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The Louisiana State University and Agricultural and Mechanical Col., 1988

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ON PRIME POWERS AND SYMBOLIC PRIME POWERS

A Dissertation

**Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy**

in

The Department of Mathematics

by

Patrick H. Phenix

B.A., University of Pennsylvania, 1975

M.A., Temple University, 1979

August, 1988

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to see this dream fulfilled.

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Abstract

We study the conditions under which the power of a prime ideal is equal to the corresponding symbolic prime power.

We begin by extending a result of Villamayor(h). We consider a smooth k -algebra S (S') which is the localization of a finite k -algebra where k is a field of characteristic zero. For a prime ideal P (P') we show that if $\frac{S}{P} \cong \frac{S'}{P'}$ then $P^n = P^{(n)}$ if and only if $P'^{(n)} = P'^n$, for $n \geq 1$. In the proof we use a generalization of the notion of a truncated cotangent complex introduced by Illusie.

We then continue on by using the notions developed in the course of the proof to construct a new class of cohomological objects $\mathcal{Q}^{n,i}$ which play an analogous role for the higher order differentials to the role played by the cotangent complex of Lichtenbaum and Schlessinger in the case of the ordinary Kahler differentials.

Chapter 1 : Questions on Symbolic Prime Powers

The purpose of this work is to expand the results of Villamayor [V] related to a question originally posed by Hochster [Ho] on the conditions of the equality of powers of prime ideals and symbolic powers of prime ideals.

In this chapter we will begin by discussing Hochster's question and the results obtained for some special classes of rings. Then we will discuss the approach of Villamayor and present a short review of some homological constructions needed to extend these results. Finally, we will present two general theorems which we will need in the following chapters.

Section 1.1: The Basic Question and its Antecedents

We will begin with the definition of a symbolic power of a prime ideal. All of our rings will be commutative with identity. Let P be a prime ideal in a ring A . Consider the localization A_P and the ideal of A generated by the image of P^n , $P^n A_P$. The n th symbolic prime power, $P^{(n)} = P^n A_P \cap A$, is the inverse image of $P^n A_P$. If A has a primary decomposition (e.g. if A is noetherian, then we notice that $P^{(n)}$ is the P -primary component of P^n . While it is always true that $P^n \subset P^{(n)}$, it may happen that this containment is strict. Examples where the prime power and the symbolic prime are not equal are provided by Northcott [N, page 29, example 3] and Matsumura [M1, page 56].

When P^n is primary then we always have $P^n = P^{(n)}$ and Hochster [Ho, page 63] sought more general criteria for this equality to hold. He found that when the prime ideal P is a complete intersection (P is generated by an A -sequence) then $P^n = P^{(n)}$ for all n , and when P is the prime ideal generated by the $k-1$ by $k-1$ minors of a $k-1$ by k matrix of indeterminates over a field

K , then the equality also holds. Hochster then asked if some restriction on the quotient ring $\hat{A}_{\mathfrak{p}}$ might be able to guarantee the desired equality. He was able to rule out $\hat{A}_{\mathfrak{p}}$ being Cohen-Macaulay as a sufficient condition, but conjectured Gorenstein as a possibility. In a later paper, Cowsik [Co] was able to produce a counterexample to Gorenstein, but extended Hochster's original results to locally complete intersections. (For more on complete intersections, see Looijenga [Lo, section 1.B].) Eliahou was able to use the information on complete intersections to examine cases involving monomial curves in [E].

Then Villamayor slightly rephrased the question. Considering the equality as a condition of the quotient ring $\hat{A}_{\mathfrak{p}}$, Villamayor showed that for polynomial rings A and A' over a field k , if P (resp. Q) was a prime ideal of A (resp. A') such that $\hat{A}_{\mathfrak{p}} \cong \hat{A}'_{\mathfrak{q}}$ then $P^{\mathfrak{a}} = P^{(\mathfrak{a})}$ if and only if $Q^{\mathfrak{a}} = Q^{(\mathfrak{a})}$. The equality for $n=2$ was thus independent of the representation as a quotient ring. One of the results we will demonstrate (in chapter 2) is that this result may be extended to the general case of $P^{\mathfrak{a}} = P^{(\mathfrak{a})}$.

Another question related to all of this was raised by Hartshorne in [Ha, section 7]. Given a complete regular local ring A with P a prime ideal of A , when is the P -adic topology on an A -module M equivalent to the P -symbolic topology on M ? This question was answered by Schenzel [Sch] using some results of Brodmann [Br] and has been extended to some classes of primary ideals by Ratliff [Ra] and to the case of linear equivalence of general ideals by Verma [Ve1] and [Ve2].

Section 1.2: Some Basic Definitions

We begin by recalling some of the basic ideas of derivations. For a ring A and an A -module M , a derivation $D:A \rightarrow M$ is an additive map satisfying

$D(ab) = aD(b) + bD(a)$ for all a, b in A . We often consider the case where A is a k -algebra and the derivations are required to vanish on k . If we consider $B = A \otimes_k A$ and the exact sequence

$$0 \rightarrow I \rightarrow B \xrightarrow{\varepsilon} A \rightarrow 0$$

where $\varepsilon(a \otimes a') = aa'$, then the A -module $\mathcal{I}_A^1 = \Omega_{A/k}$ is called the module of (Kähler) differentials of A over k . We get the usual derivation $d: A \rightarrow \Omega_{A/k}$ and the pair $(\Omega_{A/k}, d)$ satisfies the universal property that if $D: A \rightarrow M$ is any derivation into an A -module M then there is a unique A -linear map $f: \Omega_{A/k} \rightarrow M$ such that $D = fd$. For further details, see Matsumura [M1, section 26].

This approach extends naturally to higher degrees. Again considering a k -algebra A and an A -module M , an n -derivation over k from A to M is a k -linear function L from A to M such that

- (i) $L_n(\alpha_1 \cdots \alpha_n) = \sum_{i=1}^n (-1)^{i-1} \left[\sum_{j, \ell < j_i} \alpha_j \cdots \alpha_\ell \right] L_n(\alpha_1 \cdots \hat{\alpha}_i \cdots \hat{\alpha}_j \cdots \alpha_n)$
(ii) $L_n(1) = 0$. See [V, section 1.1]

If we recall the map $\varepsilon: B \rightarrow A, \varepsilon(a \otimes a') = aa'$, $I = \ker \varepsilon$, then using the map $a \otimes 1 \otimes a^{-1}$, we have the composition $A \rightarrow I \rightarrow \mathcal{I}_A^{n+1}$

and we denote $\Omega_{A/k}^n$ (or simply \mathcal{Q}_A^n , where k is understood), which may be called the module of differentials of order n . At this point the reader is warned that, in the usual terminology, [M1, section 26, D] $\Omega_{A/k}^n$ is the n th exterior product of $\Omega_{A/k}^1$, and this is often found in books on Algebraic Geometry and Differential Geometry. Our meaning is different and throughout the remainder of this paper we will have no occasion to refer to the n th exterior product.

$\Omega_{A/k}^n$ is easily seen to be an A -module and its properties are fully discussed in Mount and Villamayor [MV] where it is denoted $D_{A/k}^n$. We will recall

the properties we need, with references to [MV] where necessary.

Ω_A^n is a ring without identity and any product of $n+1$ elements in it vanishes. The composition in (1.2.1) is denoted T^n and is a higher derivation in the sense of (i) and (ii) above. We also note that when A is a regular local ring, $\text{char}(k)=0$ and x_1, \dots, x_r is a system of parameters for A , then for any $f \in A$

we have

$$T^n f = \sum_{|j| \leq n} \frac{1}{j_1! \dots j_r!} \frac{\partial^{|j|} f}{\partial x_1^{j_1} \dots \partial x_r^{j_r}} (T^n x_1)^{j_1} \dots (T^n x_r)^{j_r} \quad (1.2.2)$$

where $j = (j_1, \dots, j_r)$, $|j| = j_1 + \dots + j_r$ and $\frac{\partial}{\partial x_j} = \delta_{ij}$, the Kronecker delta.

In the same way that Ω_A is a universal object for derivations, Ω_A^n is a universal object for n -derivations [MV, page 17, 3.3]. If S is a multiplicatively closed subset of A then $\Omega_{S^{-1}A}^n \cong S^{-1}A \otimes \Omega_A^n$

[MV, page 20, 3.4]. For computational purposes we have the following extremely useful result

$$T^n(x_1^{a_1} \dots x_j^{a_j}) = (x_1 + T^n x_1)^{a_1} \dots (x_j + T^n x_j)^{a_j} - x_1^{a_1} \dots x_j^{a_j}$$

[MV, page 16, 2.5]. Finally, if $A = k[x_1, \dots, x_n]$ is a finitely generated k -algebra and P is a prime ideal in A , then Ω_A^n is a finitely generated A -algebra with algebra generators $T^n x_1, \dots, T^n x_n$ [MV, page 21, 3.5]. In the case where A is a regular local ring with a system of parameters x_1, \dots, x_j then Ω_A^n is a free A -module.

These results are all extensions of the case where $n=1$ to the general case. It is worth pointing out that these ideas are related to the notion of "principal parts" described by Grothendieck in [EGA, chapter 4].

We will also have occasion to use the cotangent complex and the

first derived upper cotangent functor $T'(A, M)$. This was introduced by Lichtenbaum and Schlessinger [LS] and its construction is also described by Hoffman [H] and in more generality by Illusie [Ill]. Later this was extended to higher orders independently by Andre [A1] and [A2] and Quillen [Q1].

Since we only use T' we will refer to it as the cotangent functor without any resulting confusion. Here we need only recall that given the exact sequence where S and A are k -algebras, and φ is a homomorphism of k -algebras:

$$0 \rightarrow P \rightarrow S \xrightarrow{\varphi} A \rightarrow 0$$

we have an exact sequence of A -modules

$$P/P^2 \rightarrow \Omega_S \otimes_S A \rightarrow \Omega_A \rightarrow 0 \quad (1.2.3)$$

[M1, page 187, theorem 58] and, for an A -module M , $T'(A, M)$ is defined in such a way that, applying $\text{Hom}_A(_, M)$ to (1.2.3) the sequence of A -modules

$$0 \rightarrow \text{Hom}_A(\Omega_A, M) \rightarrow \text{Hom}_S(\Omega_S, M) \rightarrow \text{Hom}_A(P/P^2, M) \rightarrow T'(A, M) \rightarrow 0$$

is exact.

Later we will introduce a modification due to Iversen [I], called the truncated cotangent complex, which will play a key role in our extension of Villamayor's result.

Section 1.3: Preliminary Results

We will consider, in this section, a field k of characteristic zero, a k -algebra S of finitely generated type. This means that S is the localization of a k -algebra of finite type. P will be a prime ideal of S .

Lemma 1: Let S be a regular k -algebra of finitely generated type and P a prime ideal of S . Then

$$P^{(n+1)} = \{ F \in P : D_1 \dots D_n F \in P; \text{ where } D_j \text{ is a } k\text{-derivation} \\ S \rightarrow S, 0 \leq l \leq n \}$$

(For $l=0$, the resulting composition is the identity operator.)

Now, letting $R = S/P$, we have an exact sequence

$$0 \rightarrow P \rightarrow S \rightarrow R \rightarrow 0 \quad (1.3.1)$$

Associated to (1.3.1) there is always an exact sequence [M1, page 187]

$$P/P^2 \xrightarrow{\alpha} \Omega_S \otimes_S R \rightarrow \Omega_R \rightarrow 0 \quad (1.3.2)$$

We want to prove

Theorem 1: $P^2 = P^{(2)}$ if and only if α is injective.

To prove this we start with

$$\text{Lemma 2: } 0 \rightarrow P/P^2 \xrightarrow{\alpha'} \Omega_S \otimes_S R \rightarrow \Omega_R \rightarrow 0$$

is always an exact sequence of R -modules.

Proof of Lemma 2: It is sufficient to show that α' is injective. α' is the map induced from $P \xrightarrow{\alpha} \Omega_S \otimes_S R$ which arises from

$$P \hookrightarrow S \xrightarrow{d} \Omega_S \longrightarrow \Omega_S \otimes_S R$$

namely, $\tilde{\alpha}(g) = dg \otimes 1$. Since $R = S/P$, $\Omega_S \otimes_S R = \Omega_S / P\Omega_S$. So $\tilde{\alpha}(g) = \text{image of } dg \text{ in } \Omega_S / P\Omega_S$. Recalling that Ω_S is free on the basis dT_1, \dots, dT_r [M1, page 184] then $dg = g_{T_1} dT_1 + \dots + g_{T_r} dT_r$ where g_{T_i} is the corresponding derivation, i.e. $\frac{\partial}{\partial T_i}$.

First we show that α' is well-defined. If $g \in P^{(2)}$ then the result of Seibt implies that $g_{T_i} \in P$, for all i , hence $dg = \sum g_{T_i} dT_i \in P\Omega_S$, so α' is well-defined. Now assume, for $\bar{g} \in P/P^2$, $\alpha'(g) = 0$ in $\Omega_S / P\Omega_S$, where g is a representative for \bar{g} . This implies $\tilde{\alpha}(g) \in P\Omega_S$ so $g_{T_i} = b_i$ where $b_i \in P$ for all i , hence by Seibt, $g \in P^{(2)}$.

This yields Lemma 2.

Now to prove theorem 1. We know that $P^{\mathfrak{a}} \subset P^{(\mathfrak{a})}$ is always true, hence $P/P^{\mathfrak{a}} \rightarrow P/P^{(\mathfrak{a})}$. Thus the following diagram commutes

$$\begin{array}{ccccccc}
 P/P^{\mathfrak{a}} & \longrightarrow & \Omega_S \otimes_S R & \longrightarrow & \Omega_R & \longrightarrow & 0 \\
 \beta \downarrow & & \nearrow & & & & \\
 0 & \longrightarrow & P/P^{(\mathfrak{a})} & & & &
 \end{array} \quad (1.3.3)$$

where the bottom row is exact by Lemma 2. An easy diagram chase shows that the top row is exact if and only if β is an isomorphism if and only if $P^{\mathfrak{a}} = P^{(\mathfrak{a})}$. This proves Theorem 1.

Now we extend theorem one to the more general case. With the same conditions on S , P and R , we want to construct a map

$$\alpha^n: P/P^{n+1} \longrightarrow \Omega_S^n \otimes_S R$$

We begin with the map, $\alpha^{(n)}: P \rightarrow \Omega_S^n \otimes_S R$ given by $T^n(\rho) \otimes 1$. This is clearly k -linear and we use this to construct α^n . Let $x_1, \dots, x_{n+1} \in P^{n+1}$ where $x_i \in P$ for each i . By our observation in section 1.2 we may write

$$\begin{aligned}
 T^n(x_1, \dots, x_{n+1}) &= (x_1 + T^n x_1) \dots (x_{n+1} + T^n x_{n+1}) - x_1 \dots x_{n+1} \\
 &= [A] + T^n x_1 \dots T^n x_{n+1}
 \end{aligned}$$

where $[A]$ is a sum of terms, each of which contains at least one factor of the form x_j and at least one of the form $T^n x_k$. Each such term in $[A]$ is in $P\Omega_S^n$ and recalling $\Omega_S^n \cong P\Omega_S^n$, we see that each term in $[A]$ vanishes. The final term on the right in (1.3.4) is a product of $n+1$ terms in Ω_S^n and thus is zero by definition. Therefore the induced map

$$\alpha^n : P/P^{(n+1)} \longrightarrow \Omega_S^n \otimes_S R$$

is well-defined.

We also have the map $\alpha^{(n)} : P/P^{(n+1)} \rightarrow \Omega_S^n \otimes_S R$ given by

$$\alpha^{(n)}(\bar{p}) = \sum_{|j| \leq n} \frac{\partial^{|j|} \bar{p}}{\partial x_1^{j_1} \dots \partial x_r^{j_r}} (T_{x_1}^{j_1})^{j_1} \dots (T_{x_r}^{j_r})^{j_r} \in \Omega_S^n / P \Omega_S^n$$

from (1.2.2). It is easy to check, again using Seibt's result, that $\alpha^{(n)}$ is injective.

As in the $n=2$ case, we construct the commutative diagram

$$\begin{array}{ccccccc} P/P^{(n+1)} & \longrightarrow & \Omega_S^n \otimes_S R & \longrightarrow & \Omega_R^n & \longrightarrow & 0 \\ \pi \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & P/P^{(n+1)} & \longrightarrow & \Omega_R^n & \longrightarrow & 0 \end{array}$$

where π , induced from the inclusion $P^{(n+1)} \subset P^{(n)}$ is surjective. Once again, an easy diagram chase shows

Theorem 2: $P^{(n+1)} = P^{(n)}$ if and only if π is injective if and only if α^n is injective.

Section 1.4: The Rest of the Paper

In Chapter 2 we will extend the result of Villamayor mentioned on section 1.2 to the case $P^{(n+1)} = P^{(n)}$, and show that this remains a condition that is independent of the representation of R as a quotient ring, $R = S/\mathfrak{p}$, where S is as in section 1.3. We will then construct intrinsic objects $\mathcal{U}^{n,i}(R,M)$ which will serve as higher order analogues of the cotangent functor $T^1(R,M)$. This will require a generalization of the notion of truncated cotangent complex described by Iversen [I], and the appropriate theory will be developed there.

Then in Chapter 3 we will study the properties of the $\mathcal{U}^{n,i}(R,M)$ and use them to prove a more comprehensive version of the result we obtain in

Chapter 2.

Finally, in Chapter 4, we will discuss the restrictions we have placed on k , R , S , and P and we will outline some areas of possible future study.

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Chapter 2 : $P^{(n)} = P^{(n)}$ An Extension of Villamayor's Theorem

In this chapter we will extend the result of Villamayor mentioned in Chapter 1 to the general case $P^{(n)} = P^{(n)}$. This will require us to review first some notions on smooth algebras and infinitesimal extensions and to further these results to a more general setting. Some of this will be standard commutative algebra and some will depend on work done by Iversen [I] on truncated cotangent complexes.

After this introduction and the generalization of the result of Villamayor, we will construct a higher order analogue to the cotangent complex, our set of homological $\mathcal{U}^{n,i}$.

Section 2.1: Remarks on Formal Smoothness

For the remainder of this section we will consider k to be any commutative ring with identity. In some later uses of our work we will need more restrictive conditions, about which we will warn the reader at the appropriate time.

We begin with a concept that is well known. A k -algebra A is formally smooth if, given a solid arrow commutative diagram of k -algebras, where $I^{\wedge}(0)$

$$\begin{array}{ccc}
 A & \longrightarrow & C/I \\
 \uparrow & \dashrightarrow & \uparrow \\
 k & \longrightarrow & C
 \end{array}$$

then the dotted arrow exists and the relevant triangles commute. The properties of formal smoothness are developed in Matsumura [M1, section 28]. (Warning: what we call "formally smooth" Matsumura calls only "smooth". Thus our definition will correspond more precisely to that used by Iversen.) We also notice that, in this definition, we may replace " $I^\wedge = (0)$ " with " $I^n = (0)$ " for any $n > 2$ and the definition remains the same [M1, page 199].

We now turn to an extension of an idea developed by Iversen [I, section II.2]. If A is a k -algebra, then an n th order infinitesimal extension of A is a pair $E := (E, f)$ where E is a k -algebra and $f: E \rightarrow A$ is a surjective k -algebra morphism, and for $I = \ker f$ we have $I^{n+1} = (0)$. (With this terminology, the constructions of Iversen are first order extensions.) Further, an n th order infinitesimal extension will be considered versal if, for any other n th order extension $F := (F, g)$ of A , there will exist a k -algebra homomorphism $\varphi: E \rightarrow F$ so that the obvious diagram commutes.

Proposition 1: Let A be a k -algebra, E a formally smooth k -algebra and $f: E \rightarrow A$ a surjective such that $I = \ker f$. Consider the inclusions $I^{n+1} \subset I \subset E$ which produce the exact sequence

$$0 \rightarrow I/I^{n+1} \rightarrow E/I^{n+1} \xrightarrow{f} E/I \rightarrow 0$$

Then $(E/I^{n+1}, p)$ is a versal n th order infinitesimal extension of A .

Proof: Consider another n th order infinitesimal extension of A represented by the exact sequence

$$0 \rightarrow L \rightarrow F \rightarrow A \rightarrow 0$$

where $L^{n+1} = (0)$. Looking at the solid arrow diagram

$$\begin{array}{ccc} E & \longrightarrow & A \\ \uparrow & \searrow \psi & \uparrow \\ * & \longrightarrow & F \end{array}$$

and recalling the formal smoothness of E , we get a map ψ which completes the diagram. Thus we have

$$\begin{array}{ccccccc} 0 & \rightarrow & I & \rightarrow & E & \rightarrow & A \rightarrow 0 \\ & & \downarrow \psi & & \downarrow \psi & & \parallel \\ 0 & \rightarrow & L & \rightarrow & F & \rightarrow & A \rightarrow 0 \end{array}$$

By the universal property of the kernel, ψ exists and $\psi(I^{n+1}) \subset L^{n+1} = (0)$ implies that ψ factors through

$$\begin{array}{ccc} E/I^{n+1} & \longrightarrow & A \\ \downarrow & & \nearrow \\ F & \longrightarrow & A \end{array}$$

Since for any ring k , the polynomial ring $k[X_1, X_2, \dots, X_n, \dots]$ is formally smooth over k , then we have immediately

Corollary 1: Any k -algebra A has a versal n th order infinitesimal extension.

Now we turn our attention to the truncated cotangent complex. Given an n th order infinitesimal extension $f: E \rightarrow A$ of the k -algebra A , we consider the following complex of k -modules

$$0 \longrightarrow I \xrightarrow{d_0} \mathcal{S}_E^n \otimes_E A \longrightarrow 0$$

where $d_0(x) = T^n x \otimes_E 1$, T^n the universal n -derivation described in Chapter 1 and in [MV]. We remind the reader that this T^n is often referred to as a truncated Taylor series, where a Taylor series T possesses the property $T(xy) = xT(y) + yT(x) + T(x)T(y)$. This is the usual statement in the case of a derivation since the last term on the right $T(x)T(y) \in I^n$ and hence $= 0$ in \mathcal{S}_E .

Lemma 1: Consider the following commutative diagram of k -algebras

$$\begin{array}{ccc}
 E & & A \\
 \alpha \downarrow & \searrow \rho & \\
 F & & A \\
 N \hookrightarrow & \nearrow \varphi & \\
 & &
 \end{array}$$

where $N = \ker \varphi$. Let $T = \beta - \alpha$. Then (a) $\text{im } T \subset N$, and (b) still denoting by $T: E \rightarrow N$ the induced map, then T is a Taylor series if we regard N as an E -module "via α ". Note: N is naturally a ring without identity and also an E -algebra via $e \cdot n = \alpha(e) \cdot n$.

Proof: (We mimic [M1, page 188].) $\text{Im } T \subset N$ follows from $\varphi \alpha = \varphi \beta$ i.e. the fact that the diagram commutes. Letting $T = \beta - \alpha$, then $\beta = \alpha + T$ and, for any x, y in E we have

$$\begin{aligned}
 \beta(xy) &= \beta(x)\beta(y) = [\alpha(x) + T(x)][\alpha(y) + T(y)] \\
 &= \alpha(x)\alpha(y) + \alpha(x)T(y) + \alpha(y)T(x) + T(x)T(y) \\
 &= \alpha(xy) + xT(y) + yT(x) + T(x)T(y)
 \end{aligned}$$

so we find $T(xy) = \beta(xy) - \alpha(xy) = xT(y) + yT(x) + T(x)T(y)$

and so T is an N -valued Taylor series.

Proposition 2: (i) Consider a morphism $\alpha(E,p) \rightarrow (F,q)$ of n th order infinitesimal extensions of the k -algebra A . Then there is an induced morphism $c.(\alpha)$ of the complexes (2.1.1) constructed above. (ii) If β is another such morphism, then $c.(\alpha)$ and $c.(\beta)$ are homotopic as morphisms of complexes.

Proof: (i) This follows from the functorial properties of the module of derivations of degree $-n$, Ω_E^n . (ii) Consider $T = \beta - \alpha$ as a map from E to $L = \ker q$ which by the above lemma is an E -Taylor series and where L is regarded as an E -module as via α . By the universal property of Ω_E^n there exists a unique E -linear map such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{T^n} & \Omega_E^n \\ & \searrow T & \swarrow \delta \\ & & L \end{array}$$

Then for $K = \ker p$

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & \Omega_C^n \otimes_E A & \longrightarrow & 0 \\ & & \alpha \downarrow & & \delta \swarrow & & \downarrow \beta' \\ & & L & \longrightarrow & \Omega_F^n \otimes_F A & \longrightarrow & 0 \end{array}$$

is the required homotopy.

We now conclude this section by noting that if (E,p) is a versal n th order infinitesimal extension of A , then given any extension (F,q) there exists an $\alpha: (E,p) \rightarrow (F,q)$ and hence a map $c.(\alpha)$ of the associated complex as before. This map is unique up to homotopy, so by Proposition 2, any two such complexes (2.1.1) corresponding to a versal n th order infinitesimal extension are

homotopically equivalent.

Definition: The n th order truncated cotangent complex of A is the homotopically unique truncated cotangent complex arising from any versal n th order infinitesimal extension of the k -algebra A .

Section 2.2: $P^{n+1} = P^{(n+1)}$

In this section we will extend the result of Villamayor to arbitrary powers of a prime ideal. To be consistent with the requirements of Seibt's theorem, which we use, we will consider k as before and A a k -algebra of finitely-generated type. This means that $A = C^{-1}B$ where B is a k -algebra of finite type and C is any multiplicative subset of A . In particular B is the quotient of a polynomial ring over k , and we may include the case $C=1$ so that A is possibly just a polynomial ring over k in finitely many indeterminates. We emphasize that, unlike earlier results, we are not restricted to localization at a prime or maximal ideal.

Proposition 3: Let S be a k -algebra of finitely generated type and P a prime ideal of S such that $R = S/P$ and $E = S/P^{n+1}$. Then there is a canonical isomorphism of R -algebras

$$\Omega_S^n \otimes_S R \cong \Omega_E^n \otimes_E R$$

(Note R is an E -algebra via the surjective homomorphism $S/P^{n+1} \rightarrow S/P$ induced by the inclusions $P^{n+1} \subset P \subset S$)

Proof: We begin, as always, with an exact sequence

$$0 \rightarrow P^{n+1} \rightarrow S \rightarrow S/P^{n+1} \rightarrow 0$$

which induces the exact sequence

$$0 \rightarrow P/P^{n+1} \rightarrow S/P^{n+1} \rightarrow S/P \rightarrow 0$$

where we recognize $E = S/P^{n+1}$, $R = S/P$. From the second sequence we have the canonical sequence [MV] or [M1, page 186 for $n=0$]

$$P^{n+1}/P^{2n+2} \rightarrow \Omega_S^n \otimes_S S/P^{n+1} \rightarrow \Omega_{S/P^{n+1}}^n \rightarrow 0$$

Now apply $\otimes_{S/P^{n+1}} S/P$ to this to obtain the commutative diagram with exact top row

$$\begin{array}{ccccc}
 P^{n+1}/P^{2n+2} \otimes_{S/P^{n+1}} S/P & \longrightarrow & \Omega_S^n \otimes_S S/P^{n+1} \otimes_{S/P^{n+1}} S/P & \longrightarrow & \Omega_{S/P^{n+1}}^n \otimes_{S/P^{n+1}} S/P \\
 & & \parallel & & \parallel \\
 & & \Omega_S^n \otimes_S S/P & & \Omega_E^n \otimes_E R \\
 \alpha \searrow & & \parallel & \nearrow \gamma & \\
 & & \Omega_S^n/P \Omega_S^n & & \\
 & & \parallel & & \\
 & & \Omega_S^n \otimes_S R & &
 \end{array}$$

where α is defined by $T^n x \otimes \bar{p}$ and $x \in P^{n+1}$ implies $x = x_1 x_2 x_3 \dots x_{n+1}$ with $x_i \in P$ for all i . Then we have

$$T_S^n(x) = T_S^n(x_1 \dots x_{n+1}) = (x_1 + T_S^n x_1) \dots (x_{n+1} + T_S^n x_{n+1}) - x_1 \dots x_{n+1}$$

from our computational formula in section 1.2. The term on the right, when expanded, yields

$$\text{terms of the form } x_1 \dots x_n T_S^n x_{n+1} + (T_S^n x_1) \dots (T_S^n x_{n+1})$$

Each term of the first part is in $P\Omega_S^n$ and is thus zero and the final term is zero in Ω_S^n as a product of $n+1$ terms. Thus, $\alpha(\alpha\bar{p})=0$, so $\alpha=0$ and γ is an isomorphism.

$$\text{Hence } \Omega_S^n \otimes_S R \cong \Omega_E^n \otimes_E R$$

Remark: If R is a k -algebra, to obtain an n -truncated cotangent complex we may take any formally smooth k -algebra S of which R is a quotient, say $R = S/P$, P an ideal of S , then let $E = S/P^{n+1}$, $I = P/P^{n+1}$ and consider

$$0 \rightarrow I \rightarrow \Omega_E^n \otimes_E R \rightarrow 0 \quad (\mathcal{C}.)$$

a complex with degree one term $=I$ and degree zero term $=\Omega_E^n \otimes_E R$. As we saw in Proposition 3, this is canonically isomorphic to the complex of k -modules

$$0 \rightarrow I \rightarrow \Omega_S^n \otimes_S R \rightarrow 0 \quad (\mathcal{C}.)$$

Proposition 4: Let S and S' be smooth k -algebras, and let P and P' be prime ideals of S and S' respectively, such that $S/P \cong S'/P' = R$. Then the k -linear map

$$\alpha^n: P/P^{n+1} \rightarrow \Omega_S^n \otimes_S R$$

is injective if and only if the canonical k -linear map

$$\alpha'^n: P'/P'^{n+1} \rightarrow \Omega_{S'}^n \otimes_{S'} R$$

is injective.

Proof: The injectivity of α^n is equivalent to having $\alpha(\mathcal{C}_n) = 0$ or, equivalently

, $H_1(\bar{C}) = 0$ because C and \bar{C} are isomorphic. Similarly, α' is injective if and only if $H_1(\bar{C}') = 0$, where we construct \bar{C}' in the same manner as \bar{C} , using S' and P' in a strictly analogous construction.

Now in view of the remarks preceding the definition of the n th truncated cotangent complex in section 2.1 (concerning "homotopy uniqueness"), the versality of S and S' imply that \bar{C} and \bar{C}' , both representing n -cotangent complexes are homotopy equivalent, hence indeed $H_1(\bar{C}) \cong H_1(\bar{C}')$ and one vanishes if and only if the other vanishes.

As an immediate corollary we may generalize Villamayor's result.

Theorem 1: Let S (resp S') be smooth k -algebras of finitely generated type, and let P (resp P') be a prime ideal of S (resp S') such that $\frac{S}{P} \cong \frac{S'}{P'}$. Then $P^{(n)} = P'^{(n)}$ if and only if $P^{(n+1)} = P'^{(n+1)}$.

Proof: We have seen that $P^{(n)} = P'^{(n)}$ if and only if α^n is injective (Chapter 1, theorem 2) and α^n is injective if and only if α'^n is injective if and only if $P^{(n+1)} = P'^{(n+1)}$.

At this point it may be worthwhile to point out specific instances where the theorem holds. Because of a dependence on Seibt's results we restrict ourselves to the case where k is a field of characteristic zero.

If A is an algebra of finite type over k and P is a prime ideal in A then for $S = A_P$ where S is regular and $k \subset K = \frac{S}{\mathfrak{m}_P(S)}$ is a separable extension, then using Matsumura [M1, Theorem 64 and page 207], it follows that S is formally smooth over k .

Section 2.3: $\mathcal{U}^{n,i}(R, M)$

We begin this section by reminding the reader that $T'(R, M)$ can be considered to be the term necessary to complete an exact sequence. That is, given the sequence, where S is a smooth k -algebra

$$P/P^2 \rightarrow \Omega_S \otimes_S R \rightarrow \Omega_R \rightarrow 0 \quad (1.2.3)$$

then $T'(R, M)$ is (non-intrinsically) defined as the term used to complete the exact sequence

$$0 \rightarrow \text{Hom}_R(\Omega_R, M) \rightarrow \text{Hom}_S(\Omega_S, M) \rightarrow \text{Hom}_R(P/P^2, M) \rightarrow T'(R, M) \rightarrow 0$$

By non-intrinsic definition, we mean that it is not immediately clear that T' will be well-defined independently of the choice of representation of R as a quotient $R = S/P$.

In this definition we can write

$$T'(R, M) = \text{coker} \left[\text{Hom}_R(\Omega_S \otimes_S R, M) \rightarrow \text{Hom}_R(P/P^2, M) \right]$$

Now that we have extended ourselves to consider P/P^{n+1} for $n > 1$, we would like to have an object similar to T' . Unfortunately, P/P^{n+1} for $n > 1$ is no longer an R -module so we are no longer dealing simply with R -modules and R -module homomorphisms. In order to return to the setting of R -modules, we are going to "filter" our modules in such a way that every term of the associated graded object is again an R -module.

For the remainder of this section we will work with the case where k is a field of characteristic zero and S is the localization of a finitely generated k -algebra at a prime ideal, and we will assume that S is regular.

We consider our ring S , a prime ideal P , and $R = S/P$, $E = S/P^{n+1}$.

We have an exact sequence

$$0 \rightarrow P/P^{n+1} \rightarrow S/P^{n+1} \rightarrow S/P \rightarrow 0$$

and using proposition 1 of section 2.1 and the fact [M1, page 220] that regular implies formally smooth, then we see

$$S/P^{n+1} \longrightarrow S/P$$

is a versal n th order infinitesimal extension of R . This gives rise to the complex

(2.1.1)

$$0 \longrightarrow P/P^{n+1} \longrightarrow \Omega_E^n \otimes_E R \longrightarrow 0$$

where the degree zero term is $\Omega_E^n \otimes_E R$ and the degree one term is P/P^{n+1} and the remaining terms are zero.

We introduce the filtrations

$$P/P^{n+1} = G_1^n \supset G_2^n \supset \dots \supset G_{n+1}^n = 0 \quad \text{where } G_i^n = P^i/P^{n+1}$$

and $L_i^n =$ the submodule of $\Omega_E^n \otimes_E R$ generated by terms of the form $\sum_j T_j^i \otimes 1$ where $j \geq i$.

Now consider the induced complexes

$$0 \rightarrow G_i^n/G_{i+1}^n \longrightarrow L_i^n/L_{i+1}^n \rightarrow 0$$

This is clearly a complex of $R = S/P$ -modules for each value of i .

We now proceed with the construction of the $U^{n,i}$. For $i < n$ we begin with

$$0 \rightarrow G_i^n/G_{i+1}^n \longrightarrow L_i^n/L_{i+1}^n \rightarrow 0$$

and apply $\text{Hom}_R(\quad, M)$ to obtain

$$0 \rightarrow \text{Hom}_R(L_i^n/L_{i+1}^n, M) \rightarrow \text{Hom}_R(G_i^n/G_{i+1}^n, M) \rightarrow U^{n,i}(R, M) \rightarrow 0$$

where the $U^{n,i}$ are defined to complete the Hom sequence to exactness; i.e.

$$U^{n,i}(R, M) = \text{coker} [\text{Hom}_R(L_i^n/L_{i+1}^n, M) \rightarrow \text{Hom}_R(G_i^n/G_{i+1}^n, M)]$$

With this definition it is easy to see that the $U^{n,i}$ are well-defined, that is to say intrinsic objects independent of the representation $R = S/\rho$. In fact, given $R = S'/\rho'$ (where S' satisfies the same conditions as S) then the versality of S'/ρ^{n+1} (and also S'/ρ'^{n+1}) leads us, by Proposition 2 (ii) to see that the corresponding n -cotangent complexes are unique up to homotopy equivalence. We note, however, that the homotopy equivalence γ in the proof of proposition 2 induces a homotopy of $L_{S'/\rho'}^n/L_{S'/\rho'^{n+1}}^n$. We will consider this further in section 2.4. It remains to be seen how these objects $U^{n,i}$ will function in a role analogous to the T^i we considered before.

Section 2.4: An Alternate Construction for the

We recall that T^i arose from the consideration of the standard sequence

$$P/\rho^2 \rightarrow \Omega_S \otimes_S R \rightarrow \Omega_R \rightarrow 0$$

where T^i was defined to ensure exactness in

$$0 \rightarrow \text{Hom}_R(\Omega_R, M) \rightarrow \text{Hom}_S(\Omega_S, M) \rightarrow \text{Hom}_R(P/\rho^2, M) \rightarrow T^i(R, M) \rightarrow 0$$

a sequence involving the prime ideal P . To return explicitly to a situation involving P and its powers, we present a second definition of the $U^{n,i}$ and we will show the equivalence with the objects constructed in section 2.3.

Again we will deal with the same conditions on S , R , and k that we described at the beginning of 2.3. Let us start with the filtration

$$G_i^n = P^i/\rho^{n+1} \quad \text{from 2.3}$$

For $j < n+1$ we have $P^{n+1} \subset P^{j+1} \subset P^j$, hence we have an exact sequence

$$0 \rightarrow P^{j+1}/\rho^{n+1} \rightarrow P^j/\rho^{n+1} \rightarrow P^j/P^{j+1} \rightarrow 0$$

$$\text{i.e. } 0 \rightarrow G_{j+1}^n \rightarrow G_j^n \rightarrow G_j^n/G_{j+1}^n \rightarrow 0$$

Clearly each $\mathcal{C}_{\mathfrak{M}}^{j/m}$ is a R -module.

For any local ring S with maximal ideal \mathfrak{M} , we can define H_j^n as the submodule of Ω_S^n generated by elements of the form $(T_{x_1}^n)^{j_1} \dots (T_{x_r}^n)^{j_r}$, where $x_1, \dots, x_r \in \mathfrak{M}$, $j_1 + \dots + j_r = j$. When we consider H_j^n / H_{j+1}^n , we will have an R -module.

Now for S a regular local ring, let x_1, \dots, x_m be a regular system of parameters of S . Let H_j^m be the submodule of Ω_S^m generated by monomials of the form $(T_{x_1}^m)^{j_1} \dots (T_{x_m}^m)^{j_m}$ such that $j_1 + \dots + j_m = j$. If $j \geq m+1$ we have $H_j^m = 0$. This defines a filtration

$$\Omega_S^m = H_1^m \supset H_2^m \supset \dots \supset H_m^m \supset H_{m+1}^m = 0$$

Now we consider H_j^m / H_{j+1}^m . This gives us monomials of size exactly $= j$. Once again, this object $\Omega_S^{m,j}$ is an R -module.

Finally, we let F_j^n be the submodule of $\Omega_S^n \otimes_R \mathcal{R}$ generated by monomials of the form $T_{x_i}^n \otimes 1$ where $x_1, \dots, x_r \in \mathfrak{M}$. This is just the image of H_j^n under the map $\Omega_S^n \rightarrow \Omega_S^n \otimes_R \mathcal{R}$. We see that F_j^n / F_{j+1}^n will also be an R -module.

The next lemma and the following two propositions (Propositions 5 and 6) are technical results dealing with the filtrations and their maps. Although we will include them here, they will not be used in this section and will not be specifically needed until chapter 3.

We will need the following technical result:

Lemma 2: $(\alpha^n)^{-1}(F_j^n) = (P^{(j)} + P^{(n+1)}) / P^{n+1}$

Recall that α^n is the map $P/P^{n+1} \rightarrow \Omega_S^n \otimes_R \mathcal{R}$ defined in connection with theorem 2 of Chapter 1, section 1.3.

Proof of Lemma 2: (a) We want to show that $x \in (\alpha^n)^{-1}(F_j^n) \Rightarrow$

$$x \in (P^{(j)} + P^{(n+1)}) / P^{n+1} \quad . \quad x \in (\alpha^n)^{-1}(F_j^n) \Rightarrow \alpha^n(x) \in F_j^n$$

and this implies

$$\alpha(x) = \sum_{|h| \leq j} \frac{\partial^{|h|} \bar{x}}{\partial x_1^{h_1} \dots \partial x_r^{h_r}} (T_{x_1})^{h_1} \dots (T_{x_r})^{h_r}$$

and so, for $|h| \leq |j|$ each such term is zero in $\mathcal{R}_S^{\rho} / \rho \mathcal{R}_S^{\rho}$ which implies that each

$$\frac{\partial^{|h|} \bar{x}}{\partial x_1^{h_1} \dots \partial x_r^{h_r}} (T_{x_1})^{h_1} \dots (T_{x_r})^{h_r} \in \rho \mathcal{R}_S^{\rho}$$

since, as before, \mathcal{R}_S^{ρ} is freely generated by the T_{x_i} . This tells us that

$$\frac{\partial^{|h|} \bar{x}}{\partial x_1^{h_1} \dots \partial x_r^{h_r}} \in \rho \quad (|h| < j)$$

and so, again using the result of Seibt [S],

$$\bar{x} \in \rho^{(j)} \quad \text{or} \quad x \in \rho^{(j)} + \rho^{n+1} / \rho^{n+1}$$

(b) \supseteq Here we want to show

$$x \in \rho^{(j)} + \rho^{n+1} / \rho^{n+1} \quad \rightarrow \quad \alpha^{\rho}(x) \in F_j^{\rho}$$

Let $x = y + \rho^{n+1}$, where $y \in \rho^{(j)}$ and so by Seibt, y and all the partial derivatives of y up to order $j-1$ are in ρ . Now

$$\begin{aligned} \alpha^{\rho}(x) &= \sum_{|h| \leq n} \frac{1}{h_1! \dots h_r!} \frac{\partial^{|h|} y}{\partial x_1^{h_1} \dots \partial x_r^{h_r}} (T_{x_1})^{h_1} \dots (T_{x_r})^{h_r} \in \rho \\ &= \sum_{|j| \leq |h| \leq n} \frac{1}{h_1! \dots h_r!} \frac{\partial^{|h|} y}{\partial x_1^{h_1} \dots \partial x_r^{h_r}} (T_{x_1})^{h_1} \dots (T_{x_r})^{h_r} \in \rho \end{aligned}$$

since all terms $|h| \leq |j|$ are in $\rho \mathcal{R}_S^{\rho}$, hence $= 0$ in $\mathcal{R}_S^{\rho} / \rho \mathcal{R}_S^{\rho}$. Thus we see that $\alpha^{\rho}(x) \in F_j^{\rho}$ and so $x \in (\alpha^{\rho})^{-1}(F_j^{\rho})$ as desired.

Returning to our three sets of filtrations, we have the commuting

diagram

$$\begin{array}{ccccccc} G_{j+1}^{\rho} & \longrightarrow & F_{j+1}^{\rho} & \xrightarrow{\wedge_{j+1}^{\rho}} & H_{j+1}^{\rho} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ G_j^{\rho} & \longrightarrow & F_j^{\rho} & \xrightarrow{\wedge_j^{\rho}} & H_j^{\rho} & \longrightarrow & 0 \end{array}$$

where Λ_j^n and Λ_{j+1}^n are onto. This induces a sequence of order two

$$G_j^n/G_{j+1}^n \xrightarrow{\alpha_j^n} F_j^n/F_{j+1}^n \xrightarrow{\lambda_j^n} H_j^n/H_{j+1}^n \longrightarrow 0 \quad (2.4.1)$$

where λ_j^n is onto since Λ_j^n is onto. This final sequence is easily seen to be a sequence of R-modules and R-module homomorphisms.

Proposition 5: For a fixed index n, assume

$$\alpha_j^n: G_j^n/G_{j+1}^n \longrightarrow F_j^n/F_{j+1}^n \quad 1 \leq j \leq n+1$$

is injective. Then

$$\alpha^n: P/P^{n+1} \longrightarrow \Omega_S^n \otimes_S R$$

is injective.

Proof: We have the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & G_{j+1}^n & \longrightarrow & G_j^n & \longrightarrow & G_j^n/G_{j+1}^n \longrightarrow 0 \\ & & \downarrow \beta_{j+1}^n & & \downarrow \beta_j^n & & \downarrow \alpha_j^n \\ 0 & \rightarrow & F_{j+1}^n & \longrightarrow & F_j^n & \longrightarrow & F_j^n/F_{j+1}^n \longrightarrow 0 \end{array}$$

Of course, for $j+1 \geq n+1$ we have $G_{j+1}^n = F_{j+1}^n = 0$. For the case $j=n$ the two extreme columns are injective (the left-hand column trivially so, the right-hand column by assumption) and so we see that the middle column yields β_j^n injective. Shifting indices down by one, the next diagram gives α_{n-1}^n , β_n^n injective, and hence β_{n-1}^n is injective. Continuing by descending induction, we finally reach $\beta_1^n = \alpha^n$ is injective.

Proposition 6: Assume, for a fixed index n, that all maps

$$\alpha^L: P/P^{n+1} \longrightarrow \Omega_S^L \otimes_S R$$

are injective ($1 \leq L \leq n$). Then

$$\alpha_j^n: G_j^n/G_{j+1}^n \longrightarrow F_j^n/F_{j+1}^n \quad 1 \leq j \leq n+1$$

are injective.

Proof: Assume

$$\alpha^l: P/p^{l+1} \rightarrow \Omega_s^l \in \mathcal{R}$$

is injective for all $(1 \leq l \leq n)$. As noted in theorem 2, Chapter 1, section 1.3, α^l is injective if and only if $P^l = P^{l+1}$ so we have $P^j = P^{j+1}$ for each $1 \leq j \leq n$. Now returning to

Lemma 2 above, we find

$$(\alpha^n)^{-1}(F_j^n) = P^{(j)} + P^{n+1}/P^{n+1} = P^j + P^{n+1}/P^{n+1} = P^j/P^{n+1} = G_j^n$$

Consider the canonical map $\gamma_j^n: G_j^n/G_{j+1}^n \rightarrow F_j^n/F_{j+1}^n$. This is now

injective, so we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & G_j^n/G_{j+1}^n & \longrightarrow & G_i^n/G_{i+1}^n & & \\ & & \alpha_j^n \downarrow & & \gamma_i^n \downarrow & & \\ 0 & \rightarrow & F_j^n/F_{j+1}^n & \longrightarrow & F_i^n/F_{i+1}^n & & \end{array}$$

and γ_j^n injective leads us immediately to α_j^n injective and this completes Proposition 6.

Now for the punchline, we return to the sequence (2.4.1)

$$G_j^n/G_{j+1}^n \xrightarrow{\alpha_j^n} F_j^n/F_{j+1}^n \xrightarrow{\lambda_j^n} H_j^n/H_{j+1}^n \rightarrow 0$$

These are R-modules and R-module homomorphisms.

$$G_j^n/G_{j+1}^n \xrightarrow{\alpha_j^n} F_j^n/F_{j+1}^n \xrightarrow{\lambda_j^n} \Omega_{\mathcal{R}}^{n,j} \rightarrow 0$$

We apply the contravariant functor $\text{Hom}_{\mathcal{R}}(\quad, M)$ to this sequence and we define

so that the following sequence is exact for $\hat{U}^{n,i}(R, M)$

$$0 \rightarrow \text{Hom}_{\mathcal{R}}(\Omega_{\mathcal{R}}^{n,i}, M) \rightarrow \text{Hom}_{\mathcal{R}}(F_i^n/F_{i+1}^n, M) \rightarrow \text{Hom}_{\mathcal{R}}(G_i^n/G_{i+1}^n, M) \rightarrow \hat{U}^{n,i}(R, M) \rightarrow 0$$

i.e. $\hat{U}^{n,i}(R, M) = \text{coker} [\text{Hom}_{\mathcal{R}}(F_i^n/F_{i+1}^n, M) \rightarrow \text{Hom}_{\mathcal{R}}(G_i^n/G_{i+1}^n, M)]$

Again these objects are not obviously intrinsically defined so we want to show

that these $\hat{\mathcal{U}}^{n,i}$ agree with the $\mathcal{U}^{n,i}$ we defined above, in section 2.3. We want to check that the difference between the F_i^n defined from $\Omega_s^n \otimes_s R$ and the L_i^n defined from the $\Omega_s^n \otimes_s R$ does not affect the construction of the $\mathcal{U}^{n,i}$. But this is an immediate consequence of Proposition 3 in section 2.2 and hence we have

Proposition 7: $\hat{\mathcal{U}}^{n,i}(R, M) \cong \mathcal{U}^{n,i}(R, M)$

So we have constructed a collection of R-modules $\mathcal{U}^{n,i}$ and we have done so in two equivalent ways. In Chapter 3 we will make specific use of these modules by using the filtrations and sequences defined in this section.

Finally, we notice that for the case $n=1$ we again get the case of T^1 defined by Lichtenbaum and Schlessinger. This can be done directly as an easy consequence of letting $n=1$ in proposition 3. Thus the $\mathcal{U}^{n,i}$ are clearly an extension of T^1 to the case involving derivations of higher orders.

Chapter 3: An Application of the $\mathcal{U}^{n,i}(R, M)$

At the end of Chapter 2 we constructed a series of filtrations to return our setting to one of R -modules, then we delineated the construction of a cohomological object, $\mathcal{U}^{n,i}$ by placing it into position to complete an exact sequence analogous to that used to define the usual cotangent complex T' .

In this chapter we will begin by demonstrating a canonically defined map

$$\mathcal{S}_{n,i}^M : \text{Ext}_R^i(\Omega_R^{n,i}, M) \rightarrow \mathcal{U}^{n,i}(R, M)$$

We will then return to our filtered sequence and see how these objects are related to our previous ideas and then go on to consider the cases where the sequences possess a particularly useful structure.

In the final section we will put this material to work by proving another extension of Villamayor's result.

Section 3.1: Diagrammatics

Throughout this section we will deal with an algebraically closed field k of characteristic zero and a ring S which is the localization of a finitely generated k -algebra at a prime ideal, where S is assumed to be regular. As before, re write $R = S/P$ where P is a prime ideal of S . We want to construct a map from $\text{Ext}_R^i(\Omega_R^{n,i}, M)$ to $\mathcal{U}^{n,i}(R, M)$. The importance of this map will be the on those occasions when it allows us direct computation of the newfound $\mathcal{U}^{n,i}$ in

terms of a familiar construction applied to an only slightly unfamiliar module $\Omega_R^{n,i}$.

We want to define

$$\delta_{n,i}^M : \text{Ext}_R^1(\Omega_R^{n,i}, M) \rightarrow \mathcal{U}^{n,i}(R, M) \quad (3.1.1)$$

We begin by recalling from Chapter 2 that $\mathcal{U}^{n,i}$ has the form

$$\mathcal{U}^{n,i}(R, M) = \text{coker} \left[\text{Hom}_R(F_{i+1}^n, M) \rightarrow \text{Hom}_R(G_{i+1}^n, M) \right] \quad (2.1.2)$$

If we revisit the construction of Ext^1 as a group of extensions, (see Rotman [R, page 202ff]), we find a typical element of $\text{Ext}^1(\Omega_R^{n,i}, M)$ is an exact sequence of the form

$$0 \rightarrow M \xrightarrow{i} Q \xrightarrow{p} \Omega_R^{n,i} \rightarrow 0 \quad [Q]$$

Now by attaching (2.4.1) we find the solid-arrow diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \xrightarrow{i} & Q & \xrightarrow{p} & \Omega_R^{n,i} \rightarrow 0 \\ & & \uparrow \hat{g} & & \uparrow \hat{f} & & \parallel \\ & & G_{i+1}^n & \rightarrow & F_{i+1}^n & \rightarrow & \Omega_R^{n,i} \rightarrow 0 \end{array} \quad (3.1.3)$$

where the rightmost vertical arrow is the identity isomorphism and the rows are exact sequences of R-modules and R-module homomorphisms. By the freeness of F_{i+1}^n we have a map $f: F_{i+1}^n \rightarrow Q$ such that the right hand square commutes. We want to define a map $g: G_{i+1}^n \rightarrow M$ to complete the left hand square.

Let $e \in G_{i+1}^n$. Since $\hat{f} \alpha_i^*(e) = 0$, we have $p f \alpha_i^*(e) = 0$ and by the exactness of the top row $f \alpha_i^*(e) \in \text{im } i$ or $f \alpha_i^*(e) = i(m)$, where m is an element of M . Now we define $g: G_{i+1}^n \rightarrow M$ by $g(e) = m$. By Lemma 2 in section 2.3 and the exactness of the top row, g is well-defined and we see easily that

$$i(g)(e) = i(m) = f \alpha_i^n(e)$$

and so the left hand square, and hence all of diagram (3.1.3) commutes.

Now the freeness of F_i^n/F_{i+1}^n does not guarantee the uniqueness of f , but by a standard argument, if f' is another such map and g' is the induced map $g' : G_i^n \rightarrow M$, then we find

$$g - g' \in \alpha_i^n [\text{Hom}_R (F_i^n / F_{i+1}^n, M)]$$

and so we have

$$g \in \mathcal{U}^{n,i} (R, M) = \text{coker } \alpha_i^{n,*}$$

Thus we have a map $\delta_{n,i}^M$ taking an element $[Q]$ from $\text{Ext}'(\Omega_R^{n,i}, M)$ to $\mathcal{U}^{n,i}(R, M)$.

It is also straightforward, but tedious, to see that this map does not depend on the representation $R = \mathcal{S}_p$. Here we use the versality of $\mathcal{S}_p^{n,i}$ as an n th order versal infinitesimal extension to develop a parallel construction for the map to Ext and then use the homotopy equivalence (and corresponding homology equality to see that the same map is obtained. The details of this construction are left to the interested reader as a finger exercise.

Let's retrace our steps. We began with the (solid arrow) diagram

$$\begin{array}{ccccccc}
 G_i^n / G_{i+1}^n & \xrightarrow{\alpha_i^n} & F_i^n / F_{i+1}^n & \xrightarrow{\lambda_i^n} & \Omega_R^{n,i} & \longrightarrow & 0 \\
 \downarrow q_i & & \nearrow & & & & \\
 \circ & \longrightarrow & K_i^n & & & &
 \end{array}
 \tag{3.1.3 a}$$

where $K_i^n = \ker \lambda_i^n$, so the bottom row is exact. By the usual property of the kernel, we have a surjective map $q_i : G_i^n / G_{i+1}^n \rightarrow K_i^n$. Applying $\text{Hom}_R(\quad, M)$ for an

arbitrary R -module M , we find

$$\begin{array}{ccccccc}
 0 \rightarrow \text{Hom}_R(\Omega_R^{n,i}, M) & \rightarrow & \text{Hom}_R(F_i^n/F_{i+1}^n, M) & \rightarrow & \text{Hom}_R(K_i^n, M) & \rightarrow & \text{Ext}_R^i(\Omega_R^{n,i}, M) \rightarrow 0 \\
 \parallel & & \parallel & & q_i^* \downarrow & & \downarrow \delta_{n,i}^M \\
 0 \rightarrow \text{Hom}_R(\Omega_R^{n,i}, M) & \rightarrow & \text{Hom}_R(F_i^n/F_{i+1}^n, M) & \rightarrow & \text{Hom}_R(G_i^n/G_{i+1}^n, M) & \rightarrow & \mathcal{U}^{n,i}(R, M) \rightarrow 0
 \end{array}$$

where q_i^* is the map induced from q_i , the top row is the beginning of the long exact sequence for Ext and the two left hand vertical maps are the usual identity isomorphisms. The left hand square and the middle square commute, and we would like to complete our construction by showing that the right hand square commutes. Begin with an element $\sigma \in \text{Hom}(K_i^n, M)$ and consider the solid diagram

$$\begin{array}{ccccccc}
 0 \rightarrow K_i^n & \rightarrow & F_i^n/F_{i+1}^n & \rightarrow & \Omega_R^{n,i} & \rightarrow & 0 \\
 & & \downarrow \sigma & & \downarrow \delta & & \parallel \\
 0 \rightarrow M & \xrightarrow{f} & \tilde{F} & \xrightarrow{h} & \Omega_R^{n,i} & \rightarrow & 0
 \end{array}$$

By constructing the fibre product \tilde{F} (pushout) with the standard maps f , g , and h we obtain an element $[E]$ in $\text{Ext}_R^i(\Omega_R^{n,i}, M)$. Now continue on to construct

$$\begin{array}{ccccccc}
 0 \rightarrow M & \xrightarrow{g} & \tilde{F} & \xrightarrow{h} & \Omega_R^{n,i} & \rightarrow & 0 \\
 \uparrow \beta & & \uparrow \alpha & & \parallel & & \\
 G_i^n/G_{i+1}^n & \rightarrow & F_i^n/F_{i+1}^n & \rightarrow & \Omega_R^{n,i} & \rightarrow & 0
 \end{array}$$

again using the freeness of F_i^n/F_{i+1}^n and following the procedure above to give us an element in $\mathcal{U}^{n,i}$.

In the opposite direction, begin with $\sigma: K_i^n \rightarrow M$ and composing we obtain an element in $\text{Hom}(G_i^n/G_{i+1}^n, M)$ depending on σ , and when we compare via the combined diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & k_i^* & \rightarrow & F_{F_{i+1}}^n & \rightarrow & \Omega_R^{n,i} \rightarrow 0 \\
 & & \downarrow \sigma & & \downarrow & & \parallel \\
 0 & \rightarrow & M & \rightarrow & \tilde{F} & \rightarrow & \Omega_R^{n,i} \rightarrow 0 \\
 & & \uparrow \beta & & \uparrow \alpha & & \parallel \\
 & & G_i^*/G_{i+1}^* & \rightarrow & F_{F_{i+1}}^n & \rightarrow & \Omega_R^{n,i} \rightarrow 0
 \end{array}$$

using $\beta \circ \sigma$, α freely constructed yields the desired result.

Section 3.2: The Main Theorem.

The Main Theorem: Let S be the localization of a finitely generated k -algebra at a prime ideal, where k is an algebraically closed field of characteristic zero. Let $R = \hat{S}_P$ where P is a prime ideal of S . Assume S is regular. Then for a fixed integer $n \geq 0$, the following statements are equivalent:

- (1) $P^i = P^{(n)}$ for all $i \leq n_0 + 1$
- (2) $\delta_{n,i}^M$ is onto, for all $i \leq n$, $n \leq n_0 + 1$, for all R -modules M
- (3) $\delta_{n,i}^M$ is an isomorphism for all $i \leq n$, $n \leq n_0 + 1$, for all R -modules M .

M .

Proof of the Main Theorem: (1) \Leftrightarrow (2) From theorem 1, section 1.3, we have $P^i = P^{(n)}$ for all $i \leq n_0 + 1$ if and only if α^i is injective for all $i \leq n$. By propositions 5 and 6 of section 2.4, this is true if and only if α_i^* is injective for all $n \leq n_0 + 1$ and $i \leq n$ and from an easy diagram chase using (3.1.3a) this is true if and only if q_i is injective if and only if q_i^* is surjective if and only if $\delta_{n,i}^M$ is onto from diagram (3.1.4).

(2) \Leftrightarrow (3) $\delta_{n,i}^M$ is onto (and hence $P^i = P^{(n)}$ for all $i \leq n_0 + 1$) implies

$$G_i^*/G_{i+1}^* \rightarrow F_{F_{i+1}}^n$$

is injective, hence q_i is injective and thus q_i^* is an isomorphism and so we finally have $\delta_{n,i}^M$ is an isomorphism. The reverse implication is immediate.

This theorem gives us a new, though not particularly simple, criterion for determining the equality of prime powers and symbolic prime powers in terms of the objects \mathcal{U}^{n_i} . So the more properties of each \mathcal{U}^{n_i} we are able to determine, the easier it becomes to answer Hochster's original question. On the other hand, when we know that $P^i = P^{(i)}$ for a certain class of rings, we can then compute the \mathcal{U}^{n_i} by using the more explicit $\text{Ext}'_{\mathfrak{A}}(\Omega_{\mathfrak{A}}, M)$. This transition from one set of questions to another should facilitate the study of both types of ideas.

Chapter 4 : What to Do Until the Doctor Arrives

In this final chapter we want to examine various ways we can extend our results by relaxing the restrictions we have put on our choice of the base ring k . We would also like to discuss certain directions that appear most conducive to us for future research.

Section 4.1: Considerations on the Base Field k

We have assumed in Chapter 3 that k is an algebraically closed field of characteristic zero. In this section we would like to explore possible modifications. The first possibility might be to examine a field of characteristic p not equal to 0. One problem that arises in noncharacteristic zero fields is that a derivation might vanish on a non-zero subring of the original ring. If a ring A is of characteristic $p \neq 0$ and $A^p = \{a^p : a \in A\}$ then for an A -module M and derivation $D: A \rightarrow M$, we have $D(a^p) = pa^{p-1}D(a) = 0$. (See [M1, page 181].) It is also not clear that the key result of Seibt [S] would hold in a characteristic different from zero. And in any case, any extension to $\text{char} = p$ would certainly involve some separability assumptions we have not previously needed or considered, since they have been, in any case, automatically satisfied.

The situation grows more interesting when we consider the case of k not algebraically closed. In Chapter 3 we considered a field k and a finite k -algebra S with maximal ideal $\max(S)$ (usually considering S as a localization at a prime and assuming S to be regular as well). We then dealt with the case

where we have $k \cong \sum_{i=0}^{\infty} k[S]_i$, which is the case when k is algebraically closed, since then the freeness of Ω_S^1 , with S regular, can be proved in this case. Although the results from Chapter 2 remain quite general, there is apparently the possibility of relaxing the conditions invoked in parts of Chapter 3.

Section 4.2: Lines of Future Research

We conclude by posing several questions that have been suggested by the results we have obtained.

Foremost among these are extensions of the results described above. How much more can we modify the restrictions on k , S , and P to produce results on the equality of prime powers and symbolic prime powers? Also, how much can we relax the relationship between k and the extension $K = \sum_{i=0}^{\infty} k[S]_i$ discussed in section 4.1? What, exactly, is the optimal relationship, if any? And can we develop any such theory along the same lines for a field of prime characteristic? How much will our results change and how much will they tell us about the original problem?

We would also like to consider analogue to derivations of higher order of a result proved in Matsumura [M1, page 215, Lemma 1]. When B is smooth over k , then Ω_B^n is a projective B -module. (Matsumura proves a more general assertion for formally smooth algebras.) The proof given in Matsumura does not directly extend to Ω_B^n , but it may be possible to use the systems of filtrations described in Chapter 2 to obtain results in the higher order cases.

Another question which we would like to treat is related to the Rees algebra of the "blow-up" of a commutative noetherian ring A . If P is a

prime ideal of A , then $\mathcal{O}_{A^0} P^n$ is known to be noetherian. The question then arises, When is $S(P) = \mathcal{O}_{A^0} P^{(n)}$ noetherian? It is known to be noetherian when P is a set-theoretic complete intersection (see [Co] and [Hu]) but perhaps a more careful study of $\mathcal{U}^{n,i}$ would allow us to approach a general solution. For more work on this line, see Eliahou [E] and the results mentioned there.

Returning to the $\mathcal{U}^{n,i}$ we have constructed, several lines of inquiry are open here as well. Recalling that they arise from the work of Lichtenbaum and Schlessinger in relation to the cotangent functors T^i , we notice that several interesting things are shown to happen when the T^i and T_i are known to vanish. It would be instructive to try to extend this study to the case of the general vanishing criteria for the $\mathcal{U}^{n,i}$. Also, it seems that the $\mathcal{U}^{n,i}$ are essentially related to the T^i . Perhaps, using more general techniques, it would be possible to define analogous functors related to the higher T^i developed by Lichtenbaum-Schlessinger, Quillen, Andre, etc.

Finally we are led back to the motivating questions for this entire subject raised in the Introduction in Chapter 1. Hartshorne asked us to determine the relationship between the p -adic topology and the p -symbolic topology; in particular, when are these two topologies equivalent. As we indicated earlier, several partial answers are known and the use of the $\mathcal{U}^{n,i}$ may help us to extend these results. And at last, we would like to return to Hochster's original problem. Exactly when is $P^\wedge = P^{(n)}$ for every positive integer n ? Although a complete answer may be currently out of reach, the results we have obtained here may help us to continue to make further progress on this question also.

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