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A discretization of a continuum theory with constraints or conserved quantities is called *mimetic* if it mirrors the conserved laws or constraints of the continuum theory at the discrete level. Such discretizations have been found useful in continuum mechanics and in electromagnetism. We have recently introduced a new technique for discretizing constrained theories. The technique yields discretizations that are consistent, in the sense that the constraints and evolution equations can be solved simultaneously, but it cannot be considered mimetic since it achieves consistency by determining the Lagrange multipliers. In this paper we would like to show that when applied to general relativity linearized around a Minkowski background the technique yields a discretization that is mimetic in the traditional sense of the word. We show this using the traditional metric variables and also the Ashtekar new variables, but in the latter case we restrict ourselves to the Euclidean case. We also argue that there appear to exist conceptual difficulties to the construction of a mimetic formulation of the full Einstein equations, and suggest that the new discretization scheme can provide an alternative that is nevertheless close in spirit to the traditional mimetic formulations.

I. INTRODUCTION

Continuum theories, either mechanical systems or field theories, usually have conservation laws and sometimes constraints. When one discretizes the equations of these theories, for instance in order to solve them numerically on a computer, or for "quantization on the lattice" purposes, the resulting discrete equations will usually fail to preserve the conserved quantities of the continuum theory upon evolution. Similar comments apply to constraints. Although one may have discrete equations resulting from discretizing the constraints of the continuum theory, if one chooses initial data that solves these equations exactly, they will fail to be solved upon discrete evolution.

Mimetic discretizations are discretizations of continuum theories that preserve conserved quantities or constraints in the discrete theory that *mimic* those of the continuum theory. There is quite a body of literature [1] on mimetic discretizations in the context of continuum mechanics and electromagnetism. The literature on Hamiltonian lattice QCD implicitly considers a mimetic discretization of Yang–Mills theory, although this fact is not usually emphasized.

Some authors have considered the question of whether mimetic discretizations of general relativity can be constructed [2, 3]. It is well known that if one discretizes the Einstein equations, the Hamiltonian and momentum constraint, which should hold for all time if satisfied initially (ignoring for the moment the issue of spatial boundaries), fail to do so in the discrete theory. Although there has been success in generating mimetic formulations of linearized relativity, it appears unlikely that something similar will be available for the full theory (or even for the linearized theory on non-trivial backgrounds or slicings). This is due to the fact that discretized derivatives fail to satisfy Leibnitz' rule and therefore the nonlinear terms when discretized do not have properties that mirror those of the continuum [3].

We have recently introduced a new approach to the discretization of theories, particularly of theories with constraints [4, 5] called "consistent discretization". The technique guarantees that the resulting discrete equations are compatible, i.e., they admit a common set of solutions (something that is not generically true if one discretizes the equations of a constrained theory). The technique has been tried out in the context of cosmological solutions of the Einstein equations [6], of BF theory and of Maxwell and Yang–Mills theories on the lattice [4]. Current investigations are testing it for the Gowdy models.

In this paper we would like to show that the technique we proposed, when applied to the Einstein theory linearized around a Minkowski background yields a discrete formulation that is mimetic. That is, the discretized constraints are exactly preserved under evolution without determining the Lagrange multipliers. We first consider linearized general relativity in terms of the traditional metric variables. We then consider it in terms of Ashtekar's variables, which have the advantage of being closer to the discretizations used in Yang–Mills theories (although in this case we restrict ourselves to Euclidean general relativity).

In the consistent discretization scheme, equations are discretized with variables evaluated at two (or more) different levels in time. This includes the constraints of general relativity, which in the discrete theory therefore can only be viewed as "pseudo" constraints (we will reserve the word constraint for expressions that involve all variables evaluated at the same instant of time, as in traditional canonical terminology). These equations, with variables discretized at mixed instants of time are the equations that are solved by the consistent discretization scheme. Of course, if one is in a regime in which the time-step is small, then satisfying the pseudo-constraints implies that the usual discrete constraints (with all the variables at the same time-step) are approximately satisfied as well. Therefore the resulting scheme cannot be strictly called mimetic, although it approximately is. We will show that if one uses a discretization for general relativity that is mimetic in the linearized case, one further improves the accuracy with which the consistent scheme for the full non-linear theory satisfies the constraints. This encourages further studies of these discretization schemes in the context of numerical applications.

In the next section we will present a brief summary of the consistent discretizations scheme. In the following two sections we apply it to linearized gravity, first with the traditional variables and then with the Ashtekar variables. We end with a discussion and proposals for further research.

II. CONSISTENT DISCRETIZATION OF CONSTRAINED THEORIES

We illustrate the technique with a mechanical system for simplicity, but there is no problem working it out for field theories, since upon discretization the latter become mechanical systems. We assume we start from an action in the continuum, written in first-order form,

$$S = \int L(q,p)dt \tag{1}$$

with

$$L(q,p) = p\,\dot{q} - H(q,p) - \lambda\phi(q,p) \tag{2}$$

where λ is a Lagrange multiplier and the theory has a single (it is immediate to incorporate several) constraint $\phi(q,p) = 0$. The discretization of the action yields $S = \sum_0^N L(n, n+1)$, where

$$L(n, n+1) = p_n(q_{n+1} - q_n) - \epsilon H(q_n, p_n) - \lambda_n\phi(q_n, p_n), \tag{3}$$

where $\epsilon = t_{n+1} - t_n$ and we have absorbed an ϵ in the definition of the Lagrange multipliers.

We will now view the Lagrangian as the generator of a type 1 canonical transformation between the instant n and the instant $n+1$. In ordinary classical mechanics parlance, given a canonical transformation between a canonical pair q, p and a new canonical pair Q, P, the generating function of a type 1 canonical transformation is a function of q, Q, $F(q, Q)$ and the canonically conjugate momenta are defined by $P = \partial F/\partial Q$, $p = \partial F/\partial q$. In our case we will view q_n, p_n, λ_n and $q_{n+1}, p_{n+1}, \lambda_{n+1}$ as "configuration variables" and will assign to each of them a canonically conjugate momentum through the canonical transformation,

$$\begin{aligned}
P^q_{n+1} &= \frac{\partial L(n,n+1)}{\partial q_{n+1}}, &(4)\\
P^p_{n+1} &= \frac{\partial L(n,n+1)}{\partial p_{n+1}}, &(5)\\
P^\lambda_{n+1} &= \frac{\partial L(n,n+1)}{\partial \lambda_{(n+1)}}, &(6)\\
P^q_{n} &= -\frac{\partial L(n,n+1)}{\partial q_{n}}, &(7)\\
P^p_{n} &= -\frac{\partial L(n,n+1)}{\partial p_{n}}, &(8)\\
P^\lambda_{n} &= -\frac{\partial L(n,n+1)}{\partial \lambda_{(n)}}. &(9)
\end{aligned}$$

If one explicitly computes the partial derivatives with the Lagrangian given, one can eliminate the p, P^p and P^λ to yield a more familiar-looking set of equations,

$$P^q_{n+1} - P^q_n = -\epsilon\frac{\partial H(q_n, P^q_{n+1})}{\partial q_n} - \lambda_{nB}\frac{\partial \phi^B(q_n, P^q_{n+1})}{\partial q_n},$$

$$\begin{aligned} q_{n+1} - q_n &= \epsilon \frac{\partial H(q_n, P^q_{n+1})}{\partial P^q_{n+1}} + \lambda_{nB} \frac{\partial \phi^B(q_n, P^q_{n+1})}{\partial P^q_{n+1}}, \\ \phi^B(q_n, P^q_{n+1}) &= 0. \end{aligned} \tag{10}$$

These indeed look like a discrete version of equations for a system with constraints. However, there are important differences. First of all, notice that as an evolution system the equations are implicit. Secondly, if one solves the first two equations one obtains P^q and q as functions of the initial data and the Lagrange multipliers. The last equation however, will generically not hold. One will have to choose specific values for the Lagrange multipliers at each time-step (and if one is dealing with a field theory at each point in space) for all the equations to be solved.

Notice that there can be particular cases in which the system does not determine the Lagrange multipliers. For instance, consider a totally constrained system like general relativity. There the Hamiltonian vanishes. Suppose now that the constraint in (10) is only a function of q_n. Then the evolution equation for q_{n+1} implies that $q_{n+1} = q_n$ and the constraint is automatically preserved. Therefore the resulting formulation is mimetic in the traditional sense of the word, a constraint that is just the discrete version of the continuum constraint is preserved under evolution by the discrete evolution equations. A similar situation develops if the constraint is only a function of P^q.

If the Hamiltonian is non-vanishing, and the constraint depends only on P^q_{n+1} then the latter is not automatically preserved upon evolution, but it cannot be satisfied by choosing the Lagrange multipliers either since they drop out from the relevant evolution equations. On the other hand if the constraint is only a function of q_n, its preservation could be enforced by choosing the Lagrange multipliers (the asymmetry between P^q and q in this treatment comes from the fact that we chose to write the equations as "propagating forward" in time, if one had chosen to propagate backwards, the roles of q and P^q in this discussion would be reversed).

Summarizing, the consistent discretization technique consists of discretizing the action and working out the resulting equations of motion for the discrete theory from it through the canonical transformation that implements time evolution. The resulting evolution equations (and constraints) are made a consistent set of nonlinear algebraic equations by considering the Lagrange multipliers as dynamical variables one has to solve for. In particular situations, the Lagrange multipliers are not determined by the equations. In such cases the resulting set of equations and constraints has to be consistent since it has been derived from a variational principle and the resulting discrete theory is mimetic in the traditional sense of the word: the constraints are automatically preserved upon evolution. In the other case, when the Lagrange multipliers are determined the resulting discrete theory is based on a consistent set of algebraic equations, but as one can see in equation (10), one is enforcing the constraints with some variables evaluated at instant n and some at instant $n + 1$. For small stepsizes, this implies that the constraints with all the variables evaluated at the same instant of time are approximately preserved. The resulting theory therefore cannot be called mimetic in the traditional sense of the word, although it can do a good job of preserving (approximately) the discrete constraints.

In the next two sections we apply this technique to linearized general relativity. We will see that the resulting theories do not determine the Lagrange multipliers, preserve the constraints automatically, and therefore are mimetic in the traditional sense of the word. We will not discuss the case of full general relativity here, but in several examples we have considered elsewhere for the non-linear theory (cosmologies [6], Gowdy spacetimes) the Lagrange multipliers are determined. Therefore it is unlikely that this method will yield a mimetic formulation for full GR. However, as we argued above, it will yield a formulation that approximates general relativity well in certain regimes and in such regimes the discrete constraints are enforced approximately very well. We believe it is likely that this is "as close as one will get" to a mimetic formulation of full general relativity.

III. LINEARIZED GENERAL RELATIVITY IN TERMS OF METRIC VARIABLES

In this section we will apply the technique we described in the previous section to linearized general relativity written in terms of the traditional variables. We assume the background is the Minkowski metric.

A. Continuum formulation

We start with the Arnowitt, Deser and Misner (ADM) [7] form of the action of general relativity,

$$S = \int d^4x \left[\pi^{ab} \dot{q}_{ab} - NC - N_a C^a \right], \tag{11}$$

where

$$C = \frac{1}{\sqrt{q}}\left[\pi^{ab}\pi_{ab} - \frac{1}{2}(\pi^b_b)^2\right] - \sqrt{q}\,{}^{(3)}R \quad (12)$$

$$C^a = -2\pi^{ab}_{;b} \quad (13)$$

and the variables (q_{ab}, N, N_a) are related to the four dimensional metric ${}^{(4)}g_{\mu\nu}$ through,

$$q_{ab} = {}^{(4)}g_{ab}, \quad (14)$$

$$N = \left(-\,{}^{(4)}g_{00}\right)^{-1/2} \quad (15)$$

$$N_a = {}^{(4)}g_{0a}, \quad (16)$$

The indices a, b, c run from 1 to 3. $\sqrt{q}$ is the determinant of the spatial metric q_{ab} and ${}^{(3)}R$ is its Ricci curvature scalar. The momenta π^{ab} are related to the extrinsic curvature of the space-like surfaces $x^0 = t =$ constant through $\pi^{ab} = -\sqrt{q}\left[K^{ab} - q^{ab}K^c_c\right]$ and indices are raised and lowered with the spatial metric. The semicolon denotes covariant differentiation with respect to the Christoffel connection of the spatial metric. Variation with respect to π^{ab}, q_{ab}, N, N_a yields the Einstein equations. In particular variation of N, N_a gives rise to four constraints $C = 0, C^a = 0$ usually referred to as (super)Hamiltonian and momentum (or diffeomorphism) constraints.

We have chosen the ADM action since it is one of the most traditionally used in general relativity. Modern numerical implementations favor the use of formulations in which the evolution equations are manifestly symmetric-hyperbolic. This is not the case for the ADM equations. In principle there is no obstruction in applying our technique to any action, but it just is the case that there has been little investigations about formulating the symmetric-hyperbolic formulations as deriving from an action principle. This will require further study and therefore we decided to concentrate on this paper on the ADM action for simplicity.

We now consider that the spacetime metric is given by a static background metric plus small perturbations ${}^{(4)}g_{\mu\nu} = {}^{(4)}g^{(0)}_{\mu\nu} + h_{\mu\nu}$. For simplicity we make the further choice that the foliation is such that the zeroth order shift $N_a = 0$ and the zeroth order extrinsic curvature is therefore zero $\pi^{ab} = 0$. The constraint equations to leading order in the perturbations are given by [8],

$$C^a = -2p^{ab}{}_{;b} \quad (17)$$

$$C = -\sqrt{q}\left[h_{ab}{}^{;ab} - h_{;a}{}^{;a} - h_{ab}\,{}^{(3)}R^{ab}\right] \quad (18)$$

In these expressions p^{ab} is the linear portion of the canonical momentum π^{ab} and ${}^{(3)}R^{ab}$ is the Ricci tensor of the background metric. The action for the linearized theory is,

$$S = \int d^4x\left[p^{ab}\dot{h}_{ab} - N^{(0)}H - N^{(1)}_a C^a - N^{(1)}C\right] \quad (19)$$

where we have kept track of the order in the perturbation expansion of the lapse and the shift (recall that we assume zero shift in the background). The constraints C, C^a are given by the expressions above, where only terms up to order linear have been kept. The quantity

$$H = \frac{1}{\sqrt{q}}\left[p^{ab}p_{ab} - \frac{1}{2}p^2\right] + \frac{1}{2}\sqrt{q}\left[\frac{1}{2}h_{ab;c}h^{ab;c} - h_{ab;c}h^{ac;b} - \frac{1}{2}h_{;a}h^{;a} + 2h_{;a}h^{ab}{}_{;b} + hh^{ab}{}_{;ab} - hh_{ab}\,{}^{(3)}R^{ab}\right] \quad (20)$$

is a true Hamiltonian density (not a constraint) that is responsible for the evolution of the canonical variables, and is multiplied in the action times the lapse of the background space-time.

At this point we can make an important observation. The momentum constraint (17) is only a function of the momenta p^{ab} (the covariant derivative is with respect to the background metric) and the Hamiltonian constraint (18) is only a function of the configuration variables q^{ab}. Therefore, as we discussed in section II, our discretization technique will not determine the value of the Lagrange multipliers. The resulting theory therefore can only either be: mimetic or inconsistent. We will proceed to show that the resulting discrete theory is indeed consistent.

B. Discretization

We start by discretizing the linearized action, $S = \sum_{n=1}^{N} L(n, n+1)$, where,

$$\begin{aligned} L(n,n+1) \;=\; & \sum_{\vec m} \Bigg(\sum_{a,b=1}^{3} \{ p_{ab}(n,\vec m)\,(h_{ab}(n+1,m) - h_{ab}(n,m)) \\ & -N(n,\vec m)\,[h_{ab}(n,\vec m+\vec e_a+\vec e_b) - h_{ab}(n,\vec m-\vec e_a+\vec e_b) \\ & -h_{ab}(n,\vec m+\vec e_a-\vec e_b) + h_{ab}(n,\vec m-\vec e_a-\vec e_b) \\ & - h_{aa}(n,\vec m+2\vec e_b) + 2h_{aa}(n,\vec m) - h_{aa}(n,\vec m-2\vec e_b)] \\ & -N_a(n,\vec m)\,[2p_{ab}(n,\vec m+\vec e_b) - 2p_{ab}(n,\vec m-\vec e_b)] \} - H(n,\vec m) \Bigg) \end{aligned} \tag{21}$$

and,

$$\begin{aligned} H(n,\vec m) \;=\; & \sum_{a,b=1}^{3} \left[p_{ab}(n,\vec m)^2 - \frac{1}{2} p_{aa}(n,\vec m) p_{bb}(n,\vec m) \right] \\ & + \frac{1}{2} \sum_{a,b,c=1}^{3} \Bigg[\frac{1}{2} \left(h_{ab}(n,\vec m+\vec e_c) - h_{ab}(n,\vec m-\vec e_c)\right)^2 \\ & - \left(h_{ab}(n,\vec m+\vec e_c) - h_{ab}(n,\vec m-\vec e_c)\right)\left(h_{ac}(n,\vec m+\vec e_b) - h_{ac}(n,\vec m-\vec e_b)\right) \\ & - \frac{1}{2} \left(h_{aa}(n,\vec m+\vec e_c) - h_{aa}(n,\vec m-\vec e_c)\right)\left(h_{bb}(n,\vec m+\vec e_c) - h_{bb}(n,\vec m-\vec e_c)\right) \\ & + 2\left(h_{cc}(n,\vec m+\vec e_a) - h_{cc}(n,\vec m-\vec e_a)\right)\left(h_{ab}(n,\vec m+\vec e_b) - h_{ab}(n,\vec m-\vec e_b)\right) \\ & + h_{cc}(n,\vec m)\,[h_{ab}(n,\vec m+\vec e_a+\vec e_b) - h_{ab}(n,\vec m+\vec e_a-\vec e_b) \\ & + h_{ab}(n,\vec m-\vec e_a-\vec e_b) - h_{ab}(n,\vec m-\vec e_a+\vec e_b)] \Bigg], \end{aligned} \tag{22}$$

where we have assumed that the background metric is Minkowski and we have chosen the zeroth order lapse equal to unity and we have dropped the (1) superscript from the first order lapse and shift. We have also chosen a centered prescription for spatial derivatives, with the following conventions, i.e. $\phi(i)_{,x} = \phi(i+1) - \phi(i-1)$ and $\phi(i)_{,xx} = \phi(i+2) + \phi(i-2) - 2\phi(i)$ and similarly for higher derivatives. This choice of prescription is needed for two reasons: i) it ensures that "summation by parts" (ignoring boundaries) is satisfied, which is important when taking variations of the action; ii) it makes the successive application of two first derivatives the second derivative, etc. This is important when proving mimetism.

The Lagrangian is the generator of the canonical transformation that materializes evolution from instant n to instant $n+1$. Specifically, we will introduce the canonically conjugate momenta as we discussed in the previous section,

$$P^h_{ab}(n+1,\vec m) = p_{ab}(n,\vec m) \tag{23}$$

$$P^p_{ab}(n+1,\vec m) = 0 \tag{24}$$

$$P^N(n+1,m) = 0 \tag{25}$$

$$P^N_a(n+1,m) = 0 \tag{26}$$

$$\begin{aligned} & P^h_{ab}(n,\vec m) = p_{ab}(n,\vec m) + N(n,\vec m-\vec e_a-\vec e_b) - N(n,\vec m-\vec e_a+\vec e_b) \\ & -N(n,\vec m+\vec e_a-\vec e_b) + N(n,\vec m+\vec e_a+\vec e_b) \\ & -\delta_{ab} \sum_{c=1}^{3} \left(N(n,\vec m-2\vec e_c) - 2N(n,\vec m) + N(n,\vec m+2\vec e_c)\right) \\ & +\frac{1}{2} \sum_{c=1}^{3} \left[\left(h_{ab}(n,\vec m) - h_{ab}(n,\vec m-2\vec e_c)\right) - \left(h_{ab}(n,\vec m+2\vec e_c) - h_{ab}(n,\vec m)\right) \right] \\ & -\frac{1}{2} \sum_{c=1}^{3} \left[\left(h_{ac}(n,\vec m+\vec e_b-\vec e_c) - h_{ac}(n,\vec m-\vec e_b-\vec e_c)\right) - \left(h_{ac}(n,\vec m+\vec e_b+\vec e_c) - h_{ac}(n,\vec m-\vec e_b+\vec e_c)\right) \right] \end{aligned} \tag{27}$$

$$-\frac{1}{2}\sum_{c=1}^{3}\left[(h_{bc}(n,\vec{m}+\vec{e}_a-\vec{e}_c)-h_{bc}(n,\vec{m}-\vec{e}_a-\vec{e}_c))-(h_{bc}(n,\vec{m}+\vec{e}_a+\vec{e}_c)-h_{bc}(n,\vec{m}-\vec{e}_a+\vec{e}_c))\right]$$
$$-\frac{1}{2}\delta_{ab}\sum_{c,d=1}^{3}\left[(h_{dd}(n,\vec{m})-h_{dd}(n,\vec{m}-2\vec{e}_c))-(h_{dd}(n,\vec{m}+2\vec{e}_c)-h_{dd}(n,\vec{m}))\right]$$
$$+\frac{\delta_{ab}}{2}\sum_{c,d=1}^{3}\left[h_{cd}(n,\vec{m}+\vec{e}_d-\vec{e}_c)-h_{cd}(n,\vec{m}-\vec{e}_d-\vec{e}_c)-h_{cd}(n,\vec{m}+\vec{e}_c+\vec{e}_d)+h_{cd}(n,\vec{m}+\vec{e}_c-\vec{e}_d)\right]$$
$$+\frac{1}{2}\sum_{c=1}^{3}\left[(h_{cc}(n,\vec{m}+\vec{e}_a-\vec{e}_b)-h_{cc}(n,\vec{m}-\vec{e}_a-\vec{e}_b))-(h_{cc}(n,\vec{m}+\vec{e}_a+\vec{e}_b)-h_{cc}(n,\vec{m}-\vec{e}_a+\vec{e}_b))\right],$$
$$P^p_{ab}(n,\vec{m})=-\left(h_{ab}(n+1,\vec{m})-h_{ab}(n,\vec{m})\right)+2p_{ab}(n,\vec{m}) \tag{28}$$
$$-\sum_{c=1}^{3}p_{cc}(n,\vec{m})\delta_{ab}+N_a(n,\vec{m}-\vec{e}_b)-N_a(n,\vec{m}+\vec{e}_b)+N_b(n,\vec{m}-\vec{e}_a)-N_b(n,\vec{m}+\vec{e}_a)$$
$$P^{N_a}(n,m)=\sum_{b=1}^{3}\left[2p_{ab}(n,\vec{m}+\vec{e}_b)-2p_{ab}(n,\vec{m}-\vec{e}_b)\right] \tag{29}$$
$$P^N(n,m)=\sum_{a,b=1}^{3}\left[h_{ab}(n,\vec{m}+\vec{e}_a+\vec{e}_b)-h_{ab}(n,\vec{m}-\vec{e}_a+\vec{e}_b)\right. \tag{30}$$
$$\left.-h_{ab}(n,\vec{m}+\vec{e}_a-\vec{e}_b)+h_{ab}(n,\vec{m}-\vec{e}_a-\vec{e}_b)-h_{aa}(n,\vec{m}+2\vec{e}_b)+2h_{aa}(n,\vec{m})-h_{aa}(n,\vec{m}-2\vec{e}_b)\right]$$

The system has four primary constraints (24-27). Preserving these constraints in time implies, via (28-30) that the linearized Hamiltonian and momentum constraints are satisfied,

$$C_a = 2\sum_{b=1}^{3}\left[P^h_{ab}(n,\vec{m}+\vec{e}_b)-P^h_{ab}(n,\vec{m}-\vec{e}_b)\right]=0 \tag{31}$$
$$C = \sum_{a,b=1}^{3}\left[h_{ab}(n,\vec{m}+\vec{e}_a+\vec{e}_b)-h_{ab}(n,\vec{m}-\vec{e}_a+\vec{e}_b)-h_{ab}(n,\vec{m}+\vec{e}_a-\vec{e}_b)+h_{ab}(n,\vec{m}-\vec{e}_a-\vec{e}_b)\right.$$
$$\left.-h_{aa}(n,\vec{m}+2\vec{e}_b)+2h_{aa}(n,\vec{m})-h_{aa}(n,\vec{m}-2\vec{e}_b)\right]=0 \tag{32}$$

Constraints (24,27) can be imposed strongly, the second constraint determines the variable p_{ab}. This eliminates the variable p_{ab} and its canonically conjugate momenta from the theory.

We now combine (23) and (27) to get the evolution equation for P^h,

$$P^h_{ab}(n+1,\vec{m})=P^h_{ab}(n,\vec{m}) \tag{33}$$
$$-N(n,\vec{m}-\vec{e}_a-\vec{e}_b)+N(n,\vec{m}-\vec{e}_a+\vec{e}_b)+N(n,\vec{m}+\vec{e}_a-\vec{e}_b)-N(n,\vec{m}+\vec{e}_a+\vec{e}_b)$$
$$+\delta_{ab}\sum_{c=1}^{3}\left(N(n,\vec{m}-2\vec{e}_c)-2N(n,\vec{m})+N(n,\vec{m}+2\vec{e}_c)\right)$$
$$-\frac{1}{2}\sum_{c=1}^{3}\left[(h_{ab}(n,\vec{m})-h_{ab}(n,\vec{m}-2\vec{e}_c))-(h_{ab}(n,\vec{m}+2\vec{e}_c)-h_{ab}(n,\vec{m}))\right]$$
$$+\frac{1}{2}\sum_{c=1}^{3}\left[(h_{ac}(n,\vec{m}+\vec{e}_b-\vec{e}_c)-h_{ac}(n,\vec{m}-\vec{e}_b-\vec{e}_c))-(h_{ac}(n,\vec{m}+\vec{e}_b+\vec{e}_c)-h_{ac}(n,\vec{m}-\vec{e}_b+\vec{e}_c))\right]$$
$$+\frac{1}{2}\sum_{c=1}^{3}\left[(h_{bc}(n,\vec{m}+\vec{e}_a-\vec{e}_c)-h_{bc}(n,\vec{m}-\vec{e}_a-\vec{e}_c))-(h_{bc}(n,\vec{m}+\vec{e}_a+\vec{e}_c)-h_{bc}(n,\vec{m}-\vec{e}_a+\vec{e}_c))\right]$$
$$+\frac{1}{2}\delta_{ab}\sum_{c,d=1}^{3}\left[(h_{dd}(n,\vec{m})-h_{dd}(n,\vec{m}-2\vec{e}_c))-(h_{dd}(n,\vec{m}+2\vec{e}_c)-h_{dd}(n,\vec{m}))\right]$$
$$-\frac{\delta_{ab}}{2}\sum_{c,d=1}^{3}\left[(h_{cd}(n,\vec{m}+\vec{e}_d-\vec{e}_c)-h_{cd}(n,\vec{m}-\vec{e}_d-\vec{e}_c))-(h_{cd}(n,\vec{m}+\vec{e}_c+\vec{e}_d)-h_{cd}(n,\vec{m}+\vec{e}_c-\vec{e}_d))\right]$$

$$-\frac{1}{2}\sum_{c=1}^{3}\left[(h_{cc}(n,\vec{m}+\vec{e}_a-\vec{e}_b)-h_{cc}(n,\vec{m}-\vec{e}_a-\vec{e}_b))-(h_{cc}(n,\vec{m}+\vec{e}_a+\vec{e}_b)-h_{cc}(n,\vec{m}-\vec{e}_a+\vec{e}_b))\right],$$

and from (28) we get the evolution equation for h,

$$\begin{aligned}
h_{ab}(n+1,\vec{m}) &= h_{ab}(n,\vec{m})+2P^h_{ab}(n,\vec{m})-\delta_{ab}\sum_{f=1}^{3}P^h_{ff}(n,\vec{m}) \\
&+N_a(n,\vec{m}-\vec{e}_b)-N_a(n,\vec{m}+\vec{e}_b)+N_b(n,\vec{m}-\vec{e}_a)-N_b(n,\vec{m}+\vec{e}_a) \\
&-2N(n,\vec{m}-\vec{e}_a-\vec{e}_b)+2N(n,\vec{m}-\vec{e}_a+\vec{e}_b)+2N(n,\vec{m}+\vec{e}_a-\vec{e}_b)-2N(n,\vec{m}+\vec{e}_a+\vec{e}_b) \\
&-\sum_{c=1}^{3}\left[2h_{ab}(n,\vec{m})-h_{ab}(n,\vec{m}-2\vec{e}_c)-h_{ab}(n,\vec{m}+2\vec{e}_c)\right] \\
&+\sum_{c=1}^{3}\left[h_{ac}(n,\vec{m}+\vec{e}_b-\vec{e}_c)-h_{ac}(n,\vec{m}-\vec{e}_b-\vec{e}_c)-h_{ac}(n,\vec{m}+\vec{e}_b+\vec{e}_c)+h_{ac}(n,\vec{m}-\vec{e}_b+\vec{e}_c)\right] \\
&+\sum_{c=1}^{3}\left[h_{bc}(n,\vec{m}+\vec{e}_a-\vec{e}_c)-h_{bc}(n,\vec{m}-\vec{e}_a-\vec{e}_c)-h_{bc}(n,\vec{m}+\vec{e}_a+\vec{e}_c)+h_{bc}(n,\vec{m}-\vec{e}_a+\vec{e}_c)\right] \\
&+\frac{1}{2}\delta_{ab}\sum_{c,d=1}^{3}\left[2h_{dd}(n,\vec{m})-h_{dd}(n,\vec{m}-2\vec{e}_c)-h_{dd}(n,\vec{m}+2\vec{e}_c)\right] \\
&-\frac{1}{2}\delta_{ab}\sum_{c,d=1}^{3}\left[h_{cd}(n,\vec{m}+\vec{e}_d-\vec{e}_c)-h_{cd}(n,\vec{m}-\vec{e}_d-\vec{e}_c)-h_{cd}(n,\vec{m}+\vec{e}_c+\vec{e}_d)+h_{cd}(n,\vec{m}+\vec{e}_c-\vec{e}_d)\right] \\
&-\sum_{c=1}^{3}\left[h_{cc}(n,\vec{m}+\vec{e}_a-\vec{e}_b)-h_{cc}(n,\vec{m}-\vec{e}_a-\vec{e}_b)-h_{cc}(n,\vec{m}+\vec{e}_a+\vec{e}_b)+h_{cc}(n,\vec{m}-\vec{e}_a+\vec{e}_b)\right],
\end{aligned} \tag{34}$$

A first point to be noted is that the evolution equations have resulted in an explicit evolution scheme. This is usually not the case, it is a particularity of the linearized theory that the evolution is explicit. It should be noted that the evolution equations obtained are just a straightforward discretization of the evolution equations one would obtain in the continuum by working out the variations of the continuum action.

We have checked, using a computer algebra code, that the evolution equations (34,33) exactly preserve the constraints (31,32), or more precisely that,

$$C_a(n+1,m) = C_a(n,m), \tag{35}$$

$$C(n+1,m) = C(n,m)+\sum_{a=1}^{3}\left[C_a(n,m+\vec{e}_a)-C_a(n,m-\vec{e}_a)\right]. \tag{36}$$

This result was expected since we used differentiation operators that ensure that mixed discrete spatial derivatives commute, and that one can integrate by parts (more precisely "sum by parts"), and that is all that is needed in a linear theory on a Minkowski background to show that the constraints are preserved upon evolution. It is interesting to compare this result with that of Meier [3]. He finds a mimetic discretization of linearized general relativity around Minkowski spacetime, but using staggered grids. This is a natural approach, for instance, in electromagnetism and Yang–Mills theory (and it is the one we will take in the next section where we deal with gravity with the Ashtekar variables).

It would be interesting to generalize these results to the case of linearization around a static background. In that case it is not obvious that the formulation would result automatically mimetic. In fact, the failure of the Leibnitz rule at a discrete level implies that it will be difficult to find a mimetic formulation since the equations now will have non-constant coefficients and one will need Leibnitz' rule to show conservation. Our formalism will yield a consistent formulation, but it is possible that it will require determining the Lagrange multipliers.

C. Stability

We have discretized the time derivatives without centering them (that is, we have used a stencil that is first order accurate only). The reason for this is that the canonical theory is much cleaner with only two levels in time involved in the derivatives. It is possible to use derivatives that are second order accurate in time and use our construction by rewriting the theory in terms new variables in such a way that the resulting theory has derivatives that are first order accurate, but we will not do this here.

The spatial derivatives, on the other hand, were centered (this was required in order to have summation by parts). The resulting scheme is therefore "forward in time centered in space", a recipe that is not stable, for instance, for the advection or the wave equation. We therefore would like to check if our scheme is stable. To simplify things, we will consider (34,33) and make the following assumptions: the metric and extrinsic curvatures are diagonal and only depend on the coordinates t, x, the lapse is unity and the shift is zero. The resulting equations therefore are,

$$P^h_{11}(n+1,\vec{m}) = P^h_{11}(n,\vec{m}) - \frac{1}{2}\sum_{c=2}^{3}\left[2h_{cc}(n,\vec{m}) - h_{cc}(n,\vec{m}-2\vec{e}_1) - h_{cc}(n,\vec{m}+2\vec{e}_1)\right], \tag{37}$$

$$P^h_{22}(n+1,\vec{m}) = P^h_{22}(n,\vec{m}) + \frac{1}{2}\left[2h_{33}(n,\vec{m}) - h_{33}(n,\vec{m}-2\vec{e}_1) - h_{33}(n,\vec{m}+2\vec{e}_1)\right] \tag{38}$$

$$P^h_{33}(n+1,\vec{m}) = P^h_{33}(n,\vec{m}) + \frac{1}{2}\left[2h_{22}(n,\vec{m}) - h_{22}(n,\vec{m}-2\vec{e}_1) - h_{22}(n,\vec{m}+2\vec{e}_1)\right] \tag{39}$$

$$\begin{aligned} h_{11}(n+1,\vec{m}) \;=\;& h_{11}(n,\vec{m}) + 2P^h_{11}(n,\vec{m}) - \sum_{f=1}^{3} P^h_{ff}(n,\vec{m}) \\ &-\frac{1}{2}\sum_{d=2}^{3}\left[2h_{dd}(n,\vec{m}) - h_{dd}(n,\vec{m}-2\vec{e}_1) - h_{dd}(n,\vec{m}+2\vec{e}_1)\right] \end{aligned} \tag{40}$$

$$\begin{aligned} h_{22}(n+1,\vec{m}) \;=\;& h_{22}(n,\vec{m}) + 2P^h_{22}(n,\vec{m}) - \sum_{f=1}^{3} P^h_{ff}(n,\vec{m}) \\ &-\frac{1}{2}\left[2h_{22}(n,\vec{m}) - h_{22}(n,\vec{m}-2\vec{e}_1) - h_{22}(n,\vec{m}+2\vec{e}_1)\right] \\ &+\frac{1}{2}\left[2h_{33}(n,\vec{m}) - h_{33}(n,\vec{m}-2\vec{e}_1) - h_{33}(n,\vec{m}+2\vec{e}_2)\right] \end{aligned} \tag{41}$$

$$\begin{aligned} h_{33}(n+1,\vec{m}) \;=\;& h_{33}(n,\vec{m}) + 2P^h_{33}(n,\vec{m}) - \sum_{f=1}^{3} P^h_{ff}(n,\vec{m}) \\ &-\frac{1}{2}\left[2h_{33}(n,\vec{m}) - h_{33}(n,\vec{m}-2\vec{e}_1) - h_{33}(n,\vec{m}+2\vec{e}_1)\right] \\ &+\frac{1}{2}\left[2h_{22}(n,\vec{m}) - h_{22}(n,\vec{m}-2\vec{e}_1) - h_{22}(n,\vec{m}+2\vec{e}_2)\right] \end{aligned} \tag{42}$$

As a test case, we concentrate on a subfamily of solutions of the equations, in which $h_{11} = P_{11} = 0$ and $h_{22} = -h_{33}$ and $P_{22} = -P_{33}$. In that case, the equations reduce to,

$$P^h_{22}(n+1,\vec{m}) = P^h_{22}(n,\vec{m}) - \frac{1}{2}\left[2h_{22}(n,\vec{m}) - h_{22}(n,\vec{m}-2\vec{e}_1) - h_{22}(n,\vec{m}+2\vec{e}_1)\right], \tag{43}$$

$$h_{22}(n+1,\vec{m}) = h_{22}(n,\vec{m}) + 2P^h_{22}(n,\vec{m}) - \left[2h_{22}(n,\vec{m}) - h_{22}(n,\vec{m}-2\vec{e}_1) - h_{22}(n,\vec{m}+2\vec{e}_1)\right]. \tag{44}$$

We have performed a Von Neumann analysis of this system and confirmed that the scheme is stable provided the Courant factor is less than one. So at least for this particular subcase the scheme is stable. A more complete analysis is needed to guarantee stability in general.

IV. LINEARIZED GENERAL RELATIVITY IN TERMS OF ASHTEKAR VARIABLES

A. Continuum formulation

We will now apply the technique we outlined in the previous section to general relativity linearized around Minkowski space using the Ashtekar formulation. The formulation of linearized gravity with the new variables was first discussed by Ashtekar and Lee [9]. The discussion presented in that paper required the use of complex variables if one was to describe general relativity with metrics with a Lorentzian signature (alternatively, one could consider real variables, but then the theory described the Euclidean signature sector.) Developments that have taken place in the field since the publication of that paper that allow to consider the Lorentzian sector using real variables [10, 11], but we will see that the discretized theory is more problematic in this case and we will not discuss it in detail in this paper.

The Ashtekar canonical variables consist of a set of triads with density weight 1, E^{ai} and a (complex) $SO(3)$ connection A_{ai}. In this notation $a, b, \ldots$ are spatial vector indices and $i, j, \ldots$ range from 1 to 3. Following Ashtekar and Lee we omit using tildes to denote density weights since in this context they do not play an important role. To linearize the theory around Minkowski, we choose a fixed background ($E^{ai} = E_0^{ai}, A_{ai} = 0$) in the phase space and consider fluctuations around it. In Cartesian coordinates, $E_0^{ai} = \delta^{ai}$. The triad is therefore given by,

$$E^{ai} = \delta^{ai} + e^{ai}, \tag{45}$$

and therefore the background metric has components $q^{ab} = \delta^{ab}$ and its determinant is unity and therefore density weights are all trivial. We will denote by A_{ai} the fluctuations of the connection. The Poisson bracket of the canonical variables is $\{e^{ai}(x), A_{bj}(y)\} = i\delta^a_b\delta^i_j\delta(x-y)$.

The Ashtekar formulation has, in addition to the usual diffeomorphism and Hamiltonian constraints of the metric canonical formulation of general relativity, a set of additional constraints that make the formulation invariant under triad rotations. The additional constraints take the form of a Gauss law, which linearized will read,

$$\mathcal{G}^i_L = \partial_a e^{ai} + \epsilon^{ija} A_{aj} = 0, \tag{46}$$

where from now on the subscript L means we have kept the minimum required number of terms in the perturbative expansion. In spite of the second term, one can check that if one computes the Poisson bracket of two Gauss laws, they commute, that is, they form an Abelian algebra. The internal symmetry group of the linearized theory is therefore $U(1)^3$.

Ignoring boundary terms, the (super)Hamiltonian for general relativity can be written as,

$$H = \int d^3x \left[N E^a_i E^b_j \left(F^k_{ab}\epsilon_{ijk} - \frac{(\beta^2-\sigma)}{\beta^2}(\Gamma^i_a - \sigma\beta A^i_a)(\Gamma^j_b - \sigma\beta A^j_b)\right) + N^a E^a_i F^i_{ab}, \right] \tag{47}$$

and in the full theory it vanishes identically. The parameter β is called the Immirzi parameter and the parameter σ is equal to $+1$ for the Euclidean case and -1 for the Lorentzian signature. Classically, different values of the Immirzi parameter correspond to different representations of the same theory. The quantities Γ^i_a are the spin connections compatible with the triads, defined by,

$$\partial_{[a}\bar{E}^i_{b]} + \epsilon^i_{jk}\Gamma^j_{[a}\bar{E}^k_{b]} = 0, \tag{48}$$

where the $\bar{E}$'s are the triads (without density weight), related to the Ashtekar variables by $E^a_i = \det(\bar{E})\bar{E}^a_i$, or equivalently, $E^a_i = \bar{E}^j_b \bar{E}^k_c \epsilon^{abc}\epsilon_{ijk}$. Indices are lowered and raised with the flat Euclidean metric. One can obtain an explicit expression for the spin connection in terms of the triads,

$$\Gamma^i_c = \epsilon^{ab}_c(\partial_a e^i_b - \partial_a \mathrm{Tr}(e)\delta^i_b). \tag{49}$$

To study it in the linearized theory, we need to choose a lapse and a shift. The natural choice is to use as zeroth order lapse and shifts the ones that would preserve the spatial background metric explicitly time-independent. This corresponds to a lapse $N = 1$ and a shift $N^a = 0$. So we will write $N_L = 1 + \nu$ and $N^a_L = \nu^a$, and these will become Lagrange multipliers in the linearized theory. The super-Hamiltonian then separates into two pieces, one that acts as a Hamiltonian and another piece that is given by the Lagrange multipliers times constraints of the linearized theory. These constraints are,

$$C^L_a = -i f^b_{ab} = 0, \tag{50}$$

$$C^L = -i\epsilon^{ab}_c f^c_{ab} = 0, \tag{51}$$

where $f^i_{ab} = 2\partial_{[a}A^i_{b]}$ is the linearized field strength, and the first one is the linearized momentum constraint and the latter the linearized Hamiltonian constraint. The non-vanishing Hamiltonian for the linearized theory is given by,

$$
\begin{aligned}
H_L \;=\; & \int d^3x \left(2\epsilon^{ib}_k f^k_{ab} e^a_i + (A^a_a A^b_b - A^b_a A^a_b)\right. \\
& \left. -\frac{\beta^2-\sigma}{\beta^2}\left[(\Gamma^a_a - \sigma\beta A^a_a)(\Gamma^b_a - \sigma\beta A^b_a)) - (\Gamma^b_a - \sigma\beta A^b_a)(\Gamma^a_b - \sigma\beta A^a_b)\right]\right)
\end{aligned}
\tag{52}
$$

B. Discretizing the full theory on the lattice

In this section we review some results of reference [4, 5] where we discretized general relativity on the lattice. In the next section we will particularize these results to the case of linearized general relativity. We start by considering an action for general relativity written in terms of Ashtekar's variables (see for instance [12] and the book by Ashtekar [13] page 47),

$$L = \int E^{ai} F^i_{a0} - H \tag{53}$$

where N and N^a are the lapse and shift and H the super-Hamiltonian (47). We will particularize to the Euclidean case $\sigma = 1$ and choose the Immirzi parameter $\beta = 1$ which correspond to the original form of Ashtekar's variables, for simplicity (see section VI for more details). From now on we will not assume Einstein's summation convention and present the summations explicitly, since many expressions would otherwise be confusing. The Lagrangian can be discretized as follows,

$$
\begin{aligned}
L(n,n+1) \;=\; & -\frac{1}{4}\sum_v \mathrm{Tr}\left[\sum_a E^a_{n,v}(h^{a0}_{n,v} - h^{0a}_{n,v}) - \sum_{a,b} K^{ab}_{n,v}(h^{ab}_{n,v} - h^{ba}_{n,v})\right. \\
& \left. + \sum_a \alpha_{a,n,v}\left(h^a_{n,v}\left(h^a_{n,v}\right)^\dagger - 1\right) + \beta_{n,v}\left(h^0_{n,v}\left(h^0_{n,v}\right)^\dagger - 1\right)\right]
\end{aligned}
\tag{54}
$$

where h^a_{n+1} represents an holonomy along the a direction at instant $n+1$, h^0_n represents the "vertical" (time-like) holonomy. The holonomy associated with a plaquette in the $\alpha\beta$ ($\alpha \neq \beta$) plane ($\alpha, \beta = 0\ldots 3$) is

$$h^{\alpha\beta}_{n,v} \equiv h^\alpha_{n,v} h^\beta_{n,v+e_\alpha}(h^\alpha_{n,v+e_\beta})^\dagger (h^\beta_{n,v})^\dagger\,, \tag{55}$$

and

$$K^{ab}_{n,v} \equiv \frac{1}{2}\left[(E^a_{n,v}E^b_{n,v} - E^b_{n,v}E^a_{n,v})N_{n,v} + N^a_{n,v}E^b_{n,v} - N^b_{n,v}E^a_{n,v}\right]\,. \tag{56}$$

We will assume that the holonomies are matrices of the form $h = \sum_I h^I T^I$ where $T^0 = I$ and $T^a = -i\sigma^a$ where σ^a are the Pauli matrices. The indices n, v represent a label for "time" n and a spatial label for the vertices of the lattice v. The elementary unit vectors along the spatial directions are labeled as e_a, so for instance $n + e_1$ labels the nearest neighbor to n along the e_1 direction. The unit vector in the timelike direction is e_0 and we chose $h^\alpha_{n,v+e_0} \equiv h^\alpha_{n+1,v}$. The quantities $E^a_{n,v}$ are elements of the algebra of $su(2)$ and α and β are Lagrange multipliers, the last two terms of the Lagrangian enforcing the condition that the holonomies are elements of $SU(2)$. We use the usual conventions of lattice gauge theories in which one has oriented links and the natural variables are the holonomies in a given orientation and based at a given vertex. If we need to traverse back, as in the case of closed loops one then considers the adjoint of the holonomy based at the vertex one is ending at.

The discretization of the field tensor is based on,

$$F^i_{ab} \to -\frac{1}{4}\mathrm{Tr}\left[(h^{ab}_{n,v} - h^{ba}_{n,v})T^i\right]\,. \tag{57}$$

Instead of working out the equations of motion for this action, we will, in the next section, particularize it to the linearized case and work out the relevant equations of motion, which is equivalent to working the equations first and then linearizing if appropriate perturbative orders are kept.

C. The linearized theory on the lattice

We now proceed to linearize the action. We start with the holonomies. The explicit form of the linearized holonomy is

$$h_v^\alpha = 1 + \sum_i \phi_v^{\alpha i} T^i \tag{58}$$

where we have dropped the subscript n we used in the last section to indicate the time level in order to make the notation more compact (but we will make it explicit when things are evaluated at $n+1$). In this equation $\phi_v^{\alpha i} T^i$ is an element of the algebra that can be viewed as a "phase" (it corresponds to the logarithm of the path-ordered exponential of the connection along the direction α). The holonomy of a plaquette in the plane $\alpha\beta$ is (neglecting higher order terms),

$$h_v^{\alpha\beta} = h_v^\alpha h_{v+e_\alpha}^\beta (h_{v+e_\beta}^\alpha)^\dagger (h_v^\beta)^\dagger = (1 - \sum_i \Phi_{2v}^{\alpha\beta\, ii}) 1 + \sum_i (\Phi_{1v}^{\alpha\beta i} + \Phi_{2v}^{\alpha\beta i}) T^i. \tag{59}$$

The first order contribution is,

$$\Phi_{1v}^{\alpha\beta k} \equiv +\phi_v^{\alpha k} - \phi_v^{\beta k} - \phi_{v+\hat{e}_\beta}^{\alpha k} + \phi_{v+\hat{e}_\alpha}^{\beta k}\,, \tag{60}$$

and the second order contribution is,

$$\Phi_{2v}^{\alpha\beta\, ij} \equiv -\phi_v^{\alpha i}\phi_v^{\beta j} + \phi_v^{\alpha i}\phi_{v+\hat{e}_\alpha}^{\beta j} + \phi_{v+\hat{e}_\beta}^{\alpha i}\phi_v^{\beta j} - \phi_{v+\hat{e}_\beta}^{\alpha j}\phi_{v+\hat{e}_\alpha}^{\beta i} - \phi_v^{\alpha i}\phi_{v+\hat{e}_\beta}^{\alpha j} - \phi_v^{\beta j}\phi_{v+\hat{e}_\alpha}^{\beta i} \tag{61}$$

$$\Phi_{2v}^{\alpha\beta\, k} \equiv \sum_{ij} \epsilon^{ijk} \Phi_{2v}^{\alpha\beta\, ij}\,. \tag{62}$$

We now linearize the expression for K defined in the previous section, by noting that to first order,

$$e_v^a = \sum_i (\delta^{ai} + e_v^{ai}) T^i \tag{63}$$

and ignoring higher order terms we get that

$$K_v^{ab} = \sum_i (\epsilon^{abi} + K_{1,v}^{abi}) T^i\,. \tag{64}$$

with

$$K_1^{abk} \equiv \epsilon^{abk}\nu_v + \frac{1}{2}(\nu_v^a \delta^{bk} - \nu_v^b \delta^{ak}) + \sum_i (e_v^{ai}\epsilon^{ibk} - e_v^{bi}\epsilon^{iak}). \tag{65}$$

We now consider the first term in the discretized Lagrangian (54). Substituting the expression for the holonomy around a plaquette (59) we get the following identity, valid up to second order,

$$-\frac{1}{4}\mathrm{Tr}\left[\sum_a E_{n,v}^a (h_{n,v}^{a0} - h_{n,v}^{0a})\right] = \sum_a \Phi_{1v}^{a0a} + \sum_a \Phi_{2v}^{a0a} + \sum_{ak} e_v^{ak}\Phi_{1v}^{a0k} \tag{66}$$

and we note that when one considers the sum over all vertices, the first term of the right hand side yields a total derivative with respect to time that can be ignored in the Lagrangian.

For the second term in (54) we use (59) and (64), getting the following identity, valid up to second order,

$$\frac{1}{4}\mathrm{Tr}[\sum_{ab} K_{n,v}^{ab}(h_{n,v}^{ab} - h_{n,v}^{ba})] = -\sum_{abk}(\epsilon^{abk}\Phi_{1v}^{abk} + \epsilon^{abk}\Phi_{2v}^{abk} + K_{1v}^{abk}\Phi_{1v}^{abk})\,. \tag{67}$$

When one considers the sum over all vertices the first term on the right hand side of this expression vanishes. The resulting Lagrangian therefore can be written as,

$$\begin{aligned} L = & -\sum_v \Bigg\{ \sum_{ai} e_v^{ai}(\phi_{n+1,v}^{ai} - \phi_{n,v}^{ai}) + \sum_{ijk}\epsilon_{ijk}(\phi_v^{ij}\phi_{n+1,v}^{ik} + \Phi_{2v}^{ijk} + \nu_v \Phi_{1v}^{ijk}) \\ & + \sum_{aijk} 2e_v^{ai}\epsilon^{ijk}\Phi_{1v}^{ajk} + \sum_{ab}\nu_v^a \Phi_{1v}^{abb} + \sum_{ij}(\phi_v^{0i} - \phi_{v+\hat{e}_j}^{0i})e_v^{ji} \\ & + \sum_{ijk}\epsilon_{ijk}\phi_v^{0i}(\phi_{n+1,v}^{kj} + \phi_{v+\hat{e}_j}^{0k} + \phi_v^{jk}) - \sum_{ijk}\phi_{v+\hat{e}_j}^{0i}\epsilon_{ijk}(\phi_v^{jk} + \phi_{n+1,v}^{jk}) \Bigg\} \end{aligned} \tag{68}$$

Now that we have an explicit expression for the Lagrangian we can proceed to identify the various terms. The theory has the following Lagrange multipliers: ϕ_v^{0i} the "vertical component of the phase" (which plays a role analogous to the time component of the vector potential in Maxwell theory) and the linearized lapse and shift. These quantities multiply times the constraints of the linearized theory. Explicitly, the momentum and Hamiltonian constraint read,

$$C_v^a = \sum_b \Phi_{1v}^{abb}, \tag{69}$$

$$C_v = \sum_{ijk} \epsilon_{ijk} \Phi_{1v}^{ijk}. \tag{70}$$

In order to get Gauss' law, we first take the variation of the Lagrangian with respect to the Lagrange multiplier ϕ_v^{0i} to get,

$$G_v^i \equiv \sum_a (\xi_{n+1,v}^{ai} + \xi_{n+1,v}^{\overline{a}i}) = 0 \tag{71}$$

where

$$\xi_{n+1,v}^{ai} \equiv +\delta^{ai} + e_{n,v}^{ai} + \sum_k \epsilon_{aik} \left(-\phi_{n,v}^{ak} - \phi_{n,v+\hat{e}_a}^{0k} - \phi_{n,v}^{0k} + \phi_{n+1,v}^{ak}\right) \tag{72}$$

$$\xi_{n+1,v}^{\overline{a}i} \equiv -\delta^{ai} - e_{n,v-\hat{e}_a}^{ai} + \sum_k \epsilon_{aik} \left(\phi_{n,v-\hat{e}_a}^{ak} + \phi_{n,v}^{0k} + \phi_{n,v-\hat{e}_a}^{0k} + \phi_{n+1,v-\hat{e}_a}^{ak}\right) \tag{73}$$

At the moment this does not appear to be a true constraint since it involves variables at instant n and at instant $n+1$. To see that it actually is a constraint, we will call $\xi_{n+1,v}^{ai}$ the component in the direction $\hat{e}_a$ of a quantity that we will think of as an "electric field" (in the sense that it is the quantity that satisfies the usual form of Gauss law) and we will call $\xi_{n+1,v}^{\overline{a}i}$ the component in the direction $-\hat{e}_a$, both at point $(n+1, v)$. To make this more transparent, we need to see how they transform under gauge transformations. To leading order the field $e_{n,v}^a$ is $e_{0,n,v}^a = \sum_i \delta^{ai} T^i$. We then define

$$\breve{e}_{n,v}^{\,a} = e_{n,v}^a + \frac{1}{4}[\, h_{n,v}^{0a}\, e_{0,n,v}^a\, (h_{n,v}^{0a})^\dagger - h_{n,v}^{a0}\, e_{0,n,v}^a\, (h_{n,v}^{a0})^\dagger\,] = e_{n,v}^a + \sum_{jk} \epsilon_{ajk} \Phi_{1v}^{0ak}\, T^j\,, \tag{74}$$

with the second equality valid up to second order. By inspection one sees that the field $\breve{e}_{n,v}^{\,a}$ is an element of the algebra that transforms like an electric field at (n, v) under gauge transformation. One can also show the following identities, valid to first order,

$$\xi_{n+1,v}^a \equiv \sum_i \xi_{n+1,v}^{ai}\, T^i = (h_{n,v}^0)^\dagger\, \breve{e}_{n,v}^{\,a}\, h_{n,v}^0 \tag{75}$$

$$\xi_{n+1,v}^{\overline{a}} \equiv \sum_i \xi_{n+1,v}^{\overline{a}i}\, T^i = -(h_{n+1,v-\hat{e}_a}^a)^\dagger\, \xi_{n+1,v-\hat{e}_a}^a\, h_{n+1,v-\hat{e}_a}^a \tag{76}$$

from which one immediately sees that the quantities we identified as components of the electric field have the appropriate transformation properties under gauge transformations.

Therefore we can identify (71) as the usual intuitive expression of Gauss' law stating that field lines cannot emanate from a point in vacuum.

We now turn our attention to the equations of motion. Given the Lagrangian (68) we work out the equations of motion from the canonical transformation. We start by computing the canonical conjugate momentum to e,

$$P_{n+1,v}^{(e)ak} \equiv \frac{\partial L(n, n+1)}{\partial e_{n+1,v}^{ak}} = 0\,, \tag{77}$$

$$P_v^{(e)ak} \equiv -\frac{\partial L(n, n+1)}{\partial e_v^{ak}} = -\Phi_{1v}^{a0k} + 2\sum_{ij} \epsilon^{kij} \Phi_{1v}^{aij} = 0. \tag{78}$$

Therefore the dynamics of $P^{(e)}$ is trivial. However, the last equation can be viewed as an evolution equation for ϕ through (60),

$$\phi_{n+1,v}^{ak} = \phi_v^{ak} - \phi_v^{0k} + \phi_{v+\hat{e}_a}^{0k} - 2\sum_{ij} \epsilon^{kij} \Phi_{1v}^{aij}\,. \tag{79}$$

Notice that by adding over indices a belonging to a given plaquette equation (79), one effectively gets an evolution equation for all the horizontal ϕ's in the plaquette. This is due to the fact that the vertical contributions in (60) will cancel out in pairs when adding through the plaquette. Explicitly,

$$\Phi^{abk}_{1,n+1,v} = \Phi^{abk}_{1v} - 2\sum_{ij}\epsilon^{kij}(\Phi^{aij}_{1v} + \Phi^{bij}_{1v+\hat{e}_a} - \Phi^{aij}_{1v+\hat{e}_b} - \Phi^{bij}_{1v}) \equiv \Phi^{abk}_{1v} - 2\sum_{ij}\epsilon^{kij}\sum_{d\in P_{ab}}\Phi^{dij}_{1v_d} \tag{80}$$

where in the last term v_d is the vertex in which the link d originates and P_{ab} is the plaquette spanned by a and b.

We now consider the momentum canonically conjugate to ϕ. We start by computing the canonical conjugate momentum at instant $n+1$

$$P^{(\phi)ak}_{n+1,v} \equiv \frac{\partial L(n,n+1)}{\partial \phi^{ak}_{n+1,v}} = -e^{ak}_v + \sum_i \epsilon^{aki}(\phi^{ai}_v + \phi^{0i}_v + \phi^{0i}_{v+\hat{e}_a})\,. \tag{81}$$

The momentum can be written in terms of the electric field in an expression that sees parallels the usual relation between the electric field and the canonical momentum in the lattice ξ^{ak} as,

$$P^{(\phi)ak}_{n+1,v} = \delta_{ak} - \xi^{ak}_{n+1,v} + \sum_i \epsilon_{aki}\,\phi^{ai}_{n+1,v} \tag{82}$$

and in terms of it, Gauss' law (71) can be written as,

$$G^k_{n+1,v} = -\sum_a \left(P^{(\phi)ak}_{n+1,v} - P^{(\phi)ak}_{n+1,v-\hat{e}_a} - \sum_i \epsilon^{aki}(\phi^{ai}_{n+1,v} + \phi^{ai}_{n+1,v-\hat{e}_a})\right)\,. \tag{83}$$

This final expression for Gauss' law is a genuine constraint, in the sense that all variables are expressed at the same instant of time.

We now compute the momentum conjugate to ϕ at instant n,

$$\begin{aligned} P^{(\phi)ab}_v \;=\; & -\frac{\partial L(n,n+1)}{\partial \phi^{ab}_v} = -e^{ab}_v + \nu^a_v - \nu^a_{v-\hat{e}_b} + \delta_{ab}\sum_i\left(-\nu^i_v + \nu^i_{v-\hat{e}_i}\right) \\ & +2\left(-\phi^{aa}_{v-\hat{e}_b} + \phi^{aa}_{v+\hat{e}_b} + \phi^{ba}_v - \phi^{ba}_{v+\hat{e}_a} - \phi^{ba}_{v-\hat{e}_b} - \phi^{ba}_{v+\hat{e}_a-\hat{e}_b}\right) \\ & +2\delta_{ab}\sum_i\left(\phi^{ai}_{v-\hat{e}_i} - \phi^{ai}_{v+\hat{e}_i} - \phi^{ii}_v + \phi^{ii}_{v+\hat{e}_a} + \phi^{ii}_{v-\hat{e}_i} + \phi^{ii}_{v+\hat{e}_a-\hat{e}_i}\right) \\ & +\sum_i \epsilon_{abi}\left(-2\nu_v + 2\nu_{v-\hat{e}_i} + \phi^{0i}_v - \phi^{0i}_{v+\hat{e}_a} - 2e^{aa}_v + 2e^{aa}_{v-\hat{e}_i} - 2e^{bi}_v + 2e^{bi}_{v-\hat{e}_b}\right. \\ & \left.-2e^{ii}_v + 2e^{ii}_{v-\hat{e}_i} + \phi^{ai}_{n+1,v}\right) + 2\delta_{ab}\sum_{i,j}\epsilon_{aij}\left(e^{ai}_v - e^{ai}_{v-\hat{e}_j}\right)\,. \end{aligned} \tag{84}$$

One still needs to replace the expressions for the e's and for the ϕ's evaluated at instant $n+1$. The resulting substitutions lead to lengthy expressions that are not particularly illuminating, and will not be needed in what follows, so we will not display them here. We point out however, that the resulting scheme is not an explicit one for the $P^{(\phi)}$'s. Since these variables do not arise in the constraints, we do not need this evolution equation to show mimetism.

One now needs to show that the evolution is mimetic, that is, it preserves the discrete constraints (69, 70, 83). Using the evolution equation (80) one gets that,

$$C^a_{n+1,v} \;=\; C^a_v + C_v - C_{v+\hat{e}_a}\,, \tag{85}$$

$$C_{n+1,v} \;=\; C_v + 4\sum_a (C^a_{v+\hat{e}_a} - C^a_v). \tag{86}$$

To study the time evolution of Gauss' law one needs equations (79,81,84) and one gets that

$$G^k_{n+1,v} = G^k_{n,v}\,. \tag{87}$$

As in the previous section, we have checked these identities using computer algebra.

V. DISCUSSION AND CONCLUSIONS

The consistent discretization scheme is such that it yields a set of discrete equations for the evolution equations and the constraints of general relativity that is compatible, that is, all can be solved simultaneously. It does so at the price of determining the Lagrange multipliers (the lapse and the shift). In the linearized case we have shown that one can discretize the theory in such a way that the Lagrange multipliers are not determined and nevertheless the theory is consistent.

When one discretizes a theory there is always an ambiguity in how to proceed. Among the ambiguities we have the dependence on how one chooses to represent the derivatives. What we have found is that in the linearized case one can choose certain derivative operators for which the Lagrange multipliers are not determined. It should be noted that the consistent discretization scheme would work even if one did not choose the derivatives this way, but the Lagrange multipliers will be determined in order to have a consistent set of equations. This is true both in the case of the traditional variables and also in the Ashtekar variables. In the latter case there is an additional element that is the presence of an extra constraint: the Gauss law. We have also chosen a specific way of discretizing the theory in such a way that Gauss' law is implemented exactly in the discrete theory (this is standard in Yang–Mills theory on the lattice, and implies that the discrete formulation is gauge invariant, and also that these discretizations are mimetic, though this is rarely emphasized in the Yang–Mills literature). In the case of the traditional variables, one can also associate mimetism with gauge invariance. The action we chose to work with is invariant under linearized coordinate transformations of the form $h'_{\mu\nu} = h_{\mu\nu} + \xi_{(\mu,\nu)}$. The discrete action, if one chooses a derivative operator such that the second derivatives coincide with the derivative of a first derivative and satisfies summation by parts, is invariant under a discrete version of the above symmetry. This symmetry is generated canonically by the discrete constraints. This explains in a geometrically nice way why mimetism was possible in the linearized case.

In the case of Lorentzian general relativity written in terms of Ashtekar's variables, the presence of the terms $(\Gamma^i_a - \sigma\beta A^i_a)$ in the Hamiltonian make it more difficult to discretize the action in such a way that the Gauss law is preserved exactly. This is because the Γ^i_a's have to be written in terms of the triads and the resulting expressions are not easy to discretize on the lattice preserving the internal symmetry (unlike the A^i_a's which are readily discretized by considering a parallel transport operator along the elementary links). It may be possible using dual lattices to discretize such terms in an invariant way, but this will require further study. There is no problem applying our discretization technique to this case directly, but what will happen is that the internal symmetry will be broken, the Lagrange multipliers associated with the Gauss law will be determined and the resulting discretization will not be mimetic in the traditional sense of the word. It is clear that further work is needed before this kind of discretizations will be useful numerically.

In the full nonlinear case either with the traditional or the new variables, the constraints involve both coordinates and momenta and therefore the application of our technique will determine the Lagrange multipliers and will therefore not furnish a mimetic discretization in the traditional sense. The resulting discrete theory is consistent, but it does so at the expense of determining the Lagrange multipliers. Based on what we learned from the linearized case, we can conclude that the only remaining possibility for a formulation that is mimetic in the traditional sense would be to implement the symmetries implied by the constraints exactly in the lattice. Since the symmetries implied by the diffeomorphism constraint are broken by the introduction of the lattice it appears unlikely that such a formulation would ever be found.

The conclusion we can draw from this is that for the case of full nonlinear general relativity, the closest one can come to a formulation that preserves the constraints under evolution is the proposal of consistent discretizations we have introduced. Such proposal is not mimetic in the traditional sense in that it imposes the constraints by determining the Lagrange multipliers. This proposal has many new aspects that are currently in investigation. It has been successfully applied in cosmological examples and is now being studied in detail for the Gowdy space-times. If it works for this example, it is likely that it could be applied successfully in general, but this obviously requires further study.

Something to be noticed is that it is not clear that the formulations we presented are going to be useful numerically. In particular, the fact that they are not based on manifestly hyperbolic equations. We have presented a first step towards showing stability of the scheme in a particular situation, but a fuller analysis should be carried out to determine if the scheme is stable in general. Numerical relativity codes also use more sophisticated time stepping techniques than the one we use. It is clear, however, that our method can accommodate more elaborate discretizations of the time derivatives and the calculations in this paper could be repeated in that case. Another interesting point would be to attempt to apply the techniques in this paper to the several manifestly hyperbolic formulations of the Einstein equations that have been proposed in the last few years. Unfortunately, few of them have been worked out in the context of an action principle, but this difficulty could presumably be remedied. This would also allow to study within our framework manifolds with boundaries.

Another element of interest is the impact of the choice of the derivative operators on the construction of consistent discrete theories. The consistent technique will work no matter what derivative operators are chosen. But here we have learnt that one can choose them in such a way that the linearized theory is automatically mimetic. We would like to argue that the level of accuracy with which the consistent discretization enforces the constraints is improved when one chooses a formulation that is mimetic at the linear level, at least for weak fields. The argument is simple. In the consistent discretization scheme the constraints that are enforced exactly have the form $\phi(q_n, p_{n+1}) = 0$. The constraints one would like to see enforced are of the form $\phi(q_n, p_n) = 0$. Starting from the former, and using the equations of motion one has that $\phi(q_n, p_n + O(p^2)) = 0$ where the terms that correct p_n are of order p^2 (or q^2 or mixed but quadratic). This is true if the theory is mimetic in the linearized level. Otherwise one would have $\phi(q_n, p_n + O(p)) = 0$. Therefore choosing a discretization that is mimetic at the linearized level, at least for weak fields, implies that the constraints are tracked more accurately in the full non-linear theory when one discretizes consistently.

Summarizing, we have shown that the consistent discretization scheme we have introduced recently, when applied to general relativity discretized around Minkowski spacetime yields a formulation that is mimetic. That is, a formulation in which the discrete constraints are exactly preserved upon discrete evolution. We have also argued that for the full nonlinear case, the use of consistent discretizations appears as a possibility to yield a formulation that is close to the intention of mimetic formulations, although only approximately.

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