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On Roundedness in Matroid Theory.

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ON ROUNDEDNESS IN MATROID THEORY

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ABSTRACT

This thesis studies the relationship between subsets and specified minors in a 3-connected matroid. For positive integers k and m, a set S of k-connected matroids is (k,m)-rounded if it satisfies the following condition. Whenever M is a k-connected matroid having an S-minor and X is a subset of E(M) with at most m elements, then M has an S-minor using X.

Oxley characterized the (3,2)-rounded sets that contain a single matroid. In Chapter 2, we obtain an analog of this result for binary matroids. In Chapter 3, we use this result to characterize the pairs of matroids which form (3,2)-rounded sets.

The methods of Chapter 3 are generalized to 4-connected matroids in Chapter 4 to determine the (4,2)-rounded sets that contain a single matroid. This extends results of Coullard and Kahn.

For a 3-connected minor N of a 3-connected matroid M, the following question arises from roundedness theory. Let X be a subset of E(M). How small a 3-connected minor of M can we find which both uses X and has an N-minor? Seymour answered this question for $|X| = 1$ and 2. We answer this question for $|X| \geq 3$ in Chapter 5.

Finally, in Chapter 6, results from roundedness theory are applied to the study of 3-element circuits in 3-connected matroids. An extension of a result of Asano, Nishizeki, and Seymour is obtained for binary matroids which are non-regular.

CHAPTER 1

Introduction to Roundedness Theory

1.1 Notation and Terminology

The study of the property of roundedness in matroids involves such matroid-theoretic concepts as connectivity, extensions, and representability. We shall first discuss these concepts before beginning our investigation of roundedness theory in Section 1.6.

We start with some notation and terminology. Most of the matroid terminology used follows Welsh [47], while most of the graph terminology used follows Bondy and Murty [5]. Let M be a matroid. The ground set of M is denoted by E(M). Let N be a minor of M. If E(N) is a proper subset of E(M), then N is said to be a *proper minor* of M. If Y is a subset of E(M), then we say that M *uses* Y. An N-*minor* of M is a minor of M that is isomorphic to N. Let $\mathcal{S}$ be a set of matroids. We say that M has an $\mathcal{S}$-*minor using* Y if M has an N-minor using Y for some member N of $\mathcal{S}$.

The deletion and contraction of Y from M are denoted by $M\backslash Y$ and M/Y, respectively. The restriction of M to Y, $M\backslash(E(M)-Y)$, is denoted by $M|Y$. Distinct elements e and f of M are said to be in *parallel* in M if {e,f} is a circuit of M. We shall say that e and f are in <u>series</u> in M if {e,f} is a cocircuit of M. If P is a maximal subset of E(M) such that every pair of elements of P are in parallel in M, then P is said to be a <u>parallel</u> <u>class</u> of M. We say that S is a <u>series</u> <u>class</u> of M if it is a parallel class of M^*. The <u>simplification</u> of M is obtained by deleting all but one element from each parallel class of M and deleting all loops. The <u>cosimplification</u> of M is obtained by contracting all but one element from each series class of M and deleting all coloops. Note that these matroids are only defined up to isomorphism. Let $\tilde{M}$ denote the simplification of M. The cosimplification of M is denoted by $\hat{M}$.

The rank and closure of Y in M are denoted by $rk_M Y$ and $\sigma_M(Y)$. We will sometimes write rk Y for $rk_M Y$ and rk M for $rk_M E(M)$. Three-element circuits and cocircuits of M are called <u>triangles</u> and <u>triads</u>, respectively. Flats of M of rank two and three are called <u>lines</u> and <u>planes</u>, respectively. The property that M cannot possess a circuit and a cocircuit which meet in one element is referred to as <u>orthogonality</u>.

We now give some graphs and matroids which are referred to in the subsequent chapters. We shall only consider graphs with a finite number of edges in this dissertation. The complete graph on n vertices is denoted by K_n. Let K_5-a denote the graph which is obtained from K_5 by deleting an edge. $K_{3,3}$ is the complete bipartite graph with two vertex classes of size three. The wheel graph with n spokes and 2n edges is denoted by $\mathcal{W}_n$ for each integer n exceeding two [47, p.80]. We shall let $\mathcal{W}^n$ denote the whirl matroid of rank n for each integer n exceeding one [47, p.81].

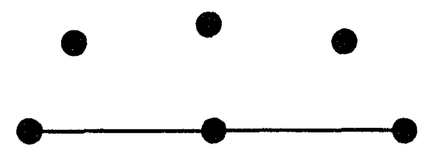
The uniform matroid of rank r with n elements is denoted $U_{r,n}$ and the Fano matroid is denoted by F_7[47]. We shall denote the r-dimensional vector space over GF(q) by V(r,q). We let $V(r,q)^{\bullet}$ denote the set of non-zero elements of the vector space V(r,q). The rank-(n+1) affine geometry over GF(q) is denoted by AG(n,q) [47].

Euclidean representations for some rank-three and rank-four matroids are given in Table 1.

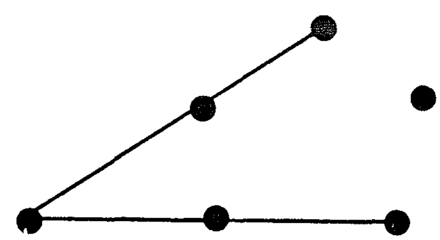
Table 1 Some Rank-3 and Rank-4 Matroids

Matroid | Euclidean Representation

P_6



Q_6



J_{10}

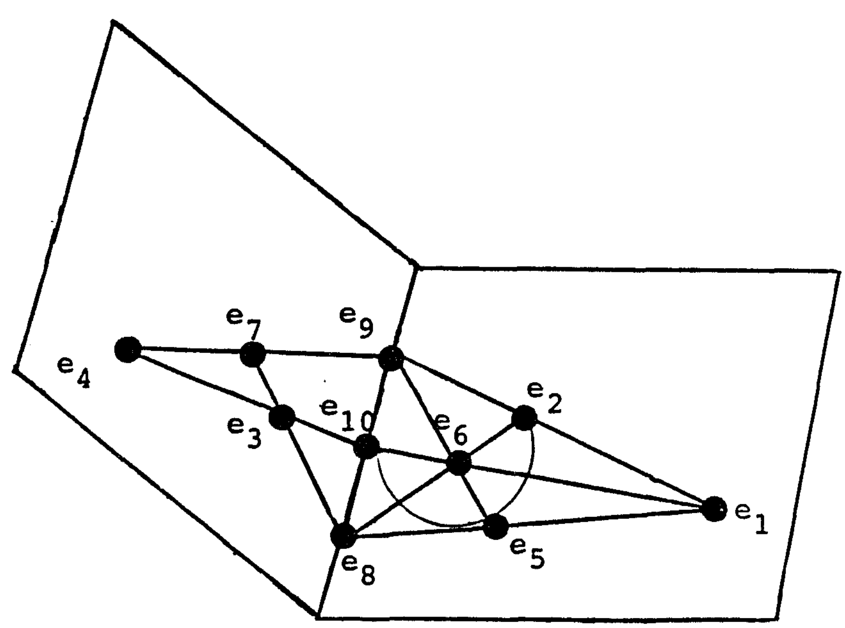
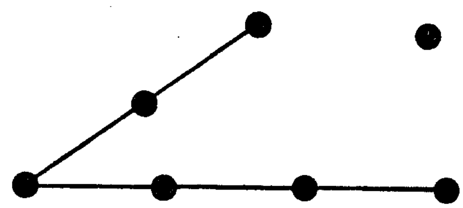
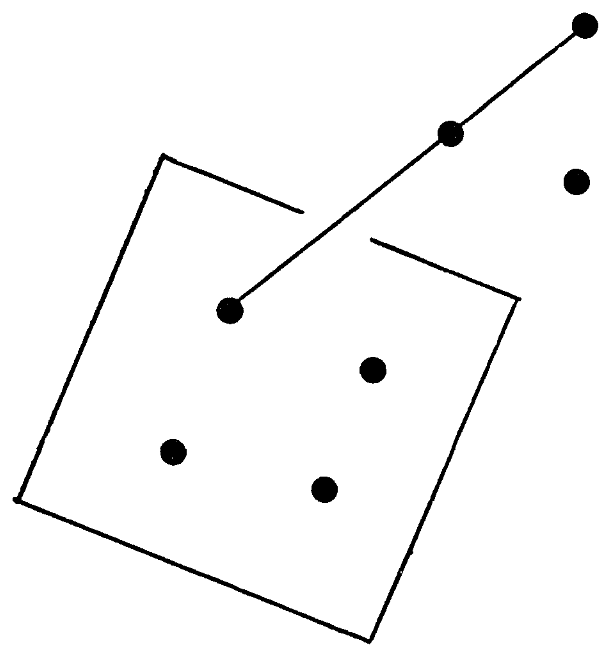


Table 1 cont. Some Rank-3 and Rank-4 Matroids

Matroid	Euclidean Representation
$P_9 = J_{10} \setminus e_{10}$	$J_{10} \setminus e_{10}$
$S_8 = J_{10} \setminus \{e_9, e_{10}\}$	$J_{10} \setminus \{e_9, e_{10}\}$
Q_7	
Q_7^*	

1.2 Connectivity in Matroids and Graphs

The property of n-connectivity in matroids was conceived by Tutte [46] as a generalization of vertex connectivity in graphs [5]. This property plays an essential role in the theory of roundedness in matroids. We shall begin with the definition of n-connectivity in a matroid and then give some useful facts about this concept.

If k is a positive integer, then a bipartition(A,B) of E(M) is a k-separation of the matroid M if A and B both have at least k elements and $rk_M A + rk_M B - rk\ M \leq k - 1$ [46]. For an integer n which is at least two, M is n-connected if M has no k-separation for any $k < n$.

We say that M is connected if, whenever e and f are distinct elements of M, there is a circuit of M which contains both e and f [47]. M is connected if and only if it is 2-connected [47, p. 71, (4)].

We shall mostly be concerned with the class of 3-connected matroids in this dissertation. Tutte's wheels and whirls theorem is given next. This is the result which began the study of 3-connectivity in matroids. An element e of a 3-connected matroid is said to be essential if both $M\backslash e$ and M/e are not 3-connected.

1.2.1 Theorem [46]. *Let M be a 3-connected matroid in which every element is essential. Then M is either the cycle matroid of a wheel graph or is a whirl of rank at least three.* □

An easy extension of Tutte's wheels and whirls theorem is the following result. This result is well known (see, for example, [23,(4.1)]). Recall that $U_{2,4}$ is the whirl of rank two.

1.2.2 Theorem. *Let M be a 3-connected matroid with at least four elements that is neither a wheel-matroid nor a whirl. Then there is a sequence $M_1, M_2, \ldots, M_n$ of 3-connected matroids such that M_1 is a wheel-matroid or a whirl, $M_n = M$, and, for each i in {1,2,...,n-1}, M_i is a minor of M_{i+1} obtained by deleting or contracting a single element.* □

Seymour strengthened the previous theorem with the next result.

1.2.3 Theorem [36,(7.3)]. *Let M and N be 3-connected matroids having at least four elements such that N is a* *minor of* M. *Further suppose that if* N *is isomorphic to* $M(\mathcal{W}_k)$, *then* M *has no* $M(\mathcal{W}_{k+1})$-*minor, while if* N *is isomorphic to* $\mathcal{W}^k$, *then* M *has no* $\mathcal{W}^{k+1}$-*minor. Then there is a sequence* $M_0, M_1, M_2, \ldots, M_n$ *of* 3-*connected matroids such that* M_0 *is isomorphic to* N, $M_n = M$, *and, for each* i *in* $\{1,2,\ldots,n\}$, M_{i-1} *is obtained from* M_i *by deleting or contracting an element.* □

The following connectivity results will be frequently used. For a subset A of E(M), the next fact is easily checked.

(1.2.4) $\quad rk_M A + rk_M(E(M)-A) - rk\, M = rk_M A + rk_{M^*} A - |A|.$ □

Suppose M is 3-connected with at least five elements. It follows from (1.2.4) that M has no 3-element subset which is both a triangle and a triad. The following result is also a direct consequence of (1.2.4).

1.2.5 *Lemma* [23]. *If* M *is an* n-*connected matroid with at least* 2(n-1) *elements, then every circuit and cocircuit of* M *contains at least* n *elements.* □

The next lemma of Oxley is often used.

1.2.6 Lemma [23,(2.1)]. *Let M be a matroid having at least two elements and n be an integer which is at least two. Suppose that $M\backslash e$ is n-connected and e is not a coloop of M. If e is not contained in a circuit of M with fewer than n elements, then M is also n-connected.* □

We may determine when the cycle matroid of a graph is 3-connected by using the following well-known result (see, for example, [47,pp. 78-79]).

1.2.7 Lemma. *Let G be a graph without isolated vertices. If G has at least four vertices, then M(G) is 3-connected if and only if G is 3-connected and simple.* □

The next result is an immediate consequence of Hassler Whitney's 2-isomorphism theorem [49].

1.2.8 Theorem [49]. *Let G and H be loopless 3-connected graphs. Then M(G) and M(H) are isomorphic if and only if G and H are isomorphic.* □

This result will be used implicitly in our investigation of roundedness in 3-connected graphic matroids. It allows us to conclude that there is, up to isomorphism, only one graph representing a 3-connected graphic matroid.

1.3 Extensions of Matroids

In our study of roundedness we shall need to produce n-connected matroids which have a given n-connected matroid as a minor. Results of Brylawski and Crapo on constructing such matroids are given in this section. We begin with some notation.

Let N be a matroid. Suppose M is a matroid with ground set $E(N) \cup \{e\}$ such that $M\backslash e = N$. We denote this by $M = N+e$ and say that M is an *extension* of N. Note that N+e is not uniquely determined. If e is not in any circuit of M of size one or two, and e is not a coloop of M, then M is called a *non-trivial extension* of N.

Suppose $M/e = N$. Then M is said to be a *lift* of N. Suppose e is not in any cocircuit of M of size one or two, and e is not a loop of M. Then M is said to be a *non-trivial lift* of N. Lemma 1.2.6 is now restated in terms of 3-connected matroids.

1.3.1 Lemma. *Let N be a 3-connected matroid with at least three elements and M be an extension of N. Then M is 3-connected if and only if M is a non-trivial extension of N.* □

Crapo's theory of modular cuts is used to construct extensions of a matroid. A pair of distinct flats (F_1, F_2)

of a matroid M is said to be a modular pair if $rkF_1 + rkF_2 = rk(F_1 \cup F_2) + rk(F_1 \cap F_2)$. Let F be a flat of M such that if G is any other flat of M, then (F,G) is a modular pair of flats of M. Then we say that F is a modular flat of M.

A modular cut $\mathcal{M}$ of M is a subset of the set of flats of M satisfying the following two conditions.

(1) If $F_1 \varepsilon \mathcal{M}$ and F_2 is a flat of M containing F_1, then $F_2 \varepsilon \mathcal{M}$.

(2) If (F_3,F_4) is a modular pair of flats in $\mathcal{M}$, then $F_3 \cap F_4$ is also in $\mathcal{M}$.

Evidently the intersection of two modular cuts in a matroid is also a modular cut of that matroid. If $\{F_1,F_2,\ldots,F_n\}$ is a set of flats of a matroid, then the modular cut generated by this set is the intersection of all modular cuts containing $\{F_1,F_2,\ldots,F_n\}$. A principal modular cut is a modular cut generated by a set containing a single flat.

A modular cut of a simple matroid gives an extension of M with flats as specified in the next result.

1.3.2 Theorem [14]. *Let $\mathcal{M}$ be a modular cut of a simple matroid* M *and suppose* e *is not in* E(M). *Then* $\mathcal{M}$ *determines a unique extension of* M *on* $E(M) \cup \{e\}$. *The* *flats of this extension*, M + e, *are as follows*.

(1) *Those sets* F *such that* F *is a flat of* M *not in* $\mathcal{M}$.

(2) *Those sets* $F \cup e$ *such that* $F \in \mathcal{M}$.

(3) *Those sets* $F \cup e$ *such that* F *is a flat of* M *that is not in* $\mathcal{M}$ *and is not covered in* M *by a flat of* $\mathcal{M}$. □

If M + e, M, and $\mathcal{M}$ are as given in Theorem 1.3.2, then we shall refer to M + e as the *extension of* M *determined by* $\mathcal{M}$.

Now, let M and N be matroids such that E(M) and E(N) meet in the set F. Suppose that F is a flat of both M and N. Further suppose that F is a modular flat of M. Then the *generalized parallel connection of* M *and* N *across* F is denoted by $P_F(M,N)$ [7,Sect. 5]. This is the matroid on $E(M) \cup E(N)$ such that a subset A of $E(M) \cup E(N)$ is a flat of $P_F(M,N)$ if and only if $A \cap E(M)$ is a flat of M and $A \cap E(N)$ is a flat of N. We now list some properties of $P_F(M,N)$ that we will use later. Let $P = P_F(M,N)$.

1.3.3 *Theorem* [7,(5.5)]. *If* A *is a flat of* P, *then* $rk_P A = rk_M(A \cap E(M)) + rk_N(A \cap E(N)) - rk_M(A \cap F)$. *In particular* $rkP = rk\ M + rk\ N - rk_M F$. □

1.3.4 Theorem [7,(5.11)]. Let $m \in E(M) - F$, $n \in E(N) - F$, and $f \in F$.

(1) $P\backslash m = P_F(M\backslash m, N)$.

(2) $P\backslash n = P_F(M, N\backslash n)$.

(3) $P/m \cong P_G(M/m, N)$ where G is the ground set of $(M|\sigma_M(F \cup m))/m$.

(4) $P/n \cong P_F(M, N/n)$.

(5) $P/f \cong P_H(M/f, N/f)$ where H is the ground set of $(M|F)/f$. □

1.4 Representability

We shall investigate roundedness in certain classes of representable matroids in Chapters 2 and 6. Some notation and fundamental observations about representable matroids are given in this section.

Let A be a matrix with entries in a field F. The dependence matroid on the columns of A is denoted by D(A). If M = D(A), then we say that M is *representable* over F. In particular, when F = GF(2), we shall call A a *binary matrix* and D(A) a *binary matroid*. If column e is adjoined to A, then A + e will denote the resulting matrix. If M = D(A), then M + e will denote D(A+e).

We shall use the following characterizations of binary matroids.

1.4.1 *Theorem* [47,p.162]. *The following statements about a matroid* M *are equivalent*.

(1) M *is binary*.

(2) *Any circuit* C *and cocircuit* C* *meet in an even number of elements*.

(3) *If* C_1 *and* C_2 *are distinct circuits of* M, *then their symmetric difference* $C_1 \Delta C_2$ *contains a circuit* C.

(4) *If* C_1 *and* C_2 *are distinct circuits of* M, *then their symmetric difference* $C_1 \Delta C_2$ *is a disjoint union of circuits*. □

1.4.2 Theorem [45]. *A matroid is binary if and only if it has no $U_{2,4}$-minor.* □

The fact that a graphic matroid is representable over every field will be used [32]. We shall also implicitly use the following well-known fact [9,(3.7)]. Binary matroids are uniquely representable in the following sense. If A and B are binary matrices with the same dimensions such that D(A) and D(B) are isomorphic, then A can be transformed into B by a sequence of elementary row operations followed by a permutation of the columns.

The binary matroids given below will be referred to in the subsequent chapters.

Table 2 Some Binary Matroids

Matroid	Representing Binary Matrix
$S_8 = D(A_1)$	$A_1 \left[\begin{array}{c|cccc} I_4 & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \end{array} \right]$
$AG(3,2)=D(A_2)$	$A_2 \left[\begin{array}{c|cccc} I_4 & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 0 \end{array} \right]$
$P_9=D(A_3)$	$\begin{array}{c} e_1\ e_2\ e_3\ e_4\ e_5\ e_6\ e_7\ e_8\ e_9 \end{array}$ $A_3 \left[\begin{array}{c|ccccc} I_4 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \end{array} \right]$
$Z_4=D(A_4)$	$\begin{array}{c} a_1\ a_2\ a_3\ a_4\ b_1\ b_2\ b_3\ b_4\ c_4 \end{array}$ $A_4 \left[\begin{array}{c|ccccc} I_4 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 0 & 1 \end{array} \right]$

$$P_9^* = D(A_3^*) \qquad A_3^* \begin{array}{c} \\ \left[\begin{array}{ccccc|cccc} e_5 & e_6 & e_7 & e_8 & e_9 & e_1 & e_2 & e_3 & e_4 \\ \hline & & & & & 0 & 1 & 1 & 1 \\ & & & & & 1 & 0 & 1 & 1 \\ & & I_5 & & & 1 & 1 & 0 & 1 \\ & & & & & 1 & 1 & 1 & 1 \\ & & & & & 1 & 1 & 0 & 0 \end{array}\right] \end{array}$$

$$Z_4^* = D(A_4^*) \qquad A_4^* \left[\begin{array}{ccccc|cccc} b_1 & b_2 & b_3 & b_4 & c_4 & a_1 & a_2 & a_3 & a_4 \\ \hline & & & & & 0 & 1 & 1 & 1 \\ & & & & & 1 & 0 & 1 & 1 \\ & & I_5 & & & 1 & 1 & 0 & 1 \\ & & & & & 1 & 1 & 1 & 0 \\ & & & & & 1 & 1 & 1 & 1 \end{array}\right]$$

$$Z_r = D(A_r) \qquad A_r \left[\begin{array}{cccc|cccccc} a_1 & a_2 & \cdots & a_r & b_1 & b_2 & b_3 & \cdots & b_r & c_r \\ \hline & & & & 0 & 1 & 1 & \cdots & 1 & 1 \\ & & & & 1 & 0 & 1 & \cdots & 1 & 1 \\ & I_r & & & 1 & 1 & 0 & \cdots & 1 & 1 \\ & & & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & & 1 & 1 & 1 & \cdots & 0 & 1 \end{array}\right]$$

1.5 Free Elements

The concept of a free element in a matroid is introduced in this section. The properties of these elements will be particularly useful in our study of roundedness.

Let M be a matroid with at least two elements. An element e of M is said to be *free* if it is in no circuit of size less than rkM+1 and it is not a coloop of M.

Suppose that M is simple and f is not an element of E(M). Let F be a flat of M. Suppose $\mathcal{M}$ is the principal modular cut of M generated by F and M+f is the extension of M determined by $\mathcal{M}$. Then we say that M+f is the extension of M obtained by *freely adding* f to F. In particular, if F = E(M), then M+f is said to be obtained by *freely adding* f to M.

Evidently if f is freely added to M, then f is free in M+f. The relationship between free elements and duality will be exploited. This relationship is explained in the next theorem of Oxley.

1.5.1 *Lemma* [24,(2.2)]. *Let* e *be an element of a connected matroid* M *with at least two elements. Then* e *is free in* M^* *if and only if* e *is in every dependent flat of* M. □

In light of the above lemma, if e is an element in a connected matroid M that has at least two elements, and e is in every dependent flat of M, then e will be called a cofree element of M. The next lemma is an immediate consequence of Lemma 1.5.1.

1.5.2 Lemma. *Let M be a connected matroid with at least two elements. Then M has an element which is both free and cofree if and only if M is isomorphic to* $U_{r,n}$ *for some integer r such that* $1 \leq r \leq n - 1$. □

For integers r and n with $1 \leq r \leq n - 1$, each element of the matroid $U_{r,n}$ is both free and cofree. Let M be a connected matroid with at least two elements. The next lemma is used several times in Chapters 3 and 4.

1.5.3 Lemma. *Suppose M possesses at least m free elements and at least n cofree elements. If* $|E(M)| \geq m + n$, *then there exist disjoint subsets* S_1 *and* S_2 *of E(M) having m and n elements, respectively, such that each element of* S_1 *is free in M and each element of* S_2 *is cofree in M.*

Proof. Suppose e is both free and cofree in M. Then, by Lemma 1.5.2, M is isomorphic to $U_{r,n}$ for integers r and n with $1 \leq r \leq n - 1$. Thus all elements of M are both free and cofree. □

We next show that, in general, a binary matroid does not have any free elements. Let B be a base of a matroid M and e be an element of E(M) which is not included in B. The <u>fundamental circuit</u> of e in B is denoted by C(e,B) [47]. The graph which is a cycle on n edges is denoted C_n.

1.5.4 <u>Lemma</u>. *Let M be a simple binary matroid with at least three elements. Then M has a free element if and only if M is isomorphic to $M(C_n)$ for some $n \geq 3$.* □

<u>Proof</u>. Let f be a free element of M and suppose that M is not isomorphic to $M(C_n)$. Evidently M has rank at least two. Let B be a base of M\f. Now $B \cup \{f\}$ is a circuit in the binary matroid M, and M is not isomorphic to $M(C_n)$. Thus there exists an element e of E(M) which is not in $B \cup \{f\}$.

Now, by Lemma 1.4.1(3), there exists a circuit C contained in $C(e,B) \Delta C(f,B) = C(e,B) \Delta (B \cup \{f\})$ $= (B - C(e,B)) \cup \{e,f\}$. Since M is simple, C(e,B) has at least three elements. Thus C has fewer than rkM + 1 elements. Hence f is not in C and C is a circuit other than C(e,B) which is contained in $B \cup \{e\}$; a contradiction. Thus M is isomorphic to $M(C_n)$.

Conversely, it is easily checked that, for n at least three, each element of $M(C_n)$ is free. □

1.6 Roundedness in Matroids

The central theme of this dissertation, the theory of roundedness in matroids, is discussed in this section. We begin by examining the terminology and development of this theory. Questions of the following type are addressed by the theory of roundedness. Suppose we are given structural information including connectivity about a matroid M. Can we say, for an arbitrary subset T of E(M), that M has a specified minor using T? Particular cases of this question have been addressed by several authors including Asano, Nishizeki, and Seymour [1], Bixby [2], Bixby and Coullard [4], Coullard [10,11], Coullard and Reid [13], Kahn [18], Oxley [24,25,27], Oxley and Reid [30], Oxley and Row [31], Seymour [35,37,38,39,40,41], and Tseng and Truemper [42].

The role of the theory of roundedness in the study of matroid structure was surveyed by Seymour [41 , Section 3].

Let k and m be positive integers with k at least two. The following definition is due to Bixby and Coullard [4].

1.6.1 Definition. *Let* $\mathcal{S}$ *be a set of* k-*connected matroids. Further suppose that each matroid in* $\mathcal{S}$ *has at least four elements. The set* $\mathcal{S}$ *is* (k,m)-*rounded if and only if it satisfies the following condition.*

(i) *If M is a k-connected matroid having an S-minor and X is a subset of E(M) with at most m elements, then M has an S-minor using X.*

This definition generalized an earlier definition of Seymour who called a set of matroids m-rounded when it is (m+1, m)-rounded in the above sense [38]. Seymour developed an efficient test for the property of (3,2)-roundedness. The set S is a collection of 3-connected matroids with each matroid in S having at least four elements.

1.6.2 *Theorem* [38]. *The set S is (3,2)-rounded if and only if S satisfies the following condition.*

(i) *If M is a 3-connected extension or lift of a matroid in S, and X is a subset of E(M) with at most two elements, then M has an S-minor using X.* □

Oxley noted that there is a similar test for the property of (3,1)-roundedness.

1.6.3 *Theorem* [24]. *The set S is (3,1)-rounded if and only if S satisfies the following condition. If M is a 3-connected extension or lift of a matroid in S and e is an element of E(M), then M has an S-minor using e.* □

Bixby and Coullard provided an analogous, but less efficient, test for the property of (3,m)-roundedness if m exceeds two [4].

The result which provided the impetus for the study of roundedness in matroids is the next theorem of Bixby.

1.6.4 Theorem [2]. *The set* $\{U_{2,4}\}$ *is* (2,1)-*rounded*. □

The above theorem extends Theorem 1.4.2, Tutte's excluded minor characterization of the binary matroids. Seymour strengthened Bixby's result as follows.

1.6.5 Theorem [38,(3.1)]. *The set* $\{U_{2,4}\}$ *is* (3,2)-*rounded*. □

Oxley extended this result with the next two theorems. The first theorem presented is an example of the type of results which are given in Chapters 2,3, and 4. It characterizes, for particular values of k and m, when certain sets of matroids can be (k,m)-rounded.

1.6.6 Theorem [24,(1.5)]. *Let* M *be a matroid. The set* {M} *is* (3,2)-*rounded if and only if* M *is isomorphic to* $U_{2,4}$. □

1.6.7 _Theorem_ [27, (1.9)]. _The set_ $\{U_{2,4}, \mathcal{W}^3\}$ _is_ (3,3)-_rounded_. □

The singleton (2,1)- and (3,1)-rounded sets were also characterized by Oxley. The matroid Q_6 is listed Table 1. Let Q_4 be the cycle matroid of the graph obtained by adding an edge in parallel to one of the edges of a triangle.

1.6.8 _Theorem_ [24, (1.4)]. _Let_ M _be a matroid_. _The set_ {M} _is_ (2,1)-_rounded if and only if_ M _is isomorphic to one of_ $U_{2,4}$, Q_4, _and_ Q_6. _Moreover, the set_ {M} _is_ (3,1)-_rounded if and only if_ M _is isomorphic to_ $U_{2,4}$ _or_ Q_6. □

We conclude the section by listing some sets which were shown to be rounded by Seymour and Oxley. The matroid S_8 is given in Table 2.

1.6.9 _Theorem_ [38, (3.1)]. _The sets_ $\{U_{2,4}, M(\mathcal{W}_3)\}$ _and_ $\{U_{2,4}, F_7, F_7^*, S_8\}$ _are_ (3,2)-_rounded_. □

1.6.10 *Theorem* [35]. *The set* $\{U_{2,5}, U_{3,5}, F_7, F_7^*\}$ *is both* (2,1)- *and* (3,1)-*rounded*. □

1.6.11 *Theorem* [27, (3.6)]. *The set* $\{U_{3,6}, P_6, Q_6, \mathcal{W}^3\}$ *is* (3,2)-*rounded*. *The set* $\{U_{3,6}, P_6, Q_6, \mathcal{W}^3, M(\mathcal{W}_3)\}$ *is* (3,3)-*rounded*. □

1.7 Observations on Roundedness

Some elementary facts about rounded sets are presented in this section. These facts will be used in our study of roundedness theory which begins in the next chapter.

The following fact is easily checked.

1.7.1 Lemma. *A set* $\{M_1, M_2, \ldots, M_n\}$ *of matroids is (k,m)-rounded if and only if* $\{M_1^*, M_2^*, \ldots, M_n^*\}$ *is (k,m)-rounded.* □

This lemma is frequently used to invoke duality in the subsequent chapters. The next elementary fact will also be useful.

1.7.2 Lemma. *Let* S *be a (k,m)-rounded set of matroids. If M is a k-connected matroid having an* S*-minor, then the set* $S \cup \{M\}$ *is (k,m)-rounded.* □

The lemma below will allow us to conclude that certain rounded sets must contain a matroid which possesses some free èlements. This information will be of particular use in classifying certain rounded sets of matroids in Chapters 2, 3, and 4.

1.7.3 Lemma. *Let S be a (k,m)-rounded set of matroids. Further suppose that S contains a matroid N with rank at least $k-1$. Then S contains a matroid which has at least m free elements. Moreover, if S contains a matroid with corank at least $k-1$, then S contains a matroid which has at least m cofree elements.*

Proof. Let M be the matroid formed by freely adding m elements to M. Then M is k-connected by Lemma 1.2.6. Let A be a set of m free elements in M. Now M has an S-minor using A. This S-minor possesses at least m free elements. The second part of the result follows by applying Lemma 1.7.1. □

Recall that C_n denotes a cycle on n edges. The next corollary suggests that the property of roundedness is not a natural property for the class of binary matroids.

1.7.4 Corollary. *Let k be an integer exceeding two. Suppose S is a (k,m)-rounded set of matroids and some member of S has rank at least $k-1$. Then S contains at least one non-binary matroid.*

Proof. S contains a matroid M which possesses a free element by Lemma 1.7.3. Since S is (k,m)-rounded, M is 3-connected and has at least four elements. Thus M is not isomorphic to $M(C_n)$ for any n. It follows from Lemma 1.5.4 that M is non-binary. □

CHAPTER 2

Roundedness in Binary Matroids

2.1 Introduction

In this chapter we shall concentrate on the classes of binary and graphic matroids. These are natural classes to consider for roundedness as they are both closed under minors. The results on roundedness in binary matroids are used in Chapter 3 in the characterization of the pairs of matroids which form (3,2)-rounded sets. This chapter is the result of joint work with James G. Oxley.

It follows from Corollary 1.7.4 that a set of binary matroids is not (k,m)-rounded for k exceeding two. However, there is an obvious generalization of the property of roundedness to the class of binary matroids, or any other minor-closed class of matroids. Let k and m be positive integers with k exceeding one.

2.1.1 Definition. *Let $\mathcal{F}$ be a minor-closed class of matroids. Suppose $\mathcal{S}$ is a set of k-connected matroids in $\mathcal{F}$ each having at least four elements. The set $\mathcal{S}$ is (k,m)-rounded within the class $\mathcal{F}$ if $\mathcal{S}$ satisfies the following condition.*

(i) *If M is a k-connected matroid in $\mathcal{F}$ having an $\mathcal{S}$-minor and X is a subset of E(M) with at most m elements, then M has an $\mathcal{S}$-minor using X.*

Note that condition 2.1.1(i) is obtained by adding the restriction that M is in $\mathcal{F}$ to condition 1.6.1(i). In this chapter we are only concerned with roundedness within the classes of binary and graphic matroids. The main results of the chapter are now stated.

2.1.2 Theorem. *Let M be a 3-connected binary matroid with at least four elements. The set {M} is (3,2)-rounded within the class of binary matroids if and only if M is isomorphic to $M(\mathcal{W}_3)$ or $M(\mathcal{W}_4)$.*

The methods used in the proof of Theorem 2.1.2 will be adapted to the class of graphic matroids to obtain an analog of this theorem for graphic matroids.

2.1.3 Theorem. *Let M be a 3-connected graphic matroid with at least four elements. The set {M} is (3,2)-rounded within the class of graphic matroids if and only if M is isomorphic to $M(\mathcal{W}_3)$ or $M(\mathcal{W}_4)$.*

The proofs of these theorems are given in Sections 2.2 and 2.4 respectively. An extension of Theorem 2.1.2 to pairs of binary matroids is proved in Section 2.3. This result is stated below. The binary matroid $Z_r \backslash b_r$ is given in Table 2.

2.1.4 Theorem. *Let M and N be 3-connected binary matroids each having at least four elements. The set {M,N} is (3,2)-rounded within the class of binary matroids if and only if either:*

(i) *at least one of M and N is isomorphic to* $M(\mathcal{W}_3)$; *or*

(ii) *at least one of M and N is isomorphic to* $M(\mathcal{W}_4)$ *and the other either has an* $M(\mathcal{W}_4)$*-minor or is isomorphic to* $Z_r \backslash b_r$ *for some r exceeding three.*

The next theorem is the result corresponding to Theorem 2.1.4 for graphic matroids. This result is proved in Section 2.4. The graph P is given below.

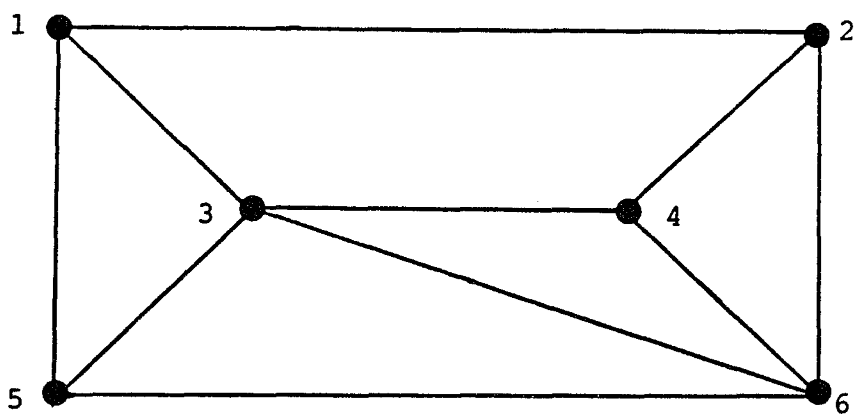


Figure 1

P

2.1.5 Theorem. *Let M and N be 3-connected graphic matroids having at least four elements. The set {M,N} is (3,2)-rounded within the class of graphic matroids if and only if {M,N} is* $\{M(\mathcal{W}_5), M(P)\}$, *or at least one of M and N is isomorphic to* $M(\mathcal{W}_3)$ *or* $M(\mathcal{W}_4)$.

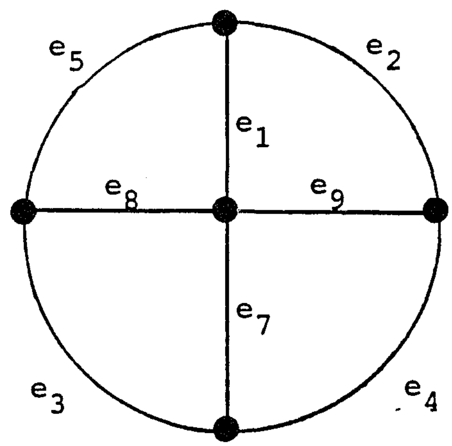
2.2 Binary Rounded Sets

The proof of Theorem 2.1.2 is given in this section. The section begins with results which are used in the proof of this theorem.

Let $P_9 = D(A_3)$ be the matroid on $\{e_1, e_2, \ldots, e_9\}$ given in Table 2. Now $P_9 \backslash e_6$ is isomorphic to $M(\mathcal{W}_4)$ where the latter is labelled as below.

Figure 2

$P_9 \backslash e_6 \cong M(\mathcal{W}_4)$



The next three lemmas will be used to extend Theorem 1.6.9(i).

2.2.1 Lemma. The group of automorphisms of P_9 is transitive on both $\{e_1, e_2, e_5, e_6\}$ and $\{e_8, e_9\}$.

Proof. Let A_3 be the binary matrix representing P_9 that is given in Table 2. In A_3, replace row i by row i + row 2 for i = 3 and 4. Then interchange rows 3 and 4 in the resulting matrix. This gives a matrix which can be transformed into A_3 by a suitable permutation of its columns. These operations induce an automorphism ϕ of A_3 such that $\phi(e_2) = e_5$ and $\phi(e_8) = e_9$. Let ψ be the automorphism of A_3 induced by interchanging rows 1 and 2 of A_3. Evidently $\psi(e_1) = e_2$ and $\psi(e_5) = e_6$. The result follows from considering compositions of these two automorphisms.

Suppose r is an integer exceeding two. The graph H_r illustrated below is referred to several times in the remainder of the chapter.

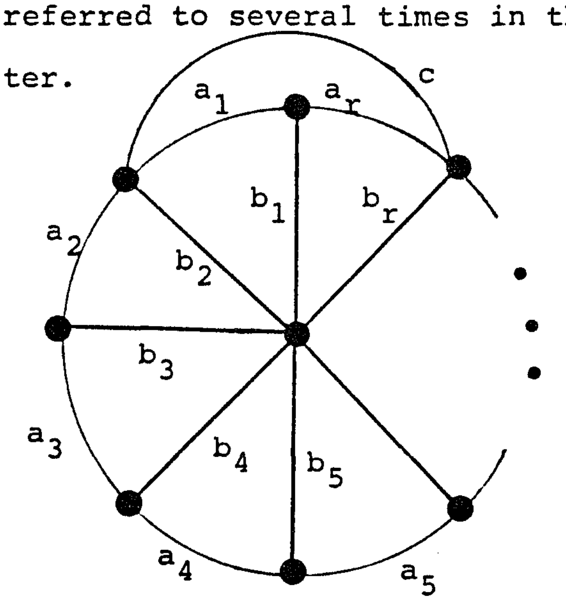


Figure 3

H_r

Evidently H_4 is isomorphic to K_5-a. The graph $H_5 \backslash b_2$ is isomorphic to the graph P given in Figure 1.

2.2.2 <u>Lemma</u>. <u>Let</u> n <u>be an integer exceeding four</u>. <u>Then</u> $M(H_n)$ <u>does not have an</u> $M(W_n)$<u>-minor using</u> c.

<u>Proof</u>. Let G be a graph obtained from H_n by deleting any edge other than c. Then either G has a degree-2 vertex,

or G does not have a degree-n vertex. Thus G is not isomorphic to $\mathcal{W}_n$. The result follows by Theorem 1.2.8. □

Although the next lemma is not explicitly stated in [28], it is not difficult to check that it can be obtained from the proof of Lemma 2.6 of that paper.

2.2.3 Lemma. *Let M be a 3-connected binary extension of $M(\mathcal{W}_4)$. Then M is isomorphic to $P_9, M(K_5-a)$, or $M^*(K_{3,3})$.*

The next result is an extension of Theorem 1.6.9(i).

2.2.4 Lemma. *Let n be an integer exceeding two. The set $\{U_{2,4}, M(\mathcal{W}_n)\}$ is (3,2)-rounded if and only if n is three or four.*

Proof. The set $\{U_{2,4}, M(\mathcal{W}_3)\}$ is (3,2)-rounded by Theorem 1.6.9(i). Let M be a 3-connected binary extension of $M(\mathcal{W}_4)$. Then, by Lemma 2.2.3, M is isomorphic to $P_9, M(K_5-a)$, or $M^*(K_{3,3})$. We show that each pair of elements in M is in an $M(\mathcal{W}_4)$-minor.

By Lemma 2.2.1, if e is in $\{e_1, e_2, e_5, e_6\}$, then $P_9 \backslash e \cong P_9 \backslash e_6 \cong M(\mathcal{W}_4)$. Consider the graph $H_4 \cong K_5-a$ given in Figure 3. The deletion of an edge in $\{b_2, b_4, c\}$ from H_4 produces a $\mathcal{W}_4$-minor. The deletion of any element from $M^*(K_{3,3})$ produces an $M(\mathcal{W}_4)$-minor. It follows from these comments that M has an $M(\mathcal{W}_4)$-minor using any specified pair of elements. Hence, by duality and Theorems 1.6.2 and 1.6.5, the set $\{U_{2,4}, M(\mathcal{W}_4)\}$ is (3,2)-rounded.

Suppose that n exceeds four. Consider the 3-connected graph H_n given in Figure 3. The deletion of the edge c from H_n produces a $\mathcal{W}_n$-minor. However, by Lemma 2.2.2, $M(H_n)$ does not have an $M(\mathcal{W}_n)$-minor using c. Thus $\{U_{2,4}, M(\mathcal{W}_n)\}$ is not (3,1)-rounded. □

The following result is an immediate corollary of Lemma 2.2.4. It is one direction of Theorem 2.1.2.

2.2.5 Corollary. *The set* $\{M(\mathcal{W}_n)\}$ *is* (3,2)-*rounded within the class of binary matroids if and only if* n *is three or four*.

We pause to note a consequence of the above corollary. It contains one direction of Theorem 2.1.3.

2.2.6 Corollary. *The set* $\{M(\mathcal{W}_n)\}$ *is* (3,2)-*rounded within the class of graphic matroids if and only if* n *is three or four*.

Proof. It follows, from Corollary 2.2.5 and the fact that a graphic matroid is also binary, that the sets $\{M(\mathcal{W}_3)\}$ and $\{M(\mathcal{W}_4)\}$ are (3,2)-rounded within the class of graphic matroids. Suppose n exceeds four. Let H_n be the graph given in Figure 3. By Lemma 2.2.2, $M(H_n)$ has no $M(\mathcal{W}_n)$-minor using c. Thus $\{M(\mathcal{W}_n)\}$ is not (3,1)-rounded within the class of graphic matroids. □

We shall use the concept of a chain in a matroid in the proofs of Theorems 2.1.2 through 2.1.5.

2.2.7 Definition. Let $(T_i)_{1,k}$ be a non-empty sequence of subsets of a matroid M. Suppose that, for all i in $\{1,2,\ldots,k-1\}$,

(i) one of T_i and T_{i+1} is a triangle and the other is a triad;

(ii) $|T_i \cap T_{i+1}| = 2$; and

(iii) $(T_{i+1} - T_i) \cap (T_1 \cup T_2 \cup \ldots \cup T_i)$ is empty.

Then we shall call $(T_i)_{1,k}$ a chain of M of length k.

Evidently $(T_i)_{1,k}$ is a chain of M if and only if it is a chain of M*. The following observations concerning chains in a 3-connected binary matroid are used in the proofs of Theorems 2.1.2 through 2.1.5.

Let N be a 3-connected binary matroid with at least six elements. Let $r = \text{rk}N$. Evidently we may identify N with the restriction to some set S of the matroid induced on $V(r,2)$. Let $(T_i)_{1,k}$ be a chain of N and suppose that T_k is a triad of N. By (2.2.7) (ii) and (iii), $(T_i)_{1,k}$ has $k + 2$ distinct elements. Order these elements so that, for each i in $\{1,2,\ldots,k\}$, $T_i = \{a_i, a_{i+1}, a_{i+2}\}$. Take a_{k+3} to be the element $a_{k+1} + a_{k+2}$ of $V(r,2)$. Let $T_{k+1} = \{a_{k+1}, a_{k+2}, a_{k+3}\}$. The next three lemmas are used in the proofs of Theorems 2.1.2 through 2.1.5.

2.2.8 Lemma. Suppose a_{k+3} is not in S. Let M be the restriction $V(r,2)|(S \cup a_{k+3})$. The following are true.

(1) $(T_i)_{1,k+1}$ is a chain of M.

(2) Let N_1 be a 3-connected single-element deletion or contraction of M which uses a_1 and a_{k+3}. Then $(T_i)_{1,k+1}$ is a chain of N_1.

(3) Suppose that $M\backslash\{f,g\}$ is 3-connected for some elements f and g of $E(M) - \{a_1, a_{k+3}\}$. Then $(T_i)_{1,k+1}$ is a chain of $M\backslash\{f,g\}$.

(4) Suppose that $M\backslash f/g$ is 3-connected for some elements f and g of $E(M) - \{a_1, a_{k+3}\}$.
Then $M\backslash f/g$ has a chain of length at least k.

Proof of (2.2.8)(1). Suppose T_i is a triad of N for some i in $\{1,2,\ldots,k\}$. Then T_i or $T_i \cup a_{k+3}$ is a cocircuit of M. Suppose the latter and assume that $i < k$. Since T_k is a triad, $i \leq k - 2$. Hence $T_i \cup \{a_{k+3}\}$ meets the triangle T_{k+1} in one element in N. This contradicts orthogonality. Thus $i = k$ and $T_i \cup \{a_{k+3}\}$ meets T_{k+1} in three elements. This contradicts Theorem 1.4.1(2). It follows that T_i is a triad of M. Hence $(T_i)_{1,k+1}$ is also a chain of M. □

Proof of (2.2.8)(2) and (3). Each element of $(T_1 \cup T_2 \cup \ldots \cup T_k) - \{a_1\}$ is in both a triangle and a triad of M by (2.2.8)(1). By Lemma 1.2.5, N_1, N_1^*, and the dual of $M\backslash\{f,g\}$ are simple. From using these facts, both (2.2.8)(2) and (2.2.8)(3) follow. □

Proof of (2.2.8)(4). Both $M\backslash f/g$ and its dual are simple by Lemma 1.2.5. It follows that there is a chain of $M\backslash f/g$ of length at least k whose elements are in $T_1 \cup T_2 \cup \ldots \cup T_{k+1}$. □

Now take $(T_i)_{1,k}$ to be a maximum-length chain of N.

2.2.9 Lemma. Suppose a_{k+3} is in S. Then N is a wheel-matroid.

Proof. Since T_{k+1} is a triangle of N and $(T_i)_{1,k}$ is a maximum-length chain, a_{k+3} is in $T_1 \cup T_2 \cup \ldots \cup T_k$.

Every element of $(T_1 \cup T_2 \cup \ldots \cup T_{k-2}) - \{a_1\}$ is in a triad of N which does not contain a_{k+1} or a_{k+2}. Thus, by orthogonality, a_{k+3} is not in $(T_1 \cup T_2 \cup \ldots \cup T_{k-2}) - \{a_1\}$ $= \{a_2, a_3, \ldots, a_k\}$. Since a_{k+3} is clearly not a_{k+1} or a_{k+2}, we conclude as a_{k+3} is in $\{a_1, a_2, \ldots, a_{k+2}\}$, that $a_{k+3} = a_1$. Moreover, T_1 is a triangle of N and k is even.

Now let $A = \{a_1, a_2, \ldots, a_{k+2}\}$. Then A is spanned in N and N* by $\{a_1, a_3, a_5, \ldots, a_{k+1}\}$ and $\{a_2, a_4, a_6, \ldots, a_{k+2}\} \cup \{a_1\}$, respectively. Thus

$$rk_N A + rk_{N^*} A - |A| \leq 1.$$

Rewriting the left hand side here, we have

$$rk_N A + rk_N (E(N) - A) - rkN \leq 1.$$

Therefore, as N is 3-connected, $E(N) - A$ has at most one element and so

(2.2.10) $\quad rkN = rk_N A \leq (k/2) + 1.$

Now, for each j in $\{1,2,\ldots,k/2\}$, T_{2j} is a triad in N. The intersection of the complements of these k/2 triads is a flat F such that

(2.2.11) $rk_N F \leq rk\, N - (k/2)$.

As a_1 is in F, $rk_N F \geq 1$. Combining this with (2.2.10) and (2.2.11), we deduce that

$$rk\, N = (k/2) + 1 \text{ and } rk_N F = 1.$$

Therefore F has exactly one element. As E(N)-A is contained in $F-\{a_1\}$, it follows that E(N)-A is empty, that is, A = E(N). Finally, we note that the closure of $\{a_3,a_5,a_7,\ldots,a_{k+1}\}$ is a hyperplane of N whose complement is $\{a_1,a_2,a_{k+2}\}$. Hence $\{a_1,a_2,a_{k+2}\}$ is a triad of N. Thus every element of the 3-connected matroid N is in both a triangle and a triad and so, by Theorem 1.2.1, Tutte's wheels and whirls theorem, N is a wheel-matroid.□

2.2.12 *Lemma*. *Let* M_1 *and* M_2 *be* 3-*connected binary matroids each having at least six elements such that* $|E(M_1)| = |E(M_2)|$. *Suppose that, whenever* e *is an element of a* 3-*connected binary matroid* M_3 *which is an extension of* M_1 *or* M_2, M_3 *has an* M_1- *or* M_2-*minor using* e. *Then either* M_1 *or* M_2 *has a triangle*.

Proof. Let $C = \{c_1,c_2,\ldots,c_j\}$ be a circuit of minimum size among all the circuits of M_1 and M_2 and suppose that j exceeds three. Suppose, without loss of generality, that C is a subset of $E(M_1)$. Let $r = rk\, M_1$ and identify

M_1 with the restriction to some set S of the matroid induced on V(r,2). Let e denote the element $c_1 + c_2$ of V(r,2). Evidently e is not in S. Let $M_1 + e$ denote the restriction $V(r,2)|(S \cup e)$. Both $\{c_1,c_2,e\}$ and $\{c_3,c_4,\ldots,c_j,e\}$ are circuits of M_1+e, and M_1+e has an M_1- or M_2-minor using e. Thus M_1 or M_2 contains a circuit of size less than j; a contradiction. □

The last lemma will often be applied in the special case that $M_1 = M_2$. We now begin the proof of the main result of the chapter.

<u>Proof of Theorem 2.1.2.</u> By Corollary 2.2.5, both $\{M(\mathcal{W}_3)\}$ and $\{M(\mathcal{W}_4)\}$ are (3,2)-rounded within the class of binary matroids. For the converse, suppose that N is a 3-connected binary matroid such that the set {N} is (3,2)-rounded within the class of binary matroids. Let r = rkN and identify N with the restriction to some set S of V(r,2).

We conclude from Lemma 2.2.12 that N has a triangle and hence N has a chain. Let $(T_i)_{1,k}$ be a chain of N of maximum length where, for each i in {1,2,...,k}, T_i is $\{a_i,a_{i+1},a_{i+2}\}$. T_k is a triad of N or N*. Without loss of generality suppose the former.

Take a_{k+3} to be the element $a_{k+1} + a_{k+2}$ of V(r,2). Let $T_{k+1} = \{a_{k+1},a_{k+2},a_{k+3}\}$. Suppose a_{k+3} is not in S. Let M be the restriction $V(r,2)|(S \cup a_{k+3})$. By Lemma 1.3.1, M is 3-connected. Thus M has an N-minor using both a_1 and a_{k+3}. By Lemma 2.2.8(2), $(T_i)_{1,k+1}$ is a chain of this N-minor. Hence, N has a chain of length k+1; a contradiction. Thus a_{k+3} is in S. It follows from Lemma 2.2.9 that N is a wheel-matroid. Since the set $\{U_{2,4},N\}$ is (3,2)-rounded, the result follows by Lemma 2.2.4. □

2.3 Applications

Several consequences of the proof of Theorem 2.1.2 are noted in this section. Theorem 2.1.4 will follow immediately from the next result, the main result of the section. The matroid $Z_r \backslash b_r$ is given in Table 2.

2.3.1 Theorem. *Let M and N be 3-connected matroids with at least four elements. The set $\{U_{2,4}, M, N\}$ is (3,2)-rounded if and only if either:*

(i) *both M and N are non-binary; or*

(ii) *at least one of M and N is isomorphic to $M(\mathcal{W}_3)$; or*

(iii) *at least one of M and N is isomorphic to $M(\mathcal{W}_4)$ and the other is either non-binary, has an $M(\mathcal{W}_4)$-minor, or is isomorphic to $Z_r \backslash b_r$ for some r exceeding three.*

The proof of this theorem is given at the end of the section. We will first consider some special cases of this result.

2.3.2 Lemma. *Let N be a 3-connected matroid with at least four elements. Then the set $\{U_{2,4}, M(\mathcal{W}_3), N\}$ is (3,2)-rounded.*

Proof. By Theorem 1.2.2, N must have a $U_{2,4}$- or $M(\mathcal{W}_3)$-minor. The lemma follows by Theorem 1.6.9(i) and Lemma 1.7.2. □

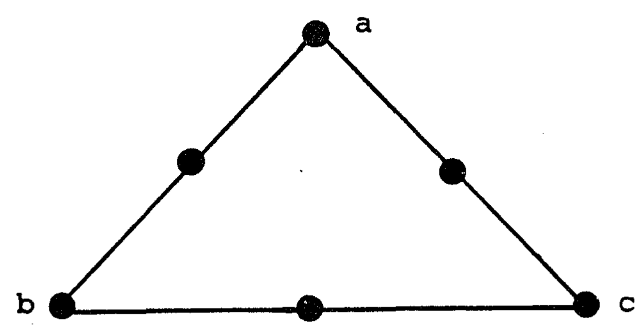
Lemma 2.3.2 and the next result will be used in Theorem 2.3.4 to characterize certain (3,3)-rounded collections containing $U_{2,4}$ and $M(\mathcal{W}_3)$. We shall then continue with results used in the proof of Theorem 2.3.1.

The following result is an immediate consequence of Theorem 1.6.11.

2.3.3 Theorem. The set $\{M(\mathcal{W}_3)\}$ is (3,3)-rounded within the class of binary matroids.

A Euclidean representation for the rank-3 whirl is given below.

Figure 4 $\mathcal{W}^3$



We next give an analog of Theorem 2.3.1(ii) for (3,3)-roundedness.

2.3.4 Theorem. Let N be a 3-connected matroid with at least four elements. Then the set $\{U_{2,4}, M(\mathcal{W}_3), N\}$ is (3,3)-rounded if and only if N is isomorphic to $\mathcal{W}^3$.

Proof. The fact that $\{U_{2,4}, M(\mathcal{W}_3), \mathcal{W}^3\}$ is (3,3)-rounded follows immediately from Theorems 1.6.7 and 2.3.3. For the converse, suppose that $\{U_{2,4}, M(\mathcal{W}_3), N\}$ is (3,3)-rounded. Let a,b, and c be the elements of $\mathcal{W}^3$ marked in Figure 4.

W^3 does not have a 3-connected proper minor that both uses {a,b,c} and has at least four elements. Thus N is isomorphic to W^3. □

Results similar to Theorems 2.3.3 and 2.3.4 with the rank-4 wheel replacing the rank-3 wheel are given next. We shall use the following decomposition theorem in the proof of these results. The binary matroid Z_r is given in Table 2.

2.3.5 <u>Theorem</u> [28,(2.1)]. <u>Let</u> M <u>be</u> <u>a</u> 3-<u>connected</u> <u>binary</u> <u>matroid</u> <u>with</u> <u>at</u> <u>least</u> <u>four</u> <u>elements</u>. <u>Then</u> M <u>has</u> <u>no</u> $M(W_4)$-<u>minor</u> <u>if</u> <u>and</u> <u>only</u> <u>if</u> M <u>is</u> <u>isomorphic</u> <u>to</u> <u>one</u> <u>of</u> <u>the</u> <u>following</u>:

(i) Z_r, Z_r^*, $Z_r\backslash b_r$, <u>or</u> $Z_r\backslash c_r$ <u>for</u> <u>some</u> r <u>exceeding</u> <u>three</u>; <u>or</u>

(ii) F_7, F_7^*, <u>or</u> $M(W_3)$. □

Let A_r be the binary matrix which represents Z_r and is given in Table 2.

2.3.6 <u>Lemma</u>. <u>Let</u> r <u>be</u> <u>an</u> <u>integer</u> <u>exceeding</u> <u>three</u>. <u>Then</u> <u>the</u> <u>set</u> $\{U_{2,4}, M(W_4), Z_r\backslash b_r\}$ <u>is</u> (3,2)-<u>rounded</u>.

<u>Proof</u>. Let M be a 3-connected binary extension or lift of $Z_r\backslash b_r$, and e and f be elements of E(M). If M has an $M(W_4)$-minor, then, by Lemma 2.2.4, M has such a minor using both e and f. Suppose that M does not have an $M(W_4)$-minor.

It follows from Theorem 2.3.5, and the fact that M has 2r elements, that M is isomorphic to Z_r or Z_r^*.

Oxley showed that the group of automorphisms of Z_r is transitive on the columns $\{a_1,a_2,\ldots,a_r,b_1,b_2,\ldots,b_r\}$ of $A_r[28,(2,3)]$. Thus $Z_r\backslash x$ is isomorphic to $Z_r\backslash b_r$ for each x in $\{a_1,a_2,\ldots,a_r,b_1,b_2,\ldots,b_r\}$. Hence, if M is isomorphic to Z_r, then there is a $(Z_r\backslash b_r)$-minor of M using both e and f. Moreover, as $Z_r\backslash b_r$ is self-dual, if M is isomorphic to Z_r^*, then M has a $(Z_r\backslash b_r)$-minor using both e and f. The result follows by Lemmas 1.6.2 and 2.2.4. □

We are now ready to prove an analog of Theorem 2.3.2. This result is used in the proof of Theorem 2.3.1.

2.3.7 Theorem. *Let N be a 3-connected matroid with at least four elements. The set $\{U_{2,4},M(\mathcal{W}_4),N\}$ is (3,2)-rounded if and only if either:*

(i) *N is non-binary; or*

(ii) *N is binary and has an $M(\mathcal{W}_4)$-minor; or*

(iii) *N is isomorphic to $M(\mathcal{W}_3)$ or $Z_r\backslash b_r$ for some integer r exceeding three.*

Proof. If N is listed in (i), (ii), or (iii), then, by Lemmas 1.7.2, 2.2.4, and 2.3.6, $\{U_{2,4},M(\mathcal{W}_4),N\}$ is (3,2)-rounded. For the converse, suppose that N is binary, has no $M(\mathcal{W}_4)$-minor, and is not isomorphic to $M(\mathcal{W}_3)$ or

$Z_r \backslash b_r$. It follows from Theorem 2.3.5 that N is isomorphic to F_7, F_7^*, Z_r, Z_r^*, or $Z_r \backslash c_r$. To complete the proof we will show that the set $\{U_{2,4}, M(\mathcal{W}_4), N\}$ is not (3,1)-rounded.

Consider the Euclidean representation for the matroid S_8 given in Table 1. The element e_4 is the only element of S_8 whose contraction produces a Fano-minor. Thus S_8 has no F_7-minor using e_4. Hence $\{U_{2,4}, M(\mathcal{W}_4), F_7\}$ is not (3,1)-rounded.

If x is an element of Z_r other than c_r, then, by counting triangles, we see that $Z_r \backslash x$ is not isomorphic to $Z_r \backslash c_r$. Hence Z_r has no $(Z_r \backslash c_r)$-minor which uses c_r. Also, by Theorem 2.3.5, Z_r has no $M(\mathcal{W}_4)$-minor. It follows that the set $\{U_{2,4}, M(\mathcal{W}_4), Z_r \backslash c_r\}$ is not (3,1)-rounded.

$Z_{r+1} \backslash b_{r+1}, c_{r+1}$ is isomorphic to Z_r^*[28,Sect. 2]. If x and y are elements of Z_{r+1} other than c_{r+1}, then it is easily checked that $Z_{r+1} \backslash x,y$ has a triangle. Thus $Z_{r+1} \backslash x,y$ cannot be isomorphic to Z_r^* since the latter has no triangles. Hence Z_{r+1} has no Z_r^*-minor using c_{r+1}. We have shown that if N is isomorphic to F_7, Z_r^*, or $Z_r \backslash c_r$, then the set $\{U_{2,4}, M(\mathcal{W}_4), N\}$ is not (3,1)-rounded. The result follows by duality. □

The preceding theorem states that there are many matroids N for which the set $\{U_{2,4}, M(\mathcal{W}_4), N\}$ is (3,2)-rounded. The next theorem shows that quite a different result is true for (3,3)-rounded sets of this type.

2.3.8 Theorem. Let N be a 3-connected matroid with at least four elements. Then the set $\{U_{2,4}, M(\mathcal{W}_4), N\}$ is not (3,3)-rounded.

Proof. Assume the contrary. Let a, b, and c be the elements of $\mathcal{W}^3$ marked in Figure 4. Since $\mathcal{W}^3$ has no $U_{2,4}$-minor using {a,b,c}, N is isosomorphic to $\mathcal{W}^3$. The graph H_4 of Figure 3 has a $\mathcal{W}_4$-minor, but does not have such a minor using a_1, a_r, and c. Since $M(H_4)$ is binary, it has neither a $U_{2,4}$-minor nor a $\mathcal{W}^3$-minor. Hence, $\{U_{2,4}, M(\mathcal{W}_4), N\}$ is not (3,3)-rounded; a contradiction. □

We next give some technical lemmas before proving Theorem 2.3.1. Let $\mathcal{F}$ be a minor-closed class of matroids. In the next lemma, Seymour's quick test for (3,2)-roundedness is adapted to test a set of matroids for the property of being (3,2)-rounded within the class $\mathcal{F}$.

2.3.9 Lemma. Let $\mathcal{S}$ be a set of 3-connected matroids in $\mathcal{F}$ each having at least four elements. The set $\mathcal{S}$ is (3,2)-rounded within the class $\mathcal{F}$ if and only if $\mathcal{S}$ satisfies the following condition.

(i) If M is a 3-connected member of $\mathcal{F}$ which is an extension or lift of a member of $\mathcal{S}$, and X is a subset of E(M) with at most two elements, then M has an $\mathcal{S}$-minor using X.

Proof. Note that condition (2.3.9)(i) is obtained by adding the restriction that M is in $\mathcal{F}$ to condition (1.6.2)(i).

The class $\mathcal{F}$ is closed under minors. Hence, we may prove this result by modifying the proof of Theorem 1.6.2 given in [37] by requiring that each matroid in the proof be in $\mathcal{F}$. □

We require three more lemmas before beginning the proof of Theorem 2.3.1.

For each integer r exceeding four, let G_r be the 3-connected graph with 2r + 1 edges given below.

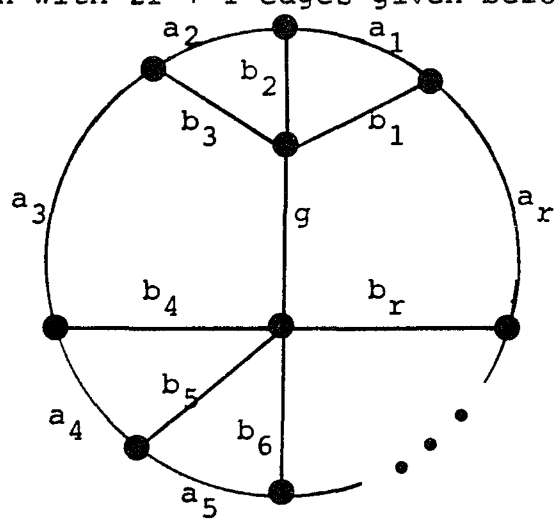


Figure 5

G_r

Evidently G_r/g is isomorphic to $\mathcal{W}_r$.

2.3.10 <u>Lemma</u>. <u>Let</u> n <u>be an integer exceeding four</u>. <u>Then</u> $M(G_n)$ <u>does not have an</u> $M(\mathcal{W}_n)$<u>-minor using</u> g.

<u>Proof</u>. Each element of $M(G_n)$ other than a_2, a_n, and g is in a triangle. Thus, the only simple single-element contractions of G_n are G_n/a_2, G_n/a_n, and $G_n/g \cong \mathcal{W}_n$. Neither G_n/a_2 nor G_n/a_n possesses a vertex of degree n. Hence, neither is isomorphic to $\mathcal{W}_n$. It follows that G_n has no $\mathcal{W}_n$-minor using g. □

The graph H_n is given in Figure 3.

2.3.11 Lemma. *Let n be an integer exceeding four. The set $\{M(\mathcal{W}_n), M(H_n)\}$ is not (3,1)-rounded within the class of graphic matroids.*

Proof. $M(G_n)$ has an $M(\mathcal{W}_n)$-minor as $G_n/g \cong \mathcal{W}_n$. By Lemma 2.3.10, $M(G_n)$ has no $M(\mathcal{W}_n)$-minor using g. The matroids $M(G_n)$ and $M(H_n)$ have the same number of elements, but different ranks, and hence are not isomorphic. Thus $M(G_n)$ has no minor in $\{M(\mathcal{W}_n), M(H_n)\}$ which uses g. □

The binary matrix F_r which represents $M(H_r)$ is given below.

Figure 6

F_r

$$
\begin{array}{c|cccccccc}
b_1\ b_2\ \dots\ b_r & a_1 & a_2 & a_3 & \dots & a_{r-2} & a_{r-1} & a_r & c \\
 & 1 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\
 & 1 & 1 & 0 & \dots & 0 & 0 & 0 & 1 \\
 & 0 & 1 & 1 & \dots & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\
I_r & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & \dots & 1 & 1 & 0 & 0 \\
 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1
\end{array}
$$

2.3.12 Lemma. *Let n be an integer exceeding four. The set $\{M(\mathcal{W}_n), M(H_n \backslash b_2)\}$ is not (3,2)-rounded within the class of binary matroids.*

Proof. Let e be the vector in V(n,2) with a one in each position. Let $F_n \backslash b_2$ be the binary matrix which represents $M(H_n \backslash b_2)$ and is given in Figure 6. Suppose B is the binary matrix formed by adjoining the column vector e to $F_n \backslash b_2$. By Lemma 1.3.1, D(B) is 3-connected. Neither a_2 nor e is in a triangle of D(B). Hence, any single-element deletion of D(B) which uses a_2 and e has at least two elements which are not in a triangle. It follows that D(B) has no $M(\mathcal{W}_n)$- or $M(H_n \backslash b_2)$ -minor which uses a_2 and e. □

We are now ready to prove the main result of the section.

Proof of Theorem 2.3.1. Suppose that the set $\{U_{2,4}, M, N\}$ is of the form given in (i), (ii), or (iii) of Theorem 2.3.1. It follows immediately from Theorems 1.6.5 and 2.3.7 and Lemmas 1.7.2 and 2.3.2 that $\{U_{2,4}, M, N\}$ is (3,2)-rounded.

For the converse, suppose that $\{U_{2,4}, M, N\}$ is a (3,2)-rounded set which is not listed in (i), (ii), or (iii) of Theorem 2.3.1. Then, as M and N are 3-connected and binary, M and N must have at least six elements.

If either of M and N is isomorphic to $M(\mathcal{W}_3)$ or $M(\mathcal{W}_4)$, then, by Theorem 2.3.7, the set $\{U_{2,4}, M, N\}$ is of the form listed in (ii) or (iii) of Theorem 2.3.1; a contradiction. It follows that

(2.3.13) <u>neither</u> M <u>nor</u> N <u>is</u> <u>isomorphic</u> <u>to</u> $M(\mathcal{W}_3)$ <u>or</u> $M(\mathcal{W}_4)$.

We show in the next three lemmas that at least one of M and N must be a wheel-matroid.

2.3.14 <u>Lemma</u>. $\big||E(M)| - |E(N)|\big| \leq 1$. <u>Moreover</u>, <u>if</u> $\big||E(M)| - |E(N)|\big| = 1$, <u>then</u> M <u>or</u> N <u>has</u> <u>a</u> <u>minor</u> <u>isomorphic</u> <u>to</u> <u>the</u> <u>other</u>.

<u>Proof</u>. Suppose that $|E(M)| \leq |E(N)| - 2$. It follows from Lemma 2.3.9 that $\{M\}$ is (3,2)-rounded within the class of binary matroids. Thus, by Theorem 2.1.2, M is the wheel of rank three or four. This contradicts (2.3.13). Thus
$|E(M)| \not\leq |E(N)| - 2$, and likewise,
$|E(N)| \not\leq |E(M)| - 2$. Hence $\big||E(M)| - |E(N)|\big| \leq 1$.
The second part of the lemma follows by a similar argument. □

2.3.15 <u>Lemma</u>. <u>Suppose</u> $|E(M)| = |E(N)|$. <u>Then</u> <u>either</u> M <u>or</u> N <u>is</u> <u>a</u> <u>wheel-matroid</u>.

<u>Proof</u>. $\{M,N\}$ is (3,1)-rounded within the class of binary matroids. By Lemma 2.2.12, M or N possesses a triangle

and hence a chain. Let $(T_i)_{1,k}$ be a chain of maximum length among all the chains of M and N. From following the proof of Theorem 2.1.2 we obtain that M or N is a wheel-matroid. □

2.3.16 Lemma. *Suppose* $||E(M)| - |E(N)|| = 1$. *Then either* M *or* N *is a wheel-matroid*.

Proof. Assume the contrary. Suppose, without loss of generality, that $|E(N)| < |E(M)|$. By Lemma 2.3.14, N has an extension or lift which is isomorphic to M. By duality, we may assume, without loss of generality, that there is an element e of E(M) such that $M\backslash e = N$.

Let $r = \text{rk } N$ and identify N with the restriction to some set S of V(r,2). Since $M^*/e = N^*$, it follows from Lemma 2.2.12 that N^*, and hence N, possesses a chain. Let $(T_i)_{1,k}$ be a maximum-length chain of N. It follows from applying Lemmas 2.2.8(2) and 2.2.9 to N^* that neither T_1 nor T_k is a triad of N^*. Hence

(2.3.17) *both* T_1 *and* T_k *are triads of* N.

We next show that M has a chain. Order the elements of the chain $(T_i)_{1,k}$ of N so that $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ for each i in $\{1,2,\ldots,k\}$. Let a_{k+3} be the element $a_{k+1} + a_{k+2}$ of V(r,2). By Lemma 2.2.9, a_{k+3} is not in S. Let $N + a_{k+3}$ denote the matroid $V(r,2)|(S \cup a_{k+3})$. By Lemma 2.2.8(2), $N + a_{k+3}$ has no N-minor using a_1 and a_{k+3}. Thus $N + a_{k+3}$ is isomorphic to M. We have shown that

(2.3.18) M <u>has a chain of length at least</u> $k + 1$.

Let $(R_i)_{1,m}$ be a chain of M of maximum length. By (2.3.18), $m \geq k + 1$. Order the elements of the chain so that $R_i = \{c_i, c_{i+1}, c_{i+2}\}$ for each i in $\{1,2,\ldots,m\}$. Since $M \backslash e = N$ and $m \geq k + 1$, e must be in $R_1 \cup R_2 \cup \ldots \cup R_m$. Since N is 3-connected, e is either c_1 or c_{m+2}. Hence, either $(R_i)_{2,m}$ or $(R_i)_{1,m-1}$ is a chain of N. It follows that $m = k + 1$. By (2.3.17), R_1 or R_m is a triad of N. Since M is a 3-connected binary matroid we obtain:

(2.3.19) <u>Either</u> R_1 <u>or</u> R_m <u>is a triad of</u> M.

It follows from Lemmas 2.2.8 and 2.2.9 and (2.3.19) that M or N has a chain of length m+1; a contradiction. This completes the proof of Theorem 2.3.16. □

It follows from Lemmas 2.3.13 through 2.3.16 that

(2.3.20) <u>either</u> M <u>or</u> N <u>is isomorphic to</u> $M(\mathcal{W}_r)$ <u>for some</u> r <u>exceeding four</u>.

Suppose, without loss of generality, that M is isomorphic to $M(\mathcal{W}_r)$ for some r exceeding four. We require two more lemmas before completeing the proof of Theorem 2.3.1. The graph H_r is given in Figure 3.

2.3.21 <u>Lemma</u>. N <u>is isomorphic to</u> $M(H_r)$, $M(H_{r-1})$, $M(H_r) \backslash b_2$, <u>or</u> $M(H_r) \backslash b_2, b_r$.

Proof. By Lemmas 2.2.2 and 2.3.14, N is isomorphic to $M(H_r)$, or to some (2r-1)- or (2r)-element minor of $M(H_r)$ which uses c. Suppose N is a proper minor of $M(H_r)$. Let x be an edge of H_r other than c. The simplification of $M(H_r)/x$ has at least 2r - 1 elements if and only if x is in $\{a_2,a_3,\ldots,a_{r-1}\}$. The cosimplification of $M(H_r)\backslash x$ has at least 2r - 1 elements if and only if x is in $\{b_2,b_3,\ldots,b_r\}$. The lemma follows from these facts. □

2.3.22 Lemma. N is not isomorphic to $M(H_{r-1})$ or $M(H_r)\backslash b_2,b_r$.

Proof. As $M = M(\mathcal{W}_r)$, the only 3-connected minors of M with at least four elements are wheel-matroids. Thus, M has neither $M(H_{r-1})$ nor $M(H_r)\backslash b_2,b_r$ as a minor. The result follows from Lemma 2.3.14. □

We now complete the proof of Theorem 2.3.1. It follows from Lemmas 2.3.21 and 2.3.22 that N is isomorphic to either $M(H_r)$ or $M(H_r\backslash b_2)$. Thus {M,N} is either $\{M(\mathcal{W}_r),M(H_r)\}$ or $\{M(\mathcal{W}_r),M(H_r\backslash b_2)\}$. By Lemmas 2.3.11 and 2.3.12, {M,N} is not (3,2)-rounded within the class of binary matroids. This contradiction completes the proof of Theorem 2.3.1. Note that Theorem 2.1.4 is an immediate consequence of Theorems 1.6.5 and 2.3.1. □

2.4 Roundedness in Graphic Matroids

In this section we shall adapt the methods used in Sections 2.2 and 2.3 to the class of graphic matroids. Proofs will be given for Theorems 2.1.3 and 2.1.5. We first give some graph terminology which is used in these proofs.

Let G be a loopless graph with at least four vertices. Let w_1 and w_2 be vertices of G. Then (w_1,w_2) will denote the edge of the complete graph on $|V(G)|$ vertices which contains G as a subgraph. Suppose w_1 and w_2 are not adjacent in G and $e = (w_1,w_2)$. Then G + e denotes the graph with edge set $E(G) \cup \{e\}$ formed by adding e to G[5,p.9].

Let v be a vertex of G. Then $d_G(v)$ denotes the degree of v in G. Suppose that $d_G(v)$ exceeds three. Let H be a graph constructed from G as follows. Replace v by two new vertices v_1 and v_2 that are joined by a new edge e. Every edge of G that was incident with v is incident with exactly one of v_1 and v_2 in H so that both v_1 and v_2 have degree at least three. The rest of G is left unchanged. Then we say that H has been obtained from G by splitting v. Evidently H/e = G and H is a lift of G. We will let $G(v,e)$ denote the set of all graphs obtained from G by splitting the vertex v into two new vertices v_1 and v_2 joined by e. The following result of Tutte [44] will be used in the proofs of Theorems 2.1.3 and 2.1.5.

2.4.1 Lemma. *Let G be a simple 3-connected graph and suppose H is a lift of G. The following are equivalent.*

(i) *H is simple and 3-connected.*

(ii) *H is obtained from G by splitting a vertex of degree at least four.* □

Theorems 2.1.3 and 2.1.5 are the graphic analogs of Theorems 2.1.2 and 2.1.4, respectively. However, the class of graphic matroids is not closed under duality. Thus, duality cannot be invoked in the proofs of Theorems 2.1.3 and 2.1.5. It follows that the proofs of these theorems are somewhat longer than the proofs of the corresponding binary results given in the last section.

We next give some technical lemmas used in the proofs of Theorems 2.1.3 and 2.1.5. Let H_1 and H_2 be 3-connected simple graphs with at least four vertices. Identify the elements of $M(H_1)$ and $M(H_2)$ with the edges of H_1 and H_2, respectively. Let $(T_i)_{1,k}$ be a chain of maximum length among all the chains of H_1 and H_2. Let H be the member of $\{H_1,H_2\}$ containing $(T_i)_{1,k}$. Order the elements of $(T_i)_{1,k}$ so that $T_i = \{a_i,a_{i+1},a_{i+2}\}$ for each i in $\{1,2,\ldots,k\}$. Suppose T_k is a triad of M(H). We can apply Lemma 2.2.8 to the class of graphic matroids if and only if a_{k+1} and a_{k+2} are incident with a common vertex of H. We next investigate when this occurs.

2.4.2 Lemma. *Suppose* k *exceeds one. Then each triad* T *of* $(T_i)_{1,k}$ *is a set of edges incident with a vertex of* H *of degree three.*

Proof. Let T = {e,f,g}. T meets some triangle of H in two elements. Suppose, without loss of generality, that e and f are in a triangle of H. Let v be the vertex of H incident with both e and f. Suppose g is not incident with v. Let w be an endvertex of g. Then {v,w} is a vertex cut of H; a contradiction. Thus g is incident with v. If $d_H(v) > 3$, then H - {e,f,g} is connected; a contradiction. □

The following assumption will be made throughout the section whenever H_1 or H_2 has a vertex of degree three.

(2.4.3) *Both* a_{k+1} *and* a_{k+2} *are incident with a common vertex.*

If k exceeds one, then, by Lemma 2.4.2, (2.4.3) must hold. If K is one, then choose T_1 to be a set of edges incident with a vertex of degree three. It follows from (2.4.3) that if T_k is a triad and H_1 or H_2 possesses a vertex of degree three, then we may apply Lemma 2.2.8 to the graph H and chain $(T_i)_{1,k}$.

Suppose T_k is a triangle. We next give an analog of Lemma 2.2.8 for this case. We require to following lemma to prove this analog. Let v be the vertex of H incident with a_{k+1} and a_{k+2}.

2.4.4 Lemma. If H is not a wheel, then $d_H(v) > 3$.

Proof. Suppose $d_H(v) = 3$. Let e be the edge of H incident with v other than a_{k+1} and a_{k+2}. Since $(T_i)_{1,k}$ is a maximum-length chain, e is in $T_1 \cup T_2 \cup \ldots \cup T_k$. By orthogonality, $e = a_1$. It is now easily checked using Lemma 2.4.2 that H is a wheel. □

Let G be the graph obtained from H by splitting v into vertices v_1 and v_2 joined by a_{k+3} so that $d_G(v_1) = 3$ and a_{k+1}, a_{k+2}, and a_{k+3} are incident with v_1. Let $T_{k+1} = \{a_{k+1}, a_{k+2}, a_{k+3}\}$. The next lemma is the dual of Lemma 2.2.8.

2.4.5 Lemma. The following are true.

(1) Let G_1 be a 3-connected simple single-edge deletion or contraction of G using a_1 and a_{k+3}. Then $(T_i)_{1,k+1}$ is a chain of G_1.

(2) Suppose that $G\backslash f/g$ is 3-connected and simple for some edges f and g of G other than a_1 and a_{k+3}. Then $G\backslash f/g$ has a chain of length at least k.

(3) Suppose that $G/f,g$ is 3-connected and simple for some edges f and g of G other than a_1 and a_{k+3}. Then $(T_i)_{1,k+1}$ is a chain of $G/f,g$. □

We require one more lemma before beginning the proof of Theorem 2.1.3. Let v be a vertex of minimum degree among all the vertices of H_1 and H_2. Suppose, without loss of generality, that v is a vertex of H_1.

2.4.6 Lemma. *Suppose* $d_{H_1}(v) > 3$ *and* $|E(H_1)| = |E(H_2)|$. *Let* G *be a graph in* $H_1(v,e)$. *Then* G *has neither an* H_1*-minor nor an* H_2*-minor using* e.

Proof. Let f be an edge of G other than e. Since f is not incident with both v_1 and v_2 in G, the degree of at least one of v_1 and v_2 is unchanged by deleting or contracting f from G. Thus $G\backslash f$ and G/f both possess a vertex of degree less than $d_{H_1}(v)$. Hence, neither $G\backslash f$ nor G/f is isomorphic to H_1 or H_2. □

We are now ready to prove the graphic analog of Theorem 2.1.2.

Proof of Theorem 2.1.3. As a graphic matroid is necessarily binary, it follows from Lemma 2.2.4 that $\{M(W_3)\}$ and $\{M(W_4)\}$ are (3,2)-rounded within the class of graphic matroids. For the converse, suppose that M is a graphic matroid such that {M} is (3,2)-rounded within the class of graphic matroids, but M is not isomorphic to $M(W_3)$ or $M(W_4)$. By Lemma 2.2.2,

(2.4.7) M *is not a wheel-matroid*.

Let G be a graph such that M = M(G). By Theorem 1.2.8, up to isomorphism, G is uniquely determined. Identify the elements of M with the edges of G. The following result is an immediate consequence of Lemma 2.4.6.

(2.4.8) *G possesses a vertex of degree three*.

By (2.4.8), G has a triad and hence a chain. Let $(T_i)_{1,k}$ be a chain G of maximum length. Let $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ for each i in $\{1,2,...,k\}$. It follows from Lemma 2.2.8(2), (2.4.3), and (2.4.8), that

(2.4.9) <u>both</u> T_1 <u>and</u> T_k <u>are</u> <u>triangles</u>.

Let v be the vertex of G which is incident with both a_{k+1} and a_{k+2}. It follows from Lemma 2.4.4, (2.4.7), and (2.4.9) that

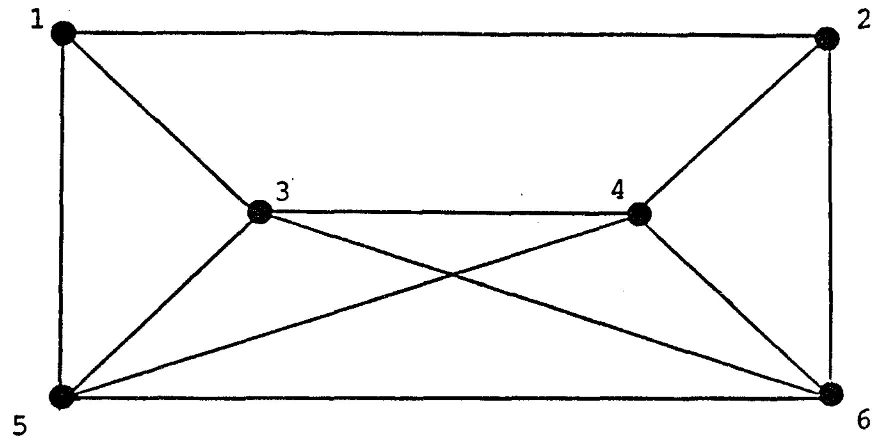
(2.4.10) $d_G(v) > 3$.

However, Lemmas 2.4.1 and 2.4.5(1) and (2.4.8), (2.4.9), and (2.4.10) imply that {M} is not (3,2)-rounded within the class of graphic matroids; a contradiction. This completes the proof of Theorem 2.1.3. □

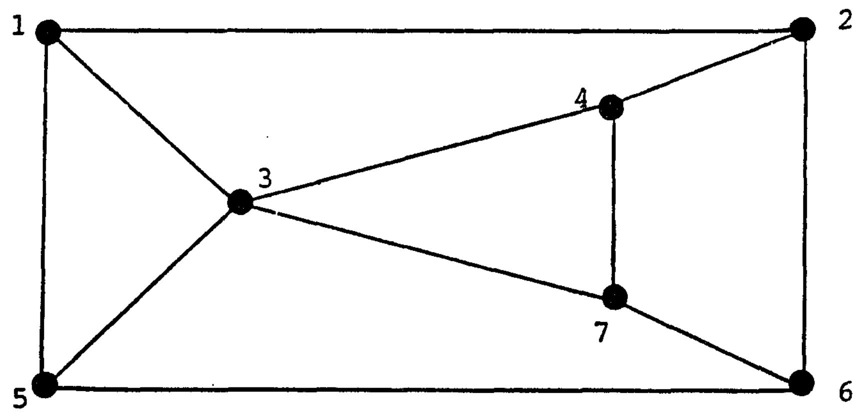
We now give some preliminary lemmas which are used in the proof of Theorem 2.1.5. In Figure 7 we give some eleven-edge graphs which are referred to in the subsequent lemmas.

Figure 7

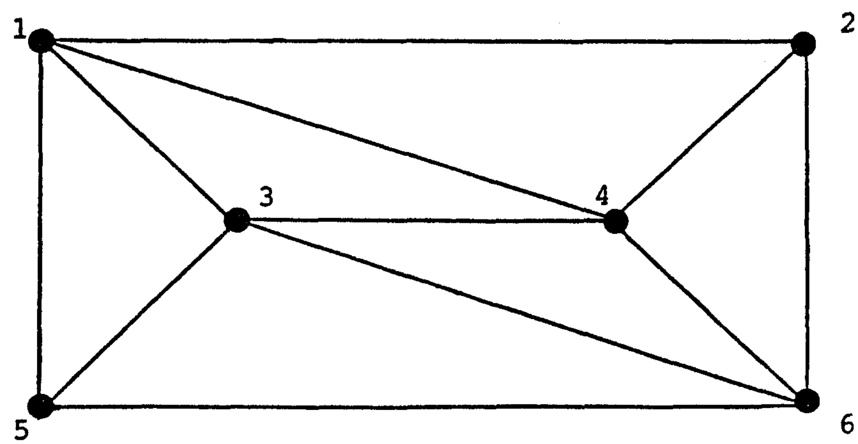
J_1



J_2



J_3



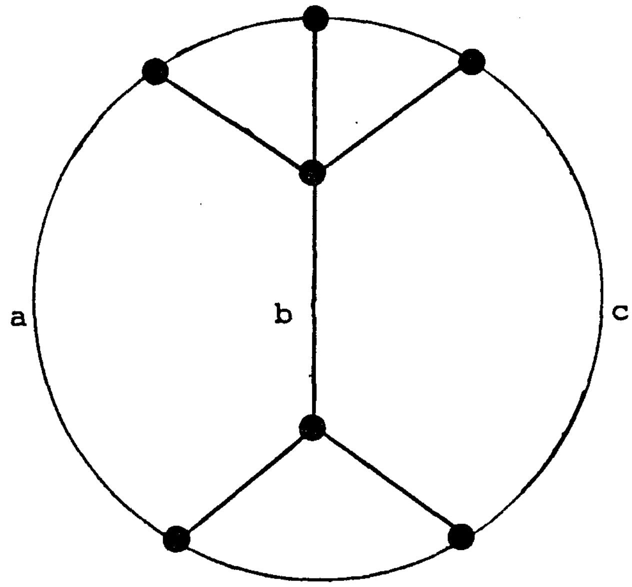
The graph P is given in Figure 1. In the next three results we show that the set $\{M(\mathcal{W}_5), M(P)\}$ is (3,2)-rounded within the class of graphic matroids.

2.4.11 Lemma [29,(Table 1)]. *Let* G *be an eleven-edge 3-connected simple graph with a* P-*minor but no* $\mathcal{W}_5$-*minor. Then* G *is isomorphic to* J_1, J_2, *or* J_3. □

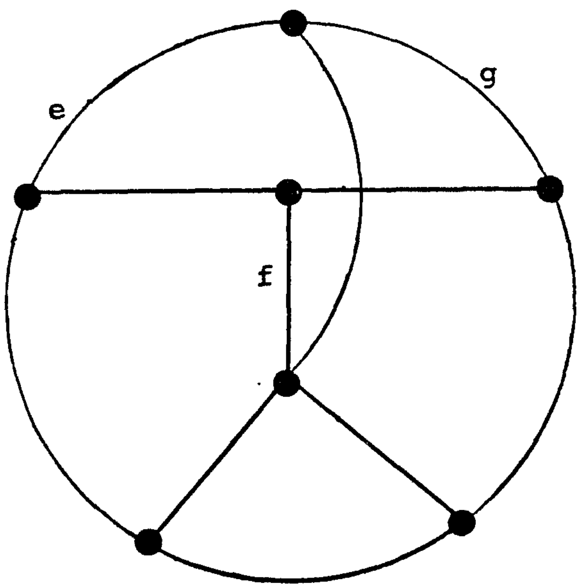
The graphs in the next figure are both lifts of $\mathcal{W}_5$. Note that L_1 is isomorphic to the graph G_5 given in Figure 5.

Figure 8

L_1



L_2



2.4.12 Lemma. *Let G be a 3-connected simple lift of W_5. Then G is isomorphic to L_1 or L_2.*

Proof. Suppose v is the vertex of W_5 of degree five. By Lemma 2.4.1, G must be obtained from W_5 by splitting v. It is easily checked that G must possess a triangle. If G has one triangle, then G must be isomorphic to L_2. If G has more than one triangle, then G must be isomorphic to L_1. □

The graphs P, H_r, and G_r are given in Figures 1, 3, and 5, respectively. Evidently, the graphs P and $H_5\backslash b_2$ are isomorphic.

2.4.13 Lemma. *Let n be an integer exceeding four. Then the set $\{M(W_n), M(H_n\backslash b_2)\}$ is (3,2)-rounded within the class of graphic matroids if and only if n is five.*

Proof. Suppose that n exceeds five. By Lemma 2.3.10, G_n does not have a W_n-minor using g. Any simple single-edge contraction of G_n which uses g has no vertex of degree n-1. Thus G_n does not have an $(H_n\backslash b_2)$-minor using g. It follows that $\{M(W_n), M(H_n\backslash b_2)\}$ is not (3,2)-rounded within the class of graphic matroids.

We next show that the set $\{M(W_5), M(P)\}$ is (3,2)-rounded within the class of graphic matroids. This will complete the proof, as P and $H_5\backslash b_2$ are isomorphic. Let G be a

3-connected simple graph which is an extension or lift of $\mathcal{W}_5$ or P.

Suppose G has no $\mathcal{W}_5$-minor. Then, by Lemma 2.4.11, G is isomorphic to J_1, J_2, or J_3. The deletion from J_1 of an edge in {(3,4), (3,6), (4,5)} produces a graph which is isomorphic to P. The contraction from J_2 of an edge in {(1,2), (2,4), (6,7)} produces a graph which is isomorphic to P. By deleting from J_3 an edge in {(1,3),(1,4),(3,6)}, we obtain a P-minor. It follows that each pair of edges of G is in some P-minor.

Now suppose G has a $\mathcal{W}_5$-minor. If G is an extension of $\mathcal{W}_5$, then G is isomorphic to H_5. The minors $H_5 \backslash b_2$ and $H_5 \backslash b_5$ are isomorphic to P, while the minor $H_5 \backslash c$ is isomorphic to $\mathcal{W}_5$. Thus, every pair of edges of H_5 appears in either a P- or $\mathcal{W}_5$-minor. Suppose G is a lift of $\mathcal{W}_5$. Then, by Lemma 2.4.12, G is isomorphic to L_1 or L_2. Now L_1/b and L_2/f are isomorphic to $\mathcal{W}_5$, while L_1/a, L_1/c, L_2/e, and L_2/g are isomorphic to P. It follows that each pair of edges of G appears in either a P- or $\mathcal{W}_5$-minor. Thus, by Lemma 2.3.9, the set $\{M(\mathcal{W}_5), M(P)\}$ is (3,2)-rounded within the class of graphic matroids. Since P and $H_5 \backslash b_2$ are isomorphic, the result follows. □

We require one more lemma before beginning the proof of Theorem 2.1.5.

2.4.14 <u>Lemma</u>. <u>Let</u> M <u>be</u> <u>a</u> 3-<u>connected</u> <u>graphic</u> <u>matroid</u>

with at least four elements. Then either M *is isomorphic to* $M(W_3)$ *or* M *has an* $M(W_4)$*-minor*.

Proof. Suppose that M is not isomorphic to $M(W_3)$. Then, by Theorem 1.2.2, M must have $M(W_3)$ as a proper minor. Suppose M does not have an $M(W_4)$-minor. Then, by Theorem 1.2.3, there is a 3-connected minor of M which is an extension or lift of $M(W_3)$. HOwever, M has no 3-connected graphic extensions. Moreover, by Lemma 2.4.1, M has no 3-connected graphic lifts; a contradiction. □

The methods used in the proofs of Theorems 2.1.3 and 2.1.4 are now generalized to pairs of graphic matroids.

Proof of Theorem 2.1.5. Suppose that $M(W_3)$ is in the set $\{M,N\}$, say $N = M(W_3)$. Then M has $M(W_3)$ as a minor by Theorem 1.2.2. It follows from Lemma 2.2.4 that the set $\{M,N\}$ is (3,2)-rounded within the class of graphic matroids. Likewise, Lemmas 2.2.4 and 2.4.14 can be used to show that if $M(W_4)$ is in $\{M,N\}$, then this set is (3,2)-rounded within the class of graphic matroids. Also, by Lemma 2.4.13, the set $\{M(W_5),M(P)\}$ is (3,2)-rounded within the class of graphic matroids.

For the converse, suppose that $\{M,N\}$ is a set other than $\{M(W_5),M(P)\}$ which is (3,2)-rounded within the class of graphic matroids and which contains neither $M(W_3)$ nor $M(W_4)$. The next lemma is the graphic analog of Lemma 2.3.14.

2.4.15 Lemma. $||E(M)| - |E(N)|| \leq 1$. *Moreover, if* $||E(M)| - |E(N)|| = 1$, *then one of* M *and* N *has a minor which is isomorphic to the other.*

Proof. By Theorem 2.1.3, neither {M} nor {N} is (3,2)-rounded within the class of graphic matroids. Thus, the result is an immediate consequence of the proof of Lemma 2.3.14. □

The next lemma is a key step in the proof. The graph H_r is given in Figure 3.

2.4.16 Lemma. *Neither* M *nor* N *is a wheel-matroid.*

Proof. Suppose that M is isomorphic to $M(\mathcal{W}_r)$ for some r exceeding four. Then, by Lemma 2.3.21, 2.3.22, and 2.4.15, N is isomorphic to $M(H_r)$ or $M(H_r \backslash b_2)$. It follows from Lemmas 2.3.11 and 2.4.13 that {M,N} is not (3,2)-rounded within the class of graphic matroids; a contradiction. Thus M, and similarly N, is not a wheel-matroid. □

Let G_1 and G_2 be graphs such that $M = M(G_1)$ and $N = M(G_2)$ and identify the elements of M and N with the edges of G_1 and G_2, respectively. We next show that E(M) and E(N) do not have the same number of elements.

2.4.17 Lemma. $||E(M)| - |E(N)|| = 1$.

Proof. By Lemma 2.4.15, it suffices to show that $|E(M)|$ and $|E(N)|$ are different. Suppose $|E(M)| = |E(N)|$. It follows from Lemma 2.4.6 that

(2.4.18) <u>at least one of</u> G_1 <u>and</u> G_2 <u>possesses a vertex of degree three</u>.

It follows from (2.4.18) that G_1 or G_2 has a triad and hence a chain. Let $(T_i)_{1,k}$ be a chain of maximum length among all the chains of M and N. By Lemmas 2.2.8(2) and 2.4.16, (2.4.3) and (2.4.18), T_k is a triangle. However, Lemmas 2.4.4, 2.4.5, and 2.4.16 imply that {M,N} is not (3,2)-rounded within the class of graphic matroids; a contradiction. This completes the proof of Lemma 2.4.17. □

By Lemmas 2.4.15 and 2.4.17, either M or N has an extension or lift which is isomorphic to the other. Without loss of generality, suppose that g is an element of E(M) such that either M\g or M/g is N. We first show that the former cannot occur.

2.4.19 <u>Lemma</u>. M/g = N.

<u>Proof</u>. Suppose $G_1\backslash g = G_2$. We now show that N has a chain.

(2.4.20) G_2 <u>has a vertex of degree three</u>.

<u>Proof</u>. Let v be a vertex of G_2 of minimum degree and suppose this degree exceeds three. Let $H \varepsilon G_2(v,e)$. By Lemma 2.4.6, H has no G_2-minor using e. By Lemma 2.4.1, H is 3-connected and simple. Thus, H must have a G_1-minor using e. However,

$|E(H)| = |E(G_1)|$, but $rkM(H) > rkM(G_2) = rkM(G_1)$. Thus H is not isomorphic to G_1; a contradiction. □

It follows from (2.4.20) that G_2 has a chain. Let $(T_i)_{1,k}$ be a maximum-length chain of G_2, and $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ for each i in $\{1,2,\ldots,k\}$. By Lemmas 2.4.4, 2.4.5, and 2.4.16.

(2.4.21) T_1 _and_ T_2 _are triads of_ G_2.

By Lemma 2.4.3 and (2.4.20), we may assume that a_{k+1} and a_{k+2} are incident with a common vertex v. We next show that G_1 has a chain.

(2.4.22) G_1 _has a chain of length at least_ $k + 1$.

Proof. Form the graph H from G_2 by adding the edge a_{k+3} so that $\{a_{k+1}, a_{k+2}, a_{k+3}\}$ is a triangle of H. Let $T_{k+1} = \{a_{k+1}, a_{k+2}, a_{k+3}\}$. By Lemma 2.2.8(2), H has no G_2-minor using a_1 and a_{k+3}. Thus H is isomorphic to G_1, and $(T_i)_{1,k+1}$ is a chain of H. □

Let $(R_i)_{1,m}$ be a maximum-length chain of G_1. By (2.4.22), m exceeds k. Recall that $G_1 \backslash g = G_2$. By (2.3.19), (2.4.21), and (2.4.22) we obtain that

(2.4.23) _either_ R_1 _or_ R_m _is a triad of_ G_1.

By Lemma 2.2.8(2) and (3), either G_1 possesses a chain of length $m + 1$, or G_2 possesses a chain of length at least

$k + 1$; a contradiction. This completes the proof of Lemma 2.4.19. □

It follows from Lemma 2.4.19 that $G_1/g = G_2$. It follows from Lemma 2.2.12 that G_2 has a triangle. Let $(T_i)_{1,k}$ be a chain of G_2 of maximum length. We next show that we may assume

(2.4.24) *neither* T_1 *nor* T_k *is a triad*.

If k exceeds one, then, by Lemma 2.2.8(2) and (2.4.16), (2.4.24) must hold. If k is one, then choose T_1 to be a triangle. By Lemmas 2.4.4 and 2.4.5 and (2.4.16) we obtain that

(2.4.25) G_1 *has a chain of length at least* $k + 1$.

Let $(R_i)_{1,m}$ be a chain of G_1 of maximum length. By (2.4.24), (2.4.25), and the dual of (2.3.19) we obtain that

(2.4.26) *either* R_1 *or* R_m *is a triangle of* G_1.

It follows from (2.4.16), (2.4.26), and Lemmas 2.4.4 and 2.4.5 that G_1 or G_2 has a chain of length at least $m + 1$; a contradiction. This completes the proof of Theorem 2.1.5. □

CHAPTER 3

Rounded Pairs of Matroids

3.1 Introduction

The main result of this chapter is a characterization of all two-element sets which are (3,2)-rounded. This is the result of joint work with J.G. Oxley. It extends Theorem 1.6.6 of Oxley who proved the corresponding result for one-element sets. The motivation for studying small rounded sets is that, intuitively, these are the rounded sets which provide the most structural information.
The main result is now given.

3.1.1 *Theorem*. *Let* M *and* N *be* 3-*connected matroids*. *The set* {M,N} *is* (3,2)-*rounded if and only if* $\{M,N\} = \{U_{2,4},N'\}$ *where either*

(i) N' *is non-binary, or*

(ii) N' *is isomorphic to* $M(\mathcal{W}_3)$ *or* $M(\mathcal{W}_4)$.

The proof of this result will be given in Section 3.2. In Section 3.3 the definition of a (k,m)-rounded set is modified to allow such a set to contain matroids on fewer than four elements. The effect of this modification on the above theorem and the results of Chapter 2 is discussed in that section.

The following consequence of Theorem 3.1.1 is proved in Section 3.2.

3.1.2 *Corollary*. *Let* M *and* N *be* 3-*connected matroids. The set* {M,N} *is* (3,3)-*rounded if and only if* {M,N} *is* $\{U_{2,4}, \mathcal{W}^3\}$.

We next show that there are no one-element sets which are (3,3)-rounded. Thus the last corollary classifies the smallest (3,3)-rounded sets.

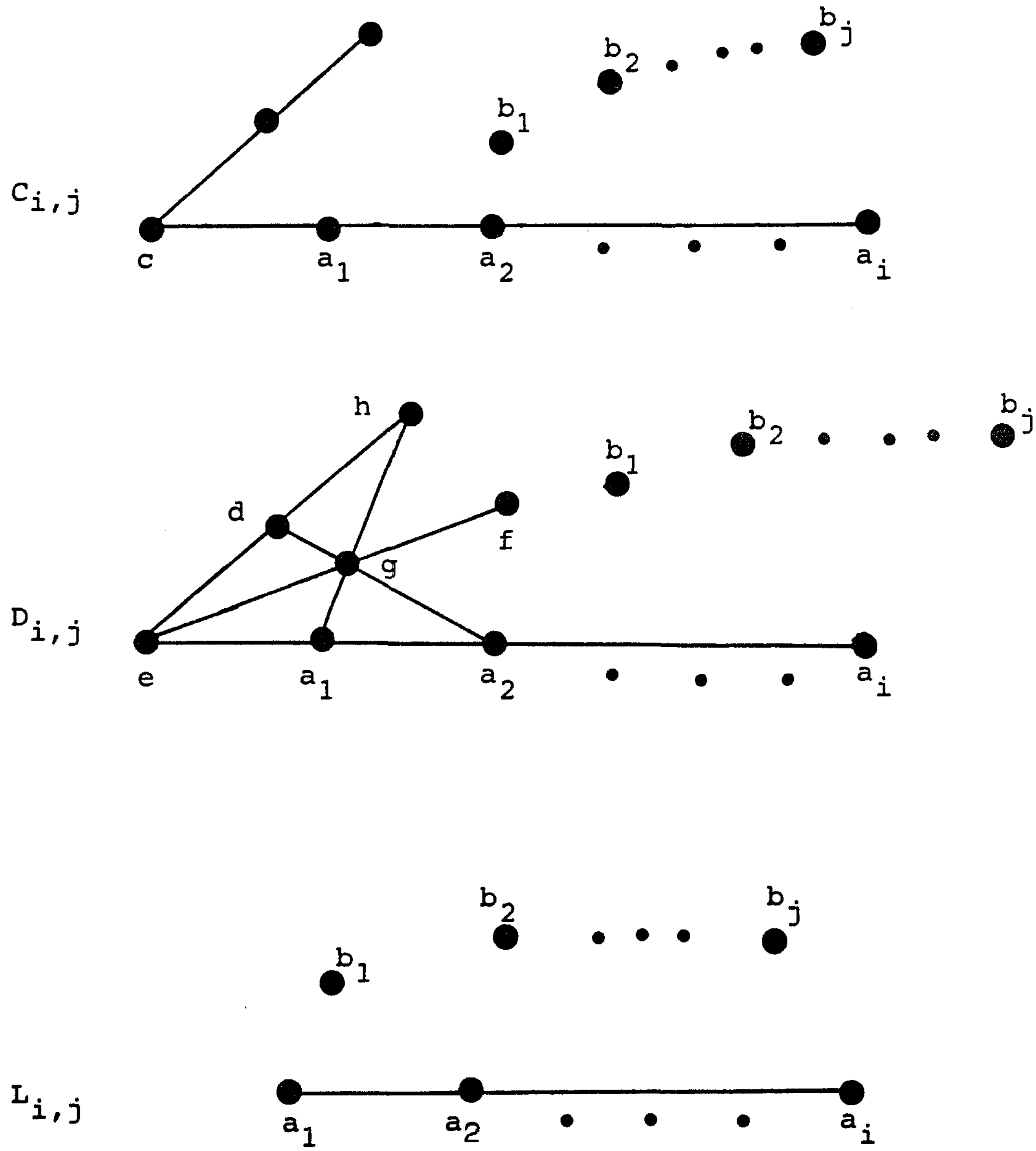
3.1.3 *Theorem*. *Let* M *be a matroid. The set* {M} *is not* (3,3)-*rounded*.

Proof. By Theorem 1.6.6, it suffices to show that the set $\{U_{2,4}\}$ is not (3,3)-rounded. However, this follows from considering the elements a, b, and c of the matroid $\mathcal{W}^3$ given in Figure 4. □

3.2 The Proofs

The proofs of Theorem 3.1.1 and Corollary 3.1.2 are given in this section. Figure 9 gives Euclidean representations for some rank-3 matroids that will be referred to in the proofs which follow. Let i and j be non-negative integers.

Figure 9



Evidently $C_{2,1}$ is isomorphic to the matroid Q_6 of Table 1, while $C_{3,1}$ is the matroid Q_7 of Table 1.

The next result of Oxley is frequently used throughout the proof of Theorem 3.1.1 to construct extensions of matroids.

3.2.1 Lemma [24,(2.5)]. *Let $\{x_1,x_2,\ldots,x_n\}$ be a circuit in a matroid M and suppose that x_1 is in every dependent flat of M. Then a flat F of M is in the modular cut $\mathcal{M}$ generated by $\sigma_M\{x_1,x_2\}$ and $\sigma_M\{x_3,x_4,\ldots,x_n\}$ if and only if F contains one of the two generating flats. Moreover, the generating flats are disjoint.* □

Proof of Theorem 3.1.1. Suppose N' is a 3-connected non-binary matroid. Then the set $\{U_{2,4},N'\}$ is (3,2)-rounded by Theorem 1.6.5 and Lemma 1.7.2. If N' is isomorphic to $M(\mathcal{W}_3)$ or $M(\mathcal{W}_4)$, then the set $\{U_{2,4},N'\}$ is (3,2)-rounded by Lemma 2.2.4.

Now suppose that M and N are 3-connected matroids such that $\{M,N\}$ is a (3,2)-rounded set. If M is isomorphic to $U_{2,4}$, then we may assume that N is binary. Thus $\{N\}$ is (3,2)-rounded within the class of binary matroids. It follows from Theorem 2.1.2 that N is isomorphic to $M(\mathcal{W}_3)$ or $M(\mathcal{W}_4)$. Hence we may suppose that neither M nor N is isomorphic to $U_{2,4}$.

The remainder of the proof is devoted to obtaining the contradiction that $\{M,N\}$ is not (3,2)-rounded. We begin with the following lemma.

3.2.2 <u>Lemma</u>. <u>Both</u> M <u>and</u> N <u>have rank and corank at least three</u>.

<u>Proof</u>. By duality, it suffices to show that neither M nor N has rank two. We shall prove a stronger result. The matroid $C_{i,j}$ is as given in Figure 9.

3.2.3 <u>Lemma</u>. <u>If</u> n <u>is at least five</u>, <u>then neither</u> M <u>nor</u> N <u>is isomorphic to</u> $U_{2,n}$ <u>or</u> $C_{n-3,1}$.

<u>Proof</u>. Assume the contrary and let $m = \min\{n\colon M$ or N is isomorphic to $U_{2,n}$ or $C_{n-3,1}\}$. Evidently m is at least five. Suppose that M is isomorphic to $U_{2,m}$. Then $C_{m-3,1}$ has an M-minor but has no such minor using both b_1 and c. Hence $C_{m-3,1}$ has an N-minor using both b_1 and c. By the choice of m, it follows that N is isomorphic to $C_{m-3,1}$. But now the matroid $D_{m-3,0}$ of Figure 9 has an N-minor, yet has no M- or N-minor using both e and g. This contradiction implies that M is not isomorphic to $U_{2,m}$. Similarly N is not isomorphic to $U_{2,m}$.

We may now assume that M is isomorphic to $C_{m-3,1}$. It follows that $D_{m-3,0}$ has an N-minor using e and g. By the choice of m, N must have rank 3. Thus $D_{m-3,0}$ has a restriction N_1 that uses both e and g and is isomorphic to N. Since N_1 has no 2-element cocircuits, $E(N_1)$ uses at least two of d, h, and f. It follows, since N_1 is 3-connected, that it has at most one free element.

Next consider the matroid $C_{m-3,2}$. This matroid has no $C_{m-3,1}$-minor using both b_1 and b_2, and so must have a restriction isomorphic to N using both b_1 and b_2. In such a restriction, both b_1 and b_2 are still free. Hence N_1 has at least two free elements. This is a contradiction as we showed that N_1 has at most one such element. This completes the proof of Lemma 3.2.3 and thereby that of Lemma 3.2.2. □

The next three results are used in the proof of Lemma 3.2.7 where it will be shown that M and N have the same number of elements. Let Q_6, Q_7, and Q_7^* be as given in Table 1. Evidently $C_{2,1} \cong Q_6$ and $C_{3,1} \cong Q_7$ where $C_{2,1}$ and $C_{3,1}$ are as given in Figure 9. Thus the next lemma follows immediately from Lemma 3.2.3 and its dual.

3.2.4 Lemma. *Neither M nor N is isomorphic to Q_6 or Q_7^*.* □

Although the next lemma is not explicitely stated in [24], it is not difficult to see that it may be obtained from the proof of Lemma 2.6 of that paper.

3.2.5 Lemma. *Let N_1 be a 3-connected matroid having rank and corank at least three and assume that N_1 has both a free and a cofree element. Suppose that, whenever N_2 is a non-trivial extension of N_1, each element of N_2 appears in an N_1-minor. Then N_1 is isomorphic to Q_6 or Q_7^*.* □

3.2.6 <u>Lemma</u>.

(i) M <u>or</u> N <u>has at least two free elements; and</u>

(ii) <u>neither</u> M <u>nor</u> N <u>is a lift or an extension of the other</u>.

<u>Proof</u>. Part (i) follows immediately from Lemmas 1.7.3 and 3.2.2. To prove (ii), suppose that M/e is isomorphic to N for some e in E(M). Let N + f be formed by freely adding f to N. Now rk(N+f) = rk N < rk M and so N + f has no M-minor. Thus N + f has an N-minor using f and hence N has a free element. As {M*,N*} is (3,2)-rounded, we may apply part (i) to it to get that M* or N* has at least two free elements. Since N* is isomorphic to M*\e, it follows, in either case, that N* has a free element. Thus N has both a free and a cofree element. Thus, by Lemma 3.2.5, N is isomorphic to Q_6 or Q_7^*. But, by Lemma 3.2.4, this is a contradiction. We conclude that M is not a lift of N and, by duality, M is not an extension of N. Similarly, N is neither an extension nor a lift of M. □

We are now ready to show that M and N have the same number of elements. Recall that, by Lemma 3.2.2, M and N each have rank and corank exceeding two.

3.2.7 <u>Lemma.</u> $|E(M)| = |E(N)|$.

Proof. By Theorem 1.6.6 and Lemma 3.2.2, neither of the sets {M} and {N} is (3,2)-rounded. Thus, if $|E(N)|<|E(M)|$, then, by Theorem 1.6.2, M is an extension or lift of N. But this contradicts Lemma 3.2.6(ii). It follows that $|E(N)| \geq |E(M)|$ and likewise, $|E(M)| \geq |E(N)|$. □

The next step in the proof of Theorem 3.1.1 is to show that M and N have the same rank. To prove this we shall need the following lemma which is also used in the proof of Theorem 3.2.12.

3.2.8 Lemma. *At least one of* M, N, M*, *and* N* *has at least one free element and at least two cofree elements.*

Proof. By Lemma 3.2.6(i) and duality, at least one member of each of {M,N} and {M*,N*} has two or more free elements. Thus either the lemma holds or, without loss of generality, we may assume that both M and N* have at least two free elements.

Let N + f be formed by freely adding f to N. If N + f has an N-minor using f, then N has the required property. Thus we may assume that N + f has no such minor. Then N + f has an M-minor using f. By Lemma 3.2.7, E(M) and E(N) have the same number of elements. Hence at least one of the two cofree elements of N + f is in the M-minor of N + f. Thus M has a cofree element and M* has the required property. □

3.2.9 Lemma. rk M = rk N.

Proof. Assume, without loss of generality, that rk N < rk M. Then the fact that $|E(M)| = |E(N)|$ implies that rk $M^* <$ rk N^*. By Lemma 3.2.8, either N or M^* must possess both a free and a cofree element. Since rk N < rk M and rk $M^* <$ rk N^*, it follows that at least one of N and M^* satisfies the hypothesis of Lemma 3.2.5. It follows that N or M^* must be isomorphic to one of Q_6 and Q_7^*. However, this is a contradiction to Lemma 3.2.4 or its dual. □

We next give a technical lemma before showing that M and N must have rank and corank at least four. The matroids $C_{i,j}$, $D_{i,j}$, and $L_{i,j}$ are as given in Figure 9.

3.2.10 Lemma. Let m and n be integers exceeding two. Neither M nor N is isomorphic to $L_{m,n}$.

Proof. Assume the contrary and let j = min {n: M or N is isomorphic to $L_{m,n}$}. We may assume that M is isomorphic to $L_{m,j}$ without loss of generality. The deletion of c from $C_{m,j-2}$ produces an M-minor. However, $C_{m,j-2}$ has no M-minor using c. It follows from Lemmas 3.2.2 and 3.2.7 that N is isomorphic to a single-element deletion of $C_{m,j-2}$ which uses c. The only such deletions are $C_{m-1,j-2}$, $C_{m,j-3}$, and $L_{m+1,j-1}$. By the choice of j, N is not isomorphic to $L_{m+1,j-1}$. Thus N is isomorphic to

$C_{m-1,j-2}$ or $C_{m,j-3}$. Suppose the former holds.

Now $D_{m-1,j-3}$ has a $C_{m-1,j-2}$-minor, but has no such minor using both e and g. It also has no M-minor. Thus {M,N} is not (3,2)-rounded; a contradiction. It follows that N is isomorphic to $C_{m,j-3}$. By the 3-connectivity of N, j must be at least four. Now $D_{m,j-4}$ has an N-minor, but has no such minor using both e and g. As $D_{m,j-4}$ has no M-minor, we obtain a contradiction. □

We require one more lemma before showing that the set {M,N} is not (3,2)-rounded.

3.2.11. Lemma. Both the rank and corank of M and N are at least four.

Proof. Assume the lemma is false. Then by duality and Lemmas 3.2.2 and 3.2.9, we may assume that rk M = rk N = 3 and M and N have the same number, say n, of elements. By Lemmas 1.5.1 and 3.2.6(i) and duality, M or N, say N, has at least two elements that are in every dependent flat. Therefore N has at most one dependent line. Thus either N is isomorphic to $U_{3,n}$, or N is isomorphic to $L_{i,j}$ for some i and j. However, the latter cannot occur by Lemma 3.2.10. Thus N is isomorphic to $U_{3,n}$ and n exceeds four.

Let $C_{2,n-4}$ be as given in Figure 9. This matroid has an N-minor, but has no N-minor using c. Thus, by Lemmas 3.2.2 and 3.2.7, M is isomorphic to a single-element deletion of $C_{2,n-4}$ which uses c. The only such deletions are $C_{2,n-5}$ and $L_{3,n-3}$. By Lemma 3.2.10, M is not isomorphic to $L_{3,n-3}$. Thus M is isomorphic to $C_{2,n-3}$ and n is at least six. Now $D_{2,n-6}$ has an M-minor but has no such minor using both e and g. Also $D_{2,n-6}$ has no N-minor. It follows that the set {M,N} is not (3,2)-rounded; a contradiction. □

3.2.12 Theorem. The set {M,N} is not (3,2)-rounded.

Proof. By duality and Lemmas 1.5.3 and 3.2.8 we may assume that

(3.2.13) M has a free element f together with elements d_1 and d_2 which are in every dependent flat.

We remark that throughout this proof condition (3.2.13) will provide the sole feature distinguishing M from N.

As the rank of M is not two, f is not included in $\sigma_M\{d_1,d_2\}$. Now augment $\{d_1,d_2\}$ to a base $\{d_1,d_2,a_1,a_2,\ldots,a_{r-2}\}$ of M\f. Let $\mathcal{M}$ be the modular cut of M generated by the flats $\sigma_M\{d_1,d_2\}$ and $\{a_1,a_2,\ldots,a_{r-2},f\}$ and let $M + e_1$ be the extension determined by $\mathcal{M}$. Evidently $M + e_1$ is 3-connected by Lemmas 3.2.1 and 1.3.1. Moreover, by Lemma 3.2.1 we have:

(3.2.14) The dependent flats of $M + e_1$ are the circuit-hyperplane $\{a_1, a_2, \ldots, a_{r-2}, f, e_1\}$ together with the sets $F \cup e_1$ for which F is a flat of M containing both d_1 and d_2.

As $\{M,N\}$ is (3,2)-rounded, there is an element g_1 of $E(M + e_1) - \{e_1, f\}$ such that $(M + e_1)\backslash g_1$ is isomorphic to M or N. We now eliminate the first possibility. Thus assume that $(M + e_1)\backslash g_1$ is isomorphic to M. We shall show that this implies the contradiction that $(M + e_1)\backslash g_1$ has more dependent flats than M. First note that, as d_1 and d_2 are in every dependent flat of M, no line of M has more elements than $\sigma_M\{d_1, d_2\}$. Thus g_1 is included in $\sigma_M\{d_1, d_2\}$. Using this, it is not difficult to check that for every dependent flat F of M, $(F - g_1) \cup e_1$ is a dependent flat of $(M + e_1)\backslash g_1$. Moreover, $\{a_1, a_2, \ldots, a_{r-2}, f, e_1\}$ is also a dependent flat of $(M + e_1)\backslash g_1$ since g_1 is not included in this set. Thus $(M+e_1)$ g_1 does indeed have more dependent flats than M. We conclude that

(3.2.15) $(M + e_1)\backslash g_1$ is isomorphic to N.

As e_1 is in every dependent flat of $(M + e_1)\backslash g_1$, it follows by (3.2.15) that

(3.2.16) N has an element that is in every dependent flat.

We next show that

3.2.17 Lemma. N has a unique dependent line.

Proof. We shall first show that M or N has a triangle. Among all the circuits of M and N, let $\{c_1, c_2, \ldots, c_j\}$ be one of minimum size and suppose that j is at least four. Let P be the member of {M,N} that contains $\{c_1, c_2, \ldots, c_j\}$. As both M and N have an element in every dependent flat, we may assume that c_1 is in every dependent flat of P.

Let $\mathcal{P}$ be the modular cut of P generated by $\sigma_P\{c_1, c_2\}$ and $\sigma_P\{c_3, c_4, \ldots, c_j\}$ and let $P + e_2$ be the extension determined by $\mathcal{P}$. By Lemma 3.2.1, both $\{c_1, c_2, e_2\}$ and $\{c_3, c_4, \ldots, c_j, e_2\}$ are circuits of $P + e_2$. Thus any single-element deletion of $P + e_2$ which uses e_2 contains a circuit of size less than j. Hence $P + e_2$ has no M- or N-minor using e_2; a contradiction. We conclude that M or N has a triangle. □

Now, as d_1 and d_2 are in every dependent flat of M, by (3.2.14), the only possible dependent line of $(M + e_1)\backslash g_1$ is $(\sigma_M\{d_1, d_2\} \cup \{e_1\}) - \{g_1\}$. Since M or N has a triangle and $(M + e_1)\backslash g_1$ is isomorphic to N, we deduce that $(M+e)\ g_1$, and hence N, has exactly one dependent line.

3.2.18 Lemma. g_1 is in $\{a_1, a_2, \ldots, a_{r-2}\}$.

Proof. Assume the contrary and let $N' = (M + e_1)\backslash g_1$. Then N' has $\{a_1, a_2, \ldots, a_{r-2}, f, e_1\}$ as a circuit-hyperplane. Since N' is isomorphic to N, the former has a unique dependent line L. By (3.2.14) and Lemma 3.2.11, it follows that $L = (\sigma_M\{d_1,d_2\} \cup \{e_1\}) - \{g_1\}$. Moreover, e_1 is in every dependent flat of N'.

Now let $N' + e_3$ be the extension determined by the modular cut generated by the flats $\{e_1,f\}$ and $\{a_1,a_2,\ldots,a_{r-2}\}$ of N'. By Lemma 3.2.1, $\{e_1,f,e_3\}$, $\{a_1,a_2,\ldots,a_{r-2},e_3\}$ and L are all dependent flats of $N' + e_3$. Moreover, $\{e_1,f,e_3\} \cap L = \{e_1\}$ and $\{a_1,a_2,\ldots,a_{r-2},e_3\} \cap L$ is empty. As $\{M,N\}$ is (3,2)-rounded, there is an element g_3 of $E(N' + e_3) - \{e_1,e_3\}$ such that $(N' + e_3)\backslash g_3$ is isomorphic to M or N. Since $(N' + e_3)\backslash g_3$ clearly does not have two elements in every dependent flat, (3.2.13) implies that $(N' + e_3)\backslash g_3$ is not isomorphic to M.

We may now assume that $(N' + e_3)\backslash g_3$ is isomorphic to N. By Lemma 3.2.17, g_3 is in $L \cup \{e_1,f,e_3\}$. But g_3 is neither e_1 nor e_3 and, by (3.2.16), g_3 is not f. Hence g_3 is in $L-e_1$. Thus $\{a_1,a_2,\ldots,a_{r-2},e_3\}$ is both a circuit and a flat of $(N' + e_3)\backslash g_3$. But $(N' + e_3)\backslash g_3 \cong N \cong (M + e_1)\backslash g_1 = N'$ and $(\sigma_M\{d_1,d_2\} \cup \{e_1\}) - \{g_1\}$ is a dependent line of N'. Thus, by (3.2.14), the only circuit-flats that $(M + e_1)\backslash g_1$ can contain are a triangle and a hyperplane. Since $\{a_1,a_2,\ldots,a_{r-2},e_3\}$ has $\mathrm{rk}N - 1$ elements, this set is not a circuit-hyperplane. It must therefore be a triangle, so $r = 4$ and both $\{a_1,a_2,e_3\}$ and $\{e_1,f,e_3\}$ are lines of $(N' + e_3)\backslash g_3$. Since this matroid is isomorphic to N, this contradicts the fact that N has a unique dependent line. □

By (3.2.14), the only circuit of $M + e_1$ containing f and having fewer than r + 1 elements is $\{a_1, a_2, \ldots, a_{r-2}, f, e_1\}$. Now g_1 is in $\{a_1, a_2, \ldots, a_{r-2}\}$ by Lemma 3.2.18. It follows that f is free in $(M + e_1)\backslash g_1$. Also, by (3.2.14), $(M + e_1)\backslash g_1$ has at least two elements which are in every dependent flat. Since N is isomorphic to $(M + e_1)\backslash g_1$, we deduce that N satisfies condition (3.2.13). Thus M and N obey the same hypotheses. Therefore we may interchange them from (3.2.13) onward to deduce from Lemma 3.2.17 that M has a unique dependent line L_M. Evidently $L_M = \sigma_M\{d_1, d_2\}$. As g_1 is in $\{a_1, a_2, \ldots, a_{r-2}\}$, $\sigma_M\{d_1, d_2\} \cup \{e_1\}$ is a dependent line of $(M + e_1)\backslash g_1$. Since the last matroid is isomorphic to N, and N has a unique dependent line L_N, we deduce that $|L_N| > |L_M|$. But again, since M and N obey the same hypotheses, we may interchange them from (3.2.13) onward to get that $|L_M| > |L_N|$. This contradiction completes the proof of Theorem 3.2.12 as well as that of Theorem 3.1.1. □

The next proof concludes the section.

Proof of Corollary 3.1.2. The set $\{U_{2,4}, \mathcal{W}^3\}$ is (3,3)-rounded by Theorem 1.6.7. For the converse, suppose the set $\{M,N\}$ is (3,2)-rounded. Then, by Theorem 3.1.1, the set must contain $U_{2,4}$. Suppose, without loss of generality, that M is isomorphic to $U_{2,4}$. Consider the elements a, b, and c of $\mathcal{W}^3$ as marked in Figure 4. Since $\mathcal{W}^3$ has no M-minor using $\{a,b,c\}$, it must have an N-minor using {a,b,c}. This implies that N is isomorphic to ω^3. □

3.3 Small Matroids in Rounded Sets

Matroids with fewer than four elements are excluded from (k,m)-rounded sets in the definition. In this section we investigate the implications of dropping this condition from the definition.

Let k and m be positive integers with k at least two.

3.3.1 Definition. Let S be a set of k-connected matroids. The set S is $(k,m)_0$-rounded if and only if it satisfies the following condition.

(i) If M is a k-connected matroid having an S-minor and X is a subset of E(M) with at most m elements, then M has an S-minor using X.

Let S be a set of matroids. Evidently S is (k,m)-rounded if and only if it is $(k,m)_0$-rounded and each matroid in S has at least four elements. Using Lemma 1.2.5, the next fact is easily checked.

(3.3.2) The only 2-connected matroids with fewer than four elements are $U_{0,1}$, $U_{1,1}$, $U_{1,2}$, $U_{1,3}$, and $U_{2,3}$. □

Let S be a set of k-connected matroids. If S contains any of the matroids listed in (3.3.2), then S is easily shown to be $(k,1)_0$-rounded. We next show that the inclusion of $U_{1,2}$, $U_{1,3}$, and $U_{2,3}$ in S does not provide structural information.

3.3.3 Lemma. *If S contains at least one of $U_{1,2}$, $U_{1,3}$, and $U_{2,3}$, then S is $(k,2)_0$-rounded.*

Proof. This follows from (3.3.2) and the fact that any specified pair of elements in a 2-connected matroid is in some circuit of that matroid. □

However, we next show that the inclusion of $U_{1,3}$ or $U_{2,3}$ in a $(k,3)_0$-rounded set does provide structural information about a matroid. We shall use the next result of Oxley in investigating such sets.

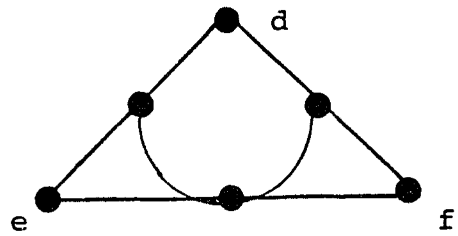
3.3.4 Lemma [8,p.56,ex9]. *A matroid with at least three elements is 2-connected if and only if every three-element subset is contained in either a circuit or a cocircuit.* □

An immediate consequence of this theorem is the following result.

3.3.5 Corollary. *The set $\{U_{1,3}, U_{2,3}\}$ is $(k,3)_0$-rounded for each integer exceeding one.* □

We will use the next two results in investigating the effect on Corollary 3.1.3 of relaxing the definition of a (3,3)-rounded set. A Euclidean representation for the rank-three wheel is given below.

Figure 10 $M(W_3)$



3.3.6 Lemma. The set $\{M,U_{2,3}\}$ is $(3,3)_0$-rounded if and only if M is isomorphic to $U_{1,3}$.

Proof. If M is isomorphic to $U_{1,3}$, then the set $\{M,U_{2,3}\}$ is $(3,3)_0$-rounded by Corollary 3.3.5. Conversely, suppose $\{M,U_{2,3}\}$ is $(3,3)_0$-rounded. Consider the elements a, b, and c of W^3 as marked in Figure 4. Now W^3 has a $U_{2,3}$-minor, but has no such minor using a, b, and c. Thus M is isomorphic to $U_{1,3}$ or W^3. From considering the subset {d,e,f} of $M(W_3)$ given in Figure 10 we see that the latter cannot occur. □

The next lemma is the dual of Lemma 3.3.6.

3.3.7 Lemma. The set $\{M,U_{1,3}\}$ is $(3,3)_0$-rounded if and only if M is isomorphic to $U_{2,3}$. □

We now obtain the following analog to Corollary 3.1.2 using Definition 3.3.1 instead of Definition 1.6.1.

3.3.8 Corollary. Let M and N be 3-connected matroids. The set {M,N} is $(3,3)_0$-rounded if and only if {M,N} is either $\{U_{2,4},W^3\}$ or $\{U_{1,3},U_{2,3}\}$.

Proof. If both M and N have at least four elements, then the result is true by Corollary 3.1.2. Suppose that M or N has fewer than four elements. By Theorem 3.1.3, both M and N have at least three elements. It follows from (3.3.2) that the set {M,N} contains $U_{1,3}$ or $U_{2,3}$. The result follows by Lemmas 3.3.6 and 3.3.7. □

CHAPTER 4

Roundedness in 4-Connected Matroids

4.1 Introduction

In this chapter we investigate the property of roundedness in 4-connected matroids. Seymour conjectured that the set $\{U_{2,4}\}$ is (4,3)-rounded [37]. This is a natural conjecture in light of Theorems 1.6.4 and 1.6.5. The next result, obtained independently by Coullard [11] and Kahn [18], shows that this conjecture is false.

4.1.1 Theorem. *The set $\{U_{2,4}\}$ is not (4,3)-rounded.* □

We extend their result by showing that, for any matroid M, the set {M} is not (4,3)-rounded. This result will follow from a characterization of the matroids M for which the set {M} is (4,2)-rounded.

The main result of the chapter is now given. It is a generalization to 4-connected matroids of Theorem 1.6.6.

4.1.2 Theorem. *Let M be a 4-connected matroid with at least four elements. The set {M} is (4,2)-rounded if and only if M is isomorphic to $U_{2,4}$.*

It follows from Lemma 3.3.3 that the sets $\{U_{1,2}\}$, $\{U_{1,3}\}$, and $\{U_{2,3}\}$ are $(4,2)_0$-rounded. However, it is easily checked that these sets are not $(4,3)_0$-rounded.

An immediate corollary of Theorems 4.1.1 and 4.1.2 is now given.

4.1.3 Corollary. *Let* M *be a matroid. The set* {M} *is not* (4,3)-*rounded*. □

The proof of Theorem 4.1.2 as well as the following extension of Theorems 1.6.6 and 4.1.2 are given in the next section.

4.1.4 Theorem. *Let* k *be an integer exceeding three. Let* M *be a* k-*connected matroid with rank at least* k. *Then the set* {M} *is not* (k,2)-*rounded*.

4.2 The Proofs

The proofs of Theorems 4.1.2 and 4.1.4 are given in this section. We begin with a preliminary lemma that is used in the proof of Theorem 4.1.4 to construct extensions of a matroid.

4.2.1 Lemma. *Let H be a hyperplane of a simple matroid N. Let f_1 and f_2 be free elements of N which are not in H, and F be a flat of N containing f_1 and f_2. Then a flat of N is in the modular cut generated by F and H if and only if it contains one of the two generating flats.*

Proof. Suppose G is a flat of N containing F such that (G,H) is a modular pair of flats. Thus $rk(G \cap H) = rk\, G + rk\, H - rk(G \cup H) = rk\, G + rk\, N - 1 - rk\, N = rk\, G - 1$.

Suppose $G \cap H$ is not a hyperplane of N. The elements f_1 and f_2 are free in G and are not contained in $G \cap H$ From combining this with the fact that $rk(G \cap H) = rk\, G - 1$, we obtain that

$$\begin{aligned} rk\, G &\geq rk((G \cap H) \cup \{f_1, f_2\}) \\ &= rk(G \cap H) + 2 \\ &= rk\, G - 1 + 2 \\ &= rk\, G + 1; \text{ a contradiction.} \end{aligned}$$

Thus $G \cap H$ is a hyperplane and hence $G \cap H = H$. So $G = E(N)$ and the modular cut generated by F and H consists only of those flats containing F or H. □

We first prove Theorem 4.1.4 as this result is used in deriving Theorem 4.1.2.

Proof of Theorem 4.1.4. Suppose that the set {M} is (k,2)-rounded.

Let H_0 be a hyperplane of M. Now
$rk\ H_0 + rk(E(M) - H_0) - rk\ M = rk(E(M) - H_0) - 1 \leq |E(M) - H_0| - 1.$
Since M is k-connected, it has no j-separations for any j less than k. Thus $E(M) - H_0$ must have at least k elements.

Observe by Lemma 1.7.3 that M possesses free elements f_1 and f_2. Let H be a hyperplane of M with the maximum number of elements. Since $|E(M) - H| \geq k$, we may choose H so that f_1 and f_2 are not in H. Let F be a set of k-1 elements of E(M)-H with f_1 and f_2 being members of F. We shall show that F is a flat of M. Assume the contrary. Let x be in the closure of F but not in F. Then there is a circuit C contained in $F \cup \{x\}$. Since C has at most k elements, f_1 and f_2 are not in C. Thus C has at most k-2 elements contradicting Lemma 1.2.5. Thus F is a flat of M.

Let $\mathcal{M}$ be the modular cut of M generated by F and H, and M + e be the extension of M determined by $\mathcal{M}$. Evidently M + e is k-connected by Lemmas 1.2.6 and 4.2.1. Thus there is an element g in E(M+e) - {e} such that $(M+e)\backslash g$ is isomorphic to M.

Now $H \cup \{e\}$ is a hyperplane of $M + e$ which is larger than the largest hyperplane of M. Thus g must be in H as $(M+e)\backslash g$ is isomorphic to M. Therefore $F \cup e$ is a circuit of $(M+e)\backslash g$ as F and H are disjoint sets. Hence M has a circuit with fewer than rk M + 1 elements. It follows that M possesses a dependent hyperplane. Hence, by the choice of H, it is dependent in M.

We next show that $(H \cup \{e\}) - \{g\}$ is dependent in $(M+e)\backslash g$. Assume the contrary. By Theorem 1.3.2, there is a circuit C_1 of M + e that contains e and is contained in $H \cup \{e\}$. Evidently g is also in C_1. Since H is dependent in M, there exists a circuit C_2 of M contained in H. Thus C_2 is a circuit of M + e distinct from C_1. Now g must be in C_2. By circuit elimination, we see that M + e has a circuit C_3 contained in $(C_1 \cup C_2) - \{g\}$. Thus C_3 is a circuit of $(M+e)\backslash g$ which is contained in $(H \cup \{e\}) - \{g\}$; a contradiction. We conclude that $(H \cup \{e\}) - \{g\}$ is a dependent flat of $(M+e)\backslash g$.

Now $(H \cup \{e\}) - \{g\}$ and $F \cup e$ are dependent flats of $(M+e)\backslash g$ which meet in e. Thus $(M+e)\backslash g$ has at most one element in every dependent flat. However, this is a contradiction as M is isomorphic to $(M+e)\backslash g$, and M has at least two such elements by Lemma 1.7.3. This contradiction completes the proof of Theorem 4.1.4. □

The following lemma is used in the proof of Theorem 4.1.2.

4.2.2 Lemma. *Let N be a 4-connected matroid with at least four elements. If N has rank less than four, then N is isomorphic to one of $U_{2,4}$, $U_{2,5}$, and $U_{3,n}$ for some n at least five.*

Proof. It follows from Lemma 1.2.5 that both N and its dual are simple. Moreover, if N has at least six elements, then N has no dependent lines. The result follows immediately from these facts. □

We now begin the proof of the main result of the chapter.

Proof of Theorem 4.1.2. From Theorem 1.6.5 and the fact that $U_{2,4}$ is 4-connected, it follows that the set $\{U_{2,4}\}$ is (4,2)-rounded. We prove the converse of Theorem 4.1.2 in the remainder of the section. Suppose the set {M} is (4,2)-rounded for some 4-connected matroid M that has at least four elements and is not isomorphic to $U_{2,4}$. We shall derive a contradiction to complete the proof of Theorem 4.1.2.

The next two lemmas are used to prove Lemma 4.2.5 where it is shown that M has rank at least four.

4.2.3 Lemma. *The sets $\{U_{3,5}\}$ and $\{U_{3,6}\}$ are not (4,1)-rounded.*

Proof. Let N be isomorphic to $U_{4,7}$ with the ground set of N being $\{1,2,\ldots,7\}$. Evidently N is 4-connected (see, for example, [17]). Let $\mathcal{N}$ be the modular cut of N generated by the hyperplanes $\{1,2,3\}$, $\{1,4,5\}$, $\{1,6,7\}$, $\{2,4,6\}$, $\{2,5,7\}$, $\{3,4,7\}$, and $\{3,5,6\}$ of N. Observe that any two such hyperplanes meet in one element. Suppose that F_1 and F_2 are distinct flats of N other than E(N) each containing one of the generating hyperplanes of $\mathcal{N}$. Then F_1 and F_2 are both hyperplanes. Thus $\text{rk } F_1 + \text{rk } F_2 = 6$ but $\text{rk}(F_1 \cup F_2) + \text{rk}(F_1 \cap F_2) = 4 + 1 = 5$. Hence (F_1, F_2) is not a modular pair of flats and $F_1 \cap F_2$ is not in $\mathcal{N}$. Thus $\mathcal{N}$ consists only of the seven generating hyperplanes together with the flat E(N).

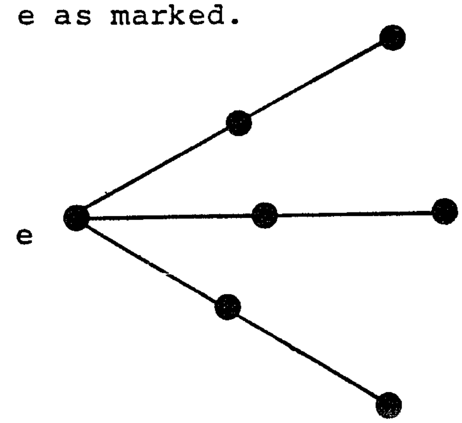
Let N + e be the extension of N determined by $\mathcal{N}$. It follows from Lemma 1.2.6 and Theorem 1.3.2 that N + e is 4-connected. Since N + e has a $U_{4,7}$-minor, it also has both $U_{3,5}$ and $U_{3,6}$ as minors. We next show that N + e has no $U_{3,5}$- or $U_{3,6}$-minor using e. This will complete the proof of the lemma.

As N + e is a 4-connected matroid with at least six elements, it has no triads by Lemma 1.2.5. Thus the deletion of any three elements from N + e produces a rank-4 matroid. Hence N + e has no restriction isomorphic to $U_{3,5}$ or $U_{3,6}$.

Let g be any element of N + e other than e. Then g is in exactly three circuits with four elements. A

Euclidean representation for the rank-three matroid (N+e)/g is given below with e as marked.

Figure 11 (N+e)/g



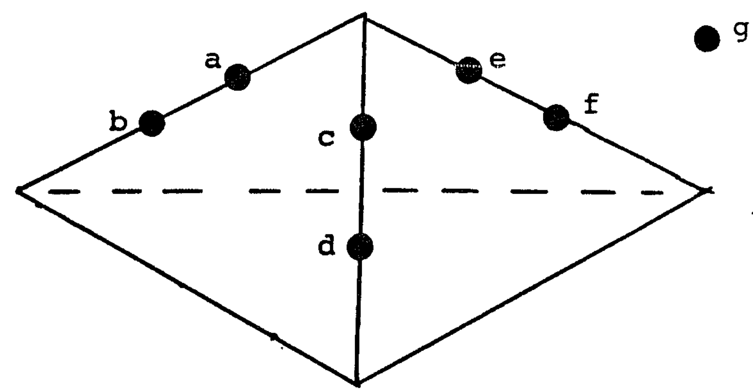
Evidently (N+e)/g has no $U_{3,5}$- or $U_{3,6}$-minor using e. Hence N + e has no $U_{3,5}$- or $U_{3,6}$-minor using e. □

4.2.4 Lemma. Let n be an integer exceeding six. The set $\{U_{3,n}\}$ is not (4,1)-rounded.

Proof. Let K be the rank-4 matroid whose Euclidean representation is given below.

Figure 12

K



We note that K is formed by freely adding the element g to the matroid $M(K_{2,3})$. Let N be the (n+2)-point matroid which is formed by freely adding an element to the flat {a,b,c,d} of K, and then freely adding n-6 elements to the flat {c,d,e,f}.

If P is a plane of N, then $rk_N(E(N)-P) = 4$. Using this fact it is easily checked that N is 4-connected. Now the contraction of g from N is isomorphic to $U_{3,n+1}$ and hence N has a $U_{3,n}$-minor. We shall show that N has no $U_{3,n}$-minor using g to complete the proof.

Let x be an element of E(N) other than g. Then N/x is an (n+1)-point matroid which has a line with at least four elements. Thus N/x has no $U_{3,n}$-minor. Clearly N has no restriction isomorphic to $U_{3,n}$. Hence N has no $U_{3,n}$-minor using g. Thus the set $\{U_{3,n}\}$ is not (4,1)-rounded. □

Since M is not isomorphic to $U_{2,4}$ we obtain, from Lemmas 4.2.2, 4.2.3, and 4.2.4, and duality:

4.2.5 Lemma. rk $M \geq 4$. □

From this result and Theorem 4.1.4, it follows that the set {M} is not (4,2)-rounded. This contradiction completes the proof of Theorem 4.1.2. □

CHAPTER 5

Subsets of 3-Connected Matroids

5.1 Introduction

This chapter is the result of joint work with Collette R. Coullard. We answer the following natural question. Let M be a 3-connected matroid. Suppose N is a 3-connected minor of M and S is a subset of E(M). How small a 3-connected minor of M can we find that both uses S and also has N as a minor? This question is answered in Theorems 5.1.1 and 5.1.2 for both the non-binary and binary cases, respectively.

A structure result relating a three-element subset in a 3-connected matroid to a 3-connected minor of that matroid is given in Theorem 5.1.3. This result is used in investigating the question mentioned above. The main results of this chapter are now given.

5.1.1 Theorem. *Let N be a 3-connected minor of the 3-connected matroid M. Suppose S is a subset of E(M) with at least three elements. Then there exists a 3-connected minor M_1 of M which uses S and has a minor N_1 that is isomorphic to N with $|E(M_1) - E(N_1)| \leq 3|S|-3$.*

In [35] and [38] Seymour provided results corresponding to the above theorem in the case that S has one or two elements. If M is binary, then the bound of $3|S|-3$ given in Theorem 5.1.1 can be improved as shown by the next result.

5.1.2 Theorem. *Let N be a 3-connected minor of a 3-connected binary matroid M. Suppose S is a subset of E(M) with at least three elements. Then there exists a 3-connected minor M_1 of M which uses S and has a minor N_1 that is isomorphic to N with* $|E(M_1) - E(N_1)| \leq 3|S|-4$.

The proofs of the last two results are given in Section 5.2. In Section 5.3 the bounds of $3|S| - 3$ and $3|S| -4$ given in these theorems are shown to be best-possible.

Let N be a 3-connected minor of a 3-connected matroid M. Suppose S is a subset of E(M). If M has no 3-connected proper minor that both uses S and has an N-minor, then M is said to be *minimal with respect to* N *and* S.

The following result is used in the proof of Theorem 5.1.2. This result is also proved in the next section.

5.1.3 Theorem. *Let N be a 3-connected minor of a 3-connected matroid M with a, b, and c being members of E(M). Let* $Z = E(M) - E(N)$ *and* $Y = \{a,b,c\} \cup Z$. *Suppose that M is minimal with respect to N and {a,b,c}. Then one of the following holds.*

(1) $|Z| = 6$ *and* $M|Y$ *or* $M^*|Y$ *is isomorphic to* $\mathcal{W}^3$.

(2) $|Z| < 6$ *and* $M|Y$ *or* $M^*|Y$ *is isomorphic to* $U_{3,5}$.

(3) $|Z| < 6$ *and* $M|Y$ *or* $M^*|Y$ *is isomorphic to a minor of* $\mathcal{W}^3$.

The chapter concludes in Section 5.3 with some applications of the results of this chapter to the theory of roundedness in matroids.

5.2 The Proofs

The proofs of Theorems 5.1.1, 5.1.2, and 5.1.3 are given in this section. Several results which are used in these proofs are now given. The first of these is due to Bixby.

5.2.1 Lemma [3,(1)]. Let M be a 3-connected matroid and e be a member of E(M). Then at least one of M\e and M/e is 3-connected. □

The following two results of Seymour are used in the proof of Theorem 5.1.3 as well as in Chapter 6.

5.2.2 Lemma [35, p.290]. Let N be a 3-connected minor of a 3-connected matroid M and a be a member of E(M). If M is minimal with respect to N and {a}, then either M = N or one of M\a and M/a is isomorphic to N. □

5.2.3 Lemma [38, (2.11)]. Let N be a 3-connected minor of a 3-connected matroid M and a and b be distinct elements of M with a being a member of E(N). Suppose M is minimal with respect to N and {a,b}. Then one of the following holds.

(i) M = N.

(ii) One of M\a, M\b, M/a, and M/b is isomorphic to N.

(iii) *For some* f *in* $E(M)$ *such that* $\{a,b,f\}$ *is a circuit of* M, *the minor* $M\backslash b/f$ *is isomorphic to* N.

(iv) *For some* f *in* $E(M)$ *such that* $\{a,b,f\}$ *is a cocircuit of* M, *the minor* $M\backslash f/b$ *is isomorphic to* N. □

The next result of Bixby and Coullard is a key component of the proofs of Theorems 5.1.1, 5.1.2, and 5.1.3.

5.2.4 *Theorem* [4,(5.1)]. *Let* N *be a* 3-*connected minor of a* 3-*connected matroid* M. *Suppose* M *and* N *have at least four elements, and* c *is a member of* $E(M)$. *If* M *has no* 3-*connected proper minor using* c *which has* N *as a minor, then, up to duality, one of the following holds.*

(i) $|E(M) - E(N)| \leq 1$.

(ii) *For some* f *in* $E(M)$ *and* n *in* $E(N)$ *such that* $\{c,f,n\}$ *is a circuit of* M, $N = M\backslash c/f$.

(iii) *For some* f *and* g *in* $E(M)$ *and* n *in* $E(N)$ *such that* $\{c,f,n\}$ *is a circuit, and* $\{f,g,n\}$ *is a cocircuit of* M, $N = M\backslash\{c,g\}/f$.

(iv) *For some* f *and* g *in* $E(M)$ *and distinct* n *and* m *in* $E(N)$ *such that* $\{c,f,g\}$ *is a cocircuit, and* $\{c,f,n\}$ *and* $\{c,g,m\}$ *are circuits of* M, $N = M\backslash\{c,g\}/f$.

(v) *For some* f, g, *and* h *in* $E(M)$ *and* n *in* $E(N)$ *such that* $\{f,g,n\}$ *is a cocircuit, and* $\{c,f,n\}$ *and* $\{g,h,n\}$ *are circuits of* M, $N = M\backslash\{c,g\}/\{f,h\}$. *Moreover,* $M\backslash c/f$ *and* $M\backslash h/g$ *are isomorphic.* □

We shall first prove Theorem 5.1.3 as this result is used in the proof of Theorem 5.1.2.

Proof of Theorem 5.1.3. We obtain 3-connected minors $N=N_0, N_1, N_2$, and N_3 of M with $a \in E(N_1)$, $\{a,b\} \subseteq E(N_2)$, and $\{a,b,c\} \subseteq E(N_3)$ as follows. First apply Lemma 5.2.2 to N and $a \in E(M)$. We obtain a 3-connected minor N_1 of M which uses a and has an N-minor with $|E(N_1)| - |E(N)| \leq 1$. Then apply Lemma 5.2.3 to N_1 and the set $\{a,b\}$. We obtain a 3-connected minor N_2 of M which uses $\{a,b\}$ and has an N_1-minor with $|E(N_2)| - |E(N_1)| \leq 2$. Finally, apply Lemma 5.2.4 to N_2 and c. We obtain a 3-connected minor N_3 of M which uses $\{a,b,c\}$ and has an N_2-minor.

M is minimal with respect to N and $\{a,b,c\}$, and N_3 is a 3-connected minor of M using $\{a,b,c\}$. Hence $N_3 = M$. It follows from the minimality of M with respect to N and $\{a,b,c\}$, that N_3 is obtained from N_2 by one of cases (i), (ii), (iii), and (iv) in Lemma 5.2.4.

For each j in $\{1,2,3\}$, let i_j be $|E(N_j)| - |E(N_{j-1})|$. We obtain from Lemmas 5.2.2 and 5.2.3 and Theorem 5.2.4 that i_j is at most j for each j in $\{1,2,3\}$. Hence,

(5.2.5) $\quad |E(M)| - |E(N)| \leq 6.$

The next structure result forms the core of the proof of Theorem 5.1.3. This result is a generalization of Lemma 5.2.3 to three-element subsets of a matroid.

For this reason, a more extensive list of cases is needed to describe the structure of M than was given in Lemma 5.2.3.

5.2.6 Lemma. The structure of M, up to duality and permutations of the set {a,b,c} , is as given in one of the following cases.

(1) $|Z| \leq 3$.

(2) For some f and g in E(M) such that {c,f,g} is a cocircuit, and {a,c,f} and {b,c,g} are circuits of M, the minor M \{c,g}/f is isomorphic to N_1 or N_2.

(3) For some f and g in E(M) such that {b,c,f} is a circuit of M, the minor M \{b,c} /{f,g} is isomorphic to N or N_1. Moreover, {a,b,g} is a circuit of the minor M\c/f which is 3-connected.

(4) For some f and g in E(M) such that {a,b,c,g} is a cocircuit, and {b,c,f} is a circuit of M, the minor M\{c,g}/{b,f} is isomorphic to N or N_1. Moreover, M\c/f is 3-connected.

(5) For some f in E(M) such that {b,c,f} is a circuit of M, the minor M\c/f is isomorphic to N_2. Moreover, $E(N_2) - E(N) = \{a,b\}$.

(6) For some f in E(M) such that either {a,b,f} or {a,b,c,f} is a circuit of M, the minor M\b/{c,f} is isomorphic to N_1. Moreover, M/c is 3-connected.

(7) <u>For some</u> f <u>in</u> E(M) <u>such that</u> {a,b,f} <u>is a circuit of</u> M, <u>the minor</u> M\{b,c}/f <u>is isomorphic to</u> N_1. <u>Moreover</u>, M\c <u>is</u> 3-<u>connected and</u> $E(N_1) - E(N) = \{a\}$.

(8) <u>For some</u> f,g, <u>and</u> h <u>in</u> E(M) <u>such that</u> {c,f,g} <u>is a cocircuit</u>, <u>and</u> {a,c,f} <u>and</u> {b,c,g}, <u>are circuits of</u> M, <u>the minor</u> M\{c,g}/f <u>is isomorphic to</u> N_2. <u>Either</u> $N_2\backslash b/h$ <u>is isomorphic to</u> N_1 <u>and</u> {a,b,h} <u>is a circuit of</u> M <u>while</u> {f,g,h} <u>is not</u>, <u>or</u> $N_2\backslash h/b$ <u>is isomorphic to</u> N_1 <u>and</u> {a,b,c,g,h} <u>is a cocircuit of</u> M.

<u>Proof</u>. Recall that, for each j in {1,2,3}, $i_j = |E(N_j)| - |E(N_{j-1})|$ and i_j is at most j. Also, $N_0=N$ and $N_3=M$. Thus $i_1 + i_2 + i_3$ is at most six.

If $i_1 + i_2 + i_3 \leq 3$, then M is as given in (5.2.6)(1). Suppose $i_1 + i_2 + i_3$ exceeds three. Then (i_1, i_2, i_3) is a member of the set {(1,2,3), (0,2,3), (1,1,3), (0,1,3), (1,0,3), (1,2,2), (0,2,2), (1,1,2), (1,2,1)}. We shall show that for such (i_1, i_2, i_3), the matroid M is as given in one of the cases (2) through (8) of Lemma 5.2.6. This will conclude the proof of Lemma 5.2.6.

We first show that, up to permutations of the set {a,b,c}, M has the same structure if it is obtained from N_2 by either of cases (iii) and (iv) of Theorem 5.2.4.

5.2.7 <u>Lemma</u>. <u>If</u> $i_3=3$, <u>then we may assume that</u> M <u>is obtained from</u> N_2 <u>by case</u> (iv) <u>of Theorem</u> 5.2.4 <u>with</u> n=a <u>and</u> m=b.

<u>Proof</u>. Suppose M is obtained from N_2 by case (iii) of Theorem 5.2.4. Then there are elements f and g in $E(M)$ and n in $E(N_2)$ such that $\{c,f,n\}$ is a circuit, and $\{f,g,n\}$ is a cocircuit of M, and $M\backslash\{c,g\}/f \cong N_2$. Moreover, n is in $\{a,b\}$ by the minimality of M with respect to N and $\{a,b,c\}$. We may assume that $n=b$.

It follows from Lemma 5.2.1 that $\widetilde{(M/g)}$ is 3-connected as $\widehat{(M\backslash g)}$ is not. Now $\widetilde{(M/g)}$ has an N-minor. Thus $\{a,b,c\} \not\subseteq E\widetilde{(M/g)}$ by the minimality of M. Hence one of $\{a,b,g\}$, $\{a,c,g\}$, and $\{b,c,g\}$ is a circuit of M. By orthogonality, $\{a,c,g\}$ is not a circuit. If $\{b,c,g\}$ is a circuit, then, by circuit elimination, $\{b,f,g\}$ is a circuit of M. This contradicts the 3-connectivity of M since $\{b,f,g\}$ is also a cocircuit of M, and M has at least five elements. Thus $\{a,b,g\}$ is a circuit of M.

Now suppose M is obtained from N_2 by case (iv) of Theorem 5.2.4. Then there are elements f and g in $E(M)$ and n and m in $E(N_2)$ such that $\{c,f,g\}$ is a cocircuit, and $\{c,f,n\}$ and $\{c,g,m\}$ are circuits of M with $M\backslash\{c,g\}/f \cong N$. Now $\{n,m\} = \{a,b\}$ by the minimality of M with respect to N and $\{a,b,c\}$. Thus, allowing permutations of $\{a,b,c\}$, M has the same structure as obtained when case (iii) of Theorem 5.2.4 was used. This completes the proof of Lemma 5.2.7. □

It follows from Lemma 5.2.7 that if (i_1,i_2,i_3) is one of (0,1,3), (1,0,3), and (1,1,3), then M is as given in Lemma 5.2.6 (2).

Suppose that $(i_1,i_2,i_3) = (1,2,3)$. Then N_2 is obtained from N_1 by Lemma 5.2.3 (iii) or (iv). Suppose the latter occurs. Then, for some h in $E(N_2)$ such that {a,b,h} is a cocircuit of N_2, we have $N_2\backslash h/b \cong N_1$. By Lemma 5.2.7, as {c,f,g} is a cocircuit of M,

(5.2.8) $N_2 = M \{c,g\}/f = M \{c,f\}/g = M \{f,g\}/c.$

By orthogonality and (5.2.8), either {a,b,c,h} or {a,b,c,g,h} is a cocircuit of M. The former cannot occur by the minimality of M. Hence 5.2.6(8) holds. Suppose N_2 is obtained from N_1 by 5.2.3(iii).

Let h' be in $E(N_2)$ such that {a,b,h'} is a circuit of N_2 with $N_2\backslash b/h' \cong N_1$. Evidently, {a,b,h'} is also a circuit of M by (5.2.8). If {f,g,h'} is a circuit of M, then $(\widehat{M\backslash h})$ is 3-connected, uses {a,b,c}, and has an N-minor; a contradiction. Thus {f,g,h'} is not a circuit of M. Hence M must be as given in Lemma 5.2.6 (8). Similarly, if $(i_1,i_2,i_3) = (0,2,3)$, then M is as given in Lemma 5.2.6(8).

The cases where (i_1,i_2,i_3) is in {(1,2,2), (0,2,2), (1,1,2), (1,2,1)} remain to be checked. We first consider the cases with $i_3=2$. Suppose i_2 is also two. M is obtained from N_2 by Theorem 5.2.4(ii). Thus $M\backslash c/f \cong N_2$ for some f in E(M) and some n in $E(N_2)$ such that $\{c,f,n\}$ is a circuit of M. Evidently n is in $\{a,b\}$ by the minimality of M. We will assume that $n=b$ without loss of generality.

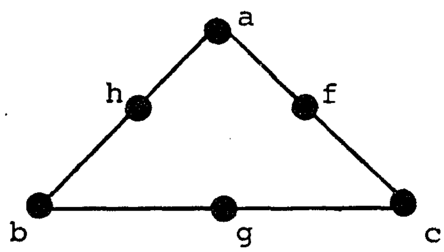
As $i_2 = 2$, N_2 is obtained from N_1 by case (iii) or (iv) of Lemma 5.2.3. Suppose the former holds. Then M is as given in Lemma 5.2.6(3). Suppose the latter holds. Then there is an element g in $E(N_2)$ such that $\{a,b,g\}$ is a cocircuit of N_2 and $N_2\backslash g/b \cong N_1$. By orthogonality, $\{a,b,c,g\}$ is a cocircuit of M. It follows that M is as given in Lemma 5.2.5(4). Thus the lemma is true if (i_1,i_2,i_3) is $(1,2,2)$ or $(0,2,2)$. If $(i_1,i_2,i_3)=(1,1,2)$, then M is as given in Lemma 5.2.6(5).

Finally, suppose that $(i_1,i_2,i_3) = (1,2,1)$. We may apply duality to assume that N_2 is obtained from N_1 by Lemma 5.2.3(iii). Thus, for some f in $E(N_2)$ such that $\{a,b,f\}$ is a circuit of N_2, the minor $N_2\backslash b/f$ equals N_1. Since $i_3=1$, either $M\backslash c$ or M/c equals N_2. If the former holds, then M is as given in Lemma 5.2.6(7). Suppose $M/c = N_2$. Then M is as given in Lemma 5.2.6(6). This completes the proof of Lemma 5.2.6. □

We now complete the proof of Theorem 5.1.3. Recall that $Z = E(M) - E(N)$ and $Y = Z\cup\{a,b,c\}$. We have shown that M is as given in one of cases (1) through (8) of Lemma 5.2.6. Suppose $|Z| = 6$. Then M is as given in case (8) of Lemma 5.2.6. Thus $\{a,c,f\}$, $\{b,c,g\}$, and $\{a,b,h\}$ are circuits of M. Also, $\{f,g,h\}$ is not a circuit of M. A

Euclidean representation for $M|Y = M|\{a,b,c,f,g,h\}$ is given below.

Figure 13 $M|Y$



We observe from Figure 13 that $M|Y$ is the rank-three whirl. It is easily checked that if M is as given in one of cases (1) through (7) of Lemma 5.2.6, then $M|Y$ is either isomorphic to $U_{3,5}$, or is isomorphic to a minor of $\mathcal{W}^3$. This completes the proof of Theorem 5.1.3.□

We now derive Theorem 5.1.1 from Lemmas 5.2.2 and 5.2.3 and Theorem 5.2.4.

Proof of Theorem 5.1.1. The result is proved by induction on $|S|$. Suppose N has at least four elements. Then, by (5.2.5), the theorem is true if S has exactly three elements. Assume that S has more than three elements, and that the theorem is true for sets with fewer elements than S.

Let $s \in S$. By the induction hypothesis, there is a 3-connected minor M_0 of M that uses $S-\{s\}$, and has an N-minor, N_0, with $|E(M_0) - E(N_0)| \leq 3|S - \{s\}| - 3 = 3|S| - 6$. Now apply Theorem 5.2.4 to M_0 and s. We obtain from cases

(i) through (iv) of this theorem that there is a 3-connected minor M_1 of M that uses S, has an isomorphic copy of M_0 as a minor, and has at most three more elements than M_0. Thus M, possesses an N-minor, and has at most $3|S| - 6 + 3 = 3|S| - 3$ more elements than N. It follows that the theorem is true if N has at least four elements.

Suppose N has fewer than four elements. Then, by (3.3.2), N is isomorphic to one of $U_{0,1}$, $U_{1,1}$, $U_{1,2}$, $U_{1,3}$, and $U_{2,3}$. In particular, N is a minor of the matroid $U_{2,4}$. Clearly, the theorem is true if M has fewer than six elements. Hence we may assume that M has at least six elements.

Suppose M is non-binary. Let e and f be elements of S. It follows from Theorem 1.6.5 that M has a $U_{2,4}$-minor using both e and f. Apply Theorem 5.2.4 to this $U_{2,4}$-minor and the elements of S- {e,f}. It is an easy induction argument to show that M has a 3-connected minor M_1 using S such that M_1 has at most $3|S| - 6$ more elements than some $U_{2,4}$-minor of M. Thus M_1 has an N-minor, and M_1 has at most $3|S| - 6 + 3 = 3|S| - 3$ more elements than N.

Suppose M is binary. Let e, f, and g be elements of M. Now M has $M(\mathcal{W}_3)$ as a minor by Theorem 1.2.2. Moreover, by Theorem 2.3.3, M has an $M(\mathcal{W}_3)$-minor using {e,f,g}. Apply Theorem 5.2.4 to this $M(\mathcal{W}_3)$-minor and the elements of S-{e,f,g}. Again, it is easy to show by induction that M has a 3-connected minor M_1 using S such that M_1 has at most

$3|S| - 9$ more elements than some $M(\mathcal{W}_3)$-minor of M_1. Now N has at most five fewer elements than $M(\mathcal{W}_3)$. Thus we obtain:

(5.2.9) M _has a_ 3-_connected minor_ M_1 _using_ S. _The minor_ M_1 _has at most_ $3|S| - 4$ _more elements than some_ N-_minor of_ M_1.

This completes the proof of Theorem 5.1.1. □

The section concludes with the next proof.

Proof of Theorem 5.1.2. If N has fewer than four elements, then the theorem is true by (5.2.9). Assume that N has at least four elements. The result is proved by induction on $|S|$.

Suppose that S has exactly three elements. Assume that M is minimal with respect to N and S. Then M is as given in one of cases (1) through (8) of Lemma 5.2.6. If M is of the form given in case (8) of Lemma 5.2.6, then, by Theorem 5.1.3(1), M is non-binary; a contradiction. Hence, M is of the form given in one of cases (1) through (7) of Lemma 5.2.6. Thus M has at most five more elements than N. Hence the theorem is true if S has three elements.

Suppose S has more than three elements and the theorem is true for sets with fewer than $|S|$ elements. Let $s \in S$. By the induction hypothesis, there is a 3-connected minor M_0 of M that uses $S-\{s\}$ and has an N-minor, N_0, with $|E(M_0) - E(N_0)| \leq 3|S - \{s\}| - 4 = 3|S| - 7$. Now apply Theorem 5.2.4 to M_0 and s. Again, by cases (i) through (iv) of Theorem 5.2.4, there is a 3-connected minor M_1

of M that uses S, has an M_0-minor, and has at most $3|S| - 7 + 3 = 3|S| - 4$ more elements than N. The result follows by induction. □

5.3 Examples and Applications

In this section we show that the bounds given in Theorems 5.1.1 and 5.1.2 are best-possible. Then the result of this chapter are used to obtain a method for embedding a matroid in a rounded set.

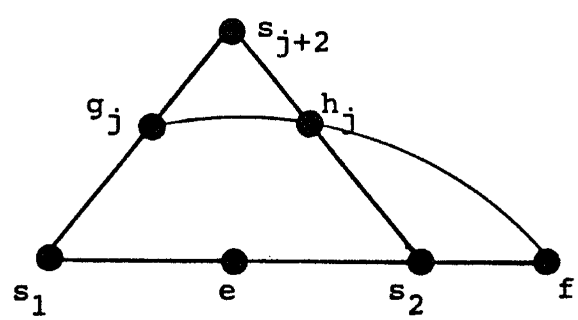
Let X and Y be disjoint subsets of a matroid M. Then $k_M(X,Y)$ is defined to be $\min\{rk_M A + rk_M B - rkM:$ (A,B) is a bipartition of $E(M)$ with $X \subseteq A$ and $Y \subseteq B\}$. The following results of Seymour and Brylawski, respectively, are used to show that the bound given in Theorem 5.1.1 is best-possible.

5.3.1 Lemma [38,(2.3)]. *If N is a minor of M and X and Y are disjoint subsets of E(N), then* $k_N(X,Y) \leq k_M(X,Y)$.

5.3.2 Lemma [7,(3.4)]. *A hyperplane of a matroid is a modular flat if and only if it meets every line.*

Let j be a positive integer. Let T_j be the rank-three matroid whose Euclidean representation is given below.

Figure 14 T_j



By Lemma 5.3.2, $\{e,f,s_1,s_2\}$ is a modular flat of T_j.

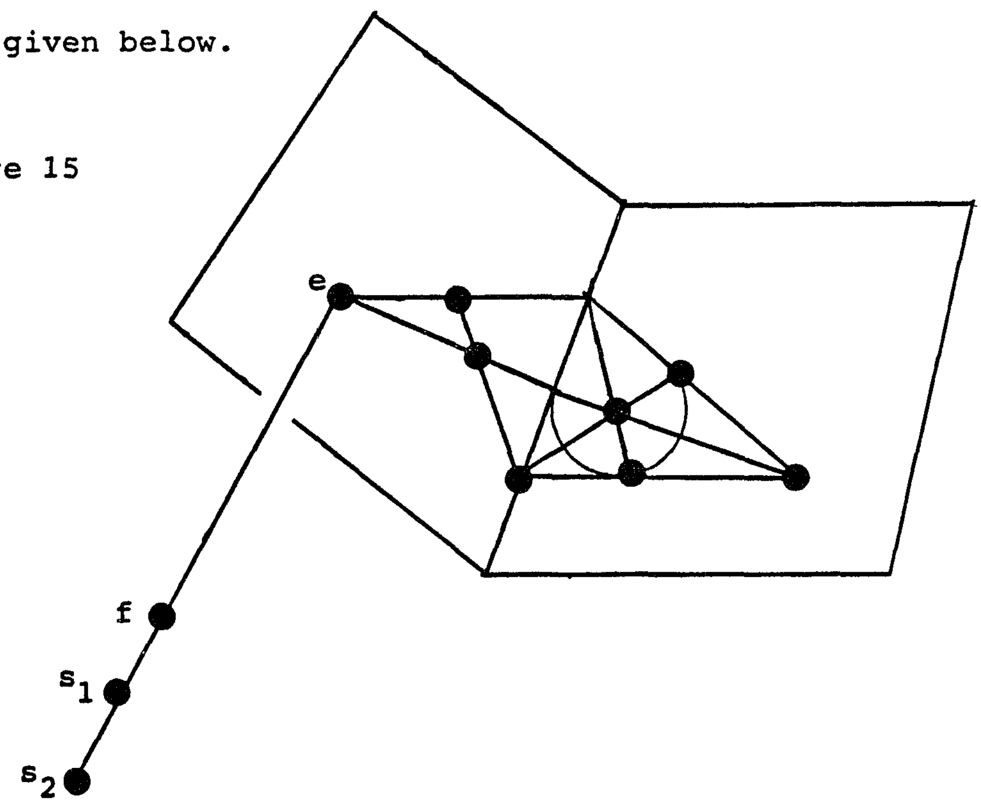
We next construct an eleven-element matroid N. Then this matroid will be combined with the matroid T_j using the operation of generalized parallel connection mentioned in Section 1.3.

Consider the representation $J_{10}\backslash\{e_9,e_{10}\}$ of S_8 given in Table 1. Let $e = e_4$. Note that e is the unique element of S_8 whose contraction produces a Fano-minor. Freely add f to S_8 to form the matroid N_1. Then freely add s_1 to $\sigma_{N_1}\{e,f\}$ in N_1 to form N_2. Finally, freely add s_2 to $\sigma_{N_2}\{e,f\}$ to form N. Note that both N and N\f are 3-connected by Lemma 1.3.1 and Theorem 1.3.2.

A Euclidean representation for the rank-four matroid N is given below.

Figure 15

N



We will use the next lemma in the proof of Lemma 5.3.5.

5.3.3 Lemma. The circuits of N containing s_1 or s_2 and having fewer than five elements are the subsets of $\{e,f,s_1,s_2\}$ with three elements.

Proof. We first show that

(5.3.4) if C is a circuit of N_2 that contains f or s_1 and has fewer than five elements, then $C = \{e,f,s_1\}$.

Proof. Suppose that $C \neq \{e,f,s_1\}$.

If $f \in C$, then, as f is free in N_1, we must have that $s_1 \in C$. Thus we may suppose that $s_1 \in C$. Then, by Theorem 1.3.2, $\sigma_{N_2}(C) = F \cup \{s_1\}$ where F is a flat of N_1 containing $\sigma_{N_1}\{e,f\} = \{e,f\}$. Thus f is in $\sigma_{N_2}(C) = \sigma_{N_2}(C - \{s_1\})$. It follows that if f is not in C, then f is in a circuit of N_2 which does not contain s_1, and has fewer than five elements. This contradicts the fact that f is free in N_1. Thus f is in C. By circuit elimination, there is a circuit of N_2 contained in $(C \cup \{e,f,s_1\}) - \{s_1\}$. This circuit has fewer than five elements again contradicting the fact that f is free in N_1. Thus $s_1 \notin C$; a contradiction. □

Let C be a circuit of N that is not contained in $\{e,f,s_1,s_2\}$ and has fewer than five elements. Suppose s_1 is in C. Then s_2 is also in C by (5.3.4). Thus, by Theorem 1.3.2, $\sigma_N(C) = F \cup \{s_2\}$ where F is a flat of N_2 containing $\sigma_{N_2}\{e,f\} = \{e,f,s_1\}$. Evidently f is not in C as C is not a subset of $\{e,f,s_1,s_2\}$. However, f is in $\sigma_N(C) = \sigma_N(C - \{s_2\})$. This contradicts (5.3.4). Thus $s_1 \notin C$.

Suppose s_2 is in C. Then s_1 is in $\sigma_N(C - \{s_2\})$ but not C. This contradicts (5.3.4). Thus $s_2 \notin C$. This completes the proof of Lemma 5.3.3. □

Let n be an integer exceeding two. We recursively define the matroid $P = P_F(T_1,T_2,\ldots,T_{n-2},N)$ as follows. Let $P_1 = P_F(T_1,N)$. If n exceeds three, then, for each i in $\{1,2,\ldots,n-3\}$, let $P_{i+1} = P_F(T_{i+1},P_i)$. Define P to be P_{n-2}. Now N, and hence P, has a Fano-minor. We shall show in Lemmas 5.3.5 through 5.3.8 that $P\backslash f$ is minimal with respect to F_7 and $\{s_1,s_2,\ldots,s_n\}$. Since $P\backslash f$ has $(3n+4) - 7 = 3n - 3$ more elements than F_7, this will show that the bound given in Theorem 5.1.1 is best-possible.

5.3.5 Lemma. $P\backslash f$ *is* 3-*connected*.

Proof. We argue by induction on n. Suppose n is 3 and (A,B) is a k-separation of $P\backslash f$ for some $k<3$. Now both $T_1\backslash f$ and $N\backslash f$ are 3-connected minors of $P\backslash f$ and hence have no k-separations. By Lemma 5.3.1,

$$\begin{aligned}
&rk_{N\backslash f}(A\cap E(N\backslash f)) + rk_{N\backslash f}(B\cap E(N\backslash f)) - rk(N\backslash f)\\
&= k_{N\backslash f}(A\cap E(N\backslash f),\ B\cap E(N\backslash f))\\
&\le k_{P\backslash f}(A\cap E(N\backslash f),\ B\cap E(N\backslash f))\\
&\le rk_{P\backslash f}A + rk_{P\backslash f}B - rk(P\backslash f)\\
&< k.
\end{aligned}$$

Thus A or B meets $E(N\backslash f)$ in fewer than two elements. Without loss of generality, suppose the former. A similar argument shows that A or B meets $E(T_1\backslash f)$ in fewer than two elements.

Now $F = \{e,f,s_1,s_2\} = E(T_1) \cap E(N)$. Since A meets $F - \{f\}$ in at most one element, B meets $F - \{f\}$ in at least two elements. Hence, as B meets $E(T_1\backslash f)$ in at least two elements, A meets $E(T_1\backslash f)$ in at most one element. Thus A has at most two elements. It is easily checked that both $P\backslash f$ and $(P\backslash f)^*$ are simple. Thus, by (1.2.4),

$$\begin{aligned} k &\leq |A| \\ &= rk_{P\backslash f}A + rk_{(P\backslash f)^*}A - |A| \\ &= rk_{P\backslash f}A + rk_{P\backslash f}B - rk(P\backslash f) \\ &< k; \text{ a contradiction.} \end{aligned}$$

Thus the lemma is true if n is 3. Suppose n exceeds three and the lemma is true for integers m with $3 \leq m < n$. Then a similar argument shows the result still holds. We conclude that $P\backslash f$ is 3-connected. □

We require two more lemmas before showing that $P\backslash f$ is minimal with respect to F_7 and $\{s_1,s_2,\ldots,s_{n-2}\}$.

5.3.6 Lemma. $N\backslash e$ has no Fano-minor. Let $x \in E(N) - F$. Then neither $N\backslash x$ nor N/x has a Fano-minor.

Proof. Suppose Q is a Fano-minor of N. Evidently N has no restriction which is isomorphic to Q. Thus Q is a minor of N/x for some x in E(N). By Lemma 5.3.3, if x is not e, then N/x is a rank-three matroid which does not have a Fano-minor; a contradiction. Thus x=e. Hence Q is a minor of N/e. By Lemma 5.3.3, none of f, s_1 and s_2 is in a triangle of N/e. Thus $Q = N\backslash\{f,s_1,s_2\}/e$. □

5.3.7 Lemma. *Suppose Q is a Fano-minor of* P\f. *Then Q is a minor of* N\f.

Proof. Clearly, for each i in {1,2,...,n-2}, Q is not a minor of $T_i\backslash f$. Now Q is a 3-connected rank-three matroid. It follows from Theorem 1.3.4 that Q is a minor of N\f. □

We now show that the bound given in Theorem 5.1.1 is best-possible.

5.3.8 Lemma. P\f *is a 3-connected matroid which is minimal with respect to* F_7 *and* $\{s_1,s_2,\ldots,s_{n-2}\}$.

Proof. P\f is 3-connected by Lemma 5.3.5. Suppose M is a 3-connected minor of P\f that uses $\{s_1,s_2,\ldots,s_n\}$ and has a Fano-minor. Moreover, suppose M is minimal with respect to F_7 and $\{s_1,s_2,\ldots,s_n\}$.

Suppose $(P\backslash f)\backslash X/Y = M$. By Lemma 5.3.6, $e \notin X$. As $\{s_1,s_2\}$ is independent in M, $e \notin Y$. Thus, by Lemma 5.3.6, $E(N) \subseteq E(M)$.

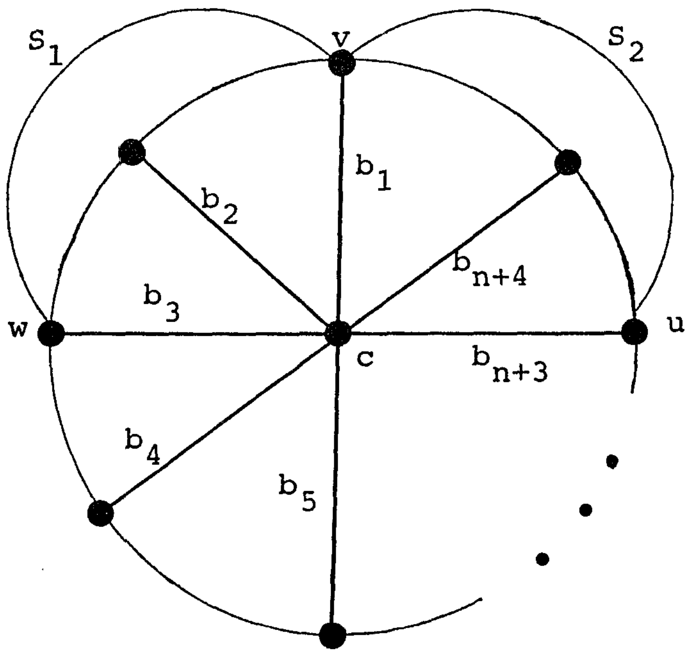
For each j in $\{1,2,\ldots,n-2\}$, let g_j and h_j be the elements of the matroid T_j given in Figure 14. As M is 3-connected, for each such j, neither g_j nor h_j is in X. Moreover, as M is simple, for each such j, neither g_j nor h_j is in Y. Thus $E(M) = E(P\backslash f)$ and $P\backslash f$ is minimal with respect to F_7 and $\{s_1,s_2,\ldots,s_{n-2}\}$. □

We next show that the bound given in Theorem 5.1.2 is best-possible. Let n be an integer exceeding two. We shall construct a 3-connected graphic matroid $M(G)$ with $5n + 4$ elements. This matroid possesses an n-element subset S such that $M(G)$ is minimal with respect to an $M(\mathcal{W}_{n+4})$-minor and S. This matroid has $(5n+4) - (2n+8) = 3n-4$ more elements than $M(\mathcal{W}_{n+4})$. This will show that the bound given in Theorem 5.1.2 is best-possible.

Let H be the graph given below.

Figure 16

H



Form the graph G from H as follows. Add new vertices $v_1, v_2, \ldots,$ and v_{n-2} to H so that these vertices are isolated. Then add the edges $s_{i+2} = (v_i, v)$, $c_{i+2} = (v_i, w)$, and $d_{i+2} = (v_i, u)$ for each i in $\{1, 2, \ldots, n-2\}$. Evidently M(G) is 3-connected by Lemma 1.2.7. The next lemma is used in showing that M(G) is minimal with respect to $M(\mathcal{W}_{n+4})$ and $\{s_1, s_2, \ldots, s_n\}$.

5.3.9 Lemma. Let e be an edge in $E(H) - \{s_1, s_2\}$. Then neither $M(G)\backslash e$ nor $M(G)/e$ has an $M(\mathcal{W}_{n+4})$-minor.

Proof. Let Q be a $\mathcal{W}_{n+4}$-minor of G. We will show that E(Q) consists of the edges of $E(H) - \{s_1, s_2\}$.

Let i and j be distinct members of $\{3, 4, \ldots, n\}$. Suppose that both $\{c_i, d_i, s_i\}$ and $\{c_j, d_j, s_j\}$ are in E(Q). Then v_i and v_j are degree-three vertices of Q having three common neighbors. This is a contradiction as $\mathcal{W}_{n+4}$ does not possess two such vertices. Hence there exist no such i and j. Thus we may assume, without loss of generality, that Q is a minor of the subgraph G_0 of G induced by $V(H) \cup \{v_1\}$.

Let X and Y be subsets of $V(G_0)$ such that $G_0\backslash X/Y = Q$. Now $\text{rk}\, M(G_0) = \text{rk}\, Q + 1$. Thus Y has at most one element.

Suppose Y is empty. Then c is the only vertex of G_0 of degree at least n+4. Thus c is not in X and c is the unique vertex of Q of degree n+4. Hence $E(Q) = E(H) - \{s_1, s_2\}$.

Suppose $|Y| = 1$. Let $Y = \{e\}$. Then, as G_0/e must have a vertex of degree $n+4$, e is in $\{c_3, d_3, s_3, b_1, b_3, b_{n+3}\}$. If e is in $\{c_3, d_3, s_3\}$, then it is immediate that $E(Q) = E(H) - \{s_1, s_2\}$. If e is in $\{b_1, b_3, b_{n+3}\}$, then it is easily checked that G_0 has no $M(\mathcal{W}_{n+4})$-minor. Thus $E(Q) = E(H) - \{s_1, s_2\}$. □

We next show that $M(G)$ is a 3-connected matroid which is minimal with respect to $M(\mathcal{W}_{n+4})$ and $\{s_1, s_2, \ldots, s_n\}$ thereby showing that the bound given in Theorem 5.1.2 is best-possible.

5.3.10 Lemma. $M(G)$ *is minimal with respect to* $M(\mathcal{W}_{n+4})$ *and* $\{s_1, s_2, \ldots, s_n\}$.

Proof. Let M be a 3-connected minor of $M(G)$ using $\{s_1, s_2, \ldots, s_n\}$ which is minimal with respect to an $M(\mathcal{W}_{n+4})$-minor and $\{s_1, s_2, \ldots, s_n\}$. By Lemma 5.3.9, $E(H) - \{s_1, s_2\}$ is in $E(M)$. As $\{s_1, s_2, \ldots, s_n\}$ is both independent and coindependent in M, $E(G) - E(H) \subseteq E(M)$. Thus $M = M(G)$. □

We conclude the chapter with some applications to roundedness. Specifically, we show how to embed a 3-connected matroid into a (3,1)- or (3,2)-rounded set. An alternate method for constructing (3,1)-rounded sets was given by Oxley and Row [31].

5.3.11 Theorem. *Let N be a 3-connected matroid with at least four elements. Suppose $\mathcal{S}$ = {K: K is a 3-connected extension or lift of N, and K possesses an element which is in no N-minor of K}. Then $\mathcal{S} \cup \{N\}$ is (3,1)-rounded.*

Proof. Let M be a 3-connected matroid having a minor in $\mathcal{S} \cup \{N\}$. Evidently M has N as a minor. Let $e \in E(M)$. By Theorem 5.2.2, there exists a 3-connected minor M_1 of M using e such that either M_1 is isomorphic to N, or M_1 is an extension or lift of an N-minor. If M_1 is not isomorphic to a member of $\mathcal{S} \cup \{N\}$, then M_1 has an N-minor using e. It follows that $\mathcal{S} \cup \{N\}$ is (3,1)-rounded. □

Note that the rounded sets listed in Theorems 1.6.7 through 1.6.11 are all closed under duality. By using Theorem 5.3.11 we next show that this is not always the case. The matroids P_6 and Q_6 are as given in Table 1.

5.3.12 Theorem. *The set $\{U_{2,5}, Q_6\}$ is (3,1)-rounded.*

Proof. It is easily checked that the matroids $U_{2,6}$, $U_{3,6}$, P_6, and Q_6 are the only 3-connected extensions or lifts of $U_{2,5}$. Now Q_6 is the only such matroid that possesses an element which is in no $U_{2,5}$-minor. Let $N = U_{2,5}$ and $\mathcal{S} = \{Q_6\}$. The set $\mathcal{S} \cup \{N\}$ is (3,1)-rounded by Theorem 5.3.11. □

A similar construction is given for (3,2)-rounded sets in the next result.

5.3.13 Theorem. *Let N be a 3-connected matroid with at least four elements. Suppose $\mathcal{S}$ = {M: M is a 3-connected matroid having N as a minor, $|E(M) - E(N)| \leq 3$, and M possesses a pair of elements which are in no N-minor.}. Then $\mathcal{S} \cup \{N\}$ is (3,2)-rounded.*

Proof. Let M be a 3-connected matroid having a minor in $\mathcal{S} \cup \{N\}$ and e and f be elements of M. Thus M has N as a minor. By Lemmas 5.2.2 and 5.2.3, M has a minor M' using {e,f} such that M' has an N-minor, and $|E(M')| - |E(N)| \leq 3$. If M' is not isomorphic to a member of $\mathcal{S} \cup \{N\}$, then M' has an N-minor using {e,f}. Thus M has an $(\mathcal{S} \cup \{N\})$-minor using e and f. □

CHAPTER 6

Triangles in 3-Connected Matroids

6.1 Introduction

The relationship between a three-element subset S of a 3-connected matroid, and a 3-connected minor of that matroid was studied in Lemma 5.2.6 of Chapter 5. In this chapter this relationship is investigated in the special case that S is a triangle. We begin with the following consequence of Lemma 5.2.6.

6.1.1 Theorem. *Let {a,b,c} be a triangle of a 3-connected matroid M, and N be a 3-connected minor of M. Then M has a 3-connected minor M' using {a,b,c} such that M' has an N-minor, and M' has at most four more elements than N.*

The proof of this result will be given in Section 6.2. In particular, if M is binary, then a somewhat sharper result is obtained.

6.1.2 Theorem. *Let {a,b,c} be a triangle of a 3-connected binary matroid M, and N be a 3-connected minor of M. Then M has a 3-connected minor M' using {a,b,c} such that M' has an N-minor, and M' has at most three more elements than N.*

Theorem 6.1.2 can be used to give a proof of the following theorem of Asano, Nishizeki, and Seymour. The original proof of this result used Seymour's decomposition for regular matroids [36].

6.1.3 Theorem [1,(9)]. *Let* $\{e,f,g\}$ *be a triangle of a 3-connected non-graphic matroid* M. *Then* M *has a minor* N *using* $\{e,f,g\}$ *where*

(i) $N \cong M^*(K_{3,3})$ *if* M *is regular;*

(ii) $N \cong F_7$ *if* M *is binary but not regular; and*

(iii) $N \cong U_{2,4}$ *if* M *is non-binary.* □

The next theorem is a strengthening of Theorem 6.1.2. The proof of this result is also given in the next section.

6.1.4 Theorem. *Let* $\{e,f,g\}$ *be a triangle of a 3-connected binary matroid* M, *and* N *be a 3-connected minor of* M *using* e. *Then* M *has a 3-connected minor* M' *using* $\{e,f,g\}$ *such that* M' *has an* N-*minor, and* M' *has at most two more elements than* N.

We will use this result to obtain the next theorem which is a strengthening of Theorem 6.1.3 of Asano, Nishizeki, and Seymour in the case that M is binary but not regular. The matroids S_8 and J_{10} are as given in Table 1. Evidently J_{10} is the generalized parallel connection across $\{e_8,e_9,e_{10}\}$ of the Fano-matroid and the cycle matroid of the complete graph on four vertices. Accordingly, we shall call $\{e_8, e_9, e_{10}\}$ the *join-triangle* of J_{10}.

6.1.5 *Theorem. Let* $\{e,f,g\}$ *be a triangle of a* 3-*connected binary non-regular matroid* M *with at least eight elements. Then* M *has a minor* N *using* $\{e,f,g\}$ *such that one of the following holds.*

(i) $N \cong S_8$;

(ii) $N \cong J_{10}$ *and* $\{e,f,g\}$ *is the join-triangle of* J_{10}.

The next result is an analog of Theorem 6.1.5 for the class of binary matroids.

6.1.6 *Theorem. Let* $\{e,f,g\}$ *be a triangle of a* 3-*connected binary matroid* M *with at least eight elements. Then* M *has a minor using* $\{e,f,g\}$ *that is isomorphic to* $S_8, M(\mathcal{W}_4)$, *or* $M(K_5 - a)$.

6.2 Roundedness and the Splitter Theorem

In this section the proofs of the theorems stated in the previous section will be given. The main tools used are results from roundedness theory, and the splitter theorem. We begin with some consequences of Lemma 5.2.6.

Proofs of Theorems 6.1.1 and 6.1.2. Assume that M is minimal with respect to N and {a,b,c}. Then M or M* is of the form given in one of cases (1) through (8) of Lemma 5.2.6. From using orthogonality and the fact that a 3-connected matroid with at least four elements is simple, we obtain that {a,b,c} can only be a triangle of M in case(1), case (7), and the dual of case (6). Note that M has at most four more elements than N in these cases. This completes the proof of Theorem 6.1.1.

Suppose M is binary. If M is of the form given in case (7) of Lemma 5.2.6, then M|{a,b,c,f} is isomorphic to $U_{2,4}$ and this contradicts the fact that M is binary. We next show that M is not of the form given in the dual of case (6) of Lemma 5.2.6. It will then follow that M is as given in case (1) of Lemma 5.2.6. This will complete the proof of Theorem 6.1.2.

Assume that M is of the form given in the dual of case(6) of Lemma 5.2.6. The set {a,b,c,f} meets the circuit {a,b,c} in three elements. Thus, by Theorem 1.4.1(2), {a,b,c,f} is not a cocircuit of M. It follows from the dual of 5.2.6(6) that {a,b,f} is a cocircuit of M. Now $\widehat{M}\backslash f$ is not simple and hence not 3-connected. Thus, by Lemma 5.2.1, $\widetilde{M/f}$ is 3-connected. Since {a,b,c} is a circuit, and {a,b,f} is a cocircuit of M,
$N_1 = M\backslash\{c,f\}/b \cong M\backslash\{a,f\}/b = M\backslash\{a,b\}/f$.
Thus M/f and hence $\widetilde{M/f}$ has an N-minor. Moreover, as M is binary, a, b, and c are elements of $\widetilde{M/f}$. This contradicts the minimality of M with respect to N and {a,b,c} thereby completing the proof of Theorem 6.1.2. □

We shall use the following lemma several times in the proof of Theorem 6.1.4. Let M_1 and N_1 be 3-connected matroids with at least four elements and X and Y be subsets of $E(M_1)$ such that $M_1\backslash X/Y = N_1$. Suppose that {x,y,z} is a triangle of M_1 with {y,z} in $E(N_1)$ and x in $E(M_1) - E(N_1)$. Evidently x is contained in X.

6.2.1 <u>Lemma</u>. <u>Either</u> N_1+x <u>is</u> 3-<u>connected</u> <u>or</u> N_1+x <u>has</u> <u>an</u> N_1-<u>minor</u> <u>using</u> {x,y,z}.

<u>Proof</u>. Suppose N_1+x is not 3-connected. Then, by Lemma 1.3.1, x is contained in a circuit of N_1+x of size one or two, or x is a coloop of N_1+x. The latter case clearly cannot occur. Suppose that x is a loop of N_1+x, or x

is contained in a two-element circuit of N_1+x with one of y and z. Then, as $\{x,y,z\}$ is a triangle of M_1, circuit elimination implies that $\{y,z\}$ is dependent in N_1+x. This contradicts the 3-connectivity of N_1. Thus $\{x,x'\}$ is a circuit of N_1 for some x' distinct from y and z. Hence $(N_1+x)\backslash x'$ is a minor of N_1+x that is isomorphic to N_1 and uses $\{x,y,z\}$.

Proof of Theorem 6.1.4. If f or g is in E(N), then from Lemma 6.2.1 we obtain M' as desired. Suppose that neither f nor g is in E(N). Apply Lemma 5.2.3 to $\{e,f\}$ and N. There exists a 3-connected minor M_1 of M using $\{e,f\}$ such that M_1 has at most two more elements than some minor N_1 which is isomorphic to N. If g is contained in $E(M_1)$, then let $M' = M_1$. Suppose g is not an element of M_1.

Now, by Lemma 6.2.1, either M_1+g is 3-connected, or M_1+g has a minor isomorphic to M_1 using $\{e,f,g\}$. In the latter case, the result holds. Suppose the former case holds.

If M_1 has exactly one more element than N_1, let $M' = M_1+g$. Suppose that M_1 has two more elements than N_1. Then M_1 is as given in case (iii) or (iv) of Lemma 5.2.3. Suppose case (iii) holds. Then, for some element g' of M_1, $M_1\backslash f/g' = N_1$ and $\{e,f,g'\}$ is a circuit of M_1. Thus $\{e,f,g\}$ and $\{e,f,g'\}$ are triangles of the 3-connected binary matroid M_1+g. Hence $g = g'$; a contradiction. Thus case (iv) of Lemma 5.2.3 holds.

Let g' be an element of M_1 such that $M_1\backslash g'/f = N_1$ and $\{e,f,g'\}$ is a cocircuit of M_1.

Now $\{e,f,g'\}$ or $\{e,f,g,g'\}$ is a cocircuit of M_1+g. As $\{e,f,g,g'\}$ meets the circuit of $\{e,f,g\}$ in three elements in the binary matroid M_1+g, the former occurs.

Apply Lemma 5.2.3 to the elements e of $E(N_1)$ and g of $E(M_1+g) - E(N_1)$. There exists a 3-connected minor M_2 of $M_1 + g$ using $\{e,g\}$ such that M_2 has a minor $N_2 \cong N_1 \cong N$ with M_2 having at most two more elements than N_2. If M_2 has at most one more element than N_2, or f is in $E(M_2)$, then, as before, the result holds. Suppose M_2 has exactly two more elements than N_2 and $f \varepsilon E(M_1+g) - E(M_2)$. Then $(M_1+g)/f = M_2$ or $(M_1+g)\backslash f = M_2$. However, $\{e,f,g\}$ is a circuit, and $\{e,f,g'\}$ a cocircuit of M_1+g. Hence M_2 is not 3-connected; a contradiction. □

Several results which are used in the proof of Theorem 6.1.5 are given next. The following result of Oxley is used in the proofs of Lemmas 6.2.3 and 6.2.7. The matroids S_8, P_9, and Z_4 are as given in Table 2.

6.2.2 Lemma [28,(2.6)]. If Q is a 3-connected binary extension or lift of S_8, then Q is isomorphic to one of P_9, P_9^*, Z_4, and Z_4^*. □

6.2.3 Lemma. The set $\{U_{2,4}, S_8\}$ is (3,1)-rounded.

Proof. Suppose M is a 3-connected binary extension of S_8. Then M is isomorphic to P_9 or Z_4 by Lemma 6.2.2. By Lemma 2.2.1, both $P_9 \backslash e_8$ and $P_9 \backslash e_9$ are isomorphic to S_8. Hence each element of P_9 is in some S_8-minor of P_9.

Let A_4 be the binary matrix which represents Z_4 and is given in Table 2. From considering the automorphisms induced by interchanging any two of the rows of A_4, we see that the group of automorphisms of Z_4 is transitive on $\{b_1,b_2,b_3,b_4\}$. Hence, for each x in $\{b_1,b_2,b_3,b_4\}$, $Z_4 \backslash x \cong Z_4 \backslash b_4 = S_8$. Thus each element of Z_4 is in some S_8-minor of Z_4. The result follows by duality and Theorem 1.6.4. □

The binary matrices A_1 and A_2 which represent S_8 and AG(3,2), respecitvely, are given in Table 2. The next lemma is due to Seymour.

6.2.4 Lemma [38,p.375]. S_8 and AG(3,2) are the only eight-element 3-connected binary non-regular matroids. □

We next restate Lemma 6.2.4 in a form that will be used in the proof of Theorem 6.1.5. Let B be the binary matrix given below.

Figure 17

$$B \quad \left[\begin{array}{c|cccc} & 0 & 1 & 1 & x_1 \\ I_4 & 1 & 0 & 1 & x_2 \\ & 1 & 1 & 0 & x_3 \\ & 1 & 1 & 1 & x_4 \end{array} \right]$$

6.2.5 Corollary. *If* $(x_1, x_2, x_3, x_4)^T$ *has exactly two entries which are equal to one, then* D(B) *and* $D(A_1) = S_8$ *are isomorphic.*

Proof. D(B) is a non-trivial extension of F_7^* and hence is 3-connected and non-regular. Since D(B) contains a triangle, it is not isomorphic to AG(3,2). Hence, by Lemma 6.2.4, $D(B) \cong S_8$. □

The next lemma may be proved using Theorem 1.2.3 and Lemma 6.2.4. This lemma is used in the proof of Lemma 6.2.7.

6.2.6 Lemma. *Let* M *be a* 3-*connected binary non-regular matroid with at least eight elements. Then* M *has an* S_8- *or an* AG(3,2)-minor. □

The investigation of the relationship between triangles in 3-connected binary matroids and the matroid S_8 was motivated by the following result.

6.2.7 Lemma. *Let* M *be a* 3-*connected binary non-regular matroid with at least nine elements. Then* M *has an* S_8-*minor*.

Proof. By Lemma 6.2.6, M has an S_8- or an AG(3,2)-minor. Suppose the latter holds. Then, by Theorem 1.2.3, M has, as a minor, a 3-connected binary extension or lift M' of AG(3,2). By duality, we may assume the former. Let

A_2 be the binary matrix representing AG(3,2) that is given in Table 2. Suppose the binary vector $v = (x_1,x_2,x_3,x_4)^T$ is adjoined to A_2 to give a representation for M'. Evidently exactly two or four of the coordinates of v are one. It follows from considering A_2+v and applying Corollary 6.2.5 that M' has an S_8-minor. □

We now give some notation and observations which are used in the proof of Theorem 6.1.5. Let A_1, A_2, A_3, A_3^*, A_4 and A_4^* be the binary matrices given in Table 2 that represent S_8, AG(3,2), P_9, P_9^*, Z_4, and Z_4^*, respectively.

The following notation is used. Let $v_{i_1,i_2,\ldots,i_j}$ and $w_{i_1,i_2,\ldots,i_j}$ denote the non-zero vectors in V(4,2) and V(5,2), respectively, with a one in positions $i_1,i_2,\ldots,$ and i_j and a zero in all other positions. Hence $v_{124} = (1,1,0,1)^T$ and $w_{235} = (0,1,1,0,1)^T$. Computations such as $(P_9^* + w_{125})/e_9\backslash e_4 \cong S_8$ are made as follows. Note that $(A_3^* + w_{125})/e_9\backslash e_4$ is the matrix B of Figure 17 with $x_1=x_2=1$ and $x_3=x_4=0$. Then, by Corollary 6.2.5, we see that $(P_9^* = w_{125})/e_9\backslash e_4 \cong S_8$.

<u>Proof of Theorem 6.1.5</u>. If M has eight elements, then, by Lemma 6.2.4, M is isomorphic to S_8 or AG(3,2). As M possesses a triangle the former occurs and the result holds. Suppose M has at least nine elements. Then M has an S_8-minor using e by Lemmas 6.2.3 and 6.2.7.

Hence, by Theorem 6.1.4, M has a 3-connected minor M' using {e,f,g} such that M' has a minor N' which is isomorphic to S_8 with $|E(M') - E(N')| \leq 2$. Thus M' has at most ten elements. If M' has eight elements, then, as above, the result holds.

6.2.8 Lemma. If M' has nine elements, then each triangle of M' is in some S_8-minor of M'.

Proof. From Lemma 6.2.2 and the fact that M' possesses a triangle, M' is isomorphic to one of P_9, P_9^*, and Z_4. From considering the matrix A_3 representing P_9, we see that $P_9 \backslash e_8 \cong P_9 \backslash e_9 \cong S_8$. Each triangle of P_9 appears in an S_8-minor as $\{e_8, e_9\}$ is not contained in a triangle of P_9. As $P_9^*/e_9 \cong S_8$, each triangle of P_9^* appears in an S_8-minor. If x is in $\{b_1, b_2, b_3, b_4\}$, then $Z_4 \backslash x$ is isomorphic to S_8. Hence every triangle of Z_4 appears in an S_8-minor. □

Now suppose that M' has ten elements. Then, by Theorem 1.2.3, for some x in E(M'), either M'\x or M'/x is isomorphic to a member of $\{P_9, P_9^*, Z_4, Z_4^*\}$. In the latter case, as x is not contained in a triangle of M', Lemma 6.2.8 implies that each triangle of M' appears in an S_8-minor. In the former case, by Lemma 6.2.8, it will suffice to show the following. If M' is not isomorphic to J_{10}, then each triangle of M' containing x appears in an S_8-minor. If M' is isomorphic to J_{10}, then each triangle of M' other than the join triangle appears in an S_8-minor. These cases are treated in Lemmas 6.2.9 through 6.2.12. In light of the above remarks, to show, for example, that all triangles of $P_9 + v_{123}$ are contained in some S_8-minor, we merely provide enough information to show that all triangles of $P_9 + v_{123}$ containing v_{123} appear in some S_8-minor.

6.2.9 Lemma. *If* $M'\backslash x$ *is isomorphic to* P_9, *but* M' *is not isomorphic to* J_{10}, *then each triangle of* M' *appears in an* S_8-*minor. Every triangle of* J_{10} *other than* $\{e_8, e_9, e_{10}\}$ *appears in an* S_8-*minor.*

Proof. Suppose the non-zero column vector x of V(4,2) is adjoined to the binary matrix A_3 to obtain a representation for M'. Evidently x is in $\{v_{123}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34}\}$. From the symmetry of A_3 induced by interchanging rows 1 and 2, we may assume that x is contained in $\{v_{123}, v_{13}, v_{14}, v_{34}\}$. In $A_3 + v_{13}$, replace row i by row i + row 2 for each i in {3,4}. After interchanging rows 3 and 4 and suitably reordering the columns, we obtain $A_3 + v_{14}$. Hence $P_9 + v_{13}$ and $P_9 + v_{14}$ are isomorphic. Thus M' is isomorphic to P_9 + x for some x in $\{v_{123}, v_{13}, v_{34}\}$.

Now $(P_9+v_{123})\backslash e_7 \cong P_9$ and hence each triangle of $P_9 + v_{123}$ containing v_{123} appears in an S_8-minor. Thus, by Lemma 6.2.7, each triangle of $P_9 + v_{123}$ appears in some S_8-minor. Similarly, as $(P_9+v_{13})\backslash\{e_8, e_9\}$ is isomorphic to S_8, each triangle of $P_9 + v_{13}$ appears in an S_8-minor.

Let $e_{10} = v_{34}$. Then the binary matrix $A_3 + e_{10}$ represents J_{10} with the representation as given in Table 1. Thus $P_9 + v_{34} = J_{10}$. Since $(P_9+v_{34})\backslash\{e_8,e_9\}$ is isomorphic to S_8, it follows that each triangle of $P_9 + v_{34}$ other than $\{e_8,e_9,v_{34}\}$ appears in an S_8-minor. □

6.2.10 Lemma. _If_ $M'\backslash x$ _is isomorphic to_ Z_4, _then each triangle of_ M' _appears in an_ S_8-_minor_.

Proof. M' is represented by $A_4 + x$, where x is in $\{v_{12},v_{13},v_{14},v_{23},v_{24},v_{34}\}$. From the symmetry of A_4 induced by interchanging any two of its rows, we may assume that $x = v_{12}$. As $A_4 + v_{12}$ can be obtained from $A_3 + v_{123}$ by reordering columns, and $A_3 + v_{123}$ represents $P_9 + v_{123}$, we deduce that $Z_4 + v_{12}$ and $P_9 + v_{123}$ are isomorphic. Thus the result follows by Lemma 6.2.8. □

6.2.11 Lemma. _If_ $M'\backslash x$ _is isomorphic to_ Z_4^*, _then each triangle of_ M' _appears in an_ S_8-_minor_.

Proof. M' is represented by $A_4^* + x$, where x is one of the twenty-two non-zero column vectors of $V(5,2)$ that are different from those vectors which are columns of A_4^*. From the symmetry of A_4^* induced by interchanging any two of rows 1, 2, 3, and 4, we may assume x is in $\{w_{12345}, w_{1234}, w_{123}, w_{125}, w_{14}, w_{15}\}$. In $A_4^* + w_{123}$, replace row i by row i + row 1 + row 4 for each i in $\{2,3,5\}$.

After suitably reordering the columns, we obtain $A_4^* + w_{15}$. Thus $Z_4^* + w_{123}$ and $Z_4^* + w_{15}$ are isomorphic. In $A_4^* + w_{125}$, replace row i by row i + row 1 + row 3 for i = 2,4, and 5. After suitably reordering the columns we obtain $A_4^* + w_{14}$. Hence $Z_4^* + w_{125}$ and $Z_4^* + w_{14}$ are isomorphic. It follows that M' is isomorphic to $Z_4^* + x$ for some x in $\{w_{12345}, w_{1234}, w_{123}, w_{125}\}$.

Now $(Z_4^* + w_{12345})/c_4\backslash a_i$ is isomorphic to S_8 for i = 1, 2, 3, and 4. Hence each triangle of $Z_4^* + w_{12345}$ appears in an S_8-minor. Note that $Z_4^* + w_{1234}$ has no triangle. Now $(Z_4^* + w_{123})/b_4$ is isomorphic to Z_4. Thus by Lemma 6.2.7, each triangle of $Z_4^* + w_{123}$ appears in an S_8-minor. Since $(Z_4^* + w_{125})/b_1\backslash a_1$ is isomorphic to S_8, every triangle of $Z_4^* + w_{124}$ appears in an S_8-minor. □

6.2.12 <u>Lemma</u>. <u>If</u> M'\x <u>is isomorphic to</u> P_9^*, <u>then each triangle of</u> M' <u>appears in an</u> S_8-<u>minor</u>.

<u>Proof</u>. M' is represented by $A_3^* + x$, where x is in V(5,2). By the symmetry of A_3^* induced by interchanging rows 1 and 2, we may assume that x is in $\{w_{12345}, w_{1245}, w_{1235}, w_{123}, w_{125}, w_{134}, w_{135}, w_{145}, w_{345}, w_{12}, w_{13}, w_{14}, w_{15}, w_{34}, w_{35}, w_{45}\}$.

Replace row i by row i + row 1 in A_3^* for i = 3, 4, and 5. After reordering the columns we obtain A_3^* again. From performing the same row operations on x we may assume that x is not one of w_{12}, w_{123}, w_{125}, w_{14}, w_{15}, and w_{13}.

Replace row i by row i + row 2 in A_3^* for i = 3, 4, and 5 and then interchange rows 4 and 5. We obtain A_3^* again after a suitable reordering of the columns. From performing the same row operations on x, we may suppose x is not w_{135} or w_{35}. Hence M' is isomorphic to $P_9^* + x$ for some x in $\{w_{12345}, w_{1245}, w_{1235}, w_{134}, w_{145}, w_{345}, w_{34}, w_{45}\}$.

Now, w_{1235} appears in no triangle of $P_9^* + w_{1235}$. The following computations show that each triangle of $P_9^* + x$ appears in an S_8-minor for these x. Each of the following matroids is isomorphic to S_8:

$(P_9^* + w_{12345})/e_7\backslash e_3$, $(P_9^* + w_{1245})/e_8\backslash e_1$,
$(P_9^* + w_{134})/e_8\backslash e_3$, $(P_9^* + w_{145})/e_9\backslash e_4$,
$(P_9^* + w_{345})/e_9\backslash e_4$, $(P_9^* + w_{34})/e_9\backslash e_4$,
$(P_9^* + w_{45})/e_7\backslash e_3$. □

It follows from Lemmas 6.2.9 through 6.2.12 that if M' has ten elements, then each triangle of M' appears in an S_8-minor of M'. This completes the proof of Theorem 6.1.5. □

The next lemma is used in the proof of Theorem 6.1.6.

6.2.13 Lemma. Let {e,f,g} be a triangle of a 3-connected binary matroid M which has an $M(\mathcal{W}_4)$-minor. Then M has an $M(\mathcal{W}_4)$- or an $M(K_5-a)$-minor using {e,f,g}.

Proof. By Lemma 2.2.4, M has a minor N_1 which is isomorphic to $M(\mathcal{W}_4)$ and uses $\{e,f\}$. If g is in $E(N_1)$, then the result holds. Otherwise, by Lemma 6.2.1, we may suppose that $N_1 + g$ is 3-connected. It follows from Lemma 2.2.3 that $N_1 + g$ is isomorphic to one of $M(K_5-a)$, $M^*(K_{3,3})$, and P_9. The contraction of any edge of $K_{3,3}$ produces a $\mathcal{W}_4$-minor. Hence as $M(\mathcal{W}_4)$ is self-dual, each triangle of $M^*(K_{3,3})$appears in an $M(\mathcal{W}_4)$-minor. By Lemma 2.2.1, $P_9\backslash x$ is isomorphic to $M(\mathcal{W}_4)$ if x is in $\{e_1,e_2,e_5,e_6\}$. Thus each triangle of P_9 appears in an $M(\mathcal{W}_4)$-minor. It follows that $\{e,f,g\}$ appears in an $M(\mathcal{W}_4)$-minor of $N_1 + g$. □

We now use Lemma 6.2.13 to generalize Theorem 6.1.5 to the class of binary matroids.

Proof of Theorem 6.1.6. It follows from Theorem 1.2.2 that M has an $M(\mathcal{W}_3)$-minor. If M also has an $M(\mathcal{W}_4)$-minor, then the result holds by Lemma 6.2.13. Suppose that M has no $M(\mathcal{W}_4)$-minor. Then, by Theorem 1.2.3, M has a 3-connected binary extension or lift of an $M(\mathcal{W}_3)$-minor as a minor. Hence, M has an F_7- or F_7^*-minor and therefore is non-regular. By Theorem 6.1.5, M has an S_8- or J_{10}-minor using $\{e,f,g\}$. In the first case, the result holds. Suppose the second case holds. Observe from the representation of J_{10} given in Table 1, that if one of e_1, e_2, e_5, and e_6 is deleted from J_{10} we obtain the generalized parallel connection across a triangle of two $M(\mathcal{W}_3)$ matroids. This is the matroid $M(K_5-a)$. Hence each triangle of J_{10} appears in an $M(K_5-a)$-minor. □

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Appendix 1 Index of Notation

The following is a list of frequently used notation and the page on which it was introduced.

Appendix 2 Index of Definitions

Curriculum Vitae

Talmage James Reid was born in Luling, Louisiana on November 4, 1961. He received a Bachelor of Science degree from Southeastern Louisiana University in May 1983. He entered Louisiana State University in June 1983 and received a Master of Science degree in May 1985.

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Candidate: Talmage James Reid

Major Field: Mathematics

Title of Dissertation: On Roundedness in Matroid Theory

Approved:

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

Date of Examination:

April 27, 1988