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## Second order perturbations of a Schwarzschild black hole

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We study the even-parity  $\ell = 2$  perturbations of a Schwarzschild black hole to second order. The Einstein equations can be reduced to a single linear wave equation with a potential and a source term. The source term is quadratic in terms of the first order perturbations. This provides a formalism to address the validity of many first order calculations of interest in astrophysics.

Black hole perturbation theory has been a ubiquitous tool in the analysis of astrophysical situations without symmetries. It has played an important role in the study of the expected gravitational radiation from processes like the infall of matter into a hole [1], and more recently the collision of two black holes [2]. For black holes, perturbations to linear order in a dimensionless expansion parameter  $\epsilon$  can be described through a simple scalar wave equation in two spacetime dimensions, and yet can describe complex situations of no particular symmetry. In contrast, a full numerical simulation of the nonlinear Einstein equations is at present prohibitive for most situations of physical interest.

In spite of these advantages there is an important limitation in the use of linearized theory: there is no information within linearized theory to determine its range of applicability, i.e., to determine what values of  $\epsilon$  are sufficiently small. Largely due to this last problem, linearized theory has usually been limited in use to situations departing only very slightly from an undisturbed black hole. However, recent comparisons between linearized theory and full numerical simulations for the head-on collisions of two black holes [2] have shown that linearized theory can work remarkably well in domains in which it would be supposed to fail. The qualitative explanation for this is that strong nonlinearities near the horizon can be absorbed by the black hole and therefore not affect the outgoing radiation.

In order to promote linearized theory to a serious tool for predicting astrophysical answers and in particular to provide benchmarks for difficult full numerical simulations, we need a reliable measure of the errors in a perturbation result for a given value of  $\epsilon$ . One can try to construct simply implemented analytic *a priori* indices of the validity of perturbation theory, e.g., the violation of the exact Hamiltonian constraint by the linearized data [3], or the violation of the linearized Hamiltonian constraint by the exact initial data [4]. Such measures are interesting because of their simplicity, but are not unique and worse, they are not guaranteed to work in all situations. For instance, their success for head-on collisions of two black holes gives no reliable information about their value for less symmetric collisions.

There is, in fact, only one generally reliable index of the accuracy of linearized perturbation theory: to calculate answers to the next order in  $\epsilon$ . Where the higher-order results and the linearized results differ by (say) 10% is a point at which one has some confidence that either answer is accurate to within around 10%. Higher order perturbation calculations seem at the outset to be an obvious extension of the familiar techniques of linearized perturbation theory, with a guarantee of simple results. It turns out — and this is one of the major points we wish to make — that there are some subtle issues of gauge that must be recognized. Despite these issues, and despite some necessarily lengthy calculations, second order computations are far more easily done than the development of full numerical relativity codes. And the difficulty of the higher order calculations is a necessary price well worth paying for the advantages provided by reliable perturbation theory results.

The basic structure of higher order perturbation theory starts with the same basic formalism as linearized theory. We write the metric as  $g_{\mu\nu}^{(0)} + \epsilon g_{\mu\nu}^{(1)} + \epsilon^2 g_{\mu\nu}^{(2)} + \mathcal{O}(\epsilon^3)$  where  $g_{\mu\nu}^{(0)}$ , the “background,” is a known solution of the vacuum field equations. We write the sourceless Einstein equations as

$$\mathcal{G}(g_{\mu\nu}^{(0)} + \epsilon g_{\mu\nu}^{(1)} + \epsilon^2 g_{\mu\nu}^{(2)}) = 0, \quad (1)$$

where  $\mathcal{G}$  represents the actions of taking partial derivatives and algebraic combinations to form the components of the Einstein tensor. If we expand (1) in  $\epsilon$ , the term of order  $\epsilon^0$  automatically vanishes if  $g_{\mu\nu}^{(0)}$  is a background solution. The terms first order in  $\epsilon$  can be written in the form

$$\epsilon \mathcal{L}(g_{\mu\nu}^{(1)}) = 0, \quad (2)$$

where  $\mathcal{L}$  is a set of differentiations and combinations with details that depends on  $g_{\mu\nu}^{(0)}$ . These operations are all linear in  $g_{\mu\nu}^{(1)}$ , and eqs. (2) constitute linearized perturbation theory, a reduction of the problem to a system of linearized

equations. If the background is the Schwarzschild geometry, it is well known that these linear equations can be decomposed into multipole moments, can be separated into two sets of independent functions (the even-parity and odd-parity perturbations), and can be rearranged into two simple linear wave equations in the variables  $r, t$ : the Regge-Wheeler [5] equation for the odd-parity perturbations, and the Zerilli [6] equation for even-parity.

The part of (1) that is proportional to  $\epsilon^2$  has two kinds of terms. There are terms that are linear in  $g_{\mu\nu}^{(2)}$ , and terms that are quadratic in  $g_{\mu\nu}^{(1)}$ . It is clear that the former terms occur in precisely the same form as do the  $g_{\mu\nu}^{(1)}$  terms in (2). The set of  $\epsilon^2$  terms can then be written as

$$\epsilon^2 \mathcal{L}(g_{\mu\nu}^{(2)}) = \epsilon^2 \mathcal{J}(g_{\mu\nu}^{(1)}, g_{\mu\nu}^{(1)}), \quad (3)$$

where  $\mathcal{J}$  is quadratic in the first order perturbations. In solving for the second order perturbations, one treats the first order perturbations as already known, so  $\mathcal{J}$  plays the role of a source term in (3).

The  $\mathcal{L}$  operator in (3) is precisely the same operator as in (2), so for each linearized theory equation for  $g_{\mu\nu}^{(1)}$  there is a corresponding equation for  $g_{\mu\nu}^{(2)}$ , differing only in the presence of a “source” term. The very same manipulations that lead to the Regge-Wheeler and Zerilli equations, must therefore lead to Regge-Wheeler and Zerilli equations of precisely the same form for second order perturbations, except for the presence of source terms. We therefore have a guarantee at the outset that, as in first order theory, we can derive a simple wave equation for the perturbations.

The form of the metric functions, to any order in  $\epsilon$ , depend on how the spacetime coordinates are defined, to that order in  $\epsilon$ . We assume that we have a set of coordinates  $t, r, \theta, \phi$  in terms of which the metric, to zero order in  $\epsilon$ , takes on the standard Schwarzschild form. The background metric  $g_{\mu\nu}^{(0)}$  then is the Schwarzschild metric. We can transform to new coordinates  $x^{\alpha'}$  with  $\epsilon$ -dependent coordinate transformations

$$x^{\alpha'} = x^\alpha + \epsilon X_{(1)}^\alpha(x^\beta) + \epsilon^2 X_{(2)}^\alpha(x^\beta) + \mathcal{O}(\epsilon^3). \quad (4)$$

The form of the background metric  $g_{\mu\nu}^{(0)}$  is unchanged by the transformation, and we call such a coordinate change a gauge transformation. If the first order perturbation functions  $g_{\mu\nu}^{(1)}$  are changed by the coordinate change (i.e., if  $X_{(1)}^\alpha(x^\beta) \neq 0$ ) we call it a first order gauge transformation. Note that a first order gauge transformation will in general also change the form of the second order perturbations  $g_{\mu\nu}^{(2)}$ . We can also make gauge transformations which leave  $g_{\mu\nu}^{(1)}$  invariant and which change  $g_{\mu\nu}^{(2)}$  (i.e., transformations with  $X_{(1)}^\alpha(x^\beta) = 0$ , but for which  $X_{(2)}^\alpha(x^\beta) \neq 0$ ). These we call second order gauge transformations.

We will want to take advantage of the freedom to choose coordinates in order to impose some simplifying special conditions on the metric perturbations. As an example, for even parity perturbations we will at one stage be setting  $g_{r\theta} = 0$  through second order in  $\epsilon$ . To do this one makes a first order gauge transformation to set  $g_{r\theta}^{(1)} = 0$ . This transformation will affect the second order perturbations in some way that depends on the details of the coordinate transformation. One next makes a second order transformation to set  $g_{r\theta}^{(2)} = 0$ . This leaves  $g_{r\theta}^{(1)} = 0$ , since a second order transformation cannot change the first order perturbations.

In higher order perturbation calculations new questions can arise, such as what it means for a quantity to be gauge invariant. It turns out to be useful always to think of coordinate transformation as a sequence of distinct steps, as in the example above. In this way we see that a quantity can be “second order gauge invariant” (invariant only under a second order transformation), “first order invariant” (invariant only to first order under a first order transformation) or “first and second order invariant” (invariant up to second order for a general transformation (4)).

In this paper we take the first important steps in using second order perturbation theory as a tool; we provide the formalism for calculating the second order contribution to outgoing radiation. We will, however, make important, and yet physically sensible restrictions in our formulation. First, we will restrict attention to the even-parity  $\ell = 2, m = 0$  second order contributions. This is justified by the fact that most radiation processes are dominated by quadrupolar radiation, and we are primarily interested in the “error measure” on the first order quadrupole calculation. We give here the even-parity formalism, but the odd-parity equivalent follows the same pattern and is in fact simpler. The restriction to  $m = 0$  is for simplicity only, the generalization to  $m \neq 0$  is immediate. Our second, and more subtle, restriction is on the contributions to the “source” term. Since the source is quadratic in the first order perturbations, there is an infinite number of first order multipoles contributing to the  $\ell = 2$  projection of the source. We restrict attention only to the first order  $\ell = 2$  contributions. There are several reasons for this: (i) It is primarily a simplifying assumption; the  $\ell = 2$  projection of the *complete* source is straightforward to compute either numerically or as an infinite series, but would introduce unnecessarily distracting complications here. (ii) As a practical matter, it is plausible that in most practical situations the source will be dominated by the first order quadrupole terms. (iii) The specific case to which we will first apply this formalism (details to be published elsewhere) is the head-on collision

from initially small separation [2], in which the initial separation is the expansion parameter. In that case it can be shown that the first order perturbations have *only* an  $\ell = 2$  contribution.

In terms of the usual Schwarzschild-like coordinates  $t, r, \theta, \phi$ , and following the Regge-Wheeler [5] prescription and notation, we write the general form of the  $\ell = 2$ , even parity, perturbation of the spherically symmetric black hole metric in the form,

$$\begin{aligned}
g_{tt} &= -(1 - 2M/r) \left\{ 1 - \left[ \epsilon H_0^{(1)} + \epsilon^2 H_0^{(2)} \right] P_2(\theta) \right\} \\
g_{rr} &= (1 - 2M/r)^{-1} \left\{ 1 + \left[ \epsilon H_2^{(1)} + \epsilon^2 H_2^{(2)} \right] P_2(\theta) \right\} \\
g_{rt} &= \left[ \epsilon H_1^{(1)} + \epsilon^2 H_1^{(2)} \right] P_2(\theta) \\
g_{t\theta} &= \left[ \epsilon h_0^{(1)} + \epsilon^2 h_0^{(2)} \right] P_2'(\theta) \\
g_{r\theta} &= \left[ \epsilon h_1^{(1)} + \epsilon^2 h_1^{(2)} \right] P_2'(\theta) \\
g_{\theta\theta} &= r^2 \left\{ 1 + \left[ \epsilon K^{(1)} + \epsilon^2 K^{(2)} \right] P_2(\theta) + \left[ \epsilon G^{(1)} + \epsilon^2 G^{(2)} \right] P_2''(\theta) \right\} \\
g_{\phi\phi} &= r^2 \left\{ 1 + \sin^2 \theta \left[ \epsilon K^{(1)} + \epsilon^2 K^{(2)} \right] P_2(\theta) + \sin(\theta) \cos(\theta) \left[ \epsilon G^{(1)} + \epsilon^2 G^{(2)} \right] P_2'(\theta) \right\}
\end{aligned} \tag{5}$$

where  $P_2(\theta) = 3(\cos^2 \theta - 1)/2$ ,  $P_2'(\theta) = \partial P_2(\theta)/\partial \theta$ , and  $P_2''(\theta) = \partial^2 P_2(\theta)/\partial \theta^2$ , and where the functions  $H, h, K$  and  $G$  depend on  $t$  and  $r$  only. Just as in [5], which was restricted to linearized theory, we may impose the Regge-Wheeler gauge conditions through second order,  $h_0^{(i)} = h_1^{(i)} = G^{(i)} = 0$ ,  $i = 1, 2$ , by the two-step process described above. One can show, (details will be given in a lengthier paper), that given an arbitrary perturbation of the form (5), there exists always (locally) a uniquely defined gauge transformation that takes the metric to the Regge-Wheeler gauge. (One can, in fact, write these Regge-Wheeler functions, both to first and second order, explicitly in terms of perturbations in an arbitrary gauge, and through these expressions view the Regge-Wheeler perturbations as gauge invariant.)

With the linearized equations in the Regge-Wheeler gauge, Zerilli [6] assumes time dependence  $e^{i\omega t}$  (i.e., makes a fourier transform), and works with functions of  $\omega$  and  $r$ , but the process of deriving a single wave equation can be done equally well in terms of the original  $r, t$  variables. One can repeat this Zerilli process, step by step, with the second order equations, in the Regge-Wheeler gauge, in which the only new feature is the inclusion of quadratic ‘‘source’’ terms. One finds a total of seven nontrivial Einstein equations for the four second order Regge-Wheeler perturbation functions  $H_0^{(2)}, H_1^{(2)}, H_2^{(2)}, K^{(2)}$ . These equations are linear in the second order functions, but quadratic in the first order perturbations. One of these takes the form

$$H_2^{(2)} = H_0^{(2)} + [(H_1^{(1)})^2 - (H_0^{(1)})^2]/7 \tag{6}$$

and may be used to eliminate  $H_2^{(2)}$  from the other equations. (We also have  $H_2^{(1)} = H_0^{(1)}$ , from the first order equations). We are therefore left with six equations for the three remaining second order functions, but it turns out that we may choose three of these equations so that they can be obtained by compatibility of the other three. Moreover, from one of Einstein’s equations we find,

$$H_0^{(2)},_t = -K^{(2)},_t + [(1 - 2M/r)H_1^{(2)}]_{,r} + (1/14)[2(K^{(1)})^2 + 3(H_0^{(1)})^2 - 3(H_1^{(1)})^2]_{,t} \tag{7}$$

(a comma indicates partial derivative), which may be used to eliminate  $H_0^{(2)}$  (or rather its time derivative) in the remaining two independent equations. Following Zerilli [6], we introduce now a pair of functions  $\chi(t, r)$  and  $\mathcal{R}(t, r)$ , related to  $K^{(2)}(t, r)$ , and  $H_1^{(2)}(t, r)$  by

$$K^{(2)},_t = \frac{6(r^2 + Mr + M^2)}{r^2(2r + 3M)}\chi + \mathcal{R} \tag{8}$$

$$H_1^{(2)} = \frac{2r^2 - 6Mr - 3M^2}{(r - 2M)(2r + 3M)}\chi + \frac{r^2}{r - 2M}\mathcal{R} \tag{9}$$

By substitution of these relations in Einstein’s equations we find,

$$\begin{aligned}
\mathcal{R} = & \left(1 - 2\frac{M}{r}\right) \chi_{,r} + \frac{1}{7} \frac{r-2M}{2r+3M} \left( H_0^{(1)} H_0^{(1)},_t - 2K^{(1)} H_0^{(1)},_t - rK^{(1)},_r H_0^{(1)},_t \right. \\
& - 2K^{(1)} K^{(1)},_t - \frac{4}{r} K^{(1)} H_1^{(1)} + \frac{r^2}{r-2M} K^{(1)},_t H_1^{(1)},_t - \frac{2M}{r-2M} H_0^{(1)} K^{(1)},_t \\
& \left. + \frac{2M}{r} H_1^{(1)} K_r^{(1)} - H_1^{(1)} H_1^{(1)},_t - 2\frac{r-2M}{r^2} H_0^{(1)} H_1^{(1)} \right)
\end{aligned} \tag{10}$$

while the function  $\chi(t, r)$  satisfies the single second order differential equation,

$$\frac{\partial^2 \chi(t, r)}{\partial t^2} - \frac{\partial^2 \chi(t, r)}{\partial r^{*2}} + V(r) \chi(t, r) = \mathcal{S}_{\text{RW}}. \tag{11}$$

where  $r^* = r + 2M \log(r/2M - 1)$  is the ‘‘tortoise’’ radial coordinate, and where  $V(r)$  is the  $\ell = 2$  Zerilli potential, identical to that for the linearized theory (see [6]). The ‘‘source term’’  $\mathcal{S}_{\text{RW}}$  is a quadratic expression in the first order perturbations.

As guaranteed by the procedure, we have arrived at a Zerilli equation for a second order Zerilli function, differing from the Zerilli equation of linearized theory only in the presence of the source term  $\mathcal{S}_{\text{RW}}$  quadratic in the first order perturbations. (The explicit expression for  $\mathcal{S}_{\text{RW}}$  is somewhat lengthy and will be given elsewhere.) The second order Zerilli equation (11) is then the core of second order perturbation theory. We have verified, as a consistency check, that these functions satisfy the second order Einstein equations, (or rather, their time derivatives [7]), provided that the first order Einstein equations and (11) are satisfied. In principle, one first solves the first order Zerilli equation, finds the first order perturbations, constructs the source term  $\mathcal{S}_{\text{RW}}$ , and solves (11) for  $\chi$ . In practice, however, there are complications. These are evident in the fact that the source term in (11) diverges at large radius. This divergence does not indicate singular physical behavior; the second order radiation could in principle be extracted from  $\chi$  computed in this way, but the procedure would, at best, be computationally inefficient. In connection with this, two issues concerning gauge choice should, in particular, be noted. First, it must be realized that most of the development of the second order Zerilli equation (11) did not require that the gauge choice be Regge-Wheeler for the first order perturbations; only second order Regge-Wheeler restrictions are really needed. If we had made another gauge choice, or no gauge restrictions at all, only the quadratic terms in (7), (10), and (11) would change. The ‘‘second order Zerilli equation with source’’ is, therefore, not unique. Another viewpoint on this is that we could start with (11) and introduce a new Zerilli function through

$$\chi = \chi_{\text{new}} + \text{quad} \tag{12}$$

where quad is any expression quadratic in the first order perturbations. The new Zerilli function  $\chi_{\text{new}}$  would then satisfy (11), but with a modified source term. The choice of the Regge-Wheeler first order gauge is made for convenience. Expressions in this gauge usually turn out to be most compact.

The second gauge issue of note concerns the computation of radiation. To first order all information about radiation is carried in a ‘‘Zerilli function’’  $\psi$  which we define, in the Regge-Wheeler gauge, with the notation introduced in (5), by

$$\psi = \frac{r(r-2M)}{3(2r+3M)} \left( H_0^{(1)} - r \frac{\partial K^{(1)}}{\partial r} \right) + \frac{r}{3} K^{(1)}. \tag{13}$$

Several different normalizations for the ‘‘Zerilli function’’ have been used in the literature, so it is important to specify the relationship of our choice to others. If there are no (first order) sources, the definition in (13) can be shown, for  $\ell = 2$ , to have the same appearance as the definition of the wave function  $\widehat{R}_{LM}$  of Zerilli. That is, (13) specifies the same combination of perturbation functions  $K^{(1)}$  and so forth. But in our definition (5) of the metric perturbation functions, we have expanded in Legendre polynomials, whereas Zerilli does his multipole expansions in terms of spherical harmonics. As a result the actual relationship between our  $\psi$  and Zerilli’s  $\widehat{R}$  is

$$\psi = \sqrt{\frac{2\ell+1}{4\pi}} \widehat{R} \tag{14}$$

Another normalization that is of importance is the normalization used by Cunningham et al. [8], which agrees with that of Zerilli. The normalization used in [2] and [10], called  $\psi_{\text{pert}}$  in the latter reference, is related to  $\psi$  used here by:

$$\psi = \sqrt{\frac{2\ell+1}{4\pi}} 2 \frac{(\ell-2)!}{(\ell+2)!} \psi_{\text{pert}}. \tag{15}$$

The computation of gravitational radiation power is done in a coordinate system that is asymptotically flat. In this system, which we will denote with tildes, radiation information is carried by the perturbations in  $g_{\theta\theta}$ ,  $g_{\phi\phi}$  and  $g_{\theta\phi}$ . The function  $\tilde{G}$ , in the notation of (5), falls off in this gauge as  $1/r$ , to all orders in  $\epsilon$ , and  $r\tilde{G}$  can be thought of as the amplitude of the even parity gravitational wave. In terms of our definitions, The first order part  $\tilde{G}^{(1)}$  is given by  $\tilde{G}_{,t}^{(1)} = \psi(t, r)_{,t} / r + O(1/r^2)$  and the first order radiated power can be shown to be

$$\text{Power} = \frac{1}{16} \frac{(\ell + 2)!}{(\ell - 2)!} \frac{1}{2\ell + 1} (\epsilon\dot{\psi})^2, \quad (16)$$

where  $\dot{\psi}$  is the time derivative of  $\psi$ . For  $\ell = 2$  the result is  $\text{Power} = (3/10)(\epsilon\dot{\psi})^2$ . If we want the gravitational wave amplitude to second order we must calculate  $\epsilon\tilde{G}^{(1)} + \epsilon^2\tilde{G}^{(2)}$ .

The computation of  $\tilde{G}^{(2)}$  can be approached in several ways. One could, in principle, transform to a gauge which is asymptotically flat to first order, so that the source term  $\mathcal{S}_{\text{RW}}$  is replaced by the appropriately modified source term  $\mathcal{S}_{\text{rad}}$ . One then solves for  $\chi$ , from it computes the second order Regge-Wheeler metric perturbations, and then does a second order gauge transformation to a second order asymptotically flat gauge. In this gauge one reads off  $\tilde{G}^{(2)}$ . In practice the same thing can be accomplished more conveniently with a transformation of the form (12). We start with (11) and with a gauge which has Regge-Wheeler restrictions to first and second order, and we introduce a new function  $\chi(t, r)_{\text{ren}}$ , a sort of “renormalized  $\chi$ ,” given by

$$\chi_{\text{ren}} = \chi - \frac{2}{7} \left[ \frac{r^2}{(2r + 3M)} K^{(1)} K^{(1)}_{,t} + (K^{(1)})^2 \right]. \quad (17)$$

Simple replacement in (11) gives us an equation for  $\chi(t, r)_{\text{ren}}$  with the same form as (11), but with a source term

$$\begin{aligned} \mathcal{S}_{\text{ren}} = & \frac{12}{7} \frac{\mu^3}{\lambda} \left[ -\frac{12(r^2 + Mr + M^2)^2}{r^4 \mu^2 \lambda} (\psi_{,t})^2 - 4 \frac{(2r^3 + 4r^2 M + 9rM^2 + 6M^3)}{r^6 \lambda} \psi \psi_{,rr} \right. \\ & + \frac{(112r^5 + 480r^4 M + 692r^3 M^2 + 762r^2 M^3 + 441rM^4 + 144M^5)}{r^5 \mu^2 \lambda^3} \psi \psi_{,t} - \frac{1}{3r^2} \psi_{,t} \psi_{,rrr} \\ & + \frac{18r^3 - 4r^2 M - 33rM^2 - 48M^3}{3r^4 \mu^2 \lambda} \psi_{,r} \psi_{,t} + \frac{12r^3 + 36r^2 M + 59rM^2 + 90M^3}{3r^6 \mu} (\psi_{,r})^2 \\ & + 12 \frac{(2r^5 + 9r^4 M + 6r^3 M^2 - 2r^2 M^3 - 15rM^4 - 15M^5)}{r^8 \mu^2 \lambda} \psi^2 - 4 \frac{(r^2 + rM + M^2)}{r^3 \mu^2} \psi_{,t} \psi_{,tr} \\ & - 2 \frac{(32r^5 + 88r^4 M + 296r^3 M^2 + 510r^2 M^3 + 561rM^4 + 270M^5)}{r^7 \mu \lambda^2} \psi \psi_{,r} + \frac{1}{3r^2} \psi_{,r} \psi_{,trr} \\ & - \frac{2r^2 - M^2}{r^3 \mu \lambda} \psi_{,t} \psi_{,rr} + \frac{8r^2 + 12rM + 7M^2}{r^4 \mu \lambda} \psi \psi_{,tr} + \frac{3r - 7M}{3r^3 \mu} \psi_{,r} \psi_{,tr} - \frac{M}{r^3 \lambda} \psi \psi_{,trr} \\ & \left. + \frac{4(3r^2 + 5rM + 6M^2)}{3r^5} \psi_{,r} \psi_{,rr} + \frac{\mu \lambda}{3r^4} (\psi_{,rr})^2 - \frac{2r + 3M}{3r^2 \mu} (\psi_{,tr})^2 \right] \quad (18) \end{aligned}$$

where  $\lambda = (2r + 3M)$ , where  $\mu = (r - 2M)$ , and where  $\psi$  is the solution of the first order Zerilli equation. This “renormalized” source dies off asymptotically for large  $r$ , and we find that  $\chi_{\text{ren}}$  behaves asymptotically as a function of  $t - r^*$  only. We will postpone a detailed proof to a lengthier publication.

Most important, it can be shown that  $\chi_{\text{ren}}$  and the radiation are related by

$$\tilde{G}^{(2)}_{,t} = \frac{1}{r} \left[ \chi_{\text{ren}} + \frac{1}{7} \frac{\partial}{\partial t} \left( \psi \frac{\partial \psi}{\partial t} \right) \right] + O(1/r^2) \quad (19)$$

and we see that  $\chi_{\text{ren}}$  determine completely, (and in a numerically meaningful way), the asymptotic behavior of  $\tilde{G}^{(2)}_{,t}$ . It seem appropriate, then, to refer to  $\chi_{\text{ren}}$  as the second order Zerilli function.

To conclude, we present the expression of the gravitational radiation power, accurate to second order in  $\epsilon$ , from which the total radiated energy is easily obtained, in terms of the first and second order Zerilli functions. To compute it, we just write the Landau-Lifshitz pseudotensor in terms of the metric perturbations in the asymptotically flat gauge (see [8] for details) and substitute the expression for the manifestly asymptotically flat form of the metric perturbation (19). The result is

$$\text{Power} = \frac{3}{10} \left\{ \epsilon \frac{\partial \psi}{\partial t} + \epsilon^2 \left[ \chi_{\text{ren}} + \frac{1}{7} \frac{\partial}{\partial t} \left( \psi \frac{\partial \psi}{\partial t} \right) \right] \right\}^2 . \quad (20)$$

The formalism, finally, consists of the following steps: (i) From a solution to the initial value problem, one extracts first order perturbations of three geometry and extrinsic curvature, and from them computes Cauchy data for the first order Zerilli function  $\psi$  [4]. (ii) The Zerilli equation is then solved numerically for  $\psi$  in the  $t, r$  region of interest. (iii) The solution for  $\psi$  is used to compute  $\mathcal{S}_{\text{ren}}$  in (18). (iv) From the initial value solution one next uses the definition of  $\chi$  in the Regge-Wheeler gauge

$$\chi(t, r) = \frac{r}{2r + 3m} \left[ r K_{,t}^{(2)} - \left( 1 - \frac{2M}{r} \right) H_1^{(2)} \right] , \quad (21)$$

from (8) and (9), and the other second order Einstein equations, at the initial hypersurface, to find the initial value of  $\chi$ , and its time derivative. (v) Next (17) and its time derivative are used to find the initial  $\chi_{\text{ren}}$  and  $\dot{\chi}_{\text{ren}}$  (vi) The renormalized Zerilli equation [eq. (11) with the source  $\mathcal{S}_{\text{ren}}$ ] is then solved numerically for  $\chi_{\text{ren}}$ . (vi) Finally, the outgoing radiation power is computed from (20). We will be pursuing several applications of this formalism in subsequent publication.

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