periodic semigroup** if every element of

S is periodic.

A semigroup S is said to be **a uniformly periodic semigroup** provided that there exists $n \in \mathbb{N}$ such that $x^n = x$ for every $x \in S$.

1.18 Proposition. *A finite periodic semigroup is uniformly periodic.*

Proof. Let $S = \{x_1, x_2, \ldots, x_n\}$ and let p_i denote the period of x_i for each $i = 1, 2, \ldots, n$. Let m be the least common multiple of $\{p_i - 1 : i = 1, 2, \ldots, n\}$ and let $p = m+1$. Then for each i, there exists $k_i \in \mathbb{N}$ such that $k_i(p_i - 1) = m$. Let $i \in \mathbb{N}$ with $1 \leq i \leq n$. Then $x_i^p = x_i^{m+1} = x_i^{k_i(p_i-1)+1} = x_i^{p_i+(p_i-1)(k_i-1)} = x_i$ from Lemma 1.17. ∎

A semigroup S is said to be **medial** if for each $x, y, z, w \in S$, $xyzw = xzyw$. It has been shown in [Anderson and Hunter, 1962] that this condition is equivalent to $xyzx = xzyx$.

1.19 Notes.

(1) [Stralka, 1972]. *A semilattice has the congruence extension property.* Recall that a **semilattice** is a commutative semigroup in which each element is idempotent (i.e., $e^2 = e$ for each element e).

(2) [Stralka, 1972]. *Let S be a medial semigroup and let A be a subsemigroup of the regular elements of S such that A is a band of groups. Then each congruence on A can be extended to S.*

(3) [Biró, Kiss, and Pálfy, 1977]. *A medial uniformly periodic semigroup has the congruence extension property.*

A **homomorphism** of a semigroup S is a map $\phi: S \to T$ from S to a semigroup T such that $\phi(xy) = \phi(x)\phi(y)$ for each $x, y \in S$.

If M is a subsemigroup of a semigroup S and $\phi: S \to M$ is a homomorphism of S onto M such that $\phi|M = 1_M$ (the identity of M), then ϕ is called a **homomorphic retraction** of S onto M and M is called a **homomorphic retract** of S.

1.20 Proposition. *Let S be a semigroup, M a homomorphic retract of S, and let σ be a congruence on M. Then there exists an extension of σ to S.*

Proof. Let $\phi: S \to M$ be a homomorphic retraction. Define $\sigma^* = \{(a,b) \in (S \times S): (\phi(a), \phi(b)) \in \sigma\}$. Then σ^* is an extension of σ to S. ∎

1.21 Corollary. *Let S be a commutative semigroup having a group minimal ideal M. Then each congruence on M can be extended to S.*

Proof. Let e be the identity of M. Then $x \mapsto ex$ is a homomorphic retraction of S onto M. ∎

1.22 Corollary. *If S is a finite commutative semigroup, then each congruence on the minimal ideal $M(S)$ of S can be extended to S.*

It is well-known that the minimal ideal of a compact commutative semigroup is a group.

CHAPTER 2

THE GROUP CONGRUENCE EXTENSION PROPERTY

Much of the work in this chapter is an amalgamation of the work of previous authors. These results will be used in later chapters.

A study of congruences on groups appears in [Biró,Kiss, and Pálfy, 1977] as a digression in their study of the congruence extension property for varieties of algebras. Their study was restricted to varieties (a class which is closed under subalgebras, products, and homomorphisms). The congruence extension property for semigroups is not productive (closed under products), but it is productive in the case of abelian groups.

A non-abelian group G such that every subgroup of G is a normal subgroup is called a **hamiltonian group.**

It has been shown [Rotman, 1965] that a hamiltonian group G is of the form $G = Q \times A \times B$, where Q is the quaternions, A is a (abelian) group in which each element has order 2, and B is an abelian group in which each element has odd order.

A group G is called a **t-group** if the relation *is a normal subgroup of* is transitive among the subgroups of G, i.e., if L, M, and N are subgroups of G such that $L \triangleleft M \triangleleft N$, then $L \triangleleft N$ (where $\triangleleft$ indicates normal subgroup).

Recall from Chapter 1 that a group G has the group congruence extension property (GCEP) provided that for each subgroup H of G and each congruence σ on H, then exists a congruence $\overline{\sigma}$ on G such that $\overline{\sigma} \cap (H \times H) = \sigma$, i.e., σ has an extension to G. It is immediate that a group with the congruence extension property will also have the group congruence extension property. However, the converse of this statement is not true, as will be shown later in the next chapter.

2.1 Notes. The following sequence of results is contained in [Biró,Kiss, and Pálfy, 1977], [Best and Tausky, 1942], and [Zacher, 1952]. Let G be a group and σ a congruence on G.

(1) There exists a normal subgroup N of G such that $(a, b) \in \sigma$ if and only if $ab^{-1} \in N$.

(2) If e is the identity of G, then $N = \{g \in G: (g, e) \in \sigma\}$ (in (1)).

(3) A group G has the group congruence extension property (GCEP) if and only if whenever H is a subgroup of G and K is a normal subgroup of H, there exists a normal subgroup N of G such that $N \cap H = K$.

(4) A group G has GCEP if and only if each homomorphic image of G has GCEP.

(5) [Biró,Kiss, and Pálfy, 1977] In a finite group G with GCEP, the relation *is a normal subgroup of* is transitive among the subgroups of G.

(6) If G has GCEP, then every subgroup of G has GCEP.

(7) If G has GCEP, then every subgroup of G is a t-group.

(8) [Best and Taussky, 1942] A p-group G which is a t-group is either abelian or hamiltonian.

(9) [Biró,Kiss, and Pálfy, 1977] A finite group with GCEP is solvable.

(10) [Zacher, 1952] If G is a solvable finite t-group and the prime divisors of the order of G are $p_1 < p_2 < \cdots < p_r$, then there exist Sylow p_i-subgroup P_i of G such that:

(i) For each i, $1 \leq j \leq r$, P_i is abelian or Hamiltonian; and

(ii) $1 \leq i \leq j \leq r$ implies P_i is normal in $N_G(P_j)$ (the normalizer of P_j in G), and for each $g \in P_i$, there exists $n \in \mathbb{N}$ such that $g^{-1}ag = a^n$ for all $a \in P_j$.

(11) [Biró,Kiss, and Pálfy, 1977] A finite group G has GCEP if and only if G is a solvable t-group.

(12) (see [Howie, 1976]) Let G be a group, M and N normal subgroups of G, and let σ_M and σ_N be the congruences defined on G by M and N, respectively. Then $\sigma_M \cap \sigma_N = \sigma_{M\cap N}$ and $\sigma_M \circ \sigma_N = \sigma_{MN}$.

(13) [Best and Taussky, 1942] A normal subgroup of a t-group is a t-group.

An immediate consequence of 2.1(1) and 2.1(2) is that the group defined on a group G by the universal congruence $G \times G$ is just G, and the group corresponding to the diagonal congruence Δ_G is the trivial group consisting of the identity of G.

2.2 Proposition. *A group G such that each subgroup is normal has the group congruence extension property.*

Proof. Let H be a subgroup of G and suppose that σ is a congruence on H. Then σ determines a normal subgroup $K \triangleleft H$. Let $\overline{\sigma} = \{(a,b) \in (G \times G): ab^{-1} \in K\}$. Then K is a subgroup of G, and hence, from our hypothesis, we have that $K \triangleleft G$. It follows that $\overline{\sigma}$ is a congruence on G. Clearly, $\overline{\sigma} \cap (H \times H) =$

σ. ∎

2.3 Corollary. *Abelian and hamiltonian groups have the group congruence extension property (GCEP).*

2.4 Proposition. *Let G be a finite group. Then G has the group congruence extension property (GCEP) if and only if G has the congruence extension property (CEP).*

Proof. This is immediate from the fact that a subsemigroup of a finite group is a subgroup. ∎

2.5 Proposition. *Let $\{G_\alpha : \alpha \in A\}$ be an ascending family of groups with the group congruence extension property (GCEP), and let $G = \bigcup\{G_\alpha : \alpha \in A\}$. Then G has the group congruence extension property.*

Proof. Let H be a subgroup of G and let H be a normal subgroup of H. For each $\alpha \in A$, let $H_\alpha = H \cap G_\alpha$. Then H_α is a subgroup of G_α for each $\alpha \in A$. Let $K_\alpha = K \cap G_\alpha$. Then K_α is a subgroup of G_α for each $\alpha \in A$.

We claim that K_α is a normal subgroup of H_α for each $\alpha \in A$. To prove this let $x \in K_\alpha$, and let $y \in H_\alpha$. Then $x \in K$ and $y \in H$. Since $K \triangleleft H$, $y^{-1}xy \in K$. Since x and y are both in G_α, we have $y^{-1}xy \in G_\alpha$, and thus $y^{-1}xy \in (G_\alpha \cap K) = K_\alpha$. It follows that K_α is a normal subgroup of H_α.

In view of the fact that G_α has GCEP for each $\alpha \in A$, we see that there is a normal subgroup N_α of G_α such that $N_\alpha \cap H_\alpha = K_\alpha$. Let $N = \bigcup\{N_\alpha : \alpha \in A\}$. We will prove that N is a normal subgroup of G.

To see that N is a subgroup of G, let $x, y \in N$. Since $\{G_\alpha : \alpha \in A\}$ is ascending, there exists $\beta \in A$ such that $x, y \in G_\beta$. Thus $x, y \in N \cap G_\beta = N_\beta$. It follows that $xy^{-1} \in N_\beta \subseteq N$ and N is a subgroup of G.

To establish that N is normal in G, let $x \in N$ and let $y \in G$. Then, since $\{G_\alpha : \alpha \in A\}$ is ascending, there exists $\beta \in A$ such that $x, y \in G_\beta$. In particular, $x \in N_\beta$. In view of the normality of N_β in G_β we have that $y^{-1}xy \in N_\beta \subseteq N$ and hence N is normal in G.

From the fact that $N \cap H = K$, we conclude that G has GCEP. ∎

2.6 Example. *This is an example of a finite group which does not have the congruence extension property (CEP).* Let $G = S_5$ (the symmetric group on five elements). It is well known that G is not solvable. In view of 2.1(11), we see that G does not have the group congruence extension property (GCEP). From 2.4, we conclude that G does not have CEP. It was observed in [Best and Taussky, 1942] that S_n is a t-group for each $n \neq 4$.

2.7 Example. The additive group of real numbers $\mathbb{R}$ has the group congruence extension property (GCEP) (2.3) but does not have the congruence extension property (CEP), since $\mathbb{R}$ contains the additive semigroup $\mathbb{N}$ which does not have CEP (see 1.2).

2.8 Example. *This is an example of a group Q which has the congruence extension property (CEP), but $Q \times Q$ does not have CEP.* Let Q be the quaternion group. Then Q is a finite group with the group congruence extension property (GCEP) and hence Q has CEP by 2.4. To see that $Q \times Q$ does not have CEP, we will show that it does not have GCEP (since, again $Q \times Q$ is finite).

Let $G = \langle i \rangle \times Q$, where $\langle i \rangle$ denotes the subgroup of Q generated by i. Then G is a subgroup of $Q \times Q$. We claim that G does not have GCEP. Let $H_1 = \langle i \rangle \times \langle j \rangle$. Then H_1 is a normal subgroup of G. Let

$$H_2 = \{(i,j), (-i,-j), (1,1), (-1,-1)\}.$$

Then H_2 is an normal subgroup of H_1. Now $x = (i,k) \in G$ and $y = (i,j) \in H_2$. Moreover, $x^{-1}yx = (-i,-k)(i,j)(i,k) = (-i^3, -kjk) = (i, ik) = (i, -j)$. Thus, $x^{-1}yx \notin H_2$. It follows that H_2 is not normal in G. Thus G is not a t-group, and hence does not have GCEP. Since GCEP is hereditary, we conclude that $Q \times Q$ does not have CEP.

A θ-finite group is called a **torsion group.** This conforms to the usual definition of a torsion group, i.e., a torsion group is a group in which each element has finite order.

2.9 Example. *This is an example of a torsion group which does not have the congruence extension property (CEP).* Let Q be the quaternion group of 2.8. Since $Q \times Q$ is finite, it is a torsion group. Thus $Q \times Q$ is a torsion group that does not have CEP.

2.10 Lemma. *Let G be a torsion group. Then G has the congruence extension property (CEP) if and only if G has the group congruence extension property (GCEP).*

Proof. It will be sufficient to show that each subsemigroup of G is a group. To prove this let T be a subsemigroup of G and let $x \in T$. Then $x^n \in T$ for each $n \in \mathbb{N}$. Since G is a torsion group, there exists $m \in \mathbb{N}$ such that $x^m = e$ (the identity of G). It follows that $e \in T$. Since $x \cdot x^{m-1} = e$, we see that $x^{-1} = x^{m-1}$ is also in T, and hence T is a subgroup of G. ∎

2.11 Theorem. *Let G be a group. Then G has the congruence extension property (CEP) if and only if G is a torsion group with the group congruence extension property (GCEP).*

Proof. Suppose that G is a torsion group with GCEP. Then G has CEP from 2.10.

Suppose, on the other hand, that G has CEP. Then G does not contain an infinite cyclic group, since otherwise G would contain a copy of the additive semigroup $\mathbb{N}$ of natural numbers and this would be contradictory to the fact that CEP is hereditary (1.2). It follows that G is a torsion group. Since CEP implies GCEP, we conclude that G is a torsion group with GCEP. ∎

2.12 Corollary. *Let G be an abelian group. Then G has the congruence extension property (CEP) if and only if G is a torsion group.*

2.13 Corollary. *The homomorphic image of a group with the congruence extension property (CEP) has the congruence extension property.*

Proof. Suppose that G has CEP and that H is a homomorphic image of G. Then, according to 2.11, G has the group congruence extension property (GCEP) and is a torsion group. It follows from 2.1(4) that H has GCEP. It is immediate that H is a torsion group. Thus from 2.11, we conclude that H has CEP. ∎

CHAPTER 3

CYCLIC SEMIGROUPS

Cyclic semigroups are the atoms of every semigroup, i.e., each semigroup is a union of its cyclic subsemigroups. Thus a characterization of cyclic semigroups with the congruence extension property (CEP) is an important step in searching for a general characterization. We present a characterization of cyclic semigroups with CEP in this chapter (3.8). As a consequence of this result, it is established that each semigroup with CEP must have index at most three (3.9).

A semigroup S is called a **cyclic semigroup** if S is generated by some $a \in S$. We write $\theta(a) = \{a, a^2, a^3, \ldots\}$ and denote that S is cyclic with generator a by writing $S = \theta(a)$.

Observe that a semigroup S is cyclic if and only if S is a homomorphic image of the additive semigroup of positive integers.

3.1 Proposition. *An infinite cyclic semigroup does not have the congruence extension property (CEP).*

Proof. Suppose that S is an infinite cyclic semigroup with generator a. Then $S = \{a, a^2, a^3, \ldots\}$. Let $T = \{a^2, a^3, a^4, \ldots\}$. Then T is a subsemigroup of S. Let $I = \{a^2, a^4, a^5, a^6, \ldots\}$. Then I is an ideal of T and $\rho = (I \times I) \cup \Delta_T$ is a congruence on T.

Suppose that $\overline{\rho}$ is an extension of ρ to S. Then $(a, a) \in \Delta_S \subseteq \overline{\rho}$ and $(a^2, a^4) \in \rho \subseteq \overline{\rho}$, and hence $(a^2, a^5) \in \overline{\rho}$. It follows that $(a^3, a^5) \in \overline{\rho} \cap (T \times T)$. In view of the fact that $(a^3, a^5) \notin \rho$, we see that $\overline{\rho}$ cannot be an extension of ρ. This contradiction proves that an extension cannot exist, and hence S does not have CEP. ∎

Let S be a finite cyclic semigroup. A number $s \in \mathbb{N}$ is called the **index** of S provided that $S = \{a, a^2, a^3, \ldots, a^n\}$ with these element distinct and $a^{n+1} = a^s$ for $1 \leq s \leq n$.

Observe that a finite cyclic semigroup of index s can be written

$$S = \{a, a^2, \ldots, a^{s-1}\} \cup \{a^s, a^{s+1}, \ldots, a^n\}$$

and hence

$$S = \{a, a^2, \ldots, a^{s-1}\} \cup M(S),$$

where $M(S)$ is the minimal ideal of S and is a cyclic group.

3.2 Lemma. *Let S be a finite cyclic semigroup of index 2 and let T be a proper subsemigroup of S. Then T is a subgroup of the minimal ideal $M(S)$ of S.*

Proof. Let $S = \{a\} \cup M(S)$ and let T be a subsemigroup of S. If a were in T, then we would have $T = S$. Thus $a \notin T$ and $T \subseteq M(S)$. ∎

3.3 Proposition. *Let S be a finite cyclic semigroup of index 2. Then S has the congruence extension property (CEP).*

Proof. Let $S = \{a\} \cup M(S)$ be a finite cyclic semigroup of index 2, let T be a subsemigroup of S, and let σ be a congruence on T. Since $M(S)$ is a cyclic group, it has the group congruence extension property (GCEP) from 2.3. In view of the fact that $M(S)$ is a finite group, each subsemigroup of $M(S)$ is a group, and hence $M(S)$ has CEP. Now if $a \in T$, then $T = S$ and there is nothing to prove. Thus we can assume that T is a subsemigroup of $M(S)$. Let σ^* be an extension of σ to $M(S)$.

We claim that $\overline{\sigma} = \sigma^* \cup \{(a,a)\}$ is an extension of σ^* (and hence of σ) to S. It is immediate that $\overline{\sigma}$ is reflexive, symmetric, and transitive.

It remains to show that if $(a^n, a^m) \in \sigma^*$ $(1 < m, n)$, then $(a^{n+1}, a^{m+1}) \in \sigma^*$. From 2.1(1), there exists a (normal) subgroup N of $M(S)$ such that $\sigma^* = \{(a,b) \in (M(S) \times M(S)) : ab^{-1} \in N\}$. Since $(a^n, a^m) \in \sigma^*$, we have that $a^n a^{-m} \in N$. Let $u = a^n a^{-m}$. Then $u \in N$ and $ua^m = a^n$. Thus $ua^{m+1} = a^{n+1}$, so that $u = a^{n+1}a^{-(m+1)}$. It follows that $a^{n+1}a^{-(m+1)} \in N$ and $a^{n+1}a^{-(m+1)} \in N$ and $(a^{n+1}, a^{m+1}) \in \sigma^*$. We conclude that S has CEP. ▮

3.4 Lemma. *Let $S = \{a, a^2\} \cup M(S)$ be a finite cyclic semigroup of index 3 and let T be a subsemigroup of S, then either:*

(i) $T = S$;

(ii) $T = H$ *for some (normal) subgroup of* $M(S)$;

(iii) $T = \theta(a^2)$; *or*

(iv) $T = \theta(a^2) \cup M(S)$.

Proof. Let $H = T \cap M(S)$ and observe that H is a subgroup of $M(S)$. If $a \in T$, then $T = S$. If $a \notin T$ and $a^2 \notin T$, then $T = T \cap M(S) = H \triangleleft M(S)$. We therefore, hereafter in this proof, assume that $a \notin T$ and that $a^2 \in T$.

From the fact that $a^2 \in T$, we conclude that $a^{2k} \in T$ for all $k \in \mathbb{N}$.

If $a^3 \in T$, we claim that $T = \{a^2\} \cup M(S)$. To see this, let $n \in \mathbb{N}$ with

$n \geq 2$. Then $a^{2n+1} = a^{2(n-1)} \cdot a^3$, and hence $a^{2n+1} \in T$ for $n \geq 2$. It follows in this case that $T = \{a^2\} \cup M(S)$.

If $a^{2m+1} \in T$ for some $m \in \mathbb{N}$, we claim that $a^3 \in T$. To see this, observe that there exists $r \in \mathbb{N}$ such that $a^{2m+1}a^r = a^3$, since $M(S)$ is a finite cyclic group. Now if r is even, then $a^r \in T$ and we have that $a^3 \in T$. If $r = 2j+1$ for some $j \in \mathbb{N}$, i.e., r is odd, then $a^3 = a^{2m+1}a^{2j+1} = a^{2m+2j+2} = a^{2(m+j+1)} \in T$.

From these observations, we conclude that $T = \theta(a^2)$ if no odd power of a is in T, and $T = \theta(a^2) \cup M(S)$ otherwise. ∎

3.5 Lemma. *Let $S = \{a, a^2\} \cup M(S)$ be a finite cyclic semigroup of index 3 and let T be a subgroup of $M(S)$. Then each congruence σ on T can be extended to S.*

Proof. There exists a normal subgroup H of T such that $\sigma = \{(x,y) \in (T \times T){:}\, xy^{-1} \in H\}$. Since $M(S)$ is a finite cyclic group, it has the congruence extension property. Thus σ has a extension σ^* to $M(S)$. Let

$$\overline{\sigma} = \sigma^* \cup \{(a,a),(a^2,a^2)\}.$$

Then $\overline{\sigma}$ is an equivalence relation on S and extends σ. It remains to show that $\overline{\sigma}$ is compatible with multiplication on S. For this purpose, let $(a^r, a^m) \in \overline{\sigma}$ and let $a^s \in S$, with $r, m, s \in \mathbb{N}$. We want to show that $(a^{r+s}, a^{m+s}) \in \overline{\sigma}$.

If $3 \leq s$, then $a^s \in M(S)$, and hence $(a^{r+s}, a^{m+s}) \in \sigma^* \subseteq \overline{\sigma}$.

If $s = 1$ or $s = 2$, then $(a^{r+s}, a^{m+s}) \in \sigma^*$ by the same argument used in the proof of 3.3.

Finally, we have $(a,a)(a^2,a^2) = (a^3,a^3) \in \Delta_{M(S)} \subseteq \sigma^* \subseteq \overline{\sigma}$. ∎

3.6 Lemma. *Let $S = \{a, a^2\} \cup M(S)$ be a cyclic semigroup of index 3,* *and let $T = \{a^2\} \cup M(S)$. Then each congruence σ on T can be extended to S.*

Proof. Let σ be a congruence on T and let $\overline{\sigma} = \sigma \cup \{(a,a)\}$. We claim that $\overline{\sigma}$ is an extension of σ to S. It is immediate that $\overline{\sigma}$ extends σ and is an equivalence relation on S. It remains to show that $\overline{\sigma}$ is compatible with the multiplication on S. For this purpose, let $(a^r, a^m) \in \overline{\sigma}$ and let $a^s \in S$, with $r, m, s \in \mathbb{N}$. We claim that $(a^{r+s}, a^{m+s}) \in \sigma \subseteq \overline{\sigma}$.

Suppose that $r > 1$. Then $m > 1$ and hence $(a^r, a^n) \in \sigma$. Now if $s > 1$, then $(a^{r+s}, a^{m+s}) \in \sigma$ follows from the fact that σ is a congruence on T. If $s = 1$, then $(a^{r+1}, a^{m+1}) \in \sigma$ follows by using the argument in the proof of 3.3.

If $r = 1$, then $m = 1$ and hence $(a^{1+s}, a^{1+s}) \in \Delta_T \subseteq \sigma$. ∎

3.7 Lemma. *Let $S = \{a, a^2\} \cup M(S)$ be a cyclic semigroup of index 3, and let $T = \theta(a^2)$. Then each congruence σ on T can be extended to S.*

Proof. Now $T = \{a^2\} \cup H$, where $H = T \cap M(S)$ is a subgroup of $M(S)$. Let σ be a congruence on T, and let σ_H be the restriction of σ to H. Then σ_H extends to a congruence σ_M on $M(S)$, and $\sigma^* = \{(a^2, a^2)\} \cup \sigma_M$ is an extension of σ to the subsemigroup $\{a^2\} \cup M(S)$ of S. According to 3.6, σ^* extends to a congruence $\overline{\sigma}$ on S. It is immediate that $\overline{\sigma}$ is an extension of σ to S. ∎

3.8 Theorem. *Let S be a cyclic semigroup. Then S has the congruence extension property (CEP) if and only of S is finite and S has index at most 3.*

Proof. Suppose that S is a finite cyclic semigroup with CEP and let s

denote the index of S.

Suppose that $4 \leq s$. Denote $S = \{a, a^2, a^3, \ldots, a^n\} \cup M(S)$, let $T = \{a^2, a^3, \ldots, a^n\} \cup M(S)$, and let $I = \{a^2, a^4, a^5, a^6, \ldots, a^n\} \cup M(S)$. Then T is a subsemigroup of S and I is an ideal of T. Let $\sigma = (I \times I) \cup \Delta_T$. Then σ is a congruence on T. Let $\overline{\sigma}$ be an extension of σ to S. Then $(a^2, a^4) \in \sigma \subseteq \overline{\sigma}$ and $a \in S$. Thus $(a^3, a^5) \in \overline{\sigma}$. We thus have that $(a^3, a^5) \in \overline{\sigma} \cap (T \times T) = \sigma$. From this contradiction, we conclude that $s \leq 3$.

If S is a finite cyclic semigroup of index $s \leq 3$, then it follows from 3.3, 3.4, 3.5, 3.6, and 3.7 that S has CEP. ∎

A semigroup S is said to be **θ-finite** provided that each cyclic subsemigroup of S is finite.

3.9 Theorem. *Let S be a semigroup with the congruence extension property (CEP). Then S is θ-finite and each cyclic subsemigroup of S has index at most 3.*

Proof. Each subsemigroup of S has CEP from 1.2. Thus each cyclic subsemigroup of S has CEP. The conclusion follows from 3.8. ∎

The following example shows that the converse of 3.9 is not valid.

3.10 Example. *This is an example of a finite semigroup S such that each element has index at most 2 and S does not have the congruence extension property (CEP).*

Let $S = \{1, 2, 3, 4, 5\}$ with Cayley table given by:

$$\begin{array}{ccccc} 1 & 1 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 3 & 3 & 1 & 1 & 1 \\ 3 & 4 & 1 & 1 & 1 \\ 3 & 3 & 1 & 1 & 1 \end{array}$$

Let T be the subsemigroup $\{1, 2, 4, 5\}$ and let σ be the congruence on T defined by $\sigma = \{(3,5), (5,3)\} \cup \Delta_T$.

The congruence on S generated by σ is

$$\langle\sigma\rangle_S = \{(3,4), (4,3), (3,5), (5,3), (4,5), (5,4)\} \cup \Delta_S,$$

and $\langle\sigma\rangle_S \cap (T \times T) \neq \sigma$. It follows that S does not have CEP.

Now $\theta(1) = \{1\}$, $\theta(2) = \{2\}$, $\theta(3) = \{1,3\}$ $(3^3 = 3)$, $\theta(4) = \{1,3,4\}$ $(4^4 = 4^2 = 1)$, and $\theta(5) = \{1,3,5\}$ $(5^4 = 5^2 = 1)$. It follows that each of 1,2, and 3 have index 1 and that each of 4 and 5 have index 2.

CHAPTER 4

THE IDEAL EXTENSION PROPERTY

In this chapter we discuss the important concept of the ideal extension property (IEP) of a semigroup (ideals of subsemigroups are determined by ideals in the parent semigroup). The relation between the congruence extension property (CEP) and IEP is investigated. It is established that IEP is preserved by homomorphisms (4.3).

A special class of semigroups called ideal semigroups (congruences are determined by ideals) is discussed in detail and those with CEP are characterized (4.9).

This chapter is rich in examples which yield a substantial quantity of information on semigroups with IEP.

A semigroup S is said to have the **ideal extension property** (IEP) provided that for each subsemigroup T of S and each ideal I of T there exists an ideal J of S such that $J \cap T = I$.

4.1 Proposition. *A semigroup S has the ideal extension property (IEP) if and only if each subsemigroup of S has the ideal extension property.*

Proof. If every subsemigroup of S has IEP, it is clear that S has IEP.

Suppose that S has IEP, let T be a subsemigroup of S, let K be a subsemigroup of T, and let I be an ideal of K. Since K is also a subsemigroup of S and S has IEP, there exists an ideal J of S such that $J \cap K = I$. Now $J \cap T$

is an ideal of T and $J \cap T \cap K = J \cap K = I$. It follows that T has IEP. ∎

4.2 Example. To see that the semigroup $(\mathbb{N}, +)$ does not have the ideal extension property (IEP), let $T = \{2, 3, 4, 5, \ldots\}$ and let $I = \{2, 4, 5, 6, \ldots\}$. Then T is a subsemigroup of $\mathbb{N}$ and I is an ideal of T.

Suppose there exists an ideal J of $\mathbb{N}$ such that $J \cap T = I$. Then $2 \in I \subseteq J$ and $1 \in \mathbb{N}$. Thus $2 + 1 = 3 \in J$. We obtain that $3 \in J \cap T$, but $3 \notin I$. This contradiction establishes that $(\mathbb{N}, +)$ does not have IEP.

As a consequence of 4.1, we see that the additive semigroup $\mathbb{H} = [0, \infty)$ does not have the ideal extension property, since $\mathbb{N}$ is a subsemigroup and $\mathbb{N}$ does not have the ideal extension property.

4.3 Proposition. *A homomorphic image of a semigroup with the ideal extension property (IEP) has the ideal extension property.*

Proof. Let $\phi: S \to S^*$ be a homomorphism of a semigroup S with IEP onto a semigroup S^*. Let T^* be a subsemigroup of S^* and let I^* be an ideal of T^*. Then $T = \phi^{-1}(T^*)$ is a subsemigroup of S and $I = \phi^{-1}(I^*)$ is an ideal of T. Since S has IEP, there exists an ideal J of S such that $J \cap T = I$. Let $J^* = \phi(J)$. Then, since ϕ is onto S^*, we have that J^* is an ideal of S^*.

To complete the proof, we will establish that $J^* \cap T^* = I^*$.

Let $x \in J^* \cap T^*$. Then $\phi^{-1}(x) \subseteq J$ and $\phi^{-1}(x) \subseteq T$. Therefore, $\phi^{-1}(x) \subseteq J \cap T = I$. It follows that $x \in \phi(I) = I^*$.

Let $y \in I^*$. Then $\phi^{-1}(y) \subseteq I = J \cap T$. Thus $y \in \phi(J \cap T) = \phi(J) \cap \phi(T) = J^* \cap T^*$.

We conclude that $J^* \cap T^* = I^*$, and hence S^* has IEP. ∎

For a subsemigroup T of a semigroup S and $a \in T$, we let $J_T(a)$ denote the ideal of T generated by a,i.e.,

$$J_T(a) = T^1 a T^1 = \{a\} \cup aT \cup Ta \cup TaT.$$

A semigroup S is said to have the **principal ideal extension property** (PIEP) provided that for each subsemigroup T of S and each $a \in T$, $J_T(a) = J_S(a) \cap T$.

4.4 Theorem. *Let S be a semigroup. Then S has the ideal extension property (IEP) if and only if S has the principal ideal extension property (PIEP).*

Proof. Suppose that S has the principal ideal extension property, and let T be a subsemigroup of S and I and ideal of T. Let $J = \bigcup_{a \in I} J_S(a)$. Then J is an ideal of S and $J \cap T = (\bigcup_{a \in I} J_S(a)) \cap T = \bigcup_{a \in I} (J_S(a) \cap T) = \bigcup_{a \in I} J_T(a) = I$. We conclude that S has the ideal extension property.

On the other hand, suppose that S has the ideal extension property, and let T be a subsemigroup of S and $a \in T$. Then, since $J_T(a)$ is an ideal of T, there exists an ideal I of S such that $I \cap T = J_T(a)$. Since $a \in I$, we have that $J_S(a) \subseteq I$, and hence $J_S(a) \cap T \subseteq I \cap T = J_T(a)$. To establish the other inclusion observe that $J_T(a) = T^1 a T^1 \subseteq S^1 a S^1 = J_S(a)$ and hence $J_T(a) \subseteq J_S(a) \cap T$. ∎

A congruence σ on a semigroup S is called an **ideal congruence** provided that there exists an ideal I of S such that $\sigma = (I \times I) \cup \Delta_S$.

A semigroup S is called an **ideal semigroup** if each congruence on S is an ideal congruence.

4.5 Proposition. *Let S be an ideal semigroup. Then:*

(1) S has a zero element 0; and

(2) If ρ is a congruence on S, then $\rho = (I \times I) \cup \Delta_S$, where $I = \{x \in S : (x, 0) \in \rho\}$.

Proof. To prove (1), observe that $\Delta = \{(s, s) : s \in S\}$ is a congruence on S. Since S is an ideal semigroup, there exists an ideal I of S such that $\Delta = (I \times I) \cup \Delta$. Thus $I \times I \subseteq \Delta$ and $I \times I = \{(0, 0)\}$. We conclude that $I = \{0\}$ and S has a zero element 0.

To prove (2), let ρ be a congruence on S. Then, since S is an ideal semigroup, there exists an ideal I of S such that $\rho = (I \times I) \cup \Delta_S$. Let $J = \{x \in S : (x, 0) \in \rho\}$. Then J is an ideal of S and $J \subseteq I$. To complete the proof, we will show that $I \subseteq J$. Let $x \in I$. Then $(x, 0) \in I \times I \subseteq \rho$. We conclude that $x \in J$. ∎

4.6 Proposition. *The homomorphic image of an ideal semigroup is an ideal semigroup.*

Proof. Let S be an ideal semigroup and let $\phi : S \to T$ be a homomorphism of S onto a semigroup T and let σ be a congruence on T. Define $\rho = \{(x, y) \in S \times S : (\phi(x), \phi(y)) \in \sigma\}$. Then ρ is a congruence on S. Since S is an ideal semigroup, $\rho = (I \times I) \cup \Delta_S$ for some ideal I of S. Let $J = \phi(I)$. Then J is an ideal of T.

We claim that $\sigma = (J \times J) \cup \Delta_T$. Suppose that $(\phi(x), \phi(y)) \in \sigma$. Then $(x, y) \in \rho$. If $x = y$, then $(x, y) \in \Delta_T$. If $x \neq y$, then $(x, y) \in I \times I$ and

$(\phi(x), \phi(y)) \in J \times J$. It follows that $\sigma \subseteq (J \times J) \cup \Delta_T$.

On the other hand, let $(a, b) \in J \times J$. Then $a = \phi(x)$ and $b = \phi(y)$ for some $(x, y) \in (I \times I)$. Thus $(x, y) \in \rho$ and we conclude that $(a, b) \in \sigma$. It follows that $\sigma = (J \times J) \cup \Delta_T$ and T is an ideal semigroup. ∎

4.7 Proposition. *Each ideal semigroup with the congruence extension property (CEP) has the ideal extension property (IEP).*

Proof. Let S be an ideal semigroup with CEP, T a subsemigroup of S, and let I be an ideal of T. Then $\sigma = (I \times I) \cup \Delta_T$ is a congruence on T. Since S has CEP, there exists an extension $\overline{\sigma}$ of σ to S. Since S is an ideal semigroup, there exists an ideal J of S such that $\overline{\sigma} = (J \times J) \cup \Delta_S$. Since $\sigma = \overline{\sigma} \cap (T \times T)$, we obtain that $I = J \cap T$, and hence S has IEP. ∎

4.8 Example. *This is an example of an ideal semigroup which has the ideal extension property and does not have the congruence extension property.*

Let $S = \{1, 2, 3, 4, 5\}$ with Cayley table:

1	1	1	1	1
1	1	1	1	2
1	1	1	3	1
1	1	3	4	1
1	2	1	1	5

Let $T = \{1, 2, 3\}$. Then T is a subsemigroup of S and $\sigma = \{(2, 3), (3, 2)\} \cup \Delta_T$ is a congruence on T and $\langle\sigma\rangle_S$ contains the pair $(2, 1) = (5, 5)(2, 3)$. Thus its restriction to T is not σ.

4.9 Proposition. *Let S be an ideal semigroup. Then S has the congru-*

ence extension property (CEP) if and only if S has the ideal extension property (IEP) and each subsemigroup of S is an ideal semigroup.

Proof. Suppose that S has IEP and that each subsemigroup of S is an ideal semigroup. We claim that S has CEP. For the purpose of proving this claim, let T be a subsemigroup of S and let σ be a congruence on T. Then, since T is an ideal semigroup, there exists an ideal I of T such that $\sigma = (I \times I) \cup \Delta_T$. Since S has IEP, there exists an ideal J of S such that $J \cap T = I$. Let $\overline{\sigma} = (J \times J) \cup \Delta_S$. Then $\overline{\sigma}$ is a congruence extension of σ to S, and hence S has CEP.

Suppose, on the other hand, that S has CEP. Then S has IEP from 4.7. Let T be a subsemigroup of S and let σ be a congruence on T. Then, since S has CEP, there exists an extension $\overline{\sigma}$ of σ to S. Since S is an ideal semigroup, there exists and ideal J of S such that $\overline{\sigma} = (J \times J) \cup \Delta_S$. Let $I = J \cap T$. Then I is an ideal of T and $\sigma = (I \times I) \cup \Delta_T$. We conclude that T is an ideal semigroup. ∎

4.10 Example. *This is an example of a semigroup S such that each subsemigroup of S is an ideal semigroup and S does not have the congruence extension property (CEP).*

Let $S = \{1, 2, 3, 4, 5\}$ with multiplication table:

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 4 & 5 \\ 1 & 4 & 5 & 1 & 1 \end{array}$$

Now S is congruence free and each subsemigroup of S is an ideal semigroup. The only ideals of S are S and $\{1\}$. The semigroup S does not have CEP, and hence does not have IEP.

To see that S does not have CEP, consider the subsemigroup $T = \{1,3,5\}$ and let $\alpha = \{(1,5),(5,1)\} \cup \Delta_T$. Then α is a congruence on T. Since $(2,2) \in \Delta_S$ and $(1,5) \in \alpha$, we have that $(2 \cdot 1, 2 \cdot 5) = (1,3) \in \langle\alpha\rangle_S$. Thus $(1,3) \in \langle\alpha\rangle_S \cap (T \times T)$, but $(1,3) \notin \alpha$. It follows that α does not extend to S, and we conclude that S does not have CEP.

4.11 Proposition. *Let S be an ideal semigroup. Then S is congruence free if and only if S is 0-simple.*

Proof. Suppose first that S is 0-simple and let σ be a congruence on S. Then there exists an ideal I of S such that $\sigma = (I \times I) \cup \Delta_S$. Since S is 0-simple, either $I = \{0\}$ or $I = S$. If $I = \{0\}$, then $\sigma = \Delta_S$. If $I = S$, then $\sigma = S \times S$. In any case, we conclude that S is congruence free.

Now suppose that S is congruence free and that I is an ideal of S. Then $\rho = (I \times I) \cup \Delta_S$ is a congruence on S. Since S is congruence free, either $\rho = \Delta_S$ or $\rho = S \times S$. If $\rho = \Delta_S$, then $I = \{0\}$. If $\rho = S \times S$, then $I = S$. We conclude that S is 0-simple. ∎

A commutative semigroup S which has the congruence extension property (CEP) has the ideal extension property (IEP). This is established as a corollary in chapter 6.

4.12 Example. *This example shows that the property of being an ideal semigroup is not productive.* Let $T_1 = \{1,2\}$ and let $T_2 = \{1,2,3\}$ with multiplication tables:

1	1
1	1

and

1	1	1
1	1	2
1	2	3

respectively. Let $S = T_1 \times T_2$ and rename the elements of S according to the following scheme: $1 = (1,1)$, $2 = (1,2)$, $3 = (1,3)$, $4 = (2,1)$, $5 = (2,2)$, and $6 = (2,3)$. Then the multiplication table for S is:

1	1	1	1	1	1
1	1	2	1	1	2
1	2	3	1	2	3
1	1	1	1	1	1
1	1	2	1	1	2
1	2	3	1	2	3

Then each of T_1 and T_2 are ideal semigroups. Consider $\alpha^S(5,6)$ and observe that for each ideal K of S, $1 \in K$. In particular, if $5 \in K$, then $(1,5) \in (K \times K)$. However, $(1,5) \notin \alpha^S(5,6)$. It follows that S is not an ideal semigroup.

4.13 Example. *This is an example of an ideal semigroup which contains a subsemigroup which is not an ideal semigroup.* Let S be the semigroup of Example 4.8, i.e., $S = \{1,2,3,4,5\}$ with multiplication table:

1	1	1	1	1
1	1	1	1	2
1	1	1	3	1
1	1	3	4	1
1	2	1	1	5

Let T be the subsemigroup $\{1,2,3\}$. Then S is an ideal semigroup. Observe that each ideal of T must contain 1. Let $\alpha = \{(2,3),(3,2)\} \cup \Delta_T$. Then α is a congruence on T and $\alpha \neq (I \times I) \cup \Delta_T$ for any ideal I of T. We conclude that T is not an ideal semigroup.

4.14 Example. *This example illustrates that the ideal extension property (IEP) is not productive.* Let $S = \{1,2\}$ with multiplication:

$$\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$$

and let $T = \{1,2,3\}$ with multiplication:

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{matrix}$$

Then each of S and T have IEP. In $S \times T$ we rename the elements according to the scheme: $1 = (1,1)$, $2 = (1,2)$, $3 = (1,3)$, $4 = (2,1)$, $5 = (2,2)$, and $6 = (2,3)$. The multiplication table for $S \times T$ is:

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & 2 & 3 \end{matrix}$$

Consider the subsemigroup $U = \{1,2,4,5\}$ of $S \times T$ and let $I = \{1,5\}$. Then I is an ideal of U. Let J be any ideal of $S \times T$ containing I. Then $5 \in J$ and $3 \in S \times T$, and hence $3 \cdot 5 = 2 \in J$. We see that $2 \in J \cap U$, but $2 \notin I$. We conclude that $S \times T$ does not have IEP.

4.15 Example. *This is an example of a cyclic semigroup which is not an ideal semigroup.* Let $S = \{a, a^2\} \cup M(S)$, where $M(S) = \{a^3 = a^7, a^4, a^5, a^6\}$, be a finite cyclic semigroup of index 3. Let $\sigma = \alpha^S(a^3, a^5)$. Then $\sigma = \{(a^3, a^5), (a^4, a^6), (a^5, a^3), (a^6, a^4)\} \cup \Delta_S$ and $\sigma \neq (I \times I) \cup \Delta_S$ for any ideal I of S.

CHAPTER 5

HOMOMORPHISMS

The study of congruences in semigroups is closely related to the study of homomorphisms, since the **kernel** of a homomorphism $\phi: S \to X$ defined:

$$\ker \phi = \{(a, b) \in S \times S: \phi(a) = \phi(b)\}$$

is a congruence on S, and each congruence σ on a semigroup S may be regarded as the kernel of the natural homomorphism $\pi: S \to S/\sigma$. We consider this connection in detail when congruences on subsemigroups of S can be extended to S.

It remains an unsolved problem to determine whether the homomorphic image of a semigroup with the congruence extension property (CEP) also has CEP (see [Biró,Kiss, and Pálfy, 1977]). Some partial results in this connection are obtained in 5.23 and 5.24. These are applied to show that the ideal quotient image of a semigroup with CEP also has CEP. In light of the groupoid example of [Biró,Kiss, and Pálfy, 1977] (discussed in 5.12 and 5.27) which shows that the homomorphic image of a groupoid with CEP need not have CEP, it appears that associativity has an important role in a resolution of this problem. Recall that a **groupoid** is a set with a binary operation (not necessarily associative). Indeed, the groupoid under consideration is not associative. There is yet another feature of this groupoid that comes to the attention of the reader. This groupoid is finite and contains no idempotent element. It is well known that finite semigroups have idempotents.

If $\phi: S \to T$ is a homomorphism of a semigroup S onto a semigroup T, P is an ideal [or subsemigroup] of S, and Q is an ideal [or subsemigroup] of T, then it is well-known that $\phi(P)$ is an ideal [subsemigroup] of T and $\phi^{-1}(Q)$ is an ideal [subsemigroup] of S.

5.1 Lemma. *Let $\phi: S \to T$ be a homomorphism of a semigroup S onto a semigroup T, let Q be a subsemigroup of T, and let σ be a congruence on Q. Then the relation $\rho = \{(x,y) \in \phi^{-1}(Q) \times \phi^{-1}(Q): (\phi(x), \phi(y)) \in \sigma\}$ is a congruence on the subsemigroup $\phi^{-1}(Q)$.*

Proof. It is immediate that ρ is reflexive and symmetric. To see that it is compatible with multiplication on $\phi^{-1}(Q)$, let $(x,y) \in \rho$ and let $t \in \phi^{-1}(Q)$. Then $(\phi(xt), \phi(yt)) = (\phi(x)\phi(t), \phi(y)\phi(t)) \in \sigma$, since $(x,y) \in \rho$ and σ is compatible with multiplication on Q.

To see that ρ is transitive, let $(s,t),(t,w) \in \rho$. Then $(\phi(s), \phi(t))$ and $(\phi(t), \phi(w))$ are in σ. Since σ is transitive, we have that $(\phi(s), \phi(w)) \in \sigma$. Hence $(s,w) \in \rho$ and ρ is transitive. ∎

The relation ρ in 5.1 is called the **pullback** relation of σ.

5.2 Proposition. *Let S be a semigroup. Then the following are equivalent:*

(1) The semigroup S has the congruence extension property (CEP);

(2) For each subsemigroup T of S and each homomorphism $\phi: T \to Q$ of T onto a semigroup Q, there exists a homomorphism $\gamma: S \to R$ of S onto a semigroup R and an embedding $\alpha: Q \to R$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\gamma"] & R \\ T \arrow[u, "i"] \arrow[r, "\phi"] & Q \arrow[u, "\alpha"'] \end{tikzcd}$$

where $i: T \to S$ is the inclusion map; and

(3) For each subsemigroup T of S and each congruence σ on T, the congruence $\langle\sigma\rangle_S$ has the property that $\langle\sigma\rangle_S \cap (T \times T) = \sigma$.

Proof. To prove that (1) implies (2), suppose that S has CEP, let T be a subsemigroup of S, and let $\phi: T \to Q$ be a homomorphism of T onto a semigroup Q. Let $\sigma = \ker(\phi) = \{(a, b) \in (T \times T): \phi(a) = \phi(b)\}$. Then σ is a congruence on T. Let $\overline{\sigma}$ be an extension of σ to S and let $R = S/\overline{\sigma}$. Let $\gamma: S \to R$ be the natural homomorphism. Since $\ker\phi \subseteq \ker\gamma$, there is a one-to-one homomorphism $\alpha: Q \to R$ such that the above diagram commutes.

To prove that (2) implies (1), suppose that the condition of (2) holds. Let T be a subsemigroup of S and let σ be a congruence on T. Let $Q = T/\sigma$ and let $\phi: T \to Q$ be the natural homomorphism. Let $\gamma: S \to R$ and $\alpha: Q \to R$ be the homomorphisms which the condition states exists to make the diagram above commute. Then $\overline{\sigma} = \ker\gamma$ is an extension of σ to S.

That (3) implies (1) is immediate, since (in the case of the condition of (3)), $\langle\sigma\rangle_S$ is an extension of σ to S.

To prove that (1) implies (3), suppose that S has CEP, let σ be a congruence on a subsemigroup T of S, and let $\overline{\sigma}$ be an extension of σ to S. Then $\sigma \subseteq \langle\sigma\rangle_S \subseteq \overline{\sigma}$, since $\langle\sigma\rangle_S$ is the congruence on S generated by σ. In view of the fact that $\overline{\sigma}$ is and extension of σ, we see that $\overline{\sigma} \cap (T \times T) = \sigma$, and hence $\langle\sigma\rangle_S \cap (T \times T) = \sigma$. ∎

5.3 Notes. The following is a summary of some basic facts found in

[Clifford and Preston, 1963] and in [Howie, 1976].

(1) *If ρ is a congruence on a semigroup S, then S/ρ is a semigroup with the operation defined by $a^*b^* = (ab)^*$ for each $a^*, b^* \in S/\rho$, and the map $\rho^*: S \to S/\rho$ defined by $\rho(x) = x^*$ is a homomorphism, where x^* denotes the ρ-class of x.*

(2) *Let ρ be a congruence on a semigroup S. If $\phi: S \to T$ is a homomorphism such that $\rho \subseteq \ker \phi$, then there exists a unique homomorphism $\beta: S/\rho \to T$ such that ϕ and β have the same range and the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "\rho^*"'] & T \\ S/\rho \arrow[r, "1_{S/\rho}"'] & S/\rho \arrow[u, "\beta"'] \end{tikzcd}$$

(3) Corollary. *If ρ and σ are congruences on S with $\rho \subseteq \sigma$, then there exists a unique homomorphism $\beta: S/\rho \to S/\rho$ such that the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "\sigma^*"] \arrow[d, "\rho^*"'] & S/\sigma \\ S/\rho \arrow[r, "1_{S/\rho}"'] & S/\rho \arrow[u, "\beta"'] \end{tikzcd}$$

5.4 Proposition. *Let S be a semigroup, T a subsemigroup of S, σ a congruence on T, and let $\overline{\sigma}$ be a congruence on S. Then $\overline{\sigma}$ is an extension of σ if and only if there exists an embedding $\phi: T/\sigma \to S/\overline{\sigma}$ such that the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "\alpha"] & S/\overline{\sigma} \\ T \arrow[u, "i"] \arrow[r, "\beta"] & T/\sigma \arrow[u, "\phi"'] \end{tikzcd}$$

where i is the inclusion map of T into S, and α and β are natural homomorphisms.

Proof. Suppose first that $\overline{\sigma}$ is an extension of σ. Then ϕ is induced, since $\ker\beta = \sigma \subseteq \overline{\sigma} = \ker\alpha$. To see that ϕ is one-to-one, let $s, t \in T$ such that $\phi(\beta(s)) = \phi(\beta(t))$. Then $\alpha(i(t)) = \alpha(i(s))$ and $(t, s) \in \overline{\sigma}$. Since $\overline{\sigma} \cap (T \times T) = \sigma$ and $t, s \in T$, we have that $(t, s) \in \sigma$. It follows that $\beta(t) = \beta(s)$, and ϕ is one-to-one.

On the other hand, suppose that the one-to-one homomorphism α exists which makes the diagram above commute. Let $(s, t) \in \overline{\sigma} \cap (T \times T)$. Then $i\alpha(s) = i\alpha(t)$. From the diagram, we have $\phi(\beta(s)) = \phi(\beta(t))$. Since ϕ is one-to-one, we see that $\beta(s) = \beta(t)$. Since $\beta = \ker\sigma$, we conclude that $(s, t) \in \sigma$, and $\overline{\sigma}$ is an extension of σ to S. ∎

5.5 Corollary. *Let S be a semigroup. Then S has the congruence extension property (CEP) if and only if for each subsemigroup T and each homomorphism $\phi: T \to K$ of T onto a semigroup K, there exists a homomorphism $\gamma: S \to R$ onto a semigroup R and an embedding $\alpha: K \to R$ such that the diagram*

$$\begin{tikzcd} S \arrow[r, "\gamma"] & R \\ T \arrow[u, "i"] \arrow[r, "\phi"] & K \arrow[u, "\alpha"'] \end{tikzcd}$$

commutes, where i is the inclusion of T into S.

5.6 Notes. The **bicyclic semigroup** $\mathbb{B}$ is the semigroup on two generators p and q with an identity subject to the relation $qp = 1$ (see [Clifford and Preston, 1961]).

(1) *If ϕ is a homomorphism of $\mathbb{B}$ onto a semigroup S, then either ϕ is an isomorphism or $\phi(\mathbb{B})$ is a cyclic group.*

(2) Corollary. *If σ is a congruence on $\mathbb{B}$, then $\mathbb{B}/\sigma$ is isomorphic to $\mathbb{B}$ or is a cyclic group.*

5.7 Example. *The semigroup $\mathbb{B}$ does not have the congruence extension property (CEP).*

Let T be the subsemigroup $\{p, p^2, \ldots\}$ of $\mathbb{B}$, and let $I = \{p^3, p^4, \ldots\}$. Then I is an ideal of T and T/I is a three element semigroup $\{r, r^2, 0\}$, where $rr^2 = 0$. Let σ be the congruence on T defined by $\sigma = (I \times I) \cup \Delta_T$. Suppose that σ can be extended to a congruence ρ on $\mathbb{B}$. Then the diagram:

$$\begin{tikzcd} \mathbb{B} \arrow[r] & \mathbb{B}/\overline{\sigma} \\ T \arrow[u, "i"] \arrow[r] & T/\sigma \arrow[u, "\phi"'] \end{tikzcd}$$

commutes and ϕ embeds T/σ into $\mathbb{B}/\rho$. Now $\mathbb{B}/\rho$ is either a cyclic group or is isomorphic to $\mathbb{B}$. In either of these cases we would have that $\phi(0)$ is a zero element for the image, which cannot be the case. We conclude that $\mathbb{B}$ does not have CEP.

5.8 Proposition. *Let S be a finite cyclic semigroup of index s and let $\phi: S \to T$ be a homomorphism of S onto a semigroup T. Then T is a cyclic semigroup with index t such that $t \leq s$.*

Proof. Let $S = \{a, a^2, \ldots, a^s, a^{s+1}, \ldots, a^{n+1} = a^s\}$ and let $b = \phi(a)$. Then b is a cyclic generator for T. Let t be the index of T. Then $b^{n+1} = \phi(a^{n+1}) = \phi(a^s) = b^s$ implies that $t \leq s$. ∎

5.9 Corollary. *Let S be a finite cyclic semigroup with the congruence extension property and let $\phi: S \to T$ be a homomorphism of S onto a semigroup T. Then T has the congruence extension property (CEP).*

Proof. This follows from 3.8 and 5.8. ∎

5.10 Proposition. *Let S be an ideal semigroup with the congruence extension property and let $\phi: S \to T$ be a homomorphism of S onto a semigroup T. Then T has the congruence extension property.*

Proof. In view of 4.7, we see that T is an ideal semigroup. From 4.10, it sufficisis to show that T has the ideal extension property and that each subsemigroup of T is an ideal semigroup. For this purpose, let P be a subsemigroup of T. Then $Q = \phi^{-1}(P)$ is a subsemigroup of S and hence is an ideal semigroup. Again, $P = \phi(Q)$ is an ideal semigroup. Let I be an ideal of P and let $M = \phi^{-1}(I)$. Then M is an ideal of S and there exists an ideal J of S such that $J \cap Q = M$. Let $K = \phi(J)$. Then K is an ideal of T and $K \cap P = I$. It follows that T has the ideal extension property and hence has the congruence extension property from 4.10. ∎

5.11 Note. It was established in [Biró,Kiss, and Pálfy, 1977] that if G is a finite group with the group congruence extension property (GCEP) and $\phi: G \to H$ is a homomorphism of G onto a group H, then H has GCEP. An alternate argument is provided by observing that H is solvable and hence is a t-group. A finite solvable t-group is known to have GCEP.

5.12 Example. *This is an example of a groupoid S with the congruence extension property and a groupoid X which is a homomorphic image of S and does not have the congruence extension property.*

This example appears in [Biró,Kiss, and Pálfy, 1972]. We let S be the set $\{1, 2, 3, 4, 5\}$ with multiplication table:

$$\begin{array}{ccccc} 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 4 & 4 \\ 3 & 3 & 2 & 5 & 4 \\ 4 & 4 & 5 & 5 & 4 \\ 4 & 4 & 4 & 4 & 4 \end{array}$$

It is a matter of checking to verify that S has CEP. Let $X = \{a, b, c, d\}$ with multiplication table:

$$\begin{array}{cccc} c & c & c & c \\ c & c & c & d \\ c & c & b & d \\ d & d & d & d \end{array}$$

Define $\phi: S \to X$ by $\phi(1) = a$, $\phi(2) = b$, $\phi(3) = c$ and $\phi(4) = \phi(5) = d$. Then ϕ is a homomorphism of the groupoid S onto the groupoid X.

Let Y denote the subgroupoid $\{a, b, c\}$ of X, and let $\alpha = \{(a, b), (b, a)\} \cup \Delta_Y$. Then α is a congruence on Y. However, the congruence on X generated by α is $\langle\alpha\rangle_X = X \times X$. Thus $\langle\alpha\rangle_X \cap (Y \times Y) \neq \alpha$ since $(c, b) \in \langle\alpha\rangle_X \cap (Y \times Y)$ but is not in α. It follows that X (which is a homomorphic image of S) does not have CEP.

Observe that $(c, b) \in \langle\alpha\rangle_X$ because X is not associative. Notice that $(b, a) \in \alpha$ and thus $(b, a)(d, d) = (bd, ad) = (d, c) \in \langle\alpha\rangle_X$. Also notice that

$((bd)c, (ad)c) = (d, b) \in \langle\alpha\rangle_X$. By symmetry and transitivity, we obtain that $(c, b) \in \langle\alpha\rangle_X$. On the other hand, if S were associative, then $(bdc, adc) = ((bd)c, (ad)c) = (b(dc), a(dc))$ and thus $(d, b) = (d, c)$. From this, we have $(c, b) = (c, d) * (d, b) = (c, d) * (d, c) = (c, c) \in \Delta_Y$, where $*$ is defined by $(x, y) * (y, z) = (x, z)$. In this case the congruence would extend.

5.13 Lemma. *Let S be a semigroup and $\phi: S \to X$ be a homomorphism of S onto a semigroup X, let α be a congruence on X, and let ρ be the pullback of α to S. Then $\ker\phi \subseteq \rho$. Moreover;*

(1) If $\rho = \Delta_S$, then $\alpha = \Delta_X$;

(2) If $\alpha = \Delta_X$, then $\rho = \ker\phi$;

(3) If $\alpha = \Delta_X$ and ϕ is one-to-one, then $\rho = \Delta_S$;

(4) $\rho = S \times S$ if and only if $\alpha = X \times X$; and

(5) If β is a congruence on X with pullback σ, and $\alpha \subseteq \beta$, then $\rho \subseteq \sigma$.

Proof. To see that $\ker\phi \subseteq \rho$, let $(x, y) \in \ker\rho$. Then $\phi(x) = \phi(y)$, so that $(\phi(x), \phi(y)) \in \Delta_X \subseteq \alpha$. It follows that $(x, y) \in \rho$.

To prove (1), suppose that $\rho = \Delta_S$, and let $(x, y) \in \alpha$. Let $a, b \in S$ such that $\phi(a) = x$ and $\phi(b) = y$. Then $(\phi(a), \phi(b)) \in \alpha$ and $(a, b) \in \rho$. Since $\rho = \Delta_S$, $a = b$, and hence $x = \phi(a) = \phi(b) = y$. Thus $\alpha \subseteq \Delta_X$ and $\alpha = \Delta_X$.

To prove (2), suppose that $\alpha = \Delta_X$. Let $(a, b) \in \rho$. Then $(\phi(a), \phi(b)) \in \alpha = \Delta_X$ and $\phi(a) = \phi(b)$. We conclude that $(a, b) \in \ker\phi$, and $\rho \subseteq \ker\phi$. Since we have already shown that $\ker\phi \subseteq \rho$, we see that $\rho = \ker\phi$.

To prove (3), suppose that $\alpha = \Delta_X$ and that ϕ is one-to-one. Let $(a, b) \in \rho$. Then $(\phi(a), \phi(b)) \in \alpha = \Delta_X$, and hence $\phi(a) = \phi(b)$. Since ϕ is one-to-one, we have $a = b$, and $(a, b) \in \Delta_S$. Thus $\rho \subseteq \Delta_S$ and $\rho = \Delta_S$.

To prove (4), first suppose that $\rho = S \times S$, and let $(x, y) \in X \times X$. Let $a, b \in S$ such that $\phi(a) = x$ and $\phi(b) = y$. Since $\rho = S \times S$, we have $(a, b) \in \rho$, so that $(\phi(a), \phi(b)) = (x, y) \in \alpha$. Thus $\alpha = X \times X$. Suppose, on the other hand, that $\alpha = X \times X$, and let $(a, b) \in S \times S$. Then $(\phi(a), \phi(b)) \in X \times X = \alpha$, and hence $(a, b) \in \rho$. It follows that $\rho = S \times S$.

To prove (5), suppose that β is a congruence on X with pullback σ, and that $\alpha \subseteq \beta$. Let $(a, b) \in \rho$. Then $(\phi(a), \phi(b)) \in \alpha$, and hence $(\phi(a), \phi(b)) \in \beta$. Since σ is the pullback of β, we conclude that $(a, b) \in \sigma$ and $\rho \subseteq \sigma$. ∎

5.14 Proposition. *Let $\phi: S \to X$ be a homomorphism of a semigroup S onto a semigroup X, α_2 a relation on X, ρ_2 a relation on S, $\alpha = \bigcup_{n \in \mathbb{N}} \alpha_2^{(n)}$ (the transitive closure of α_2), and let $\rho = \bigcup_{n \in \mathbb{N}} \rho_2^{(n)}$ (the transitive closure of ρ_2). Then*

(1) If ρ_2 is contained in the pullback of α_2, then ρ is contained in the pullback of α;

(2) If the pullback of α_2 is contained in ρ_2, then the pullback of α is contained in ρ; and

(3) If ρ_2 is the pullback of α_2, then ρ is the pullback of α.

Proof. To prove (1), suppose that ρ_2 is contained in the pullback of α_2 and let $(a, b) \in \rho$. Then there exist $r_0, r_1, r_2, \ldots, r_n \in S$ such that $a = r_0$, $b = r_n$, and $(r_{i-1}, r_i) \in \rho_2$ for $i = 1, 2, \ldots, n$. Since ρ_2 is contained in the pullback of α_2, we have that $(\phi(r_{i-1}), \phi(r_i)) \in \alpha_2$ for $i = 1, 2, \ldots, n$, and hence $(\phi(a), \phi(b)) \in \alpha$. We conclude that ρ is contained in the pullback of α.

To prove (2), let (a, b) be in the pullback of α. Then $(\phi(a), \phi(b)) \in \alpha$, and thus there exists $t_0, t_1, \ldots, t_n \in X$ such that $t_0 = \phi(a)$, $t_n = \phi(b)$, and

$(t_{i-1}, t_i) \in \alpha_2$ for $i = 1, 2, \ldots, n$. Let $r_0 = a$, $r_n = b$, and $r_i \in S$ such that $\phi(r_i) = t_i$ for $i = 1, 2, \ldots, n-1$. Then $(\phi(r_{i-1}, \phi(r_i)) \in \alpha_2$ for $i = 1, 2, \ldots, n$, and hence $(r_{i-1}, r_i) \in \rho_2$ for $i = 1, 2, \ldots, n$, since the pullback of α_2 is contained in ρ_2. It follows that $(a, b) \in \rho$, and we conclude that the pullback of α is contained in ρ.

The conclusion of (3) is an immediate consequence of (1) and (2). ∎

5.15 Proposition. *Let $\phi: S \to X$ be a homomorphism of a semigroup S onto a semigroup X. Let ρ be a congruence on S and define $\alpha = \bigcup_{n \in \mathbb{N}} \alpha_2^{(n)}$ where $\alpha_2 = \{(x, y) \in (X \times X): (x, y) = (\phi(a), \phi(b))$ for some $(a, b) \in \rho\}$. Then α is a congruence on X.*

Proof. Now α is the transitive closure of α_2. Hence it is sufficient to show that α_2 is reflexive, symmetric, and compatible with the multiplication on X.

Since $\Delta_S \subseteq \rho$, $\{(\phi(s), \phi(s)): s \in S\} \subseteq \alpha_2$. Let $x \in X$ and let $s \in S$ such that $\phi(s) = x$. Then $(x, x) \in \alpha_2$, and we conclude that α_2 is reflexive.

The symmetry of α_2 is immediate from the symmetry of ρ.

Finally, to show that α_2 is compatible with multiplication on X, let $(x, y) \in \alpha_2$ and let $z \in X$. Then $(x, y) = (\phi(a), \phi(b))$ for some $(a, b) \in \rho$. Let $s \in S$ such that $\phi(s) = z$. Then, since ρ is a congruence, $(sa, sb) \in \rho$ and $(zx, zy) = (\phi(s)\phi(a), \phi(s)\phi(b)) = (\phi(sa), \phi(sb))$. We conclude that $(zx, zy) \in \alpha_2$ and similarly $(xz, yz) \in \alpha_2$. ∎

The congruence α in 5.15 is called the **pushout** of the congruence ρ.

5.16 Lemma. *Let $\phi: S \to X$ be a homomorphism of a semigroup S onto a semigroup X. Let ρ be a congruence on S and define $\alpha_2 = \{(x, y) \in$*

$(X \times X)\colon (x,y) = (\phi(a), \phi(b))$ for some $(a,b) \in \rho\}$. If $\ker\phi \subseteq \rho$, then α_2 is a congruence on X, and hence α_2 is the pushout of ρ to X.

Proof. Let (x,w) and (w,y) be in α_2. Then there exist (a,c) and (d,b) in ρ such that $(x,w) = (\phi(a), \phi(c))$ and $(w,y) = \phi(d), \phi(b))$. Since $w = \phi(c) = \phi(d)$, we have that $(c,d) \in \ker\phi \subseteq \rho$. Thus $(a,c), (c,d)$ and (d,b) are all in ρ. Since ρ is transitive, we conclude that $(a,b) \in \rho$. Thus $(x,y) = (\phi(a), \phi(b))$ and $(a,b) \in \rho$. We conclude that $(x,y) \in \alpha_2$ and α_2 is transitive. The conclusion follows from 5.15. ∎

5.17 Lemma. *Let $\phi\colon S \to X$ be a homomorphism of a semigroup S onto a semigroup X, let α be a congruence on X and let ρ be the pullback of α to S. Then α is the pushout of ρ to X.*

Proof. Now $\rho = \{(a,b) \in S \times S\colon (\phi(a), \phi(b)) \in \alpha\}$. Let $\alpha_2 = \{(x,y) \in X \times X\colon (x,y) = (\phi(a), \phi(b))$ for some $(a,b) \in \rho\}$. Then $\overline{\alpha} = \bigcup\limits_{n \in \mathbb{N}} \alpha_2^{(n)}$ (transitive closure) is the pushout of ρ by definition. Now if $(a,b) \in \ker\phi$, then $\phi(a) = \phi(b)$, $(\phi(a), \phi(b)) \in \Delta_X \subseteq \alpha$ and $(a,b) \in \rho$. Thus $\ker\phi \subseteq \rho$. From 5.16, α_2 is a congruence on X and hence $\alpha_2 = \overline{\alpha}$ is the pushout of ρ.

It remains to show that $\alpha_2 = \alpha$. For this purpose, let $(x,y) \in \alpha_2$. Then $(x,y) = (\phi(a), \phi(b))$ for some $(a,b) \in \rho$, and $(x,y) \in \alpha$. We have that $\alpha_2 \subseteq \alpha$.

To prove the other inclusion, i.e., that $\alpha \subseteq \alpha_2$, let $(x,y) \in \alpha$, and let $a,b \in S$ such that $x = \phi(a)$ and $y = \phi(b)$. We see that $(\phi(a), \phi(b)) = (x,y) \in \alpha$ and hence $(a,b) \in \rho$. From this and the definiton of α_2, we have $(x,y) \in \alpha_2$. It follows that $\alpha = \alpha_2$. ∎

5.18 Lemma. *Let $\phi\colon S \to X$ be a homomorphism of a semigroup S onto* *a semigroup X. Let ρ be a congruence on S and let α be the pushout of ρ to X. Let σ be the pullback of α to S. Then $\rho \subseteq \sigma$.*

Proof. Let $(c,d) \in \rho$. Then $(\phi(c), \phi(d)) \in \alpha_2 \subseteq \alpha$. Thus $(c,d) \in \{(a,b) \in S \times S: (\phi(a), \phi(b)) \in \alpha\} = \sigma$. We conclude that $\rho \subseteq \sigma$. ∎

5.19 Lemma. *Let $\phi: S \to X$ be a homomorphism of a semigroup S onto a semigroup X, let ρ and σ be congruences on S, and let α and β be the pushouts of ρ and σ, respectively. If $\rho \subseteq \sigma$, then $\alpha \subseteq \beta$.*

Proof. Let $\alpha_2 = \{(x,y) \in X \times X: (x,y) = (\phi(a), \phi(b)) \text{ for some } (a,b) \in \rho\}$ and let $\beta_2 = \{(x,y) \in X \times X: (x,y) = (\phi(a), \phi(b)) \text{ for some } (a,b) \in \sigma\}$. Then α is the transitive closure of α_2 and β is the transitive closure of β_2. It will be sufficient to show that $\alpha_2 \subseteq \beta_2$. Let $(x,y) \in \alpha_2$. Then $(x,y) = (\phi(a), \phi(b))$ for some $(a,b) \in \rho$. Since $\rho \subseteq \sigma$, we have that $(a,b) \in \sigma$ and hence $(x,y) \in \beta_2$. ∎

5.20 Lemma. *Let $\phi: S \to X$ be a homomorphism of a semigroup S onto a semigroup X, let Y be a subsemigroup of X, let $T = \phi^{-1}(Y)$, let α be a congruence on Y, let $\overline{\alpha} = \langle \alpha \rangle_X$, let ρ be the pullback of α to T, and let $\overline{\rho} = \langle \rho \rangle_S$. Then $\overline{\alpha}$ is the pushout of $\overline{\rho}$.*

Proof. Let $\alpha_1 = \alpha \cup \Delta_X$, and let $\alpha_2 = \{(xuy, xvy): (u,v) \in \alpha_1 \text{ and } x,y \in X^1\}$. Then $\overline{\alpha}$ is the transitive closure of α_2. Let $\beta_2 = \{(\phi(a), \phi(b)): (a,b) \in \overline{\rho}\}$. To establish that $\overline{\alpha}$ is the pushout of $\overline{\rho}$, we need to show that $\overline{\alpha} = \bigcup_{n \in \mathbb{N}} \beta_2^{(n)}$.

To show that $\bigcup_{n \in \mathbb{N}} \beta_2^{(n)} \subseteq \overline{\alpha}$, it is sufficient to show that $\beta_2 \subseteq \overline{\alpha}$, since $\overline{\alpha}$ is a congruence (and hence is transitive). To establish this containment, let $(a', b') \in \beta_2$. Then $a' = \phi(a)$ and $b' = \phi(b)$ for some $(a,b) \in \overline{\rho}$ (from the definition of β_2). Now $\overline{\rho} = \bigcup_{n\in\mathbb{N}} \rho_2^{(n)}$, where $\rho_2 = \{(sct, sdt): (c,d) \in (\rho \cup \Delta_S)$ and $s,t \in S^1\}$. We consider four cases:

Case 1. If $(a,b) \in \rho$, then $(\phi(a),\phi(b)) \in \alpha \subseteq \overline{\alpha}$, since ρ is the pullback of α (and hence α is the pushout of ρ from 5.17).

Case 2. Suppose that $(a,b) = (sct, sdt)$ for some $(c,d) \in \rho$ and $s,t \in S^1$. Then $(\phi(c),\phi(d)) \in \alpha$ (again, since α is the pushout of ρ from 5.17). Thus $(\phi(a),\phi(b)) = (\phi(sct),\phi(sdt)) = (\phi(s)\phi(c)\phi(t),\phi(s)\phi(d)\phi(t)) \in \alpha_2 \subseteq \overline{\alpha}$.

Case 3. If $(a,b) = (sct, sct)$ for some $(c,c) \in \Delta_S$, then $(a,b) \in \Delta_S$. Thus $(\phi(a),\phi(a)) \in \Delta_X \subseteq \overline{\alpha}$.

Case 4. Suppose that there exists a sequence $c_0, c_1, \ldots, c_m$ such that $a = c_0$, $b = c_m$, and $(c_{i-1}, c_i) \in \rho_2$ for $i = 1, 2, \ldots, c_m$. Then $(\phi(c_{i-1}),\phi(c_i)) \in \alpha_2$ and $(\phi(a),\phi(b)) \in \alpha_2^{(m)} \subseteq \overline{\alpha}$.

To show the other inclusion, note again that since ρ is the pullback of α, α is the pushout of ρ by 5.17. Since $\rho \subseteq \overline{\rho}$, the pushout of ρ is contained in the pushout of $\overline{\rho}$ from 5.17. Thus α is contained in the pushout of $\overline{\rho}$. Now, the pushout of $\overline{\rho}$ is a congruence on X which contains α, and hence it must contain $\overline{\alpha}$. ∎

5.21 Lemma. *Let $\phi: S \to X$ be a homomorphism of a semigroup S onto a semigroup X. Let ρ be a congruence on S and let α be the pushout of ρ to S. Then ρ is the pullback of α if and only if* $\ker\phi \subseteq \rho$.

Proof. Let α be the pushout of ρ and suppose that $\ker\phi \subseteq \rho$. Then

$\alpha = \bigcup_{n \in \mathbb{N}} \alpha_2^{(n)}$, where $\alpha_2 = \{(\phi(a), \phi(b)) : (a,b) \in \rho\}$. From 5.16, we obtain that $\alpha = \alpha_2$ in this case. It follows that if $(\phi(a), \phi(b)) \in \alpha$, then $(a,b) \in \rho$, and hence ρ is the pullback of α.

On the other hand, if ρ is the pullback of α, since $\Delta_X \subseteq \alpha$, we have $\ker \phi$ (= pullback of Δ_X from 5.13) is contained in ρ from 5.14. ∎

If α and β are congruences on a semigroup S, then $\alpha \vee \beta$ denotes the congruence on S generated by $\alpha \cup \beta$.

5.22 Lemma. *Let $\phi: S \to X$ be a homomorphism of a semigroup S onto a semigroup X. Let ρ be a congruence on S and let α be the pushout of ρ to X. Then $\sigma = \rho \vee \ker \phi$ is the pullback of α to S.*

Proof. Let μ be the pullback of α to S. Then $\mu = \{(s,t) \in S \times S : (\phi(s), \phi(t)) \in \alpha\}$. Now $\ker \phi \subseteq \mu$ and $\rho \subseteq \mu$ by 5.18. Since μ is a congruence on S, we obtain that $\sigma = \rho \vee \ker \phi \subseteq \mu$.

To establish the other inclusion, let $(c,d) \in \mu$. Then $(\phi(c), \phi(d)) = (x,y) \in \alpha$. Recall that $\alpha = \bigcup_{n \in \mathbb{N}} \alpha_2^{(n)}$, where $\alpha_2 = \{(\phi(a), \phi(b)) : (a,b) \in \rho\}$.

If $(x,y) \in \alpha_2$, then $(x,y) = (\phi(a), \phi(b))$ for some $(a,b) \in \rho$. Thus $\phi(c) = \phi(a) = x$ and $\phi(d) = \phi(b) = y$. It follows that $(c,a) \in \ker \phi$, $(a,b) \in \rho$, and $(b,d) \in \ker \phi$. In view of the transitivity of $\sigma = \phi \vee \ker \phi$, we have that $(c,d) \in \sigma$.

If there exists $(x,z), (z,y) \in \alpha_2$, then

$(x,z) = (\phi(a), \phi(e))$ for some $(a,e) \in \rho$ and

$(z,y) = (\phi(f), \phi(b))$ for some $(f,b) \in \rho$.

Since $z = \phi(e) = \phi(f)$, $x = \phi(a) = \phi(c)$, and $y = \phi(d) = \phi(b)$, we have that

$(c, a) \in \ker\phi$, $(a, e) \in \rho$, $(e, f) \in \ker\phi$, $(f, b) \in \rho$, and $(b, d) \in \ker\phi$.

Thus $(c, d) \in \ker\phi \vee \rho = \sigma$.

An extension of this argument establishes that $(c, d) \in \sigma$ if we assume that $(x, y) \in \alpha_2^{(m)}$ for some $m \in \mathbb{N}$. It follows that $\mu \subseteq \sigma$.

From the arguments above, we conclude that $\mu = \sigma = \rho \vee \ker\phi$. ∎

5.23 Theorem. *Let $\phi: S \to X$ be a homomorphism of a semigroup S onto a semigroup X, let Y be a subsemigroup of X, and let α be a congruence on Y. If the pullback of $\langle\alpha\rangle_X$ is an extension of the pullback of α, then α extends to a congruence on X.*

Proof. Let $T = \phi^{-1}(Y)$, $\overline{\alpha} = \langle\alpha\rangle_X$, ρ the pullback of α to T, and let $\overline{\rho} = \langle\rho\rangle_S$. From 5.20, $\overline{\alpha}$ is the pushout of $\overline{\rho}$ and by 5.22, $\sigma = \overline{\rho} \vee \ker\phi$ is the pullback of $\overline{\alpha}$. According to our hypothesis, we have that $\sigma \cap (T \times T) = \rho$.

We will show that $\overline{\alpha}$ is an extension of α to X. For this purpose, let $(x, y) \in \overline{\alpha} \cap (Y \times Y)$. Since σ is the pullback of $\overline{\alpha}$, there is a pair $(a, b) \in \sigma$ such that $(x, y) = (\phi(a), \phi(b))$. Since $(x, y) \in (Y \times Y)$, it is also true that $(a, b) \in (T \times T)$. Thus $(a, b) \in \sigma \cap (T \times T) = \rho$. It follows that $(x, y) = (\phi(a), \phi(b)) \in \alpha$, since ρ is the pullback of α. We conclude that $\overline{\alpha} \cap (Y \times Y) \subseteq \alpha$, and $\overline{\alpha}$ is an extension of α. ∎

5.24 Theorem. *Let $\phi: S \to X$ be a homomorphism of a semigroup S with the congruence extension property (CEP) onto a semigroup X, let Y be a subsemigroup of X and let $T = \phi^{-1}(Y)$. Suppose that $\langle\ker\phi \cap (T \times T)\rangle_S = \ker\phi$. Then each congruence on Y can be extended to X.*

Proof. Let α be a congruence on Y and let ρ be the pullback of α to $T = \phi^{-1}(Y)$. Let $\overline{\alpha} = \langle\alpha\rangle_X$ and let $\overline{\rho} = \langle\rho\rangle_S$. Then, according to 5.20, $\overline{\alpha}$ is the pushout of $\overline{\rho}$. Since ρ is the pullback of α, $\ker\phi \cap (T \times T) \subseteq \rho$. From our hypothesis that $\langle\ker\phi \cap (T \times T)\rangle_S = \ker\phi$, we have that $\ker\phi \subseteq \overline{\rho}$. Thus, by 5.21, $\overline{\rho}$ is the pullback of $\overline{\alpha}$.

To see that $\overline{\alpha}$ is an extension of α to X, let $(x, y) \in \overline{\alpha} \cap (Y \times Y)$. Then $(x, y) = (\phi(a), \phi(b))$ for some $(a, b) \in \overline{\rho}$, since $\overline{\rho}$ is the pullback of $\overline{\alpha}$. As a consequence of the fact that S has CEP, we have that $\overline{\rho} \cap (T \times T) = \rho$. Since $(x, y) \in Y \times Y$, we have that $(a, b) \in T \times T$ and hence $(a, b) \in \overline{\rho} \cap (T \times T) = \rho$. We conclude that $(x, y) = (\phi(a), \phi(b)) \in \alpha$ from the fact that ρ is the pullback of α. It follows that $\overline{\alpha} \cap (Y \times Y) \subseteq \alpha$. ∎

5.25 Corollary. *Let $\phi: S \to X$ be a homomorphism of a semigroup S onto a semigroup X, let Y be a subsemigroup of X, and let $T = \phi^{-1}(Y)$. If $\phi|(S\backslash T)$ is one-to-one, then each congruence on Y can be extended to X.*

Proof. Let α be a congruence on Y and let ρ be the pullback of α to T.

Observe that $\ker\phi = (\ker\phi \cap (T \times T)) \cup \Delta_S$, and hence $\langle\ker\phi \cap (T \times T)\rangle_S = \ker\phi$. Therefore, by 5.24, α can be extended to X. ∎

5.26 Corollary. *Let S be a semigroup with the congruence extension property (CEP) and let I be an ideal of S. Then S/I has the congruence extension property.*

Proof. Let $X = S/I$, $\phi: S \to X$ the natural homomorphism, let Y be a subsemigroup of X, $T = \phi^{-1}(Y)$, and let $0 = \phi(I)$ be the zero element of X. We consider two cases:

Case 1. If $0 \in Y$, then $I \subseteq T$, so that $(I \times I) \cap (T \times T) = I \times I$ and hence $\langle \ker \phi \cap (T \times T) \rangle_S = (I \times I) \cup \Delta_S = \ker \phi$. It follows from 5.24, that each congruence on Y can be extended to X.

Case 2. If $0 \notin Y$, then $T \cap I = \emptyset$. Let $\overline{\alpha} = \langle \phi \rangle_X$, let ρ be the pullback of α to T and let $\overline{\rho} = \langle \rho \rangle_S$. Since S has CEP, $\overline{\rho} \cap (T \times T) \doteq \rho$. Let $\sigma = \overline{\rho} \vee \ker \phi$ be the pullback of $\overline{\alpha}$. Then $\sigma = \overline{\rho} \cup (I \times I)$ and $\sigma \cap (T \times T) = (\overline{\rho} \cup (I \times I)) \cap (T \times T) = \overline{\rho} \cap (T \times T) = \rho$. The conclusion that $\overline{\alpha}$ extends α now follows from 5.23. ∎

5.27 Example. We investigate the groupoid example of 5.12 in view of the results of 5.23 and 5.24. Recall that $S = \{1, 2, 3, 4, 5\}$ and $X = \{a, b, c, c\}$ with $\phi: S \to X$ defined by $\phi(1) = a$, $\phi(2) = b$, $\phi(3) = c$, and $\phi(4) = \phi(5) = d$. Now $Y = \{a, b, c\}$ is a subgroupoid of X and $T = \phi^{-1}(Y) = \{1, 2, 3\}$ is a subgroupoid of S. We list some relations determined by α below by exhibiting their classes:

α	$\{a, b\}$, $\{c\}$
$\rho =$ the pullback of α	$\{1, 2\}$, $\{3\}$
$\overline{\rho} = \langle \rho \rangle_S$	$\{1, 2, 5\}$, $\{3, 4\}$
$\langle \alpha \rangle_X$	$\{a, b, c, d\}$
$\sigma =$ the pullback of $\langle \alpha \rangle_X$	$\{1, 2, 3, 4, 5\}$

Since σ is not an extension of ρ, the hypothesis of 5.23 does not hold. Morover, $\langle \ker \phi \cap (T \times T) \rangle_S = \Delta_S \neq \ker \phi$, and thus the hypothesis of 5.24 does not hold. It is also evident that $\langle \alpha \rangle_X$ is not an extension of α.

CHAPTER 6

DISRUPTIVE ELEMENTS AND DISRUPTIVE PAIRS

One of the features of this chapter is a characterization of semigroups with the congruence extension property (CEP) in terms of disruptive pairs (6.11). Unfortunately, disruptive pairs are difficult to identify in a semigroup. Disruptive elements are more generally accessible in semigroups, but their absence does not guarentee that the semigroup has CEP (6.13). It is true however, that a commutative semigroup with CEP contains no disruptive elements.

An element a of a semigroup S is said to be a **disruptive element** if there exists a subsemigroup T of S such that $a \in T$ and $J_T(a) \subset J_S(a) \cap T$ (proper containment). Recall that $J_T(a) = T^1aT^1$ is the ideal of T generated by a.

In view of 4.4, we see that if S is a semigroup, then these are equivalent:

(1) S has the ideal extension property (IEP);

(2) S has the principal ideal extension property (PIEP); and

(3) S contains no disruptive elements.

A element r of a semigroup S is called a **regular element** provided there exists $t \in S$ such that $rtr = r$. The element t is called an **inverse** for r. Observe that if e is an idempotent element of S, i.e., $e^2 = e$, then e is regular.

6.1 Proposition. *Let S be a commutative semigroup and let T be a subsemigroup of S. Then no regular element of T is disruptive in T.*

Proof. Let r be a regular element of T and let $t \in T$ be an inverse of r. Let $p \in J_S(r) \cap T$. Then $p = sr$ for some $s \in S^1$ and $p \in T$. Therefore, $ptr = srtr = sr = p$. Since $p, t \in T$, we have $p \in J_T(r)$, and hence $J_T(r) = J_S(r) \cap T$. We conclude that r is not disruptive in T. ∎

6.2 Corollary. *No idempotent element of a commutative semigroup is a disruptive element.*

A commutative semigroup which consists entirely of idempotents is called a **semilattice.**

6.3 Corollary. *Each semilattice has the ideal extension property.*

6.4 Theorem. *Let S be a commutative semigroup. If S has the congruence extension property (CEP), then S has the ideal extension property (IEP).*

Proof. Suppose, for the purpose of proof by contradiction, that S does not have IEP. Then S contains a disruptive element a, and consequently there exists a subsemigroup T of S containing a such that $J_T(a) \subset J_S(a) \cap T$. Let $s \in S$ such that $sa \in T$ and $sa \neq ta$ for all $t \in T^1$.

According to 6.2, $a \neq a^2$. Since T is a subsemigroup of S, we have $(a, a^2) \in (T \times T)$. Now $sa \in T$ and $sa^2 = (sa)a \in T$. Thus $(sa, sa^2) \in \alpha^S(a, a^2) \cap (T \times T)$. In view of the characterization of $\alpha^T(a, a^2)$ following 1.8, and the facts that $sa \neq ta$ for all $t \in T^1$ and $sa \neq sa^2$, there is no transition in T linking sa to sa^2, and hence $(sa, sa^2) \notin \alpha^T(a, a^2)$. We obtain that $\alpha^T(a, a^2) \neq \alpha^S(a, a^2) \cap (T \times T)$. By 1.11, this contradicts that S has

CEP. ∎

6.5 Corollary. *Let S be a commutative semigroup with the congruence extension property (CEP). Then S contains no disruptive elements.*

6.6 Proposition. *Let $\phi\colon S \to X$ be a homomorphism of a commutative semigroup S onto a semigroup X. If b is a disruptive element of X, then each element of $\phi^{-1}(b)$ is a disruptive element of S.*

Proof. Let Y be a subsemigroup of X such that $b \in Y$ and $J_Y(b) \subset J_X(b) \cap Y$. Let $a \in \phi^{-1}(b)$. Now there exists $r \in X$ such that $rb \in Y$ and $rb \neq qb$ for all $q \in Y^1$. Let $s \in \phi^{-1}(r)$, and let $T = \phi^{-1}(Y)$. Then $\phi(sa) = \phi(s)\phi(a) = rb \in Y$, and hence $sa \in T$.

Suppose that $sa = ta$ for some $t \in T^1$, and let $q = \phi(t)$. Then $\phi(s)\phi(a) = \phi(t)\phi(a)$, so that $rb = qb$ and $q \in Y^1$. This contradiction proves that $sa \neq ta$ for all $t \in T^1$, and hence a is disruptive. ∎

6.7 Corollary. *Each homomorphic image of a commutative semigroup with the ideal extension property has the ideal extension property.*

We consider some examples of commutative semigroups which do not have the ideal extension property (IEP) due to the presence of a disruptive element. It follows from 6.4 that these examples do not have the congruence extension property (CEP).

6.8 Example. Consider the semigroup $(\mathbb{N}, \cdot)$, where $\cdot$ is the usual multiplication on $\mathbb{N}$. The element 6 is disruptive. To see this, let $T = \{2, 4, 6, \ldots\}$. Then $3 \cdot 2 = 6 \in T$, but no product of elements of T is 6. We conclude that

$(\mathbb{N}, \cdot)$ does not have IEP and does not have CEP.

6.9 Example. The semigroup $(\mathbb{N}, +)$ does not have CEP. We claim that 4 is disruptive. Let $T = \{4, 6, 8, \ldots\}$. Observe that $2 + 4 = 6 \in T$, but no two elements of T have sum 6.

6.10 Examples. From 6.8, 6.9, and in view of 1.2, we see that each of the following do not have CEP: $\mathbb{H} = ([0, \infty), +)$, $(\mathbb{R}, +)$, and $(\mathbb{R}, \cdot)$, where $\mathbb{R}$ denotes the real numbers.

Let S be a semigroup, T a subsemigroup of S and let $x, y \in T$. A **Malcev chain** from x to y in T is a sequence of pairs $(p_{i-1}, p_i) \in (T \times T)$ for $i = 1, 2, \ldots, m$ such that $p_0 = x$ and $p_m = y$. Each pair (p_{i-1}, p_i) is called a **link** of the chain and we say that x and y are **linked in** T by the Malcev chain $\{(p_{i-1}, p_i) : i = 1, 2, \ldots, m\}$. This is denoted

$$(x, y) = (p_0, p_1) * (p_1, p_2) * \cdots * (p_{m-1}, p_m)$$

Let S be a semigroup. A pair $(a, b) \in (S \times S)$ is called a **disruptive pair** provided that there exists a subsemigroup T of S and elements $x \in T \cap J_S(a)$ and $y \in T \cap J_S(b)$ such that x is linked to y in S be a Malcev chain with links of the form $(s_1 u s_2, s_1 v s_2)$, where $s_1, s_2 \in S^1$ and $u, v \in \{a, b\}$, but x is not linked to y in T by a Malcev chain with links of the form $(t_1 u t_2, t_1 v t_2)$, where $t_1, t_2 \in T^1$ and $u, v \in \{a, b\}$.

If S is a semigroup, T is a subsemigroup of S, and $(a, b) \in (T \times T)$, then $\alpha^T(a, b) = \{(x, y) \in (T \times T) : x$ is linked to y by a Malcev chain in T with links of the form $(t_1 u t_2, t_3 v t_4)$, where $t_1, t_2, t_3, t_4 \in T^1$ and $u, v \in \{a, b\}\}$. This is a consequence of the characterization of $\alpha^T(a,b)$ as the transitive closure of $\alpha_2 = \{(rct, rdt): (c,d) \in \alpha_1, r, t \in T^1\}$, where $\alpha_1 = \Delta_T \cup \{(a,b),(b,a)\}$, which follows 1.8 in Chapter 1.

6.11 Theorem. *A semigroup S has the congruence extension property (CEP) if and only if $S \times S$ contains no disruptive pairs.*

Proof. Suppose that $S \times S$ contains a disruptive pair (a,b). Then there exists a subsemigroup T of S and elements $x \in T \cap J_S(a)$ and $y \in T \cap J_S(b)$ such that x in linked to y in S by a Malcev chain with links of the form (s_1us_2, s_1vs_2), where $s_1, s_2 \in S^1$, and $u, v \in \{a,b\}$, but x is not linked to y in T by a Malcev chain of the form (t_1ut_2, t_1vt_2), where $t_1, t_2 \in T^1$, and $u, v \in \{a,b\}$. Thus $(x,y) \in \alpha^S(a,b) \cap (T \times T)$, but $(x,y) \notin \alpha^T(a,b)$. It follows from 1.11 that S does not have CEP.

The converse follows from the fact that if $(x,y) \in \alpha^S(a,b) \cap (T \times T)$, where $a, b \in T$ for some subsemigroup T of S, and $(x,y) \notin \alpha^T(a,b)$, then (x,y) is a disruptive pair. ∎

6.12 Proposition. *Let S be a commutative semigroup and let a be a disruptive element of S. Then (a, a^2) is a disruptive pair in $S \times S$.*

Proof. Since a is a disruptive element of S, there exists a subsemigroup T of S such that $J_T(a) \subset J_S(a) \cap T$. Let $s \in S^1$ such that $sa \in T$ but $t \in T$. Now $a \neq a^2$ from 6.2 and $sa \neq sa^2$ (otherwise $sa = sa \cdot a$). We also have $sa^2 = sa \cdot a \in T$, since $sa \in T$ and $a \in T$. Thus sa and sa^2 are linked by a Malcev chain with one link (sa, sa^2) in S, but sa and sa^2 are not linked in T by a Malcev chain with links of the form (ta, ta^2) or (ta^2, ta), since $sa \neq ta$

and $sa \neq ta^2$ (otherwise $sa = ta \cdot a$ and $ta \in T$) for all $t \in T$. ▮

6.13 Example. *This is an example of a semigroup which has no disruptive elements and does not have the congruence extension property (CEP).*

Let $S = \{1, 2, 3, 4, 5\}$ be the semigroup of 4.8. The multiplication table for S is

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 3 & 4 & 1 \\ 1 & 2 & 1 & 1 & 5 \end{matrix}$$

We claim that S has no disruptive elements. Observe that 1,4, and 5 are idempotent elements and hence are not disruptive.

Let T be a subsemigroup of S such that $2 \in T$. Then $2^2 = 1 \in T$, and so $\{1, 2\} \subseteq J_T(2)$. Now $J_T(2) \subseteq J_S(2) = \{1, 2\}$. Thus $J_S(2) = J_T(2)$ for all subsemigroups T of S containing 2. It follows that 2 is not disruptive. The same type of argument used to show that 2 is not disruptive also works to show that 3 is not disruptive.

Now S does not have CEP, since $T = \{1, 2, 3\}$ is a subsemigroup of S, $\sigma = \{(2, 3), (3, 2)\} \cup \Delta_T$ is a congruence on T and $\overline{\sigma} = \langle \sigma \rangle_S$ contains the pair $(2, 1) = (5, 5)(2, 3)$. We conclude that $\overline{\sigma} \cap (T \times T) \neq \sigma$ and S does not have CEP.

CHAPTER 7

FURTHER RESULTS AND EXAMPLES

This chapter is devoted to the expansion of the class of known examples of semigroups with the congruence extension property (CEP). There are numerous examples of finite semigroups with CEP. Indeed, every semigroup of order 3 or less has CEP. This is due to the fact that the only congruences on an order two semigroup S are Δ_S and $S \times S$, and these always extend. The following table details the number of semigroups with CEP for orders 3 through 6.

Order	Semigroups	CEP	% CEP
3	18	18	100
4	126	112	88
5	1,160	773	66
6	15,973	6,490	40

The initial example of this chapter shows that left trivial semigroups (dually right trivial semigroups) have CEP. Thus, an infinite set given left trivial multiplication is an example of an infinite semigroup with CEP. That a left trivial semigroup has CEP is also a consequence of the theorem of [Biró, Kiss, and Pálfy, 1977] that a uniformly periodic medial semigroup has CEP. Our approach to the example is to demonstrate the extension.

Example 7.2 is yet another example of an infinite semigroup (group) with CEP. This semigroup is commutative (and hence medial), but it is not uniformly periodic. That it has CEP is a consequence of 2.11.

Example 7.5 demonstrates that semigroups do not generally have maximal CEP homomorphic images. This question arises naturally from the fact that semigroups have maximal semilattice homomorphic images and semilattices have CEP by [Stralka, 1972].

An attempt was made to determine which completely simple semigroups have CEP (see [Clifford and Preston, 1961]). We conjecture that (independent of the sandwich function) if the Schutzenberger group has CEP, then so does the completely simple semigroup. Using a special result regarding the structure of subsemigroups (7.7) when the sandwich function maps the entire domain the the identity of the group (and thus the semigroup is a threefold product), it was established that completely simple semigroups of this type having Shutzenberger groups with CEP do have CEP (7.8).

7.1 Example. *A left trivial semigroup has the congruence extension property (CEP).* Let S be any set with left trivial multiplication, i.e., $xy = x$ for all $x, y \in S$. It is well known and simple to show that any equivalence relation on S is a congruence. We claim that S has CEP. Let T be a subsemigroup of S and let σ be a congruence on T. Then $\overline{\sigma} = \sigma \cup \Delta_S$ is an equivalence relation on S and hence is a congruence extension of σ.

7.2 Example. The group $\mathbb{Z}(p^\infty)$, where $p \in \mathbb{N}$ is a prime, which consists of the p th. roots of unity for $n = 0, 1, 2, \cdots$ as a subgroup of $\mathbb{R}/\mathbb{Z}$ is an example of a group with the congruence extension property (CEP). This is a consequence of 2.11.

If S is a semigroup, then $S_1 = S \cup \{1\}$ is the semigroup S with an identity adjoined (even if S already has an identity), and $S_0 = S \cup \{0\}$ is the semigroup S with a zero adjoined (even if S already has a zero).

If σ is a congruence on S, then $\sigma_1 = \sigma \cup \{(1,1)\}$ is a congruence on S_1 and $\sigma_0 = \sigma \cup \{(0,0)\}$ is a congruence on S_0. For the congruence extension property (CEP), these are equivalent:

(1) S has CEP;

(2) S_1 has CEP; and

(3) S_0 has CEP.

If S is a nondegenerate semigroup with an identity e such that S is an ideal semigroup, then S_1 is not an ideal semigroup, since the congruence which identifies 1 and e is not determined by an ideal of S_1. However, if S is an ideal semigroup, then so is S_0.

7.3 Proposition. *Let S be an ideal semigroup. Then S_0 is an ideal semigroup.*

Proof. Let σ be a congruence on S_0. Then $\sigma \cap (S \times S)$ is a congruence on S. In view of the fact that S is an ideal semigroup, we see that there exists an ideal I of S such that $\sigma \cap (S \times S) = (I \times I) \cup \Delta_S$. Let

$$J = I \cup \{x \in S_0 : (x,0) \in \sigma\}.$$

Then J is an ideal of S_0.

We claim that $\sigma = (J \times J) \cup \Delta_{S_0}$.

Let $(a,b) \in \sigma$. If $a = 0$ or $b = 0$, then $a, b \in J$. If $a \neq 0 \neq b$, then $(a,b) \in \sigma \cap (S \times S) \subseteq (I \times I) \cup \Delta_S \subseteq (J \times J) \cup \Delta_{S_0}$. In any case, we have that

$\sigma \subseteq (J \times J) \cup \Delta_{S_0}$.

To show the other inclusion, let $(a,b) \in J \times J$. If $(a,b) \in I \times I$, then $(a,b) \in \sigma$. If $(a,0),(b,0) \in \sigma$, then $(a,b) \in \sigma$, since σ is symmetric and transitive. It remains to show that if $(a,0) \in \sigma$, $a \in S$, and $b \in I$, then $(a,b) \in \sigma$ (or the dual case). Since $(a,0) \in \sigma$, we have that $(at,0) \in \sigma$ for all $t \in S$, so that again by symmetry and transitivity, we have $(a,at) \in \sigma$ for all $t \in S$. If $at = a$ for all $t \in S$, then a is a zero for S and hence $a \in I$. It $at \neq a$ for some $t \in S$, then $(a,at) \in \sigma \cap (S \times S) \subseteq (I \times I) \cup \Delta_S$, and hence $a, at \in I$, since $at \neq t$. Again, we have that $a \in I$, and $(a,b) \in (I \times I) \cap (S \times S) \subseteq \sigma$. ∎

7.4 Example. *A zero semigroup has the congruence extension property (CEP).* If S is a zero semigroup, i.e., S has a zero element 0 and $xy = 0$ for all $x, y \in S$, then S has CEP. To see this, let T be a subsemigroup of S. Then $0 \in T$. The congruences on a zero semigroup are the equivalence relations. Thus if σ is a congruence on T, then σ is an equivalence on T and $\overline{\sigma} = \sigma \cup \Delta_S$ is an equivalence (hence a congruence) relation on S. It follows that S has CEP. Moreover, for each $(a,b) \in S \times S$, $\alpha^S(a,b) = \{(a,b),(b,a)\} \cup \Delta_S$.

7.5 Example. *The additive semigroup* $\mathbb{N}$ *of natural numbers does not have a maximal homomorphic image with the congruence extension property (CEP).* Suppose that $\mathbb{N}$ has a maximal CEP homomorphic image H. Then H is a cyclic semigroup with CEP. Hence, according to 3.8, H is a finite cyclic semigroup with index at most 3. Let T be a finite cyclic semigroup with index at most 3 and order greater than the order of H. Again, by 3.8, T has CEP and hence is a CEP homomorphic image of $\mathbb{N}$. In view of the maximality of H, we see that T is a homomorphic image of H. This is not possible, since

H has fewer elements than T. We conclude that $\mathbb{N}$ has no maximal CEP homomorphic image.

7.6 Lemma. *Let G be a θ-finite group, L a left trivial semigroup, R a right trivial semigroup, and let e denote the identity of G. Let $S = L \times G \times R$, let T be a subsemigroup of S, and suppose that $(a, g, b) \in T$. Then $(a, e, b) \in T$.*

Proof. Let $(a, g, b) \in T$. Then $(a, g, b)^n = (a, g^n, b)$ for each $n \in \mathbb{N}$. Thus $(a, g^n, b) \in T$ for each $n \in \mathbb{N}$. Since G is θ-finite, there exists $m \in \mathbb{N}$ such that $g^m = e$. We conclude that $(a, e, b) \in T$. ∎

7.7 Lemma. *Let L be a left trivial semigroup, R a right trivial semigroup, G a θ-finite group, and let T be a subsemigroup of $S = L \times G \times R$. Then there exist $A \subseteq L$, $B \subseteq R$, and a subgroup H of G such that $T = A \times H \times B$.*

Proof. Let $H = \{g \in G: (a, g, b) \in T$ for some $a \in L$ and $b \in R\}$. Then H is a subsemigroup of G. As it was observed in the proof of 2.10, each subsemigroup of a torsion group is a subgroup, and hence H is a subgroup of G. Let $A = \{a \in L: (a, g, b) \in T$ for some $g \in G$ and $b \in R\}$ and let $B = \{b \in R: (a, g, b) \in T$ for some $a \in L$ and $g \in G\}$. Then $A \subseteq L$, $B \subseteq R$, and $T \subseteq A \times H \times B$.

To show that $A \times H \times B \subseteq T$, let $(a, h, b) \in A \times H \times B$. Since $a \in A$, there exists $g \in G$ and $y \in R$ such that $(a, g, y) \in T$. Since $h \in H$, there exist $r \in L$ and $s \in R$ such that $(r, h, s) \in T$. From the fact that $b \in B$, we have that $(p, w, b) \in T$ for some $p \in L$ and $w \in G$. Since G is θ-finite, $(p, e, b) \in T$, and $(a, e, y) \in T$ from 7.6. In view of the facts that $(a, e, y), (r, h, s), (p, e, b) \in T$, we see that their product $(a, h, b) \in T$. ∎

7.8 Theorem. *Let L be a left trivial semigroup, R a right trivial semigroup, and let G be a group. Then $S = L \times G \times R$ has the congruence extension property (CEP) if and only if G has the congruence extension property.*

Proof. If S has CEP, then for $d \in L$ and $f \in R$, the subsemigroup $\{d\} \times G \times \{f\}$ has CEP by 1.2. Since this subsemigroup (subgroup) is isomorphic to G, we conclude that G has CEP.

Suppose, on the other hand, that G has CEP. Let T be a subsemigroup of S and let σ be a congruence on T. Since G has CEP, G is θ-finite from 3.9. From 7.7, there exist $A \subseteq L$, $B \subseteq R$, and a subgroup H of G such that $T = A \times H \times B$.

Let $\sigma_H = \{(u, v) \in H \times H : ((a, u, b), (x, v, y)) \in \sigma$ for some $a, x \in L$ and some $b, y \in R\}$.

It is immediate that σ_H is reflexive and symmetric. To see that σ_H is transitive, suppose that $(u, v), (v, w) \in \sigma_H$. Then there exist $a, x, r, s \in L$ and $b, y, t, p \in R$ such that

$$((a, u, b), (x, v, y)) \in \sigma$$

and

$$((r, v, t), (s, w, p)) \in \sigma.$$

Multiplying both of these on the left and on the right by (a, e, b) (where e is the identity of G) and observing that $(a, e, b) \in T$, since $(a, u, b) \in T$ by 7.6, we obtain that

$$((a, u, b), (a, v, b)) \in \sigma$$

and

$$((a, v, b), (a, w, b)) \in \sigma.$$

Since σ is transitive, we have that

$$((a,u,b),(a,w,b)) \in \sigma,$$

and hence $(u,w) \in \sigma_H$. We conclude that σ_H is transitive and is therefore an equivalence relation on H.

To show that σ_H is compatible with the multiplication on H, let $(u,v) \in \sigma_H$ and let $g \in H$. Then there exist $a,x \in L$ and $b,y \in R$ such that $((a,u,b),(x,v,y)) \in \sigma$ and $(a,g,b) \in T$. Since σ is a congruence on T, we have that $((a,u,b)\cdot(a,g,b),(x,v,y)\cdot(a,g,b)) = ((a,ug,b),(x,vg,b)) \in \sigma$, so that $(ug,vg) \in \sigma_H$. This (together with its dual) proves that σ_H is a congruence relation on H.

Let $\sigma_A = \{(a,x) \in A \times A\colon ((a,u,b),(x,v,y)) \in \sigma$ for some $u,v \in H$ and some $b,y \in B\}$.

Let $\sigma_B = \{(b,y) \in B \times B\colon ((a,u,b),(x,v,y)) \in \sigma$ for some $a,x \in A$ and some $u,v \in H\}$.

Then σ_A is a congruence on A and σ_B is a congruence on B. To varify that σ_A is a congruence on A first observe that σ_A is reflexive and symmetric. For the purpose of proving that σ_A is transitive, let $(a,x),(x,t) \in \sigma_A$. Then there exist $u,v,w,z \in H$ and $b,y,c,k \in B$ such that

$$((a,u,b),(x,v,y)) \in \sigma$$

and

$$((x,w,c),(t,z,k)) \in \sigma.$$

Now (as in the argument for σ_H), we have

$$((a,e,b),(x,e,y)) \in \sigma$$

and

$$((x,e,c),(t,e,k)) \in \sigma.$$

Multiplication of the latter by (x,e,y) on the right yields,

$$((x,e,y),(t,e,y)) \in \sigma.$$

Since σ is transitive, we have

$$((a,e,b),(t,e,y)) \in \sigma$$

and hence $(a,t) \in \sigma_A$. It follows that σ_A is transitive and is therefore an equivalence on A. Since A is left trivial, we conclude that σ_A is a congruence on A.

Let σ_L be an extension of σ_A to L, let σ_R be an extension of σ_B to R (see 7.1), and let σ_G be an extension of σ_H to G.

Define $\overline{\sigma} = \{((a,u,b),(x,v,y)) : (a,x) \in \sigma_L, (u,v) \in \sigma_G \text{ and } (b,y) \in \sigma_R\}$. Then $\overline{\sigma}$ is a congruence on S and $\sigma \subseteq \overline{\sigma}$. It remains to show that $\overline{\sigma} \cap (T \times T) = \sigma$.

Let $((a,u,b),(x,v,y)) \in \overline{\sigma} \cap (T \times T)$. Then $(a,x) \in \sigma_L \cap (A \times A) = \sigma_A$, $(u,v) \in \sigma_G \cap (H \times H) = \sigma_H$, and $(b,y) \in \sigma_R \cap (B \times B) = \sigma_B$.

Since $(a,x) \in \sigma_A$, there exist $h,g \in H$ and $s,t \in B$ such that

$$((a,h,s),(x,g,t)) \in \sigma.$$

Since σ is compatible with multiplication on T, we have $((a,h^n,s),(x,g^n,t)) \in \sigma$ for each $n \in \mathbb{N}$. Since T is θ-finite, we have that $((a,e,s),(x,e,t)) \in \sigma$, where e is the identity of G.

Since $(b,y) \in \sigma_B$, there exist $z,w \in H$ and $p,m \in A$ such that

$$((p,z,b),(m,w,y)) \in \sigma.$$

As in the preceeding paragraph, we have $((p, e, b), (m, e, y)) \in \sigma$.

From the fact that $(u, v) \in \sigma_H$, we have that $((r, u, c), (k, v, j)) \in \sigma$ for some $r, k \in A$ and $c, j \in B$. Thus each of the pairs

$$P_1 = ((a, e, s), (x, e, t))$$

$$P_2 = ((r, u, c), (k, v, j))$$

and

$$P_3 = ((p, e, b), (m, e, y))$$

is in σ. Since σ is a congruence on T, we have that

$$P_1 \cdot P_2 \cdot P_3 = ((a, u, b), (x, v, y))$$

is in σ. It follows that $\overline{\sigma} \cap (T \times T) \subseteq \sigma$. ∎

CHAPTER 8

SUMMARY AND OPEN QUESTIONS

In the table and diagrams below, we summarize some of the attributes of:

(1) The congruence extension property (CEP);

(2) The principal congruence extension property (PCEP);

(3) The ideal extension property (IEP);

(4) The principal ideal extension property (PIEP);

(5) Ideal semigroups (IS); and

(6) The group congruence extension property (GCEP).

The term "hereditary" refers to subsemigroups in all cases except for GCEP. In this case it refers to subgroups. The term "homomorphic" means preserved by homomorphisms.

	Hereditary	Productive	Homomorphic
CEP	yes (1.2)	no (1.15)	?
IEP	yes (4.1)	no (4.14)	yes (4.3)
IS	no (4.13)	no (4.12)	yes (4.6)
GCEP	yes (2.1)	no (2.8)	yes (2.1)

Whether CEP is preserved by homomorphisms remains an open question. Chapter 5 gives some partial results for this problem. For quotient homomorphisms (with ideal kernel), the answer is yes (5.25). If the domain semigroup is an ideal semigroup, then the answer is also yes (5.10). For groupoids, the answer (in general) is no (5.12).

The following implication diagram summarizes the relations between these properties for semigroups:

$$\begin{tikzcd}
& \text{Index} \leq 3 \arrow[d, shift right, "3.10"'] & \\
\text{PCEP} \arrow[r, leftrightarrow, "1.11"] & \text{CEP} \arrow[u, shift right, "3.9"'] \arrow[r, leftrightarrow, "6.10"] \arrow[d, shift right, "\text{IS}\ 4.7"'] & \text{No disruptive pairs} \\
\text{PIEP} \arrow[r, leftrightarrow, "4.4"'] & \text{IEP} \arrow[u, shift right, "4.8"'] \arrow[r, leftrightarrow, "\text{def}"'] & \text{No disruptive elements}
\end{tikzcd}$$

(1) A commutative semigroup with CEP has IEP (6.4);

(2) For an ideal semigroup, CEP$\longleftrightarrow$IEP plus each subsemigroup is an ideal semigroup (4.9); and

(3) For a cyclic semigroup, CEP$\longleftrightarrow$ index≤ 3 (3.8).

(4) A commutative semigroup with no disruptive pairs has no disruptive elements (6.11).

The following diagram summarizes the relations between these properties for groups:

$$\begin{tikzcd}
\text{CEP} \arrow[r, shift left, "\text{def}"] \arrow[d, leftrightarrow, "2.11"'] & \text{GCEP} \arrow[l, shift left, "2.7", "/" marking] \arrow[r, leftrightarrow, "2.1"] & \text{Solvable t-group} \\
\text{GCEP + Torsion group}
\end{tikzcd}$$

An abelian group has CEP if and only if it is a torsion group (2.12).

OPEN QUESTIONS.

(1) *The main problem still remaining is whether the congruence extension property (CEP) is preserved by homomorphisms.*

We conjecture that this is true. An extensive computer search for a counter-example among the lower order semigroups was conducted. None were discovered. In view of the groupoid example of 5.12, it appears that a proof must involve some consequence of associativity.

(2) *A complete characterization of semigroups with the congruence extension property (CEP) is not known.*

We are searching for a characterization which involves properties of semigroups which are easily detected. The characterization which we presented in Chapter

6 involving disruptive pairs does not enjoy this feature. A place to start is perhaps the condition that the index be less than four. In view of the example of a semigroup with index 2 that does not have CEP (3.10), it is immediate that this condition will not stand alone (although the converse holds with just this condition (3.9)).

(3) *It is not known whether each completely simple semigroup whose Shutzenberger group has the congruence extension property (CEP) has CEP.*

We conjecture that this is true. The special case, where the sandwich function maps the entire domain to the identity of the group is established in Chapter 7. The technique which was employed in that argument does not extend to the case where the sandwich function is more complicated, since subsemigroups do not generally have a product structure.

(4) *It is yet to be determined whether the structure of the lattice of congruences of a semigroup reveals information regarding the congruence extension property (CEP) for the semigroup.*

We have not considered this problem here. It appears that there is some potential for such a connection, and that this could lead to a resolution of open question (1).

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Index of Definitions

Vita

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Dean of the Graduate School

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