1-1-1999

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(Received September 6, 1999)

This is the writeup of the talk I gave at the Yukawa International Symposium at Kyoto, Japan on June 29, 1999. The talk summarizes the present status of the close limit approximation for colliding black holes.

§1. Introduction

1.1. *The three regimes of a black hole collision*

The collision of binary black holes is one of the primary expected sources of gravitational waves to be detected by the broadband interferometric gravitational wave telescopes currently under construction, like the LIGO project in the US, the British/German GEO project, the TAMA project in Japan and the French/Italian VIRGO project.

A collision of two binary black holes can be divided into three distinct regimes. Initially, the black holes spiral around each other in quasi-Newtonian orbits. The radius of the orbits decrease due to the emission of gravitational radiation. Let us call this period the "inspiral" phase. The gravitational waves produced during this phase are well described by the post-Newtonian approximation. Notice that such an approximation does not provide a good description of the whole spacetime, since it breaks down close to each hole (to first approximation, the holes are singular point particles), but as long as the holes are far apart, this is not expected to be relevant from the point of view of the waveforms observed at infinity.[*] The gravitational waves from this phase of the collision correspond to a quasi-regular sinusoid whose frequency and amplitude increases with time as the holes start getting closer to each other, known as the "chirp". Good descriptions of this approximation applied to the binary black hole case and appropriate references can be found in Blanchet et al.[2]

When the separation of the holes is around 10 to 12 times the mass of each individual holes, it is expected that the post-Newtonian approximation breaks down. It is not completely clear what is the extent of the breakdown, since the post-Newtonian approximation leads to an asymptotic perturbation series. In fact, attempts are currently being made to extend the validity of the domain of the approximation using Padé approximants.[3] In any event, there will be a limiting separation of the holes such that if they are any closer, one cannot use the post-Newtonian approximation. The domain that starts at that point and continues up to the point in which the

[*] Technically, one can ignore the vicinity up to third order post-Newtonian level.[1]

black holes form a single black hole is expected to only be treatable by implementing the evolution of the Einstein equations numerically. This has proven to be a notoriously difficult problem. The state of the art of three dimensional simulations of black hole collisions is such that at present the codes can rarely evolve more than 30 or 40 in units of the final black hole mass. One would need at least two orders of magnitude more to be able to follow the black holes in the supposedly rapidly decaying orbit below 10 m in separation. Given that additional resolution in 3D is very expensive, it is unlikely that the required increase will be obtained simply by using more powerful computers; new ideas appear to be needed.

Finally, when the black holes are close to each other, one can treat the problem as a single distorted black hole that "rings down" into equilibrium, evolving the distortions using perturbation theory. This is called the "close limit approximation" and will be the main subject of this talk.

1.2. *Why study the close limit?*

The study of the final ringdown can be approached with three different perspectives. All of them have their own appeal, so I will describe them in some detail:

a) *As a code check* Whenever we finally have available a three dimensional numerical code to integrate the Einstein equations for colliding holes, one could start the evolutions with the black holes close to each other.[*] The results should therefore coincide with those of the close limit approximation. This point of view has actually been pursued successfully in the head-on collisions. It turns out that even for this case the full numerical simulations have certain difficulties, and the close limit approximation can be used as a guiding principle to build numerical codes.[5]

b) *To reach astrophysical conclusions* It is usually assumed that the ringdown waveforms play no role in gravitational wave detection. This assumption is based on the fact that expectations are that most black holes will occur in a mass range of a few solar masses. For such mass range, the ringdown occurs at too high a frequency to be detectable by interferometric detectors, whereas the inspiral phase sweeps the frequency range at which the detectors have the peak sensitivity. However, if the mass of the colliding holes is higher, the inspiral's frequency becomes too low to be detected whereas the ringdown is more easily detectable. In fact, given that larger masses also imply more radiated energy, these collisions become easier to detect (for a detailed discussion see the papers by Flanagan and Hughes[6]). In fact, for an optimal mass range of about 300 Solar masses these collisions could even be visible by the initial LIGO interferometers up to a distance of 200 Mpc. In fact, they are likely to be the *only ones visible by the initial interferometer*. That such collisions might occur is not completely out of the question, given our current ignorance about the population of black holes. Recent suggestions[7] that black holes in the significant mass range might exist only reinforce this possibility, although the existence of such holes is being currently debated. Even if one assumes that collisions like these take place, the detectability of ringdowns is technically more involved than that of the

[*] As the experience in head-on collisions shows[4] numerical codes can develop additional problems when the black holes are close, let us ignore this detail here, since these problems can usually be dealt with.

inspiral (essentially since many noises in the detector look like ringdowns and also because template matching is hard since the ringdowns are short lived in terms of number of cycles of oscillation). Jolien Creighton discusses these issues in detail.[8)]

The main drawback of using the "close limit approach" in this context is that one does not have the appropriate initial data to start the problem. The initial data one would need to have "astrophysically meaningful" estimates of waveforms and radiated energies would correspond to the endpoint of a black hole merger. But this is precisely what we are unable to compute! The families of initial data for colliding black holes usually considered are not supposed to be physically realistic when one makes the separation parameters too small. This is essentially due to the fact that they are constructed via ad-hoc superpositions based on mathematical convenience. If one still insists on using them in the close regime (as we will do) one has to admit that the results will not have a definite physical justification. Our experience after trying several families of initial data is that the results (in terms of radiated energy) in the end rarely differ by more than a factor of order unity. Therefore — at the level of an art-form more than that of a scientific prediction —, one may be able to trust the results we present here physically as order of magnitude estimates. This is the point of view we will adopt from now on.

c) *To supplement numerical evolutions* If the state of the art of numerical relativity remains limited to few dozens of m in terms of the time length of the evolution, it would be useful to spend the precious three dimensional evolution time of the codes "coalescing" the holes rather than following the ringdown of a single formed hole. This approach has already been implemented in the case of collapse of disks by Abrahams, Shapiro and Teukolsky.[9)] This study used perturbations of Schwarzschild. Currently under study is the more general approach based on perturbations of Kerr for the case of colliding holes by Baker, Campanelli and Lousto (the "Lazarus/Zorro" project at the Albert Einstein Institute in Potsdam).

1.3. *Initial data*

To evolve a collision of black holes in the close limit one has to start with a given family of initial data. As we mentioned in the previous section, the physically correct initial data for close black holes arising from an inspiral and merger is not available. The usual families of initial data for binary black holes are obtained by ad-hoc mathematical prescriptions. We will discuss in this section some of the issues involved in such constructions. To have initial data for general relativity means to have a three dimensional spatial metric and an extrinsic curvature that solve the constraint equations of the initial value problem of general relativity, i.e., the "G_{00}" and "G_{0i}" components of the Einstein equations.

A popular method of constructing solutions for these equations is the Lichnerowicz-York conformal approach. In this approach one assumes that the three dimensional metric is conformally related to a given fixed metric. To simplify things, let us assume (as was done for instance by Bowen and York[10)]) that it is conformally flat. If in addition one assumes that the trace of the extrinsic curvature vanishes, the constraint equations are simplified significantly. The momentum constraint simply becomes the flat space divergence of a tensor that is up to a factor the extrinsic curvature, and the Hamiltonian constraint becomes an equation stating that the Laplacian of conformal factor is related to the square of the extrinsic curvature divided by the conformal factor to a given power.

The momentum constraint equations are easy to solve, and solutions were introduced by Bowen and York.[10] In this approach the tensor related to the extrinsic curvature completely determines the ADM momentum and angular momentum of the slice. The solutions constructed by Bowen and York depend on two vectors that coincide with the angular momentum and linear momentum of the slice.

One then is supposed to solve the remaining nonlinear elliptic equation for the conformal factor prescribing certain boundary conditions. This is usually achieved numerically, as discussed by Cook.[11] Since the Bowen-York extrinsic curvatures are linear in the momentum (linear or angular), for slow moving (or rotating) holes, one can also seek approximate solutions for the conformal factor expanding in powers of the momentum. To zeroth order the solution simply corresponds to the vanishing of the Laplacian of the conformal factor. This is the same equation one would have for a time-symmetric situation ($K_{ab} = 0$). Since solutions to this case with the topology of two holes are known (the Misner[12] and Brill-Lindquist[13] solutions), one immediately has approximate solutions for moving holes, to zeroth order of approximation. It turns out that this is all we will really need for the close limit (in first order perturbation theory). For higher orders in perturbation theory, one can iterate the construction and explicitly obtain a solution for the conformal factor as a power series in the momentum. These approximations work remarkably well, as

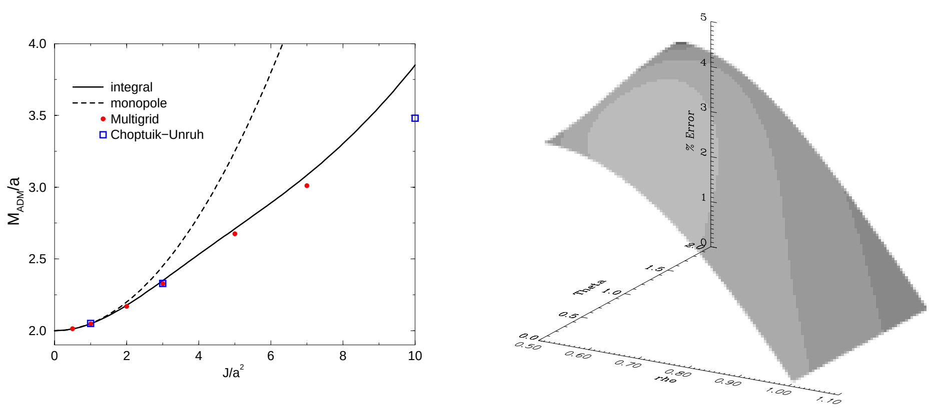

Fig. 1. Comparison of approximate solutions to the initial value problem with a full numerical integration performed with a multi-grid method, for the case of a single spinning Bowen-York black hole. The figure at the left compares the ADM mass of approximate and numerical solutions. The "monopole" and "integral' curves correspond to two different ways of computing the mass, looking at the monopole term of the conformal factor and using a Komar-type integral. The figure on the right shows the percentile difference between the full numerical solution for ψ (the fourth root of the conformal factor) and the second order approximation, for $J/a^2 = 4$, as a function of ρ, θ, where a is the conformal radius of the black hole ($M_{\rm ADM} = 2a + J^2/20a^3$ to second order in J). We see that the approximation works very well even for moderately large spins.

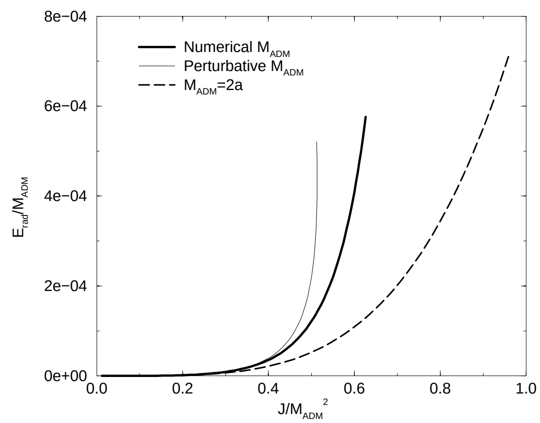

Fig. 2. The amount of energy radiated by a single spinning Bowen-York black hole as it "relaxes" to a Kerr black hole, computed treating the spacetime as a perturbation of Schwarzschild (one can view Kerr as a stationary perturbation of Schwarzschild, and the radiated energy is given by the non-stationary part of the perturbation). The amount becomes comparable to that produced by a collision for values of the spin larger than 0.5 in terms of the Kerr parameter a. The three curves correspond to different choices in how to compute the ADM mass. As we argue in the text, the perturbative calculations of the mass are not very accurate, but one can recourse to numerical calculations to get good estimates.

shown in Fig. 1 for the case of a single spinning hole.[14)]

An important drawback of the Bowen-York family of solutions is the conformally flat nature of the spatial metric. This is especially troublesome since neither a boosted Schwarzschild black hole nor a spinning Kerr hole[15)] appear to admit slicings with spatially flat sections. This means that the Bowen-York solutions will not represent purely boosted or spinning black holes, but there will be "additional radiation", which in general will be larger, the larger the momenta of the holes. Since for realistic collisions one expects the holes to be rapidly spinning, this is a serious impediment. In fact, one can consider[14)] a single Bowen-York hole and study its behavior treating it as a perturbation of a Schwarzschild hole. This has been done both for boosted and spinning holes, and as shown in Fig. 2 for the spinning case, the total radiated energy is low for small values of the spin. As we will see, collisions of black holes rarely radiate more than 1% of the holes' mass, therefore one sees that the extra radiation in the Bowen-York family is tolerable even for moderate values of the momenta.

Attempts have been made to generalize the Bowen-York ansatz to better accommodate especially the spinning cases. Krivan and Price[16)] and independently, Baker and Puzio,[17)] have proposed methods of solutions of the constraint equations. The Krivan-Price approach is based on the fact that for solutions that are "conformally Kerr" one can also find ways of superposing holes. Their solutions develop some undesired singularities, which in the case of close black holes can be hidden by the common horizon. These families of initial data have indeed been evolved successfully in the close limit. [16] The Baker-Puzio method is based on choosing an ansatz for the spatial geometry that is able to accommodate what one would intuitively consider the superposition of the spatial metrics of two Kerr black holes, and then solving an eikonal equation for the extrinsic curvature. This is a quite novel approach in that one prescribes metric and solves for the extrinsic curvature. The eikonal equations might however develop caustics and other singularities, and the method has not completely been implemented in practice. Both methods are up to present restricted to axisymmetry, and therefore are not yet applicable for the most interesting cases of inspiralling holes. It appears that the only solutions that one can construct that can reasonably accommodate spinning holes in inspiralling situations will have to be built numerically.

Other initial data proposals involve the use of Kerr-Schild ansatze for the metric. They have both been pursued in the Cauchy [18] and null [19] formulations. The close limit of these families has not been explored yet.

§2. Evolution

Once one has the initial data, one can proceed to evolve. To achieve this, the usual procedure has been to expand the initial data in terms of an expansion parameter that goes to zero as the separation of the holes goes to zero, and to identify the radial coordinate in conformal space with the radial coordinate in isotropic Schwarzschild coordinates. For cases involving boost or spin, one also keeps the leading terms in the linear momentum P and the angular momentum S of the individual holes, and assumes that P and S are of the same order as the separation in conformal space d in order to keep mixed terms. The first order departures from Schwarzschild are used to evaluate the initial data for the Zerilli function, which is then evolved using the Zerilli equation. It should be noticed that the first order departures are of order d^2 in terms of the conformal separation for non-moving holes and also have terms of order Pd for boosted holes. Here we face an inevitable difficulty, which in the end becomes problematic, at least for inspiralling holes. When one considers collisions of black holes with arbitrary boosts and spins, the problem is really multi-parametric, and one is really pushing things by insisting on fitting the problem into the usual framework of black hole perturbation theory, where one starts assuming a one-parameter family of space-times.

To evolve first order perturbations of black holes one has available several formalisms. They are all equivalent, but the details are significantly different. Let me concentrate on two of the most popular approaches. One of them is based on the Newman-Penrose formulation and leads (in the case in which the background space-time corresponds to a rotating black hole) to the so-called Teukolsky equation. This equation is a (complex) equation for the linearized part of one of the components of the Weyl spinor. In the case of a non-rotating background the Teukolsky equation reduces to the so-called Bardeen-Press equation. For a variety of historical reasons we have not used these formalisms in our approach, but rather used a different formalism which we broadly call Regge-Wheeler-Zerilli (RWZ) formalism. This formalism was constructed by treating separately the even and odd parity portions of the linearized perturbations. In both cases a real function is constructed out of the linearized components of the metric and satisfies a linear equation. For the case of even-parity perturbations the equation is called the Zerilli equation and for the odd-parity perturbations it is called the Regge-Wheeler equation. They are both equations for a single real function enconding the relevant gravitational degree of freedom. In the even parity case, the so-called "Zerilli function" is constructed with the components of the perturbative metric and its first time derivatives. Formulas for its construction are available that are invariant under first order coordinate transformations (gauge transformations).[20] Similar formulations are available for the odd-parity perturbations.

For the case of head-on collisions of momentarily stationary and boosted black holes, as well as for non-head-on collisions of non-spinning holes, the first order perturbations are only even-parity, and therefore the whole problem can be treated solely with the Zerilli equation. This was in part the historical reason for looking at this formalism, since it is somewhat simpler than the Teukolsky one for these initially important cases, and yet applicable. At first order in perturbation theory, the Zerilli and Regge-Wheeler formalisms use functions that involve only first order time derivatives of the initial metric. The Bardeen-Press approach requires one further derivative, which means that one has to use the Einstein equations in addition to the initial value equations to construct the initial data. This is a bit more cumbersome, and in fact, can introduce differences[21] depending on how one keeps orders in solving the Einstein equations, so it should be kept in mind in comparing the formalisms.

Figure 3 shows the radiated energy in a collision of two momentarily stationary black holes (Misner problem), as a function of the initial separation. We see here that the close approximation predicts very well the radiated energy up to separations of about six times the mass of each individual hole, when compared with full numerical simulations.

The figure also includes results for second order perturbation theory. The formalism is described in detail in Ref. 22). What is clear from the figure is that the perturbative formalism is self-consistent, that is, it is able to predict via recourse to higher order perturbations when it will fail to agree with the numerical results. The results are even more impressive for waveforms as shown in Fig. 4. This figure corresponds to a region of parameters in which the second order correction is maximal, that is, just before perturbation theory breaks down. It should be emphasized that the numerical results have certain uncertainties as well, as discussed in Ref. 5), so the agreement is probably even better than depicted.

These results are illustrative of what one can achieve in the head-on collision case with the close limit. One can also discuss collisions of boosted black holes[23),24)] to first and second order in perturbation theory, and the agreement with the numerical results is even more attractive. In Fig. 5 we show the agreement with numerical results for the energy. We see two remarkable things in the energy plot: first of all, the approximation works very well for large values of the momentum. Remember we argued initially that we would be considering a "slow" approximation, in the fact that we ignored the right-hand side of the Hamiltonian constraint. Why is it then that the energies keep on agreeing well for large values of P? The answer lies in the structure of the equations of the initial value problem. The equation satisfied by the extrinsic curvature is linear. Therefore as we increase the momentum, the extrinsic curvature grows without bound. In fact it grows linearly. The Hamiltonian constraint, however, due to its non-linear structure, implies that the conformal factor has a weak dependence on the momentum. As a consequence, for large values of the momentum, the initial data is completely "dominated" by the extrinsic curvature portion. We are doing a very poor job of accounting for the conformal factor with the "slow" approximation, but since the conformal factor is in norm very small in comparison with the extrinsic curvature, the evolution is completely dominated by the extrinsic curvature, for which we have an exact solution! One should be a bit cautious with this statement on one occasion: in the calculation of the ADM mass. The ADM mass is *completely* dominated by the conformal factor, therefore it is very poorly approximated by our technique. But the ADM mass can be computed numerically in the initial slice, and since the collisions radiate a comparatively small amount, the approximation is good throughout the evolution.

The second interesting aspect of the energy plot for the boosted collision is given by the "dip" in the energy that occurs as one increases the momentum. This is related to the previous point. If one starts the plot from the left, initially there is zero momentum, so one is simply recovering the results of the Misner case. In

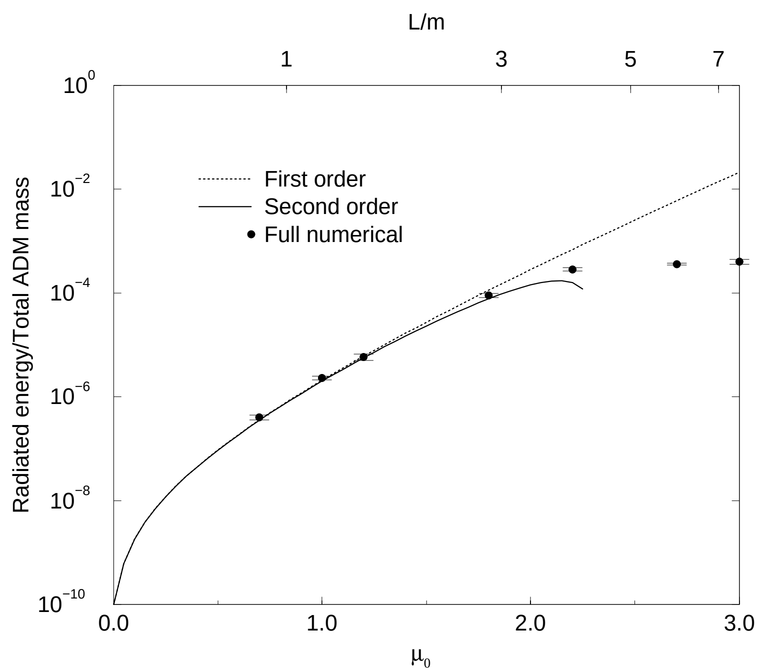

Fig. 3. The radiated energy in a collision of two momentarily-stationary black holes (the Misner problem) compared with the results of full numerical simulations of the NCSA/Potsdam/WashU group. We see that the approximation works well for black holes that are closer than about six times the mass of each hole.

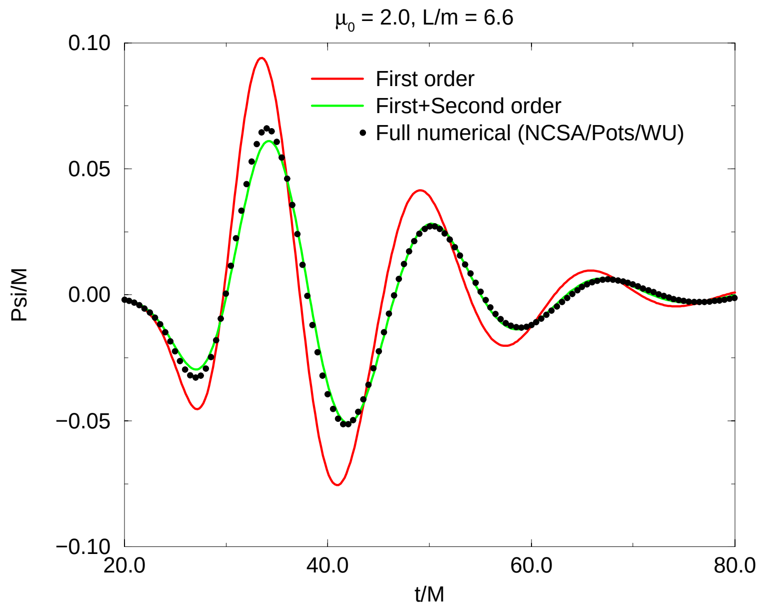

Fig. 4. First and second order waveforms. Because the first and second order Zerilli functions are not the coefficients of an expansion of a function, it makes no sense to compare them. We therefore present the time derivative of the first order Zerilli function and a second order correction to it. These quantities convey information about the gravitational waveform, since their square is proportional to the radiated power.

such case, all the radiation is produced by the conformal factor since the extrinsic curvature vanishes identically. As one increases the momentum, the portion of the initial data coming from the conformal factor and that of the extrinsic curvature "compete" with each other, and actually cancel each other, giving rise to the dip. As the momentum is increased further, the extrinsic curvature dominates. The cancellation at the dip implies that first order perturbation theory actually does not work too well, in spite of the fact that nominally we are in the optimal regime of applicability. Since there is a cancellation occurring one needs higher orders to account properly for things, as the energy plot shows.

§3. Collisions with net angular momentum

The most interesting collisions of black holes are not the head-on ones of course but the ones with angular momentum. In such case the immediate reaction is to think that the space-time should be approximated as a perturbation of a Kerr black hole. We shall see that this is not necessarily the case, however. There are several reasons why it might not be better to consider perturbations of Kerr.

To begin with, the whole perturbative paradigm consists in assuming one has a background metric and then "small departures" characterized by a dimensionless

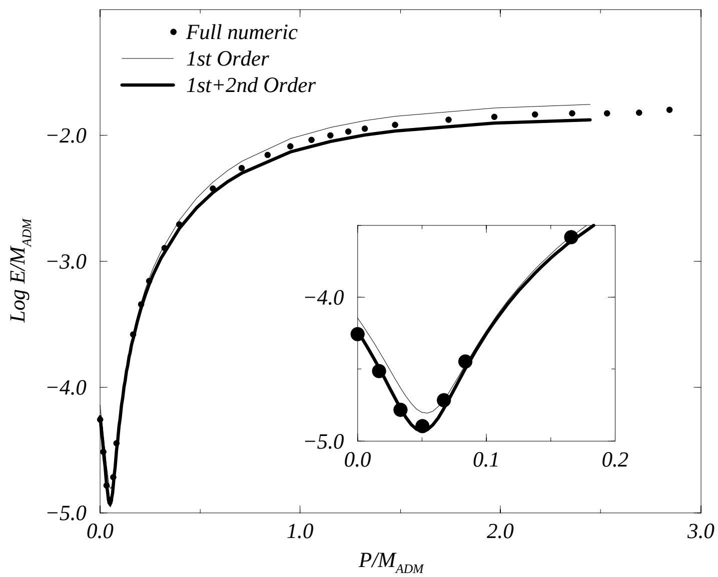

Fig. 5. Radiated energy in head-on black hole collisions as function of the momentum for a separation of $\mu_0 = 1.5$, $L_{\rm phys}/(0.5M_{ADM}) = 5.5$. Depicted are the close-slow approximation and the full numerical results of the Potsdam/NCSA/WashU group. Even for large values of the momentum, the first order results overshoot and the first plus second order undershoot the numerical results by only 20%. The inset shows the "dip" region.

parameter ϵ. Consider the collision of two non-spinning holes. In the "close limit" approximation the way we have set up things is to assume that both the separation of the holes d and their linear momenta P are small. As a consequence the total angular momentum of the holes $L = Pd$ will be small. In the "close limit" the angular momentum goes to zero. That is, when we make the perturbative parameter small in such a family of initial data, we recover the Schwarzschild spacetime and not the Kerr spacetime. We shall in fact see that for this case (non-spinning holes) one is indeed better off not using Kerr perturbations in practice. Moreover, if one insists on using Kerr perturbations, the perturbative formalism one sets up is at best peculiar. This is due to the fact that *the perturbative parameter* (essentially the angular momentum) *now appears in the background spacetime*, and to all orders in perturbation theory. This is not the usual way perturbation theory is set up. We have carried out calculations of this sort but we will shortly see that these conceptual difficulties eventually lead to confusions.

What if the holes are spinning? In such a situation one would presumably be better off considering the problem as a perturbation of a Kerr spacetime, but there are caveats. Is one going to consider the spins as fixed and determining the background and then use linear momenta and separation as "small" and "comparable" perturbative parameters? One might, but it would be odd in the sense that the angular momentum of the system should be taken into account when computing the total angular momentum. However, the orbital angular momentum is a significant component of the total angular momentum, we are back at the same problem as before: the background will depend (maybe more mildly) on the perturbative parameter. To add to the difficulties, the Bowen-York family of solutions does not represent Kerr black holes well individually, so if one is interested in studying situation with high spins in the individual holes, one will be adding a lot of spurious radiation.

One might be facing an unsolvable problem in the sense that the Schwarzschild solution has a more "robust" nature than the Kerr solution. That is, all concentrations of energy that are roughly spherical are close to the Schwarzschild solution outside. Rotating configurations only have exterior Kerr fields if there is a precisely tuned set of multipoles in the field. Two distinct concentrations of energy, like black holes, that inspiral towards each other might simply not look from the outside like a single rotating black holes, multipole-wise.

In the end perhaps the best way of sorting out these issues is to attempt to apply the perturbative formalism for these problems, and see what is the outcome. One can be conservative: in parameter regions where all formalisms agree, one can be quite confident in the results, and discard other results until confirmed in other ways. We are in the process of doing so. Currently we have only completed the non-head-on collision of two non-spinning Bowen-York holes. [25] We have evolved it with both the Zerilli and the Teukolsky formalism. To achieve the evolutions with the Teukolsky formalism we needed several intermediate results. To begin with, there was virtually no experience with the Teukolsky equation in the time domain, largely because it is a $2+1$-dimensional problem. Krivan, Laguna, Papadopoulos and Andersson have now written a code [26] that integrates the Teukolsky equation in the time domain. That is the code we are using for evolution. Given the lack of experience with the Teukolsky equation in the time domain, we had to set up formulas relating the metric and extrinsic curvature to the initial data for the Teukolsky function. This is somewhat complicated technically, but it can be achieved. [27]

Figure 6 depicts the waveforms and energy radiated for the non-head-on collision of two black holes. The result shown in this figure is for two black holes initially separated in conformal flat space by $d = 1.8$ in terms of the mass of each hole. (If one were considering a Misner type geometry, the proper separation measured along the geodesic threading the throats would be 5.5 in in the same units. [4]) The curve labeled Z shows results for linearized perturbation calculations using the Zerilli equation; the curve T shows the result of "hybrid perturbation" calculations using the Teukolsky equation. The two results diverge around parameter values of $J/M^2 = 0.4$ to 0.5, and this is a reasonable limit to take for the applicability of perturbation estimates. We note that the Teukolsky results lie above the Zerilli results, and this weakly suggests that the Zerilli-based estimates are more accurate. (In close limit estimates for head-on collisions, linearized results always overestimated the nonlinear — i.e., numerical relativity — results.)

Summarizing, we see that in the close limit the collisions do not seem to radiate more than 1% of the mass of the holes. This limitation is robust, in the sense that we already saw it in the boosted head-on collisions: if one attempts to increase the

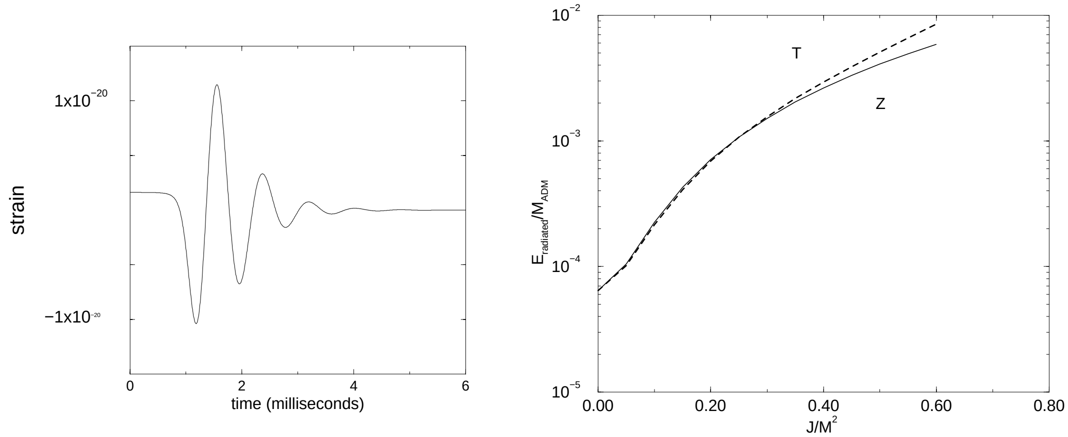

Fig. 6. The figure on the left shows the strain amplitude in the equatorial plane as a function of time, producted by a $10M_\odot$ black hole binary going through its ringdown phase at a distance of 100 Mpc from the detector. We assume that the detector is oriented for maximum sensitivity to the radiation in the orbital plane. (For an L shaped laser interferometer, one arm of the L would have to have the orientation just described for the bar; the other arm would have to be perpendicular to the orbital plane.) On the right is shown the fraction of the mass of the system radiated as gravitational waves as a function of the normalized initial angular momentum of the collision.

radiation by boosting the black holes harder, one also increases the initial ADM mass and therefore the radiated fraction of the energy in the end does not increase. The one percent figure is smaller than estimates that have been traditionally used for data analysis purposes.[6)]

Having at hand collisions without axisymmetry, one can ask questions about the radiation of angular momentum. A priori these are very interesting questions since it is expected that black holes will inspiral towards each other with too much angular momentum, in the sense of possessing more angular momentum than that needed to make the final resulting black hole extremal. Presumably this excess angular momentum has to be radiated somehow. It is not expected that this would happen in the final instants of the collision, but nevertheless it would be instructive to see what happens in these final moments.

We have computed the radiated angular momentum in both the Zerilli and the Teukolsky formalisms. At the moment however, it is not clear if these calculations are appropriate. We find that the radiated angular momentum disagrees in both calculations. We have eliminated all possible sources of errors by simply evolving the same initial data with the Teukolsky and the Zerilli codes and checking that if one eliminates the angular momentum dependent terms from the Teukolsky evolution equation (but not from the initial data), the results agree with those of the Zerilli evolutions. It appears that the addition of those (inconsistent perturbatively, as we argued above) small terms changes the predictions dramatically. This is not entirely surprising. The radiated angular momentum is a more subtle quantity to compute than the energy (where both formalisms agree quite well). The latter is basically a sum of squares, whereas the angular momentum is given by a correlation of modes. A small phase shift in one of the modes will therefore have no impact on the calculation of the energy, but would change dramatically the angular momentum radiated. Apparently this is the effect of the extra terms in the Teukolsky equation, and therefore the calculation in this formalism predicts much more radiated angular momentum.

There clearly is more to be understood in the comparison of Teukolsky and Zerilli calculations for the close limit of colliding black holes. This will require working both formalisms to higher order. Progress in setting up a second order formalism for the Teukolsky equation is being made by Campanelli and Lousto.[28]

§4. Summary

The close limit of black hole collisions has taught us several things about black hole collisions. The formalism is not capable of addressing the most interesting questions in the subject, but it allows us to tackle certain issues in a degree of concreteness that the full numerical simulations are currently lacking. Further work is needed to complete the understanding of the close limit of inspiralling black holes with spin. The whole subject has spawned interest in the initial data problem and progress is being made on this front too. The application of perturbative techniques to extend the life of full numerical codes also opens a new avenue for synergy between numerical and analytical work. In my opinion this synergy will be vital to allow the final tackling of the problem of two colliding black holes.

Acknowledgements

I wish to thank the organizers for the invitation to this wonderful conference. This paper summarizes work done with many collaborators: among the ones who have contributed the most are John Baker, Reinaldo Gleiser, Gaurav Khanna, Pablo Laguna, Hans-Peter Nollert, Richard Price. I am also grateful to Pete Anninos, Andrew Abrahams, Greg Cook, Steve Brandt and Ed Seidel for help with comparisons with full numerical results. I am indebted to many other people for insights and discussions. This work was supported in part by the National Science Foundation under grants NSF-INT-9512894, NSF-PHY-9423950, NSF-PHY-9407194, research funds of the Pennsylvania State University, the Eberly Family research fund at PSU. JP acknowledges support of the Alfred P. Sloan and John Simon Guggenheim foundations. I wish to thank the Institute for Theoretical Physics of the University of California at Santa Barbara for hospitality during the completion of this work.

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