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Mixed Categories of Sheaves on Toric Varieties

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MIXED CATEGORIES OF SHEAVES ON TORIC VARIETIES

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Abstract

In [BGS96], Beilinson, Ginzburg, and Soergel introduced the notion of mixed categories. This idea often underlies many interesting "Koszul dualities." In this paper, we produce a mixed derived category of constructible complexes (in the sense of [BGS96]) for any toric variety associated to a fan. Furthermore, we show that it comes equipped with a t-structure whose heart is a mixed version of the category of perverse sheaves. In chapters 2 and 3, we provide the necessary background. Chapter 2 concerns the categorical preliminaries, while chapter 3 gives the background geometry. This concerns both some basics of toric varieties as well as basics of constructible sheaves in this setting. In chapter 4, we introduce the primary category of interest, $\mathrm{D}^{\mathrm{mix}}(X_0)$ for a toric variety X_0 defined over some finite field. We prove that this is a mixed version of $\mathrm{D}^b_c(X)$, the bounded derived category of constructible complexes over $X = X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}}_q)$, the variety obtained by extension of scalars. In chapter 5, we introduce the standard suite functors associated to a locally closed inclusion of toric varieties, $h : Y_0 \to X_0$, between the mixed categories $\mathrm{D}^{\mathrm{mix}}(X_0)$ and $\mathrm{D}^{\mathrm{mix}}(Y_0)$. We provide some functors associated to other special types of toric maps as well. Finally, we prove that some of these functors commute with the realization functor $\mathfrak{r} : \mathrm{D}^{\mathrm{mix}}(X_0) \to \mathrm{D}^b_{T,m}(X_0)$. We call this being genuine.

Chapter 1

Introduction

In [BGS96], Beilinson, Ginzburg, and Soergel made their seminal contribution to the project of relating various blocks of BGG category $\mathcal{O}$ via Koszul duality. In their paper, they construct a pair of Koszul dual rings for which the blocks in question can be realized as categories of modules over these rings. The surprising fact is precisely that the rings constructed are Koszul; in particular, they are graded. There was no obvious grading on category $\mathcal{O}$, so a natural question becomes, "Where did the grading come from?" Another way to phrase it might be, "Where did the grading go?"

This is the motivation for what we pursue in this paper. In [Bra07] and [BL06], the authors begin to uncover the phenomenon of Koszul duality in the setting of toric varieties. To speak of Koszul duality, one needs just such a strict grading on the category of constructible sheaves on toric varieties. It would be nice to have this more naturally and fully in this context.

1.1 Mixed Categories

In that paper, the authors make note that the "correct" proof relies on mixed geometry and ends up involving what they call **mixed categories**. These are special types of mixed categories, in the above sense of geometry. The authors in this original paper work exclusively with the abelian category of perverse sheaves on a flag variety. Let us be precise and give a definition of this notion of mixed:

Definition 1.1.1. Let $\mathcal{M}$ be a finite-length abelian category. A mixed structure on $\mathcal{M}$ is a function

$$\mathrm{wt} : \mathrm{Irr}(\mathcal{M}) \to \mathbb{Z}$$

from the simple objects of $\mathcal{M}$ to $\mathbb{Z}$ such that for any two simple objects, S, S', such that $\mathrm{wt}(S') \geq \mathrm{wt}(S)$,

$$\mathrm{Ext}^1(S, S') = 0.$$

In the case of a triangulated category, we need another definition.

Definition 1.1.2. Let $\mathcal{D}$ be a triangulated category and suppose that $\mathcal{D}$ has a bounded t-structure on it whose heart is the abelian category $\mathcal{M}$. A mixed structure on $\mathcal{D}$ is a mixed structure on $\mathcal{M}$ such that for any $S, S' \in \mathrm{Irr}(\mathcal{M})$ with $\mathrm{wt}(S') > \mathrm{wt}(S) - i$,

$$\mathrm{Hom}^i_{\mathcal{D}}(S, S') = 0.$$

In the case of abelian categories, an important example to keep in mind is that of a graded ring. If $A = \oplus_{i \geq 0} A_i$ is a graded ring and A_0 is semisimple, then we can consider two different categories. First, there is Mod-A, the category of A-modules where A is considered without its graded structure. Secondly, there is mod-A, the category of graded modules over the (graded) ring A. In this case, mod-A is a mixed version of Mod-A.

1.2 "Mixed" Structures in Geometry

In the world of geometry, the term "mixed" has been in frequent use for more than thirty-five years. In this context, however, it means something different than the definition given above. It can still be heuristically thought of as a sort of a grading. The first use of this term comes from Deligne's proof of the Weil conjectures. In that context, he developed his category of "mixed" constructible complexes for varieties defined over a finite field. In this case, the "mixed" structure was that of the action of the geometric Frobenius on stalks of constructible $\overline{\mathbb{Q}}_\ell$-sheaves at geometric points. In this setting, any constructible sheaf will have such an action, however, Deligne enforces rather stringent conditions on this action for a sheaf to be considered “mixed.” For the reader not familiar with this theory, it is worth noting that this essentially ends up being an invertible linear operator acting on a vector space and Deligne essentially restricts the eignevalues of this linear operator. These eigenvalues are referred to as the “weights” of the sheaf to which they are attached by Deligne.

Results in [BBD82], in particular the much celebrated decomposition theorem, suggested that such a theory seemed likely to exist for varieties over $\mathbb{C}$. Finally, in [Sai88] and [Sai90], Saito accomplished finding this structure via his category of “mixed” Hodge modules. In this case, the notion of weight has to do with mixed Hodge structures that exist on the stalks of sheaves of $\mathbb{C}$-vector spaces. In both this case and in Deligne's, the authors provide a collection of functors for maps $f : X \to Y$ between varieties that correspond to Grothendieck's six functors in the non-“mixed” setting.

Recall from above that for an abelian category to be mixed, it must satisfy the condition that if S and S' are two simple objects with $\mathrm{wt}(S') \geq \mathrm{wt}(S)$, then

$$\mathrm{Ext}^1(S, S') = 0.$$

This immediately implies that any pure object must also be semisimple. This shows that, despite what we would have hoped, Deligne's category of “mixed” perverse sheaves is not a mixed category in the sense of Definition 1.1.1. This is simply because there are sheaves that are pure, but that are not semisimple; that is, the geometric Frobenius does not act semisimply on them. Similarly, we see that $\mathrm{D}^b_m(X_0)$ (or $\mathrm{D}^b_{G,m}(X_0)$ in the equivariant case) is not mixed as a triangulated category.

In the case that we have category (abelian or triangulated) that is not mixed, there is a notion of finding a mixed category that can act as a suitable substitute for the original (non-mixed) category. To do this is essentially the same as adding a grading to a category of modules. We call these mixed substitutes "mixed versions" of the original category. See Definitions 2.1.4 and 2.1.5 for the precise meaning of this.

1.3 Problems and Results

In the present paper, our context is that of toric varieties over finite fields. In particular, we will only consider the étale topology and the theory of $\overline{\mathbb{Q}}_\ell$-sheaves. We will also work in the context of T-equivariant constructible sheaves, where T is the algebraic torus of a given toric variety.

In particular, suppose that $X_0 = X_0(\Delta)$ is a toric variety over a finite field, $\mathbb{F}_q$. Let $X = X_0 \times_{\mathbb{F}_q} \mathrm{Spec}(\overline{\mathbb{F}}_q)$ be the toric variety obtained by extension of scalars, then we have the two categories $\mathrm{D}^b_{T,m}(X_0)$ and $\mathrm{D}^b_{T,c}(X)$ along with the functor

$$\chi : \mathrm{D}^b_{T,m}(X_0) \to \mathrm{D}^b_{T,c}(X),$$

pullback along the canonical map $X \to X_0$. For exactly the same reasons as stated above, $\mathrm{D}^b_{T,m}(X_0)$ is not a mixed category. However, we would still like to have a mixed version of $\mathrm{D}^b_{T,c}(X)$. We then can formalize our problem as:

Problem 1.3.1. *Find a category $D^{mix}(X_0)$ with a realization functor to $D^b_{T,m}(X_0)$ such that the composition*

$$D^{mix}(X_0) \xrightarrow{\mathfrak{r}} D^b_{T,m}(X_0) \xrightarrow{\chi} D^b_{T,c}(X)$$

turns $D^{mix}(X_0)$ into a mixed version of $D^b_{T,c}(X)$.

In the paper [AR13], the authors provide an answer to this problem for $\mathrm{D}^b_c(X)$, the bounded derived category of constructible complexes, for flag varieties. Note that in this case, the sheaves under consideration were not equivariant. Unlike the paper [BGS96], but like our own setting here, their construction deals exclusively with varieties over finite fields. This involves a great deal of categorical machinery outside of standard homological algebra.

The basic strategy is to replace $\mathrm{D}^b_{T,m}(X_0)$ with a category of sheaves that only have semisimple Frobenius action. The simplicity and fundamental nature of the deficiency of these categories not being mixed–that it is as simple as there being nonsemisimple operators–means that it can be very difficult to provide appropriate categories to serve as their mixed substitutes. In the paper [AR13], the authors note that, in the case of the derived category, if one attempts to do the most obvious thing–that is, simply take out those pure objects that are not semisimple–then the resulting category is not triangulated, because it does not contain cones for all morphisms. It is to this end that they are forced to develop a theory of infinitesimal extensions of triangulated categories as well as a theory of Orlov categories.

We define the additive category $\mathrm{Pure}(X_0)$ to be the category whose objects are direct sums of various $\tilde{\mathcal{L}}_\sigma[m](m/2)$. Then we have the following:

Theorem 1.3.2. *Let $X_0 = X_0(\Delta)$ be a toric variety over a finite field and let X be the toric variety obtained by extension of scalars as above. Then the category $D^{mix}(X_0) \coloneqq K^b Pure(X_0)$ is a mixed version of $D^b_{T,c}(X)$.*

At this point, were we to stop, if we are unsure if there are at least some of the standard functors between the mixed categories for various toric varieties, it might be unclear as to how useful such a category is. It was mentioned above that in the settings of Deligne and Saito, they provide analogues of Grothendieck's six functors for their "mixed" settings. We thus arrive at the following problem:

Problem 1.3.3. *Let X_0, Y_0 and X, Y be toric varieties as above. Let $f : X_0 \to Y_0$ be a toric map. That is, we would like to produce functors*

$$f_{(*)}, f_{(!)} : D^{mix}(X_0) \to D^{mix}(Y_0)$$

and

$$f^{(*)}, f^{(!)} : D^{mix}(Y_0) \to D^{mix}(X_0)$$

that adhere to the normal adjunction properties. Furthermore, we would like for these functors to commute with the realization functor in the sense that the diagram

$$\begin{tikzcd} D^{mix}(X_0) \arrow[r, "F'"] \arrow[d, "\mathfrak{r}"] & D^{mix}(Y_0) \arrow[d, "\mathfrak{r}"] \\ D^b_{T,m}(X_0) \arrow[r, "F"] & D^b_{T,m}(Y_0) \end{tikzcd}$$

commutes (where F' is either $f_{(*)}$ or $f_{(!)}$ and F is, correspondingly, f_* or $f_!$) and similarly for $f^{(*)}$ and $f^{(!)}$.

This ultimately has to do with understanding which of the normal functors preserve semisimplicty of Frobenius. This is, however, far too lofty of a goal. We do not arrive at such a general theory, but we do arrive at a solution to this problem for some of the most important toric morphisms. To this end, and beginning with the case of open and closed inclusions, we show that, for any locally closed inclusion of toric varieties

$$h : Y_0 \hookrightarrow X_0,$$

there are functors

$$h_*, h_! : \mathrm{D}^{\mathrm{mix}}(Y_0) \to \mathrm{D}^{\mathrm{mix}}(X_0)$$

and

$$h^*, h^! : \mathrm{D}^{\mathrm{mix}}(X_0) \to \mathrm{D}^{\mathrm{mix}}(Y_0)$$

that satisfy the usual adjoint relationships as hoped. We also show, for some types of toric maps $f : X_0 \to Y_0$, that there are some functors between the mixed categories.

In particular, we arrive at the following:

Theorem 1.3.4. *Let $h : Y_0 \hookrightarrow X_0$ be a locally closed inclusion of toric varieties. Then the desiderata in Problem 1.3.3 are satisfied.*

We also prove that the pushforward of a proper, smooth toric map is genuine. We believe that more than this should be true. However, at this time, it is not clear how to prove such a thing.

1.4 Outline of the Paper

We now proceed to give a general outline of the structure of this paper. In chapter 2, we present an introduction to the necessary categorical preliminaries. That is, we introduce mixed categories, followed by infinitesimal extensions of triangulated categories, then finally the notion of Orlov categories. In chapter 3, we briefly present the basics of toric varieties, sheaf theory, and, in particular, perverse sheaves.

We begin in the first section of chapter 4 by defining $\mathrm{Pure}(X_0) \subseteq \mathrm{D}^b_{T,m}(X_0)$. If for all $\sigma \in \Delta$, we denote by $\tilde{\mathcal{L}}_\sigma$ the simple equivariant perverse sheaf associated to $O(\sigma)$ and Tate twisted to have weight 0, then we define $\mathrm{Pure}(X_0)$ to be the category whose objects are direct sums of various $\tilde{\mathcal{L}}_\sigma[n](n/2)$. This still has weight 0. By [dC15, Theorem 1.4.1], it is known that these sheaves, in addition to being pure, are actually pointwise pure with semisimple Frobenius action.

From this point, we define $\mathrm{D}^{\mathrm{mix}}(X_0)$ to be $\mathrm{K}^b\mathrm{Pure}(X_0)$. Next, we prove that it actually is mixed and that it is a mixed version of $\mathrm{D}^b_{T,c}(X)$. For this step, we employ some general machinery developed in [Rid13]. In her paper, the author develops some fairly general conditions under which just such a category will be a mixed version of $\mathrm{D}^b_{T,c}(X)$.

To apply this machinery, however, we need to know that for any $\sigma, \tau \in \Delta$,

$$\underline{\mathrm{Hom}}^i_T(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau) \coloneqq \mathrm{H}^i(\mathrm{R}a_* \, \mathcal{RH}\mathrm{om}(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau)),$$

where $a : X_0 \to \mathrm{Spec}(\mathbb{F}_q)$ is the canonical map, has semisimple Frobenius action pure of weight i. This turns out to be a rather delicate procedure and relies in great part to results from the paper [Lun95]. With this in hand, the machinery from [Rid13] can be applied and we are finally able to show that $\mathrm{D}^{\mathrm{mix}}(X_0)$ is indeed a mixed version of $\mathrm{D}^b_{T,c}(X)$. This also provides a mixed version, $\mathcal{P}^{\mathrm{mix}}(X_0)$, of $\mathcal{P}_T(X)$, the category of equivariant perverse sheaves on X.

The next step, in chapter 5, is to consider the case when there is a functor $\tilde{F} : \mathrm{D}^{\mathrm{mix}}(X_0) \to \mathrm{D}^{\mathrm{mix}}(Y_0)$ and another functor $F : \mathrm{D}^b_{T,m}(X_0) \to \mathrm{D}^b_{T,m}(Y_0)$ from which $\tilde{F}$ is "induced" as above in Problem 1.3.3. This notion of one functor being induced from another will be made precise in the course of the paper, but for now, one should imagine something like the case of pullback or pushforwards along locally closed inclusions as above. That is, for any locally closed inclusion and, say, considering the pushforward, there is the functor going between Deligne's categories and there is the functor going between the mixed categories. We should think of the mixed version as being induced from the Deligne's functor in some suitable sense.

In these cases, we will say that F is **genuine** when we have a diagram of the form

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{mix}}(X_0) & \xrightarrow{\tilde{F}} & \mathrm{D}^{\mathrm{mix}}(Y_0) \\ \downarrow{\scriptstyle \mathfrak{r}} & & \downarrow{\scriptstyle \mathfrak{r}} \\ \mathrm{D}^b_m(X_0) & \xrightarrow{F} & \mathrm{D}^b_m(Y_0) \end{array}$$

and it commutes. Intuitively, this tells us that $\tilde{F}$ behaves as closely as possible to a true extension of F to the mixed setting.

In this final section, we prove that the standard functors above coming from locally closed inclusions of toric varieties are genuine.

Chapter 2

Categorical Preliminaries

In this section, we recall some of the relevant definitions and theory of mixed categories as well as some aspects of homological algebra that are not necessarily well known in the literature. The framework that we use here is taken from the beautiful paper [AR13]. Therein, the authors interpret the phenomenon of mixed categories and Koszul duality in a rather broad context.

We fix a field $\Bbbk$. We will now assume, unless explicitly stated to the contrary, that all additive categories are $\Bbbk$-linear. We also assume that all functors between additive categories are additive and $\Bbbk$-linear.

If $\mathcal{A}$ is an additive category, then we write $\mathrm{Ind}(\mathcal{A})$ for the set of isomorphism classes of indecomposable objects in $\mathcal{A}$. By abuse of notation, we may even mean by $\mathrm{Ind}(\mathcal{A})$ a collection of chosen representatives of these isomorphisms classes. It should be clear from context which usage is intended.

Likewise, if $\mathcal{M}$ is an abelian category, we denote by $\mathrm{Irr}(\mathcal{M})$ the set of isomorphism classes of simple objects in $\mathcal{M}$ or, by abuse of notation, a set of chosen representatives of these classes. For any $S \in \mathrm{Irr}(\mathcal{M})$, $\mathrm{End}(S)$ is a division ring over $\Bbbk$.

We would also like to recall what it means for an abelian category to be split. We say that an abelian category $\mathcal{M}$ is **split** if for all $S \in \mathrm{Irr}(\mathcal{M})$,

$$\mathrm{End}(S) \simeq \Bbbk. \tag{2.1}$$

Finally, we say that an abelian category $\mathcal{M}$ is **finite-length** if $\mathcal{M}$ is both noetherian and artinian.

2.1 Mixed Categories

In this section, we introduce the concept of a mixed category. This will be done at both the abelian and triangulated levels. While much of this theory is well known by experts, we feel it is worth writing this up not only to fix notation, but also as background for readers who are new to these notions.

Let $\mathcal{M}$ be a finite-length abelian category. As in [AR13], a mixed structure on $\mathcal{M}$ is a function

$$\mathrm{wt} : \mathrm{Irr}(\mathcal{M}) \to \mathbb{Z} \tag{2.2}$$

such that for any $S, S' \in \mathrm{Irr}(\mathcal{M})$ with $\mathrm{wt}(S') \geq \mathrm{wt}(S)$,

$$\mathrm{Ext}^1(S, S') = 0. \tag{2.3}$$

Such a function is called a **weight function**. For $X \in \mathcal{M}$, the weights of X are simply the set of numbers $\{\mathrm{wt}(X_i)\}$ where $\{X_i\}$ are the composition factors of X. An object is said to be pure if all its simple composition factors have the same weight. In consequence of 2.3, any pure object is semisimple. Each object $X \in \mathcal{M}$ is endowed with a canonical **weight filtration**

$$\mathrm{W}_\bullet X$$

such that for each k, $W_k X$ is the unique maximal subobject with highest weights $\leq k$.

To define the notion of a mixed structure on a triangulated category, we assume that we have a triangulated category $\mathcal{D}$ with a bounded t-structure whose heart is the finite-length abelian category $\mathcal{M}$.

Definition 2.1.1. A **mixed structure on** $\mathcal{D}$ is a mixed structure on $\mathcal{M}$ that satisfies a stronger version of 2.3. Namely, for any $S, S' \in \mathrm{Irr}(\mathcal{M})$ such that

$\mathrm{wt}(S') > \mathrm{wt}(S) - i$,

$$\mathrm{Hom}^i_{\mathcal{D}}(S, S') = 0. \tag{2.4}$$

We say that an object $X \in \mathcal{D}$ has weights $\leq w$ (resp. $\geq w$, pure of weight w) if for all i, $\mathrm{H}^i(X)$ has weights $\leq w + i$ (resp. $\geq w + i$, $= w + i$).

In the special case that $\mathcal{D} = \mathrm{D}^b(\mathcal{M})$, then 2.3 implies 2.4. In general, however, 2.4 is strictly stronger. We now recall some well known facts.

Lemma 2.1.2. *Let $\mathcal{M}$ be the heart of a t-structure on the triangulated category $\mathcal{D}$ and suppose that $\mathcal{D}$ has a mixed structure on it.*

1. *If $X, Y \in \mathcal{D}$, X has weights $\leq w$, and Y has weights $> w$, then $Hom_{\mathcal{D}}(X, Y) = 0$.*

2. *Let $X \in \mathcal{D}$ be an object with weights $\geq a$ and $\leq b$. For any $w \in \mathbb{Z}$, there is a distinguished triangle*

$$X' \to X \to X'' \xrightarrow{+1}$$

   where X' has weights $\geq a$ and $\leq w$ and X'' has weights $\leq b$ and $> w$.

3. *Every pure object $X \in \mathcal{D}$ is semisimple. That is, if $X \in \mathcal{D}$ is pure, then $X \simeq \oplus_i H^i(X)[-i]$ where each $H^i(X)[-i] \in \mathcal{M}$ is pure (and so semisimple) of weight $w + i$.*

It is worth noting that neither the distinguished triangles in (2) nor the direct sum in (3) above are canonical in general.

Finally, we wish to introduce a few more notions. The first is that of a Tate twist.

Definition 2.1.3. Suppose that $\mathcal{M}$ is a mixed abelian category. A **Tate twist** on $\mathcal{M}$ is an autoequivalence

$$\langle 1 \rangle : \mathcal{M} \to \mathcal{M}$$

such that $\mathrm{wt}(\mathcal{M}\langle 1\rangle) = \mathrm{wt}(\mathcal{M}) + 1$.

We can now define a key notion:

Definition 2.1.4. Suppose that $\mathcal{M}$ and $\mathcal{M}'$ are two finite-length abelian categories. Further, suppose that $\mathcal{M}$ is a mixed category with weight function $\mathrm{wt} : \mathrm{Irr}(\mathcal{M}) \to \mathbb{Z}$ and Tate twist $\langle 1\rangle : \mathcal{M} \to \mathcal{M}$. Suppose that there exists an exact functor

$$\zeta : \mathcal{M} \to \mathcal{M}'$$

and an isomorphism

$$\epsilon : \zeta \circ \langle 1\rangle \to \zeta.$$

Assume that all simple objects of $\mathcal{M}'$ lie in the essential image of ζ. Then we say that $\mathcal{M}$ is a **mixed version of $\mathcal{M}'$** if for every $M, N \in \mathcal{M}$, ζ induces an isomorphism

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{M}}(M, N\langle n\rangle) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{M}'}(\zeta M, \zeta N). \tag{2.5}$$

This is the notion of a mixed version of an abelian category. There are, however, two ways of generalizing this to the setting of a triangulated category.

Definition 2.1.5. Let $\mathcal{D}$ and $\mathcal{D}'$ be two triangulated categories. Let $\mathcal{D}$ be equipped with an autoequivalence $\langle 1\rangle : \mathcal{D} \to \mathcal{D}$. Furthermore, suppose that we have a functor $\zeta : \mathcal{D} \to \mathcal{D}'$ such that the essential image of ζ generates $\mathcal{D}'$ as a triangulated category as well as an isomorphism $\epsilon : \zeta \circ \langle 1\rangle \xrightarrow{\sim} \zeta$. Then $\mathcal{D}$ is called a **graded version of $\mathcal{D}'$** if (2.5) holds for all $M, N \in \mathcal{D}$.

Suppose, in addition, that $\mathcal{D}$ and $\mathcal{D}'$ are equipped with t-structures such that $\mathcal{D}$ is a mixed triangulated category and that ζ and $\langle 1\rangle$ are t-exact with respect to these t-structures. Then we say that $\mathcal{D}$ is a **mixed version of $\mathcal{D}'$**.

2.2 Infinitesimal Extensions of Triangulated Categories

In this section we describe a way to "infinitesimally thicken" a triangulated category. This theory was originally developed in [AR13]. The main reason we need this theory is because of its positive interactions with the theory of Orlov categories that will be recalled in the sequel.

The first thing to be noted about infinitesimal extensions of triangulated categories is that they are not themselves triangulated. This is, in fact, the reason why Achar and Riche needed to develop this machinery: To have a method for interacting with categories that were close to being triangulated, but did not have cones for all morphisms. It turns out that one can still develop a reasonable amount of homological algebra to interact with these objects. We will need to begin with a definition.

Definition 2.2.1. Let $\mathscr{D}$ be a triangulated category. The infinitesimal extension of $\mathscr{D}$, $\mathscr{I}\mathscr{D}$, is the category with the same objects as $\mathscr{D}$ but with

$$\mathrm{Hom}_{\mathscr{I}\mathscr{D}}(X,Y) = \mathrm{Hom}_{\mathscr{D}}(X,Y) \oplus \mathrm{Hom}_{\mathscr{D}}(X,Y[-1]).$$

Here composition is given by the rule

$$(g_0, g') \circ (f_0, f') = (g_0 \circ f_0, g_0[-1] \circ f' + g' \circ f_0).$$

We now need to recall some basic functors between a triangulated category $\mathscr{D}$ and its infinitesimal extension. First, there is the obvious inclusion functor

$$\iota : \mathscr{D} \hookrightarrow \mathscr{I}\mathscr{D}.$$

This sends objects to themselves and for which

$$\mathrm{Hom}_{\mathscr{D}}(X,Y) \to \mathrm{Hom}_{\mathscr{I}\mathscr{D}}(X,Y)$$

is the inclusion map. We also have the map

$$\varpi : \mathscr{I}\mathscr{D} \to \mathscr{D}.$$

This also maps objects to themselves, but the map

$$\mathrm{Hom}_{\mathscr{I}\mathscr{D}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(\iota X, \iota Y)$$

is projection. Finally, we have the inclusion map

$$v : \mathrm{Hom}_{\mathscr{D}}(X, Y[-1]) \to \mathrm{Hom}_{\mathscr{I}\mathscr{D}}(\iota X, \iota Y).$$

The formula for this is, for $f \in \mathrm{Hom}_{\mathscr{D}}(X, Y[-1])$, $v(f) = (0, f)$. This is a natural transformation.

Definition 2.2.2. We say that a morphism $f \in \mathrm{Hom}_{\mathscr{I}\mathscr{D}}(X, Y)$ is **infinitesimal** if $\varpi(f) = 0$. We say that f is **genuine** if $f = \iota(f_0)$ for some $f_0 \in \mathrm{Hom}_{\mathscr{D}}(X, Y)$.

It is important to note that the property of being genuine is **not** natural. The property of being infinitesimal is, however. (Cf. [AR13] Remark 3.2.)

We say that a triangle $X \to Y \to Z \xrightarrow{+1}$ is **distinguished** if there exists a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ \iota(X') \arrow[r] & \iota(Y') \arrow[r] & \iota(Z') \end{tikzcd}$$

such that $X' \to Y' \to Z' \xrightarrow{+1}$ is a distinguished triangle in $\mathscr{D}$.

Definition 2.2.3. Let $\mathscr{C}$ and $\mathscr{D}$ be triangulated categories. Suppose that $\mathscr{I}\mathscr{C}$ and $\mathscr{I}\mathscr{D}$ are their infinitesimal extensions. An additive functor $F : \mathscr{I}\mathscr{C} \to \mathscr{I}\mathscr{D}$ is said to be **pseudotriangulated** if and only if

1. It commutes with [1] and takes distinguished triangles to distinguished triangles.

2. It commutes with $\upsilon \circ \varpi$.

Lemma 2.2.4. *[AR13, Lemma 3.8] Let $F : \mathscr{I}\mathscr{C} \to \mathscr{I}\mathscr{D}$ be a pseudotriangulated functor. Then there exists a functor, unique up to isomorphism, $\tilde{F} : \mathscr{C} \to \mathscr{D}$ such that*

$$\varpi \circ F = \tilde{F} \circ \varpi.$$

In the situation of this lemma, we say that $\tilde{F}$ is **induced** by F.

Definition 2.2.5. Let $F : \mathscr{C} \to \mathscr{D}$ be a pseudotriangulated functor. We say that F is **genuine** if and only if

$$F \circ \iota = \iota \circ \tilde{F}.$$

There is much more to be said about infinitesimal extensions of triangulated categories and pseudotriangulated functors between them. For that, we encourage the reader to see especially [AR13, Section 3]. For now, we mention only one more result that will be significant for us in the sequel.

Theorem 2.2.6. *[AR13, Theorem 3.16] Let $F : \mathscr{I}\mathscr{C} \to \mathscr{I}\mathscr{D}$ be a genuine pseudotriangulated functor. If F has a right adjoint (respectively left adjoint) pseudotriangulated functor $G : \mathscr{I}\mathscr{D} \to \mathscr{I}\mathscr{C}$, then G is genuine as well.*

2.3 Orlov Categories

In [AR13], the authors introduced the notion of an Orlov category. Their motivation was to generalize a proof technique of Orlov for showing that two functors are isomorphic that he used in [Orl97]. In their paper Achar and Riche also link Orlov categories and Koszul categories. We wish to give a review of their theory here. For a complete introduction, see [AR13]. In this section, we will only assume that $\mathcal{A}$ is an additive category.

Definition 2.3.1. Let $\mathcal{A}$ be an additive category enriched over a field $\Bbbk$ and let $\mathrm{Ind}(\mathcal{A})$ be the collection of isomorphism classes of indecomposable objects. Suppose that $\mathrm{Ind}(\mathcal{A})$ is finite and suppose that there is a function

$$\deg : \mathrm{Ind}(\mathcal{A}) \to \mathbb{Z}.$$

Then we say that $\mathcal{A}$ is an **Orlov category** if the following are satisfied:

1. All Hom-spaces in $\mathcal{A}$ are finite-dimensional.

2. For all $S \in \mathrm{Ind}(\mathcal{A})$, $\mathrm{End}(S) \simeq \Bbbk$.

3. For all $S, S' \in \mathrm{Ind}(\mathcal{A})$ such that $S \not\simeq S'$ and $\deg(S) \leq \deg(S')$, $\mathrm{Hom}(S, S') = 0$.

We say that an object $X \in \mathcal{A}$ is **homogeneous of degree** n if $X \simeq \bigoplus_{i \in I} S_i$ such that $S_i \in \mathrm{Ind}(\mathcal{A})$ and $\deg(S_i) = n$ for all $i \in I$. We say that a functor $F : \mathcal{A} \to \mathcal{B}$ between two Orlov categories is a **homogeneous functor** if it takes homogeneous objects of degree n to homogeneous objects of degree n.

Definition 2.3.2.

- We say that an additive category is **Karoubian** if every idempotent endomorphism splits.

- We say that an additive category is **Krull–Schmidt** if every objects is isomorphic to a finite direct sum of indecomposable objects whose isomorphism classes and multiplicities are determined uniquely.

We can now recall from [AR13] the following:

Corollary 2.3.3. *Let $\mathcal{A}$ be an Orlov category. Then $\mathcal{A}$ is both Karoubian and Krull-Schmidt.*

Now, suppose that $\mathcal{A}$ is an Orlov category and let $\mathrm{K}^b(\mathcal{A})$ denote its bounded homotopy category.

Definition 2.3.4. For $X^\bullet \in \mathrm{K}^b(\mathcal{A})$, the **support of** $X^\bullet$, with $\operatorname{supp} X^\bullet \subset \mathbb{Z} \times \mathbb{Z}$ is defined as follows:

We say that $(i,j) \in \operatorname{supp} X^\bullet$ if and only if X^i contains a non-zero direct summand of degree j.

It is very important to note that this notion is **not** homotopy-invariant. That is, two isomorphic objects in $\mathrm{K}^b(\mathcal{A})$ can have different supports. Even in spite of this significant limitation, this will be an important and useful concept. For any subset $\Sigma \subset \mathbb{Z} \times \mathbb{Z}$, we define the full subcategory

$$\mathrm{K}^b(\mathcal{A})_\Sigma := \{X \in \mathrm{K}^b(\mathcal{A}) \mid X \simeq X' \text{ such that } \operatorname{supp}(X') \subseteq \Sigma\}.$$

It is clear that every $X \in \mathrm{K}^b(\mathcal{A})$ belongs to $\mathrm{K}^b(\mathcal{A})_\Sigma$ for some finite $\Sigma \subset \mathbb{Z} \times \mathbb{Z}$. We will give $\mathbb{Z} \times \mathbb{Z}$ the lexicographic order. Namely, $(i,j) \leq (i',j')$ if and only if either $i \leq i'$ or $i = i'$ and $j \leq j'$. With respect to the lexicographic order, any finite subset $\Sigma \in \mathbb{Z} \times \mathbb{Z}$ has a largest element. Now, consider the following two subsets of $\mathbb{Z} \times \mathbb{Z}$:

$$\triangleleft := \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid i \leq -j\}$$

$$\triangleright := \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid i \geq -j\}$$

To these two subsets, we can, of course, associate two full subcategories, $\mathrm{K}^b(\mathcal{A})_\triangleleft$ and $\mathrm{K}^b(\mathcal{A})_\triangleright$, of $\mathrm{K}^b(\mathcal{A})$. It is further clear that these two full subcategories "cover" $\mathrm{K}^b(\mathcal{A})$. We now have the following lemmas:

Lemma 2.3.5. *[AR13, Lemma 5.1] If $X \in K^b(\mathcal{A})_\triangleleft$ and $Y[1] \in K^b(\mathcal{A})_\triangleright$, then $Hom(X,Y) = 0$.*

Lemma 2.3.6. *[AR13, Lemma 5.2] Let $S \in Ind(\mathcal{A})$.*

1. *If $X \in K^b(\mathcal{A})_\triangleright$, then the cone of any non-zero morphism $S[\deg S] \to X$ lies in $K^b(\mathcal{A})_\triangleright$.*

2. *If $X \in K^b(\mathcal{A})_\triangleleft$, then the cocone of any non-zero morphism $X \to S[\deg S]$ lies in $K^b(\mathcal{A})_\triangleleft$.*

Lemma 2.3.7. *[AR13, Lemma 5.3] For any $X \in K^b(\mathcal{A})$ there exists a distinguished triangle $A \to X \to B \xrightarrow{+1}$ with $A \in K^b(\mathcal{A})_{\triangleleft}$ and $B[1] \in K^b(\mathcal{A})_{\triangleright}$.*

We can now state the most important of these results (for our purposes at least):

Theorem 2.3.8. *[AR13, Proposition 5.4] For any Orlov category $\mathcal{A}$, the pair $(K^b(\mathcal{A})_{\triangleleft}, K^b(\mathcal{A})_{\triangleright})$ is a bounded t-structure. Moreover, the heart of this t-structure,*

$$Kos(\mathcal{A}) \coloneqq K^b(\mathcal{A})_{\triangleleft} \cap K^b(\mathcal{A})_{\triangleright},$$

is a split finite-length abelian category. The simple objects in $Kos(\mathcal{A})$ are those isomorphic to objects in the set

$$Irr(Kos(\mathcal{A})) = \{S[deg\, S] \mid S \in Ind(\mathcal{A})\}.$$

Furthermore, $Kos(\mathcal{A})$ has the structure of a mixed category. The weight function $wt : Irr(Kos(\mathcal{A})) \to \mathbb{Z}$ is given by

$$wt(S[deg\, S]) = deg\, S.$$

The proof of this theorem in [AR13] relies on Lemmas 5.1 and 5.3 that we have restated above. We next have the corollary

Corollary 2.3.9. *Consider $K^b(\mathcal{A})_{\triangleleft\cap\triangleright} = Kos(\mathcal{A})$. Then, for $X = (X^\bullet, d_X) \in K^b(\mathcal{A})$ such that $supp(X) \subset \triangleleft\cap\triangleright$, the associated graded with respect to the weight filtration on X is given by*

$$gr_i^W X = X^{-i}[i].$$

We also have the following result that will be relevant for our purposes:

Theorem 2.3.10. *Let $\mathcal{A}$ be an Orlov category. Then the mixed structure on $Kos(\mathcal{A})$ gives $K^b(\mathcal{A})$ the structure of a mixed triangulated category. Furthermore, if $X, Y \in K^b(\mathcal{A})$ such that $wt(X) \leq w$ and $wt(Y) > w$, then $Hom(X, Y) = Hom(Y, X) = 0$.*

Orlov categories are also important, because they lead to rather "rigid" conditions on functors between their bounded homotopy categories.

Theorem 2.3.11. *[AR13, Theorem 4.7] Suppose that $\mathcal{C}$ and $\mathcal{D}$ are Orlov categories. If $F, F' : K^b(\mathcal{C}) \to K^b(\mathcal{D})$ are two triangulated functors between their bounded homotopy categories. Suppose that $F(\mathcal{C}) \subseteq \mathcal{D}$ and $F'(\mathcal{C}) \subseteq \mathcal{D}$. Also, suppose that $F|_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ and $F'|_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ are homogeneous functors. Then any natural transformation of additive functors*

$$\theta^\circ : F|_{\mathcal{C}} \to F'|_{\mathcal{C}}$$

can be extended to a natural transformation

$$\theta : F \to F'$$

in such a way so that if $\theta^\circ : F|_{\mathcal{C}} \xrightarrow{\sim} F'|_{\mathcal{C}}$ is an isomorphism, then so is $\theta : F \xrightarrow{\sim} F'$.

We also have the following:

Theorem 2.3.12. *[AR13, Theorem 4.9] Suppose that $\mathcal{C}$ and $\mathcal{D}$ are Orlov categories. Suppose that $F, F' : K^b(\mathcal{C}) \to \mathscr{I} K^b(\mathcal{D})$ are two pseudotriangulated functors. Furthermore, suppose that $F|_{\mathcal{C}}(\mathcal{C}) \subseteq \mathcal{D}$ and $F'|_{\mathcal{C}}(\mathcal{C}) \subseteq \mathcal{D}$ and that these restricted functors are homogeneous. Then any natural transformation of additive functors*

$$\theta^\circ : F|_{\mathcal{C}} \to F'|_{\mathcal{C}}$$

can be extended to a natural transformation

$$\theta : F \to F'$$

in such a way so that if θ° is an isomorphism, then θ is as well.

We also need the following:

Theorem 2.3.13. *[AR13, Theorem 4.11] Let $\mathcal{A}$ and $\mathcal{B}$ be two Orlov categories and let $F : \mathscr{I}K^b(\mathcal{A}) \to \mathscr{I}K^b(\mathcal{B})$ be a pseudotriangulated functor. If $F(\mathcal{A}) \subseteq \mathcal{B}$ and the restricted functor $F|_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ is homogeneous, then F is genuine.*

These theorems will be extremely important for us later in proving genuineness of some of the functors we construct.

Chapter 3

Preliminaries From Toric Geometry

3.1 Basic Toric Geometry

In this section, we wish to introduce toric varieties associated to fans and give some general facts about them. This is an area of much interest in algebraic geometry, because of (a) the nice properties associated with them and (b) the fact that there are (paradoxically) so many of them makes them a good testing ground for many theorems. For the reader who might not have come in contact with these beautiful spaces yet, we would like to make it so that the paper could still be read after this introduction with much profit. We would also like to introduce some standard facts about sheaves, their derived categories, and perverse sheaves. We will only briefly describe some of the most important results without proof. For proofs of the statements herein as well as many other details about toric varieties, see [Ful93] or [CLS11] for a full introductory account.

A geometric definition of a toric variety is that it is a variety with an algebraic torus as an open, dense subvariety such that the natural action of the torus on itself extends to an action on the entire variety. Some natural examples to keep in mind are $(\mathbb{G}_m)^n$, $\mathbb{A}^n$, and $\mathbb{P}^n$. (Here $\mathbb{G}_m$ stands for the multiplicative algebraic group.) Given in this form, it is not immediately clear how to construct new and interesting toric varieties, but thankfully for the theory there is a combinatorial way to construct toric varieties. This not only makes it easy to find examples of them, but it also allows the combinatorics to interact with the algebraic geometry of the varieties.

By Δ we will always mean a collection of strongly convex rational cones in a real n-dimensional vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ where N is an n-dimensional lattice.

Finally, by M we mean the dual lattice to N. If σ is a cone in N, then we will denote by $\sigma^\vee$ the dual lattice in M via the non-degenerate pairing between N and M. From the single cone σ, the associated toric variety $X(\sigma)$ is defined as $\mathrm{Spec}(k[\sigma^\vee \cap M])$. For a fan, Δ, we form the variety $X(\Delta)$ by forming the varieties $X(\sigma)$ for all cones inside Δ and gluing them together along their shared faces. Not all toric varieties arise from fans in this way, but we will follow the convention of essentially ignoring those cases. So, henceforth, by "toric variety" we will mean "toric variety arising from a fan." It is well known that toric varieties arising from fans are always normal varieties.

The next natural step after defining the objects of study is to describe morphisms between those objects. First of all, toric varieties are, in particular, algebraic varieties, so if we are only thinking of them with that structure, then we can clearly choose any morphism of varieties. However, if we wish to think of them with respect not only to their variety structure, but also with respect to the action of the torus, then we need something else. To this end, we require the morphisms to respect that structure:

Definition 3.1.1. Let $T_1 \subseteq X(\Delta_1)$ and $T_2 \subseteq X(\Delta_2)$ be two toric varieties with corresponding tori T_1, T_2. Given a map $\Phi : X(\Delta_1) \to X(\Delta_2)$, we say that it is a toric map if it is equivariant with respect to the two torus actions. That is, if

$$\Phi(t_1 \cdot x) = t_2 \cdot \Phi(x)$$

for all $t_1 \in T_1$ and where $t_2 = \Phi(t_1) \in T_2$. Here $t_i \cdot$ is understood as the action of T_i for $i = 1, 2$.

Given our point of view of toric varieties as varieties combinatorially/functorially made from fans, we should expect a characterization of toric maps in terms of fans. Indeed, we have such a thing.

Theorem 3.1.2. *Let X_1 and X_2 be two toric varieties arising from the fans $\Delta_1 \subset N_1$ and $\Delta_2 \subset N_2$ respectively. Then a morphism*

$$f : X_1 \to X_2$$

is a toric map, that is a map that is equivariant with respect to the torus actions on X_1 and X_2, if and only if there exists a map of lattices

$$f_N : N_1 \to N_2$$

such that for any $\sigma_1 \in \Delta_1$, there exists a $\sigma_2 \in \Delta_2$ so that $f_N(\sigma_1) \subseteq \sigma_2$.

Another fundamental result is the cone-orbit correspondence for toric varieties. Before we state this though, we introduce a relation between two cones $\sigma, \tau \in \Delta$. We say that $\tau < \sigma$ if τ is a face of σ. Now we can state the following:

Theorem 3.1.3. *Let $X(\Delta)$ be a toric variety with torus $T = T(\Delta)$. There is a one-to-one correspondence between T-orbits and cones $\sigma \in \Delta$. Furthermore, there are the following relations between T-orbits $O(\sigma)$, distinguished affine open sub-toric varieties U_σ, and orbit closures $V(\sigma)$:*

1. $U_\sigma = \bigsqcup_{\tau \leq \sigma} O(\tau)$.

2. $V(\sigma) = \bigsqcup_{\tau \geq \sigma} O(\sigma)$.

3. $O(\sigma) = V(\sigma) \setminus \bigsqcup_{\tau > \sigma} O(\tau)$.

This theorem is especially important for the study of perverse sheaves, since it gives us a combinatorial calculus for the torus stratification. This will always be the stratification with respect to which we consider the category of perverse sheaves on a toric variety.

Another fact that will be incredibly important for us is the "local product structure" of toric varieties. In order to do this though, we must first introduce the notion of an (affine) **toric variety of contractible type**.

We say that an affine toric variety is of contractible type if its fan $\sigma \subset N$ has full dimension. In symbols, if $\dim(\sigma) = \dim(N)$. In this case, it is a theorem that $X(\sigma)$ is actually a contractible space with a unique torus fixed point.

We can finally give the local product strucure of a toric variety. Specifically, we have that

Lemma 3.1.4. *Let $X(\Delta)$ be a toric variety and let $\sigma \in \Delta$ be any cone. Then there is a non-canonical isomorphism*

$$U_\sigma \simeq O(\sigma) \times U'_\sigma$$

where U'_σ is a toric variety of contractible type for torus $T_\sigma = stab(O(\sigma))$. Furthermore, $O(\sigma) \cap U'_\sigma = w_\sigma$ where w_σ is the unique T_σ-fixed point in U'_σ.

Note that in the above setup, $O(\sigma)$ is a $T' \simeq T/T_\sigma \simeq O(\sigma)$ toric variety. Note that each orbit $O(\sigma)$ is isomorphic to a torus. Given a cone $\sigma \in \Delta$, we will often use the notation

$$T \simeq T' \times T_\sigma \tag{3.1}$$

with the meaning as given above.

Suppose that $f : X \to Y$ is a proper toric fibration. For $\sigma \in \Delta(Y)$, there is a non-canonical, equivariant splitting as in 3.1 and a non-canonical equivariant isomorphism of toric maps, compatible with 3.1:

$$(f^{-1}(U_\sigma) \to U_\sigma) \simeq (f^{-1}(U'_\sigma) \times O(\sigma) \xrightarrow{f_{\sigma'} \times Id} U_{\sigma'} \times O(\sigma)). \tag{3.2}$$

The resulting, natural map $f_{\sigma'} = f|_{U_{\sigma'}}$ is a toric fibration onto a base of contractible type. Furthermore, we have a natural identification $f^{-1}(y_\sigma) = f_{\sigma'}^{-1}(y_{\sigma'})$.

This implies that we have a non-canonical, $(T_X \to T_Y(\sigma))$-equivariant decomposition:

$$f^{-1}(O(\sigma)) \simeq f^{-1}(y_\sigma) \times O(\sigma).$$

For the next result, we will still consider $f : X \to Y$, a toric map. Fix $\xi \in \Delta(X)$. Consider the natural map of tori induced by f:

$$\phi : (O(\xi), x_\xi) \to (O(\bar(\xi)), y_{\overline{\xi}}).$$

The image,

$$i : (O'(\xi), y_{\overline{\xi}}) \to O(\overline{\xi}), y_{\overline{\xi}}),$$

is a closed subtorus. There is the canonical factorization into maps of tori:

$$\phi : O(\xi) \xrightarrow{a} A \xrightarrow{b} B \xrightarrow{c} O'(\xi) \xrightarrow{i} O(\overline{\xi}). \tag{3.3}$$

Here, a is a toric fibration, non-canonically a product projection, b is a geometric quotient map, étale and Galois by the action of a finite subgroup of the torus A, i is the natural closed inclusion above, and c is a universal homeomorphism.

Next, we let $f : X \to Y$ be a proper toric map again. There is a canonical toric Stein factorization

$$f : X \xrightarrow{g} Z \xrightarrow{h} Y.$$

In this setup, g is a proper toric fibration and h is a toric finite map. The normalization of $f(X) \subset Y$ is a toric variety.

Finally, we wish to make mention of an incredibly important fact concerning toric varieties of contractible type. If X is a toric variety of contractible type with fixed point x, then we will often write it as (X, x). Let $f : X \to Y$ be a proper toric fibration onto a toric variety (Y, y) of contractible type. Let $\mathcal{F}$ be an equivariant complex on X. We are assuming here that the ground field is either finite or algebraically closed. Then the natural, graded map

$$\mathrm{H}^\bullet(X, \mathcal{F}) = \mathrm{H}^\bullet(Y, f_*\mathcal{F}) \to (\mathrm{R}^\bullet f_*\mathcal{F})_y \tag{3.4}$$

is an isomorphism. (Here, as elsewhere, we write f_* for the right derived functor.) If the ground field is finite, then it is understood that we have passed to the algebraic closure. In this case, the isomorphism is compatible with the action of Frobenius on both sides.

This is a very specific instance of a much more general theorem. For the general statement, see either [Spr84] or the discussion following [Bra03, Lemma 6]. We will refer to this as the **homotopy trick**.

3.2 Sheaf Theory

Let X_0 be a variety over a finite field $\mathbb{F}_q$. We simply write X for the variety $X_0 \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}(\overline{\mathbb{F}}_q)$. This is the variety we get by extending scalars to an algebraic closure. It is common to abuse notation and write this as $X = X_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. The variety X_0 comes equipped with a geometric Frobenius map that acts on the stalks of constructible sheaves at geometric points. We will now give a brief introduction to this. For more information concerning this morphism, see [FK88], [KW01], and [Mil80] for a detailed explanation.

We denote by $\overline{x}$ a geometric point on X_0. For any geometric point $\overline{x}$, we can consider the pullback of $\mathbb{F}_0$ to $\overline{x}$. This has an action of the group $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$. In particular, it carries an action of Fr, the **geometric Frobenius** element. This is the inverse of the usual (or arithmetic) Frobenius automorphism of the Galois group. We only consider constructible complexes $\mathbb{F}_0$ such that, after choosing an isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$, the eigenvalues of Fr on the stalks are algebraic numbers with complex conjugates all having absolute value equal to $q^{n/2}$ for any integer n. This must be independent of the isomorphism chosen.

If only one such n appears for a sheaf $\mathcal{F}_0$, then we say that the sheaf is **pointwise pure**. (This is also called **punctual purity** in some places in the literature.) If $\mathcal{F}_0$ has a filtration by subsheaves such that the subquotients appearing in the filtration are pure, then we say that the $\mathcal{F}_0$ is **mixed**. The n's that appear here are referred to as the "weights" of the sheaf. If $\mathcal{F}_0^\bullet$ is a complex of sheaves, then $\mathcal{F}_0^\bullet$ is pointwise pure of weight n if $\mathcal{H}^i(\mathcal{F}_0^\bullet)|_{\overline{x}}$ is pointwise pure of weight $n+i$ for all i and for all geometric points $\overline{x}$.

For a complex $\mathcal{F}_0^\bullet$, we say that $\mathcal{F}_0^\bullet$ has weights $\leq n$ if for each i, $\mathcal{H}^i(\mathcal{F}_0^\bullet)$ has weights $\leq n+i$. However, we say that $\mathcal{F}_0^\bullet$ has weights $\geq n$ if $\mathbb{D}(\mathcal{F}_0^\bullet)$ has weights $\leq -n$. (Note that the weights of these cohomology sheaves are in the sense of pointwise purity.)

In particular, the constant sheaf on a smooth variety is pure of weight 0. The constant sheaf is always pointwise pure of weight 0 on any variety, but for singular varieties, it is not pure as a complex in the derived category. Another example is that for any stratum X_α, $IC(X_\alpha)$ is pure of weight $n_\alpha = \dim(X_\alpha)$. It is worth noting that it is not, however, pointwise pure in general.

We should also mention the notion of the **Tate twist**. Recall that Tate twisting is (non-canonically) isomorphic to tensoring by the constant sheaf. That is to say, Tate twisting is non-canonically isomorphic to the identity functor. This is why Tate twist commutes with all sheaf functors. The fact that this is non-canonical is just the fact that there is no distinguished unit in the ℓ^nth roots of unity. This is also why this procedure changes the action of Fr. If $\mathcal{F}$ is pure of weight n (has weights $\leq n$, has weights $\geq n$), then $\mathcal{F}(1)$ is pure of weight $n-2$ (has weights $\leq n-2$, has weights $\geq n-2$). Here and now, we will choose a square root of the Tate sheaf and consider it fixed for the rest of this paper.

Definition 3.2.1. A complex $\mathcal{F}_0^\bullet$ is pure, even, and Tate if for all geometric points $\overline{x}$, we have

$$\mathcal{H}^\bullet(\mathcal{F}_0^\bullet)|_{\overline{x}} \simeq \bigoplus_{i\in\mathbb{Z}} \overline{\mathbb{Q}}_\ell^{\oplus n(i)}[2i](i).$$

The notion of being pure, odd, and Tate is the same except for having only odd degrees. That is, a complex $\mathcal{F}_0^\bullet$ is pure, odd, and Tate if for all geometric points $\overline{x}$, we have

$$\mathcal{H}^\bullet(\mathcal{F}_0^\bullet)|_{\overline{x}} \simeq \bigoplus_{i\in\mathbb{Z}} \underline{\bar{\mathbb{Q}}}_\ell^{\oplus n(i)}[2i+1](\frac{2i+1}{2}).$$

This makes sense, since we have chosen a square root of the Tate sheaf.

In this section, we recall some basics concerning $\mathrm{D}_c^b(X)$ and $\mathrm{D}_m^b(X_0)$, the bounded derived category of constructible complexes and the bounded derived category of mixed constructible complexes, respectively. In this paper, we will be working with varieties over $\mathbb{F}_q$ and $\overline{\mathbb{F}}_q$. In this context, constructible complexes are always with respect to the étale topology. Cohomology also always refers to étale cohomology. Obviously, there are occasions in this setting where different topologies or cohomologies are used, but they are explicitly mentioned when applicable. For the reader unfamiliar with this setting, one may (for the most part) treat it like a black box and pretend that these are varieties over $\mathbb{C}$ endowed with the Euclidean topology. This is at least true enough of the time to allow one to read through most details on a first approach. For introductions to étale cohomology and the étale topology as well as the theory of sheaves on étale sites, see [Mil80], [FK88], or [KW01].

Furthermore, we will mean constructible $\overline{\mathbb{Q}}_\ell$ sheaves. This is yet another construction that is far too long to describe here. The above references for information on the various Frobenius morphisms cover this topic adequately and in great detail. For simplicity, the reader is encouraged to imagine that these are sheaves of $\overline{\mathbb{Q}}_\ell$

vector spaces or even sheaves of $\mathbb{C}$ vector spaces. This is not true, but the intuition that it provides is the correct way of thinking about these things.

We recall that a locally constant sheaf is a sheaf $\mathcal{F}$ such that there exists an open covering $\{U_i\}_{i\in I}$ of X such that $\mathcal{F}_{U_i}$ is a constant sheaf for all i. For suitably nice topological spaces, the category of locally constant sheaves is equivalent to the category of representations of $\pi_1(X)$, the fundamental group (or of $\pi_1^{Et}(X)$, the étale fundamental group if in the étale topology).

A constructible sheaf is a sheaf $\mathcal{F}$ such that there exists a stratification of X into a disjoint union of locally closed smooth subvarieties

$$X = \bigsqcup_{\lambda\in\Lambda} X_\lambda$$

with the property that $\mathcal{F}_{X_\lambda}$ is a locally constant sheaf for all $\lambda \in \Lambda$.

To define $\mathrm{D}_c^b(X)$, the bounded derived category of constructible complexes, or any of its variants, we do not require that for such an $\mathcal{F}^\bullet$, each $\mathcal{F}^i$ is a constructible sheaf. Instead, the correct notion ends up being that a complex $\mathcal{F}$ is in $\mathrm{D}_c^b(X)$ if for all i $\mathcal{H}^i(\mathcal{F})$ is a constructible sheaf.

Similarly to our notation of dropping the subscript "0″, X_0 to X, when moving from the variety over the finite field to that over the algebraic closure, for a sheaf (or complex of sheaves) on X_0, $\mathcal{F}_0$, we will include a subscript 0 and drop the subscript, $\mathcal{F}$, to indicate that the sheaf/complex of sheaves has been pulled back to X. This same notation is also used when talking about maps and functors in these two different settings. We will use χ to denote this pullback, so that we have $\chi(\mathcal{F}_0) = \mathcal{F}$. If it is clear from the context that a complex of sheaves must be on one variety or the other, we may drop the 0 in an abuse of notation.

This action on stalks at geometric points is an automorphism of a $\bar{\mathbb{Q}}_\ell$ vector space, so one can think of the extra ("mixed") structure as that of having an automorphism attached to the vector space (not unlike the situation in the setting of quiver representations) or as a type of grading. We will see soon that making this notion precise is exactly the notion of a mixed version of a category as defined above.

By $\mathrm{D}^b_m(X_0)$ or $\mathrm{D}^{\mathrm{Del}}(X_0)$ we indicate Deligne's category of mixed constructible complexes from [Del80]. Likewise, we use $\mathcal{P}_m(X_0)$ to denote the category of mixed perverse sheaves (cf. [BBD82]). We will say more in the next section about the specifics of this category. It is very important to note that $\mathcal{P}_m(X_0)$ is *not* a mixed category in the sense described in the section above. This is easily seen by noting that there are pure non-semisimple objects in $\mathcal{P}_m(X_0)$ (cf. [BGS96]). For this reason, we will not refer to this as the "mixed category of perverse sheaves." If we have to, we will refer to it "the category of mixed perverse sheaves" or "Deligne's category of mixed perverse sheaves." Even this will be avoided where reasonable. Denote by $\underline{\bar{\mathbb{Q}}}$ the "constant sheaf" with value of $\bar{\mathbb{Q}}_\ell$ over connected étale neighborhoods. Let us assume that we have an algebraic stratification (cf. [CG10]):

$$X_0 = \sqcup_{\lambda \in \Lambda} X_\lambda$$

and, for all $\lambda \in \Lambda$, let $j_\lambda : X_\lambda \hookrightarrow X$ denote the locally closed embedding. We will always assume that the following condition (cf. [BBD82] 2.2.10(c)) is in force whenever we discuss constructible sheaves: For all $\lambda, \kappa \in \Lambda$, $\mathrm{H}^i(\mathrm{R}j_{\lambda *}\underline{\bar{\mathbb{Q}}}_\ell)|_{X_\kappa}$ is a local system with irreducible subquotients of the form $\underline{\bar{\mathbb{Q}}}_\ell(-n/2)$. We will prove that, in the equivariant setting, toric varieties satisfy a stronger condition. To indicate the simple perverse sheaves we write

$$\mathcal{L}_\lambda \coloneqq j_{\lambda !*}\underline{\bar{\mathbb{Q}}}_\ell[\dim(X_\lambda)]$$

in the non-mixed setting and

$$\tilde{\mathcal{L}}_\lambda \coloneqq j_{\lambda !*}\underline{\bar{\mathbb{Q}}}_\ell[\dim(X_\lambda)](\dim(X_\lambda)/2)$$

in the mixed setting. Note that this convention means that $\tilde{\mathcal{L}}_\lambda$ has weight 0. This is different than the convention in [BGS96]. This has the effect of making it so that, if $\mathbb{D}(-)$ denotes the Verdier duality functor, then

$$\mathbb{D}(\tilde{\mathcal{L}}_\lambda) \simeq \tilde{\mathcal{L}}_\lambda.$$

We also denote by $\mathrm{D}^{\mathrm{Weil}}(X_0)$ the full triangulated subcategory of $\mathrm{D}^b_m(X_0)$ generated by the $\tilde{\mathcal{L}}_\lambda$ and Tate twists thereof. The extension of scalars functor then restricts to a functor

$$\chi : \mathrm{D}^{\mathrm{Weil}}(X_0) \to \mathrm{D}^b_c(X)$$

that we will still denote by χ. We analogously use $\mathcal{P}^{\mathrm{Weil}}(X_0)$ (resp. $\mathcal{P}(X)$) to denote the the abelian category of perverse sheaves in $\mathrm{D}^{\mathrm{Weil}}(X_0)$ (resp. $\mathrm{D}^b_c(X)$). From [BBD82, 5.1.2] we know that $\mathcal{P}^{\mathrm{Weil}}(X_0)$ is equivalent to a certain category of sheaves on X with a "Weil structure" or an accompanying sheaf automorphism. It is not true, however, that the same is true for the relationship between $\mathrm{D}^{\mathrm{Weil}}(X_0)$ and $\mathrm{D}^b_c(X)$. By [BBD82, Thérèome 5.3.5], every $\mathcal{F}_0 \in \mathcal{P}^{\mathrm{Weil}}(X_0)$ comes with a canonical weight filtration $\mathrm{W}_\bullet \mathcal{F}_0$. The subquotients $\mathfrak{r}\mathcal{F}_0$ are pure, but they are not necessarily semisimple (cf. [BBD82, Proposition 5.3.9]). (This is what was alluded to earlier in regard to the failure of $\mathrm{D}^{\mathrm{Del}}(X_0)$ to be mixed as a category.) Furthermore, all the morphisms in $\mathcal{P}^{\mathrm{Weil}}(X_0)$ are strictly compatible with the weight filtration. If $a : X_0 \to \mathrm{Spec}(\mathbb{F}_q)$ is the canonical structure map to $\mathrm{Spec}(\mathbb{F}_q)$, then we make the following definition:

$$\underline{\mathrm{RHom}}(\mathcal{F}_0, \mathcal{G}_0) := \mathrm{R}a_* \mathcal{RH}\mathrm{om}(\mathcal{F}_0, \mathcal{G}_0).$$

This is an ℓ-adic sheaf over $\mathrm{Spec}(\mathbb{F}_q)$, so it still has a natural action of Fr. Similarly, we let

$$\underline{\mathrm{Hom}}^i(\mathcal{F}_0, \mathcal{G}_0) := \mathrm{H}^i(\mathrm{R}a_* \mathcal{RH}\mathrm{om}(\mathcal{F}_0, \mathcal{G}_0)).$$

Since χ is compatible with all the usual sheaf functors,

$$\mathrm{Hom}^i_{\mathrm{D}^{\mathrm{Weil}}(X)}(\chi(\mathcal{F}_0), \chi(\mathcal{G}_0)) \simeq \chi(\underline{\mathrm{Hom}}^i(\mathcal{F}_0, \mathcal{G}_0)).$$

That is, if we forget the Fr action on $\underline{\mathrm{Hom}}^i(\mathcal{F}_0, \mathcal{G}_0)$, then we get the homomorphisms between $\chi(\mathcal{F}_0) = \mathcal{F}$ and $\chi(\mathcal{G}_0) = \mathcal{G}$ in $\mathrm{D}^b_c(X)$. However, within $\mathrm{D}^{\mathrm{Weil}}(X_0)$, the relationship between the Hom-groups and the $\underline{\mathrm{Hom}}$'s is more complicated. By [BBD82, 5.1.2.5], there is a short exact sequence of vector spaces:

$$0 \to \underline{\mathrm{Hom}}^{i-1}(\mathcal{F}_0, \mathcal{G}_0)_{\mathrm{Fr}} \to \mathrm{Hom}^i_{\mathrm{D}^{Weil}(X_0)}(\mathcal{F}_0, \mathcal{G}_0) \to \underline{\mathrm{Hom}}^i(\mathcal{F}_0, \mathcal{G}_0)^{\mathrm{Fr}} \to 0, \qquad (3.5)$$

where $(\cdot)^{\mathrm{Fr}}$ are the invariants of Fr (i.e. the kernel of $\mathrm{Fr} - \mathrm{id}$) and $(\cdot)_{\mathrm{Fr}}$ are the coinvariants of Fr (i.e. the cokernel of $\mathrm{Fr} - \mathrm{id}$). Also, note that the natural morphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{Weil}}(X_0)}(\mathcal{F}_0, \mathcal{G}_0) \to \mathrm{Hom}_{\mathrm{D}^{\mathrm{Weil}}(X)}(\chi(\mathcal{F}_0), \chi(\mathcal{G}_0))$$

factors through the map

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{Weil}}(X_0)}(\mathcal{F}_0, \mathcal{G}_0) \to \underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{G}_0)^{\mathrm{Fr}}$$

of (3.5). We now list (without proof) three results from [AR13]:

Lemma 3.2.2. *[AR13, Lemma 6.1] The following two conditions are equivalent:*

1. $\mathcal{F}_0 \in D^{Weil}(X_0)$.

2. *For all* $\lambda \in \Lambda$, $j^*_\lambda \mathcal{F}_0 \in D^{Weil}(X_0)$.

Lemma 3.2.3. *[AR13, Lemma 6.2] Let X_0 be a stratified variety and $h : Y_0 \to X_0$ the locally closed inclusion of a union of strata. Then $h^*, h^!, h_*, h_!$ all preserve the the Weil categories. That is, $h^* : D^{Weil}(X_0) \to D^{Weil}(Y_0)$ and similarly for the other three.*

Lemma 3.2.4. *[AR13, Lemma 6.3] For any stratified variety, the functors $\mathbb{D}, \otimes^L$, and $\mathcal{RH}\mathrm{om}$ send objects in $D^{Weil}(X_0)$ to objects of $D^{Weil}(X_0)$.*

We also note that all of the above is transportable to the equivariant setting. This leads us naturally to the notion of the equivariant derived category. We will give the briefest of introductions to this. We need to first know what equivariant sheaves are before doing that. Everything that is said here comes from [BL94] and that monograph should be referenced for a more detailed account of this material.

Suppose

$$act : G \times X \to X$$

gives X the structure of a G-space. Then we also have the map

$$pr_2 : G \times X \to X$$

that is simply projection along the second factor. A sheaf $\mathcal{F}$ on X is G-equivariant if there exists an isomorphism

$$\alpha : act^* \mathcal{F} \xrightarrow{\sim} pr_2^* \mathcal{F}$$

and if this isomorphism satisfies a cocycle condition.

First, one should note that a sheaf $\mathcal{F}$ being equivariant is not intrinsic to it. In fact, it is a piece of extra data. So, in general it does not make sense to say that a sheaf "is equivariant" or "is not equivariant." One exception to this rule is that is a perverse sheaf has a structure as an equivariant complex with respect to a connected group, then such an isomorphism is unique.

If X is free as a G-space, then we have an equivalence of categories

$$\mathrm{Sh}_G(X) \xrightarrow{\sim} \mathrm{Sh}(\overline{X})$$

where $\overline{X} = X/G$ is the quotient space. It is also worth noting that, in general, if one is willing to think in stack language, then

$$\mathrm{Sh}_G(X) \simeq \mathrm{Sh}([X/G])$$

where $[X/G]$ is the quotient stack.

Definition 3.2.5. If X is a G-space, then a resolution of X is a map

$$p : P \to X$$

where P is a free G-space. We write $Res(X)$ for the category of resolutions of X with morphisms given by the obvious notion.

We now need to associate a diagram to any resolution $p : P \to X$. More specifically, given such a resolution, we obtain a diagram

$$Q(p) : X \xleftarrow{p} P \xrightarrow{q} \overline{P} = P/G.$$

Now, given a resolution as above, we can define the category $\mathrm{D}^b_G(X, P)$ by saying that an object $\mathcal{F} \in \mathrm{D}^b_G(X, P)$ is a triple $(\mathcal{F}_X, \overline{\mathcal{F}}, \beta)$ where

- $\mathcal{F}_X \in \mathrm{D}^b(X)$,
- $\overline{F} \in \mathrm{D}^b(\overline{P})$, and
- $\beta : p^*\mathcal{F}_X \xrightarrow{\sim} q^*\overline{\mathcal{F}}$ is an isomorphism in $\mathrm{D}^b(P)$.

A morphism $\alpha : \mathcal{F} \to \mathcal{G}$ is a pair $(\alpha_X, \overline{\alpha})$ where $\alpha_X : \mathcal{F}_X \to \mathcal{G}_X$ and $\overline{\alpha} : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ satisfy the relation

$$\beta \circ p^*(\alpha_X) = q^*(\overline{\alpha}) \circ \beta.$$

Finally, we can specify a subcategory $\mathrm{D}^I(X, P) \subset \mathrm{D}^b(X, P)$ by saying that $\mathcal{F} \in \mathrm{D}^I(X, P)$ if $\mathrm{For}(\mathcal{F}) \in \mathrm{D}^I(X)$, where For is the forgetful functor.

Next we need the definition of an n-acyclic map.

Definition 3.2.6. A continuous map of topological spaces $f : X \to Y$ is n-acyclic if

1. For any $\mathcal{F} \in \mathrm{Sh}(Y)$, the adjunction morphism $\mathcal{F} \to \mathbb{R}^0 f_* f^* \mathcal{F}$ is an isomorphism and $\mathbb{R}^i f_* f^* \mathcal{F} = 0$ for all $i \in \{1, \ldots, n\}$.

2. For any base change $Z \to Y$, the base change morphism $X \times_Y Z \to Z$ satisfies property (1).

We say that a resolution $p : P \to X$ is n-acyclic if the continuous map p is. We say that a map f is ∞-acyclic if it is n-acyclic for all n.

Finally, due to some propositions in [BL94], we can make the following definition:

Definition 3.2.7. For any $I \subset \mathbb{Z}$, define $\mathrm{D}_G^I(X) \coloneqq \mathrm{D}^I(X, P)$ for some n-acyclic resolution $p : P \to X$ where $n \geq |I|$. Then we can define

$$\mathrm{D}_G^b(X) \coloneqq \varinjlim_I \mathrm{D}_G^I(X).$$

This is only the briefest introduction to the formalism of the equivariant derived category. In particular, we will not go through any of the definitions of the equivariant versions of the sheaf functors. The important thing to know here is that these equivariant sheaf functors obey the same formalism as their non-equivariant counterparts.

3.3 Perverse Sheaves

We like to consider $\mathcal{P}(X)$ (also written $\mathrm{Perv}(X)$), the category of perverse sheaves on an algebraically stratified variety

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda.$$

To speak about perverse sheaves, one must consider either the additional datum of an appropriate stratification or, else, consider the direct limit of the categories of sheaves for all such stratifications. For many purposes, the stratification one cares about comes from the action of an algebraic group. In this case, the stratification is known to be an algebraic stratification (cf. [CG10]).

In [BBD82], the authors use the notion of a t-structure on a triangulated category to construct abelian subcategories of triangulated categories. In the case when the

t-structure is the "standard" t-structure on a bounded derived category, then this category is the original abelian category.

We will not recall the generalities of t-structures, but we will just list the conditions on a complex that makes it a perverse sheaf with respect to what is called the middle perversity. For the reader who has not encountered these before, these conditions come from the heart of a specific t-structure. Being the heart of a t-structure, $\mathcal{P}(X)$ is an abelian subcategory of $\mathrm{D}^b_c(X)$. The fact that makes it interesting is that it is not equivalent to the original abelian category of constructible sheaves.

In particular, $\mathcal{F} \in \mathrm{D}^b_c(X)$ is a perverse sheaf on a stratified variety X if and only if

- For all i, dim supp $\mathcal{H}^i\mathcal{F} \leq -i$.
- For all i, dim supp $\mathcal{H}^i\mathbb{D}\mathcal{F} \leq -i$.

It is an important fact that $\mathcal{P}(X)$ is also a finite-length abelian category. Having said that, it is proved in [BBD82] that the simple objects are parametrized by pairs $(X_\lambda, \mathcal{L})$ where X_λ is a stratum of X. These will be denoted by $\mathrm{IC}(X_\lambda, \mathcal{L})$, also known as an intersection cohomology complex. Any $\mathrm{IC}(X_\lambda, \mathcal{L})$ has support on $\overline{X}_\lambda$, the closure of X_λ and $\mathrm{IC}(X_\lambda, \mathcal{L})|_{X_\lambda} \simeq \mathcal{L}_{X_\lambda}[\mathrm{dim} X_\lambda]$.

In particular, if X_μ is the open dense stratum, then $\mathrm{IC}(X_\mu, \underline{\bar{\mathbb{Q}}}_{\ell, X_\mu})$, usually written as $\mathrm{IC}(X)$ is sometimes simply called the intersection cohomology complex of X, has support on $\overline{X}_\mu = X$, and its hypercohomology groups, $\mathbb{H}^\bullet(\mathrm{IC}(X))$, are the intersection cohomology groups of X, $\mathrm{IH}^\bullet(X)$.

We now wish to recall a few of the important lemmas that will be key in our arguments in the following sections. The first couple of these come from the paper

[Lun95]. This is ultimately where we will get the semisimplicity of the action on on the $\underline{\mathrm{Hom}}^\bullet_T(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau)$.

We need to set the stage for these. Let X be a toric variety (under our assumptions from above). Let $\{\mathcal{L}_1, \dots, \mathcal{L}_k\}$ be a complete collection of simple, equivariant perverse sheaves on X with respect to the toric action. Let $\mathcal{L} = \oplus_{i=1}^k \mathcal{L}_i$ and let $A^\circ = \mathrm{Ext}^\bullet_{\mathrm{D}_T(X)}(\mathcal{L}, \mathcal{L})$ be the corresponding graded algebra. Let A be the opposite algebra. Denote by $e_i : \mathcal{L} \to \mathcal{L}_i$ the projection and by $Q_i = Ae_i$ the corresponding projective A-module. We denote by $\mathcal{A} = (A, d = 0)$ the DG-algebra with zero differential. Following [Lun95], we can construct $\mathrm{D}_{\mathcal{A}}$, the derived category of DG-modules over $\mathcal{A}$. We let $\mathrm{D}^f_{\mathcal{A}} \subset \mathrm{D}_{\mathcal{A}}$ be the full triangulated subcategory generated by the DG-modules $(Q_i, d = 0)$.

Theorem 3.3.1. *[Lun95, Theorem 0.1.1] Assume that X is affine or projective. Then there exists a natural equivalence of triangulated categories*

$$D^b_{T,c}(X) \simeq D^f_{\mathcal{A}}.$$

Lemma 3.3.2. *[Lun95, Lemma 4.0.1] Suppose that in the Theorem 3.3.1, the toric variety X is affine. Then we may assume that X has a fixed point.*

The proof of this lemma is particularly short, so we will repeat it here for use later:

Proof. Let $O(\sigma) \subset X_0$ be the orbit of minimal dimension and let T_σ be its stabilizer. Then $X_0 \simeq T \times_{T_\sigma} X_\sigma$ where X_σ is an affine toric variety of contractible type with respect to the torus T_σ. By the induction equivalence of [BL94], the categories $\mathrm{D}^b_{T,m}(X_0)$ and $\mathrm{D}^b_{T_\sigma,m}(X_\sigma)$ are naturally equivalent. Furthermore, this equivalence preserves simple perverse sheaves. Therefore, we may replace X by X_σ. $\square$

Lemma 3.3.3. *[Lun95, Lemma 4.0.3] The natural map*

$$Ext^{\bullet}(\mathcal{L}_i, \mathcal{L}_j) \to Hom^{\bullet}_{\mathcal{A}_T}(H^{\bullet}(\mathcal{L}_i), H^{\bullet}(\mathcal{L}_j))$$

is injective.

The next lemma we wish to present is the primary one from [dC15]. We will need more notation before we can state this result. Let X_0 and Y_0 be two toric varieties over a finite field and let $f_0 : X_0 \to Y_0$ be a proper toric map. Let $f_0 = h_0 \circ g_0 : X_0 \to Z_0 \to Y_0$ be the toric Stein factorization. For every $\zeta \in \Delta(Z_0)$, define

$$\begin{aligned}
&\mathrm{Ev}_\zeta := \{b \in \mathbb{Z} | b + \dim(X_0) - \dim(V(\zeta) \text{ is even}\},\\
&\beta_\zeta := \frac{b + \dim(X) - \dim(V(\zeta))}{2},\\
&O'_0(\zeta) := h_0(O_0(\zeta)),\\
&L_{0,\zeta} := h_{0*}\underline{\bar{\mathbb{Q}}}_{\ell, O_0(\zeta)}.
\end{aligned} \tag{3.6}$$

Theorem 3.3.4. *[dC15, Theorem 1.4.1] Let $f_0 : X_0 \to Y_0$ be a proper toric map.*

1. *There is an isomorphism in $D^b_m(Y_0)$:*

$$f_{0*}\, IC(X_0) \simeq \bigoplus_{\zeta \in \Delta(Z_0)} \bigoplus_{b \in Ev_\zeta} IC(\overline{O'_0(\zeta)}, L_{0,\zeta})^{\oplus s_{\zeta,b}}(-\beta_\zeta)[-b].$$

   The sheaves $L_{0,\zeta}$ are locally constant, semisimple, and pure of weight 0. The $s_{\zeta,b} \in \mathbb{Z}_{\geq 0}$ are subject to:

   (a) For all $b \in Ev_\zeta$, $s_{\zeta,b} = s_{\zeta,-b}$.

   (b) If f_0 is projective, then $s_{\zeta,b} \geq \sum_{l \geq 1} s_{\zeta,b+2l}$ for every $b \geq 0$ in Ev_ζ.

   Here f_{0*} is understood to be the right derived functor.

2. *In particular, the pure weight 0 $f_{0*}\, IC(X_0)[-dim(X_0)]$ is pointwise pure, even, and Tate. That is, for every $y \in Y_0(\overline{\mathbb{F}}_q)$, the Fr-module $(R^{\bullet}\, f_{0*}\, IC(X_0)[-dim(X_0)])_y = H^{\bullet}(f_0^{-1}(y), IC(X_0)[-dim(X_0)])$ is pure, even, and Tate.*

3. *Let f_0 be a proper toric fibration. For $\sigma \in \Delta(Y_0)$, let Ev_σ and β_σ be as above. There is an isomorphism in $D^b_m(Y_0)$:*

$$f_{0*}\, IC(X_0) \simeq \bigoplus_{\sigma \in \Delta(Y_0)} \bigoplus_{b \in Ev_\sigma} IC(V_0(\sigma))^{\oplus s_{\sigma,b}}(-\beta_\sigma)[-b].$$

Here, the $s_{\sigma,b} \in \mathbb{Z}_{\geq}$ are subject to the conditions analogous to those in (a) and (b) above.

Chapter 4

$\mathbf{K}^b\mathbf{Pure}(X_0)$ as a Mixed Category

In this section, we introduce $\mathrm{Pure}(X_0)$ and show that its bounded homotopy category, $\mathrm{K}^b(\mathrm{Pure}(X_0))$, is a suitable replacement for $\mathrm{D}^b_{T,m}(X_0)$. To do this we will construct a realization functor between them.

Let Δ be a fan in the n-dimensional real vector space $N_{\mathbb{R}} = N \otimes \mathbb{R}$, where N is an n-dimensional lattice. Let $X_0(\Delta)$ be the associated toric variety over $\mathbb{F}_q$ and $X(\Delta) = X_0(\Delta) \otimes \overline{\mathbb{F}}_q$. We will refer to the torus $X_0(N_{\mathbb{R}})$ simply as T. Recall that there is a one-to-one correspondence between cones $\sigma \in \Delta$ and T-orbits $O(\sigma)$ that is order reversing, i.e. τ is a face of σ corresponds to $O(\sigma) \subseteq \overline{O(\tau)}$. We will denote by $i_\sigma : O_0(\sigma) \hookrightarrow X_0(\Delta)$ the (locally closed) inclusion of the T-orbit into the larger toric variety. This stratification by the algebraic group T will be the one that we will work with for the rest of the paper. Furthermore, let $\mathcal{L}_\sigma = i_{\sigma !*}\mathbb{Q}_{l,\mathcal{O}(\sigma)}[n_\sigma]$ be the intersection cohomology sheaf of X_σ. Note that $\mathcal{L}_\sigma$ is a mixed sheaf, pure of weight n_σ and we will use the notation

$$\tilde{\mathcal{L}}_\sigma \coloneqq \mathcal{L}_\sigma(n_\sigma/2).$$

This now has weight 0.

Now, we introduce the subcategory from which our study departs and about which it is concerned. This will actually involve several categories. In what follows, we will denote by $\mathrm{D}^{\mathrm{Del}}_T(X_0)$ or $D^b_{T,m}(X_0)$ the category of mixed (T-equivariant) constructible complexes given in the sense of [Del80] or [BBD82]. We note that, as is well known by now, this category is not mixed in the sense of [BGS96] (which we have defined above), because pure objects fail to always be semisimple. This is why we will attempt to refrain from calling it the "mixed category" from here on. Also, following [AR13], we will denote by $\mathrm{D}^{\mathrm{Weil}}_T(X_0)$ the full triangulated subcategory of mixed T-equivariant constructible complexes generated by the $\tilde{\mathcal{L}}_\sigma$ and Tate twists of these complexes. Whenever it is clear which torus is acting, we will feel free to drop the T from the notation.

Definition 4.0.1. We let $\mathrm{Pure}(X_0)$ be the category whose objects are isomorphic to direct sums of various $\tilde{\mathcal{L}}_\sigma[i](i/2)$ for $\sigma \in \Delta$. Furthermore, we define $\mathrm{D}^{\mathrm{mix}}(X_0) \coloneqq \mathrm{K}^b\mathrm{Pure}(X_0)$.

Remark 4.0.2.

- Note that since $\tilde{\mathcal{L}}_\sigma$ is pure of weight 0, the above direct summands in this category are all pure of weight zero.
- It is important to point out that $\mathrm{Pure}(X_0)$ has a shift functor of its own, namely $[m](m/2)$. This is different than the shift functor $[n]$ in $\mathrm{K}^b(\mathrm{Pure}_T(X_0))$, so to distinguish between the two, we denote the shift in $\mathrm{Pure}(X_0)$ by

$$\{m\} \coloneqq [m](m/2).$$

Before moving on to provide a recollement structure on this category (in the sense of [BBD82]), we will first take time to prove some basic lemmas about morphisms between objects in this category. Strictly speaking, these could be noted as needed in the proofs below, but the author has personally felt it enlightening to understand them by themselves. We begin with a theorem of crucial importance.

Theorem 4.0.3. *Let $X_0(\Delta)$ be a toric variety defined over a finite field $\mathbb{F}_q$. Let $\sigma, \tau \in \Delta$ denote cones of the associated fan with inclusion maps*

$$h_\sigma : O_0(\sigma) \hookrightarrow X_0$$

$$h_\tau : O_0(\tau) \hookrightarrow X_0.$$

Then

$$\underline{RHom}_T(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau) \in D^b_m(pt)$$

is either pure of weight 0*, even, and Tate or pure of weight* 0*, odd, and Tate depending on whether* $n_\tau - n_\sigma$ *is even or odd, respectively. Furthermore, for all* i*,* $\underline{Hom}^i_T(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau)$ *is pure and Tate of weight* i*.*

Proof. This proof will consist of two parts. By Theorem 3.3.4, we see that simple perverse sheaves on toric varieties are pure, even, and Tate (peT), **but only once they have been shifted to start in degree 0**. Therefore, a simple perverse sheaf will be pure, even, and Tate, respectively pure, odd, and Tate (poT), if and only if the dimension of its support is even, respectively odd. Therefore, until the end of this proof, we work with the shifted complexes. In particular, these shifted simple perverse sheaves are no longer perverse. For $\sigma, \tau \in \Delta$, we will write L_σ, L_τ for the complexes of sheaves $\tilde{\mathcal{L}}_\sigma[-n_\sigma], \tilde{\mathcal{L}}_\tau[-n_\tau]$ respectively. In this first part, we will simply prove that $\underline{\mathrm{Hom}}^\bullet_T(L_\sigma, L_\tau)$ is pure and even. In particular, we will not say anything about the actual form of Fr in this first part.

We induce on the number of strata in X_0. Assume that Δ consists of a single cone, so that there is only one stratum in $X_0(\Delta)$. In this case, $L_\sigma = L_\tau = \underline{\bar{\mathbb{Q}}_\ell}[\dim(X_0)]$. Then the statement is true by Theorem 3.3.4, since

$$\underline{\mathrm{Hom}}^i_T(\underline{\bar{\mathbb{Q}}_\ell}, \underline{\bar{\mathbb{Q}}_\ell}) \simeq \underline{\mathrm{H}}^i_T(X_0).$$

Now, assume that the claim is true when the number of strata is n and let $X_0(\Delta)$ have $n+1$ strata. We will consider the distinguished triangle associated to the following map of spaces:

$$i : Z_0 \hookrightarrow X_0(\Delta) \hookleftarrow U_0 : j$$

where Z_0 is a closed orbit and $U_0 = X_0(\Delta) - Z_0$. Namely, we have the distinguished triangle

$$i_* i^! L_\tau \to L_\tau \to j_* j^* L_\tau \xrightarrow{+1}$$

with all functors understood to be equivariant and derived. Then, applying $\underline{\mathrm{Hom}}(L_\sigma, -)$ and using the standard six functor formalism, we arrive at the following long exact sequence:

$$\cdots \underline{\mathrm{Hom}}_T^{i-1}(j^* L_\sigma, j^* L_\tau) \to \underline{\mathrm{Hom}}_T^{i}(i^* L_\sigma, i^! L_\tau) \to \underline{\mathrm{Hom}}_T^{i}(L_\sigma, L_\tau) \to$$
$$\underline{\mathrm{Hom}}_T^{i}(j^* L_\sigma, j^* L_\tau) \to \underline{\mathrm{Hom}}_T^{i+1}(i^* L_\sigma, i^! L_\tau) \to \cdots .$$

In particular, $j^* = j^!$ and, for $i_\alpha : O_0(\alpha) \hookrightarrow Y_0$ an inclusion of an orbit into $Y_0 = U_0$ or Z_0,

$$(? \circ i_\alpha)^* \simeq i_\alpha^* \circ ?^*$$

and

$$(? \circ i_\alpha)^! \simeq i_\alpha^! \circ ?^!$$

where $? = i$ or j from above. Thus, by induction and by Theorem 3.3.4,

$$\underline{\mathrm{Hom}}_T^{i}(i^* L_\sigma, i^! L_\tau) = \underline{\mathrm{Hom}}_T^{i}(j^* L_\sigma, j^! L_\tau) = 0$$

for all odd i, which implies that $\underline{\mathrm{Hom}}_T^{i}(L_\sigma, L_\tau) = 0$ for all odd i. This also means that $\underline{\mathrm{Hom}}_T^{i}(i^* L_\sigma, i^! L_\tau)$ and $\underline{\mathrm{Hom}}_T^{i}(j^* L_\sigma, j^! L_\tau)$ are pure of weight i for all even i, so there are short exact sequences

$$0 \to \underline{\mathrm{Hom}}_T^{i}(i^* L_\sigma, i^! L_\tau) \to \underline{\mathrm{Hom}}_T^{i}(L_\sigma, L_\tau) \to \underline{\mathrm{Hom}}_T^{i}(j^* L_\sigma, j^! L_\tau) \to 0.$$

Therefore, $\underline{\mathrm{Hom}}^i_T(L_\sigma, L_\tau)$ is pure of weight i for all even i and our claim is proved.

Alternatively, we could have proved the above claim by considering spectral sequences. Recall that given a filtered chain complex, there is a spectral sequence attached. Now, let X be a topological space and suppose that X has a finite decreasing filtration by closed sets:

$$X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq X_{n+1} = \varnothing.$$

This induces an increasing finite filtration on $\Gamma(\mathcal{F})$ for any complex $\mathcal{F}$ on X:

$$0 = \Gamma_{X_{n+1}}(\mathcal{F}) \subseteq \Gamma_{X_n}(\mathcal{F}) \subseteq \cdots \subseteq \Gamma_{X_0}(\mathcal{F}) = \Gamma(\mathcal{F})$$

(Here all functors are understood to be derived and all complexes are elements of the derived category.) Thus we have a spectral sequence

$$E_1^{p,q} = \mathrm{H}^{p+q}_{X_p - X_{p+1}}(\mathcal{F}) \Rightarrow \mathrm{H}^{p+q}(\mathcal{F}).$$

We can simplify the presentation of this further, because in the language of the six-functor formalism, if $h : Z \hookrightarrow X$ is a locally closed inclusion, then

$$\mathrm{H}^i_Z(\mathcal{F}) = \mathrm{H}^i(h^!\mathcal{F}).$$

So, if we denote by $j_p : X_p - X_{p+1} \hookrightarrow X$ the open inclusion, then our spectral sequence becomes

$$E_1^{p,q} = \mathrm{H}^{p+q}(j_p^!\mathcal{F}) \Rightarrow \mathrm{H}^{p+q}(\mathcal{F}).$$

Now, we consider our particular situation. We can certainly stratify a toric variety by such a finite filtration by closed toric subvarieties. We begin the filtration by picking a closed orbit (which always exists) and then at each level add an orbit that is open in the union. This produces an increasing filtration. so we can simply flip it to produce a decreasing one. That is,

$$X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq X_{n+1} = \varnothing$$

such that for any p, $X_p - X_{p+1} = O(\sigma)$ for some $\sigma \in \Delta$. The sheaf whose cohomology we wish to calculate is $\mathcal{RH}\mathrm{om}(L_\sigma, L_\tau)$, so we get the following:

$$E_1^{p,q} = \mathrm{H}_T^{p+q} j_p^! \mathcal{RH}\mathrm{om}_T(L_\sigma, L_\tau) \Rightarrow \underline{\mathrm{Hom}}_T^{p+q}(L_\sigma, L_\tau).$$

We know that $j_p^! \mathcal{RH}\mathrm{om}(L_\sigma, L_\tau) \simeq \mathcal{RH}\mathrm{om}(j_p^* L_\sigma, j_p^! L_\tau)$, so we arrive at the final form of our spectral sequence. This is precisely what one could have used instead of the formalism of the derived category for the above proof if the reader is less familiar with triangulated categories.

Now, for the second part, we will show that this action of Fr on these $\underline{\mathrm{Hom}}_T^\bullet(L_\sigma, L_\tau)$ is that of a semisimple operator. We will prove this in two steps. We remind the reader that L_σ and L_τ are still the shifted (non-perverse) complexes $\tilde{\mathcal{L}}_\sigma[-n_\sigma]$ and $\tilde{\mathcal{L}}_\tau[-n_\tau]$ respectively.

We proceed by induction on the number of (total) cones in the fan. If there is one cone in the fan, then X_0 is an affine toric variety. In this case, the argument in the proof of Lemma 3.3.2 applies, so we may assume that X_0 has a fixed point, i.e. is an affine toric variety of contractible type.

Therefore, Lemma 3.3.3 shows that we may calculate the Frobenius action on $\underline{\mathrm{Hom}}_T^i(L_\sigma, L_\tau)$ by considering the action of Frobenius instead on

$$\mathrm{Hom}^i_{\mathrm{H}_T^\bullet(X)}(\mathrm{H}_T^\bullet L_\sigma, \mathrm{H}_T^\bullet L_\tau).$$

On this group, Frobenius acts by conjugation. Since X_0 is affine of contractible type (and so is any closed sub-toric variety of X_0), we have by the "homotopy trick" that $\mathrm{H}_T^\bullet L_\sigma = \mathcal{H}^\bullet(L_{\sigma,w})$ where $w \in X_0$ is the unique torus fixed point. (Recall that we have reviewed the homotopy trick above in the review of toric geometry.) Therefore, by Theorem 3.3.4 again, we know that this Frobenius action is semisimple.

Now, assume that the number of cones is $n > 1$. Then we may write $X_0 = U_0 \cup V_0$ where U_0 and V_0 are open sub-toric varieties with less than n cones. The Mayer-Vietoris sequence then gives us the following long exact sequence:

$$\begin{aligned}
&\cdots \to 0 \to \underline{\mathrm{Hom}}_T^{2i}(L_\sigma, L_\tau) \\
&\to \underline{\mathrm{Hom}}_T^{2i}(L_\sigma, L_\tau)|_{U_0} \oplus \underline{\mathrm{Hom}}_T^{2i}(L_\sigma, L_\tau)|_{V_0} \\
&\to \underline{\mathrm{Hom}}_T^{2i}(L_\sigma, L_\tau)|_{U_0 \cap V_0} \to 0 \to \cdots
\end{aligned} \tag{4.1}$$

The odd terms are 0 by the first part of this proof. Since U_0 and V_0 (and consequently $U_0 \cap V_0$) are open, $\underline{\mathrm{Hom}}_T^{2i}(L_\sigma, L_\tau)|_{U_0} = \underline{\mathrm{Hom}}_T^{2i}(L_\sigma|_{U_0}, L_\tau|_{U_0})$ (and similarly for the other two open sets). The restriction of a simple perverse sheaf to an open subset is still a simple perverse sheaf, so we see that by induction the middle term of the short exact sequence in (4.1) has semisimple Frobenius action. Since $\underline{\mathrm{Hom}}_T^{2i}(L_\sigma, L_\tau)$ injects into the middle term, we are done.

We must now consider original perverse sheaves $\tilde{\mathcal{L}}_\sigma$ and $\tilde{\mathcal{L}}_\tau$. We see that

$$\underline{\mathrm{Hom}}_T^i(L_\sigma, L_\tau) = \underline{\mathrm{Hom}}_T^i(\tilde{\mathcal{L}}_\sigma[n_\sigma], \tilde{\mathcal{L}}_\tau[n_\tau]) = \underline{\mathrm{Hom}}_T^i(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau)[n_\tau - n_\sigma].$$

Therefore, it is obvious that $\underline{\mathrm{Hom}}_T^i(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau)$ is pure, even, and Tate if $n_\tau - n_\sigma$ is even and pure, odd, and Tate if $n_\tau - n_\sigma$ is odd. □

Now, we would like to set up some terminology that will hopefully be suggestive.

Definition 4.0.4. We say that an object $\mathcal{F} \in \mathrm{Pure}(X_0)$ is **even** if it is isomorphic to a direct sum

$$\mathcal{F} \simeq \bigoplus_{i=1}^{n} \tilde{\mathcal{L}}_{\sigma_i}^{\oplus s(i)}\{n_i\}$$

with all $n_i + n_{\sigma_i} \in 2\mathbb{Z}$. An object $\mathcal{G} \in \mathrm{Pure}(X_0)$ is **odd** if it is isomorphic to a direct sum

$$\mathcal{G} \simeq \bigoplus_{j=1}^{m} \tilde{\mathcal{L}}_{\sigma_j}^{\oplus s(j)}\{n_j\}$$

with all $n_i + n_{\sigma_j} \in 2\mathbb{Z}+1$. We denote by $\mathrm{Pure}(X_0)^{even}$ (respectively $\mathrm{Pure}(X_0)^{odd}$) the full subcategory of $\mathrm{Pure}(X_0)$ of all even (respectively odd) objects.

Note that for $m \in 2\mathbb{Z}$, $\{m\} : \mathrm{Pure}(X_0)^{even} \to \mathrm{Pure}(X_0)^{even}$ is an endofunctor. Likewise, for $m \in 2\mathbb{Z}+1$, $\{m\} : \mathrm{Pure}(X_0)^{odd} \to \mathrm{Pure}(X_0)^{odd}$ is an endofunctor.

Now, we would like to describe how morphisms interact with the even and odd structures of the category.

Lemma 4.0.5. *Let $\mathcal{F} \in Pure(X_0)^{even}$ and $\mathcal{G} \in Pure(X_0)^{odd}$. Then $Hom(\mathcal{F}, \mathcal{G}) = Hom(\mathcal{G}, \mathcal{F}) = 0$.*

Proof. It clearly suffices to prove this when $\mathcal{F} = \tilde{\mathcal{L}}_\sigma$ with n_σ even and $\mathcal{G} = \tilde{\mathcal{L}}_\tau$ with n_τ odd. Then, because we are working within $\mathrm{Pure}(X_0)$, $\mathrm{Hom}_T(\mathcal{F}, \mathcal{G}) = \underline{\mathrm{Hom}}_T(\mathcal{F}, \mathcal{G})^{\mathrm{Fr}}$. However, we know from Theorem 4.0.3 that for $\mathcal{F}$ and $\mathcal{G}$ as above, $\underline{\mathrm{Hom}}_T(\mathcal{F}, \mathcal{G}) = \underline{\mathrm{Hom}}_T(\mathcal{G}, \mathcal{F}) = 0$, because for $\mathcal{F}$ and $\mathcal{G}$ as above, $\underline{\mathrm{RHom}}_T(\mathcal{F}, \mathcal{G})$ is pure, odd, and Tate. □

4.1 $\mathbf{D}^{\mathbf{mix}}(X_0)$ as a Mixed Version of $\mathbf{D}^b_T(X_0)$

For our proof that $\mathrm{K}^b(\mathrm{Pure}(X_0)$ is a mixed version of $\mathrm{D}^b_m(X_0)$, we will essentially repeat the argument made in [Rid13]. The primary difference will be that in [Rid13], Rider considers only even shifts of simple objects while noting that the same construction works for integral shifts. The extra structure that Rider obtains by restricting to even shifts, however, does not play a role in the arguments that we make here.

For completeness, we will describe Rider's construction with modifications.

Lemma 4.1.1. *Let $\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau$ be two IC sheaves on a toric variety $X_0(\Delta)$ over $\mathbb{F}_q$. Then $\underline{Hom}_T(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau[n](n/2))^{Fr} \simeq Hom_{Pure(X)}(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau[n](n/2))$.*

Proof. Recall the short exact sequence (3.5). In this case, this becomes

$$0 \to \underline{\mathrm{Hom}}_T^{-1}(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau[n](n/2))_{\mathrm{Fr}} \to \mathrm{Hom}(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau[n](n/2))$$
$$\to \underline{\mathrm{Hom}}_T(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau[n](n/2))^{\mathrm{Fr}} \to 0.$$

Now, from Theorem 4.0.3, we know that

$$\underline{\mathrm{Hom}}_T^{-1}(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau[n](n/2)) = \underline{\mathrm{Hom}}_T^{n-1}(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau)(n/2)$$

is pure of weight $(n-1)-2(n/2) = -1$. Since $(\cdot)_{\mathrm{Fr}}$ is a quotient of the zero-weight space, the left hand object is 0. Therefore, $\mathrm{Hom}(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau[n](n/2)) \simeq \underline{\mathrm{Hom}}_T(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau[n](n/2))^{\mathrm{Fr}}$. □

Next, we need the a connection to Orlov categories:

Lemma 4.1.2. *Define a function deg* : *Ind*(*Pure*(X_0)) $\to \mathbb{Z}$ *by*

$$deg(\tilde{\mathcal{L}}_\sigma[n](n/2)) = -n.$$

Then Pure(X_0) *is an Orlov category.*

Proof. Let $\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau \in \mathrm{Ind}(\mathrm{Pure}(X_0))$ and suppose that $\tilde{\mathcal{L}}_\sigma[n](n/2) \not\simeq \tilde{\mathcal{L}}_\tau[m](m/2)$. Then we have that

$$\mathrm{Hom}_{\mathrm{Pure}(X_0)}(\tilde{\mathcal{L}}_\sigma[n](n/2), \tilde{\mathcal{L}}_\tau[m](m/2)) = \mathrm{Hom}^{m-n}_{\mathrm{D}^b_{T,m}(X_0)}(\tilde{\mathcal{L}}_\sigma(n/2), \tilde{\mathcal{L}}_\tau(m/2)).$$

If $-n < -m$, then $m-n$ is negative. Therefore, this vanishes since $\tilde{\mathcal{L}}_\sigma(n/2)$ and $\tilde{\mathcal{L}}_\tau(m/2)$ are objects in the heart of a t-structure on $\mathrm{D}^b_{T,m}(X_0)$. If $-n = -m$, then we must have that $\tilde{\mathcal{L}}_\sigma$ and $\tilde{\mathcal{L}}_\tau$ are two non-isomorphic simple objects. Therefore, there are no morphisms in this case either. □

In [Rid13], some stronger assumptions are satisfied. However, by [Rid13, Remark 3.8], the results of the above lemma are enough for the construction presented there to work. We will recall her theorems as they are originally stated while keeping this in mind. We recall a lemma relevant in this context:

Lemma 4.1.3. *[Rid13, Lemma 2.4] For all $M, N \in Pure(X_0)$, $Hom(M, N[n]) = 0$ if $n \neq 0, 1$.*

Proof. We begin by noting that M is pure of weight 0 and $N[n]$ is pure of weight n. For $n > 1$, $\mathrm{Hom}(M, N[n]) = 0$ by the results on mixed perverse sheaves in [BBD82, Section 5]. Suppose that $M \simeq \tilde{\mathcal{L}}_\sigma[i](i/2)$ and $N \simeq \tilde{\mathcal{L}}_\tau[j](j/2)$ for some $i, j \in \mathbb{Z}$ and that $n < 0$. However,

$$\mathrm{Hom}(\tilde{\mathcal{L}}_\sigma[i](i/2), \tilde{\mathcal{L}}_\tau[j](j/2)) \simeq \mathrm{Hom}(\tilde{\mathcal{L}}_\sigma[i-j](\frac{i-j}{2}), \tilde{\mathcal{L}}_\tau[n]).$$

So, it clearly suffices to consider the case when $M \simeq \tilde{\mathcal{L}}_\sigma[i](i/2)$ and $N \simeq \tilde{\mathcal{L}}_\tau$. We know that

$$\underline{\mathrm{Hom}}_T^j(\tilde{\mathcal{L}}_\sigma[i](i/2), \tilde{\mathcal{L}}_\tau[n]) \simeq \underline{\mathrm{Hom}}_T^{n+j-i}(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau)(-i/2)$$

is pure of weight $n_j - i + i = n + j$ by Theorem 4.0.3. In particular, for $j = 0, -1$, we have that $\underline{\mathrm{Hom}}_T^j(M, N[n])$ is pure of non-zero weight. Therefore, by (3.5), $\mathrm{Hom}(M, N[n]) = 0$. For more general objects in $\mathrm{Pure}(X)$, the result follows since $\mathrm{Hom}(-,-)$ commutes with direct sums. □

There are a few more corollaries and lemmas that could be cited in the context of Frobenius invariance, but we will choose to only recall those as needed. What one needs next is a realization functor. That is, we need a triangulated functor

$$\mathrm{real} : \mathrm{K}^b\mathrm{Pure}(X_0) \to \mathrm{D}^b_{T,m}(X_0)$$

that restricts to the identity on $\mathrm{Pure}(X_0)$. Again, we follow [Rid13] very closely. This, in turn, is inspired by Beilinson's work in [Bei87]. In that work, however, Beilinson constructed a realization functor when there is another t-structure readily lying about. Rider attributes the idea of using one in this context to Achar and Kitchen. First, we need the notion of a filtered (triangulated) category.

Definition 4.1.4. A **filtered triangulated category** is the data of a triangulated category, $\mathcal{D}$, along with a collection of pairs of strictly full triangulated subcategories $\{(F^{\leq n}\mathcal{D}, F^{\geq n}\mathcal{D})\}_{n\in\mathbb{Z}}$ such that the following hold:

1. If $X \in F^{\leq n}\mathcal{D}$ and $Y \in F^{\geq n+1}\mathcal{D}$, then $\mathrm{Hom}(X, Y) = 0$.

2. There is containment $F^{\leq n}\mathcal{D} \subset F^{\leq n+1}\mathcal{D}$ and $F^{\geq n}\mathcal{D} \supset F^{\geq n+1}\mathcal{D}$.

3. For all $Z \in \mathcal{D}$ and $n \in \mathbb{Z}$, there exists a distinguished triangle
$$X \to Z \to Y \xrightarrow{+1}$$
such that $X \in F^{\leq n}\mathcal{D}$ and $Y \in F^{\geq n+1}\mathcal{D}$.

4. The filtration is **bounded**. This means that $\cup_{n\in\mathbb{Z}} F^{\leq n}\mathcal{D} = \cup_{n\in\mathbb{Z}} F^{\geq n}\mathcal{D} = \mathcal{D}$.

5. There exists a **shift of filtration**. That is, there is a pair, (s, α), with $s : \mathcal{D} \to \mathcal{D}$ an autoequivalence that shifts the filtration up by one, i.e. $s(F^{\leq n}\mathcal{D}) = F^{\leq n+1}\mathcal{D}$ and $s(F^{\geq n}\mathcal{D}) = F^{\geq n+1}\mathcal{D}$, and α is a natural transformation $s \to id_{\mathcal{D}}$ such that $\alpha_X = s(\alpha_{s^{-1}X})$.

6. For all $X \in F^{\geq 1}\mathcal{D}$ and $Y \in F^{\leq 0}\mathcal{D}$, α induces isomorphisms
$$\mathrm{Hom}(X, Y) = \mathrm{Hom}(X, sY) = \mathrm{Hom}(s^{-1}X, Y).$$

The inclusion functors $F^{\leq n}\mathcal{D} \hookrightarrow \mathcal{D}$ and $F^{\geq n}\mathcal{D} \hookrightarrow \mathcal{D}$ admit right and left adjoints, $w_{\leq n} : \mathcal{D} \to F^{\leq n}\mathcal{D}$ and $w_{\geq n} : \mathcal{D} \to F^{\geq n}\mathcal{D}$, respectively. In [Bei87], Proposition A.3, Beilinson shows that for all $n \in \mathbb{Z}$ and all $Z \in \mathcal{D}$, the distinguished triangle from (3) above is canonically isomorphic to
$$w_{\leq n}Z \to Z \to w_{\geq n+1}Z \xrightarrow{+1} .$$

If we denote by $\mathcal{D}^n = F^{\leq n}\mathcal{D} \cap F^{\geq n}\mathcal{D}$, the "$n$th graded piece," then $w_{\leq n} w_{\geq n}$ is naturally equivalent to $w_{\geq n} w_{\leq n}$ and we denote it by

$$gr_n : \mathcal{D} \to \mathcal{D}^n.$$

Definition 4.1.5. Let $X \in \mathcal{D}$. We say that the **filtered support** is the minimal interval $[a, b]$ such that $X \in F^{\leq b}\mathcal{D} \cap F^{\geq a}\mathcal{D}$.

That every object $X \in \mathcal{D}$ has a filtered support with finite length is a consequence of the fact that we have assumed that the filtration is bounded.

Lemma 4.1.6. *[Rid13, Lemma 3.2] Let $X \in \mathcal{D}$. The morphism $\alpha_{sX} : sX \to X$ induced by the natural transformation above has the property that $gr_n(\alpha_{sX}) = 0$ for all $n \in \mathbb{Z}$.*

We say that $\tilde{\mathcal{D}}$ is a filtered version of $\mathcal{D}$ if there is an equivalence $\tilde{\mathcal{D}}^0 \to \mathcal{D}$. In [Bei87], it is shown that there exists a unique functor–up to unique isomorphism– $\omega : \tilde{\mathcal{D}} \to \mathcal{D}$ such that

1. The restriction $\omega|_{F^{\geq 0}\tilde{\mathcal{D}}}$ is left adjoint to the inclusion functor $\mathcal{D} \hookrightarrow F^{\geq 0}\tilde{\mathcal{D}}$.

2. The restriction $\omega|_{F^{\leq 0}\tilde{\mathcal{D}}}$ is right adjoint to the inclusion functor $\mathcal{D} \hookrightarrow F^{\leq 0}\tilde{\mathcal{D}}$.

3. The morphism $\omega(\alpha_{sX}) : \omega(sX) \to \omega(X)$ is an isomorphism.

Intuitively, ω is the functor of "forgetting the grading." It is also known that if $X \in F^{\geq 0}\tilde{\mathcal{D}}$ and $Y \in F^{\leq 0}\tilde{\mathcal{D}}$, then ω induces an isomorphism

$$\mathrm{Hom}_{\tilde{\mathcal{D}}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{D}}(\omega(X), \omega(Y)).$$

Henceforth, we will denote by $\tilde{\mathrm{D}} = \tilde{\mathrm{D}}(X_0)$ a filtered version of $\mathrm{D}^b_{T,m}(X_0)$. We denote by $\tilde{A}$ the full subcategory of $\tilde{D}$ consisting of objects X such that for all $n \in \mathbb{Z}$, $gr_n(X) \in s^n\mathrm{Pure}(X_0)[n]$. That is, $\tilde{A}$ consists of objects whose graded pieces are objects of $\mathrm{Pure}(X_0)$ lying "on the diagonal." We can note that this is at least very suggestive of the same sort of setup that one often gets in the heart of a t-structure.

It is now important to recall the following important fact:

Lemma 4.1.7. *[Rid13, Remark 3.3] If there is some $n \in \mathbb{Z}$ such that $w_{\leq n}X$ and $w_{\geq n+1}X$ are in $\tilde{A}$ for an object $X \in \tilde{D}$, then $X \in \tilde{A}$ as well. Furthermore, $X \in \tilde{A}$ implies that $s^n X[n] \in \tilde{A}$. This implies that if $f : X \to Y$ is a morphism in $\tilde{A}$, then the cone of the composition $sX \xrightarrow{\alpha_{sX}} X \xrightarrow{f} Y$ is in $\tilde{A}$ as well.*

Proof. This is due to the fact that Lemma 4.1.6 implies that the graded pieces of the cone are given by

$$\mathrm{gr}_n\, cone(f \circ \alpha_{sX}) = \mathrm{gr}_n\, sX[1] + \mathrm{gr}_n\, Y.$$

□

Now, we make the definition of $\beta : \tilde{\mathcal{A}} \to \mathrm{C}^b(\mathrm{Pure}(X_0))$ by saying that for $M \in \tilde{\mathcal{A}}$, $\beta(M)^\bullet \in \mathrm{C}^b(\mathrm{Pure}(X))$ is the complex with ith piece $\beta(M)^i = \omega(\mathrm{gr}_{-i} M)[i] = \mathrm{gr}_0(s^i M)[i]$ and with differential $\delta^i : M^i \to M^{i+1}$ given by the third morphism in the functorial distinguished triangle

$$\omega(\mathrm{gr}_{-i-1} M)[i] \to \omega(w_{\geq -i-1} w_{\leq -i} M)[i] \to \omega(\mathrm{gr}_{-i} M)[i] \xrightarrow{\delta^i} \omega(\mathrm{gr}_{-i-1} M)[i+1]. \quad (4.2)$$

It is worth pointing out where this distinguished triangle comes from. For $M \in \tilde{\mathcal{A}}$, we know from above that there is a canonical distinguished triangle

$$w_{\leq -i-1}M \to M \to w_{\geq -i}M \xrightarrow{+1} \quad (4.3)$$

for all $i \in \mathbb{Z}$. Now, applying the functor $w_{\geq -i-1} w_{\leq -i}[i]$ to (4.3) we obtain the distinguished triangle

$$\mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}{}_{-i-1}\, M[i] \to w_{\geq -i-1} w_{\leq -i} M[i] \to \mathrm{gr}^{\mathrm{W}}_{\mathrm{i}}{}_{-i}\, M[i] \xrightarrow{+1}. \quad (4.4)$$

Applying ω to (4.4) we obtain (4.2).

Lemma 3.6 of [Rid13] tells us that β takes $M \in \tilde{\mathcal{A}}$ to a complex $M^\bullet \in \mathrm{C}^b(\mathrm{Pure}(X))$. Moreover, we have the following:

Theorem 4.1.8. *[Rid13, Proposition 3.7, Theorem 4.3, and Corollary 4.4] Let $\tilde{\mathcal{A}}$ be defined as above. Then the following hold:*

1. *The functor $\beta : \tilde{\mathcal{A}} \to C^b(Pure(X_0))$ is an equivalence of additive categories.*

2. *The composition $\omega \circ \beta^{-1} : C^b(Pure(X_0)) \to D^b_{T,m}(X_0)$ factors through $K^b Pure(X_0)$ and, therefore, induces a functor $\mathfrak{r} : K^b(Pure(X_0)) \to D^b_{T,m}(X_0)$ such that the restriction*

$$\mathfrak{r}|_{Pure(X_0)} : Pure(X_0) \to D^b_{T,m}(X_0)$$

is isomorphic to the inclusion functor. That is, $\mathfrak{r}$ is a realization functor.

3. *The category $K^b Pure(X)$ is a mixed version of the category $D^b_{T,c}(X_0)$ through the maps*

$$K^b Pure(X_0) \xrightarrow{\mathfrak{r}} D^b_{T,m}(X_0) \xrightarrow{\chi} D^b_{T,c}(X).$$

Here, as stated earlier in the review of sheaf theory, $\chi : D^b_{T,m}(X) \to D^b_{T,c}(X)$ is extension of scalars, i.e. the functor "forget the Frobenius."

4. *The heart of the t-structure $(K^b Pure(X_0)_{\triangleleft}, K^b Pure(X_0)_{\triangleright})$, $\mathcal{P}^{mix}(X_0) := K^b Pure(X_0)_{\triangleleft\cap\triangleright}$, is a mixed version of the category of perverse sheaves, $\mathcal{P}(X)$.*

As stated above, Rider's original statements are proved when $\mathrm{Pure}(X_0)$ satisfies a stronger condition. However, in [Rid13, Remark 3.8], it is commented that her proofs work in the weaker setting where $\underline{\mathrm{Hom}}^i_T(\tilde{\mathcal{L}}_\sigma, \tilde{\mathcal{L}}_\tau$ is merely pure (and Tate) of weight i. Our setting is actually stronger than that, so the proofs in [Rid13] apply for us as well. Before defining one of the primary full subcategories of interest, we wish to recall the $*$ operation.

Definition 4.1.9. Let D be a triangulated category. By $[X]$, we mean the isomorphism class of objects isomorphic to X. Let $\mathcal{S}(D)$ be the collection of isomorphisms classes of objects in D with representatives $[X]$. Then for $[A],[B] \in \mathcal{S}(D)$,

$$
\begin{aligned}
&[A] * [B] \\
&:= \{[Z] \mid \exists X \to Z \to Y \xrightarrow{+1} \text{ a distinguished triangle with } [X] = [A], [Y] = [B]\}.
\end{aligned}
$$

By [BBD82, Lemma 1.3.10], $*$ is associative. Now, we may proceed:

Definition 4.1.10. We wish to define the full subcategory $\mathrm{D}^{\mathrm{misc}}(X_0) \subseteq \mathrm{D}_T^{\mathrm{Weil}}(X_0)$. Let $\mathcal{F} \in \mathrm{D}_T^{\mathrm{Weil}}(X_0)$. We say that $\mathcal{F} \in \mathrm{D}^{\mathrm{misc}}(X_0)$ if there exist distinct integers $a, b \in \mathbb{Z}$ with $a \leq b$ such that

$$\mathcal{F} \in \mathrm{Pure}(X_0)[a] * \mathrm{Pure}(X_0)[a+1] * \cdots * \mathrm{Pure}(X_0)[b].$$

Furthermore, we say that a functor $F : \mathrm{D}_T^{\mathrm{Weil}}(X_0) \to \mathrm{D}_T^{\mathrm{Weil}}(Y_0)$ is **miscible** if it sends miscible objects to miscible objects.

In the above definition, we do mean to allow for any $\mathcal{F} \in \mathrm{Pure}(X_0)[a]$. However, we do not allow $\mathcal{F} \in \mathrm{Pure}(X_0)[a] * \mathrm{Pure}(X_0)[a]$. The category $\mathrm{D}^{\mathrm{misc}}(X_0)$ is actually the original category of interest, in a sense. That is, if one initially tries to find a mixed version of $\mathrm{D}_T^{\mathrm{Weil}}(X_0)$, then one is naturally lead to $\mathrm{D}^{\mathrm{misc}}(X_0)$. However, $\mathrm{D}^{\mathrm{misc}}(X_0)$ is not a triangulated category. This is precisely because we have defined $\mathrm{D}^{\mathrm{misc}}(X_0)$ to be a full subcategory of $\mathrm{D}_T^{\mathrm{Weil}}(X_0)$. We know, however, that there are certainly nonsplit extensions of sheaves with semisimple Frobenius action; given that, there will be morphisms whose cones are not in $\mathrm{D}^{\mathrm{misc}}(X_0)$. However, $\mathrm{D}^{\mathrm{misc}}(X_0)$ still has a structure that we can exploit. In particular, we will prove the following:

Theorem 4.1.11. *There is a natural equivalence of additive categories*

$$I : \mathscr{I} D^{mix}(X_0) \xrightarrow{\sim} D^{misc}(X_0).$$

This theorem means that we have a canonical functor

$$\iota : \mathrm{D}^{\mathrm{mix}}(X_0) \to \mathrm{D}^{\mathrm{misc}}(X_0).$$

By abuse of notation, we will also write ι for the composition

$$\mathrm{D}^{\mathrm{mix}}(X_0) \to \mathrm{D}^{\mathrm{misc}}(X_0) \to \mathrm{D}^{\mathrm{Weil}}_T(X_0).$$

Furthermore, let $\zeta = \chi \circ \iota$. Thus, we have a commutative diagram:

$$\begin{tikzcd} \mathrm{D}^{\mathrm{mix}}(X_0) \arrow[r, "\iota"] \arrow[dr, "\zeta"'] & \mathrm{D}^{\mathrm{Weil}}_T(X_0) \arrow[d, "\chi"] \\ & \mathrm{D}^b_{T,c}(X) \end{tikzcd}$$

The next lemma will follow from Theorem 4.1.11. However, it will follow also for any situation that is formally the same as that in Theorem 4.1.11. Due to the structure of the proof of Theorem 4.1.11, it is better to have a proof of this result first.

Lemma 4.1.12. *For $\mathcal{F}, \mathcal{G} \in D^{mix}(X_0)$, ι induces an isomorphism*

$$Hom_{D^{mix}(X_0)}(\mathcal{F}, \mathcal{G}) \simeq \underline{Hom}_T(\iota\mathcal{F}, \iota\mathcal{G})^{\mathrm{Fr}}.$$

There is a natural isomorphism

$$Hom_{D^{Weil}_T(X_0)}(\iota\mathcal{F}, \iota\mathcal{G}) \simeq Hom_{D^{mix}(X_0)}(\mathcal{F}, \mathcal{G}) \oplus Hom_{D^{mix}(X_0)}(\mathcal{F}, \mathcal{G}[-1]).$$

Proof. We begin the proof by recalling, for any $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{Weil}}_T(X_0)$, the short exact sequence

$$0 \to \underline{\mathrm{Hom}}^{i-1}_T(\mathcal{F}, \mathcal{G})_{\mathrm{Fr}} \to \mathrm{Hom}^i_{\mathrm{D}^{\mathrm{Weil}}_T(X_0)}(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{Hom}}^i_T(\mathcal{F}, \mathcal{G})^{\mathrm{Fr}} \to 0. \tag{4.5}$$

Here, as in the section on sheaf theory above, $\underline{\mathrm{Hom}}_T(\mathcal{F}, \mathcal{G}) = a_* \mathcal{RH}\mathrm{om}_T(\mathcal{F}, \mathcal{G})$ as above and $(\cdot)^{\mathrm{Fr}}$ and $(\cdot)_{\mathrm{Fr}}$ are the invariants and coinvariants of Frobenius, respectively. It is known that the natural map

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{Weil}}_T(X_0)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathrm{D}^b_{T,c}(X)}(\chi(\mathcal{F}), \chi(\mathcal{G}))$$

factors through the map $\mathrm{Hom}_{\mathrm{D}_T^{\mathrm{Weil}}(X_0)}(\mathcal{F},\mathcal{G}) \twoheadrightarrow \underline{\mathrm{Hom}}_T(\mathcal{F},\mathcal{G})^{\mathrm{Fr}}$ from (4.5) (cf. [BBD82]). Now, if we set $\zeta = \chi \circ \iota$, then there is a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}(X_0)}(\mathcal{F},\mathcal{G}) \arrow[r] \arrow[d, hook] & \mathrm{Hom}_{\mathrm{D}^b_{T,c}(X)}(\zeta(\mathcal{F}),\zeta(\mathcal{G})) \\ \mathrm{Hom}_{\mathrm{D}_T^{\mathrm{Weil}}(X_0)}(\iota\mathcal{F},\iota\mathcal{G}) \arrow[r] & \underline{\mathrm{Hom}}_T(\iota(\mathcal{F}),\iota(\mathcal{G}))^{\mathrm{Fr}} \arrow[u, hook] \end{tikzcd} \tag{4.6}$$

Now, since $\mathrm{D}^{\mathrm{mix}}(X_0)$ is a mixed version of $\mathrm{D}^b_c(X)$, we can sum up over all n and also obtain the following commutative diagram:

$$\begin{tikzcd} \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}(X_0)}(\mathcal{F},\mathcal{G}\langle n\rangle) \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{Hom}_{\mathrm{D}^b_{T,c}(X)}(\zeta(\mathcal{F}),\zeta(\mathcal{G})) \\ \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{D}_T^{\mathrm{Weil}}(X_0)}(\iota\mathcal{F},\iota\mathcal{G}\langle n\rangle) \arrow[r, two heads] & \bigoplus_{n\in\mathbb{Z}} \underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G}\langle n\rangle)^{\mathrm{Fr}} \arrow[u, "\sim"] \end{tikzcd} \tag{4.7}$$

We know that the left and right arrows are injections and isomorphisms, respectively, because the composition is an isomorphism and the right map in (4.6) is an injection, while summing over n gives a surjection on the right in (4.7). We know that the bottom map in (4.7) is a surjection, because the composition is of all three maps around the bottom of the diagram is an isomorphism. We also know that $\mathrm{Hom}_{\mathrm{D}^b_{T,c}(X)}(\chi(\mathcal{F}),\chi(\mathcal{G})) \simeq \chi(\underline{\mathrm{Hom}}_T(\mathcal{F},\mathcal{G}))$; i.e. it is isomorphic to $\underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G})$ with the Frobenius action forgotten. Since Fr acts semisimply on $\underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G})$, we know that

$$\underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G}) \simeq \oplus_{n\in\mathbb{Z}} \underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G})^n$$

where by $\underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G})^n$ is meant the weight n part of $\underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G})$. We also know, by the pure, even, Tate-ness or pure, odd, Tate-ness of the Fr action that

$$\underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G}(-n/2))^{\mathrm{Fr}} = \underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G})^n.$$

Since we know the domains and codomains of our maps, this shows that $\mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}(X_0)}(\mathcal{F},\mathcal{G}) \simeq \underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G})^{\mathrm{Fr}}$ canonically.

Using the isomorphism in Lemma 4.1.12, we see that the composition

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}(X_0)}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}_{\mathrm{D}_T^{\mathrm{Weil}}(X_0)}(\iota\mathcal{F},\iota\mathcal{G}) \to \underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G})^{\mathrm{Fr}} \tag{4.8}$$

is an isomorphism. Since $\mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}(X_0)}(\mathcal{F},\mathcal{G}) \simeq \underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G})^{\mathrm{Fr}}$, this provides a splitting of (4.5). Therefore,

$$\mathrm{Hom}_{\mathrm{D}_T^{\mathrm{Weil}}(X_0)}(\iota\mathcal{F},\iota\mathcal{G}) \simeq \underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G}[-1])_{\mathrm{Fr}} \oplus \underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G})^{\mathrm{Fr}}$$

canonically. However, the action of Fr on $\underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G})$ is pure, even, and Tate or pure, odd, and Tate, so $\underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G}[-1])_{\mathrm{Fr}} \simeq \underline{\mathrm{Hom}}_T(\iota\mathcal{F},\iota\mathcal{G})^{\mathrm{Fr}}$ canonically. Therefore, our claim is proven. □

Proof. (Of Theorem (4.1.11)) We begin by considering the functor

$$\mathfrak{r} : \mathrm{D}^{\mathrm{mix}}(X_0) \to \mathrm{D}_T^{\mathrm{Weil}}(X_0).$$

So that the following is more intelligible, we will use the notation $\mathrm{Pure}_M(X_0)$ to mean the category as considered within $\mathrm{D}^{\mathrm{mix}}(X_0)$ and $\mathrm{Pure}_W(X_0)$ to mean the category as considered within $\mathrm{D}_T^{\mathrm{Weil}}(X_0)$. We have already shown that $\mathfrak{r}$ is a functor of triangulated categories that commutes with Tate twists and restricts to the inclusion functor

$$\mathfrak{r}|_{\mathrm{Pure}_M(X_0)} : \mathrm{Pure}_M(X_0) \hookrightarrow \mathrm{Pure}_W(X_0) \subset \mathrm{D}_T^{\mathrm{Weil}}(X_0).$$

In particular, for any $\sigma \in \Delta$, $\mathfrak{r}(\tilde{\mathcal{L}}_\sigma\{n\}) \simeq \tilde{\mathcal{L}}_\sigma[n](n/2)$. Since we know that the composition

$$\mathrm{D}^{\mathrm{mix}}(X_0) \xrightarrow{\mathfrak{r}} \mathrm{D}_T^{\mathrm{Weil}}(X_0) \xrightarrow{\chi} \mathrm{D}^b_{T,c}(X)$$

realizes $\mathrm{D}^{\mathrm{mix}}(X_0)$ as a mixed version of $\mathrm{D}^b_{T,c}(X)$, the results of Lemma (4.1.12) apply to the functor $\mathfrak{r}$. Therefore this shows that $\mathfrak{r}$ extends in a canonical way to a functor

$$\tilde{\mathfrak{r}} : \mathscr{I}\mathrm{D}^{\mathrm{mix}}(X_0) \to \mathrm{D}_T^{\mathrm{Weil}}(X_0).$$

The results of Lemma 4.1.12 also tell us that $\tilde{\mathfrak{r}}$ is fully faithful and makes the following diagram commute:

$$\begin{tikzcd} \mathrm{D}^{\mathrm{mix}}(X_0) \arrow[r, "\iota"] \arrow[dr, "\mathfrak{r}"'] & \mathscr{I}\mathrm{D}^{\mathrm{mix}}(X_0) \arrow[d, "\tilde{\mathfrak{r}}"] \\ & \mathrm{D}^{\mathrm{Weil}}_T(X_0) \end{tikzcd} \tag{4.9}$$

The claim we wish to prove is that the essential image of $\tilde{\mathfrak{r}}$ is $\mathrm{D}^{\mathrm{misc}}(X_0)$. By the commutativity of (4.9), however, we know that an object is in the essential image of $\tilde{\mathfrak{r}}$ if and only if it is in the essential image of $\mathfrak{r}$. Thus, we will work with the latter. Suppose that $\mathcal{F} \in \mathrm{D}^{\mathrm{misc}}(X_0)$. Then for some $a, b \in \mathbb{Z}$ with $a \leq b$,

$$\mathcal{F} \in \mathrm{Pure}_W(X_0)[a] * \mathrm{Pure}_W(X_0)[a+1] * \cdots * \mathrm{Pure}_W(X_0)[b].$$

We will prove by induction on $|b-a|$ that there exists some

$$\tilde{\mathcal{F}} \in \mathrm{Pure}_M(X_0)[a] * \mathrm{Pure}_M(X_0)[a+1] * \cdots * \mathrm{Pure}_M(X_0)[b] \subset \mathrm{K}^b\mathrm{Pure}(X_0)$$

such that $\mathfrak{r}(\tilde{\mathcal{F}}) = \mathcal{F}$. If $\mathcal{F} \in \mathrm{Pure}_W(X_0)[a]$, then we know from the discussion above that $\mathfrak{r}(\mathcal{F}) = \mathcal{F}$. Now, assume that $|b-a| = n > 1$. Then there exists a distinguished triangle

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \xrightarrow{+1}$$

such that $\mathcal{F}' \in \mathrm{Pure}_W(X_0)[a]$ and $\mathcal{F}'' \in \mathrm{Pure}_W(X_0)[a+1] * \cdots \mathrm{Pure}_W(X_0)[b]$. By induction, there exists some $\tilde{\mathcal{F}}' \in \mathrm{Pure}_M(X_0)[a]$ such that $\mathfrak{r}(\tilde{\mathcal{F}}' = \mathcal{F}'$ and some $\tilde{\mathcal{F}}'' \in \mathrm{Pure}_M(X_0)[a+1] * \cdots * \mathrm{Pure}_M(X_0)[b]$ such that $\mathfrak{r}(\tilde{\mathcal{F}}'') = \mathcal{F}''$. We claim that $\mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}(X_0)}(\tilde{\mathcal{F}}'', \tilde{\mathcal{F}}') = 0$. This can be seen by an easy induction on $|b-a|$. Indeed, if $\mathcal{F}[a] \in \mathrm{Pure}_W(X_0)[a]$ and $\mathcal{G}[b] \in \mathrm{Pure}_W(X_0)[b]$ with $a \neq b$, then

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}(X_0)}(\mathcal{G}[b], \mathcal{F}[a]) = \mathrm{Hom}^{b-a}_{\mathrm{D}^{\mathrm{mix}}(X_0)}(\mathcal{G}, \mathcal{F}) = \underline{\mathrm{Hom}}^{b-a}_T(\iota\mathcal{G}, \iota\mathcal{F})^{\mathrm{Fr}} = 0.$$

Alternatively, we could see that this is 0 by noting that these are two complexes in a homotopy category of chain complexes that are concentrated in different degrees.

Now, if $\mathcal{F} \in \mathrm{Pure}_W(X_0)[a]$ and $\mathcal{G} \in \mathrm{Pure}_W(X_0)[a+1] * \cdots * \mathrm{Pure}_W(X_0)[b]$ with $b > a+1$, there exists a distinguished triangle

$$\mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \xrightarrow{+1} \tag{4.10}$$

such that $\mathcal{G}' \in \mathrm{Pure}_W(X_0)[a+1]$ and $\mathcal{G}'' \in \mathrm{Pure}_W(X_0)[a+2]*\cdots*\mathrm{Pure}_W(X_0)[b]$. Applying $\mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}(X_0)}(-, \mathcal{F})$ to (4.10), we see that $\mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}(X_0)}(\mathcal{G}, \mathcal{F}) = 0$ as claimed. Therefore, by Lemma 4.1.12 above, we see that $\mathfrak{r}$ induces an isomorphism

$$\mathrm{Hom}_{\mathrm{D}_T^{\mathrm{Weil}}(X_0)}(\mathcal{F}'', \mathcal{F}'[1]) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}(X_0)}(\tilde{\mathcal{F}}'', \tilde{\mathcal{F}}'[1]).$$

Therefore, a fortiori, to $\delta : \mathcal{F}'' \to \mathcal{F}'$ corresponds a $\tilde{\delta} : \tilde{\mathcal{F}}'' \to \tilde{\mathcal{F}}'[1]$. Denoting by $\tilde{\mathcal{F}}$ the cocone of $\tilde{\delta}$, we have that $\mathfrak{r}(\tilde{\mathcal{F}}) \simeq \mathcal{F}$ (non-canonically). Therefore, every $\mathcal{F} \in \mathrm{D}^{\mathrm{misc}}(X_0)$ is in the essential image of $\mathfrak{r}$ and, thus, also of $\tilde{\mathfrak{r}}$.

Conversely, let $\mathcal{F} \in \mathrm{K}^b\mathrm{Pure}(X_0)$. Then there exists $a, b \in \mathbb{Z}$ with $a \leq b$ such that $\mathcal{F} \in \mathrm{Pure}_M(X_0)[a] * \mathrm{Pure}_M(X_0)[a+1] \cdots * \mathrm{Pure}_M(X_0)[b]$. We know that $\mathfrak{r}(\mathrm{Pure}_M(X_0)[n]) \subseteq \mathrm{Pure}_W(X_0)[n]$. Therefore, the essential image of $\mathfrak{r}$ is contained in $\mathrm{D}^{\mathrm{misc}}(X_0)$. We now see that the essential image of $\tilde{\mathfrak{r}}$ is $\mathrm{D}^{\mathrm{misc}}(X_0)$ and our theorem is proved. $\square$

4.2 Functors Between Mixed Categories

We will now move on to the next important task. Namely, for any toric variety X_0 defined over $\mathbb{F}_q$ and extended as X to $\overline{\mathbb{F}}_q$, we now have categories, $\mathrm{D}^{\mathrm{mix}}(X_0)$, that we know can serve the job as "mixed versions" for our original categories of interest, $\mathrm{D}^b_{T,c}(X)$. It is then a generally interesting thing to produce functors between these categories, lest they seem useless. We will also show that $\mathrm{D}^{\mathrm{mix}}(X_0(\Delta))$ can be equipped with a recollement structure as in [BBD82], thus providing a second path to obtaining a perverse t-structure. Once we have our genuineness results, it will be clear that this construction agrees with the previous one. More precisely, we begin by showing that

Theorem 4.2.1. *Let $j : U_0 \hookrightarrow X_0$ be an open inclusion of a union of strata and let $i : Z_0 \hookrightarrow X_0$ be its closed complement. Then the functor $j^* : D^{mix}(X_0) \to D^{mix}(U_0)$ admits a left adjoint $j_{(!)}$ and a right adjoint $j_{(*)}$. The functor $i_* : D^{mix}(Z_0) \to D^{mix}(X_0)$ also admits a left adjoint $i^{(*)}$ and a right adjoint $i^{(!)}$. Together, these functors produce a recollment structure on $D^{mix}(X_0)$.*

Remark 4.2.2. As in [AR16], the parentheses in the notation of these functors helps to separate them from the normal functors. Once this proposition has been proven, we will often drop the parentheses from the notation.

We first recall that, since i_* is fully faithful, we know that i_* restricts to a functor

$$i_* : \mathrm{Pure}(Z_0) \to \mathrm{Pure}(X_0).$$

Therefore, it naturally induces a functor between the mixed derived categories. Next we note that by Theorem 4.0.3, the same thing is true about j^* and, in the case when Z_0 is a single stratum, i^*. To prove this proposition, we use the same strategy as in [AR16] and induce on the number of strata in Z_0.

Note that another way to proceed would have been to use the results on genuineness of adjoint functors (as we will use later for other functors). This way, however, needs none of the machinery of infinitesimal extensions of triangulated categories and Orlov categories, so the proof seemed worthwhile to show a different path to some of the results.

Lemma 4.2.3. *Let $O_0(\sigma) \subseteq X_0$ be a closed stratum. Then we have the following:*

1. *Let $Z_0 = O_0(\sigma)$ and $U_0 = X_0 \backslash Z_0$. Then j^* admits a left adjoint $j_{(!)}$ and a right adjoint $j_{(*)}$ such that the adjunction morphisms $j^* j_{(*)} \to id$ and $id \to j^* j_{(!)}$ are isomorphisms. Furthermore, we have that*

$$D^{mix}(X_0) = j_{(!)}(D^{mix}(U_0)) * i_*(D^{mix}(Z_0))$$

and

$$D^{mix}(X_0) = i_*(D^{mix}(Z_0)) * j_{(*)}(D^{mix}(U_0))$$

where $*$ is the associative operation defined in [BBD82, 1.3.9] and recalled in Definition 4.1.9 above.

2. *Let $O_0(\sigma) \subseteq Z_0 \subseteq X_0$ is a chain of a closed stratum (as above) sitting inside a closed union of strata. Let $j : U_0 = X_0 \setminus O_0(\sigma) \hookrightarrow X_0$, $j_{Z_0} : Z_0 \setminus O_0(\sigma) \hookrightarrow Z_0$, $k : Z_0 \hookrightarrow X_0$, and $k_{Z_0} : Z_0 \setminus O_0(\sigma) \hookrightarrow X_0 \setminus O_0(\sigma)$ be the inclusion maps. Then the functors $j_{(*)}$, $j_{(!)}$, $j_{Z_0(*)}$ and $j_{Z_0(!)}$ from (1) satisfy the following:*

$$j_{(!)}k_{Z_0*} \simeq k_* j_{Z_0(!)},$$

and

$$j_{(*)}k_{Z_0*} \simeq k_* j_{Z_0(*)}.$$

Proof. We prove the case of $j_{(!)}$ in detail. The case of $j_{(*)}$ can either be treated in a parallel manner or can be seen to follow from the first case via Verdier duality. We have the object $\tilde{\mathcal{L}}_\sigma = i_*\underline{\bar{\mathbb{Q}}}_{\ell,O(\sigma)}[n_\sigma](n_\sigma/2) \in \mathrm{Pure}_M(X_0)$, since $O(\sigma)$ is closed and smooth of dimension n_σ. For $\sigma \neq \tau \in \Delta$, we denote by $\tilde{\mathcal{L}}_\tau^+$ the complex

$$\tilde{\mathcal{L}}_\tau \to i_*i^*\tilde{\mathcal{L}}_\tau$$

with non-zero entries in the "0" and "1" positions and with the morphism being that of adjunction. We view this complex as an object in $\mathrm{D}^{\mathrm{mix}}(X_0)$. We know that it is in this category by the comments before this lemma. Now, we wish to show that for any $n, m \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}(X_0)}(\tilde{\mathcal{L}}_\tau^+, \tilde{\mathcal{L}}_\sigma\{m\}[n]) = 0. \tag{4.11}$$

To show this, we begin with the natural distinguished triangle

$$i_*i^*\tilde{\mathcal{L}}_\tau[-1] \to \tilde{\mathcal{L}}_\tau^+ \to \tilde{\mathcal{L}}_\tau \xrightarrow{+1} .$$

This is the "extensions" distinguished triangle, i.e. the triangle that realizes the middle term as an extension of the two outer terms. Now, after applying Hom(-$-, \tilde{\mathcal{L}}_\sigma\{m\}[n]$) to this triangle, we arrive at the following long exact sequence:

$$\cdots \to \mathrm{Hom}(i_*i^*\tilde{\mathcal{L}}_\tau, \tilde{\mathcal{L}}_\sigma\{m\}[n]) \to \mathrm{Hom}(\tilde{\mathcal{L}}_\tau, \tilde{\mathcal{L}}_\sigma\{m\}[n]) \to \mathrm{Hom}(\tilde{\mathcal{L}}_\tau^+, \tilde{\mathcal{L}}_\sigma\{m\}[n]) \to \cdots.$$

To complete this computation, we note that the natural morphism

$$\mathrm{Hom}(i_*i^*\tilde{\mathcal{L}}_\tau, \tilde{\mathcal{L}}_\sigma\{m\}[n]) \to \mathrm{Hom}(\tilde{\mathcal{L}}_\tau, \tilde{\mathcal{L}}_\sigma\{m\}[n])$$

is an isomorphism. This simply follows from considering the support of $\tilde{\mathcal{L}}_\sigma$. Therefore, (4.11) is 0.

Next, let D^+ be the triangulated subcategory of $\mathrm{D}^{\mathrm{mix}}(X_0)$ generated by the $\tilde{\mathcal{L}}_\tau^+\{m\}$ for all $\sigma \neq \tau \in \Delta$ and $m \in \mathbb{Z}$. Let $\iota : \mathrm{D}^+ \hookrightarrow \mathrm{D}^{\mathrm{mix}}(X_0)$ be the inclusion functor. We will now show that for all $\mathcal{F}^+ \in \mathrm{D}^+$ and $\mathcal{G} \in \mathrm{D}^{\mathrm{mix}}(X_0)$, the morphism induced by j^*

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}(X_0)}(\iota\mathcal{F}^+, \mathcal{G}) \to \mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}(U_0)}(j^*\iota\mathcal{F}^+, j^*\mathcal{G})$$

is an isomorphism. By a standard application of the five-lemma, it suffices to consider the cases when $\mathcal{F}^+ = \tilde{\mathcal{L}}_\tau^+$ for $\tau \in \Delta \setminus \sigma$ and when $\mathcal{G} = \tilde{\mathcal{L}}_\mu\{m\}[n]$ for some $\mu \in \Delta$ and $n, m \in \mathbb{Z}$. If $\mu = \sigma$, then this follows from 4.11. So, let $\mu \neq \sigma$. If $n \notin \{-1, 0\}$, then both sides are 0 and we are done. Assume that $n = -1$. In this case, the right hand side is 0, since $j^*i_*i^* = 0$. The left hand side consists of morphisms

$$\varphi : i_*i^*\tilde{\mathcal{L}}_\tau \to \tilde{\mathcal{L}}_\mu\{m\}$$

such that composition with the adjunction morphism $\tilde{\mathcal{L}}_\tau \to i_*i^*\tilde{\mathcal{L}}_\tau$ is 0. Then we have two scenarios: Either $\{m\}$ causes $i_*i^*\tilde{\mathcal{L}}_\tau$ and $\tilde{\mathcal{L}}_\mu$ to have the same parity or it causes them to have opposite parity. If $\{m\}$ causes them to have opposite parity, then the only morphism is 0 and we are done. Assume that $\{m\}$ causes the two to have the same parity. In this case, as a morphism in $\mathrm{D}^b_{T,m}(X_0)$, φ factors through a map $j_!j^*\tilde{\mathcal{L}}_\tau\{1\} \to \tilde{\mathcal{L}}_\mu\{m\}$. This is a for a combination of two reasons. First, in $\mathrm{D}^b_{T,m}(X_0)$, we still have the exact sequence above that gives us

$$\begin{aligned}\cdots \mathrm{Hom}(j_!j^*\tilde{\mathcal{L}}_\tau\{1\}, \tilde{\mathcal{L}}_\mu\{m\}) &\to \mathrm{Hom}(i_*i^*\tilde{\mathcal{L}}_\tau, \tilde{\mathcal{L}}_\mu\{m\}) \\ &\to \mathrm{Hom}(\tilde{\mathcal{L}}_\tau, \tilde{\mathcal{L}}_\mu\{m\}) \to \cdots .\end{aligned}$$

Secondly, we know that these particular maps go to 0 once composed with the adjunction morphism, i.e. they are in the kernel of the last map above and, so, must factor as claimed. However, we know that

$$\mathrm{Hom}_{\mathrm{D}^b_{T,m}(X_0)}(j_!j^*\tilde{\mathcal{L}}_\tau\{1\}, \tilde{\mathcal{L}}_\mu\{m\}) \simeq \mathrm{Hom}_{\mathrm{D}^b_{T,m}(U_0)}(j^*\tilde{\mathcal{L}}_\tau, j^!\tilde{\mathcal{L}}_\mu\{m-1\}) = 0.$$

Therefore, $\varphi = 0$.

Finally, we assume that $n = 0$. As before, if $\{m\}$ causes $\tilde{\mathcal{L}}_\tau$ and $\tilde{\mathcal{L}}_\mu$ to have different parities, then both sides are 0 and we are done. Therefore, we assume that $\{m\}$ causes $\tilde{\mathcal{L}}_\tau$ and $\tilde{\mathcal{L}}_\mu$ to have the same parity. Then the left-hand side of (4.11) is equal to the quotient of $\mathrm{Hom}(\tilde{\mathcal{L}}_\tau, \tilde{\mathcal{L}}_\mu\{m\})$ by the image of $\mathrm{Hom}(i_*i^*\tilde{\mathcal{L}}_\tau, \tilde{\mathcal{L}}_\mu\{m\})$ via the map above. (These maps we are quotienting out by are precisely all the possible homotopies.) However, going back to the long exact sequence from above, we have

$$\begin{aligned}\mathrm{Hom}(i_*i^*\tilde{\mathcal{L}}_\tau, \tilde{\mathcal{L}}_\mu\{m\}) &\to \mathrm{Hom}(\tilde{\mathcal{L}}_\tau, \tilde{\mathcal{L}}_\mu\{m\}) \\ &\to \mathrm{Hom}(j_!j^*\tilde{\mathcal{L}}_\tau, \tilde{\mathcal{L}}_\mu\{m\}) \\ &\to \mathrm{Hom}(i_*i^*\tilde{\mathcal{L}}_\tau\{-1\}, \tilde{\mathcal{L}}_\mu\{m\}) \to \cdots .\end{aligned}$$

By adjunction, Theorem 4.0.3, and the fact stipulation on $\{m\}$, we know that

$$\mathrm{Hom}(i_*i^*\tilde{\mathcal{L}}_\tau\{-1\}, \tilde{\mathcal{L}}_\mu\{m\}) = \mathrm{Hom}(i^*\tilde{\mathcal{L}}_\tau, i^!\tilde{\mathcal{L}}_\mu\{m-1\}) = 0.$$

Therefore, the right-hand side of (4.11)

$$\mathrm{Hom}(j^*\tilde{\mathcal{L}}_\tau, j^*\tilde{\mathcal{L}}_\mu\{m\}) = \mathrm{Hom}(j_!j^*\tilde{\mathcal{L}}_\tau, \tilde{\mathcal{L}}_\mu\{m\})$$

is identified with the the left-hand side via this exact sequence.

Now, (4.11) tells us, in particular, that $j^* \circ \iota$ is fully faithful. The objects $j^*\iota\tilde{\mathcal{L}}_\tau^+$ generate the triangulated category $\mathrm{D}^{\mathrm{mix}}(U_0)$, however, so $j^* \circ \iota$ is actually an equivalence of categories. Define the functor

$$j_{(!)} := \iota \circ (j^* \circ \iota)^{-1} : \mathrm{D}^{\mathrm{mix}}(U_0) \to \mathrm{D}^{\mathrm{mix}}(X_0).$$

We see that, by definition, the identity $j^*j_{(!)} = id$ holds. Now, using adjunction and the fact that $j^*i_* = 0$, we see that the equality in (1) holds.

Moving on to (2), we begin by considering the diagram:

$$\begin{tikzcd} \mathrm{D}^+_{Z_0} \arrow[r, dashed, "k_*^+"] \arrow[d, "\iota_Z"] & \mathrm{D}^+_{X_0} \arrow[d, "\iota_X"] \\ \mathrm{D}^{\mathrm{mix}}(Z_0) \arrow[r, "k_*"] & \mathrm{D}^{\mathrm{mix}}(X_0) \end{tikzcd} \tag{4.12}$$

We wish to prove that the dotted arrow, k_*^+, exists so that the resulting diagram commutes. Furthermore, we claim that it is the unique such functor. Consider the functor $k_*^+ := k_*|_{\mathrm{D}^+_{Z_0}} : \mathrm{D}^+_{Z_0} \to \mathrm{D}^{\mathrm{mix}}(X_0)$. We wish to prove that the essential image of this functor actually lands in $\mathrm{D}^+_{X_0}$. Since $\mathrm{D}^+_{Z_0}$ is generated by a finite collection of objects and $\mathrm{D}^+_{X_0}$ is a full subcategory, it suffices to check what k_*^+ does to these generating objects. To this end, let us make explicit some of the notation (now that we have two different D^+ categories). In this section, we write $i : O_0(\sigma) \hookrightarrow Z_0$ as the inclusion into Z_0 instead of the inclusion into the whole space. We know that Z_0 is a closed union of strata that contains $O_0(\sigma)$, so we will write it as

$$Z_0 = \cup_{\tau \in \mathcal{T}} O_0(\tau).$$

Here $\mathcal{T}$ is a subset of Δ. If $\tau \in \mathcal{T}$, then $\tilde{\mathcal{L}}_\tau$ is a simple perverse sheaf on both Z_0 and X_0 with support contained in Z_0. Technically, $\tilde{\mathcal{L}}_\tau$ considered as a perverse sheaf on X_0 is an extension by zero of $\tilde{\mathcal{L}}_\tau$ considered as a perverse sheaf on Z_0. In the following arguments, we will, however, make no distinction between the two and use the symbol $\tilde{\mathcal{L}}_\tau$ to refer to the simple perverse sheaf on either Z_0 or X_0. Having said that, we can now explicitly state that

$$D^+_{Z_0} = \langle (\tilde{\mathcal{L}}_\tau \to i_* i^* \tilde{\mathcal{L}}_\tau)\{m\} \mid \tau \in \mathcal{T} \setminus \{\sigma\}, m \in \mathbb{Z} \rangle_\triangle .$$

Here we are using the notation $\langle - \rangle_\triangle$ to mean that it is generated as a triangulated category by the objects in the angle brackets. Similarly,

$$\mathrm{D}^+_{X_0} = \langle (\tilde{\mathcal{L}}_\tau \to k_* i_* i^* k^* \tilde{\mathcal{L}}_\tau)\{m\} \mid \tau \in \Delta \setminus \{\sigma\}, m \in \mathbb{Z} \rangle_\triangle .$$

Since k_* commutes with $\{m\}$, it is enough to check this when $m = 1$. In that case, picking some arbitrary $\tau \in \mathcal{T} \setminus \{\sigma\}$, we see that

$$k^+_* (\tilde{\mathcal{L}}_\tau \to i_* i^* \tilde{\mathcal{L}}_\tau) = k_* (\tilde{\mathcal{L}}_\tau \to i_* i^* \tilde{\mathcal{L}}_\tau).$$

However, k_* is a triangulated functor. (Actually, more is true. Since $Z_0 \subseteq X_0$ is closed, k_* is actually an exact functor.) Therefore,

$$k_* (\tilde{\mathcal{L}}_\tau \to i_* i^* \tilde{\mathcal{L}}_\tau) = k_* \tilde{\mathcal{L}}_\tau \to k_* i_* i^* \tilde{\mathcal{L}}_\tau .$$

Now, $\tilde{\mathcal{L}}_\tau = k^* \tilde{\mathcal{L}}_\tau$ where we are abusing notation as warned above. We also know that $\tilde{\mathcal{L}}_\tau$ considered as a simple perverse sheaf on X_0 is just $\tilde{\mathcal{L}}_\tau$ as considered on Z_0 extended by zero to the rest of X_0. Furthermore, k_* is the extension by zero functor, so $k_* \tilde{\mathcal{L}}_\tau = \tilde{\mathcal{L}}_\tau$ (with the same abuse of notation). So, we see that

$$k^+_* \tilde{\mathcal{L}}^+_\tau = \tilde{\mathcal{L}}_\tau \to k_* i_* i^* k^* \tilde{\mathcal{L}}_\tau = \tilde{\mathcal{L}}^+_\tau \in \mathrm{D}^+_{X_0}.$$

Therefore, $k_*|_{\mathrm{D}^+_{Z_0}} : \mathrm{D}^+_{Z_0} \to \mathrm{D}^+_{X_0}$. Since our candidate for k^+_* is just the restriction of a functor $k_* : \mathrm{D}^{\mathrm{mix}}(Z_0) \to \mathrm{D}^{\mathrm{mix}}(X_0)$, it is clear that it commutes with the inclusion

$\iota : \mathrm{D}^+ \hookrightarrow \mathrm{D}^{\mathrm{mix}}(X_0)$. We only have to prove uniqueness. By the remarks above, we see that the generating objects of $\mathrm{D}^+_{Z_0}$ are simply a subset of the generating objects of $\mathrm{D}^+_{X_0}$, once they are extended by zero, that is. Since the two vertical functors are both the respective inclusion functors, we see that any other functor k^+_{2*} making diagram (4.12) commute must also map $\tilde{\mathcal{L}}^+_\tau \mapsto \tilde{\mathcal{L}}^+_\tau$. Since ι_Z and ι_X are fully faithful, this is unique. Therefore, k^+_* exists and is unique.

Now, consider the following diagram of functors:

$$\begin{tikzcd}
Z_0 \setminus O_0(\sigma) \arrow[r, hook, "k_Z"] \arrow[d, hook, "j_Z"] & X_0 \setminus O_0(\sigma) \arrow[d, hook, "j"] \\
Z_0 \arrow[r, hook, "k"] & X_0
\end{tikzcd}$$

This is clearly a Cartesian square and k (hence also k_Z) is proper, so by the proper base change theorem we see that $j^* k_* = k_{Z*} j^*_Z$. This implies that

$$j^* \circ \iota_X \circ k^+_* = k_{Z*} \circ j^*_Z \circ \iota_Z.$$

Now, we compose with $j_{(!)} = \iota_X \circ (j^* \circ \iota_X)^{-1}$ on both sides to obtain

$$\iota_X \circ (j^* \circ \iota_X)^{-1} \circ j^* \circ \iota_X \circ k^+_* = \iota_X \circ (j^* \circ \iota_X)^{-1} \circ k_{Z*} \circ j^*_Z \circ \iota_Z. \tag{4.13}$$

The left-hand side immediately cancels to

$$\iota_X \circ (j^* \circ \iota_X)^{-1} \circ j^* \circ \iota_X \circ k^+_* = \iota_X \circ k^+_* = k_* \circ \iota_Z.$$

Therefore, (4.13) becomes

$$k_* \circ \iota_Z = j_{(!)} \circ k_{Z*} \circ j^*_Z \circ \iota_Z.$$

Now, we can compose each side with $(j^*_Z \circ \iota_Z)^{-1}$ to arrive at the following:

$$k_* j_{Z(!)} = j_{(!)} k_{Z*} j^*_Z j_{Z(!)} = j_{(!)} k_{Z*}.$$

This is precisely the first isomorphism in the statement of the lemma. $\square$

Remark 4.2.4. In the above proof, we often use explicit facts about what chain maps between chain complexes look like. We are able to do this, because we are actually working in the homotopy category $\mathrm{K}^b\mathrm{Pure}(X_0)$ and not in a more abstract "derived category."

Now we can proceed to the proof of Theorem 4.2.1.

Proof. (Theorem 4.2.1) In this proof, we will show the following items:

1. We will provide the construction for $j_{(!)}$ and $i^{(*)}$.

2. We will show that the adjunctions morphisms $id \to j^* j_{(!)}$ and $i^{(*)} i_* \to id$ are isomorphisms.

3. We will prove that for any $\mathcal{F} \in \mathrm{D}^{\mathrm{mix}}(X_0)$, there is a morphism $i_* i^{(*)}\mathcal{F} \to j_{(!)} j^* \mathcal{F}[1]$ such that the triangle

$$j_{(!)} j^* \mathcal{F} \to \mathcal{F} \to i_* i^{(*)} \mathcal{F} \xrightarrow{+1} \tag{4.14}$$

is a distinguished triangle in $\mathrm{D}^{\mathrm{mix}}(X_0)$.

Recollement (cf. [BBD82, 1.4.3]) requires these statements along with analogous statements for $j_{(*)}$ and $i^{(!)}$ and the statement that $j^* i_* = 0$. As in the above lemma, the statements for $j_{(*)}$ and $i^{(!)}$ are similar and will therefore be left to the reader. We begin by showing that $j_{(!)}$ exists via induction on the number of strata in Z_0, that $id \to j^* j_{(!)}$ is an isomorphism, and that

$$\mathrm{D}^{\mathrm{mix}}(X_0) = j_{(!)}(\mathrm{D}^{\mathrm{mix}}(U_0) * i_*(\mathrm{D}^{\mathrm{mix}}(Z_0)). \tag{4.15}$$

If Z consists of one stratum, then the claim follows by Lemma 4.2.3. Now, assume that Z has $n > 0$ strata. Assume, as before, that $O_0(\sigma) \subset Z_0$ is a closed stratum. We set $X_0' = X_0 \setminus O_0(\sigma)$ and $Z_0' = Z_0 \setminus O_0(\sigma)$. We let $k_Z : Z_0' \hookrightarrow X_0'$, $k : Z_0 \hookrightarrow X_0$,

$i : O_0(\sigma) \hookrightarrow X_0$, $i_\sigma : O_0(\sigma) \hookrightarrow Z_0$, $j' : X_0' \hookrightarrow X_0$, $j_{X'} : U_0 \hookrightarrow X_0'$, $j_{Z'} : Z_0' \hookrightarrow X_0$, $j_Z : Z_0' \hookrightarrow Z_0$, and $j : U_0 \hookrightarrow X_0$ the inclusion maps. Note that $i = k \circ i_\sigma$. By induction, $j_{X'}^*$ and j'^* have left adjoints. Therefore, the same is true of the composition $j_{X'}^* \circ j'^* = j^*$. It is precisely the fact that $j' \circ j_{X'} = j$ that again implies that the adjunction map $id \to j^* j_{(!)}$ is an isomorphism by induction. We also see that (4.15) holds for $X_0 = X_0' \sqcup O_0(\sigma)$ and $X_0' = U_0 \sqcup Z_0'$ by induction. We can, therefore, write the equality

$$\mathrm{D}^{\mathrm{mix}}(X_0) = j'_{(!)}(\mathrm{D}^{\mathrm{mix}}(X_0')) * i_*(\mathrm{D}^{\mathrm{mix}}(O_0(\sigma))) \tag{4.16}$$

$$= j'_{(!)}(j_{X'(!)}(\mathrm{D}^{\mathrm{mix}}(U_0)) * k_{Z*}(\mathrm{D}^{\mathrm{mix}}(Z_0'))) * i_*(\mathrm{D}^{\mathrm{mix}}(O_0(\sigma)). \tag{4.17}$$

We now note that, if $F : \mathcal{C} \to \mathcal{D}$ is a fully faithful triangulated functor between two triangulated categories and $\mathcal{A}, \mathcal{B} \in \mathcal{C}$ are two triangulated subcategories, then

$$F(\mathcal{A} * \mathcal{B}) = F(\mathcal{A}) * F(\mathcal{B}). \tag{4.18}$$

This is because F being a triangulated functor means that it preserves distinguished triangles and being fully faithful means that $\mathrm{Hom}^1_{\mathcal{C}}(B, A) \simeq \mathrm{Hom}^1_{\mathcal{D}}(F(B), F(A))$ for any $A, B \in \mathcal{C}$. Now, we can apply this to (4.16) to see that

$$\mathrm{D}^{\mathrm{mix}}(X_0) = j_{(!)}(\mathrm{D}^{\mathrm{mix}}(U_0)) * (j'_{(!)} \circ k_{Z*}(\mathrm{D}^{\mathrm{mix}}(Z_0'))) * i_*(\mathrm{D}^{\mathrm{mix}}(O_0(\sigma))). \tag{4.19}$$

Since $*$ is associative, we are justified in writing the above unambiguously without parentheses. Recalling from Lemma 4.2.3 part (2) that $j_{(!)} k_{Z*} = k_* j_{Z(!)}$ we see that $j'_{(!)} k_{Z*}(\mathrm{D}^{\mathrm{mix}}(Z_0') = k_* j_{Z*}(\mathrm{D}^{\mathrm{mix}}(Z_0'))$. Also, since $i = k \circ i_\sigma$, we find that $i_*(\mathrm{D}^{\mathrm{mix}}(O_0(\sigma))) = k_* i_{\sigma*}(\mathrm{D}^{\mathrm{mix}}(O_0(\sigma)))$. So, again using (4.18), we have that

$$\mathrm{D}^{\mathrm{mix}}(X_0) = j_{(!)}(\mathrm{D}^{\mathrm{mix}}(U_0)) * k_*(j_{Z*}(\mathrm{D}^{\mathrm{mix}}(Z_0')) * i_{\sigma*}(\mathrm{D}^{\mathrm{mix}}(O_0(\sigma)))).$$

By induction, this says that

$$\mathrm{D}^{\mathrm{mix}}(X_0) = j_{(!)}(\mathrm{D}^{\mathrm{mix}}(U_0)) * k_*(\mathrm{D}^{\mathrm{mix}}(Z_0)).$$

So our induction is finished.

We move on now to the case of $i^{(*)}$ as well as the existence of (4.14). We see from (4.15) that for any $\mathcal{F} \in \mathrm{D}^{\mathrm{mix}}(X_0)$, there exists $\mathcal{F}' \in \mathrm{D}^{\mathrm{mix}}(U_0)$, $\mathcal{F}'' \in \mathrm{D}^{\mathrm{mix}}(Z_0)$ and a distinguished triangle

$$j_{(!)}\mathcal{F}' \to \mathcal{F} \to k_*\mathcal{F}'' \xrightarrow{+1} . \tag{4.20}$$

In this situation, $\mathcal{F}'$ and $\mathcal{F}''$ are unique due to the fully faithfulness nature of $j_{(!)}$ and k_*. However, more is true. By calculating $\mathrm{Hom}^{-1}(j_{(!)}\mathcal{F}', k_*\mathcal{F}'') \simeq \mathrm{Hom}^{-1}(\mathcal{F}', j^*k_*\mathcal{F}'') = 0$, we can apply [BBD82] Corollary 1.1.10 to see that (4.20) is unique as well. We obviously have the identity $\mathcal{F}' \simeq j^*\mathcal{F}$. Therefore, we set $i^{(*)}\mathcal{F} \coloneqq \mathcal{F}''$. Then, by definition, the desired functorial properties of $i^{(*)}$ are satisfied by construction. $\square$

We now wish to turn our attention briefly to the Verdier duality functor, $\mathbb{D}$. It is clear that $\mathbb{D} : \mathrm{D}_T^{\mathrm{Weil}}(X_0) \to \mathrm{D}_T^{\mathrm{Weil}}(X_0)$ restricts to an antiequivalence

$$\mathbb{D} : \mathrm{Pure}(X_0) \xrightarrow{\sim} \mathrm{Pure}(X_0).$$

Therefore, it also induces an antiequivalence

$$\mathbb{D} : \mathrm{D}^{\mathrm{mix}}(X_0) \to \mathrm{D}^{\mathrm{mix}}(X_0).$$

Note that, by abuse of notation, we refer to all of these functors by the same symbol, $\mathbb{D}$. The standard six-functor yoga tells us that

$$\mathbb{D}_{X_0} \circ \{n\} \simeq \{-n\} \circ \mathbb{D}_{X_0}, \mathbb{D}_{X_0} \circ [n] \simeq [-n] \circ \mathbb{D}_{X_0}, \mathbb{D}_{X_0} \circ \langle n \rangle \simeq \langle -n \rangle \circ \mathbb{D}_{X_0}.$$

Since $i_* : \mathrm{D}^{\mathrm{mix}}(Z_0) \to \mathrm{D}^{\mathrm{mix}}(X_0)$ and $j^* : \mathrm{D}^{\mathrm{mix}}(X_0) \to \mathrm{D}^{\mathrm{mix}}(U_0)$ are obtained by restricting the usual functors to $\mathrm{Pure}(Z_0)$ and $\mathrm{Pure}(X_0)$ respectively and then inducing from there to the homotopy category, the usual identities

$$\mathbb{D}_{U_0} \circ j^* \simeq j^* \circ \mathbb{D}_{X_0}, \mathbb{D}_{X_0} \circ i_* \simeq i_* \circ \mathbb{D}_{Z_0}$$

still hold in this context. This allows us to deduce the following canonical isomorphisms:

$$\mathbb{D}_{X_0} \circ j_{(!)} \simeq j_{(*)} \circ \mathbb{D}_{U_0}, \mathbb{D}_{Z_0} \circ i^{(!)} \simeq i^{(*)} \circ \mathbb{D}_{X_0}.$$

4.3 Locally Closed Inclusions

We now wish to extend the results of the previous section so that we pullback and push-forward functors for locally closed inclusions as well.

Lemma 4.3.1. *Let $h : W_0 \hookrightarrow X_0$ be a locally closed inclusion of toric varieties. Consider the commutative diagram*

$$\begin{tikzcd} W_0 \arrow[r, hook, "i'"] \arrow[d, hook, "j'"] & Z_0 \arrow[d, hook, "j"] \\ Y_0 \arrow[r, hook, "i"] & X_0 \end{tikzcd}$$

Here W_0, Z_0, and Y_0 are unions of strata. We assume that i and i' are closed inclusions and that j and j' are open inclusions. Then there are natural isomorphisms of functors

$$j_{(!)}i'_* \simeq i_* j'_{(!)}, j_{(*)}i'_* \simeq i_* j'_{(*)}, i'^{(!)}j^* \simeq j'^* i^{(!)}, i'^{(*)}j^* \simeq j'^* i^{(*)}. \tag{4.21}$$

Proof. As in [AR16], it is enough to show the first isomorphism. The second follows from the first by Verdier duality, the third follows from the first by adjunction, and the fourth follows from the second by adjunction. We proceed by induction on the number of strata. If $Y_0 = Z_0 = X_0$, then W_0 is both open and closed inside of X_0, so it is a union of connected components. In this case, $j = i = id$, so the left and right-hand sides of the first formula can be calculated and directly seen to be isomorphic. Now, assume that $Z_0 \neq X_0$ and choose a closed stratum $O_0(\sigma) \in X_0 \setminus Z_0$. Setting

$X_0' = X_0 \setminus O_0(\sigma)$ and $Y_0' = Y_0 \cap X_0'$, we have the following diagram:

$$\begin{tikzcd} W_0 \arrow[r, hook] \arrow[d, hook] & Z_0 \arrow[d, hook] \\ Y_0' \arrow[r, hook] \arrow[d, hook] & X_0' \arrow[d, hook] \\ Y_0 \arrow[r, hook] & X_0 \end{tikzcd}$$

We know, by induction, that the claim is true for the upper square. We proceed to prove it for the lower square. If $O_0(\sigma) \subseteq Y_0$, then this has already been shown in Lemma 4.2.3 part (2). Suppose that $O_0(\sigma) \nsubseteq Y_0$. Then $Y_0 = Y_0'$. $\square$

This lemma now allows us to unambiguously define, for any locally closed inclusion of strata $h : Y_0 \hookrightarrow X_0$, the push-forward functors

$$h_{(*)}, h_{(!)} : \mathrm{D}^{\mathrm{mix}}(Y_0) \to \mathrm{D}^{\mathrm{mix}}(X_0)$$

as well as the pullback functors

$$h^{(*)}, h^{(!)} : \mathrm{D}^{\mathrm{mix}}(X_0) \to \mathrm{D}^{\mathrm{mix}}(Y_0).$$

We will explicitly demonstrate this for $h_{(!)}$. Let $i : Z_0 \hookrightarrow X_0$ be a closed inclusion and $j : Y_0 \hookrightarrow Z_0$ an open inclusion. Let $h = i \circ j$. Then, up to isomorphism, $i_* \circ j_{(!)}$ does not depend on Z_0. To see this, we apply Lemma 4.3.1 to the following diagram:

$$\begin{tikzcd} Y_0 \arrow[r, equal] \arrow[d, hook, "j_Y"] & Y_0 \arrow[d, hook, "j"] \\ \overline{Y_0} \arrow[r, hook, "i_Y"] & Z_0 \end{tikzcd}$$

This gives us that $j_{(!)} = i_{Y*} j_{Y(!)}$, so $i_* j_{(!)} = i_* i_{Y*} j_{Y(!)}$. So, this does not depend on Z_0. Similarly, if $j : U_0 \hookrightarrow X_0$ is an open inclusion and $i : Y_0 \hookrightarrow U_0$ is a closed inclusion, then we see that, if $h = j \circ i$, $j_{(!)} \circ i_*$ does not depend on the choice of

U_0. In this case, we consider the diagram

$$\begin{tikzcd} Y_0 \arrow[r, equal] \arrow[d, hook] & Y_0 \arrow[d, hook] \\ U_0 \setminus (U_0 \cap (\overline{Y_0} \setminus Y_0)) \arrow[r, hook] & U_0 \end{tikzcd}$$

This case then proceeds the same as the previous one. Another application of Lemma 4.3.1 shows us that all of these functors are canonically isomorphic to each other. This is the functor we define to be $h_{(!)}$. If $h : Y_0 \hookrightarrow Z_0$ and $k : Z_0 \hookrightarrow X_0$ are two locally closed inclusions, then $(k \circ h)_{(!)} = k_{(!)} h_{(!)}$ by Lemma 4.3.1 again. This is also true for the other three functors for exactly the same reason. Also, we see directly by definition that $\mathbb{D} \circ h_{(!)} \simeq h_{(*)} \circ \mathbb{D}$ and $\mathbb{D} \circ h^{(*)} \simeq h^{(!)} \circ \mathbb{D}$.

4.4 Proper and Smooth Toric Maps

Suppose that $X_0(\Delta) = X_0$ and $Y_0(\Delta') = Y_0$ are two toric varieties. We are here viewing $\Delta \subset N$ and $\Delta' \subset N'$ as living within these two respective lattices. A map between X_0 and Y_0 is called toric if it is equivariant with respect to the two torus actions. Recall that this is the case if and only if the map is one arising from a map between the respective fans. That is, $f : X_0 \to Y_0$ is induced by a map $f_N : N \to N'$ such that $f_{N_{\mathbb{R}}} : N_{\mathbb{R}} \to N'_{\mathbb{R}}$ is linear and for all cones $\sigma \in \Delta$, there is a cone $\sigma' \in \Delta'$ so that $f_{N_{\mathbb{R}}}(\sigma) \subseteq \sigma'$. If we denote by $|\Delta|$ the support of Δ in $N_{\mathbb{R}}$, then a toric map $f : X_0(\Delta) \to X'_0(\Delta')$ is proper if and only if $f^{-1}_{N_{\mathbb{R}}}(|\Delta'|) = |\Delta|$. This is another amazing example of the combinatorics of fans and the geometry of toric varieties determining each other.

It is an amazing fact that by Theorem 3.3.4, if $f : X_0 \to Y_0$ is proper toric, then $f_* : \mathrm{D}^b_{T,m}(X_0) \to \mathrm{D}^b_{T,m}(Y_0)$ restricts to a functor

$$f_* : \mathrm{Pure}(X_0) \to \mathrm{Pure}(Y_0).$$

Therefore, it induces a functor

$$f_* : \mathrm{D}^{\mathrm{mix}}(X_0) \to \mathrm{D}^{\mathrm{mix}}(Y_0).$$

We also see that if $f : X_0 \to Y_0$ is a smooth toric morphism of relative dimension d, then $f^* \simeq f^!\{-2d\}$. We want to show that $f^*[d](d/2) : \mathrm{D}^b_{T,m}(Y_0) \to \mathrm{D}^b_{T,m}(X_0)$ also restricts to a functor $f^*[d](d/2) : \mathrm{Pure}(Y_0) \to \mathrm{Pure}(X_0)$. To see this, we recall that

$$\begin{aligned}&f^*[d](d/2) j_{\sigma!*}\underline{\bar{\mathbb{Q}}}_{\ell,O_0(\sigma)}[n_\sigma](n_\sigma/2)\\ &= j_{\sigma!*}f^*\underline{\bar{\mathbb{Q}}}_{\ell,O_0(\sigma)}[\dim(f^{-1}(O_0(\sigma)))](\dim(f^{-1}(O_0(\sigma)))/2)\end{aligned}$$

by [BBD82, Section 4.2.6]. We know that $f^{-1}(O_0(\sigma))$ is a union of strata and that the pullback of the constant sheaf is the constant sheaf. Therefore, Theorem 3.3.4 tells us that this is in $\mathrm{Pure}(X_0)$. When $f : X_0 \to Y_0$ is a smooth toric morphism of relative dimension d, we will sometimes denote the functor

$$f^\dagger \coloneqq f^*\{d\} \simeq f^!\{-d\} : \mathrm{D}^{\mathrm{mix}}(Y_0) \to \mathrm{D}^{\mathrm{mix}}(X_0).$$

Theorem 4.4.1. *Let $f : X_0 \to Y_0$ be a proper toric map. Let $h : Z_0 \hookrightarrow Y_0$ be a locally closed inclusion of a union of strata. Consider the Cartesian square:*

$$\begin{array}{ccc} f^{-1}(Z_0) & \xrightarrow{h'} & X_0 \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Z_0 & \xrightarrow{h} & Y_0 \end{array}$$

Then f' is a proper toric map. Furthermore, we have the following natural isomorphisms:

$$f_* \circ h'_{(*)} \simeq h_{(*)} \circ f'_{(*)}, \qquad f_* \circ h'_{(!)} \simeq h_{(!)} \circ f'_*, \tag{4.22}$$

$$h'^{(*)} \circ f^\dagger \simeq f^{\dagger *} \circ h^{(*)}, \qquad h'^{(!)} \circ f^\dagger \simeq f'^\dagger \circ h^{(!)}, \tag{4.23}$$

$$f^\dagger \circ h_{(*)} \simeq h'_{(*)} f'^\dagger, \qquad f^\dagger \circ h_{(!)} \simeq h'_{(!)} \circ f'^\dagger, \tag{4.24}$$

$$h^{(*)} \circ f_* \simeq f'_* \circ h'^{(*)}, \qquad h^{(!)} \circ f_* \simeq f'_* \circ h'^{(!)}. \tag{4.25}$$

Proof. We begin by showing that f' is proper toric. In this context, $f' = f|_{f^{-1}(Z_0)}$. In general, the restriction of a toric map is still toric. The same is true for proper maps. Therefore, f' is proper toric.

As for the isomorphisms, we can see from Section 2.3 above that it suffices to consider these functors when h is either an open or a closed inclusion. If h is a closed inclusion, then $h_{(*)} = h_*$, so (4.22) follows from the fact that $f_* \circ h'_*, h_* \circ f'_* :$ $\mathrm{Pure}(f^{-1}(Z_0)) \to \mathrm{Pure}(X_0)$ are isomorphic. In fact, the same reasoning also proves (4.24) in the same case, i.e. when h is a closed inclusion. By entirely analogous reasoning, (4.23) and (4.25) hold when h is an open inclusion. We can now handle the remaining cases by adjunction. $\square$

Chapter 5

Genuineness of Some Functors

In this section, we turn to the question of the "genuineness" of some functors. First, let us explain what is meant by this term. Suppose that we have two toric varieties X_0, Y_0 and some functor

$$F : \mathrm{D}^b_{T,m}(X_0) \to \mathrm{D}^b_{T,m}(Y_0).$$

Now, suppose that there is a functor

$$F' : \mathrm{D}^{\mathrm{mix}}(X_0) \to \mathrm{D}^{\mathrm{mix}}(Y_0)$$

such that F' "comes from" F in some suitable sense. One example would be if $F|_{\mathrm{Pure}(X_0)}$ restricts to a functor

$$F|_{\mathrm{Pure}(X_0)} : \mathrm{Pure}(X_0) \to \mathrm{Pure}(Y_0).$$

Then, by restricting to $\mathrm{Pure}(X_0)$ and then inducing up, F gives rise to a functor between the mixed categories. Another example that we will care about is when we have functors in both the mixed setting and in Deligne's setting that are adjoints of a functor arising in the above manner. Then we have a diagram as follows:

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{mix}}(X_0) & \xrightarrow{\mathfrak{r}} & \mathrm{D}^b_{T,m}(X_0) \\ \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F} \\ \mathrm{D}^{\mathrm{mix}}(Y_0) & \xrightarrow{\mathfrak{r}} & \mathrm{D}^b_{T,m}(Y_0) \end{array}$$

Here $\mathfrak{r}$ is, as in the above sections, our realization functor $\mathfrak{r} : \mathrm{D}^{\mathrm{mix}}(X_0) \to \mathrm{D}^b_{T,m}(X_0)$. A natural question is: When does this diagram commute? More generally one could ask the question: If there is a functor $F : \mathrm{D}^b_{T,m}(X_0) \to \mathrm{D}^b_{T,m}(Y_0)$ (respectively $F' : \mathrm{D}^{\mathrm{mix}}(x_0) \to \mathrm{D}^{\mathrm{mix}}(Y_0)$), when does there exist a functor $F' : \mathrm{D}^{\mathrm{mix}}(x_0) \to$

$\mathrm{D}^{\mathrm{mix}}(Y_0)$ (respectively $F : \mathrm{D}^b_{T,m}(X_0) \to \mathrm{D}^b_{T,m}(Y_0)$) such that the analogous diagram commutes? In the special case that F and F' are related by either F' being induced from F or being an adjoint of such a functor, then we call the functor **genuine** if and only if the above diagram commutes. We suggest that the above more general scenario should be thought of as **commuting pairs** of functors.

To do this, we will recast the terms defined when talking about infinitesimal extensions into specific notions concerning toric varieties. Then we will see that the two notions of genuineness are the same in this scenario. Before going straight to talking about genuine morphisms, however, we need a few weaker notions.

Definition 5.0.1. Let $F : \mathrm{D}^{\mathrm{Weil}}_T(X_0) \to \mathrm{D}^{\mathrm{Weil}}_T(Y_0)$ be a functor. We say that F is **geometric** if it is a functor of triangulated categories that comes with a natural transformation

$$\underline{\mathrm{RHom}}_T(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{RHom}}_T(F(\mathcal{F}), F(\mathcal{G})), \tag{5.1}$$

and commutes with χ. That is, there exists a triangulated functor $\tilde{F} : \mathrm{D}^b_{T,c}(X) \to \mathrm{D}^b_{T,c}(Y)$ such that

$$\chi \circ F = \tilde{F} \circ \chi.$$

For a geometric functor, the natural transformation (5.1) combined with the short exact sequence (4.5) gives the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathrm{Hom}}^{i-1}_T(\mathcal{F}, \mathcal{G})_{\mathrm{Fr}} \arrow[r] \arrow[d] & \mathrm{Hom}^i(\mathcal{F}, \mathcal{G}) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}^i_T(\mathcal{F}, \mathcal{G})^{\mathrm{Fr}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{\mathrm{Hom}}^{i-1}_T(F(\mathcal{F}), F(\mathcal{G}))_{\mathrm{Fr}} \arrow[r] & \mathrm{Hom}^i(F(\mathcal{F}), F(\mathcal{G})) \arrow[r] & \underline{\mathrm{Hom}}^i_T(F(\mathcal{F}), F(\mathcal{G}))^{\mathrm{Fr}} \arrow[r] & 0
\end{tikzcd} \tag{5.2}$$

Given a toric variety X_0, we have defined the miscible category $\mathrm{D}^{\mathrm{misc}}(X_0)$, so we can say that we know what "miscible objects" are. We can define the notion of a miscible morphism. Before defining this though, we note that, in the terminology of infinitesimal extensions, v can be identified with the first map in (3.5), so we have the following:

Definition 5.0.2. A morphism $f : \mathcal{F} \to \mathcal{G}$ in $\mathrm{D}_T^{\mathrm{Weil}}(X_0)$ is **infinitesimal** if $\chi(f) = 0$.

Note that this corresponds in this case to the more general concept defined in the section on infinitesimal extensions of triangulated categories. We now define miscible morphisms:

Definition 5.0.3. A morphism $f : \mathcal{G} \to \mathcal{G}$ in $\mathrm{D}^{\mathrm{misc}}(X_0)$ is said to be **miscible** if there exists a commutative diagram

$$\begin{tikzcd}
\iota\tilde{\mathcal{F}} \arrow[r] \arrow[d, "\wr"] & \iota\tilde{\mathcal{G}} \arrow[d, "\wr"] \\
\mathcal{F} \arrow[r] & \mathcal{G}
\end{tikzcd}$$

Here $\tilde{f} : \tilde{\mathcal{F}} \to \tilde{\mathcal{G}}$ is a morphism in $\mathrm{D}^{\mathrm{mix}}(X_0)$ and the vertical arrows are isomorphisms.

It turns out that there are two different notions of distinguished triangles.

Definition 5.0.4. Let $\mathcal{F} \to \mathcal{H} \to \mathcal{G} \xrightarrow{+1}$ be a diagram in $\mathrm{D}^{\mathrm{misc}}(X_0)$.

- We say that it is a **Weil distinguished triangle** if it is a distinguished triangle in the triangulated category $\mathrm{D}_T^{\mathrm{Weil}}(X_0)$.
- We say that it is a **miscible distinguished triangle** if it is isomorphic to a diagram obtained by applying ι to a distinguished triangle in $\mathrm{D}^{\mathrm{mix}}(X_0)$.

We know that $\iota : \mathrm{D}^{\mathrm{mix}}(X_0) \to \mathrm{D}_T^{\mathrm{Weil}}(X_0)$ is a triangulated functor, so every miscible distinguished triangle is also a Weil distinguished triangle. We will define the notion of a miscible functor as follows:

Definition 5.0.5. Let $F : \mathrm{D}_T^{\mathrm{Weil}}(X_0) \to \mathrm{D}_T^{\mathrm{Weil}}(Y_0)$ be a triangulated functor. We say that F is **miscible** if $F(\mathrm{D}^{\mathrm{misc}}(X_0)) \subseteq \mathrm{D}^{\mathrm{misc}}(Y_0)$.

We now wish to "recall" a lemma that will be useful:

Lemma 5.0.6. *[AR13, Lemma 7.19] Let $F : D_T^{Weil}(X_0) \to D_T^{Weil}(Y_0)$ be a miscible functor. The following are conditions on F are equivalent:*

1. *The functor F takes every miscible morphism in $D^{misc}(X_0)$ to a miscible morphism in $D^{misc}(Y_0)$.*

2. *The functor F takes every miscible distinguished triangle in $D^{misc}(X_0)$ to a miscible distinguished triangle in $D^{misc}(Y_0)$.*

3. *the functor F restricts to a pseudotriangulated functor $F : D^{misc}(X_0) \to D^{misc}(Y_0)$.*

Proof. We will follow the proof in [AR13] quite closely. Given that F is a triangulated functor that takes miscible objects to miscible objects, it follows that F takes Weil distinguished triangles to Weil distinguished triangles. Since a Weil distinguished triangle is miscible if and only if one of its morphisms is miscible, (1) and (2) above are seen to be equivalent. We now observe that, due to Lemma 4.1.12, (5.2) implies that any miscible functor commutes with $\upsilon \circ \varpi$ (as in Definition 2.2.3). From Definition 2.2.3, we see that (2) and (3) are equivalent. $\square$

Definition 5.0.7. Let $F : \mathrm{D}_T^{\mathrm{Weil}}(X_0) \to \mathrm{D}_T^{\mathrm{Weil}}(Y_0)$ be a miscible functor. We say that F is **genuine** if there exists a triangulated functor $\tilde{F} : \mathrm{D}^{\mathrm{mix}}(X_0) \to \mathrm{D}^{\mathrm{mix}}(Y_0)$ such that the diagram

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{mix}}(X_0) & \xrightarrow{\tilde{F}} & \mathrm{D}^{\mathrm{mix}}(Y_0) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\ \mathrm{D}_T^{\mathrm{Weil}}(X_0) & \xrightarrow{F} & \mathrm{D}_T^{\mathrm{Weil}}(Y_0) \end{array}$$

commutes. We will then say that $\tilde{F}$ is **induced** by F.

For the rest of this section, we will prove that certain functors are genuine. We begin by proving a general result that we will apply many times.

Theorem 5.0.8. *Let X_0 and Y_0 be toric varieties and let $F : D_T^{Weil}(X_0) \to D_T^{Weil}(Y_0)$ be a geometric functor.*

1. *If $F(Pure(X_0)) \subseteq Pure(Y_0)$, then F is a miscible functor. In this case, F also takes miscible morphisms to miscible morphism.*

2. *If it is also true that $F|_{Pure(X_0)} : Pure(X_0) \to Pure(Y_0)$ is homogeneous, then F is genuine.*

Proof. By Lemma 5.0.6, condition (1) implies that $F|_{\mathrm{D}^{\mathrm{misc}}(X_0)} : \mathrm{D}^{\mathrm{misc}}(X_0) \to \mathrm{D}^{\mathrm{misc}}(Y_0)$ is a pseudotriangulated functor. Therefore, condition (2) follows from condition (1) by Theorem 4.1.11 and Theorem 2.3.13. We see, thus, that it is enough to prove condition (1). For this, we will show that F sends miscible objects to miscible objects and miscible morphisms to miscible morphisms at the same time.

Suppose that $f : \mathcal{F} \to \mathcal{G}$ is a miscible morphism between miscible objects. We will prove this theorem by induction on I where I is the smallest interval in $\mathbb{Z}$ so that $\mathcal{F}^i = \mathcal{G}^i = 0$ if $i \notin I$. If $|I| = 1$, then $\mathcal{F}, \mathcal{G} \in \mathrm{Pure}(X_0)[a]$ for some $a \in \mathbb{Z}$. In this case, F sends $\mathcal{F}$ and $\mathcal{G}$ to miscible objects by assumption and the condition for $F(f)$ to be a miscible morphism is obviously true.

Now, let $|I| = n > 1$. We will denote by k the largest element in I. We consider the distinguished triangles

$$\mathcal{F}^k[-k] \to \mathcal{F} \to \mathcal{F}' \xrightarrow{+1} \tag{5.3}$$

and

$$\mathcal{G}^k[-k] \to \mathcal{G} \to \mathcal{G}' \xrightarrow{+1} \tag{5.4}$$

obtained by injecting $\mathcal{F}^k[-k]$ into $\mathcal{F}$ and letting $\mathcal{F}'$ be the complex $\mathcal{F}$ except with $\mathcal{F}^k = 0$. The map $\mathcal{F} \to \mathcal{F}'$ is then the natural one. (We, of course, do the same thing for the second triangle.) It can be checked by hand that these are maps of triangles and that they are distinguished triangles. Now, we have the diagram

$$\begin{tikzcd}
\mathcal{F}^k[-k] \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, "f"] & \mathcal{F}' \\
\mathcal{G}^k[-k] \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G}'
\end{tikzcd}$$

We want a morphism between these two triangles, so we consider (5.4) and apply the functor $\mathrm{Hom}(\mathcal{F}^k[-k], -)$ to it to obtain

$$\cdots \to \mathrm{Hom}(\mathcal{F}^k[-k], \mathcal{G}^k[-k]) \to \mathrm{Hom}(\mathcal{F}^k[-k], \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}^k[-k], \mathcal{G}') \to \cdots .$$

However, it is easy to see that $\mathrm{Hom}(\mathcal{F}^k[-k], \mathcal{G}') = 0$, so the above long exact sequence tell us that the map $f : \mathcal{F} \to \mathcal{G}$ factors through some other map $f^k[-k] : \mathcal{F}^k[-k] \to \mathcal{G}^k[-k]$. Now, completing this to a morphism of distinguished triangles we arrive at some map f' giving the diagram

$$\begin{tikzcd}
\mathcal{F}^k[-k] \arrow[r] \arrow[d, "f^k[-k]"] & \mathcal{F} \arrow[r] \arrow[d, "f"] & \mathcal{F}' \arrow[r] \arrow[d, "f'"] & \mathcal{F}^k[-k+1] \arrow[d, "f^k[-k+1]"] \\
\mathcal{G}^k[-k] \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{G}^k[-k+1]
\end{tikzcd}$$

All of the objects in the rightmost square have non-zero terms only in degrees $I \setminus \{k\}$. Since F is a triangulated functor, we arrive at the diagram

$$\begin{tikzcd}
F(\mathcal{F}^k[-k]) \arrow[r] \arrow[d, "F(f^k[-k])"] & F(\mathcal{F}) \arrow[r] \arrow[d, "F(f)"] & F(\mathcal{F}') \arrow[r] \arrow[d, "F(f')"] & F(\mathcal{F}^k[-k+1]) \arrow[d, "F(f^k[-k+1])"] \\
F(\mathcal{G}^k[-k]) \arrow[r] & F(\mathcal{G}) \arrow[r] & F(\mathcal{G}') \arrow[r] & F(\mathcal{G}^k[-k+1])
\end{tikzcd}$$

By induction, we know that all of the objects and morphisms in the square

$$\begin{tikzcd}
F(\iota\mathcal{F}') \arrow[r] \arrow[d, "F(\iota f')"] & F(\iota\mathcal{F}^k[-k+1]) \arrow[d, "F(\iota f^k[-k+1])"] \\
F(\iota\mathcal{G}') \arrow[r] & F(\iota\mathcal{G}^k[-k+1])
\end{tikzcd}$$

are miscible. Since $F(\iota f) : F(\iota\mathcal{F}) \to F(\iota\mathcal{G})$ comes from completing this to a morphism of distinguished triangles, then $F(\iota f)$, $F(\iota\mathcal{F})$, and $F(\iota\mathcal{G})$ are miscible as well. $\square$

5.1 Open and Closed Inclusions

We can now proceed to deal with functors arising from open and closed inclusions.

Lemma 5.1.1. *Suppose that X_0 is a toric variety and that $j : U_0 \hookrightarrow X_0$, respectively $i : Z_0 \hookrightarrow X_0$, is an open inclusion of toric varieties, respectively a closed inclusion of toric varieties. Then the functors*

$$j^* : D^{mix}(X_0) \to D^{mix}(U_0)$$

and

$$i_* : D^{mix}(Z_0) \to D^{mix}(X_0)$$

are genuine.

Proof. For this we use the language or Orlov categories and infinitesimal extensions of triangulated categories. Recall that we now know that $\mathrm{D}^{\mathrm{misc}}(X_0) \simeq \mathscr{I}\mathrm{K}^b\mathrm{Pure}(X_0)$ for any toric variety X and that the realization functor $\mathfrak{r} : \mathrm{D}^{\mathrm{mix}}(X_0) \to \mathrm{D}^{\mathrm{Weil}}_T(X_0)$ has essential image $\mathrm{D}^{\mathrm{misc}}(X_0)$. Let us re-phrase this somewhat. We know from above that

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{Weil}}_T(X_0)}(\iota\mathcal{F}, \iota\mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}(X_0)}(\mathcal{F}, \mathcal{G}) \oplus \mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}(X_0)}(\mathcal{F}, \mathcal{G}[-1]).$$

This tells us that on $\mathrm{Pure}(X_0)$, there is an isomorphism between $\mathfrak{r}$ and ι. Since they are both homogeneous, this tells us by Theorem 2.3.12 from above that they are isomorphic. So, being genuine is the same as commuting with either $\mathfrak{r}$ or ι. It has been proved that

$$\mathfrak{r}|_{\mathrm{Pure}(X_0)} : \mathrm{Pure}(X_0) \to \mathrm{Pure}(X_0)$$

is isomorphic to the inclusion functor. We also know that

$$j^*|_{\mathrm{Pure}(X_0)} : \mathrm{Pure}(X_0) \to \mathrm{Pure}(U_0)$$

is a homogeneous functor. This follows by the fact that, by Deligne's theory of weights, $j^* = j^!$ preserves weights and since a shift of j^* is perverse t-exact. But clearly for $\mathcal{F} \in \mathrm{Pure}(X_0)$, $\mathfrak{r} \circ j^*\mathcal{F} \simeq j^* \circ \mathfrak{r}\mathcal{F}$. Therefore, j^* is genuine.

As for i_*, we also know that it is homogeneous because $i_* = i_!$, so by Deligne's theory, it preserves weights. (Note that we already knew that these functors preserved semisimple actions of Fr from earlier considerations.) Thus, i_* is genuine by exactly the same considerations. □

Next, we must consider the Verdier duality function, $\mathbb{D}$, in order to proceed further.

Theorem 5.1.2. *Let X_0 be a toric variety. Then the Verdier duality functor*

$$\mathbb{D} : D^{mix}(X_0)^{op} \to D^{mix}(X_0)$$

is genuine.

Proof. We know that $\mathbb{D}$ fixed the $\tilde{\mathcal{L}}_\sigma$. Therefore, $\mathbb{D}|_{\mathrm{Pure}(X_0)^{\mathrm{op}}}$ is homogeneous. Therefore, the claim follows as above. □

With this result in hand, we now wish to prove the following:

Theorem 5.1.3. *Let X_0 be a toric variety. Suppose that*

$$j : U_0 \hookrightarrow X_0 \hookleftarrow Z_0 : i$$

is an open-closed pair. Then the following functors are genuine:

$$j_{(!)} : D^{mix}(U_0) \to D^{mix}(X_0) \tag{5.5}$$

$$j_{(*)} : D^{mix}(U_0) \to D^{mix}(X_0) \tag{5.6}$$

$$i^{(!)} : D^{mix}(X_0) \to D^{mix}(Z_0) \tag{5.7}$$

$$i^{(*)} : D^{mix}(X_0) \to D^{mix}(Z_0) \tag{5.8}$$

$$\tag{5.9}$$

Proof. This follows directly from the fact that these are adjoints of j^* and i_*, both of which are genuine functors. □

5.2 Locally Closed Inclusions and a Final Result

Theorem 5.2.1. *Let $h : Y_0 \hookrightarrow X_0$ be a locally closed inclusion of toric varieties. Then all of the following are genuine:*

$$h_* : D^{mix}(Y_0) \to D^{mix}(X_0) \tag{5.10}$$

$$h_! : D^{mix}(Y_0) \to D^{mix}(X_0) \tag{5.11}$$

$$h^* : D^{mix}(X_0) \to D^{mix}(Y_0) \tag{5.12}$$

$$h^! : D^{mix}(X_0) \to D^{mix}(Y_0) \tag{5.13}$$

Proof. If $h : Y_0 \hookrightarrow X_0$ is a locally closed inclusion, then it can be factored into the composition of an open inclusion and a closed inclusion. However, pushforward, pullback, proper pushforward, and extraordinary pullback along both of those maps are all genuine. Therefore, these functors are genuine as well. □

We obtain one more result in this vein. Namely, we obtain genuineness results for proper toric maps.

Theorem 5.2.2. *Let $f : X_0 \to Y_0$ be a smooth toric map. Then f^* and $f^!$ are genuine. In addition, if f is proper, then f_* is a genuine functor.*

Proof. In this case, we know that a f^* preserves weights and that a shift of it is perverse t-exact. This tells us that it is homogeneous and, therefore, genuine by the reasoning above. In this case, $f^!$ is a shift of a genuine functor, so it is also genuine. Now, assume that f is proper. Then, since $f_* \simeq f_!$ has a genuine adjoint, it is also genuine. $\square$

It seems that more should be true. In particular, it seems likely that pushforward along proper toric maps are genuine. However, if that is true, then the above strategy will not work.

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Vita

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