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# The large cosmological constant approximation to classical and quantum gravity: model examples

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We have recently introduced an approach for studying perturbatively classical and quantum canonical general relativity. The perturbative technique appears to preserve many of the attractive features of the non-perturbative quantization approach based on Ashtekar's new variables and spin networks. With this approach one can find perturbatively classical observables (quantities that have vanishing Poisson brackets with the constraints) and quantum states (states that are annihilated by the quantum constraints). The relative ease with which the technique appears to deal with these traditionally hard problems opens several questions about how relevant the results produced can possibly be. Among the questions is the issue of how useful are results for large values of the cosmological constant and how the approach can deal with several pathologies that are expected to be present in the canonical approach to quantum gravity. With the aim of clarifying these points, and to make our construction as explicit as possible, we study its application in several simple models. We consider Bianchi cosmologies, the asymmetric top, the coupled harmonic oscillators with constant energy density and a simple quantum mechanical system with two Hamiltonian constraints. We find that the technique satisfactorily deals with the pathologies of these models and offers promise for finding (at least some) results even for small values of the cosmological constant. Finally, we briefly sketch how the method would operate in the full four dimensional quantum general relativity case.

## I. INTRODUCTION

### A. The problem

In the last few years, important developments in canonical quantum gravity have taken place, culminating with the recent formulation of two mathematically consistent [1,2], non-trivial canonical quantizations of general relativity in terms of Ashtekar new variables and spin networks. Instrumental in these developments have been the underlying advances in mathematical techniques for dealing with infinite dimensional nonlinear spaces, like the theory of cylindrical functions and associated measures [3], and the introduction of spin networks to eliminate the over-completeness of the Wilson loop basis [4] (for a recent summary, see the review article by Rovelli [5]). In spite of the advances, most of the results obtained from these theories up to now concern statements made *at a kinematical level* (without imposing the Hamiltonian constraint).

In order to start discussing if these theories contain the correct semiclassical physics and are therefore physically viable theories of quantum gravity, we need methods to introduce semiclassical states (encouraging recent kinematical results can be seen in [6]) but also to probe their dynamics. In particular a current open issue is if the semiclassical physics can be discussed at a kinematical level only or requires the imposition of the correct dynamics. Probing the dynamics has always been a problem in canonical general relativity [7]. In any gauge theory, the only physically relevant quantities are those that are invariant under the gauge symmetries of the theory, or in the canonical language, that have vanishing Poisson brackets with the constraints that represent the gauge symmetries. At a quantum mechanical level, the physical states are those that are annihilated by the constraints. In general relativity we do not have any example of a quantity that classically has vanishing Poisson brackets with the constraints<sup>1</sup>, at least in the case of a compact manifold. In fact, it is strongly suspected that such quantities will never be constructed in closed form [9]. Finding quantum states that are annihilated by the constraints is also a challenge.

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<sup>1</sup>An exception to this may be the quantity constructed in [8], corresponding to the holonomy group of the spin connection.

## B. An approximation scheme

Most researchers in this area do not consider these problems a fundamental obstruction, since in order to do meaningful physics one only needs approximate expressions for the observables, as was advocated long ago by Bergmann and Newman [10]. Finding an approximation method that does not destroy the non-perturbative nature of the canonical treatment is, however, a challenge. We have recently proposed one such method [11]. It is based on considering general relativity coupled to a cosmological constant and taking the  $\Lambda \rightarrow \infty$  limit [12]. The construction goes as follows:

For general relativity with a cosmological constant, the Hamiltonian constraint reads,

$$H(N) = H_{\Lambda=0}(N) + \Lambda \int d^3x N(x) \sqrt{\det q(x)} \quad (1)$$

where  $N(x)$  is a smearing function,  $H_{\Lambda=0}(N)$  is the Hamiltonian constraint without a cosmological constant and  $\det q$  is the determinant of the spatial metric. If one now considers the limit  $\Lambda \rightarrow \infty$ , and re-scales the constraint by  $1/\Lambda$ , one is left with a theory, which we will call “zeroth order” theory for which the Hamiltonian constraint is,

$$H^{(0)}(N) = \int d^3x N(x) \sqrt{\det q(x)}, \quad (2)$$

that is, the Hamiltonian constraint is just the square root of the determinant of the spatial metric. In addition to this the theory has the ordinary diffeomorphism constraint and if one uses Ashtekar variables there will also be a Gauss law constraint (both are independent of  $\Lambda$ ). Imposing classically the Hamiltonian constraint of the zeroth order theory, one immediately sees that it corresponds to metrics of identically vanishing determinant, that is, degenerate metrics.

It is easy to construct quantities that have vanishing Poisson bracket with the Hamiltonian constraint of the zeroth order theory. For instance, one can consider any function depending only on the three metric (or if one is using Ashtekar variables  $(\tilde{E}_i^a, A_b^j)$ , the densitized triads  $\tilde{E}_i^a$ ) and not on its canonically conjugate momenta. To have a genuine observable the quantity should also have vanishing Poisson bracket with the diffeomorphism constraint. There are standard ways of achieving this coupling the theory to matter [13,14]. A novel point is that in our approach, since the extra matter couplings in the Hamiltonian are higher order in  $\Lambda^{-1}$  these techniques yield genuine observables for the zeroth order theory, whereas normally they just construct quantities that have vanishing Poisson bracket only with the diffeomorphism constraint but not with the Hamiltonian constraint. Since we will not need this technique for the examples we consider in this paper, we do not include further details here (see a discussion in [11]).

If one assumes that the observables of the theory are power series in the inverse cosmological constant,

$$O_\Lambda(\pi, \tilde{E}) = O^{(0)}(\pi, \tilde{E}) + \Lambda^{-1} O^{(1)}(\pi, \tilde{E}) + \dots, \quad (3)$$

and one requests that these observables have vanishing Poisson brackets with the Hamiltonian of the theory (more precisely, one would like that the Poisson bracket be proportional to the constraints of the theory, for simplicity we will just demand that they vanish, but it is immediate to extend the construction to the case in which they vanish on-shell) and we expand such requirement in powers of  $\Lambda^{-1}$  one gets

$$\{O^{(0)}, H^{(0)}\} = 0 \quad (4)$$

$$\{O^{(1)}, H^{(0)}\} + \{O^{(0)}, H^{(1)}\} = 0. \quad (5)$$

The first equation determines  $O^{(0)}$  and the second one leads to a linear partial differential equation for  $O^{(1)}$ . The construction can be readily continued to higher orders. In all cases one obtains a linear partial differential equation, albeit with a more and more complex inhomogeneous term. It should be noted that one can obtain many observables starting with different solutions to the first equation. The linear partial differential equations are not hard to solve, given that the coefficients of the derivatives are functions of the triads, whereas the derivatives are with respect to the connections. Given the simplicity of it, the system is always integrable (although the solutions might be pathological, as we see in the examples later in the paper) and therefore yields all the observables of the theory.

This amazing simplicity also has a quantum counterpart, at the time of finding states that are annihilated by the Hamiltonian constraint. If we now focus on the formulation of the quantum constraints in terms of the spin network representation of quantum gravity, for instance as discussed by Thiemann [1], one starts by considering our zeroth order Hamiltonian, which is closely related to the well-understood volume operator. It is easy to find eigenstates for

this operator, in particular ones with vanishing eigenvalues. When one considers the first order corrections, because the zeroth and first order Hamiltonians have a well defined algebraic action in the space of spin network states, one gets a completely manageable (albeit complicated) system of equations. We will not go into details of the quantum theory of full general relativity in this paper, since it would make the paper much less readable as we would have to introduce quite a bit of (readily available but quite detailed) technology of spin networks. We will only present, for completeness, a brief of what the issues in the full quantum theory are in the last section of the paper.

Let us however describe in general lines how the quantum perturbative approach would be for a generic system with a *single* Hamiltonian constraint (we will discuss the case of more than one Hamiltonian as a model of what can happen in a field theory in a subsequent section).

Consider a system where the quantum Hamiltonian constraint is of the form  $\hat{H}(\lambda) = \hat{H}^{(0)} + \lambda\hat{H}^{(1)}$  where  $\lambda$  is a parameter that we consider small. We then consider quantum states that admit a power series expansion in powers of  $\lambda$ ,

$$|\psi(\lambda)\rangle = |\psi^{(0)}\rangle + \lambda|\psi^{(1)}\rangle + \lambda^2|\psi^{(2)}\rangle + \dots \quad (6)$$

What one wants to do now is to solve the theory perturbatively. This requires some comments. What does it mean to solve perturbatively a quantum mechanical constrained system? The approach we will take is the following: consider the eigenvalue equation for the Hamiltonian constraint,

$$\hat{H}(\lambda)|\psi(\lambda)\rangle = E(\lambda)|\psi\rangle(\lambda), \quad (7)$$

and consider its solution up to a certain order in  $\lambda$ , finding the zero-energy eigenstate within such a theory. That is, what we end up doing is bound-state perturbation theory (provided the spectrum of  $\hat{H}^{(0)}$  is discrete) and finding one set of particular states, those with zero eigenvalue. A good example to bear in mind about the method is to consider an atom in a (weak) magnetic field and think that one is looking for states with a given energy. If one goes to a certain order in perturbation theory in terms of the magnetic field, one will be able to achieve the given energy level only for certain values of the magnetic field. That is, in this approach the perturbative parameter is quantized. It is also clear that the perturbed states that achieve a certain energy level for a given value of the perturbative parameter do not necessarily stem from perturbing the zeroth order theory level of that particular energy. It is important to notice this in the gravitational case, where one looks for states with vanishing eigenvalue. The perturbed states of interest will in general not have  $\hat{H}^{(0)}|\psi^{(0)}\rangle = 0$ .

So we have cast the problem of solving quantum mechanically a constrained system as finding perturbatively an eigenstate for a given Hamiltonian in a quantum mechanical system. This problem is well defined and, as we will see in the examples, solvable. For the full general relativity case, the zeroth order Hamiltonian is quite close to the volume operator, so we do have a discrete spectrum to start with. However, because it is a field theory, one really has infinitely many Hamiltonians and certain consistency requirements have to be met. Therefore the quantum mechanical analogy we made may turn out to be too naïve. The last example we consider in this paper is geared towards clarifying that point.

Why should one care about a regime in which  $\Lambda$  is large in Planck units, given that the current observables value in such units is  $10^{-132}$ ?. Our attitude is the following: we have now at least two candidates for theories of quantum gravity. We need to probe them in any regime we can get a handle on in order to help decide if the theories are plausible or not from a physical stand point. If we can do this, we can then see in detail what regime of  $\Lambda$  we can really access. Perhaps it is a bit better than what one initially expects and the results can be of interest to early universe cosmology. Or perhaps one is lucky enough to encounter a situation similar to other power series expansions (like the strong coupling limit in the lattice or the large  $N$  expansion in QCD) where the results are robust and useful way outside the naïvely expected regime. We will see some hints of this in one of the examples of this paper.

### C. Potential difficulties of the method and layout of answers provided in this paper

The above observations make the whole approach worthwhile exploring further. However, the surprising way in which it appears to deal in a simple way with questions that in the full theory seem unapproachable raises a certain level of skepticism. Several pointed questions can be asked that make the whole approach look questionable. Among them,

- a) Does the approximation have any hope of working for values of  $\Lambda$  smaller than the Planck scale?
- b) General relativity is a complex enough dynamical system, that some pathologies might be expected. For instance, solutions might exhibit some level of chaos. It could also happen that other pathologies appear, like the full theory having less observables than the zeroth order one. Does our method always produce  $2n - 2m$  observables (where  $n$

is the number of degrees of freedom and  $m$  the number of constraints)? With a method that reduces everything to linear ODE's, is there not a risk of "papering over" these subtleties.

c) What sort of approximation is one getting at a quantum level? What is meant by an approximate state, or an approximate solution to a quantum constraint?

d) When one applies the quantum perturbative approach in a field theory, one effectively is dealing with infinitely many Hamiltonian constraints. Is the method still viable or one gets an inconsistent set of conditions on the perturbed states, leading to an empty theory?

e) The zeroth order Hamiltonian constraints we consider seem to admit many more quantum states than the full Hamiltonian constraints. Does the method produce spurious states for the full theory or correctly notices the further limitations that should appear at higher orders?

This paper attempts to provide (at least partial) answers to these questions by probing the perturbative method in model systems. In section II we will consider the application of the technique to Bianchi models. This will allow us to probe questions a) and b). From Bianchi models we will learn that, surprisingly, when one evaluates the approximate power series that we get for the observables on solutions to the equations of motion, the cosmological constant drops off from the expressions. Evaluated on the solutions to the equations of motion, the expression for the observables become a sum of terms that approximate better and better a constant of motion without reference to the cosmological constant. For a Bianchi I recollapsing universe we will explicitly demonstrate that with only a few terms one finds an expression that maintains an almost constant (less than 10% variation) value across 80% of the life of the universe. We will also see that in the case of Bianchi IX one can easily construct observables, but it is likely that they will remain good approximations only for short periods of time when one enters the chaotic domain. We will also see that one does not obtain more observables than the required ones. One can obtain many *apparently different* power series with the method but they just represent reparametrizations of the basic observables. We will also learn how to deal with possible singularities in the expressions obtained for the observables as functions of the phase space.

In section III we will discuss the heavy asymmetric top in three dimensions. This is a Hamiltonian system that can be viewed as a pathological constrained system if one considers it for a fixed value of the energy. It is pathological in that it has less observables than four. We will see that our method *only apparently* generates too many observables. On close inspection we will find that several of the expressions are really not good candidates for observables because they are expressions that have support only on limited regions of phase space, that are in general not preserved by evolution. In the end, the method produces the correct number of observables for the model.

In section IV we explore a model consisting of two harmonic oscillators with constant energy difference. This system has received quite a bit of attention, for instance see [15]. Unless the ratio of the frequencies of the two oscillators is a rational number, the system is pathological in that it only admits one quantum state. We will show that our perturbative approach indeed notices this fact and produces correct approximations for the quantum states in all cases.

In section V we explore a model of coupled harmonic oscillators with two "Hamiltonian" constraints. General relativity being a field theory one has infinitely many Hamiltonian constraints (one per spatial point). Therefore the other models we consider in the paper seem a bit short in mimicking this feature, since they are mechanical systems with only one constraint. In this section we increase modestly this feature by considering a model with two Hamiltonians. We will see that the method deals with no problem with such a system.

We will end with a short reflection on the implications of the results of this paper and what challenges remain for the application of the approximate scheme in the context of full classical and quantum general relativity.

## II. BIANCHI MODELS

As a first example of the application of the technique, we will discuss the issue of finding observables for Bianchi models. Bianchi models are homogeneous cosmologies, and as a consequence the Einstein equations become ordinary differential equations. This allows to achieve, in some cases, quite a bit of progress towards their solution.

Following [16], we represent the Bianchi models using as variables the (diagonal) components of the triads  $E^i$  and the canonically conjugate momenta  $A_i$ . All class-A Bianchi models admit a simple Hamiltonian formulation in terms of these variables, but we will concentrate our attention in two models, the Bianchi I model and the Bianchi IX model. The former is the simplest model and the latter the one with the richest dynamics. Bianchi I models start from an initial singularity and expand anisotropically. Depending on the sign of the cosmological constant they may expand forever or recollapse. Bianchi IX models approach the singularity through an infinite sequence of oscillations of increasing complexity.

In terms of the Ashtekar variables the Bianchi I and IX models with a cosmological constant are described by a Hamiltonian constraint,

$$H = E^1 E^2 E^3 - \frac{1}{\Lambda} (E^1 E^2 (A_1 A_2 - \epsilon A_3) + E^1 E^3 (A_1 A_3 - \epsilon A_2) + E^2 E^3 (A_2 A_3 - \epsilon A_1)) \quad (8)$$

where  $\epsilon = 1$  corresponds to the Bianchi IX model and  $\epsilon = 0$  to the Bianchi I model. The Gauss law and diffeomorphism constraints are identically satisfied.

To apply our construction we need to start by choosing an observable for the zeroth order theory. In general such quantities are given by  $O^{(0)} = F(E^1, E^2, E^3, E^1 A_1 - E^2 A_2, E^1 A_1 - E^3 A_3)$ . If one sets out to find four independent observables one could start, for instance, with the following independent choices for zeroth order observables,

$$O_1^{(0)} = E^1 \quad (9)$$

$$O_2^{(0)} = E^2 \quad (10)$$

$$O_3^{(0)} = E^1 A_1 - E^2 A_2 \quad (11)$$

$$O_4^{(0)} = E^1 A_1 - E^3 A_3. \quad (12)$$

Let us now concentrate on the study of the Bianchi I cosmology. In this case  $O_{3,4}^{(0)}$  are both exact observables for the full model with a cosmological constant. Starting with  $O_1^{(0)}$  one finds the following first order correction,

$$O_1^{(1)} = -\frac{A_1}{E^2 E^3} (-E^1 E^2 A_2 + (E^1)^2 A_1 - E^1 E^3 A_3), \quad (13)$$

and in addition to this one has the solution to the homogeneous equation, which is given by an arbitrary function that commutes with  $H^{(0)}$ , that is, similar to  $O^{(0)}$ . The expression for the Bianchi IX model is very similar to the one we just introduced, it has an extra term  $-\epsilon A_1$ .

In order to investigate the behavior of the approximate observable we are considering we will evaluate it on an exact solution of the equations of motion, the metric of reference [17],

$$ds^2 = -dt^2 + V(t)^{2/3} \left[ \Sigma(t)^{\frac{4}{3} \cos(\gamma)} dx^2 + \Sigma(t)^{\frac{4}{3} \cos(\gamma - \frac{2}{3}\pi)} dy^2 + \Sigma(t)^{\frac{4}{3} \cos(\gamma + \frac{2}{3}\pi)} dz^2 \right] \quad (14)$$

where,

$$V(t) = \frac{\sin \omega t}{\omega} \quad (15)$$

$$\Sigma(t) = \frac{2(1 - \cos \omega t)}{\omega \sin \omega t} \quad (16)$$

and where  $\omega = \sqrt{-3\Lambda}$  and where  $\Lambda < 0$ . This solution corresponds to a Bianchi I universe that expands out of a Big Bang at  $t = 0$  but after a while the cosmological constant leads it to recollapse at a time  $\omega t_F = \pi$ . The Ashtekar variables for this metric (particularized to  $\gamma = 0$ ) read,

$$E^1 = (\cos(1/2 \omega t))^{4/3} \quad (17)$$

$$E^2 = E^3 = \frac{\sqrt[3]{2} \sqrt[3]{\tan(1/2 \omega t)} (\sin(\omega t))^{2/3}}{\omega} \quad (18)$$

$$A_1 = 1/6 \frac{\sqrt[3]{2} \left( 2 \sin(\omega t) (\cos(1/2 \omega t))^2 - 3 \sin(\omega t) - 4 \sin(1/2 \omega t) \cos(\omega t) \cos(1/2 \omega t) \right)}{\sqrt[3]{\sin(\omega t)} (\sin(1/2 \omega t))^{2/3} (\cos(1/2 \omega t))^2} \quad (19)$$

$$A_2 = A_3 = 1/3 \frac{\sin(1/2 \omega t) \omega}{\sqrt[3]{\cos(1/2 \omega t)}}. \quad (20)$$

As we can see, for the solution in question  $E_2$  and  $E_3$  vanish at  $t = 0$ . If we look at the expression for the first order correction we found, this implies that it diverges for  $t = 0$ . This is an example of what we stated earlier, namely that the expressions for the observables we found can sometimes be singular in certain points of phase space. One can construct a first order correction that is well behaved at  $t = 0$  by making use of the free function  $F^{(1)}$  that solves the homogeneous equation. If one adds to the first order correction the solution to the homogeneous equation given by  $(E^1 A_1 - E^2 A_2)(E^1 A_1 - E^3 A_3)(E^2 E^3)^{-1}$ , it becomes,

$$O^{(1)} = A_2 A_3 \quad (21)$$

and this quantity is obviously well behaved at  $t = 0$ .

If one now proceeds to compute the second order correction, one finds a similar situation with respect to the divergence at  $t = 0$ . This can be corrected by adding the function  $F_2 = -1/3((E^1 A_1 - E^2 A_2)(E^1 A_1 - E^3 A_3))^2 (E^2 E^3)^{-2} (E^1)^{-1}$  and one obtains for the correction,

$$O^{(2)} = \frac{1}{6(E^2 E^3)^2 E^1} (-2(E^1)^2 A_1^2 E^2 E^3 A_2 A_3 + (E^1)^2 A_1^2 A_3^2 (E^3)^2) \quad (22)$$

$$\begin{aligned} & -2E^1 A_1 E^2 (E^3)^2 A_2 (A_3)^2 + (E^1)^2 A_1^2 A_2^2 (E^2)^2 \\ & -2E^1 A_1 (E^2)^2 E^3 A_2^2 A_3 - 2A_2^2 (E^2)^2 A_3^2 E_3^2 \end{aligned} \quad (23)$$

and a similar subtraction yields the third order correction,

$$\begin{aligned} O^{(3)} = & -\frac{1}{30(E^2 E^3)^3 (E^1)^2} (-6(E^1)^3 A_1^3 (E^2)^2 E^3 A_2^2 A_3 - 6(E^1)^3 A_1^3 E^2 (E^3)^2 A_2 A_3^2 - 12E^1 A_1 (E^2)^3 (E^3)^2 A_2^3 A_3^2 \\ & -12E^1 A_1 (E^2)^2 (E^3)^3 A_2^2 A_3^3 - 14(E^1)^2 A_1^2 (E^2)^2 (E^3)^2 A_2^2 A_3^2 - 3(E^1)^2 A_1^2 E^2 (E^3)^3 A_2 A_3^3 \\ & -3(E^1)^2 A_1^2 (E^2)^3 E^3 A_2^3 A_3 - 4(E^1)^4 A_1^4 A_3 E^3 A_2 E^2 + 2(E^1)^4 A_1^4 A_3^2 (E^3)^2 + 2(E^1)^4 A_1^4 (E^2)^2 A_2^2 \\ & -6A_2^3 (E^2)^3 A_3^3 (E^3)^3 + 6A_2^3 (E^2)^3 (E^1)^3 A_1^3 + 6(E^1)^3 A_1^3 A_3^3 (E^3)^3) \end{aligned} \quad (24)$$

If we now evaluate the above expressions for the exact solution we discussed above one gets ( $x = \omega t$ ),

$$O_1^{(0)} = \cos^{4/3}\left(\frac{x}{2}\right) \quad (25)$$

$$\frac{O_1^{(1)}}{\Lambda} = 1/3 \frac{2 \cos^2\left(\frac{x}{2}\right) + 1}{\cos^{2/3}\left(\frac{x}{2}\right)} \quad (26)$$

$$\begin{aligned} \frac{O_1^{(2)}}{\Lambda^2} = & 1/27 \left[ -19 \cos^4\left(\frac{x}{2}\right) + 19 \cos^4\left(\frac{x}{2}\right) \cos^2(x) - 6 \cos^2\left(\frac{x}{2}\right) + 6 \cos^2\left(\frac{x}{2}\right) \cos^2(x) \right. \\ & \left. -4 \cos\left(\frac{x}{2}\right) \sin(x) \cos(x) \sin\left(\frac{x}{2}\right) - 2 + 4 \cos^3\left(\frac{x}{2}\right) \sin(x) \cos(x) \sin\left(\frac{x}{2}\right) + 2 \cos^2(x) \right] \left[ \cos^{8/3}\left(\frac{x}{2}\right) (-1 + \cos^2(x)) \right]^{-1} \end{aligned} \quad (27)$$

$$\begin{aligned} \frac{O_1^{(3)}}{\Lambda^3} = & \frac{1}{810} \left[ -568 \cos^6\left(\frac{x}{2}\right) - 15 + 15 \cos^2(x) - 194 \cos^4\left(\frac{x}{2}\right) + 488 \cos^6\left(\frac{x}{2}\right) \cos^2(x) + 354 \cos^4\left(\frac{x}{2}\right) \cos^2(x) \right. \\ & \left. -47 \cos^2\left(\frac{x}{2}\right) \cos^2(x) - 33 \cos^2\left(\frac{x}{2}\right) - 16 \cos^3\left(\frac{x}{2}\right) \sin(x) \cos(x) \sin\left(\frac{x}{2}\right) - 72 \cos\left(\frac{x}{2}\right) \sin(x) \cos(x) \sin\left(\frac{x}{2}\right) \right. \\ & \left. + 88 \cos^5\left(\frac{x}{2}\right) \sin(x) \cos(x) \sin\left(\frac{x}{2}\right) \right] \left[ \cos^{14/3}\left(\frac{x}{2}\right) (-1 + \cos^2(x)) \right]^{-1}. \end{aligned} \quad (28)$$

The first observation is that the above expressions for the correction have lost the dependence on  $\Lambda$  stemming from the perturbative approach (there is a trivially re-scalable dependence on  $\Lambda$  through  $x = \omega t$ ). Therefore, if the above expressions converge, the convergence will be independent of  $\Lambda$ . This can be seen in the following plot, in which we show the observables as functions of  $x$ . We see that in spite of the lack of dependence in  $\Lambda$ , the curves do converge, the third order one maintaining an almost constant value for the longest period of time (more than half the lifetime of the universe). The various corrections are not well behaved near the big crunch.

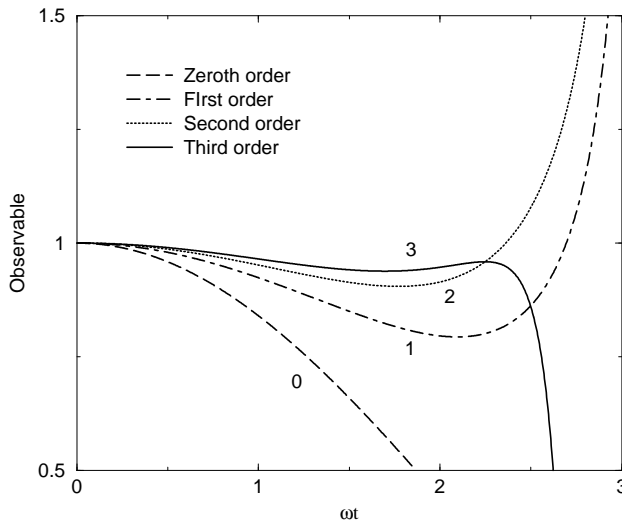


FIG. 1. The perturbative observable evaluated for a Bianchi I cosmology with negative cosmological constant. The time runs from zero, the Big Bang, to  $\pi$ , the Big Crunch. The four curves correspond to better and better approximations.

It should be remembered that the above calculations were particularized for  $\gamma = 0$ . The value of the observable will in general be  $\gamma$ -dependent. One way of understanding this is that the “physical” information content of the observable is to give a measure of the  $\gamma$  parameter, which is associated to the anisotropy of the model. For instance, the exact observable  $O_3 = A^1 E^1 - A^2 E^2$  takes the value  $O_3 = -\cos \gamma + \frac{\sqrt{3}}{3} \sin \gamma$ . Since the solution to the equations of motion we are considering is parameterized by a single constant, all observables become given functions of that constant.

As was shown in the above calculations, the Bianchi IX model appears as only slightly more complex, and similarly any class-A Bianchi model can be treated in the same way. Because of the local nature of our calculations (in time), it is not surprising to find observables in the Bianchi IX case. In that cosmology, because of the chaotic oscillations the dynamics undergoes, it is unlikely one will be able to find expressions for the observables that are global, but expressions like the ones we find should face no difficulty.

As we saw, the zeroth order observable one starts from has a huge degree of ambiguity. There is also a lot of ambiguity in each perturbative step, due to the presence of the solution of the homogeneous equation. It is evident that certain choices will lead to more useful (or in some cases shorter or even polynomial) expressions, as we exhibited when we manipulated the homogeneous solutions to avoid expressions that are singular at certain points in phase space.

Summarizing, we learn the following lessons from the Bianchi example we have just discussed. First of all, we see that the approximation for the observables we get, when evaluated on a solution of the equations of motion, becomes independent of the cosmological constant. This suggests that the perturbative approach will remain good even when the constant is small. In fact, we see in the example, that the observable is well approximated for a significant fraction of the lifetime of the universe in question. We also see that the presence of chaos in the Bianchi IX model does not preclude us from finding approximate expressions for the observable, though it is likely that it will impose further restrictions on the domain of validity (in terms of evolution time) of the approximate observable. It is possible to test these hypotheses with numerical evolutions of the Bianchi IX cosmology.

### III. THE ASYMMETRIC HEAVY TOP WITH A FIXED ENERGY VALUE

As another example of the application of the perturbative Hamiltonian ideas we are proposing, we analyze a model which in spite of its simplicity, adds pathological behaviors that are interesting to analyze under the proposed scheme. The model is an asymmetric heavy top with constant energy. This system only has one observable, the momentum canonically conjugate to the angle around the vertical axis. We will assume that the asymmetry in the top is small and treat this model as a perturbation of a symmetric heavy top, which has two observables. Will our technique notice that one loses an observable when one loses the symmetry?

Consider a heavy symmetric top with moments of inertia  $I_1 = I_2$  and  $I_3$ . Now suppose one sticks on two small masses at opposite sides of the surface of the top, forming a line perpendicular to and intersecting the top’s third axis at its center of mass. Then the moments of inertia along directions 1 and 2 will be slightly different. Let us quantify this as  $I_2 = I_1 + \lambda$ , with  $\lambda$  a small perturbation (we use the lowercase to avoid confusion with the cosmological



constant, although the parameter plays the same perturbative role in this problem as the inverse cosmological constant in full general relativity).

The Lagrangian for such a system will be given, in terms of the traditional Euler angles (see for instance [18]),

$$L = \frac{1}{2}I_1 \left( \dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta \right) + \frac{1}{2}I_3 \left( \dot{\psi}^2 + \dot{\varphi}^2 \cos^2 \theta \right) + \frac{\lambda}{2} \left( \dot{\varphi} \sin \theta \cos \psi - \dot{\theta} \sin \psi \right)^2 - M g \ell \cos \theta \quad (29)$$

where  $\ell$  is the length along the top third axis from the top base up to the center of mass. To introduce a canonical formulation, we compute the canonical momenta,

$$p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = I_1 \dot{\theta} + \lambda \dot{\theta} \sin^2 \psi - \lambda \dot{\varphi} \sin \theta \cos \psi \sin \psi \quad (30)$$

and similar equations for the other momenta, which yields an invertible linear system of equations from where one can compute the velocities as functions of the canonical momenta. One can then perform a Legendre transform and obtain a Hamiltonian  $H(p_\theta, p_\varphi, p_\psi, \theta, \psi, \lambda)$ . For small values of the perturbative parameter, one can expand,

$$H = H(\lambda = 0) + \lambda \frac{\partial H}{\partial \lambda}(\lambda = 0) \equiv H_0 + \lambda H_1, \quad (31)$$

where the general form of the Hamiltonians is,

$$H_0 = \frac{(p_\theta)^2}{2I_1} + \sum_{i,j=2}^3 c_{ij}(\theta) p_i p_j, \quad (32)$$

$$H_1 = = \sum_{i,j=1}^3 d_{ij}(\theta, \psi) p_i p_j \quad (33)$$

with  $p_1 = p_\theta, p_2 = p_\varphi, p_3 = p_\psi$ . We would like to treat the system as an example of constrained system. Therefore we request that the energy have a given value  $H = E_0$ , that is,

$$H^{(0)} - E^{(0)} + \lambda H^{(1)} = 0. \quad (34)$$

Before treating the case of the top with constant energy, let us first recall how one formulates the usual asymmetric top, but as a parameterized system. This allows a more natural contact with the constrained systems we will handle later. The parameterized top is obtained by introduce a time  $q^0$  and its canonically conjugate momentum  $p_0$ , such that,

$$p_0 - H(p_i, q^i) = 0 \quad (35)$$

where  $q^1 = \theta, q^2 = \varphi, q^3 = \psi$ . The observables are defined by,

$$\{O, p_0 - H(p_i, q^i)\} = 0, \quad (36)$$

which yields,

$$\frac{\partial O}{\partial q^0} - \frac{\partial O}{\partial q^i} \frac{\partial H}{\partial p_i} + \frac{\partial O}{\partial p_i} \frac{\partial H}{\partial q^i} = 0. \quad (37)$$

This system is solved by finding the solution to the following set of differential equations,

$$\frac{dq^0}{ds} = 1, \frac{dq^i}{ds} = -\frac{\partial H}{\partial p_i}, \frac{dp_i}{ds} = \frac{\partial H}{\partial q^i}. \quad (38)$$

The first of these equations can be integrated immediately,  $q^0 = s + C$ . The other two equations are just Hamilton's equations of motion, which integrated yield,

$$q^i = f^i(p_i^0, q_0^i, s), p_i = g_i(p_i^0, q_0^i, s), \quad (39)$$

if one substitutes  $s$  by  $-q^0$  in the above expressions, one obtains the six constants of motion,

$$q_i^0 = f_i(p_i, q^i, -q^0), \quad p_i^0 = g_i(p_i, q^i, -q^0), \quad (40)$$

which are observables of the parameterized system. A crucial element in the solution of this system, that allows to obtain the six constants of motion explicitly is that the solution of the equation for  $s$  in terms of  $q^0$  is straightforward. We will see that this poses problems in the next example.

Let us now study the case of interest, the (slightly) asymmetric heavy top, with Hamiltonian constraint (34) and assuming observables of the form  $O = O_0 + \lambda O_1 + \dots$ . Let us start with the equations to zeroth order in  $\lambda$ ,

$$\{O^{(0)}, H^{(0)} - E^{(0)}\} = 0, \quad (41)$$

with

$$H^{(0)} = \frac{(p_1)^2}{2I_1} + \sum_{i,j=2}^3 c_{ij}(\theta) p_i p_j, \quad (42)$$

which leads to the PDE,

$$\frac{p_1}{I_1} \frac{\partial O_0}{\partial q^1} + \sum_{i,j=2}^3 c_{ij}(\theta) p_j \frac{\partial O_0}{\partial q^i} + \frac{\partial O_0}{\partial p_1} \sum_{i,j=2}^3 \frac{dc_{ij}}{d\theta} p_i p_j = 0 \quad (43)$$

which again can be solved by the substitution technique,

$$\frac{dq^1}{ds} = \frac{p_1}{I_1}, \quad \frac{dq^i}{ds} = \sum_{i,j=2}^3 c_{ij}(\theta) p_j, \quad i = 2, 3 \quad (44)$$

$$\frac{dp_1}{ds} = \sum_{i,j=2}^3 c'_{ij} p_i p_j, \quad \frac{dp_i}{ds} = 0, \quad i = 2, 3. \quad (45)$$

At zeroth order the system has two trivial constants of motion,

$$p_i = p_i^0, \quad i = 2, 3. \quad (46)$$

One can in principle integrate the remaining equations to yield,

$$\theta = \theta(p_i^0, q_0^i, s), \quad p_\theta = p_\theta(p_i^0, q_0^i, s) \quad (47)$$

$$\varphi = \varphi(p_i^0, q_0^i, s), \quad \psi = \psi(p_i^0, q_0^i, s), \quad i = 1, 3. \quad (48)$$

As before, one can (formally) invert the solutions to obtain the constants of motion as functions of phase space by solving for  $s$  and substituting back in the following equations:

$$q_0^i = f_i(p_i, q^i, s), \quad p_i^0 = g_i(p_i, q^i, s), \quad i = 1, 3 \quad (49)$$

$$p_i = p_i^0, \quad i = 2, 3. \quad (50)$$

We therefore appear to have found again six constants of motion as in the case we studied before. However, the above equations involve periodic functions with arguments with incommensurate periods. One will only determine  $s$  up to multiples of the period and when substituting in the other equations will end up with multivalued functions. If one chooses a particular branch, the resulting observables only take values on a portion of the accessible phase space. They are therefore not acceptable as observables for the system.

This is an important observation, since we will see that this is the mechanism which prevents our technique from generating observables for systems that do not have them.

Let us now consider the first order corrections. The equation to be solved are,

$$\{O_1, H_0 - E_0\} + \{O_0, H_1\} = 0 \quad (51)$$

and if we start from  $O_0 = p_\psi$ , we are left with,

$$\frac{p_1}{I_1} \frac{\partial O_1}{\partial q^1} + \sum_{i,j=2}^3 c_{ij}(\theta) p_j \frac{\partial O_1}{\partial q^i} + \frac{\partial O_1}{\partial p_1} \sum_{i,j=2}^3 c'_{ij}(\theta) p_j = \sum_{i,j=2}^3 \frac{\partial d_{ij}}{\partial \psi} p_i p_j. \quad (52)$$

Now the PDE is non-homogeneous. We can still solve it with our parameterization technique via,

$$\frac{dq^1}{ds} = \frac{p_1}{I_1}, \quad (53)$$

$$\frac{dq^i}{ds} = \sum_{j=2}^3 c_{ij} p_j, \quad i = 2, 3 \quad (54)$$

$$\frac{dp_1}{ds} = \sum_{j=2}^3 c'_{ij} p_i p_j, \quad i = 2, 3 \quad (55)$$

$$\frac{dp_i}{ds} = 0 \quad (56)$$

$$\frac{dZ}{ds} = \sum_{i,j=2}^3 \frac{\partial d_{ij}(\theta, \psi)}{d\psi} p_i p_j \quad i = 1, 3. \quad (57)$$

As before, we will need to solve for  $s$  and replace it in the equations to obtain expressions for the initial values as functions of phase-space. We do run, however, into the same difficulty as before, we cannot obtain a uniquely defined expression for  $s$  and therefore we cannot construct a first order correction for the observable chosen. The procedure therefore fails to work.

What happens if we choose  $O_0 = p_\varphi$  instead? In that case  $\{O_0, H_1\} = 0$  and therefore the equation for  $O_1$  indeed has a solution. We will therefore recover, order by order, an expression for the *single* observable the system admits.

The general conclusion seems to be that, although the linear PDE's of our procedure in principle can always be solved, requiring that the solution be a well defined function on phase space (and not only on a portion of it) severely restricts the observables that can be obtained by the technique, and apparently prevents the technique from generating spurious observables in systems with less observables than degrees of freedom.

## IV. TWO OSCILLATORS WITH CONSTANT ENERGY DIFFERENCE

### A. The model

Let us now consider examples of the application of the method at a quantum level. Hajíček and others [15] have considered a model of constrained system, consisting of two harmonic oscillators with a fixed energy difference. The possible eigenstates of the system therefore consist (if one uses the standard inner product for the oscillators) in situations in which oscillators have the appropriate quanta of energy to satisfy the constraint that the difference of their energies be constant. It is clear that this can only happen if the ratio of the frequencies of the oscillators is rational. Otherwise, the system would admit at most one quantum state. This is the pathology of this system: if the ratio of the frequencies is not rational, the quantum system only has one state satisfying the energy constraint. If it is rational, the system admits infinitely many states. One could therefore consider starting with a zeroth order model consisting of two coupled oscillators with rational ratio of frequencies and perturb the system by adding the Hamiltonian of two oscillators with an irrational ratio. The zeroth order system has infinitely many quantum states and the perturbed system has only one state. Can our method detect and correctly handle this situation? This is what we attempt to probe in this section.

### B. Zeroth order Hamiltonian

The Hamiltonian constraint for the system in question simply reads,

$$H = \frac{1}{2}(p_1^2 + \omega_1^2 x_1^2) - \frac{1}{2}(p_2^2 + \omega_1^2 x_2^2) + 1 = 0 \quad (58)$$

where we have chosen the energy difference equal to one. The quantization of this model, viewed as a constrained system, has order ambiguities. Consider the transformation

$$\bar{x}_i = \sqrt{\frac{\omega_i}{\hbar}} x_i, \quad \bar{p}_i = \sqrt{\frac{\hbar}{\omega_i}} p_i, \quad (59)$$

the constraint reads,

$$H = \frac{1}{2}\hbar\omega_1(\bar{p}_1^2 + \bar{x}_1^2) - \frac{1}{2}\hbar\omega_2(\bar{p}_2^2 + \bar{x}_2^2) + 1 = 0 \quad (60)$$

and introducing the complex variables,

$$\alpha_j = \frac{1}{\sqrt{2}}(\bar{x}_j + i\bar{p}_j), \quad \alpha_j^* = \frac{1}{\sqrt{2}}(\bar{x}_j - i\bar{p}_j), \quad (61)$$

the constraint takes the form,

$$H = \hbar\omega_1\alpha_1^*\alpha_1 - \hbar\omega_2\alpha_2^*\alpha_2 + 1 = 0. \quad (62)$$

We can then proceed to the quantization,

$$\bar{x}_j \rightarrow \hat{x}_j, \quad \bar{p}_j \rightarrow \hat{p}_j \quad (63)$$

$$[\hat{x}_j, \hat{p}_k] = i\delta_{jk} \quad (64)$$

$$\alpha_j \rightarrow \hat{a}_j, \quad \alpha_j^* \rightarrow \hat{a}_j^\dagger, \quad (65)$$

and the quantum constraint can be written as  $\hat{H} = \hat{H}_1 - \hat{H}_2 + 1 = 0$ , with,

$$\hat{H}_j = \hbar\omega_j \left( \mu_j \hat{a}_j^\dagger \hat{a}_j + (1 - \mu_j) \hat{a}_j \hat{a}_j^\dagger \right) \quad (66)$$

with  $\mu_j$  representing the ordering ambiguities. To obtain the quantum states that are annihilated by this constraint, let us consider separately the spectra of both  $\hat{H}_j$ 's. Since they are harmonic oscillators, we have,

$$H_j = \hbar\omega_j n_j + \hbar K_j, \quad \text{eigenvector } |n_j\rangle \quad (67)$$

with  $K_{1,2}$  arbitrary ordering-dependent constants. In order to solve the constraint, the values of  $n_1, n_2$  must satisfy,

$$\hbar\omega_1 n_1 + \hbar K_1 - \hbar\omega_2 n_2 - \hbar K_2 + 1 = 0. \quad (68)$$

To produce solutions, one can choose the constants  $K_j$ . For instance, for  $n_1 = n_2 = 0$ , we get  $\hbar(K_1 - K_2) + 1 = 0$ .

In order to have more than one solution, one needs to have,

$$\omega_1 n_1 - \omega_2 n_2 = 0, \quad (69)$$

or,

$$\frac{n_1}{n_2} = \frac{\omega_2}{\omega_1}, \quad (70)$$

that is, the ratio of the frequencies of the oscillators must be a rational number. If  $\frac{n_1}{n_2} = \frac{N}{M}$  then  $n_1 = Nr$ ,  $n_2 = Mr$  with  $r$  an integer and the wavefunctions of the physical space are  $|Nr, Mr\rangle = |Nr\rangle |Mr\rangle$ .

### C. Perturbed model: exact treatment

We will now construct a model that is a perturbation of the above considered one but is still exactly solvable. We can then solve the model exactly and perturbatively and compare the results. Let us consider the system <sup>2</sup>,

$$H = H_1 - H_2 - \lambda \frac{\omega_2^2}{2} x_2^2 + 1 = 0. \quad (71)$$

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<sup>2</sup>Notice that this is equivalent, up to an overall rescaling, to add a perturbation corresponding to the energy difference of two oscillators, as we proposed doing in the introduction.

This system is equivalent to the original Hamiltonian constraint if we define  $\omega'_2 = \omega_2(1 + \lambda)^{1/2}$ . We can therefore readily solve it. For system to have more than one state, we have,

$$\omega_1 n_1 + \omega'_2 n_2 = 0, \quad (72)$$

or,

$$\frac{n_1}{n_2} = \frac{\omega'_2}{\omega_1} = \frac{\omega_2 \sqrt{1 + \lambda}}{\omega_1}, \quad (73)$$

therefore  $n_1 = Pr$  and  $n_2 = Qr$  with  $r$  integer, and remembering that  $\frac{\omega_2}{\omega_1} = \frac{N}{M}$ , in order to have a solution the perturbative parameter has to have a given value,

$$\lambda = \frac{P^2 M^2}{Q^2 N^2} - 1. \quad (74)$$

In order to compare with the perturbative calculation we will perform in the next paragraphs, let us consider the condition for having more than one solution for this model,

$$\omega_1 n_1 - \omega'_2 n_2 = 0 = \omega_1 n_1 - \sqrt{1 + \lambda} \omega_2 n_2 \quad (75)$$

which in the case  $\lambda \ll 1$  reads, to first order,

$$\omega_1 n_1 - \left(1 + \frac{\lambda}{2}\right) \omega_2 n_2 = 0, \quad (76)$$

or,

$$\lambda = 2 \left( \frac{M n_1}{N n_2} - 1 \right). \quad (77)$$

In order to have  $\lambda \ll 1$ , the ratio  $n_1/n_2$  must be a rational number close to  $N/M$ . We can therefore write it as,

$$\frac{n_1}{n_2} = \frac{Nr + \ell}{Mr + m}, \quad (78)$$

with large  $r$ . Expanding, we therefore get,

$$\lambda = 2 \left[ \frac{r + \frac{\ell}{N}}{r + \frac{m}{M}} - 1 \right] \sim 2 \frac{\ell}{Nr} - 2 \frac{m}{Mr} + O(r^{-2}). \quad (79)$$

Summarizing, the perturbed model can have infinitely many or a single quantum state, depending on the value of  $\lambda$ . Let us now see if treating the model perturbatively we recover the same result.

#### D. Perturbative treatment of the model

We consider the zeroth and first order Hamiltonians as,

$$H^{(0)} = H_1 - H_2 + 1 \quad (80)$$

$$H^{(1)} = -\frac{1}{2} \omega_2^2 x_2^2 = -\frac{1}{2} \hbar \omega_2^2 \bar{x}_2^2. \quad (81)$$

We start by considering the eigenvalue problem up to first order in  $\lambda$ ,

$$\left( \langle \phi^{(0)} + \lambda \phi^{(1)} | \right) \left( \hat{H}^{(0)} + \lambda \hat{H}^{(1)} \right) = \left( \epsilon^{(0)} + \lambda \epsilon^{(1)} \right) \left( \langle \phi_0 + \lambda \phi_1 | \right). \quad (82)$$

Let us start by considering the resulting zeroth order equation,

$$\langle \phi^{(0)} | H^{(0)} = \epsilon^{(0)} \langle \phi^{(0)} |. \quad (83)$$

The spectrum of  $H^{(0)}$  is,

$$\epsilon_{n_1, n_2}^{(0)} = \hbar (\omega_1 n_1 - \omega_2 n_2 + K_1 - K_2) + 1. \quad (84)$$

Here we need to distinguish if  $\omega_1/\omega_2$  is rational or not. If it is rational, then the spectrum is degenerate and we need to apply ordinary degenerate bound state perturbation theory as described in any quantum mechanics textbook. If it is not rational, the spectrum is not degenerate. As we discussed in the introduction, let us concentrate on the case in which the frequencies are related by  $\omega_2/\omega_1 = N/M$ . The degeneracy of the spectrum can be directly seen in that it takes the same values for  $n_1 = p + Nr$ ,  $n_2 = q + Mr$  with given  $p, q$  and arbitrary values of  $r$ , that is,

$$\epsilon_{p, q}^{(0)} = \hbar (\omega_1 p - \omega_2 q + K_1 - K_2) + 1 \quad (85)$$

The eigenstates of  $\hat{H}^{(0)}$  are,

$$\langle \phi_{p, q, r} | = \langle p + Nr, q + Mr | \quad (86)$$

in the number representation for the harmonic oscillators. We therefore find that the solution to zeroth order is given by (at the moment) an arbitrary combination of the given eigenstates,

$$\langle \phi_{p, q}^{(0)} | = \sum_r C(r) \langle \phi_{p, q, r} |. \quad (87)$$

Let us now consider the equation to first order in  $\lambda$ ,

$$\langle \phi_{p, q}^{(0)} | H^{(1)} + \langle \phi_{p, q}^{(1)} | H^{(0)} = \epsilon_{p, q}^{(0)} \langle \phi_{p, q}^{(0)} | + \epsilon^{(1)} \langle \phi_{p, q}^{(1)} |. \quad (88)$$

Since  $(\langle \phi_0 | + \lambda \langle \phi_1 |) (|\phi_0 \rangle + \lambda |\phi_1 \rangle) = 1$ , to first order in  $\lambda$  and we can choose the phase of  $|\phi_1 \rangle$  to have,

$$\langle \phi_0 | \phi_0 \rangle = 1, \quad \langle \phi_0 | \phi_1 \rangle = \langle \phi_1 | \phi_0 \rangle = 0 \forall p, q. \quad (89)$$

We now take the equation (88) and project it onto the the eigenspace with eigenvalue  $\epsilon_{p, q}^{(0)}$ . While doing this, the contribution from  $\langle \phi_{p, q}^{(1)} | H^{(0)}$  acting on this space cancels with the term  $\epsilon_{p, q}^{(0)} \langle \phi_{p, q}^{(1)} |$  acting on the same space. We are therefore left with,

$$\langle \phi_{p, q}^{(0)} | \hat{H}^{(1)} | p + Nr, q + Mr \rangle = \epsilon^{(1)} \langle \phi_{p, q}^{(0)} | p + Nr, q + Mr \rangle, \quad (90)$$

and recall that  $\hat{H}^{(1)} = 1/2 \hbar \omega_2 \hat{x}_2^2$  and in turn,

$$\hat{x}_2^2 = \frac{(\hat{a}_2 + \hat{a}_2^\dagger)^2}{2} = \frac{\hat{a}_2^2 + (\hat{a}_2^\dagger)^2 + \hat{a}_2 \hat{a}_2^\dagger + \hat{a}_2^\dagger \hat{a}_2}{2} = \frac{\hat{a}_2^2 + (\hat{a}_2^\dagger)^2 + 2\hat{a}_2^\dagger \hat{a}_2 + 1}{2}. \quad (91)$$

Since the matrix elements of  $H^{(1)}$  are between states of energy  $\epsilon_{p, q}^{(0)}$ ,  $\hat{a}_2^2$  and  $(\hat{a}_2^\dagger)^2$  do not contribute and the Hamiltonian is diagonal. Therefore of all the terms in the superposition defining  $\langle \phi^{(0)} |$  we are left with  $\langle p + Nr, q + Mr |$ , which normalizing yields,

$$\langle \phi_0 | = \langle p + Nr, q + Mr | \quad (92)$$

and energy,

$$\epsilon^{(1)} = -\frac{1}{2} \hbar \omega_2 \langle \phi_0 | \frac{2\hat{a}_2^\dagger \hat{a}_2 + 1}{2} | \phi_0 \rangle = -\frac{1}{2} \hbar \omega_2 \left( q + Mr + \frac{1}{2} \right). \quad (93)$$

Therefore the first order corrected energy is,

$$\epsilon^{(0)} + \lambda \epsilon^{(1)} = \hbar \omega_2 \left( \frac{M}{N} p - q - \frac{\lambda}{2} \left( q + Mr + \frac{1}{2} \right) \right) + \hbar (K_1 - K_2) + 1. \quad (94)$$

and as before we need to choose  $K_1, K_2$  in such a way that  $|0, 0 \rangle$  is a solution of the theory up to the order we are considering, i.e.,  $K_1 - K_2 + 1 - \frac{\lambda}{4} \hbar \omega_2 = 0$ . If we want other solutions, we need to make the energy eigenvalue vanish, up to the order we are considering this implies,

$$\lambda = 2 \frac{\frac{M}{N}p - q}{q + Mr}. \quad (95)$$

For small values of  $\lambda$  we should recover the results of the exact calculation. We notice that  $\lambda$  decreases with increasing  $r$ . In that limit we therefore have that,

$$\lambda \sim \frac{2}{r} \left( \frac{p}{N} - \frac{q}{M} \right) + O(r^{-2}), \quad (96)$$

and this agrees with the expansion in small  $\lambda$  of our exact calculation.

It should be noted that the above calculation shows that starting from a given state in the zeroth order theory and choosing a perturbative parameter, we get a first order solution to the constraint. It is immediate to see however, that this can be accomplished by starting from an infinite number of sets in the zeroth order theory. Simply consider a state obtained by multiplying  $p, q$  and  $r$  times an integer. The same value of lambda in equation (96) ensures that the first order state is a solution.

The calculation can be completed by considering the projection of (88) on the subspace of states orthogonal to the states of energy  $\epsilon_{p,q}^{(0)}$ . Such projection determines the correction to the state,  $\langle \phi^{(1)} |$ . This can be straightforwardly done.

The calculation can be repeated for irrational quotients of frequencies. In that case one is doing ordinary (non-degenerate) bound state perturbation theory. The calculations resemble very much the ones we did above, so we will not detail them here. Some comments are nevertheless in order.

The zeroth order theory has only one state when the frequency ratio is irrational and many states when it is rational. In the perturbed theory, in both cases we can find values of  $\lambda$  such that the theory has many states. That is, the perturbed model we constructed is such that depending on the value of the perturbative parameter one has either one or many states. If we do not choose the correct value of lambda (as determined by equation (96)), then the model only admits the state  $|0, 0\rangle$  as solution.

In particular, we have correctly addressed the situation posed in the introduction to this section: if the zeroth order model has rational frequencies (and therefore infinite solutions) but the perturbed model corresponds to a value of  $\lambda$  that only admits one solution, we will not satisfy the energy constraint to first order and we are therefore left with the vacuum as the only solution (in the factor ordering chosen). Therefore the method seems to handle well the drastic reduction in number of states implied by the perturbation.

It is suggestive that the model considered has a quantized value of  $\lambda$  in the full treatment if one wishes to have infinitely many states. That behavior is reproduced correctly in the perturbative approach. However, the quantization of the perturbative parameter is a feature of the perturbative method as long as the zeroth order Hamiltonian has a discrete spectrum (which is the only case in which we can apply the method). It is interesting to notice that in the other models considered by Hajicek, the spectrum of the Hamiltonian is continuous and therefore we cannot apply our method, and the full models do not have any particular requirements on the frequencies of the oscillators.

Another lesson from this example is that the lack of states in the perturbative treatment strongly suggests that the full model also lacks quantum states, unless there is some sort of non-analytical behavior in terms of the perturbative parameter.

## V. A MODEL WITH TWO HAMILTONIANS

In the case of quantum gravity, one is dealing with a field theory, therefore one has an infinite number of constraints. In the spin network representation, if one considers networks with a finite number of vertices, the number of equations resulting from the constraints might be finite, but in realistic situations of semi-classical interest it will still be large. Therefore the quantum models we have considered up to now, which are quantum mechanical systems with one constraint, may not capture some of the aspects present in the gravitational case. It might be that dealing with a larger than one number of constraints implies further relations between the perturbative coefficients that make the system incompatible. It is somewhat unlikely that this will happen, since after all, assuming the full theory is solvable, if the solution admits a power series expansion in  $\Lambda$ , one should find a solution that is acceptable perturbatively to any order in perturbation theory. Nevertheless, it might be useful to consider a simplified model with more than one constraint to make sure the technique works. It is in particular interesting to see how, if one has several constraints to satisfy, one can achieve it with the same value of the perturbative parameter for all the equations. This is what we attempt in this section.

## A. The model

Consider three different harmonic oscillators,

$$H_i = \frac{1}{2} (p_i^2 + \omega_i^2 x_i^2) \quad (97)$$

and define the two ‘‘Hamiltonian constraints’’,

$$H^{(0)}(1) = H_1 + H_2 - H_3 = 0 \quad (98)$$

$$H^{(0)}(2) = H_1 - H_2 + 2H_3 = 0. \quad (99)$$

The two constraints have vanishing Poisson brackets among themselves. We will have to request that the perturbed Hamiltonians also have vanishing Poisson brackets for all values of  $\lambda$  and this will impose restrictions on our perturbative approach.

As we discussed in the previous section, the eigenstates of the quantum version of the constraints are simply given in the number representation by  $|n_1, n_2, n_3\rangle$ ,

$$H^{(0)}(1)|n_1, n_2, n_3\rangle = \hbar(\omega_1 n_1 + K_1 + \omega_2 n_2 + K_2 - \omega_3 n_3 - K_3)|n_1, n_2, n_3\rangle \quad (100)$$

$$H^{(0)}(2)|n_1, n_2, n_3\rangle = \hbar(\omega_1 n_1 + K_1 - \omega_2 n_2 - K_2 + 2\omega_3 n_3 + 2K_3)|n_1, n_2, n_3\rangle. \quad (101)$$

It is possible to choose the  $K_i$  in such a way that  $|0, 0, 0\rangle$  has zero eigenvalue. The system admits other solutions only if

$$\omega_2 = \frac{M}{N}\omega_1, \quad \omega_3 = \frac{P}{Q}\omega_1 \quad (102)$$

which leads to a system of equations for the  $n_i$ ,

$$n_1 + \frac{M}{N}n_2 - \frac{P}{Q}n_3 = 0 \quad (103)$$

$$n_1 - \frac{M}{N}n_2 + 2\frac{P}{Q}n_3 = 0, \quad (104)$$

that has as general solution,

$$n_1 = \ell PM, \quad n_2 = -3\ell PN, \quad n_3 = -2\ell QM, \quad (105)$$

with  $\ell$  an arbitrary integer. Therefore the physical states of the theory (states annihilated by the constraints) are,

$$|\ell PM, -3\ell NP, -2\ell QM\rangle, \quad \text{integer } \ell. \quad (106)$$

We will now consider a perturbative term that still allows the system to be solved exactly,

$$H'_3 = H_3 + \frac{\lambda}{2}\omega_3^2 x_3^2, \quad (107)$$

so,

$$H(1) = H^0(1) - \frac{\lambda}{2}\omega_3^2 x_3^2, \quad (108)$$

$$H(2) = H^0(2) + \frac{\lambda}{2}\omega_3^2 x_3^2. \quad (109)$$

The modification introduced by the perturbation is equivalent to changing  $\omega_3$  by  $\omega'_3 = \sqrt{1 + \lambda}\omega_3$ . If we now consider the construction of quantum states  $|n_1, n_2, n_3\rangle$  that are annihilated by both Hamiltonian constraints, and seek for values of  $n_i$  that differ only slightly from the unperturbed ones, which is achieved by choosing  $p, q, s \ll \ell$ ,

$$n_1 = \ell PM + p, \quad n_2 = -3NP\ell + q, \quad n_3 = -2\ell QM + s \quad (110)$$

we find that,



$$p + \frac{M}{N}q - (\sqrt{1+\lambda} - 1)\frac{P}{Q}(-2\ell MQ) - \sqrt{1+\lambda}\frac{P}{Q}s = 0 \quad (111)$$

$$p - \frac{M}{N}q + 2(\sqrt{1+\lambda} - 1)\frac{P}{Q}(-2\ell MQ) + 2\sqrt{1+\lambda}\frac{P}{Q}s = 0. \quad (112)$$

The joint solution of these equations requires that  $p = Mb, q = -3Nb$ , and assuming that  $\lambda$  is small, we find that,

$$2Mb - \frac{P}{Q}s + \frac{\lambda}{2}(2\ell MP - \frac{P}{Q}s) = 0 \quad (113)$$

which implies that

$$\lambda = \frac{s}{\ell MQ} - \frac{2b}{\ell P} + O(\frac{1}{\ell^2}). \quad (114)$$

### B. Perturbative treatment

We recall the form of the unperturbed Hamiltonian and the first order perturbations,

$$H^{(0)}(1) = H_1 + H_2 - H_3, \quad H^{(1)} = -\frac{1}{2}\omega_3^2 x_3^2, \quad (115)$$

$$H^{(0)}(2) = H_1 + H_2 + 2H_3, \quad H^{(1)} = 2\frac{1}{2}\omega_3^2 x_3^2, \quad (116)$$

$$(117)$$

with  $\omega_2 = \frac{M}{N}\omega_1$  and  $\omega_3 = \frac{P}{Q}\omega_1$ , and we wish to solve for quantum states,

$$\langle \phi^{(0)} | + \lambda \langle \phi^{(1)} | \left( H^{(0)}(a) + \lambda H^{(1)}(a) \right) = \left( \epsilon_a^{(0)} + \lambda \epsilon_a^{(1)} \right) \left( \langle \phi^{(0)} | + \lambda \langle \phi^{(1)} | \right), \quad a = 1, 2. \quad (118)$$

The spectrum of  $H^{(0)}$  is given by,

$$\epsilon_1^{(0)} = \hbar\omega_1 \left[ n_1 + \frac{M}{N}n_2 - \frac{P}{Q}n_3 \right] + \hbar(K_1 + K_2 - K_3), \quad (119)$$

$$\epsilon_2^{(0)} = \hbar\omega_1 \left[ n_1 - \frac{M}{N}n_2 + 2\frac{P}{Q}n_3 \right] + \hbar(K_1 - K_2 + 2K_3). \quad (120)$$

Once more, we have a degenerate spectrum, with,

$$n_1 = \ell PM + p, \quad n_2 = -3\ell NP + q, \quad n_3 = -2\ell QM + s, \quad (121)$$

so the energies are,

$$\epsilon_1^{(0)}(p, q, s) = \hbar\omega_1 \left[ p + \frac{M}{N}q - \frac{P}{Q}s \right] + \hbar(K_1 + K_2 - K_3) \quad (122)$$

$$\epsilon_2^{(0)}(p, q, s) = \hbar\omega_1 \left[ p - \frac{M}{N}q + 2\frac{P}{Q}s \right] + \hbar(K_1 - K_2 + 2K_3), \quad (123)$$

and the eigenstates are,

$$\langle \psi_{\ell, p, q, s}^{(0)} | = \langle \ell PM + p, -3\ell NP + q, -2\ell QM + s |. \quad (124)$$

Going to next order in perturbation theory, we again have to require (as in the example we discussed in the previous section) that the first order correction to the state satisfy,  $\langle \phi^{(0)} | \phi^{(1)} \rangle = 0$ . We now compute the first order correction to the eigenvalues,

$$\langle \phi^{(0)} | H^{(1)}(a) | p, q, s, \ell \rangle = \epsilon_a^{(1)} \langle \phi^{(0)} | p, q, s, \ell \rangle, \quad a = 1, 2, \quad (125)$$

or, explicitly,

$$\epsilon_1^{(1)} = -\frac{1}{2}\hbar\omega_3 \langle \phi^{(0)} | \hat{x}_3^2 | p, q, s, \ell \rangle = -\frac{1}{2}\hbar\omega_3 \left( -2\ell QM + s + \frac{1}{2} \right), \quad (126)$$

$$\epsilon_2^{(1)} = \hbar\omega_3 \langle \phi^{(0)} | \hat{x}_3^2 | p, q, s, \ell \rangle = \hbar\omega_3 \left( -2\ell QM + s + \frac{1}{2} \right). \quad (127)$$

As before, we choose the  $K_i$ 's in such a way that  $|0, 0, 0\rangle$  is a solution,

$$K_1 + K_2 - K_3 - \frac{\lambda}{4}\omega_3 = 0 \quad (128)$$

$$K_1 - K_2 + 2K_3 + \frac{\lambda}{2}\omega_3 = 0. \quad (129)$$

We now require that the states have vanishing eigenvalues,

$$\epsilon_1^{(0)} + \lambda\epsilon_1^{(1)} = \hbar\omega_1 \left( p + \frac{M}{N}q - \frac{P}{Q}s - \frac{\lambda P}{2Q}(-2\ell QM + s) \right) = 0 \quad (130)$$

$$\epsilon_2^{(0)} + \lambda\epsilon_2^{(1)} = \hbar\omega_1 \left( p - \frac{M}{N}q + 2\frac{P}{Q}s + \lambda\frac{P}{Q}(-2\ell QM + s) \right) = 0. \quad (131)$$

For these equations to have a solution with a common value of  $\lambda$ , we need to choose carefully the states  $\langle p, q, s, \ell |$ , i.e, we must have  $q = -3Nb, p = Mb$ , and the solution for  $\lambda$  is,

$$\lambda \sim -\frac{2b}{P\ell} + \frac{s}{QM\ell} \quad (132)$$

so we see we completely reproduce the expansion of the exact solution we found in the previous subsection. As in the previous example, there are many eigenstates one can start from in the zeroth order theory that yield a solution for the same  $\lambda$ . The formula (132) is invariant if one multiplies  $b, s, \ell$  times an arbitrary integer.

We therefore see that indeed one needs extra conditions in order to have both constraints vanish on a given value of the perturbative parameter. But the conditions just imply that one starts with different zeroth order eigenvalues for the states in both constraints. The situation is schematized in the figure.

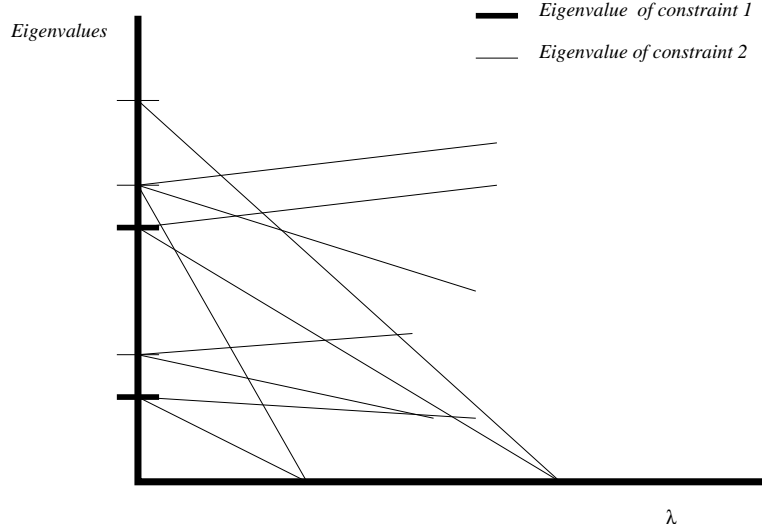


FIG. 2. A schematic diagram of how the constraints are solved. One starts from the eigenvalues of the unperturbed constraints of the system. The perturbation breaks the degeneracy of the eigenvalues and changes linearly the eigenvalues. When the corrected eigenvalues cross the zero value simultaneously for both constraints, one has a permitted value of  $\lambda$ .

Finally, if one sought to compute the first order correction of the quantum state, one get two equations for it, one per each constraint. If one studies them in detail, one finds that they admit the same solution.

## VI. SKETCHING THE FULL QUANTUM THEORY

We do not have a detailed discussion of the full general relativity quantum theory ready, but nevertheless one can sketch how the procedure would be. The calculations in this section should only be taken as a guide, and will be highly formal in nature. The details are very important in determining the final quantum theory, therefore we can only refer to these calculations as a “sketch” of what can be achieved. We will see simultaneous solutions to the diffeomorphism and Hamiltonian constraint. In order to achieve this we start with a basis consistent of equivalence classes of spin nets under subgroup of diffeomorphisms that keeps the vertices of the spin net fixed,

$$|s_{[\vec{v}]} \rangle = \sum D_{[\vec{v}]} |s \rangle, \quad (133)$$

where the  $D_{[\vec{v}]}$  are the diffeomorphisms that leave the vertices  $\vec{v}$  of the spin network fixed. This is a space very similar to the “habitat” considered by Lewandowski and Marolf [19]. The action of both the diffeomorphism constraint and Thiemann’s Hamiltonian is well defined in such a space. We can therefore proceed to study the problem of eigenvalues and eigenvectors to first order in perturbation theory,

$$\left( \langle \psi^{(0)} | + \Lambda^{-1} \langle \psi^{(1)} | \right) \left( \hat{V}(M) + \Lambda^{-1} \hat{H}(M) \right) = \left( \langle \psi^{(0)} | + \Lambda^{-1} \langle \psi^{(1)} | \right) \left( \epsilon^{(0)}(M) + \Lambda^{-1} \epsilon^{(1)}(M) \right), \quad (134)$$

Considering the equation at zeroth order we get,

$$\langle \psi^{(0)} | V(M) = \epsilon^{(0)}(M) \langle \psi^{(0)} | \quad (135)$$

the solution of this equation is given by the eigenvalues and eigenvectors of the determinant of the metric. Here we have to deal with the fact that this is a field theory. One can smear the determinant of the metric with a function  $M(x)$ . The resulting operator is closely related to the well-understood [4,20] volume operator, we call it “smeared volume” and denote it as  $V(M)$ . One will essentially get the volume associated with each vertex, which is a finite quantity, multiplied times the value of the smearing function at the vertex. Eigenstates will be reasonably easy to find. They will correspond to combinations of spin networks that yield the same total for the sum of the volumes of each vertex times the smearing function valued at the vertex. That quantity, in turn will be the eigenvalue  $\epsilon^{(0)}(M) = \sum_{v_i} V(v_i)M(v_i)$ . We can schematically write these states as  $\langle \psi_v^{(0)} | = \sum_s C(s)_v \langle \{s\}_{v, \epsilon^0(M)} |$  where we have a sum of states such that the all have the same value of  $V(M)$  for a given  $M$ . There are many states for each value of  $\epsilon^{(0)}(M)$ , to illustrate this, consider a spin network with a given number of vertices and a given value of  $\epsilon^{(0)}(M)$  and replace two lines in it by the same two lines with some knotting in between them. The value of the eigenvalue will not change, but it will be a different spin network.

Let us now consider the first order corrections. As we did in the quantum mechanical examples, we will project the perturbative equation on the space of states  $\langle s_{V(M)} |$ ,

$$\langle \psi^{(1)} | \hat{V}(M) | s_{V(M)} \rangle + \langle \psi_{\epsilon^{(0)}(M)}^{(0)} | \hat{H}(M) | s_{V(M)} \rangle = \epsilon^{(1)}(M) \langle \psi_{\epsilon^{(0)}(M)}^{(0)} | s_{V(M)} \rangle + \epsilon^{(0)}(M) \langle \psi^{(1)} | s_{V(M)} \rangle. \quad (136)$$

The first and last term in both members cancels out, since  $\hat{V}(M) | s_{V(M)} \rangle = \epsilon^{(0)}(M) | s_{V(M)} \rangle$ . One is therefore left with,

$$\langle \psi_{\epsilon^{(0)}(M)}^{(0)} | \hat{H}(M) | s_{V(M)} \rangle = \epsilon^{(1)}(M) \langle \psi^{(0)} | s_{V(M)} \rangle. \quad (137)$$

One can use this equation to explicitly compute  $\epsilon^{(1)}(M)$ , for a given proposal for the Hamiltonian constraint. One can now proceed to make the “energy” vanish at this order of perturbation theory, therefore determining the (quantized) value of the cosmological constant and the possible values of the energy of the zeroth order state  $\epsilon^0(M)$ .

To obtain the correction to the state, we need to project the above equation on states that are orthogonal to the eigenstates with eigenvalue  $\epsilon^{(0)}(M)$ . Let us call those states  $|s_{V'(M)} \rangle$ , that is, states with different values of the volume at the vertices. We therefore assign the volume at the vertices as a measure of orthogonality. We therefore write,

$$\langle \psi^{(1)} | \hat{V}(M) | s_{V'(M)} \rangle + \Lambda^{-1} \langle \psi_{\epsilon^{(0)}(M)}^{(0)} | \hat{H}(M) | s_{V'(M)} \rangle = \epsilon^{(0)}(M) \langle \psi_{\epsilon^{(0)}(M)}^{(0)} | s_{V'(M)} \rangle + \Lambda^{-1} \epsilon^{(0)}(M) \langle \psi^{(1)} | s_{V'(M)} \rangle, \quad (138)$$

Now, since  $\hat{V}(M) | s_{V'(M)} \rangle = \epsilon'^{(0)}(M) | s_{V'(M)} \rangle$ , we can write,

$$\langle \psi^{(1)} |_{s_{V'(M)}} \rangle = \frac{\langle \psi_{\epsilon^{(0)}(M)}^{(0)} | \hat{H}(M) |_{s_{V'(M)}} \rangle}{\epsilon'^{(0)}(M) - \epsilon^{(0)}(M)}. \quad (139)$$

and this equation would determine the first order correction to the quantum state. This would be only strictly true if the state of volume  $V(M)$  were non-degenerate. The degeneracy implies that there might be non-vanishing projections  $\langle \psi^{(1)} |_{s_{V(M)}} \rangle$  for vectors that belong to the space of states with eigenvalue  $\epsilon^{(0)}(M)$ . It is well known in degenerate perturbation theory that higher order calculations determine these components.

The dependence on  $M$  of the last equation might appear as surprising, since the first order quantum state should be  $M$  independent. Here we can draw on our experience on the system with two constraints studied in last section, which shows that a single correction appears no matter which Hamiltonian one uses to compute the correction. In the gravity case this would correspond to getting the same solution for different  $M$ 's. Another way to see that the equation is  $M$ -independent is to notice that one can characterize the action of the Hamiltonian independently at each vertex and in such calculations the role of  $M$  is that of a constant overall factor that drops out of the equations and one is left, as in the case of the oscillators, with a system of equations.

We have therefore constructed states such that,

$$\langle \psi_v |(\Lambda) = \langle \psi_v^{(0)} | + \Lambda^{-1} \langle \psi_v^{(1)} | \quad (140)$$

such that,

$$\langle \psi_v(\Lambda) | H(M, \Lambda) = O(\Lambda^{-2}) \forall M. \quad (141)$$

One is however, interested in states that are genuinely invariant under diffeomorphisms, whereas the above states are only invariant under diffeomorphisms that leave the vertices of the spin network fixed. The kind of states we look for are,

$$\langle \psi(\Lambda) | = \sum_{\mathcal{D}} \langle \psi_v(\Lambda) |_{\mathcal{D}}, \quad (142)$$

such that

$$\langle \psi(\Lambda) | H(M, \Lambda) = O(\Lambda^{-2}). \quad (143)$$

We shall show that with the above defined states this is indeed the case. This follows from the Poisson algebra of diffeomorphism and Hamiltonian constraints,

$$\mathcal{D}H(M, \Lambda)\mathcal{D}^{-1} = H(\mathcal{D}M, \Lambda). \quad (144)$$

Therefore,

$$\langle \psi(\Lambda) | H(M, \Lambda) = \sum_{\mathcal{D}} \langle \psi_v(\Lambda) |_{\mathcal{D}} \mathcal{D}H(M, \Lambda) = \sum_{\mathcal{D}} \langle \psi_v(\Lambda) |_{\mathcal{D}} H(\mathcal{D}M, \Lambda) \mathcal{D} = O(\Lambda^{-2}), \quad (145)$$

which is what we wished to show.

The details of these calculations can only be filled in with a given prescription for a Hamiltonian constraint and a definite space of states to operate upon. This will require much more detailed work than is appropriate for this paper. One can hint at what would happen, for instance, if one uses Thiemann's Hamiltonian. Let us start with trivalent spin networks. In such a context  $V(M)$  vanishes identically and the perturbative method has nothing to add: one is left without higher order corrections and the solution to the problem is reduced to determining which states are annihilated by the Hamiltonian constraints. To get something non-trivial, one needs four-valent vertices. Calculations on four valent vertices are readily possible, but cumbersome. In this context one could sharpen the determination of eigenstates, etc. Very schematically, one would see that the correction to the states that one gets depend on the action of the Hamiltonian constraint at each vertex. The states appear different from the ones obtained via "group averaging" that involve acting repeatedly with the Hamiltonian. However, since the action of the Hamiltonian is "local" (i.e. it does not add connections among vertices, but "dresses up" each vertex), the kind of states one gets have similar locality properties to those obtained with the "group averaging" procedure. In fact, if one works at higher order, at each order one acts with a Hamiltonian, so one can see that by going to higher orders, one is recovering something similar to the "group averaging".

An interesting point in trying to draw an analogy between quantum gravity and the simple models discussed in this paper is if the perturbation Hamiltonian has a discrete spectrum or not. We know that the zeroth order Hamiltonian

(the smeared volume) has a discrete spectrum. At the moment, it is not known if either the Thiemann or the Vassiliev Hamiltonians have a discrete spectrum. However, we have shown in this paper that the perturbative method works well in the case in which the perturbative Hamiltonian has a continuous spectrum (the case of the coupled oscillators). It is well known that the perturbative approach works well when the perturbing Hamiltonian has a discrete spectrum (for instance if one considers an atom in a magnetic field with constant energy).

It is clear that this is just the beginning of a discussion of the four dimensional quantum gravity case. It is interesting to notice, however, how one can quickly gain intuition as to the kind of states one would get and how they depend on the action of the Hamiltonian without doing the explicit computations. Since the latter must involve four valent vertices and therefore are extensive in nature, having a quick way to intuitively try out proposals for the Hamiltonian is a great asset.

## VII. CONCLUSIONS

It is obvious that the complexities that one expects in quantum gravity (in particular the infinite dimensional nature of the problem) cannot be really mimicked by finite dimensional systems. We believe, however, that the finite dimensional models we analyze in this paper help dissipate some of the most elementary skepticisms about the possibilities of the approach we are proposing. Namely: 1) that the method seems too simplistic to deal with chaotic, possibly pathological Hamiltonian systems as one expects full general relativity to contain in certain regimes; 2) that the method is only circumscribed to a range of values of the cosmological constant of no physical relevance; 3) that the application in the quantum domain of the technique is possibly inconsistent.

Applied to the full quantum theory, our method appears to reduce the problem of finding quantum states to a set of well defined, albeit complicated spin network calculations. Moreover, it appears to yield a quick intuitive handle on the properties of the quantum states of possible proposals for Hamiltonian constraints. We are currently exploring further the application of the approximation technique in full quantum gravity, as sketched in the last section.

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