8-7-1995

Is there a connection between no-hair behavior and universality in gravitational collapse?

Jorge Pullin

Center for Gravitational Physics and Geometry

CGPG-94/9-2
gr-qc/9409044

Is there a connection between no-hair behavior and universality in gravitational collapse?

Jorge Pullin
Center for Gravitational Physics and Geometry
Department of Physics, 104 Davey Lab
The Pennsylvania State University
University Park, PA 16802

We apply linear perturbation theory to the study of the universality and criticality first observed by Choptuik in gravitational collapse. Since these are essentially nonlinear phenomena our attempt is only a rough approximation. In spite of this, universal behavior of the final black hole mass is observed with an exponent of 1/2, slightly higher than the observed value of 0.367. The universal behavior is rooted in the universal form that in-falling perturbations on black holes have at the horizon.

If one considers the collapse of a spherically symmetric scalar field in general relativity, two possible end-results are expected. Either the initial data manages in the time evolution to create a big enough mass concentration such that a black hole forms or whatever scalar field was present is dispersed to infinity, leaving a flat spacetime as a result. If one considers one-parameter families of initial data, there will be in the parameter range two distinct regions, one corresponding to the formation of a black hole and another to dispersion to infinity. Christodoulou and others [1,2] had raised the question of what was the behavior of the final black hole mass as a function of one such parameter. For values of the parameter below a "critical" value for which black holes start forming, the final mass is zero, since no black hole forms. When the critical value is reached, is there a "jump" in the mass, i.e., is there a minimal mass for which one can form a black hole? This was highly unlikely, since the problem has no scale. Choptuik [3] has confirmed this through numerical simulations. He finds no "mass gap", but actually finds the final black hole mass is a *universal* function of the parameter value of the initial data,

$$M_{BH} = k(p - p_*)^\beta \tag{1}$$

where k is a constant that may be different for different one-parameter families of data, p is the parameter value chosen for the initial data and p_* is the critical parameter value at which black holes start to form. The exponent is *universal* (i.e. family-independent) and was experimentally determined by Choptuik to be $\beta = 0.367$. This value was later confirmed with a different (less sophisticated) kind of code [4].

In a separate set of experiments, Abrahams and Evans [5] considered the implosion of an axially symmetric gravitational wave (in vacuum, no scalar field present). They find a similar behavior with the same exponent. Finally, Coleman and Evans [6] considered the collapse of a radiative fluid $p = \rho/3$ and also found a similar behavior with the same exponent.

Little theoretical progress has been made towards explaining this universal behavior and in particular the value of the exponent. Several authors [7–9] have noticed that there exist in the literature exact self-similar solutions for the collapse of scalar fields. These solutions are not quite geared towards describing the effects observed by Choptuik, since they are not asymptotically flat. In spite of this they yield a critical behavior with an exponent of 1/2. An exponent of 1/2 has been also observed in a rather unrelated context, the collapse of $1+1$ dimensional black holes with quantum corrections by Strominger and Thorlacius [10].

The purpose of this note is to explore the application of perturbation theory to these phenomena. Our approach is crude, and the results can only be taken as suggestive. It is remarkable however, that we find a universal exponent that is close to the observed value (1/2 instead of 0.367), and that the root for the universality is the same as that of the no-hair behavior of black holes: the universal form that fields falling into the horizon take (quasinormal ringing).

Suppose one prepares a set of initial data for the Choptuik experiment, with a certain value of the parameter p_1 (although it is not quite needed, the reader can imagine a Gaussian of amplitude p_1 for concreteness). Let us also assume $p_1 > p_*$ and therefore, according to Choptuik, a black hole forms, with mass,

$$M_1 = k(p_1 - p_*)^\beta. \tag{2}$$

Suppose now a second set of data is prepared, of parameter value $p_2 > p_1$, with p_2 differing little from p_1, i.e., $p_2 - p_1 < p_1 - p_*$. Again, we form a black hole of mass,

$$M_2 = k(p_2 - p_*)^\beta \tag{3}$$

and we are assuming the two sets of data are in the same family, so the constant k is the same. We now would like to view the second set of data as a "perturbation" of the first set. Assuming their parameter separation to be small, i.e. $\Delta p = p_2 - p_1 << p_1 - p_*$, we can write for the final mass difference $\Delta M = M_2 - M_1$,

$$\Delta M = k\beta(p_2 - p_1)(p_1 - p_*)^{\beta-1} \tag{4}$$

which can be written as,

$$\Delta M = k^{\frac{1}{\beta}} \beta \Delta p M_1^{\frac{\beta-1}{\beta}}. \tag{5}$$

Therefore, if there would be a way of estimating the excess mass that falls into a black hole due to an increase of the parameter value of the initial data, one could estimate the exponent β. Alternatively, if one could estimate ΔM as a function of the black hole mass M_1 one could also get a value for the parameter β.

We will now provide such an estimate using perturbation theory. We start by making a rough assumption: the spacetime geometry for the collapse of the first initial data set can be approximated by the spacetime geometry of a static, ever-existing, Schwarzschild black hole of mass M_1. This is an uncontrolled and vague approximation. One expects it in general grounds to be good if p_1 is far away from criticality, so one forms a black hole with most of the initial data mass and there is little outgoing radiation. Also one should realize that approximating a time-dependent spacetime by a static one can only work in the region where the time-dependent spacetime is not rapidly changing. In fact, numerical studies of perturbations on dynamical backgrounds have shown that the approximation can work remarkably well [4]. However, there is no real justification to apply this approximation in the case we are interested since the attention should be focused near criticality, when the background differs considerably from an ever-existing black hole. What follows therefore should only be taken as suggestive and speculative.

If one assumes the spacetime determined by the initial data set with parameter p_1 is a Schwarzschild black hole of mass M_1, estimating ΔM becomes a problem in perturbations of black holes: given a perturbation incident on a black hole, how much mass falls into the hole and how much mass is radiated away? To estimate this we note that perturbations on black hole backgrounds are determined by the Zerilli function Ψ (to summarize, one takes $g_{\mu\nu} = \text{Schwarzschild}_{\mu\nu} + h_{\mu\nu}$ and finds that all the information in $h_{\mu\nu}$ can be parameterized by a function Ψ, see [11] for details). This function satisfies a Klein-Gordon-like equation coupled to a potential called the Zerilli equation. The "radial" variable in this equation is a "tortoise" coordinate r_* in terms of which the horizon is at $r_* = -\infty$ and i_0 is at $r_* = \infty$. The question is therefore, given a perturbation incoming from $r_* = \infty$, how much energy makes it to $r_* = -\infty$? Scattering in the Zerilli potential has been well studied, and one knows that for a generic incoming perturbation from $r_* = \infty$ one gets a damped oscillation known as quasinormal ringing at the horizon ($r_* = -\infty$),

$$\psi_{\text{Horizon}} = A \exp(-\omega_i t) \sin(\omega_r t) \tag{6}$$

where the quasinormal frequencies scale as inverse powers of the background spacetime mass,

$$\omega_{r,i} = \frac{\text{constant}}{M_1} \tag{7}$$

and the coefficient A depends on particular details of the initial perturbation. A study of how different perturbations generate different values of A is found in Price and Sun [12].

The total energy falling into the black hole is proportional to the integral $\int dt \dot{\psi}^2_{\text{Horizon}}$. The dependence of this integral on M_1 is determined completely by the behavior of the quasinormal frequencies and gives as a result,

$$\Delta M \sim M_1^{-1} \tag{8}$$

which implies $\beta = 1/2$.

Let me end this note with some remarks.

- The above explanation, although rough, provides a link between universality and no-hair behavior in gravitational collapse.
- In spite of the coarse approximation, the exponent comes out with a value close to the experimentally measured one. It seems that the fact that the value is the same as in the exact solutions or the Strominger-Thorlacius [10] model is a mere coincidence, since none of the features of those examples has been incorporated in this discussion.
- The above mechanism predicts the same value for the exponent for the scalar field studied by Choptuik, the gravitational waves studied by Abrahams and Evans and the fluid case studied by Coleman and Evans, since it only

depends on the relation of the quasinormal frequencies to the background spacetime mass, which does not depend on the details of the perturbations.

The outstanding question is: is there any way to refine the approximation in order to give the correct exponent? The following remarks are in order.

• The dependence of quasinormal frequencies on the background mass is a robust feature, that seems unlikely to change if one modifies the scenario. It is possible that a dynamical background could alter the picture of quasinormal ringing at the horizon and therefore alter the exponent. This would bring into question up to what extent is the link between universality and baldness a mere artifact of our rough approximation. If quasinormal ringing were still present in a dynamical background, in order to predict the right exponent, the frequencies would have to depend non-analytically on the background mass. This seems quite unlikely, even on dimensional grounds. Traschen [13] has found some non-analytic behavior in perturbations of Reissner Nordstrom. Perhaps there is a connection. It could also happen that in the non-static case there is a residual dependence of the coefficient A on M_1, which in the static case does not appear. This could allow non-analytic behavior of the energy without a counter-intuitively non-analytic behavior of the quasinormal frequencies.

• Insight into perturbations of dynamical backgrounds could be gained by studying perturbations of the Vaidya metric. This would provide a dynamical background and could help gain intuition on some of the issues raised in the previous point.

• Finally, in the case of the collapse of radiative fluid, studied by Coleman and Evans, the exact critical solution is known analytically. Studying perturbations of it should certainly allow insight into the critical exponent. Evans is currently pursuing this point, with a rather different approach than the one presented here.

I wish to to thank Doug Eardley, Charles Evans, David Garfinkle, Carsten Gundlach and Richard Price for useful comments and discussions. This work was supported by funds of the Office of Minority Faculty Development at Penn State University. I wish to thank the Aspen Center for Physics and the Max Planck Institut für Astrophysik (Munich) for hospitality during the preparation of this manuscript.

---

[1] D. Christodoulou, Commun. Math. Phys. **105**,337 (1986); **106**,587 (1986); **109**,591 and 613 (1987).
[2] D. Goldwirth, T. Piran, Phys. Rev. **D36**, 3575 (1987).
[3] M. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).
[4] C. Gundlach, R. Price, J. Pullin, Phys. Rev. **D49**, 890 (1994).
[5] A. Abrahams, C. Evans, Phys. Rev. Lett. **70**, 2980 (1993).
[6] J. Coleman, C. Evans, **72**, 1842 (1994).
[7] P. Brady, preprint gr-qc/9402023 (1994).
[8] Y. Oshiro, K. Nakamura, A. Tomimatsu, preprint gr-qc/9402017 (1994).
[9] V. Husain, E. Martinez, D. Nuñez, preprint gr-qc/9402021 (1994).
[10] A. Strominger, L. Thorlacius, Phys. Rev. Lett. **72**, 1584 (1994).
[11] C. Cunningham, R. Price, V. Moncrief, Ap. J. **230**, 870 (1979).
[12] R. Price, Y-H. Sun, Phys. Rev. **41**, 2492 (1990).
[13] J. Traschen, preprint gr-qc/9403016 (1994).